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Hölder gradient regularity for the inhomogeneous normalized $p(x)$-Laplace equation

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ABSTRACT

We prove the local gradient Hölder regularity of viscosity solutions to the inhomogeneous normalized $p(x)$-Laplace equation

$$-\Delta_{p(x)}^N u = f(x),$$

where p is Lipschitz continuous, $\inf p > 1$, and f is continuous and bounded.

1. Introduction

We study the *inhomogeneous normalized $p(x)$-Laplace equation*

$$-\Delta_{p(x)}^N u = f(x) \quad \text{in } B_1, \tag{1.1}$$

where

$$-\Delta_{p(x)}^N u := -\Delta u - (p(x) - 2)\frac{\langle D^2uDu, Du\rangle}{|Du|^2}$$

is the *normalized $p(x)$-Laplacian*, $p : B_1 \to \mathbb{R}$ is Lipschitz continuous, $1 < p_{\min} := \inf_{B_1} p \leq \sup_{B_1} p =: p_{\max}$ and $f \in C(B_1)$ is bounded. Our main result is that viscosity solutions to (1.1) are locally $C^{1,\alpha}$-regular.

Normalized equations have attracted a significant amount of interest during the last 15 years. Their study is partially motivated by their connection to game theory. Roughly speaking, the value function of certain stochastic tug-of-war games converges uniformly up to a subsequence to a viscosity solution of a normalized equation as the step-size of the game approaches zero [32,30,31,9,11]. In particular, a game with

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space-dependent probabilities leads to the normalized $p(x)$-Laplace equation [3] and games with running pay-offs lead to inhomogeneous equations [33]. In addition to game theory, normalized equations have been studied for example in the context of image processing [16,18].

The variable $p(x)$ in (1.1) has an effect that may not be immediately obvious: If we formally multiply the equation by $|Du|^{p(x)-2}$ and rewrite it in a divergence form, then a logarithm term appears and we arrive at the expression

$$-\operatorname{div}(|Du|^{p(x)-2} Du) + |Du|^{p(x)-2} \log(|Du|) Du \cdot Dp = |Du|^{p(x)-2} f(x). \tag{1.2}$$

For $f \equiv 0$, this is the so called *strong $p(x)$-Laplace equation* introduced by Adamowicz and Hästö [1,2] in connection with mappings of finite distortion. In the homogeneous case viscosity solutions to (1.1) actually coincide with weak solutions of (1.2) [35], yielding the $C^{1,\alpha}$-regularity of viscosity solutions as a consequence of a result by Zhang and Zhou [38].

In the present paper our objective is to prove $C^{1,\alpha}$-regularity of solutions to (1.1) directly using viscosity methods. The Hölder regularity of solutions already follows from existing general results, see [28,29,12,13]. More recently, Imbert and Silvestre [24] proved the gradient Hölder regularity of solutions to the elliptic equation

$$|Du|^{\gamma} F(D^2u) = f,$$

where $\gamma > 0$ and Imbert, Jin and Silvestre [25,22] obtained a similar result for the parabolic equation

$$\partial_t u = |Du|^{\gamma} \Delta_p^N u,$$

where $p > 1$, $\gamma > -1$. Furthermore, Attouchi and Parviainen [4] proved the $C^{1,\alpha}$-regularity of solutions to the inhomogeneous equation $\partial_t u - \Delta_p^N u = f(x,t)$. Our proof of Hölder gradient regularity for solutions of (1.1) is in particular inspired by the papers [25] and [4].

We point out that recently Fang and Zhang [19] proved the $C^{1,\alpha}$-regularity of solutions to the parabolic normalized $p(x,t)$-Laplace equation

$$\partial_t u = \Delta_{p(x,t)}^N u, \tag{1.3}$$

where $p \in C^1_{\mathrm{loc}}$. The equation (1.3) naturally includes (1.1) if $f \equiv 0$. However, in this article we consider the inhomogeneous case and only suppose that p is Lipschitz continuous. More precisely, we have the following theorem.

Theorem 1.1. *Suppose that p is Lipschitz continuous in B_1, $p_{\min} > 1$ and $f \in C(B_1)$ is bounded. Let u be a viscosity solution to*

$$-\Delta_{p(x)}^N u = f(x) \quad \text{in } B_1.$$

Then there is $\alpha(N, p_{\min}, p_{\max}, p_L) \in (0,1)$ such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le C(N, p_{\min}, p_{\max}, p_L, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}),$$

where p_L is the Lipschitz constant of p.

The proof of Theorem 1.1 is based on suitable uniform $C^{1,\alpha}$-regularity estimates for solutions of the regularized equation

$$-\Delta v-(p_\varepsilon(x)-2)\frac{\langle D^2vDv,Dv\rangle}{|Dv|^2+\varepsilon^2}=g(x), \tag{1.4}$$

where it is assumed that g is continuous and p_ε is smooth. In particular, we show estimates that are independent of ε and only depend on N, $\sup p$, $\inf p$, $\|Dp_\varepsilon\|_{L^\infty}$ and $\|g\|_{L^\infty}$. To prove such estimates, we first derive estimates for the perturbed homogeneous equation

$$-\Delta v-(p_\varepsilon(x)-2)\frac{\langle D^2v(Dv+q),Dv+q\rangle}{|Dv+q|^2+\varepsilon^2}=0, \tag{1.5}$$

where $q\in\mathbb{R}^N$. Roughly speaking, $C^{1,\alpha}$-estimates for solutions of (1.5) are based on "improvement of oscillation" which is obtained by differentiating the equation and observing that a function depending on the gradient of the solution is a supersolution to a linear equation. The uniform $C^{1,\alpha}$-estimates for solutions of (1.5) then yield uniform estimates for the inhomogeneous equation (1.4) by an adaption of the arguments in [24,4].

With the *a priori* regularity estimates at hand, the plan is to let $\varepsilon\to 0$ and show that the estimates pass on to solutions of (1.1). A problem is caused by the fact that, to the best of our knowledge, uniqueness of solutions to (1.1) is an open problem for variable $p(x)$ and even for constant p if f is allowed to change signs. To deal with this, we fix a solution $u_0\in C(\overline{B}_1)$ to (1.1) and consider the Dirichlet problem

$$-\Delta^N_{p(x)}u=f(x)-u_0(x)-u\quad\text{in } B_1 \tag{1.6}$$

with boundary data $u=u_0$ on ∂B_1. For this equation the comparison principle holds and thus u_0 is the unique solution. We then consider the approximate problem

$$-\Delta u_\varepsilon-(p_\varepsilon(x)-2)\frac{\langle D^2u_\varepsilon Du_\varepsilon,Du_\varepsilon\rangle}{|Du_\varepsilon|^2+\varepsilon^2}=f_\varepsilon(x)-u_{0,\varepsilon}(x)-u_\varepsilon \tag{1.7}$$

with boundary data $u_\varepsilon=u_0$ on ∂B_1 and where $p_\varepsilon,f_\varepsilon,u_{0,\varepsilon}\in C^\infty(B_1)$ are such that $p\to p_\varepsilon$, $f_\varepsilon\to f$ and $u_{0,\varepsilon}\to u_0$ uniformly in B_1 and $\|Dp_\varepsilon\|_{L^\infty(B_1)}\le\|Dp\|_{L^\infty(B_1)}$. As the equation (1.7) is uniformly elliptic quasilinear equation with smooth coefficients, the solution u_ε exists in the classical sense by standard theory. Since u_ε also solves (1.4) with $g(x)=f_\varepsilon(x)-u_{0,\varepsilon}(x)-u_\varepsilon(x)$, it satisfies the uniform $C^{1,\alpha}$-regularity estimate. We then let $\varepsilon\to 0$ and use stability and comparison principles to show that u_0 inherits the regularity estimate.

For other related results, see for example the works of Attouchi, Parviainen and Ruosteenoja [5] on the normalized p-Poisson problem $-\Delta^N_p u=f$, Attouchi and Ruosteenoja [6–8] on the equation $-|Du|^\gamma\Delta^N_p u=f$ and its parabolic version, De Filippis [15] on the double phase problem $(|Du|^q+a(x)|Du|^s)F(D^2u)=f(x)$ and Fang and Zhang [20] on the parabolic double phase problem $\partial_t u=(|Du|^q+a(x,t)|Du|^s)\Delta^N_p u$. We also mention the paper by Bronzi, Pimentel, Rampasso and Teixeira [10] where they consider fully nonlinear variable exponent equations of the type $|Du|^{\theta(x)}F(D^2u)=0$.

The paper is organized as follows: Section 2 is dedicated to preliminaries, Sections 3 and 4 contain $C^{1,\alpha}$-regularity estimates for equations (1.5) and (1.7), and Section 5 contains the proof of Theorem (1.1). Finally, the Appendix contains an uniform Lipschitz estimate for the equations studied in this paper and a comparison principle for equation (1.6).

2. Preliminaries

2.1. Notation

We denote by $B_R \subset \mathbb{R}^N$ an open ball of radius $R > 0$ that is centered at the origin in the N-dimensional Euclidean space, $N \geq 1$. The set of symmetric $N \times N$ matrices is denoted by S^N. For $X, Y \in S^N$, we write $X \leq Y$ if $X - Y$ is negative semidefinite. We also denote the smallest eigenvalue of X by $\lambda_{\min}(X)$ and the largest by $\lambda_{\max}(X)$ and set

$$\|X\| := \sup_{\xi \in B_1} |X\xi| = \sup \{|\lambda| : \lambda \text{ is an eigenvalue of } X\}.$$

We use the notation $C(a_1, \ldots, a_k)$ to denote a constant C that may change from line to line but depends only on $a_1, \ldots, a_k$. For convenience we often use $C(\hat{p})$ to mean that the constant may depend on $p_{\min}$, $p_{\max}$ and the Lipschitz constant p_L of p.

For $\alpha \in (0, 1)$, we denote by $C^\alpha(B_R)$ the set of all functions $u : B_R \to \mathbb{R}$ with finite Hölder norm

$$\|u\|_{C^\alpha(B_R)} := \|u\|_{L^\infty(B_R)} + [u]_{C^\alpha(B_R)}, \quad \text{where } [u]_{C^\alpha(B_R)} := \sup_{x,y \in B_R} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Similarly, we denote by $C^{1,\alpha}(B_R)$ the set of all functions for which the norm

$$\|u\|_{C^{1,\alpha}(B_R)} := \|u\|_{C^\alpha(B_R)} + \|Du\|_{C^\alpha(B_R)}$$

is finite.

2.2. Viscosity solutions

Viscosity solutions are defined using smooth test functions that touch the solution from above or below. If $u, \varphi : \mathbb{R}^N \to \mathbb{R}$ and $x \in \mathbb{R}^N$ are such that $\varphi(x) = u(x)$ and $\varphi(y) < u(y)$ for $y \neq x_0$, then we say that φ *touches* u *from below at* x_0.

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity supersolution* to

$$-\Delta^N_{p(x)} u \geq f(x, u) \quad \text{in } \Omega$$

if the following holds: Whenever $\varphi \in C^2(\Omega)$ touches u from below at $x \in \Omega$ and $D\varphi(x) \neq 0$, we have

$$-\Delta\varphi(x) - (p(x) - 2) \frac{\langle D^2\varphi(x) D\varphi(x), D\varphi(x)\rangle}{|D\varphi(x)|^2} \geq f(x, u(x))$$

and if $D\varphi(x) = 0$, then

$$-\Delta\varphi(x) - (p(x) - 2) \langle D^2\varphi(x)\eta, \eta\rangle \geq f(x, u(x)) \quad \text{for some } \eta \in \overline{B}_1.$$

Analogously, a lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution if the above inequalities hold reversed whenever φ touches u from above. Finally, we say that u is a *viscosity solution* if it is both viscosity sub- and supersolution.

Remark. The special treatment of the vanishing gradient in Definition 2.1 is needed because of the singularity of the equation. Definition 2.1 is essentially a relaxed version of the standard definition in [14] which is based on the so called semicontinuous envelopes. In the standard definition one would require that if φ touches a viscosity supersolution u from below at x, then

$$\begin{cases} -\Delta_{p(x)}^N \varphi(x) \geq f(x,u(x)) & \text{if } D\varphi(x) \neq 0, \\ -\Delta\varphi(x) - (p(x)-2)\lambda_{\min}(D^2\varphi(x)) \geq f(x,u(x)) & \text{if } D\varphi(x) = 0 \text{ and } p(x) \geq 2, \\ -\Delta\varphi(x) - (p(x)-2)\lambda_{\max}(D^2\varphi(x)) \geq f(x,u(x)) & \text{if } D\varphi(x) = 0 \text{ and } p(x) < 2. \end{cases}$$

Clearly, if u is a viscosity supersolution in this sense, then it is also a viscosity supersolution in the sense of Definition 2.1.

