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Jarkko Siltakoski

Department of Mathematics and Statistics, P.O.Box 35, FIN-40014, University of Jyväskylä, Finland

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Abstract

We prove the local gradient Hölder regularity of viscosity solutions to the inhomogeneous normalized $p(x)$-Laplace equation

$$-\Delta^N_{p(x)} u = f(x),$$

where $p$ is Lipschitz continuous, $\inf p > 1$, and $f$ is continuous and bounded.

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1. Introduction

We study the inhomogeneous normalized $p(x)$-Laplace equation

$$-\Delta^N_{p(x)} u = f(x) \quad \text{in } B_1, \quad (1.1)$$

where

$$-\Delta^N_{p(x)} u := -\Delta u - (p(x) - 2) \frac{\langle D^2 u Du, Du \rangle}{|Du|^2}$$

is the normalized $p(x)$-Laplacian, $p : B_1 \to \mathbb{R}$ is Lipschitz continuous, $1 < p_{\text{min}} := \inf_{B_1} p \leq \sup_{B_1} p =: p_{\text{max}}$ and $f \in C(B_1)$ is bounded. Our main result is that viscosity solutions to (1.1) are locally $C^{1,\alpha}$-regular.

Normalized equations have attracted a significant amount of interest during the last 15 years. Their study is partially motivated by their connection to game theory. Roughly speaking, the value function of certain stochastic tug-of-war games converges uniformly up to a subsequence to a viscosity solution of a normalized equation as the step-size of the game approaches zero [32,30,31,9,11]. In particular, a game with
space-dependent probabilities leads to the normalized \( p(x) \)-Laplace equation \([3]\) and games with running pay-offs lead to inhomogeneous equations \([33]\). In addition to game theory, normalized equations have been studied for example in the context of image processing \([16,18]\).

The variable \( p(x) \) in (1.1) has an effect that may not be immediately obvious: If we formally multiply the equation by \(|Du|^{p(x)-2}\) and rewrite it in a divergence form, then a logarithm term appears and we arrive at the expression

\[
- \text{div}(|Du|^{p(x)-2} Du) + |Du|^{p(x)-2} \log(|Du|) Du \cdot Dp = |Du|^{p(x)-2} f(x).
\]

(1.2)

For \( f \equiv 0 \), this is the so-called \textit{strong} \( p(x) \)-\textit{Laplace equation} introduced by Adamowicz and Hästö \([1,2]\) in connection with mappings of finite distortion. In the homogeneous case viscosity solutions to (1.1) actually coincide with weak solutions of (1.2) \([35]\), yielding the \( C^{1,\alpha} \)-regularity of viscosity solutions as a consequence of a result by Zhang and Zhou \([38]\).

In the present paper our objective is to prove \( C^{1,\alpha} \)-regularity of solutions to (1.1) directly using viscosity methods. The Hölder regularity of solutions already follows from existing general results, see \([28,29,12,13]\). More recently, Imbert and Silvestre \([24]\) proved the gradient Hölder regularity of solutions to the elliptic equation

\[
|Du|^\gamma F(D^2 u) = f,
\]

where \( \gamma > 0 \) and Imbert, Jin and Silvestre \([25,22]\) obtained a similar result for the parabolic equation

\[
\partial_t u = |Du|^\gamma \Delta_p^N u,
\]

where \( p > 1, \gamma > -1 \). Furthermore, Attouchi and Parviainen \([4]\) proved the \( C^{1,\alpha} \)-regularity of solutions to the inhomogeneous equation \( \partial_t u - \Delta_p^N u = f(x,t) \). Our proof of Hölder gradient regularity for solutions of (1.1) is in particular inspired by the papers \([25]\) and \([4]\).

We point out that recently Fang and Zhang \([19]\) proved the \( C^{1,\alpha} \)-regularity of solutions to the parabolic normalized \( p(x,t) \)-Laplace equation

\[
\partial_t u = \Delta_{p(x,t)}^N u,
\]

(1.3)

where \( p \in C^1_{\text{loc}} \). The equation (1.3) naturally includes (1.1) if \( f \equiv 0 \). However, in this article we consider the inhomogeneous case and only suppose that \( p \) is Lipschitz continuous. More precisely, we have the following theorem.

**Theorem 1.1.** Suppose that \( p \) is Lipschitz continuous in \( B_1, \) \( p_{\text{min}} > 1 \) and \( f \in C(B_1) \) is bounded. Let \( u \) be a viscosity solution to

\[
-\Delta_{p(x)}^N u = f(x) \quad \text{in } B_1.
\]

Then there is \( \alpha(N, p_{\text{min}}, p_{\text{max}}, p_L) \in (0,1) \) such that

\[
\|u\|_{C^{1,\alpha}(B_1/2)} \leq C(N, p_{\text{min}}, p_{\text{max}}, p_L, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}),
\]

where \( p_L \) is the Lipschitz constant of \( p \).

The proof of Theorem 1.1 is based on suitable uniform \( C^{1,\alpha} \)-regularity estimates for solutions of the regularized equation.
\[-\Delta v - (p_\varepsilon(x) - 2) \frac{\langle D^2vDv, Dv \rangle}{|Dv|^2 + \varepsilon^2} = g(x), \quad (1.4)\]

where it is assumed that $g$ is continuous and $p_\varepsilon$ is smooth. In particular, we show estimates that are independent of $\varepsilon$ and only depend on $N$, sup $p$, inf $p$, $\|Dp_\varepsilon\|_{L^\infty}$ and $\|g\|_{L^\infty}$. To prove such estimates, we first derive estimates for the perturbed homogeneous equation

\[-\Delta v - (p_\varepsilon(x) - 2) \frac{\langle D^2v(Dv + q), Dv + q \rangle}{|Dv + q|^2 + \varepsilon^2} = 0, \quad (1.5)\]

where $q \in \mathbb{R}^N$. Roughly speaking, $C^{1,\alpha}$-estimates for solutions of (1.5) are based on “improvement of oscillation” which is obtained by differentiating the equation and observing that a function depending on the gradient of the solution is a supersolution to a linear equation. The uniform $C^{1,\alpha}$-estimates for solutions of (1.5) then yield uniform estimates for the inhomogeneous equation (1.4) by an adaption of the arguments in [24,4].