3. Hölder gradient estimates for the regularized homogeneous equation

In this section we prove $C^{1,\alpha}$-regularity estimates for solutions to the equation

$$-\Delta u - (p(x)-2)\frac{\langle D^2u(Du+q), Du+q\rangle}{|Du+q|^2+\varepsilon^2} = 0 \quad \text{in } B_1, \tag{3.1}$$

where $p : B_1 \to \mathbb{R}$ is Lipschitz, $p_{\min} > 1$, $\varepsilon > 0$ and $q \in \mathbb{R}^N$. Our objective is to obtain estimates that are independent of q and ε. Observe that (3.1) is a uniformly elliptic quasilinear equation with smooth coefficients. Viscosity solutions to (3.1) can be defined in the standard way and they are smooth if p is smooth.

Proposition 3.1. *Suppose that p is smooth. Let u be a viscosity solution to (3.1) in B_1. Then $u \in C^\infty(B_1)$.*

It follows from classical theory that the corresponding Dirichlet problem admits a smooth solution (see [21, Theorems 15.18 and 13.6] and the Schauder estimates [21, Theorem 6.17]). The viscosity solution u coincides with the smooth solution by a comparison principle [26, Theorem 3].

3.1. Improvement of oscillation

Our regularity estimates for solutions of (3.1) are based on improvement of oscillation. We first prove such a result for the linear equation

$$-\operatorname{tr}(G(x)D^2u) = f \quad \text{in } B_1, \tag{3.2}$$

where $f \in C^1(B_1)$ is bounded, $G(x) \in S^N$ and there are constants $0 < \lambda < \Lambda < \infty$ such that the eigenvalues of $G(x)$ are in $[\lambda, \Lambda]$ for all $x \in B_1$. The result is based on the following rescaled version of the weak Harnack inequality found in [13, Theorem 4.8]. Such Harnack estimates for non-divergence form equations go back to at least Krylov and Safonov [28,29].

Lemma 3.2 *(Weak Harnack inequality). Let $u \geq 0$ be a continuous viscosity supersolution to (3.2) in B_1. Then there are positive constants $C(\lambda, \Lambda, N)$ and $q(\lambda, \Lambda, N)$ such that for any $\tau < \frac{1}{4\sqrt{N}}$ we have*

$$\tau^{-\frac{N}{q}} \left(\int_{B_\tau} |u|^q \, dx \right)^{1/q} \leq C \left(\inf_{B_{2\tau}} u + \tau \left(\int_{B_{4\sqrt{N}\tau}} |f|^N \, dx \right)^{1/N} \right). \tag{3.3}$$

Proof. Suppose that $\tau < \frac{1}{4\sqrt{N}}$ and set $S := 8\tau$. Define the function $v : B_{\sqrt{N}/2} \to \mathbb{R}$ by

$$v(x) := u(Sx)$$

and set

$$\tilde{G}(x) := G(Sx) \quad \text{and} \quad \tilde{f}(x) := S^2 f(Sx).$$

Then, if $\varphi \in C^2$ touches v from below at $x \in B_{\sqrt{N}/2}$, the function $\phi(x) := \varphi(x/S)$ touches u from below at Sx. Therefore

$$-\mathrm{tr}(G(Sx)D^2\phi(Sx)) \geq f(Sx).$$

Since $D^2\phi(Sx) = S^{-2}D^2\varphi(x)$, this implies that

$$-\mathrm{tr}(G(Sx)D^2\varphi(x)) \geq S^2 f(Sx).$$

Thus v is a viscosity supersolution to

$$-\mathrm{tr}(\tilde{G}(x)D^2 v) \geq \tilde{f}(x) \quad \text{in } B_{\sqrt{N}/2}.$$

We denote by Q_R a cube with side-length $R/2$. Since $Q_1 \subset B_{\sqrt{N}/2}$, it follows from [13, Theorem 4.8] that there are $q(\lambda, \Lambda, N)$ and $C(\lambda, \Lambda, N)$ such that

$$\begin{aligned}\left(\int_{B_{1/8}} |v|^q \, dx\right)^{1/q} \leq \left(\int_{Q_{1/4}} |v|^q \, dx\right)^{1/q} &\leq C\left(\inf_{Q_{1/2}} v + \left(\int_{Q_1} |\tilde{f}|^N \, dx\right)^{1/N}\right) \\ &\leq C\left(\inf_{B_{1/4}} v + \left(\int_{B_{\sqrt{N}/2}} |\tilde{f}|^N \, dx\right)^{1/N}\right).\end{aligned}$$

By the change of variables formula we have

$$\int_{B_{1/8}} |v|^q \, dx = \int_{B_{1/8}} |u(Sx)|^q \, dx = S^{-N} \int_{B_{S/8}} |u(x)|^q \, dx$$

and

$$\int_{B_{\sqrt{N}/2}} |\tilde{f}|^N \, dx = S^{2N} \int_{B_{\sqrt{N}/2}} |f(Sx)|^N \, dx = S^N \int_{B_{S\sqrt{N}/2}} |f(x)|^N \, dx.$$

Recalling that $S = 8\tau$, we get

$$8^{-\frac{N}{q}}\tau^{-\frac{N}{q}} \left(\int_{B_\tau} |u(x)|^q \, dx\right)^{1/q} \leq C\left(\inf_{B_{2\tau}} u + 8\tau \left(\int_{B_{S\sqrt{N}/2}} |f(x)|^N \, dx\right)^{1/N}\right).$$

Absorbing $8^{\frac{N}{q}}$ into the constant, we obtain the claim. □

Lemma 3.3 *(Improvement of oscillation for the linear equation). Let $u \geq 0$ be a continuous viscosity supersolution to (3.2) in B_1 and $\mu, l > 0$. Then there are positive constants $\tau(\lambda, \Lambda, N, \mu, l, \|f\|_{L^\infty(B_1)})$ and $\theta(\lambda, \Lambda, N, \mu, l)$ such that if*

$$|\{x \in B_\tau : u \geq l\}| > \mu |B_\tau|, \tag{3.4}$$

then we have

$$u \geq \theta \quad \text{in } B_\tau.$$

Proof. By the weak Harnack inequality (Lemma 3.2) there exist constants $C_1(\lambda, \Lambda, N)$ and $q(\lambda, \Lambda, N)$ such that for any $\tau < 1/(4\sqrt{N})$, we have

$$\inf_{B_{2\tau}} u \geq C_1 \tau^{\frac{-N}{q}} \left(\int_{B_\tau} |u|^q \, dx \right)^{1/q} - \tau \left(\int_{B_{4\sqrt{N}\tau}} |f|^N \, dx \right)^{1/N}. \tag{3.5}$$

In particular, this holds for

$$\tau := \min \left(\frac{1}{4\sqrt{N}}, \sqrt{\frac{C_1 |B_1|^{\frac{1}{q} - \frac{1}{N}} \mu^{\frac{1}{q}} l}{2 \cdot 4\sqrt{N}(\|f\|_{L^\infty(B_1)} + 1)}} \right).$$

We continue the estimate (3.5) using the assumption (3.4) and obtain

$$\begin{aligned}
\inf_{B_\tau} u \geq \inf_{B_{2\tau}} u &\geq C_1 \tau^{-\frac{N}{q}} \left(|\{x \in B_\tau : u \geq l\}| \, l^q \right)^{1/q} - \tau \left(\int_{B_{4\sqrt{N}\tau}} |f|^N \, dx \right)^{1/N} \\
&\geq C_1 \tau^{-\frac{N}{q}} \mu^{\frac{1}{q}} |B_\tau|^{\frac{1}{q}} l - \tau \left|B_{4\sqrt{N}\tau}\right|^{\frac{1}{N}} \|f\|_{L^\infty(B_1)} \\
&= C_1 |B_1|^{\frac{1}{q}} \mu^{\frac{1}{q}} l \tau^{-\frac{N}{q}} \tau^{\frac{N}{q}} - 4\sqrt{N} |B_1|^{\frac{1}{N}} \|f\|_{L^\infty(B_1)} \tau^2 \\
&= C_1 |B_1|^{\frac{1}{q}} \mu^{\frac{1}{q}} l - 4\sqrt{N} |B_1|^{\frac{1}{N}} \|f\|_{L^\infty(B_1)} \tau^2. \\
&\geq \frac{1}{2} C_1 |B_1|^{\frac{1}{q}} \mu^{\frac{1}{q}} l, \\
&=: \theta,
\end{aligned}$$

where the last inequality follows from the choice of τ. □

We are now ready to prove an improvement of oscillation for the gradient of a solution to (3.1). We first consider the following lemma, where the improvement is considered towards a fixed direction. We initially also restrict the range of $|q|$.

The idea is to differentiate the equation and observe that a suitable function of Du is a supersolution to the linear equation (3.2). Lemma 3.3 is then applied to obtain information about Du.

Lemma 3.4 *(Improvement of oscillation to direction). Suppose that p is smooth. Let u be a smooth solution to (3.1) in B_1 with $|Du| \leq 1$ and either $q = 0$ or $|q| > 2$. Then for every $0 < l < 1$ and $\mu > 0$ there exist positive constants $\tau(N, \hat{p}, l, \mu) < 1$ and $\gamma(N, \hat{p}, l, \mu) < 1$ such that*

$$|\{x \in B_\tau : Du \cdot d \le l\}| > \mu\, |B_\tau| \quad \textit{implies} \quad Du \cdot d \le \gamma \textit{ in } B_\tau$$

whenever $d \in \partial B_1$.

Proof. To simplify notation, we set

$$A_{ij}(x,\eta) := \delta_{ij} + (p(x)-2)\frac{(\eta_i+q_i)(\eta_j+q_j)}{|\eta+q|^2+\varepsilon^2}.$$

We also denote the functions $\mathcal{A}_{ij} : x \mapsto A_{ij}(x, Du(x))$, $\mathcal{A}_{ij,x_k} : x \mapsto (\partial_{x_k} A_{ij})(x, Du(x))$ and $\mathcal{A}_{ij,\eta_k} : x \mapsto (\partial_{\eta_k} A_{ij})(x, Du(x))$. Then, since u is a smooth solution to (3.1) in B_1, we have in Einstein's summation convention

$$-\mathcal{A}_{ij} u_{ij} = 0 \quad \text{pointwise in } B_1.$$

Differentiating this yields

$$\begin{aligned} 0 = (\mathcal{A}_{ij} u_{ij})_k &= \mathcal{A}_{ij} u_{ijk} + (\mathcal{A}_{ij})_k u_{ij} \\ &= \mathcal{A}_{ij} u_{ijk} + \mathcal{A}_{ij,\eta_m} u_{ij} u_{km} + \mathcal{A}_{ij,x_k} u_{ij} \quad \text{for all } k = 1, \dots N. \end{aligned} \tag{3.6}$$

Multiplying these identities by d_k and summing over k, we obtain

$$\begin{aligned} 0 &= \mathcal{A}_{ij} u_{ijk} d_k + \mathcal{A}_{ij,\eta_m} u_{ij} u_{km} d_k + \mathcal{A}_{ij,x_k} u_{ij} d_k \\ &= \mathcal{A}_{ij} (Du\cdot d - l)_{ij} + \mathcal{A}_{ij,\eta_m} u_{ij} (Du \cdot d - l)_m + \mathcal{A}_{ij,x_k} u_{ij} d_k. \end{aligned} \tag{3.7}$$

Moreover, multiplying (3.6) by $2u_k$ and summing over k, we obtain

$$\begin{aligned} 0 &= 2\mathcal{A}_{ij} u_{ijk} u_k + 2\mathcal{A}_{ij,\eta_m} u_{ij} u_{km} u_k + 2\mathcal{A}_{ij,x_k} u_{ij} u_k \\ &= \mathcal{A}_{ij}(2u_{ijk}u_k + 2u_{kj}u_{ki}) - 2\mathcal{A}_{ij} u_{kj} u_{ki} + 2\mathcal{A}_{ij,\eta_m} u_{ij} u_{km} u_k + 2\mathcal{A}_{ij,x_k} u_{ij} u_k \\ &= \mathcal{A}_{ij} (u_k^2)_{ij} - 2\mathcal{A}_{ij} u_{kj} u_{ki} + \mathcal{A}_{ij,\eta_m} u_{ij} (u_k^2)_m + 2\mathcal{A}_{ij,x_k} u_{ij} u_k \\ &= \mathcal{A}_{ij} (|Du|^2)_{ij} + \mathcal{A}_{ij,\eta_m} u_{ij} (|Du|^2)_m + 2\mathcal{A}_{ij,x_k} u_{ij} u_k - 2\mathcal{A}_{ij} u_{kj} u_{ki}. \end{aligned} \tag{3.8}$$

We will now split the proof into the cases $q = 0$ or $|q| > 2$, and proceed in two steps: First we check that a suitable function of Du is a supersolution to the linear equation (3.3) and then apply Lemma 3.3 to obtain the claim.