With the a priori regularity estimates at hand, the plan is to let $\varepsilon \to 0$ and show that the estimates pass on to solutions of (1.1). A problem is caused by the fact that, to the best of our knowledge, uniqueness of solutions to (1.1) is an open problem for variable $p(x)$ and even for constant $p$ if $f$ is allowed to change signs. To deal with this, we fix a solution $u_0 \in C(\overline{B}_1)$ to (1.1) and consider the Dirichlet problem

\[-\Delta_{p(x)}^N u = f(x) - u_0(x) - u \quad \text{in } B_1 \quad (1.6)\]

with boundary data $u = u_0$ on $\partial B_1$. For this equation the comparison principle holds and thus $u_0$ is the unique solution. We then consider the approximate problem

\[-\Delta u_\varepsilon - (p_\varepsilon(x) - 2) \frac{\langle D^2u_\varepsilon Du_\varepsilon, Du_\varepsilon \rangle}{|Du_\varepsilon|^2 + \varepsilon^2} = f_\varepsilon(x) - u_{0,\varepsilon}(x) - u_\varepsilon \quad (1.7)\]

with boundary data $u_\varepsilon = u_0$ on $\partial B_1$ and where $p_\varepsilon, f_\varepsilon, u_{0,\varepsilon} \in C^\infty(B_1)$ are such that $p \to p_\varepsilon$, $f_\varepsilon \to f$ and $u_{0,\varepsilon} \to u_0$ uniformly in $B_1$ and $\|Dp_\varepsilon\|_{L^\infty(B_1)} \leq \|p\|_{L^\infty(B_1)}$. As the equation (1.7) is uniformly elliptic quasilinear equation with smooth coefficients, the solution $u_\varepsilon$ exists in the classical sense by standard theory. Since $u_\varepsilon$ also solves (1.4) with $g(x) = f_\varepsilon(x) - u_{0,\varepsilon}(x) - u_\varepsilon(x)$, it satisfies the uniform $C^{1,\alpha}$-regularity estimate. We then let $\varepsilon \to 0$ and use stability and comparison principles to show that $u_0$ inherits the regularity estimate.

For other related results, see for example the works of Attouchi, Parviainen and Ruostekoski [5] on the normalized $p$-Poisson problem $-\Delta_p^N u = f$, Attouchi and Ruostekoski [6–8] on the equation $-|Du|^q \Delta_p^N u = f$ and its parabolic version, De Filippis [15] on the double phase problem $\|D(|Du|^q + a(x)|Du|^s)\|F(D^2u) = f(x)$ and Fang and Zhang [20] on the parabolic double phase problem $\partial_t u = (|Du|^q + a(x,t)|Du|^s)\Delta_p^N u$. We also mention the paper by Bronzi, Pimentel, Rampasso and Teixeira [10] where they consider fully nonlinear variable exponent equations of the type $|Du|^q(x) F(D^2u) = 0$.

The paper is organized as follows: Section 2 is dedicated to preliminaries, Sections 3 and 4 contain $C^{1,\alpha}$-regularity estimates for equations (1.5) and (1.7), and Section 5 contains the proof of Theorem (1.1). Finally, the Appendix contains an uniform Lipschitz estimate for the equations studied in this paper and a comparison principle for equation (1.6).
2. Preliminaries

2.1. Notation

We denote by $B_R \subset \mathbb{R}^N$ an open ball of radius $R > 0$ that is centered at the origin in the $N$-dimensional Euclidean space, $N \geq 1$. The set of symmetric $N \times N$ matrices is denoted by $S^N$. For $X, Y \in S^N$, we write $X \leq Y$ if $X - Y$ is negative semidefinite. We also denote the smallest eigenvalue of $X$ by $\lambda_{\min}(X)$ and the largest by $\lambda_{\max}(X)$ and set

$$
\|X\| := \sup_{\xi \in B_1} |X\xi| = \sup \{ |\lambda| : \lambda \text{ is an eigenvalue of } X \} .
$$

We use the notation $C(a_1, \ldots, a_k)$ to denote a constant $C$ that may change from line to line but depends only on $a_1, \ldots, a_k$. For convenience we often use $C(\hat{p})$ to mean that the constant may depend on $p_{\min}$, $p_{\max}$ and the Lipschitz constant $p_L$ of $p$.

For $\alpha \in (0, 1)$, we denote by $C^{\alpha}(B_R)$ the set of all functions $u : B_R \to \mathbb{R}$ with finite Hölder norm

$$
\|u\|_{C^\alpha(B_R)} := \|u\|_{L^\infty(B_R)} + [u]_{C^\alpha(B_R)}, \quad \text{where } [u]_{C^\alpha(B_R)} := \sup_{x,y \in B_R} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.
$$

Similarly, we denote by $C^{1,\alpha}(B_R)$ the set of all functions for which the norm

$$
\|u\|_{C^{1,\alpha}(B_R)} := \|u\|_{C^\alpha(B_R)} + \|Du\|_{C^\alpha(B_R)}
$$

is finite.

2.2. Viscosity solutions

Viscosity solutions are defined using smooth test functions that touch the solution from above or below. If $u, \varphi : \mathbb{R}^N \to \mathbb{R}$ and $x \in \mathbb{R}^N$ are such that $\varphi(x) = u(x)$ and $\varphi(y) < u(y)$ for $y \neq x_0$, then we say that $\varphi$ touches $u$ from below at $x_0$.

**Definition 2.1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution to

$$
-\Delta^N_{p(x)} u \geq f(x, u) \quad \text{in } \Omega
$$

if the following holds: Whenever $\varphi \in C^2(\Omega)$ touches $u$ from below at $x \in \Omega$ and $D\varphi(x) \neq 0$, we have

$$
-\Delta \varphi(x) - (p(x) - 2) \frac{\langle D^2 \varphi(x) D\varphi(x), D\varphi(x) \rangle}{|D\varphi(x)|^2} \geq f(x, u(x))
$$

and if $D\varphi(x) = 0$, then

$$
-\Delta \varphi(x) - (p(x) - 2) \langle D^2 \varphi(x) \eta, \eta \rangle \geq f(x, u(x)) \quad \text{for some } \eta \in \overline{B_1}.
$$

Analogously, a lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution if the above inequalities hold reversed whenever $\varphi$ touches $u$ from above. Finally, we say that $u$ is a viscosity solution if it is both viscosity sub- and supersolution.
Remark. The special treatment of the vanishing gradient in Definition 2.1 is needed because of the singularity of the equation. Definition 2.1 is essentially a relaxed version of the standard definition in [14] which is based on the so-called semicontinuous envelopes. In the standard definition one would require that if $\varphi$ touches a viscosity supersolution $u$ from below at $x$, then

\[
\begin{cases}
-\Delta^N_{p(x)}\varphi(x) \geq f(x,u(x)) & \text{if } D\varphi(x) \neq 0, \\
-\Delta \varphi(x) - (p(x) - 2)\lambda_{\min}(D^2\varphi(x)) \geq f(x,u(x)) & \text{if } D\varphi(x) = 0 \text{ and } p(x) \geq 2, \\
-\Delta \varphi(x) - (p(x) - 2)\lambda_{\max}(D^2\varphi(x)) \geq f(x,u(x)) & \text{if } D\varphi(x) = 0 \text{ and } p(x) < 2.
\end{cases}
\]

Clearly, if $u$ is a viscosity supersolution in this sense, then it is also a viscosity supersolution in the sense of Definition 2.1.

3. Hölder gradient estimates for the regularized homogeneous equation

In this section we prove $C^{1,\alpha}$-regularity estimates for solutions to the equation

\[
-\Delta u - (p(x) - 2)\frac{\langle D^2u(Du + q), Du + q \rangle}{|Du + q|^2 + \varepsilon^2} = 0 \quad \text{in } B_1, \tag{3.1}
\]

where $p : B_1 \to \mathbb{R}$ is Lipschitz, $p_{\min} > 1$, $\varepsilon > 0$ and $q \in \mathbb{R}^N$. Our objective is to obtain estimates that are independent of $q$ and $\varepsilon$. Observe that (3.1) is a uniformly elliptic quasilinear equation with smooth coefficients. Viscosity solutions to (3.1) can be defined in the standard way and they are smooth if $p$ is smooth.

Proposition 3.1. Suppose that $p$ is smooth. Let $u$ be a viscosity solution to (3.1) in $B_1$. Then $u \in C^\infty(B_1)$.

It follows from classical theory that the corresponding Dirichlet problem admits a smooth solution (see [21, Theorems 15.18 and 13.6] and the Schauder estimates [21, Theorem 6.17]). The viscosity solution $u$ coincides with the smooth solution by a comparison principle [26, Theorem 3].

3.1. Improvement of oscillation

Our regularity estimates for solutions of (3.1) are based on improvement of oscillation. We first prove such a result for the linear equation

\[
-\text{tr}(G(x)D^2u) = f \quad \text{in } B_1, \tag{3.2}
\]

where $f \in C^1(B_1)$ is bounded, $G(x) \in S^N$ and there are constants $0 < \lambda < \Lambda < \infty$ such that the eigenvalues of $G(x)$ are in $[\lambda, \Lambda]$ for all $x \in B_1$. The result is based on the following rescaled version of the weak Harnack inequality found in [13, Theorem 4.8]. Such Harnack estimates for non-divergence form equations go back to at least Krylov and Safonov [28,29].

Lemma 3.2 (Weak Harnack inequality). Let $u \geq 0$ be a continuous viscosity supersolution to (3.2) in $B_1$. Then there are positive constants $C(\lambda, \Lambda, N)$ and $q(\lambda, \Lambda, N)$ such that for any $\tau < \frac{1}{4\sqrt{N}}$ we have

\[
\tau^{-\frac{N}{q}} \left( \int_{B_\tau} |u|^q \, dx \right)^{1/q} \leq C \left( \inf_{B_{2\tau}} u + \tau \left( \int_{B_{4\sqrt{N} \tau}} |f|^N \, dx \right)^{1/N} \right). \tag{3.3}
\]
Proof. Suppose that \( \tau < \frac{1}{4 \sqrt{N}} \) and set \( S := 8\tau \). Define the function \( v : B_{\sqrt{N}/2} \to \mathbb{R} \) by

\[
v(x) := u(Sx)
\]

and set

\[
\tilde{G}(x) := G(Sx) \quad \text{and} \quad \tilde{f}(x) := S^2f(Sx).
\]

Then, if \( \varphi \in C^2 \) touches \( v \) from below at \( x \in B_{\sqrt{N}/2} \), the function \( \phi(x) := \varphi(x/S) \) touches \( u \) from below at \( Sx \). Therefore

\[
-\text{tr}(G(Sx)D^2\phi(Sx)) \geq f(Sx).
\]

Since \( D^2\phi(Sx) = S^{-2}D^2\varphi(x) \), this implies that

\[
-\text{tr}(G(Sx)D^2\varphi(x)) \geq S^2f(Sx).
\]

Thus \( v \) is a viscosity supersolution to

\[
-\text{tr}(\tilde{G}(x)D^2v) \geq \tilde{f}(x) \quad \text{in} \quad B_{\sqrt{N}/2}.
\]

We denote by \( Q_R \) a cube with side-length \( R/2 \). Since \( Q_1 \subset B_{\sqrt{N}/2} \), it follows from [13, Theorem 4.8] that there are \( q(\lambda,\Lambda,N) \) and \( C(\lambda,\Lambda,N) \) such that

\[
\left( \int_{B_{1/8}} |v|^q \, dx \right)^{1/q} \leq \left( \int_{Q_{1/4}} |v|^q \, dx \right)^{1/q} \leq C \left( \inf_{Q_{1/2}} v + \left( \int_{Q_1} |\tilde{f}|^N \, dx \right)^{1/N} \right)
\]

\[
\leq C \left( \inf_{B_{1/4}} v + \left( \int_{B_{\sqrt{N}/2}} |\tilde{f}|^N \, dx \right)^{1/N} \right).
\]

By the change of variables formula we have

\[
\int_{B_{1/8}} |v|^q \, dx = \int_{B_{1/8}} |u(Sx)|^q \, dx = S^{-N} \int_{B_{S/8}} |u(x)|^q \, dx
\]

and

\[
\int_{B_{\sqrt{N}/2}} |\tilde{f}|^N \, dx = S^{2N} \int_{B_{\sqrt{N}/2}} |f(Sx)|^N \, dx = S^N \int_{B_{S\sqrt{N}/2}} |f(x)|^N \, dx.
\]

Recalling that \( S = 8\tau \), we get

\[
8^{-\frac{N}{4}} \tau^{-\frac{N}{4}} \left( \int_{B_{\tau}} |u(x)|^q \, dx \right)^{1/q} \leq C \left( \inf_{B_{2\tau}} u + 8\tau \left( \int_{B_{S\sqrt{N}/2}} |f(x)|^N \, dx \right)^{1/N} \right).
\]

Absorbing \( 8^{\frac{N}{4}} \) into the constant, we obtain the claim. \( \square \)
Lemma 3.3 (Improvement of oscillation for the linear equation). Let \( u \geq 0 \) be a continuous viscosity supersolution to (3.2) in \( B_1 \) and \( \mu, l > 0 \). Then there are positive constants \( \tau(\lambda, A, N, \mu, l, \| f \|_{L^\infty(B_1)}) \) and \( \theta(\lambda, A, N, \mu, l) \) such that if

\[
|\{ x \in B_\tau : u \geq l \}| > \mu |B_\tau|, \tag{3.4}
\]

then we have

\[
u \geq \theta \quad \text{in } B_\tau.
\]