Case $q = 0$**, Step 1:** We denote $\Omega_+ := \{x \in B_1 : h(x) > 0\}$, where

$$h := (Du \cdot d - l + \frac{l}{2}\, |Du|^2)^+.$$

If $|Du| \le l/2$, we have

$$Du \cdot d - l + \frac{l}{2}\, |Du|^2 \le -\frac{l}{2} + \frac{l^3}{8} < 0.$$

This implies that $|Du| > l/2$ in Ω_+. Therefore, since $q = 0$, we have in Ω_+

$$
\begin{aligned}
|\mathcal{A}_{ij,\eta_m}| &= |p(x)-2| \left| \frac{\delta_{im}(u_j+q_j)+\delta_{jm}(u_i+q_i)}{|Du+q|^2+\varepsilon^2} - \frac{2(u_m+q_m)(u_i+q_i)(u_j+q_j)}{(|Du+q|^2+\varepsilon^2)^2} \right| \\
&\le 8l^{-1} \|p-2\|_{L^\infty(B_1)}, \qquad (3.9)
\end{aligned}
$$

$$
|\mathcal{A}_{ij,x_k}| = |Dp(x)| \left| \frac{(\eta_i+q_i)(\eta_j+q_j)}{|\eta+q|^2+\varepsilon^2} \right| \le p_L. \qquad (3.10)
$$

Summing up the equations (3.7) and (3.8) multiplied by $2^{-1}l$, we obtain in Ω_+

$$
\begin{aligned}
0 &= \mathcal{A}_{ij}(Du\cdot d - l)_{ij} + \mathcal{A}_{ij,\eta_m} u_{ij} (Du\cdot d - l)_m + \mathcal{A}_{ij,x_k} u_{ij} d_k \\
&\quad + 2^{-1} l \big(\mathcal{A}_{ij}(|Du|^2)_{ij} + \mathcal{A}_{ij,\eta_m} u_{ij} (|Du|^2)_m + 2\mathcal{A}_{ij,x_k} u_{ij} u_k - 2\mathcal{A}_{ij} u_{kj} u_{ki} \big) \\
&= \mathcal{A}_{ij} h_{ij} + \mathcal{A}_{ij,\eta_m} u_{ij} h_m + \mathcal{A}_{ij,x_k} u_{ij} d_k + l\mathcal{A}_{ij,x_k} u_{ij} u_k - l\mathcal{A}_{ij} u_{kj} u_{ki} \\
&\le \mathcal{A}_{ij} h_{ij} + |\mathcal{A}_{ij,\eta_m} u_{ij}|\,|h_m| + |\mathcal{A}_{ij,x_k} u_{ij}|\,|d_k + l u_k| - l\mathcal{A}_{ij} u_{kj} u_{ki}.
\end{aligned}
$$

Since $|Du| \le 1$, we have $|d_k + lu_k|^2 \le 4$ and by uniform ellipticity $\mathcal{A}_{ij} u_{kj} u_{ki} \ge \min(p_{\min}-1,1)\,|u_{ij}|^2$. Therefore, by applying Young's inequality with $\epsilon > 0$, we obtain from the above display

$$
\begin{aligned}
0 &\le \mathcal{A}_{ij} h_{ij} + N^2\epsilon^{-1}(|h_m|^2 + |d_k + lu_k|^2) + \epsilon(|\mathcal{A}_{ij,\eta_m}|^2 + |\mathcal{A}_{ij,x_k}|^2)\,|u_{ij}|^2 - l\mathcal{A}_{ij} u_{kj} u_{ki} \\
&\le \mathcal{A}_{ij} h_{ij} + N^2\epsilon^{-1}(|Dh|^2 + 4) + \epsilon C(N,\hat{p})(l^{-2}+1)\,|u_{ij}|^2 - l\min(p_{\min}-1,1)\,|u_{ij}|^2,
\end{aligned}
$$

where in the second estimate we used (3.9) and (3.10). By taking ϵ small enough, we obtain

$$
0 \le \mathcal{A}_{ij} h_{ij} + C_0(N,\hat{p}) \frac{|Dh|^2 + 1}{l^3} \quad \text{in } \Omega_+. \qquad (3.11)
$$

Next we define

$$
\overline{h} := \frac{1}{\nu}(1 - e^{\nu(h-H)}), \quad \text{where} \quad H := 1 - \frac{l}{2} \quad \text{and} \quad \nu := \frac{C_0}{l^3 \min(p_{\min}-1,1)}. \qquad (3.12)
$$

Then by (3.11) and uniform ellipticity we have in Ω_+

$$
\begin{aligned}
-\mathcal{A}_{ij}\overline{h}_{ij} &= \mathcal{A}_{ij}(h_{ij} e^{\nu(h-H)} + \nu h_i h_j e^{\nu(h-H)}) \\
&\ge e^{\nu(h-H)}\Big(-C_0 \frac{|Dh|^2}{l^3} - \frac{C_0}{l^3} + \nu \min(p_{\min}-1,1)\,|Dh|^2\Big) \\
&\ge -\frac{C_0}{l^3}.
\end{aligned}
$$

Since the minimum of two viscosity supersolutions is still a viscosity supersolution, it follows from the above estimate that $\overline{h}$ is a non-negative viscosity supersolution to

$$
-\mathcal{A}_{ij}\overline{h}_{ij} \ge \frac{-C_0}{l^3} \quad \text{in } B_1. \qquad (3.13)
$$

Case $q = 0$, Step 2: We set $l_0 := \frac{1}{\nu}(1 - e^{\nu(l-1)})$. Then, since $\overline{h}$ solves (3.13), by Lemma 3.3 there are positive constants $\tau(N,p,l,\mu)$ and $\theta(N,p,l,\mu)$ such that

$$
\left|\{x \in B_\tau : \overline{h} \ge l_0\}\right| > \mu\,|B_\tau| \quad \text{implies} \quad \overline{h} \ge \theta \quad \text{in } B_\tau.
$$

If $Du \cdot d \le l$, we have $\overline{h} \ge l_0$ and therefore

$$\left|\left\{x \in B_\tau : \overline{h} \geq l_0\right\}\right| \geq \left|\{x \in B_\tau : Du \cdot d \leq l\}\right| > \mu \left|B_\tau\right|,$$

where the last inequality follows from the assumptions. Consequently, we obtain

$$\overline{h} \geq \theta \quad \text{in } B_\tau.$$

Since $h - H \leq 0$, by convexity we have $H - h \geq \overline{h}$. This together with the above estimate yields

$$1 - 2^{-1}l - (Du \cdot d - l + 2^{-1}l\left|Du\right|^2) \geq \theta \quad \text{in } B_\tau$$

and so

$$Du \cdot d + 2^{-1}l(Du \cdot d)^2 \leq Du \cdot d + 2^{-1}l\left|Du\right|^2 \leq 1 + 2^{-1}l - \theta \quad \text{in } B_\tau.$$

Using the quadratic formula, we thus obtain the desired estimate

$$Du \cdot d \leq \frac{-1 + \sqrt{1 + 2l(1 + 2^{-1}l - \theta)}}{l} = \frac{-1 + \sqrt{(1+l)^2 - 2l\theta}}{l} =: \gamma < 1 \quad \text{in } B_\tau.$$

Case $|q| > 2$: Computing like in (3.9) and (3.10), we obtain this time in B_1

$$\left|\mathcal{A}_{ij,\eta_m}\right| \leq 4\left\|p-2\right\|_{L^\infty(B_1)} \quad \text{and} \quad \left|\mathcal{A}_{ij,x_k}\right| \leq p_L$$

Moreover, this time we set simply

$$h := Du \cdot d - l + 2^{-1}l\left|Du\right|^2.$$

Summing up the identities (3.7) and (3.8) and using Young's inequality similarly as in the case $|q| = 0$, we obtain in B_1

$$\begin{aligned} 0 &\leq \mathcal{A}_{ij}h_{ij} + N^2\epsilon^{-1}(\left|h_m\right|^2 + \left|d_k + lu_k\right|^2) + \epsilon(\left|\mathcal{A}_{ij,\eta_m}\right|^2 + \left|\mathcal{A}_{ij,x_k}\right|^2)\left|u_{ij}\right|^2 - l\mathcal{A}_{ij}u_{kj}u_{ki} \\ &\leq \mathcal{A}_{ij}h_{ij} + N^2\epsilon^{-1}(\left|Dh\right|^2 + 4) + \epsilon C(\hat{p})\left|u_{ij}\right|^2 - lC(\hat{p})\left|u_{ij}\right|^2. \end{aligned}$$

By taking small enough ϵ, we obtain

$$0 \leq \mathcal{A}_{ij}h_{ij} + C_0(N, \hat{p})\frac{\left|Dh\right|^2 + 1}{l} \quad \text{in } B_1.$$

Next we define $\overline{h}$ and H like in (3.12), but set instead $\nu := C_0/(l\min(p_{\min} - 1, 1))$. The rest of the proof then proceeds in the same way as in the case $q = 0$. □

Next we inductively apply the previous lemma to prove the improvement of oscillation.

Theorem 3.5 *(Improvement of oscillation). Suppose that p is smooth. Let u be a smooth solution to (3.1) in B_1 with $|Du| \leq 1$ and either $q = 0$ or $|q| > 2$. Then for every $0 < l < 1$ and $\mu > 0$ there exist positive constants $\tau(N, \hat{p}, l, \mu) < 1$ and $\gamma(N, \hat{p}, l, \mu) < 1$ such that if*

$$\left|\left\{x \in B_{\tau^{i+1}} : Du \cdot d \leq l\gamma^i\right\}\right| > \mu\left|B_{\tau^{i+1}}\right| \quad \text{for all } d \in \partial B_1,\ i = 0, \ldots, k, \tag{3.14}$$

then

$$|Du| \leq \gamma^{i+1} \quad \text{in } B_{\tau^{i+1}} \quad \text{for all } i = 0, \ldots, k. \tag{3.15}$$

Proof. Let $k \geq 0$ be an integer and suppose that (3.14) holds. We proceed by induction.

Initial step: Since (3.14) holds for $i = 0$, by Lemma 3.4 we have $Du \cdot d \leq \gamma$ in B_τ for all $d \in \partial B_1$. This implies (3.15) for $i = 0$.

Induction step: Suppose that $0 < i \leq k$ and that (3.15) holds for $i - 1$. We define

$$v(x) := \tau^{-i}\gamma^{-i}u(\tau^i x).$$

Then v solves

$$-\Delta v - (p(\tau^i x) - 2)\frac{\langle D^2 v(Dv + \gamma^{-i}q), Dv + \gamma^i q\rangle}{|Dv + \gamma^{-i}q|^2 + (\gamma^{-i}\varepsilon)^2} = 0 \quad \text{in } B_1.$$

Moreover, by induction hypothesis $|Dv(x)| = \gamma^{-i}\left|Du(\tau^i x)\right| \leq \gamma^{-i}\gamma^i = 1$ in B_1. Therefore by Lemma 3.4 we have that

$$|\{x \in B_\tau : Dv \cdot d \leq l\}| > \mu\,|B_\tau| \quad \text{implies} \quad Dv \cdot d \leq \gamma \text{ in } B_\tau \tag{3.16}$$

whenever $d \in \partial B_1$. Since

$$|\{x \in B_\tau : Dv \cdot d \leq l\}| > \mu\,|B_\tau| \iff \left|\left\{x \in B_{\tau^{i+1}} : Du \cdot d \leq l\gamma^i\right\}\right| > \mu\,|B_{\tau^{i+1}}|,$$

we have by (3.14) and (3.16) that $Dv \cdot d \leq \gamma$ in B_τ. This implies that $Du \cdot d \leq \gamma^{i+1}$ in $B_{\tau^{i+1}}$. Since $d \in \partial B_1$ was arbitrary, we obtain (3.15) for i. □

3.2. Hölder gradient estimates

In this section we apply the improvement of oscillation to prove $C^{1,\alpha}$-estimates for solutions to (3.1). We need the following regularity result by Savin [34].