Proof. By the weak Harnack inequality (Lemma 3.2) there exist constants \( C_1(\lambda, A, N) \) and \( q(\lambda, A, N) \) such that for any \( \tau < 1/(4\sqrt{N}) \), we have

\[
\inf_{\overline{B}_2} u \geq C_1 \tau^{-\frac{N}{p}} \left( \int_{B_\tau} |u|^q \, dx \right)^{1/q} - \tau \left( \int_{B_4 \setminus \overline{B}_\tau} |f|^N \, dx \right)^{1/N}. \tag{3.5}
\]

In particular, this holds for

\[
\tau := \min \left( \frac{1}{4\sqrt{N}}, \frac{C_1 |B_1|^\frac{1}{q} \frac{1}{2^{\frac{1}{2}}} \mu^\frac{1}{2} l}{2 \cdot 4\sqrt{N} (\| f \|_{L^\infty(B_1)} + 1)} \right).
\]

We continue the estimate (3.5) using the assumption (3.4) and obtain

\[
\inf_{\overline{B}_2} u \geq \inf_{\overline{B}_2} u \geq C_1 \tau^{-\frac{N}{p}} \left( |\{ x \in B_\tau : u \geq l \}| l^q \right)^{1/q} - \tau \left( \int_{B_4 \setminus \overline{B}_\tau} |f|^N \, dx \right)^{1/N}
\]

\[
\geq C_1 \tau^{-\frac{N}{p}} \mu^\frac{1}{2} |B_\tau|^\frac{1}{2} l - \tau |B_4 \setminus \overline{B}_\tau| \frac{1}{\sqrt{N}} \| f \|_{L^\infty(B_1)}
\]

\[
= C_1 |B_1|^\frac{1}{q} \mu^\frac{1}{2} l \tau^{-\frac{N}{p}} \tau^\frac{q}{2} - 4\sqrt{N} |B_1|^\frac{1}{q} \| f \|_{L^\infty(B_1)} \tau^2
\]

\[
= C_1 |B_1|^\frac{1}{q} \mu^\frac{1}{2} l - 4\sqrt{N} |B_1|^\frac{1}{q} \| f \|_{L^\infty(B_1)} \tau^2.
\]

\[
\geq \frac{1}{2} C_1 |B_1|^\frac{1}{q} \mu^\frac{1}{2} l, \quad \therefore \theta,
\]

where the last inequality follows from the choice of \( \tau \). \( \square \)

We are now ready to prove an improvement of oscillation for the gradient of a solution to (3.1). We first consider the following lemma, where the improvement is considered towards a fixed direction. We initially also restrict the range of \( |q| \).

The idea is to differentiate the equation and observe that a suitable function of \( Du \) is a supersolution to the linear equation (3.2). Lemma 3.3 is then applied to obtain information about \( Du \).

Lemma 3.4 (Improvement of oscillation to direction). Suppose that \( p \) is smooth. Let \( u \) be a smooth solution to (3.1) in \( B_1 \) with \( |Du| \leq 1 \) and either \( q = 0 \) or \( |q| > 2 \). Then for every \( 0 < l < 1 \) and \( \mu > 0 \) there exist positive constants \( \tau(N, \tilde{p}, l, \mu) < 1 \) and \( \gamma(N, \tilde{p}, l, \mu) < 1 \) such that
\[ \left| \{ x \in B_r : Du \cdot d \leq l \} \right| > \mu |B_r| \quad \text{implies} \quad Du \cdot d \leq \gamma \text{ in } B_r \]

whenever \( d \in \partial B_1 \).

**Proof.** To simplify notation, we set

\[ A_{ij}(x, \eta) := \delta_{ij} + (p(x) - 2) \frac{(\eta_i + q_i)(\eta_j + q_j)}{|\eta|^2 + \varepsilon^2}. \]

We also denote the functions \( A_{ij} : x \mapsto A_{ij}(x, Du(x)) \), \( A_{ij,x_k} : x \mapsto (\partial_{x_k} A_{ij})(x, Du(x)) \) and \( A_{ij,\eta_k} : x \mapsto (\partial_{\eta_k} A_{ij})(x, Du(x)) \). Then, since \( u \) is a smooth solution to (3.1) in \( B_1 \), we have in Einstein’s summation convention

\[ -A_{ij}u_{ij} = 0 \quad \text{pointwise in } B_1. \]

Differentiating this yields

\[ 0 = (A_{ij}u_{ij})_k = A_{ij}u_{ijk} + (A_{ij})_k u_{ij} = A_{ij}u_{ijk} + A_{ij,\eta_m}u_{ij}u_{km} + A_{ij,x_k}u_{ij} \quad \text{for all } k = 1, \ldots, N. \quad (3.6) \]

Multiplying these identities by \( d_k \) and summing over \( k \), we obtain

\[ 0 = A_{ij}u_{ijk}d_k + A_{ij,\eta_m}u_{ij}u_{km}d_k + A_{ij,x_k}u_{ij}d_k \]

\[ = A_{ij}(Du \cdot d - l)_{ij} + A_{ij,\eta_m}u_{ij}(Du \cdot d - l)_m + A_{ij,x_k}u_{ij}d_k. \quad (3.7) \]

Moreover, multiplying (3.6) by \( 2u_k \) and summing over \( k \), we obtain

\[ 0 = 2A_{ij}u_{ijk}u_k + 2A_{ij,\eta_m}u_{ij}u_{km}u_k + 2A_{ij,x_k}u_{ij}u_k \]

\[ = A_{ij}(2u_{ijk}u_k + 2u_{ki}u_{kj}) - 2A_{ij}u_{kj}u_{ki} + 2A_{ij,\eta_m}u_{ij}u_{km}u_k + 2A_{ij,x_k}u_{ij}u_k \]

\[ = A_{ij}(u_{jk}^2)_{ij} - 2A_{ij}u_{kj}u_{ki} + A_{ij,\eta_m}u_{ij}(u_{jk}^2)_m + 2A_{ij,x_k}u_{ij}u_k \]

\[ = A_{ij}(|Du|^2)_{ij} + A_{ij,\eta_m}u_{ij}(|Du|^2)_m + 2A_{ij,x_k}u_{ij}u_k - 2A_{ij}u_{kj}u_{ki}. \quad (3.8) \]

We will now split the proof into the cases \( q = 0 \) or \( |q| > 2 \), and proceed in two steps: First we check that a suitable function of \( Du \) is a supersolution to the linear equation (3.3) and then apply Lemma 3.3 to obtain the claim.

**Case q = 0, Step 1:** We denote \( \Omega_+ := \{ x \in B_1 : h(x) > 0 \} \), where

\[ h := (Du \cdot d - l + \frac{l}{2} |Du|^2)^+. \]

If \( |Du| \leq l/2 \), we have

\[ Du \cdot d - l + \frac{l}{2} |Du|^2 \leq -\frac{l}{2} + \frac{l^3}{8} < 0. \]

This implies that \( |Du| > l/2 \) in \( \Omega_+ \). Therefore, since \( q = 0 \), we have in \( \Omega_+ \)
\[ |\mathcal{A}_{ij,\eta_m}| = |p(x) - 2| \frac{\delta_{im}(u_j + q_j) + \delta_{jm}(u_i + q_i)}{|Du + q|^2 + \varepsilon^2} - \frac{2(u_m + q_m)(u_i + q_i)(u_j + q_j)}{|Du + q|^2 + \varepsilon^2}^2 \]
\[ \leq 8l^{-1} \|p - 2\|_{L^\infty(B_1)}, \quad \text{(3.9)} \]
\[ |\mathcal{A}_{ij,x_k}| = |Dp(x)| \frac{(|\eta + q_i|(|\eta + q_j)|}{|\eta + q|^2 + \varepsilon^2} \leq p_L. \quad \text{(3.10)} \]

Summing up the equations (3.7) and (3.8) multiplied by \(2^{-1}l\), we obtain in \(\Omega_+\)

\[ 0 = A_{ij}(Du \cdot d - l)_{ij} + A_{ij,\eta_m}u_{ij}(Du \cdot d - l)_m + A_{ij,x_k}u_{ij}d_k \]
\[ + 2^{-1}l(A_{ij}(|Du|^2)_{ij} + A_{ij,\eta_m}u_{ij}(|Du|^2)_m + 2A_{ij,x_k}u_{ij}u_k - 2A_{ij}u_ku_{ij}k_i) \]
\[ = A_{ij}h_{ij} + A_{ij,\eta_m}u_{ij}h_m + A_{ij,x_k}u_{ij}d_k + lA_{ij,x_k}u_{ij}u_k - lA_{ij}u_ku_{ij}k_i \]
\[ \leq A_{ij}h_{ij} + |A_{ij,\eta_m}u_{ij}||h_m| + |A_{ij,x_k}u_{ij}||d_k + lu_k| - lA_{ij}u_ku_{ij}k_i. \]

Since \(|Du| \leq 1\), we have \(|d_k + lu_k|^2 \leq 4\) and by uniform ellipticity \(A_{ij}u_ku_{ij}k_i \geq \min(p_{\min} - 1, 1) |u_{ij}|^2\).

Therefore, by applying Young’s inequality with \(\varepsilon > 0\), we obtain from the above display

\[ 0 \leq A_{ij}h_{ij} + N^2\varepsilon^{-1}(|h_m|^2 + |d_k + lu_k|^2) + \varepsilon(|A_{ij,\eta_m}|^2 + |A_{ij,x_k}|^2)|u_{ij}|^2 - lA_{ij}u_ku_{ij}k_i \]
\[ \leq A_{ij}h_{ij} + N^2\varepsilon^{-1}(|Dh|^2 + 4) + \varepsilon C(N, \hat{p})(l^{-2} + 1) |u_{ij}|^2 - l\min(p_{\min} - 1, 1) |u_{ij}|^2, \]

where in the second estimate we used (3.9) and (3.10). By taking \(\varepsilon\) small enough, we obtain

\[ 0 \leq A_{ij}h_{ij} + C_0(N, \hat{p}) \frac{|Dh|^2 + 1}{l^3} \quad \text{in } \Omega_. \quad \text{(3.11)} \]

Next we define

\[ \overline{h} := \frac{1}{\nu} (1 - e^{\nu(h-H)}), \quad \text{where } H := 1 - \frac{l}{2}, \quad \text{and } \nu := \frac{C_0}{l^3 \min(p_{\min} - 1, 1)}. \quad \text{(3.12)} \]

Then by (3.11) and uniform ellipticity we have in \(\Omega_+\)

\[ -A_{ij}\overline{h}_{ij} = A_{ij}(h_{ij}e^{\nu(h-H)} + \nu h_{ij}e^{\nu(h-H)}) \]
\[ \geq e^{\nu(h-H)} (-C_0 \frac{|Dh|^2}{l^3} - \frac{C_0}{l^3} + \nu \min(p_{\min} - 1, 1) |Dh|^2) \]
\[ \geq - \frac{C_0}{l^3}. \]

Since the minimum of two viscosity supersolutions is still a viscosity supersolution, it follows from the above estimate that \(\overline{h}\) is a non-negative viscosity supersolution to

\[ -A_{ij}\overline{h}_{ij} \geq - \frac{C_0}{l^3} \quad \text{in } B_1. \quad \text{(3.13)} \]

**Case q = 0, Step 2:** We set \(l_0 := \frac{1}{\nu}(1 - e^{\nu(l-1)})\). Then, since \(\overline{h}\) solves (3.13), by Lemma 3.3 there are positive constants \(\tau(N, p, l, \mu)\) and \(\theta(N, p, l, \mu)\) such that

\[ |\{x \in B_\tau : \overline{h} \geq l_0\}| > \mu |B_\tau| \quad \text{imply } \overline{h} \geq \theta \quad \text{in } B_\tau. \]

If \(Du \cdot d \leq l\), we have \(\overline{h} \geq l_0\) and therefore
\[ \left| \{ x \in B_\tau : h \geq l_0 \} \right| \geq \left| \{ x \in B_\tau : Du \cdot d \leq l \} \right| > \mu |B_\tau| , \]

where the last inequality follows from the assumptions. Consequently, we obtain
\[ h \geq \theta \quad \text{in } B_\tau. \]

Since \( h - H \leq 0 \), by convexity we have \( H - h \geq \bar{h} \). This together with the above estimate yields
\[ 1 - 2^{-1}l - (Du \cdot d - l + 2^{-1}l|Du|^2) \geq \theta \quad \text{in } B_\tau \]

and so
\[ Du \cdot d + 2^{-1}l(Du \cdot d)^2 \leq Du \cdot d + 2^{-1}l|Du|^2 \leq 1 + 2^{-1}l - \theta \quad \text{in } B_\tau. \]

Using the quadratic formula, we thus obtain the desired estimate
\[ Du \cdot d \leq \frac{-1 + \sqrt{1 + 2l(1 + 2^{-1}l - \theta)}}{l} = \frac{-1 + \sqrt{(1 + l)^2 - 2l\theta}}{l} = : \gamma < 1 \quad \text{in } B_\tau. \]