Lemma 3.6. *Suppose that p is smooth. Let u be a smooth solution to (3.1) in B_1 with $|Du| \leq 1$ and either $q = 0$ or $|q| > 2$. Then for any $\beta > 0$ there exist positive constants $\eta(N, \hat{p}, \beta)$ and $C(N, \hat{p}, \beta)$ such that if*

$$|u - L| \leq \eta \quad \textit{in } B_1$$

for some affine function L satisfying $1/2 \leq |DL| \leq 1$, then we have

$$|Du(x) - Du(0)| \leq C\,|x|^\beta \quad \textit{for all } x \in B_{1/2}.$$

Proof. Set $v := u - L$. Then v solves

$$-\Delta u - \frac{(p(x) - 2)\left\langle D^2 u(Du + q + DL), Du + q + DL\right\rangle}{|Du + q + DL|^2 + \varepsilon^2} = 0 \quad \text{in } B_1. \tag{3.17}$$

Observe that by the assumption $1/2 \leq |DL| \leq 1$ we have $|Du + q + DL| \geq 1/4$ if $|Du| \leq 1/4$. It therefore follows from [34, Theorem 1.3] (see also [37]) that $\|v\|_{C^{2,\beta}(B_{1/2})} \leq C$ which implies the claim. □

We also use the following simple consequence of Morrey's inequality.

Lemma 3.7. *Let $u : B_1 \to \mathbb{R}$ be a smooth function with $|Du| \leq 1$. For any $\theta > 0$ there are constants $\varepsilon_1(N,\theta), \varepsilon_0(N,\theta) < 1$ such that if the condition*

$$|\{x \in B_1 : |Du - d| > \varepsilon_0\}| \leq \varepsilon_1$$

is satisfied for some $d \in S^{N-1}$, then there is $a \in \mathbb{R}$ such that

$$|u(x) - a - d \cdot x| \leq \theta \text{ for all } x \in B_{1/2}.$$

Proof. By Morrey's inequality (see for example [17, Theorem 4.10])

$$\begin{aligned}
\operatorname*{osc}_{x \in B_{1/2}} (u(x) - d \cdot x) &= \sup_{x,y \in B_{1/2}} |u(x) - d \cdot x - u(y) + d \cdot y| \\
&\leq C(N)\Big(\int_{B_1} |Du - d|^{2N} \, dx \Big)^{\frac{1}{2N}} \\
&\leq C(N)(\varepsilon_1^{\frac{1}{2N}} + \varepsilon_0).
\end{aligned}$$

Therefore, denoting $a := \inf_{x \in B_{1/2}} (u(x) - d \cdot x)$, we have for any $x \in B_{1/2}$

$$|u(x) - a - d \cdot x| \leq \operatorname{osc}_{B_{1/2}} (u(x) - d \cdot x) \leq C(N)(\varepsilon_1^{\frac{1}{2N}} + \varepsilon_0) \leq \theta,$$

where the last inequality follows by taking small enough ε_0 and ε_1. □

We are now ready to prove a Hölder estimate for the gradient of solutions to (3.1). We first restrict the range of $|q|$.

Lemma 3.8. *Suppose that p is smooth. Let u be a smooth solution to (3.1) in B_1 with $|Du| \leq 1$ and either $q = 0$ or $|q| > 2$. Then there exists a constant $\alpha(N, \hat{p}) \in (0, 1)$ such that*

$$\|Du\|_{C^\alpha(B_{1/2})} \leq C(N, \hat{p}).$$

Proof. For $\beta = 1/2$, let $\eta > 0$ be as in Lemma 3.6. For $\theta = \eta/2$, let $\varepsilon_0, \varepsilon_1$ be as in Lemma 3.7. Set

$$l := 1 - \frac{\varepsilon_0^2}{2} \quad \text{and} \quad \mu := \frac{\varepsilon_1}{|B_1|}.$$

For these l and μ, let $\tau, \gamma \in (0, 1)$ be as in Theorem 3.5. Let $k \geq 0$ be the minimum integer such that the condition (3.14) does not hold.

Case $k = \infty$: Theorem 3.5 implies that

$$|Du| \leq \gamma^{i+1} \quad \text{in } B_{\tau^{i+1}} \text{ for all } i \geq 0.$$

Let $x \in B_\tau \setminus \{0\}$. Then $\tau^{i+1} \leq |x| \leq \tau^i$ for some $i \geq 0$ and therefore

$$i \leq \frac{\log |x|}{\log \tau} \leq i + 1.$$

We obtain

$$|Du(x)| \le \gamma^i = \frac{1}{\gamma}\gamma^{i+1} \le \frac{1}{\gamma}\gamma^{\frac{\log|x|}{\log \tau}} = \frac{1}{\gamma}\gamma^{\frac{\log|x|}{\log \gamma}\cdot\frac{\log \gamma}{\log \tau}} =: C\,|x|^{\alpha}\,, \tag{3.18}$$

where $C = 1/\gamma$ and $\alpha = \log\gamma/\log\tau$.

Case $k < \infty$: There is $d \in \partial B_1$ such that

$$\left|\left\{x \in B_{\tau^{k+1}} : Du \cdot d \le l\gamma^k\right\}\right| \le \mu\,|B_{\tau^{k+1}}|\,. \tag{3.19}$$

We set

$$v(x) := \tau^{-k-1}\gamma^{-k}u(\tau^{k+1}x).$$

Then v solves

$$-\Delta v - (p(\tau^{k+1}x) - 2)\frac{\left\langle D^2 v(Dv + \gamma^{-k}q), Dv + \gamma^{-k}q\right\rangle}{|Dv + \gamma^{-k}q|^2 + \gamma^{-2k}\varepsilon^2} = 0 \quad \text{in } B_1$$

and by (3.19) we have

$$\begin{aligned}|\{x \in B_1 : Dv \cdot d \le l\}| &= \left|\left\{x \in B_1 : Du(\tau^{k+1}x)\cdot d \le l\gamma^k\right\}\right| \\ &= \tau^{-N(k+1)}\left|\left\{x \in B_{\tau^{k+1}} : Du(x)\cdot d \le l\gamma^k\right\}\right| \\ &\le \tau^{-N(k+1)}\mu\,|B_{\tau^{k+1}}| = \mu\,|B_1| = \varepsilon_1.\end{aligned} \tag{3.20}$$

Since either $k = 0$ or (3.14) holds for $k-1$, it follows from Theorem 3.5 that $|Du| \le \gamma^k$ in B_{τ^k}. Thus

$$|Dv(x)| = \gamma^{-k}\left|Du(\tau^{k+1}x)\right| \le 1 \quad \text{in } B_1. \tag{3.21}$$

For vectors $\xi, d \in B_1$, it is easy to verify the following fact

$$|\xi - d| > \varepsilon_0 \implies \xi\cdot d \le 1 - \varepsilon_0^2/2 = l.$$

Therefore, in view of (3.20) and (3.21), we obtain

$$|\{x \in B_1 : |Dv - d| > \varepsilon_0\}| \le \varepsilon_1.$$

Thus by Lemma 3.7 there is $a \in \mathbb{R}$ such that

$$|v(x) - a - d\cdot x| \le \theta = \eta/2 \quad \text{for all } x \in B_{1/2}.$$

Consequently, by applying Lemma 3.6 on the function $2v(2^{-1}x)$, we find a positive constant $C(N, \hat{p})$ and $e \in \partial B_1$ such that

$$|Dv(x) - e| \le C\,|x| \quad \text{in } B_{1/4}.$$

Since $|Dv| \le 1$, we have also

$$|Dv(x) - e| \le C\,|x| \quad \text{in } B_1.$$

Recalling the definition of v and taking $\alpha' \in (0,1)$ so small that $\gamma/\tau^{\alpha'} < 1$ we obtain

$$\left|Du(x)-\gamma^k e\right| \leq C\gamma^k \tau^{-k-1}|x| \leq \frac{C}{\tau^{\alpha'}}\left(\frac{\gamma}{\tau^{\alpha'}}\right)^k |x|^{\alpha'} \leq C|x|^{\alpha'} \quad \text{in } B_{\tau^{k+1}}, \tag{3.22}$$

where we absorbed $\tau^{\alpha'}$ into the constant. On the other hand, we have

$$|Du| \leq \gamma^{i+1} \quad \text{in } B_{\tau^{i+1}} \text{ for all } i=0,\ldots,k-1$$

so that, if $\tau^{i+2} \leq |x| \leq \tau^{i+1}$ for some $i \in \{0,\ldots,k-1\}$, it holds that

$$\left|Du(x)-\gamma^k e\right| \leq 2\gamma^{i+1}\frac{|x|^{\alpha'}}{|x|^{\alpha'}} \leq \frac{2}{\tau^{\alpha'}}\left(\frac{\gamma}{\tau^{\alpha'}}\right)^{i+1}|x|^{\alpha'} \leq C|x|^{\alpha'}.$$

Combining this with (3.22) we obtain

$$\left|Du(x)-\gamma^k e\right| \leq C|x|^{\alpha'} \quad \text{in } B_\tau. \tag{3.23}$$

The claim now follows from (3.18) and (3.23) by standard translation arguments. □

Theorem 3.9. *Let u be a bounded viscosity solution to (3.1) in B_1 with $q \in \mathbb{R}^N$. Then*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(N,\hat{p},\|u\|_{L^\infty(B_1)}) \tag{3.24}$$

for some $\alpha(N,\hat{p}) \in (0,1)$.

Proof. Suppose first that p is smooth. Let $\nu_0(N,\hat{p},\|u\|_{L^\infty(B_1)})$ and $C_0(N,\hat{p},\|u\|_{L^\infty(B_1)})$ be as in the Lipschitz estimate (Theorem A.2 in the Appendix) and set

$$M := 2\max(\nu_0, C_0).$$

If $|q| > M$, then by Theorem A.2 we have

$$|Du| \leq C_0 \quad \text{in } B_{1/2}.$$

We set $\tilde{u}(x) := 2u(x/2)/C_0$. Then $|D\tilde{u}| \leq 1$ in B_1 and $\tilde{u}$ solves

$$-\Delta\tilde{u} - (p(x/2)-2)\frac{\left\langle D^2\tilde{u}(D\tilde{u}+q/C_0), D\tilde{u}+q/C_0\right\rangle}{|D\tilde{u}+q/C_0|^2+(\varepsilon/C_0)^2} = 0 \quad \text{in } B_1,$$

where $q/C_0 > 2$. Thus by Theorem 3.8 we have

$$\|D\tilde{u}\|_{C^\alpha(B_{1/2})} \leq C(N,\hat{p}),$$

which implies (3.24) by standard translation arguments.

If $|q| \leq M$, we define

$$w := u - q\cdot x.$$

Then by Theorem A.2 we have

$$|Dw| \leq C(N,\hat{p},\|w\|_{L^\infty(B_1)}) =: C'(N,\hat{p},\|u\|_{L^\infty(B_1)}) \quad \text{in } B_{1/2}.$$

We set $\tilde{w}(x) := 2w(x/2)/C'$. Then $|D\tilde{w}| \leq 1$ and so by Theorem 3.6 we have

$$\|D\tilde{w}\|_{C^{\alpha}(B_{1/2})} \leq C(N, \hat{p}),$$

which again implies (3.24).