**Case** \(|q| > 2\): Computing like in (3.9) and (3.10), we obtain this time in \( B_1 \)
\[ |A_{ij,\eta_m}| \leq 4 \| p - 2 \|_{L^\infty(B_1)} \quad \text{and} \quad |A_{ij,x}| \leq p_L \]

Moreover, this time we set simply
\[ h := Du \cdot d - l + 2^{-1}l|Du|^2. \]

Summing up the identities (3.7) and (3.8) and using Young’s inequality similarly as in the case \(|q| = 0\), we obtain in \( B_1 \)
\[ 0 \leq A_{ij}h_{ij} + N^2\epsilon^{-1}(|h_m|^2 + |d_k + l\eta_k|^2) + \epsilon(|A_{ij,\eta_m}|^2 + |A_{ij,x}|^2) |u_{ij}|^2 - lA_{ij}u_{kj}u_{ki} \]
\[ \leq A_{ij}h_{ij} + N^2\epsilon^{-1}(|Dh|^2 + 4) + \epsilon C(\hat{p}) |u_{ij}|^2 - lC(\hat{p}) |u_{ij}|^2. \]

By taking small enough \( \epsilon \), we obtain
\[ 0 \leq A_{ij}h_{ij} + C_0(N,\hat{p}) \frac{|Dh|^2 + 1}{l} \quad \text{in } B_1. \]

Next we define \( \bar{h} \) and \( H \) like in (3.12), but set instead \( \nu := C_0/(l \min(p_{\min} - 1, 1)) \). The rest of the proof then proceeds in the same way as in the case \( q = 0 \). \( \square \)

Next we inductively apply the previous lemma to prove the improvement of oscillation.

**Theorem 3.5** (Improvement of oscillation). Suppose that \( p \) is smooth. Let \( u \) be a smooth solution to (3.1) in \( B_1 \) with \(|Du| \leq 1 \) and either \( q = 0 \) or \(|q| > 2\). Then for every \( 0 < l < 1 \) and \( \mu > 0 \) there exist positive constants \( \tau(N,\hat{p},l,\mu) < 1 \) and \( \gamma(N,\hat{p},l,\mu) < 1 \) such that if
\[ \left| \{ x \in B_{\tau_{i+1}} : Du \cdot d \leq l\gamma^i \} \right| > \mu |B_{\tau_{i+1}}| \quad \text{for all } d \in \partial B_1, \; i = 0, \ldots, k, \quad (3.14) \]

then
\[ |Du| \leq \gamma^{i+1} \quad \text{in } B_{\tau_{i+1}} \quad \text{for all } i = 0, \ldots, k. \quad (3.15) \]
**Proof.** Let \( k \geq 0 \) be an integer and suppose that (3.14) holds. We proceed by induction.

**Initial step:** Since (3.14) holds for \( i = 0 \), by Lemma 3.4 we have \( Du \cdot d \leq \gamma \) in \( B_{\tau} \) for all \( d \in \partial B_1 \). This implies (3.15) for \( i = 0 \).

**Induction step:** Suppose that \( 0 < i \leq k \) and that (3.15) holds for \( i - 1 \). We define

\[
v(x) := \tau^{-i} \gamma^{-i} u(\tau^{i} x).
\]

Then \( v \) solves

\[
-\Delta v - (p(\tau^{i} x) - 2) \frac{\langle D^2 v(Dv + \gamma^{-i} q), Dv + \gamma^i q \rangle}{|Dv + \gamma^{-i} q|^2 + (\gamma^{-i} \varepsilon)^2} = 0 \quad \text{in } B_1.
\]

Moreover, by induction hypothesis \(|Dv(x)| = \gamma^{-i} |Dv(\tau^{i} x)| \leq \gamma^{-i} \gamma^i = 1 \) in \( B_1 \). Therefore by Lemma 3.4 we have that

\[
|\{x \in B_{\tau} : Dv \cdot d \leq l\}| > \mu |B_{\tau}| \quad \text{implies} \quad Dv \cdot d \leq \gamma \quad \text{in } B_{\tau}
\]

whenever \( d \in \partial B_1 \). Since

\[
|\{x \in B_{\tau} : Dv \cdot d \leq l\}| > \mu |B_{\tau}| \iff |\{x \in B_{\tau} : Dv \cdot d \leq l \gamma^i\}| > \mu |B_{\tau}^{i+1}|,
\]

we have by (3.14) and (3.16) that \( Dv \cdot d \leq \gamma \) in \( B_{\tau} \). This implies that \( Du \cdot d \leq \gamma^{i+1} \) in \( B_{\tau}^{i+1} \). Since \( d \in \partial B_1 \) was arbitrary, we obtain (3.15) for \( i \).

3.2. Hölder gradient estimates

In this section we apply the improvement of oscillation to prove \( C^{1,\alpha} \)-estimates for solutions to (3.1). We need the following regularity result by Savin [34].

**Lemma 3.6.** Suppose that \( p \) is smooth. Let \( u \) be a smooth solution to (3.1) in \( B_1 \) with \(|Du| \leq 1 \) and either \( q = 0 \) or \(|q| > 2 \). Then for any \( \beta > 0 \) there exist positive constants \( \eta(N, \tilde{p}, \beta) \) and \( C(N, \hat{p}, \beta) \) such that if

\[
|u - L| \leq \eta \quad \text{in } B_1
\]

for some affine function \( L \) satisfying \( 1/2 \leq |DL| \leq 1 \), then we have

\[
|Du(x) - Du(0)| \leq C |x|^\beta \quad \text{for all } x \in B_{1/2}.
\]

**Proof.** Set \( v := u - L \). Then \( v \) solves

\[
-\Delta u - \frac{(p(x) - 2) \langle D^2 u(Du + q + DL), Du + q + DL \rangle}{|Du + q + DL|^2 + \varepsilon^2} = 0 \quad \text{in } B_1.
\]

Observe that by the assumption \( 1/2 \leq |DL| \leq 1 \) we have \(|Du + q + DL| \geq 1/4\) if \(|Du| \leq 1/4\). It therefore follows from [34, Theorem 1.3] (see also [37]) that \( \|v\|_{C^{2,\beta}(B_{1/2})} \leq C \) which implies the claim.

We also use the following simple consequence of Morrey’s inequality.
Lemma 3.7. Let $u : B_1 \to \mathbb{R}$ be a smooth function with $|Du| \leq 1$. For any $\theta > 0$ there are constants $\varepsilon_1(N,\theta), \varepsilon_0(N,\theta) < 1$ such that if the condition

$$\{|x \in B_1 : |Du(d) - d| > \varepsilon_0\} \leq \varepsilon_1$$

is satisfied for some $d \in S^{N-1}$, then there is $a \in \mathbb{R}$ such that

$$|u(x) - a - d \cdot x| \leq \theta$$

for all $x \in B_{1/2}$.