Suppose then that p is merely Lipschitz continuous. Take a sequence $p_j \in C^{\infty}(B_1)$ such that $p_j \to p$ uniformly in B_1 and $\|Dp_j\|_{L^{\infty}(B_1)} \leq \|Dp\|_{L^{\infty}(B_1)}$. For $r < 1$, let u_j be a solution to the Dirichlet problem

$$\begin{cases} -\Delta u_j - (p_j(x) - 2)\frac{\langle D^2u(Du_j+q), Du_j+q\rangle}{|Du_j+q|^2+\varepsilon^2} = 0 & \text{in } B_r, \\ u_j = u & \text{on } B_r. \end{cases}$$

As observed in Proposition 3.1, the solution exists and we have $u_j \in C^{\infty}(B_r)$. By comparison principle $\|u_j\|_{L^{\infty}(B_r)} \leq \|u\|_{L^{\infty}(B_1)}$. Then by the first part of the proof we have the estimate

$$\|u_j\|_{C^{1,\beta}(B_{r/2})} \leq C(N, \hat{p}, \|u\|_{L^{\infty}(B_1)}).$$

By [13, Theorem 4.14] the functions u_j are equicontinuous in B_1 and so by the Ascoli-Arzela theorem we have $u_j \to v$ uniformly in B_1 up to a subsequence. Moreover, by the stability principle v is a solution to (3.1) in B_r and thus by comparison principle [27, Theorem 2.6] we have $v \equiv u$. By extracting a further subsequence, we may ensure that also $Du_j \to Du$ uniformly in $B_{r/2}$ and so the estimate $\|Du\|_{C^{1,\beta}(B_{r/2})} \leq C(N, \hat{p}, \|u\|_{L^{\infty}(B_1)})$ follows. □

4. Hölder gradient estimates for the regularized inhomogeneous equation

In this section we consider the inhomogeneous equation

$$-\Delta u - (p(x) - 2)\frac{\langle D^2u(Du+q), Du+q\rangle}{|Du+q|^2 + \varepsilon^2} = f(x) \quad \text{in } B_1, \tag{4.1}$$

where $p : B_1 \to \mathbb{R}$ is Lipschitz continuous, $p_{\min} > 1$, $\varepsilon > 0$, $q \in \mathbb{R}^N$ and $f \in C(B_1)$ is bounded. We apply the $C^{1,\alpha}$-estimates obtained in Theorem 3.9 to prove regularity estimates for solutions of (4.1) with $q = 0$. Our arguments are similar to those in [4, Section 3], see also [24]. The idea is to use the well known characterization of $C^{1,\alpha}$-regularity via affine approximates. The following lemma plays a key role: It states that if f is small, then a solution to (4.1) can be approximated by an affine function. This combined with scaling properties of the equation essentially yields the desired affine functions.

Lemma 4.1. *There exist constants $\epsilon(N, \hat{p})$, $\tau(N, \hat{p}) \in (0, 1)$ such that the following holds: If $\|f\|_{L^{\infty}(B_1)} \leq \epsilon$ and w is a viscosity solution to (4.1) in B_1 with $q \in \mathbb{R}^N$, $w(0) = 0$ and $\operatorname{osc}_{B_1} w \leq 1$, then there exists $q' \in \mathbb{R}^N$ such that*

$$\operatorname{osc}_{B_{\tau}}(w(x) - q' \cdot x) \leq \frac{1}{2}\tau.$$

Moreover, we have $|q'| \leq C(N, \hat{p})$.

Proof. Suppose on the contrary that the claim does not hold. Then, for a fixed $\tau(N, \hat{p})$ that we will specify later, there exists a sequence of Lipschitz continuous functions $p_j : B_1 \to \mathbb{R}$ such that

$$p_{\min} \leq \inf_{B_1} p_j \leq \sup_{B_1} p_j \leq p_{\max} \quad \text{and} \quad (p_j)_L \leq p_L,$$

functions $f_j \in C(B_1)$ such that $f_j \to 0$ uniformly in B_1, vectors $q_j \in \mathbb{R}^N$ and viscosity solutions w_j to

$$-\Delta w_j - (p_j(x)-2)\frac{\langle D^2 w_j(Dw_j+q_j), Dw_j+q_j\rangle}{|Dw_j+q_j|^2+\varepsilon^2} = f_j(x) \quad \text{in } B_1$$

such that $w_j(0)=0$, $\operatorname{osc}_{B_1} w_j \le 1$ and

$$\operatorname{osc}_{B_\tau}(w_j(x) - q'\cdot x) > \frac{\tau}{2} \quad \text{for all } q' \in \mathbb{R}^N. \tag{4.2}$$

By [13, Proposition 4.10], the functions w_j are uniformly Hölder continuous in B_r for any $r \in (0,1)$. Therefore by the Ascoli-Arzela theorem, we may extract a subsequence such that $w_j \to w_\infty$ and $p_j \to p_\infty$ uniformly in B_r for any $r \in (0,1)$. Moreover, p_∞ is p_L-Lipschitz continuous and $p_{\min} \le p_\infty \le p_{\max}$. It then follows from (4.2) that

$$\operatorname{osc}_{B_\tau}(w_\infty(x) - q'\cdot x) > \frac{\tau}{2} \quad \text{for all } q' \in \mathbb{R}^N. \tag{4.3}$$

We have two cases: either q_j is bounded or unbounded.

Case q_j is bounded: In this case $q_j \to q_\infty \in \mathbb{R}^N$ up to a subsequence. It follows from the stability principle that w_∞ is a viscosity solution to

$$-\Delta w_\infty - (p_\infty(x)-2)\frac{\langle D^2 w_\infty(Dw_\infty+q_\infty), Dw_\infty+q_\infty\rangle}{|Dw_\infty+q_\infty|^2+\varepsilon^2} = 0 \quad \text{in } B_1. \tag{4.4}$$

Hence by Theorem 3.9 we have $\|Dw_\infty\|_{C^{\beta_1}(B_{1/2})} \le C(N,\hat{p})$ for some $\beta_1(N,\hat{p})$. The mean value theorem then implies the existence of $q' \in \mathbb{R}^N$ such that

$$\operatorname{osc}_{B_r}(u - q'\cdot x) \le C_1(N,\hat{p}) r^{1+\beta_1} \quad \text{for all } r \le \frac{1}{2}.$$

Case q_j is unbounded: In this case we take a subsequence such that $|q_j| \to \infty$ and the sequence $d_j := d_j/|d_j|$ converges to $d_\infty \in \partial B_1$. Then w_j is a viscosity solution to

$$-\Delta w_j - (p_j(x)-2)\frac{\langle D^2 w_j(|q_j|^{-1}Dw_j+d_j), |q_j|^{-1}Dw_j+d_j\rangle}{\left||q_j|^{-1} Dw_j+d_j\right|^2+|q_j|^{-2}\varepsilon^2} = f_j(x) \quad \text{in } B_1.$$

It follows from the stability principle that w_∞ is a viscosity solution to

$$-\Delta w_j - (p_\infty(x)-2)\left\langle D^2 w_\infty d_\infty, d_\infty\right\rangle = 0 \quad \text{in } B_1.$$

By [13, Theorem 8.3] there exist positive constants $\beta_2(N,\hat{p})$, $C_2(N,\hat{p})$, $r_2(N,\hat{p})$ and a vector $q' \in \mathbb{R}^N$ such that

$$\operatorname{osc}_{B_r}(w_\infty - q'\cdot x) \le C_2 r^{1+\beta_2} \quad \text{for all } r \le r_2.$$

We set $C_0 := \max(C_1, C_2)$ and $\beta_0 := \min(\beta_1, \beta_2)$. Then by the two different cases there always exists a vector $q' \in \mathbb{R}^N$ such that

$$\operatorname{osc}_{B_r}(w_\infty - q'\cdot x) \le C_0 r^{1+\beta_0} \quad \text{for all } r \le \min(\frac{1}{2}, r_2).$$

We take τ so small that $C_0\tau^{\beta_0} \leq \frac{1}{4}$ and $\tau \leq \min(\frac{1}{2}, r_2)$. Then, by substituting $r = \tau$ in the above display, we obtain

$$\operatorname{osc}_{B_\tau}(w_\infty - q' \cdot x) \leq C_0\tau^{\beta_0}\tau \leq \frac{1}{4}\tau, \tag{4.5}$$

which contradicts (4.3).

The bound $|q'| \leq C(N, \hat{p})$ follows by observing that (4.5) together with the assumption $\operatorname{osc}_{B_1} w \leq 1$ yields $|q'| \leq C$. Thus the contradiction is still there even if (4.3) is weakened by requiring additionally that $|q'| \leq C$. □

Lemma 4.2. *Let $\tau(N, \hat{p})$ and $\epsilon(N, \hat{p})$ be as in Lemma 4.1. If $\|f\|_{L^\infty(B_1)} \leq \epsilon$ and u is a viscosity solution to (4.1) in B_1 with $q = 0$, $u(0) = 0$ and $\operatorname{osc}_{B_1} u \leq 1$, then there exists $\alpha \in (0, 1)$ and $q_\infty \in \mathbb{R}^N$ such that*

$$\sup_{B_{\tau^k}} |u(x) - q_\infty \cdot x| \leq C(N, \hat{p})\tau^{k(1+\alpha)} \quad \textit{for all } k \in \mathbb{N}.$$

Proof. Step 1: We show that there exists a sequence $(q_k)_{k=0}^\infty \subset \mathbb{R}^N$ such that

$$\operatorname{osc}_{B_{\tau^k}}(u(x) - q_k \cdot x) \leq \tau^{k(1+\alpha)}. \tag{4.6}$$

When $k = 0$, this estimate holds by setting $q_0 = 0$ since $u(0) = 0$ and $\operatorname{osc}_{B_1} \leq 1$. Next we take $\alpha \in (0, 1)$ such that $\tau^\alpha > \frac{1}{2}$. We assume that $k \geq 0$ and that we have already constructed q_k for which (4.6) holds. We set

$$w_k(x) := \tau^{-k(1+\alpha)}(u(\tau^k x) - q_k \cdot (\tau^k x))$$

and

$$f_k(x) := \tau^{k(1-\alpha)} f(\tau^k x).$$

Then by induction assumption $\operatorname{osc}_{B_1}(w_k) \leq 1$ and w_k is a viscosity solution to

$$-\Delta w_k - \frac{(p(\tau^k x) - 2)\left\langle D^2 w_k(Dw_k + \tau^{-k\alpha}q_k), Dw_k + \tau^{-k\alpha}q_k\right\rangle}{\left|Dw_k + \tau^{-k\alpha}q_k\right|^2 + (\tau^{-k\alpha}\varepsilon)^2} = f_k(x) \quad \text{in } B_1.$$

By Lemma 4.1 there exists $q_k' \in \mathbb{R}^N$ with $|q_k'| \leq C(N, \hat{p})$ such that

$$\operatorname{osc}_{B_\tau}(w_k(x) - q_k' \cdot x) \leq \frac{1}{2}\tau.$$

Using the definition of w_k, scaling to $B_{\tau^{k+1}}$ and dividing by $\tau^{-k(\alpha+1)}$, we obtain from the above

$$\operatorname{osc}_{B_{\tau^{k+1}}}(u(x) - (q_k + \tau^{k\alpha}q_k') \cdot x) \leq \frac{1}{2}\tau^{1+k(1+\alpha)} \leq \tau^{(k+1)(1+\alpha)}.$$

Denoting $q_{k+1} := q_k + \tau^{k\alpha}q_k'$, the above estimate is condition (4.6) for $k + 1$ and the induction step is complete.

Step 2: Observe that whenever $m > k$, we have

$$|q_m - q_k| \leq \sum_{i=k}^{m-1} \tau^{i\alpha} |q_i'| \leq C(N, \hat{p}) \sum_{i=k}^{m-1} \tau^{i\alpha}.$$

Therefore q_k is a Cauchy sequence and converges to some $q_\infty \in \mathbb{R}^N$. Thus

$$\sup_{x\in B_{\tau^k}} (q_k \cdot x - q_\infty \cdot x) \leq \tau^k |q_k - q_\infty| \leq \tau^k \sum_{i=k}^{\infty} \tau^{i\alpha} q_i' \leq C(N,\hat{p})\tau^{k(1+\alpha)}.$$

This with (4.6) implies that

$$\sup_{x\in B_{\tau^k}} |u(x) - q_\infty \cdot x| \leq C(N,\hat{p})\tau^{k(1+\alpha)}. \quad \square$$

Theorem 4.3. *Suppose that u is a viscosity solution to (4.1) in B_1 with $q = 0$ and $\operatorname{osc}_{B_1} \leq 1$. Then there are constants $\alpha(N,\hat{p})$ and $C(N,\hat{p},\|f\|_{L^\infty(B_1)})$ such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C.$$

Proof. Let $\epsilon(N,\hat{p})$ and $\tau(N,\hat{p})$ be as in Lemma 4.2. Set

$$v(x) := \kappa u(x/4)$$

where $\kappa := \epsilon(1 + \|f\|_{L^\infty(B_1)})^{-1}$. For $x_0 \in B_1$, set

$$w(x) := v(x + x_0) - v(x_0).$$

Then $\operatorname{osc}_{B_1} w \leq 1$, $w(0) = 0$ and w is a viscosity solution to

$$-\Delta w - \frac{(p(x/4 + x_0/4) - 2)\left\langle D^2 w Dw, Dw\right\rangle}{|Dw|^2 + \varepsilon^2\kappa^2/4^2} = g(x) \quad \text{in } B_1,$$

where $g(x) := \kappa f(x/4 + x_0/4)/4^2$. Now $\|g\|_{L^\infty(B_1)} \leq \epsilon$ so by Lemma 4.2 there exists $q_\infty(x_0) \in \mathbb{R}^N$ such that

$$\sup_{x\in B_{\tau^k}} |w(x) - q_\infty(x_0) \cdot x| \leq C(N,\hat{p})\tau^{k(1+\alpha)} \quad \text{for all } k \in \mathbb{N}.$$

Thus we have shown that for any $x_0 \in B_1$ there exists a vector $q_\infty(x_0)$ such that

$$\sup_{x\in B_r(x_0)} |v(x) - v(x_0) - q_\infty(x_0) \cdot (x - x_0)| \leq C(N,\hat{p})r^{1+\alpha} \quad \text{for all } r \in (0,1].$$

This together with a standard argument (see for example [4, Lemma A.1]) implies that $[Dv]_{C^\alpha(B_1)} \leq C(N,\hat{p})$ and so by definition of v, also $[Du]_{C^\alpha(B_{1/4})} \leq C(N,\hat{p},\|f\|_{L^\infty(B_1)})$. The conclusion of the theorem then follows by a standard translation argument. $\square$

5. Proof of the main theorem

In this section we finish the proof our main theorem.

Proof of Theorem 1.1. We may assume that $u \in C(\overline{B}_1)$. By Comparison Principle (Lemma B.2 in the Appendix) u is the unique viscosity solution to

$$\begin{cases} -\Delta v - \frac{(p(x)-2)\left\langle D^2 v Dv, Dv\right\rangle}{|Dv|^2} = f(x) + u - v & \text{in } B_1, \\ v = u & \text{on } \partial B_1. \end{cases} \tag{5.1}$$

By [21, Theorem 15.18] there exists a classical solution u_ε to the approximate problem

$$\begin{cases} -\Delta u_\varepsilon - \frac{(p_\varepsilon(x)-2)\langle D^2 u_\varepsilon Du_\varepsilon, Du_\varepsilon\rangle}{|Du_\varepsilon|^2+\varepsilon^2} = f_\varepsilon(x) + u - u_\varepsilon & \text{in } B_1, \\ u_\varepsilon = u & \text{on } \partial B_1, \end{cases}$$

where $p_\varepsilon, f_\varepsilon, u_\varepsilon \in C^\infty(B_1)$ are such that $p_\varepsilon \to p$, $f_\varepsilon \to f$ and $u_\varepsilon \to u_0$ uniformly in B_1 as $\varepsilon \to 0$ and $\|Dp_\varepsilon\|_{L^\infty(B_1)} \le \|Dp\|_{L^\infty(B_1)}$. The maximum principle implies that $\|u_\varepsilon\|_{L^\infty(B_1)} \le 2\|f\|_{L^\infty(B_1)} + 2\|u\|_{L^\infty(B_1)}$. By [13, Proposition 4.14] the solutions u_ε are equicontinuous in $\overline{B}_1$ (their modulus of continuity depends only on N, p, $\|f\|_{L^\infty(B_1)}$, $\|u\|_{L^\infty(B_1)}$ and modulus of continuity of u). Therefore by the Ascoli-Arzela theorem we have $u_\varepsilon \to v \in C(\overline{B}_1)$ uniformly in $\overline{B}_1$ up to a subsequence. By the stability principle, v is a viscosity solution to (5.1) and thus by uniqueness $v \equiv u$.

By Corollary 4.3 we have $\alpha(N, \hat{p})$ such that

$$\|Du_\varepsilon\|_{C^\alpha(B_{1/2})} \le C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}) \tag{5.2}$$

and by the Lipschitz estimate A.2 also

$$\|Du_\varepsilon\|_{L^\infty(B_{1/2})} \le C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}).$$

Therefore by the Ascoli-Arzela theorem there exists a subsequence such that $Du_\varepsilon \to \eta$ uniformly in $B_{1/2}$, where the function $\eta : B_{1/2} \to \mathbb{R}^N$ satisfies

$$\|\eta\|_{C^\alpha(B_{1/2})} \le C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}).$$

Using the mean value theorem and the estimate (5.2), we deduce for all $x, y \in B_{1/2}$

$$\begin{aligned} &|u(y) - u(x) - (y-x)\cdot\eta(x)| \\ &\le |u_\varepsilon(x) - u_\varepsilon(y) - (y-x)\cdot Du_\varepsilon(x)| \\ &\quad + |u(y) - u_\varepsilon(y) - u(x) + u_\varepsilon(x)| + |x-y|\,|\eta(x) - Du_\varepsilon(x)| \\ &\le C(N, \hat{p}, \|u\|_{L^\infty(B_1)})\,|x-y|^{1+\alpha} + o(\varepsilon)/\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, this implies that $Du(x) = \eta(x)$ for all $x \in B_{1/2}$. □

Appendix A. Lipschitz estimate

In this section we apply the method of Ishii and Lions [23] to prove a Lipschitz estimate for solutions to the inhomogeneous normalized $p(x)$-Laplace equation and its regularized or perturbed versions. We need the following vector inequality.

Lemma A.1. *Let $a, b \in \mathbb{R}^N \setminus \{0\}$ with $a \ne b$ and $\varepsilon \ge 0$. Then*

$$\left| \frac{a}{\sqrt{|a|^2+\varepsilon^2}} - \frac{b}{\sqrt{|b|^2+\varepsilon^2}} \right| \le \frac{2}{\max(|a|, |b|)} |a-b|.$$

Proof. We may suppose that $|a| = \max(|a|, |b|)$. Let $s_1 := \sqrt{|a|^2 + \varepsilon^2}$ and $s_2 := \sqrt{|b|^2 + \varepsilon^2}$. Then

$$\left|\frac{a}{s_1} - \frac{b}{s_2}\right| = \frac{1}{s_1}\left|a - b + \frac{b}{s_2}(s_2 - s_1)\right| \le \frac{1}{s_1}\left(|a-b| + \frac{|b|}{s_2}|s_2 - s_1|\right)$$
$$\le \frac{1}{|a|}(|a-b| + |s_2 - s_1|).$$

Moreover

$$|s_2 - s_1| = \left|\sqrt{|a|^2 + \varepsilon^2} - \sqrt{|b|^2 + \varepsilon^2}\right| = \frac{\left||a|^2 - |b|^2\right|}{\sqrt{|a|^2 + \varepsilon^2} + \sqrt{|b|^2 + \varepsilon^2}}$$
$$\le \frac{(|a| + |b|)\,||a| - |b||}{|a| + |b|} \le |a - b|. \quad \square$$

Theorem A.2 *(Lipschitz estimate). Suppose that $p : B_1 \to \mathbb{R}$ is Lipschitz continuous, $p_{\min} > 1$ and that $f \in C(B_1)$ is bounded. Let u be a viscosity solution to*

$$-\Delta u - (p(x) - 2)\frac{\langle D^2 u(Du + q), Du + q\rangle}{|Du + q|^2 + \varepsilon^2} = f(x) \quad \text{in } B_1,$$

where $\varepsilon \ge 0$ and $q \in \mathbb{R}^N$. Then there are constants $C_0(N, \hat{p}, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)})$ and $\nu_0(N, \hat{p})$ such that if $|q| > \nu_0$ or $|q| = 0$, then we have

$$|u(x) - u(y)| \le C_0 |x - y| \quad \text{for all } x, y \in B_{1/2}.$$

Proof. We let $r(N, \hat{p}) \in (0, 1/2)$ denote a small constant that will be specified later. Let $x_0, y_0 \in B_{r/2}$ and define the function

$$\Psi(x, y) := u(x) - u(y) - L\varphi(|x - y|) - \frac{M}{2}|x - x_0|^2 - \frac{M}{2}|y - y_0|^2,$$

where $\varphi : [0, 2] \to \mathbb{R}$ is given by

$$\varphi(s) := s - s^\gamma \kappa_0, \quad \kappa_0 := \frac{1}{\gamma 2^{\gamma+1}},$$

and the constants $L(N, \hat{p}, \|u\|_{L^\infty(B_1)}), M(N, \hat{p}, \|u\|_{L^\infty(B_1)}) > 0$ and $\gamma(N, \hat{p}) \in (1, 2)$ are also specified later. Our objective is to show that for a suitable choice of these constants, the function Ψ is non-positive in $\overline{B_r} \times \overline{B_r}$. By the definition of φ, this yields $u(x_0) - u(y_0) \le L|x_0 - y_0|$ which implies that u is L-Lipschitz in B_r. The claim of the theorem then follows by standard translation arguments.

Suppose on contrary that Ψ has a positive maximum at some point $(\hat{x}, \hat{y}) \in \overline{B_r} \times \overline{B_r}$. Then $\hat{x} \ne \hat{y}$ since otherwise the maximum would be non-positive. We have

$$0 < u(\hat{x}) - u(\hat{y}) - L\varphi(|\hat{x} - \hat{y}|) - \frac{M}{2}|\hat{x} - x_0|^2 - \frac{M}{2}|\hat{y} - y_0|^2$$
$$\le |u(\hat{x}) - u(\hat{y})| - \frac{M}{2}|\hat{x} - x_0|^2. \tag{A.1}$$

Therefore, by taking

$$M := \frac{8 \operatorname{osc}_{B_1} u}{r^2}, \tag{A.2}$$

we get

$$|\hat{x}-x_0| \leq \sqrt{\frac{2}{M}\,|u(\hat{x})-u(\hat{y})|} \leq r/2$$

and similarly $|\hat{y}-y_0| \leq r/2$. Since $x_0, y_0 \in B_{r/2}$, this implies that $\hat{x}, \hat{y} \in B_r$.

By [13, Proposition 4.10] there exist constants $C'(N, \hat{p}, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)})$ and $\beta(N, \hat{p}) \in (0,1)$ such that

$$|u(x)-u(y)| \leq C'\,|x-y|^{\beta} \quad \text{for all } x, y \in B_r. \tag{A.3}$$

It follows from (A.1) and (A.3) that for $C_0 := \sqrt{2C'}\sqrt{M}$ we have

$$\begin{aligned} M\,|\hat{x}-x_0| &\leq C_0\,|\hat{x}-\hat{y}|^{\beta/2}, \\ M\,|\hat{y}-y_0| &\leq C_0\,|\hat{x}-\hat{y}|^{\beta/2}. \end{aligned} \tag{A.4}$$

Since $\hat{x} \neq \hat{y}$, the function $(x,y) \mapsto \varphi(|x-y|)$ is C^2 in a neighborhood of $(\hat{x}, \hat{y})$ and we may invoke the Theorem of sums [14, Theorem 3.2]. For any $\mu > 0$ there exist matrices $X, Y \in S^N$ such that

$$\begin{aligned} (D_x(L\varphi(|x-y|))(\hat{x},\hat{y}), X) &\in \overline{J}^{2,+}(u - \frac{M}{2}\,|x-x_0|^2)(\hat{x}), \\ (-D_y(L\varphi(|x-y|))(\hat{x},\hat{y}), Y) &\in \overline{J}^{2,-}(u + \frac{M}{2}\,|y-y_0|^2)(\hat{y}), \end{aligned}$$

which by denoting $z := \hat{x}-\hat{y}$ and

$$\begin{aligned} a &:= L\varphi'(|z|)\frac{z}{|z|} + M(\hat{x}-x_0), \\ b &:= L\varphi'(|z|)\frac{z}{|z|} - M(\hat{y}-y_0), \end{aligned}$$

can be written as

$$(a, X+MI) \in \overline{J}^{2,+}u(\hat{x}), \quad (b, Y-MI) \in \overline{J}^{2,-}u(\hat{y}). \tag{A.5}$$