Proof. By Morrey’s inequality (see for example [17, Theorem 4.10])

$$\text{osc}_{x \in B_{1/2}} (u(x) - d \cdot x) = \sup_{x,y \in B_{1/2}} |u(x) - d \cdot x - u(y) + d \cdot y|$$

$$\leq C(N) \left( \int_{B_1} |Du(d) - d|^{2N} \, dx \right)^{\frac{1}{2N}}$$

$$\leq C(N)(\varepsilon_1^{\frac{1}{2N}} + \varepsilon_0).$$

Therefore, denoting $a := \inf_{x \in B_{1/2}} (u(x) - d \cdot x)$, we have for any $x \in B_{1/2}$

$$|u(x) - a - d \cdot x| \leq \text{osc}_{B_{1/2}} (u(x) - d \cdot x) \leq C(N)(\varepsilon_1^{\frac{1}{2N}} + \varepsilon_0) \leq \theta,$$

where the last inequality follows by taking small enough $\varepsilon_0$ and $\varepsilon_1$. $\square$

We are now ready to prove a H"older estimate for the gradient of solutions to (3.1). We first restrict the range of $|q|$.

Lemma 3.8. Suppose that $p$ is smooth. Let $u$ be a smooth solution to (3.1) in $B_1$ with $|Du| \leq 1$ and either $q = 0$ or $|q| > 2$. Then there exists a constant $\alpha(N, \hat{p}) \in (0,1)$ such that

$$\|Du\|_{C^\alpha(B_{1/2})} \leq C(N, \hat{p}).$$

Proof. For $\beta = 1/2$, let $\eta > 0$ be as in Lemma 3.6. For $\theta = \eta/2$, let $\varepsilon_0, \varepsilon_1$ be as in Lemma 3.7. Set

$$l := 1 - \frac{\varepsilon_0^2}{2} \quad \text{and} \quad \mu := \frac{\varepsilon_1}{|B_1|}.$$

For these $l$ and $\mu$, let $\tau, \gamma \in (0,1)$ be as in Theorem 3.5. Let $k \geq 0$ be the minimum integer such that the condition (3.14) does not hold.

Case $k = \infty$: Theorem 3.5 implies that

$$|Du| \leq \gamma^{i+1}$$

in $B_{\tau i+1}$ for all $i \geq 0$.

Let $x \in B_\tau \setminus \{0\}$. Then $\tau^{i+1} \leq |x| \leq \tau^i$ for some $i \geq 0$ and therefore

$$i \leq \frac{\log |x|}{\log \tau} \leq i + 1.$$

We obtain
\[ |Du(x)| \leq \gamma^i = 1 - \frac{1}{\gamma} \gamma^{i+1} \leq \frac{1}{\gamma} \gamma^{\log_{\gamma} \gamma} = \frac{1}{\gamma} \gamma^{\log_{\gamma} \gamma} = C \ |x|^\alpha, \tag{3.18} \]

where \( C = 1/\gamma \) and \( \alpha = \log \gamma / \log \tau \).

**Case** \( k < \infty \): There is \( d \in \partial B_1 \) such that
\[ \left| \{ x \in B_{\tau^{k+1}} : Du \cdot d \leq l \gamma^k \} \right| \leq \mu |B_{\tau^{k+1}}|. \tag{3.19} \]

We set
\[ v(x) := \tau^{-k-1} \gamma^{-k} u(\tau^{k+1} x). \]

Then \( v \) solves
\[ -\Delta v - (p(\tau^{k+1} x) - 2) \left< \frac{D^2 v(Dv + \gamma^{-k} q), Dv + \gamma^{-k} q}{|Dv + \gamma^{-k} q|^2 + \gamma^{-2k} \varepsilon^2} \right> = 0 \quad \text{in } B_1 \]

and by (3.19) we have
\[ \left| \{ x \in B_1 : Dv \cdot d \leq l \} \right| = \left| \{ x \in B_1 : Du(\tau^{k+1} x) \cdot d \leq l \gamma^k \} \right| = \tau^{-N(k+1)} \left| \{ x \in B_{\tau^{k+1}} : Du(x) \cdot d \leq l \gamma^k \} \right| \leq \tau^{-N(k+1)} \mu |B_{\tau^{k+1}}| = \mu |B_1| = \varepsilon_1. \tag{3.20} \]

Since either \( k = 0 \) or (3.14) holds for \( k - 1 \), it follows from Theorem 3.5 that \( |Du| \leq \gamma^k \) in \( B_{\tau^k} \). Thus
\[ |Dv(x)| = \gamma^{-k} |Du(\tau^{k+1} x)| \leq 1 \quad \text{in } B_1. \tag{3.21} \]

For vectors \( \xi, d \in B_1 \), it is easy to verify the following fact
\[ |\xi - d| > \varepsilon_0 \implies \xi \cdot d \leq 1 - \varepsilon_0^2/2 = l. \]

Therefore, in view of (3.20) and (3.21), we obtain
\[ \left| \{ x \in B_1 : |Dv - d| > \varepsilon_0 \} \right| \leq \varepsilon_1. \]

Thus by Lemma 3.7 there is \( a \in \mathbb{R} \) such that
\[ |v(x) - a - d \cdot x| \leq \theta = \eta/2 \quad \text{for all } x \in B_{1/2}. \]

Consequently, by applying Lemma 3.6 on the function \( 2v(2^{-1} x) \), we find a positive constant \( C(N, \hat{p}) \) and \( e \in \partial B_1 \) such that
\[ |Dv(x) - e| \leq C \ |x| \quad \text{in } B_{1/4}. \]

Since \( |Dv| \leq 1 \), we have also
\[ |Dv(x) - e| \leq C \ |x| \quad \text{in } B_1. \]

Recalling the definition of \( v \) and taking \( \alpha' \in (0, 1) \) so small that \( \gamma/\tau^{\alpha'} < 1 \) we obtain
\[ |Du(x) - \gamma^k e| \leq C \gamma^k r^{-\alpha' - 1} |x| \leq \frac{C}{\tau^{\alpha'}} \left( \frac{\gamma}{\tau^{\alpha'}} \right)^k |x|^\alpha' \leq C |x|^\alpha' \quad \text{in } B_{r^{k+1}}, \tag{3.22} \]

where we absorbed \( \tau^{\alpha'} \) into the constant. On the other hand, we have

\[ |Du| \leq \gamma^{i+1} \quad \text{in } B_{r^{i+1}} \text{ for all } i = 0, \ldots, k-1 \]

so that, if \( \tau^{i+2} \leq |x| \leq \tau^{i+1} \) for some \( i \in \{0, \ldots, k-1\} \), it holds that

\[ |Du(x) - \gamma^k e| \leq 2 \gamma^{i+1} |x|^\alpha' \leq \frac{2}{\tau^{\alpha'}} \left( \frac{\gamma}{\tau^{\alpha'}} \right)^{i+1} |x|^\alpha' \leq C |x|^\alpha'. \]