By assuming that L is large enough depending on C_0, we have by (A.4) and the fact $\varphi' \in \left[\frac{3}{4}, 1\right]$

$$|a|, |b| \leq L\,|\varphi'(|\hat{x}-\hat{y}|)| + C_0\,|\hat{x}-\hat{y}|^{\beta/2} \leq 2L, \tag{A.6}$$

$$|a|, |b| \geq L\,|\varphi'(|\hat{x}-\hat{y}|)| - C_0\,|\hat{x}-\hat{y}|^{\beta/2} \geq \frac{1}{2}L. \tag{A.7}$$

Moreover, we have

$$\begin{aligned} -(\mu + 2\,\|B\|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \\ &\leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \frac{2}{\mu}\begin{pmatrix} B^2 & -B^2 \\ -B^2 & B^2 \end{pmatrix}, \end{aligned} \tag{A.8}$$

where

$$B = L\varphi''(|z|)\frac{z}{|z|}\otimes\frac{z}{|z|} + \frac{L\varphi'(|z|)}{|z|}\left(I - \frac{z}{|z|}\otimes\frac{z}{|z|}\right),$$
$$B^2 = BB = L^2(\varphi''(|z|))^2\frac{z}{|z|}\otimes\frac{z}{|z|} + \frac{L^2(\varphi'(|z|))^2}{|z|^2}\left(I - \frac{z}{|z|}\otimes\frac{z}{|z|}\right).$$

Using that $\varphi''(|z|) < 0 < \varphi'(|z|)$ and $|\varphi''(|z|)| \leq \varphi'(|z|)/|z|$, we deduce that

$$\|B\| \leq \frac{L\varphi'(|z|)}{|z|} \quad \text{and} \quad \left\|B^2\right\| \leq \frac{L^2(\varphi'(|z|))^2}{|z|^2}. \tag{A.9}$$

Moreover, choosing

$$\mu := 4L\left(|\varphi''(|z|)| + \frac{|\varphi'(|z|)|}{|z|}\right),$$

and using that $\varphi''(|z|) < 0$, we have

$$\left\langle B\frac{z}{|z|}, \frac{z}{|z|}\right\rangle + \frac{2}{\mu}\left\langle B^2\frac{z}{|z|}, \frac{z}{|z|}\right\rangle = L\varphi''(|z|) + \frac{2}{\mu}L^2|\varphi''(|z|)| \leq \frac{L}{2}\varphi''(|z|). \tag{A.10}$$

We set $\eta_1 := a + q$ and $\eta_2 := b + q$. By (A.6) and (A.7) there is a constant $\nu_0(L)$ such that if $|q| = 0$ or $|q| > \nu_0$, then

$$|\eta_1|, |\eta_2| \geq \frac{L}{2}. \tag{A.11}$$

We denote $A(x, \eta) := I + (p(x) - 2)\eta\otimes\eta$ and $\overline{\eta} := \frac{\eta}{\sqrt{|\eta|^2+\varepsilon^2}}$. Since u is a viscosity solution, we obtain from (A.5)

$$\begin{aligned}
0 &\leq \operatorname{tr}(A(\hat{x}, \overline{\eta}_1)(X + MI)) - \operatorname{tr}(A(\hat{y}, \overline{\eta}_2)(Y - MI)) + f(\hat{x}) - f(\hat{y}) \\
&= \operatorname{tr}(A(\hat{y}, \overline{\eta}_2)(X - Y)) + \operatorname{tr}((A(\hat{x}, \overline{\eta}_2) - A(\hat{y}, \overline{\eta}_2))X) \\
&\quad + \operatorname{tr}((A(\hat{x}, \overline{\eta}_1) - A(\hat{x}, \overline{\eta}_2))X) + M\operatorname{tr}(A(\hat{x}, \overline{\eta}_1) + A(\hat{y}, \overline{\eta}_2)) \\
&\quad + f(\hat{x}) - f(\hat{y}) \\
&=: T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned} \tag{A.12}$$

We will now proceed to estimate these terms. The plan is to obtain a contradiction by absorbing the other terms into T_1 which is negative by concavity of φ.

Estimate of T_1: Multiplying (A.8) by the vector $(\frac{z}{|z|}, -\frac{z}{|z|})$ and using (A.10), we obtain an estimate for the smallest eigenvalue of $X - Y$

$$\begin{aligned}
\lambda_{\min}(X - Y) &\leq \left\langle (X - Y)\frac{z}{|z|}, \frac{z}{|z|}\right\rangle \\
&\leq 4\left\langle B\frac{z}{|z|}, \frac{z}{|z|}\right\rangle + \frac{8}{\mu}\left\langle B^2\frac{z}{|z|}, \frac{z}{|z|}\right\rangle \leq 2L\varphi''(|z|).
\end{aligned}$$

The eigenvalues of $A(\hat{y}, \overline{\eta}_2)$ are between $\min(1, p_{\min} - 1)$ and $\max(1, p_{\max} - 1)$. Therefore by [36]

$$\begin{aligned}
T_1 = \operatorname{tr}(A(\hat{y}, \overline{\eta}_2)(X - Y)) &\leq \sum_i \lambda_i(A(\hat{y}, \overline{\eta}_2))\lambda_i(X - Y) \\
&\leq \min(1, p_{\min} - 1)\lambda_{\min}(X - Y)
\end{aligned}$$

$$\leq C(\hat{p})L\varphi''(|z|).$$

Estimate of T_2: We have

$$T_2 = \operatorname{tr}((A(\hat{x}, \overline{\eta}_2) - A(\hat{y}, \overline{\eta}_2))X) \leq |p(\hat{x}) - p(\hat{y})| \, |\langle X\overline{\eta}_2, \overline{\eta}_2 \rangle| \leq C(\hat{p}) \, |z| \, \|X\| \, ,$$

where by (A.8) and (A.9)

$$\begin{aligned} \|X\| \leq \|B\| + \frac{2}{\mu} \|B\|^2 &\leq \frac{L\,|\varphi'(|z|)|}{|z|} + \frac{2L^2(\varphi'(|z|))^2}{4L(|\varphi''(|z|)| + \frac{|\varphi'(|z|)|}{|z|})\,|z|^2} \\ &\leq \frac{2L\varphi'(|z|)}{|z|}. \end{aligned} \tag{A.13}$$

Estimate of T_3: From Lemma A.1 and the estimate (A.11) it follows that

$$\begin{aligned} |\overline{\eta}_1 - \overline{\eta}_2| &\leq \frac{2\,|\eta_1 - \eta_2|}{\max(|\eta_1|\,, |\eta_2|)} \leq \frac{4}{L}\,|\eta_1 - \eta_2| = \frac{4}{L}\,|a-b| \\ &\leq \frac{4}{L}(M\,|\hat{x} - x_0| + M\,|\hat{y} - y_0|) \leq \frac{8C_0}{L}\,|z|^{\beta/2}\,, \end{aligned} \tag{A.14}$$

where in the last inequality we used (A.4). Observe that

$$\|\overline{\eta}_1 \otimes \overline{\eta}_1 - \overline{\eta}_2 \otimes \overline{\eta}_2\| = \|(\overline{\eta}_1 - \overline{\eta}_2) \otimes \overline{\eta}_1 - \overline{\eta}_2 \otimes (\overline{\eta}_2 - \overline{\eta}_1)\| \leq (|\overline{\eta}_1| + |\overline{\eta}_2|)\,|\overline{\eta}_1 - \overline{\eta}_2|\,.$$

Using the last two displays, we obtain by [36] and (A.13)

$$\begin{aligned} T_3 = \operatorname{tr}((A(\hat{x}, \overline{\eta}_1) - A(\hat{x}, \overline{\eta}_2))X) &\leq N\,\|A(x_1, \overline{\eta}_1) - A(x_1, \overline{\eta}_2)\|\,\|X\| \\ &\leq N\,|p(x_1) - 2|\,(|\overline{\eta}_1| + |\overline{\eta}_2|)\,|\overline{\eta}_1 - \overline{\eta}_2|\,\|X\| \\ &\leq \frac{C(N,\hat{p})C_0}{L}\,|z|^{\beta/2}\,\|X\| \\ &\leq C(N, \hat{p}, \|u\|_{L^\infty}\,, \|f\|_{L^\infty})\sqrt{M}\varphi'(|z|)\,|z|^{\beta/2-1}\,. \end{aligned}$$

Estimate of T_4 and T_5: By Lipschitz continuity of p we have

$$T_4 = M\operatorname{tr}(A(\hat{x}, \overline{\eta}_1) + A(\hat{y}, \overline{\eta}_2)) \leq 2MC(N, \hat{p}).$$

We have also

$$T_5 = f(\hat{x}) - f(\hat{y}) \leq 2\,\|f\|_{L^\infty(B_1)}\,.$$

Combining the estimates, we deduce the existence of positive constants $C_1(N, \hat{p})$ and $C_2(N, \hat{p}, \|u\|_{L^\infty(B_1)}\,, \|f\|_{L^\infty(B_1)})$ such that

$$\begin{aligned} 0 &\leq C_1 L\varphi''(|z|) + C_2\big(L\varphi'(|z|) + \sqrt{M}\varphi'(|z|)\,|z|^{\frac{\beta}{2}-1} + M + 1\big) \\ &\leq C_1 L\varphi''(|z|) + C_2(L + \sqrt{M}\,|z|^{\frac{\beta}{2}-1} + M + 1) \end{aligned} \tag{A.15}$$

where we used that $\varphi'(|z|) \in [\frac{3}{4}, 1]$. We take $\gamma := \frac{\beta}{2} + 1$ so that we have

$$\varphi''(|z|) = \frac{1-\gamma}{2^{\gamma+1}} |z|^{\gamma-2} = \frac{-\beta}{2^{\frac{\beta}{2}+3}} |z|^{\frac{\beta}{2}-1} =: -C_3 |z|^{\frac{\beta}{2}-1}.$$

We apply this to (A.15) and obtain

$$0 \leq (C_2\sqrt{M} - C_1C_3L) |z|^{\frac{\beta}{2}-1} + C_2(L+M+1) \tag{A.16}$$

We fix $r := \frac{1}{2}\left(\frac{6C_2}{C_1C_3}\right)^{\frac{1}{\frac{\beta}{2}-1}}$. By (A.2) this will also fix $M = (N, \hat{p}, \|u\|_{L^\infty(B_1)})$. We take L so large that

$$L > \max(\frac{2C_2\sqrt{M}}{C_1C_3}, M+1).$$

Then by (A.16) we have

$$\begin{aligned} 0 < -\frac{1}{2}C_1C_3L |z|^{\frac{\beta}{2}-1} + 2C_2L &\leq L(-\frac{1}{2}C_1C_3(2r)^{\frac{\beta}{2}-1} + 2C_2) \\ &= -LC_2 \leq 0, \end{aligned}$$

which is a contradiction. □

Appendix B. Stability and comparison principles

Lemma B.1. *Suppose that $p \in C(B_1)$, $p_{\min} > 1$ and that $f : B_1 \times \mathbb{R} \to \mathbb{R}$ is continuous. Let u_ε be a viscosity solution to*

$$-\Delta u_\varepsilon - (p_\varepsilon(x) - 2)\frac{\langle D^2u_\varepsilon Du_\varepsilon, Du_\varepsilon\rangle}{|Du_\varepsilon|^2 + \varepsilon^2} = f_\varepsilon(x, u(x)) \quad \textit{in } B_1$$

and assume that $u_\varepsilon \to u \in C(B_1)$, $p_\varepsilon \to p$ and $f_\varepsilon \to f$ locally uniformly as $\varepsilon \to 0$. Then u is a viscosity solution to

$$-\Delta u - (p(x) - 2)\frac{\langle D^2uDu, Du\rangle}{|Du|^2} = f(x, u(x)) \quad \textit{in } B_1.$$

Proof. It is enough to consider supersolutions. Suppose that $\varphi \in C^2$ touches u from below at x. Since $u_\varepsilon \to u$ locally uniformly, there exists a sequence $x_\varepsilon \to x$ such that $u_\varepsilon - \varphi$ has a local minimum at x_ε. We denote $\eta_\varepsilon := D\varphi(x_\varepsilon)/\sqrt{|D\varphi(x_\varepsilon)|^2 + \varepsilon^2}$. Then $\eta_\varepsilon \to \eta \in \overline{B}_1$ up to a subsequence. Therefore we have

$$\begin{aligned} 0 &\leq -\Delta\varphi(x_\varepsilon) - (p_\varepsilon(x_\varepsilon) - 2)\langle D^2\varphi(x_\varepsilon)\eta_\varepsilon, \eta_\varepsilon\rangle - f_\varepsilon(x_\varepsilon, u_\varepsilon(x_\varepsilon)) \\ &\to -\Delta\varphi(x) - (p(x) - 2)\langle D^2\varphi(x_\varepsilon)\eta, \eta\rangle - f(x, u(x)), \end{aligned} \tag{B.1}$$

which is what is required in Definition 2.1 in the case $D\varphi(x) = 0$. If $D\varphi(x) \neq 0$, then $D\varphi(x_\varepsilon) \neq 0$ when ε is small and thus $\eta = D\varphi(x)/|D\varphi(x)|$. Therefore B.1 again implies the desired inequality. □

Lemma B.2. *Suppose that $p : B_1 \to \mathbb{R}$ is Lipschitz continuous, $p_{\min} > 1$ and that $f \in C(B_1)$ is bounded. Assume that $u \in C(\overline{B}_1)$ is a viscosity subsolution to $-\Delta^N_{p(x)}u \leq f - u$ in B_1 and that $v \in C(\overline{B}_1)$ is a viscosity supersolution to $-\Delta^N_{p(x)}v \geq f - v$ in B_1. Then*

$$u \leq v \quad \textit{on } \partial B_1$$

implies

$$u \le v \quad \text{in } B_1.$$

Proof. Step 1: Assume on the contrary that the maximum of $u - v$ in $\overline{B}_1$ is positive. For $x, y \in \overline{B}_1$, set

$$\Psi_j(x, y) := u(x) - v(y) - \varphi_j(x, y),$$

where $\varphi_j(x, y) := \frac{j}{4} |x - y|^4$. Let (x_j, y_j) be a global maximum point of Ψ_j in $\overline{B}_1 \times \overline{B}_1$. Then

$$u(x_j) - v(y_j) - \frac{j}{4} |x_j - y_j|^4 \ge u(0) - v(0)$$

so that

$$\frac{j}{4} |x_j - y_j|^4 \le 2 \|u\|_{L^\infty(B_1)} + 2 \|v\|_{L^\infty(B_1)} < \infty.$$

By compactness and the assumption $u \le v$ on ∂B_1 there exists a subsequence such that $x_j, y_j \to \hat{x} \in B_1$ and $u(\hat{x}) - v(\hat{x}) > 0$. Finally, since (x_j, y_j) is a maximum point of Ψ_j, we have

$$u(x_j) - v(x_j) \le u(x_j) - v(y_j) - \frac{j}{4} |x_j - y_j|^4,$$

and hence by continuity

$$\frac{j}{4} |x_j - y_j|^4 \le v(x_j) - v(y_j) \to 0 \tag{B.2}$$

as $j \to \infty$.

Step 2: If $x_j = y_j$, then $D_x^2 \varphi_j(x_j, y_j) = D_y^2 \varphi_j(x_j, y_j) = 0$. Therefore, since the function $x \mapsto u(x) - \varphi_j(x, y_j)$ reaches its maximum at x_j and $y \mapsto v(y) - (-\varphi_j(x_j, y))$ reaches its minimum at y_j, we obtain from the definition of viscosity sub- and supersolutions that

$$0 \le f(x_j) - u(x_j) \quad \text{and} \quad 0 \ge f(y_j) - v(y_j).$$

That is $0 \le f(x_j) - f(y_j) + v(y_j) - u(x_j)$, which leads to a contradiction since $x_j, y_j \to \hat{x}$ and $v(\hat{x}) - u(\hat{x}) < 0$. We conclude that $x_j \ne y_j$ for all large j. Next we apply the Theorem of sums [14, Theorem 3.2] to obtain matrices $X, Y \in S^N$ such that

$$(D_x \varphi(x_j, y_j), X) \in \overline{J}^{2,+} u(x_j), \quad (-D_y \varphi(x_j, y_j), Y) \in \overline{J}^{2,-} v(y_j)$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2 \varphi(x_j, y_j) + \frac{1}{j} (D^2(x_j, y_j))^2, \tag{B.3}$$

where

$$D^2(x_j, y_j) = \begin{pmatrix} M & -M \\ -M & M \end{pmatrix}$$

with $M = j(2(x_j - y_j) \otimes (x_j - y_j) + |x_j - y_j|^2 I)$. Multiplying the matrix inequality (B.3) by the $\mathbb{R}^{2N}$ vector (ξ_1, ξ_2) yields

$$\begin{aligned}\langle X\xi_1,\xi_1\rangle - \langle Y\xi_2,\xi_2\rangle &\le \big\langle (M+2j^{-1}M^2)(\xi_1-\xi_2),\xi_1-\xi_2\big\rangle \\ &\le (\|M\| + 2j^{-1}\|M\|^2)\,|\xi_1-\xi_2|^2 .\end{aligned}$$

Observe also that $\eta := D_x\varphi(x_j,y_j) = -D_y(x_j,y_j) = j\,|x_j-y_j|^2\,(x_j-y_j) \neq 0$ for all large j. Since u is a subsolution and v is a supersolution, we thus obtain

$$\begin{aligned}
&f(y_j) - f(x_j) + u(x_j) - v(y_j) \\
&\le \operatorname{tr}(X-Y) + (p(x_j)-2)\left\langle X\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle - (p(y_j)-2)\left\langle Y\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle \\
&\le (p(x_j)-1)\left\langle X\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle - (p(y_j)-1)\left\langle Y\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle \\
&\le (\|M\| + 2j^{-1}\|M\|^2)\Big|\sqrt{p(x_j)-1} - \sqrt{p(y_j)-1}\Big|^2 \\
&\le Cj\,|x_j-y_j|^2\,\frac{|p(x_j)-p(y_j)|^2}{\left(\sqrt{p(x_j)-1}+\sqrt{p(y_j)-1}\right)^2} \\
&\le C(\hat{p})j\,|x_j-y_j|^4 .
\end{aligned}$$

This leads to a contradiction since the left-hand side tends to $u(\hat{x}) - v(\hat{y}) > 0$ and the right-hand side tends to zero by (B.2). □

References

[1] T. Adamowicz, P. Hästö, Mappings of finite distortion and PDE with nonstandard growth, Int. Math. Res. Not. 10 (2010) 1940–1965.

[2] T. Adamowicz, P. Hästö, Harnack's inequality and the strong $p(\cdot)$-Laplacian, J. Differ. Equ. 250 (3) (2011) 1631–1649.

[3] Á. Arroyo, J. Heino, M. Parviainen, Tug-of-war games with varying probabilities and the normalized $p(x)$-Laplacian, Commun. Pure Appl. Anal. 16 (3) (2017) 915–944.

[4] A. Attouchi, M. Parviainen, Hölder regularity for the gradient of the inhomogeneous parabolic normalized p-Laplacian, Commun. Contemp. Math. 20 (4) (2018) 1750035.

[5] A. Attouchi, M. Parviainen, E. Ruosteenoja, $C^{1,\alpha}$ regularity for the normalized p-Poisson problem, J. Math. Pures Appl. 108 (4) (2017) 553–591.

[6] A. Attouchi, E. Ruosteenoja, Remarks on regularity for p-Laplacian type equations in non-divergence form, J. Differ. Equ. 265 (5) (2018) 1922–1961.

[7] A. Attouchi, E. Ruosteenoja, Gradient regularity for a singular parabolic equation in non-divergence form, Discrete Contin. Dyn. Syst. 40 (10) (2020) 5955–5972.

[8] A. Attouchi, Local regularity for quasi-linear parabolic equations in non-divergence form, Nonlinear Anal. 199 (2020) 112051.

[9] A. Banerjee, N. Garofalo, Modica type gradient estimates for an inhomogeneous variant of the normalized p-Laplacian evolution, Nonlinear Anal. 121 (2015) 458–468.

[10] A.C. Bronzi, E.A. Pimentel, G.C. Rampasso, E.V. Teixeira, Regularity of solutions to a class of variable-exponent fully nonlinear elliptic equations, J. Funct. Anal. 279 (12) (2020) 108781.

[11] P. Blanc, J. Rossi, Game Theory and Partial Differential Equations, De Gruyter Series in Nonlinear Analysis and Applications, De Gruyter, Berlin, Boston, 2019.

[12] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. Math. (2) 130 (1) (1989) 189–213.

[13] L.A. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, American Mathematical Society Colloquium Publications, vol. 43, 1995.

[14] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Am. Math. Soc. 27 (1) (1992) 1–67.

[15] C. De Filippis, Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy, Proc. R. Soc. Edinb. A 151 (1) (2021) 110–132.

[16] K. Does, An evolution equation involving the normalized p-Laplacian, Commun. Pure Appl. Anal. 10 (1) (2011) 361–396.

[17] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, revised edition, CRC Press, 2015.

[18] A. Elmoataz, M. Toutain, D. Tenbrinck, On the p-Laplacian and ∞-Laplacian on graphs with applications in image and data processing, SIAM J. Imaging Sci. 8 (4) (2015) 2412–2451.

[19] Y. Fang, C. Zhang, Gradient Hölder regularity for parabolic normalized $p(x,t)$-Laplace equation, J. Differ. Equ. 295 (2021) 211–232.

[20] Y. Fang, C. Zhang, Regularity for quasi-linear parabolic equations with nonhomogeneous degeneracy or singularity, https://arxiv.org/abs/2105.04743, 2021.
[21] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.
[22] C. Imbert, T. Jin, L. Silvestre, Hölder gradient estimates for a class of singular or degenerate parabolic equations, Adv. Nonlinear Anal. 8 (1) (2019) 845–867.
[23] H. Ishii, P.-L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differ. Equ. 83 (1) (1990) 26–78.
[24] C. Imbert, L. Silvestre, $C^{1,\alpha}$ regularity of solutions of degenerate fully non-linear elliptic equations, Adv. Math. 233 (2012) 196–206.
[25] T. Jin, L. Silvestre, Hölder gradient estimates for parabolic homogeneous p-Laplacian equations, J. Math. Pures Appl. 108 (3) (2017) 63–87.
[26] B. Kawohl, N. Kutev, Strong maximum principle for semicontinuous viscosity solutions of nonlinear partial differential equations, Arch. Math. (Basel) 70 (6) (1998) 470–478.
[27] B. Kawohl, N. Kutev, Comparison principle for viscosity solutions of fully nonlinear, degenerate elliptic equations, Commun. Partial Differ. Equ. 32 (7–9) (2007) 1209–1224.
[28] N.V. Krylov, M.V. Safonov, An estimate for the probability of a diffusion process hitting a set of positive measure, Dokl. Akad. Nauk SSSR 245 (1) (1979) 18–20.
[29] N.V. Krylov, M.V. Safonov, A certain property of solutions of parabolic equations with measurable coefficients, Izv. Akad. Nauk SSSR, Ser. Mat. 44 (1) (1980) 161–175.
[30] J.J. Manfredi, M. Parviainen, J.D. Rossi, An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games, SIAM J. Math. Anal. 42 (5) (2010) 2058–2081.
[31] J.J. Manfredi, M. Parviainen, J.D. Rossi, On the definition and properties of p-harmonous functions, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 11 (2) (2012) 215–241.
[32] Y. Peres, S. Sheffield, Tug-of-war with noise: a game-theoretic view of the p-Laplacian, Duke Math. J. 145 (1) (2008) 91–120.
[33] E. Ruosteenoja, Local regularity results for value functions of tug-of-war with noise and running payoff, Adv. Calc. Var. 9 (1) (2016) 1–17.
[34] O. Savin, Small perturbation solutions for elliptic equations, Commun. Partial Differ. Equ. 32 (4–6) (2007) 557–578.
[35] J. Siltakoski, Equivalence of viscosity and weak solutions for the normalized $p(x)$-Laplacian, Calc. Var. Partial Differ. Equ. 57 (95) (2018).
[36] C.M. Theobald, An inequality for the trace of the product of two symmetric matrices, Math. Proc. Camb. Philos. Soc. 77 (1975) 265–267.
[37] Y. Wang, Small perturbation solutions for parabolic equations, Indiana Univ. Math. J. 62 (2) (2013) 671–697.
[38] C. Zhang, S. Zhou, Hölder regularity for the gradients of solutions of the strong $p(x)$-Laplacian, J. Math. Anal. Appl. 389 (2) (2012) 1066–1077.