Combining this with (3.22) we obtain

\[ |Du(x) - \gamma^k e| \leq C |x|^\alpha' \quad \text{in } B_r. \tag{3.23} \]

The claim now follows from (3.18) and (3.23) by standard translation arguments. \( \square \)

**Theorem 3.9.** Let \( u \) be a bounded viscosity solution to (3.1) in \( B_1 \) with \( q \in \mathbb{R}^N \). Then

\[ \|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(N, \hat{p}, \|u\|_{L^\infty(B_1)}) \tag{3.24} \]

for some \( \alpha(N, \hat{p}) \in (0,1) \).

**Proof.** Suppose first that \( p \) is smooth. Let \( \nu_0(N, \hat{p}, \|u\|_{L^\infty(B_1)}) \) and \( C_0(N, \hat{p}, \|u\|_{L^\infty(B_1)}) \) be as in the Lipschitz estimate (Theorem A.2 in the Appendix) and set

\[ M := 2 \max(\nu_0, C_0). \]

If \( |q| > M \), then by Theorem A.2 we have

\[ |Du| \leq C_0 \quad \text{in } B_{1/2}. \]

We set \( \tilde{u}(x) := 2u(x/2)/C_0 \). Then \( |D\tilde{u}| \leq 1 \) in \( B_1 \) and \( \tilde{u} \) solves

\[-\Delta \tilde{u} - (p/2 - 2) \frac{D^2 \tilde{u}(D\tilde{u} + q/C_0), D\tilde{u} + q/C_0}{|D\tilde{u} + q/C_0|^2 + (\epsilon/C_0)^2} = 0 \quad \text{in } B_1,\]

where \( q/C_0 > 2 \). Thus by Theorem 3.8 we have

\[ \|D\tilde{u}\|_{C^{\alpha}(B_{1/2})} \leq C(N, \hat{p}), \]

which implies (3.24) by standard translation arguments.

If \( |q| \leq M \), we define

\[ w := u - q \cdot x. \]

Then by Theorem A.2 we have

\[ |Dw| \leq C(N, \hat{p}, \|w\|_{L^\infty(B_1)}) =: C'(N, \hat{p}, \|u\|_{L^\infty(B_1)}) \quad \text{in } B_{1/2}. \]
We set \( \bar{w}(x) := 2w(x/2)/C' \). Then \( |D\bar{w}| \leq 1 \) and so by Theorem 3.6 we have

\[
\|D\bar{w}\|_{C^\infty(B_{1/2})} \leq C(N, \hat{\rho}),
\]

which again implies (3.24).

Suppose then that \( p \) is merely Lipschitz continuous. Take a sequence \( p_j \in C^\infty(B_1) \) such that \( p_j \to p \) uniformly in \( B_1 \) and \( \|Dp_j\|_{L^\infty(B_1)} \leq \|Dp\|_{L^\infty(B_1)} \). For \( r < 1 \), let \( u_j \) be a solution to the Dirichlet problem

\[
\begin{aligned}
\Delta u_j - (p_j(x) - 2) \frac{\langle D^2u(Du + q), Du + q \rangle}{|Du + q|^2 + \varepsilon^2} &= 0 \quad \text{in } B_r, \\
u_j &= u \quad \text{on } B_r.
\end{aligned}
\]

As observed in Proposition 3.1, the solution exists and we have \( u_j \in C^\infty(B_r) \). By comparison principle \( \|u_j\|_{L^\infty(B_r)} \leq \|u\|_{L^\infty(B_1)} \), then by the first part of the proof we have the estimate

\[
\|u_j\|_{C^{1,\gamma}(B_{r/2})} \leq C(N, \hat{\rho}, \|u\|_{L^\infty(B_1)}).
\]

By [13, Theorem 4.14] the functions \( u_j \) are equicontinuous in \( B_1 \) and so by the Ascoli-Arzelà theorem we have \( u_j \to v \) uniformly in \( B_1 \) up to a subsequence. Moreover, by the stability principle \( v \) is a solution to (3.1) in \( B_r \) and thus by comparison principle [27, Theorem 2.6] we have \( v \equiv u \). By extracting a further subsequence, we may ensure that also \( Du_j \to Du \) uniformly in \( B_{r/2} \) and so the estimate \( \|Du\|_{C^{1,\gamma}(B_{r/2})} \leq C(N, \hat{\rho}, \|u\|_{L^\infty(B_1)}) \) follows. \( \square \)

4. Hölder gradient estimates for the regularized inhomogeneous equation

In this section we consider the inhomogeneous equation

\[
-\Delta u - (p(x) - 2) \frac{\langle D^2u(Du + q), Du + q \rangle}{|Du + q|^2 + \varepsilon^2} = f(x) \quad \text{in } B_1,
\]

(4.1)

where \( p : B_1 \to \mathbb{R} \) is Lipschitz continuous, \( p_{\min} > 1, \varepsilon > 0, q \in \mathbb{R}^N \) and \( f \in C(B_1) \) is bounded. We apply the \( C^{1,\alpha} \)-estimates obtained in Theorem 3.9 to prove regularity estimates for solutions of (4.1) with \( q = 0 \). Our arguments are similar to those in [4, Section 3], see also [24]. The idea is to use the well known characterization of \( C^{1,\alpha} \)-regularity via affine approximates. The following lemma plays a key role: It states that if \( f \) is small, then a solution to (4.1) can be approximated by an affine function. This combined with scaling properties of the equation essentially yields the desired affine functions.

**Lemma 4.1.** There exist constants \( \epsilon(N, \hat{\rho}), \tau(N, \hat{\rho}) \in (0, 1) \) such that the following holds: If \( \|f\|_{L^\infty(B_1)} \leq \epsilon \) and \( w \) is a viscosity solution to (4.1) in \( B_1 \) with \( q \in \mathbb{R}^N, w(0) = 0, \text{osc}_{B_1} w \leq 1 \), then there exists \( q' \in \mathbb{R}^N \) such that

\[
\text{osc}_{B_r}(w(x) - q' \cdot x) \leq \frac{1}{2} \tau.
\]

Moreover, we have \( |q'| \leq C(N, \hat{\rho}) \).

**Proof.** Suppose on the contrary that the claim does not hold. Then, for a fixed \( \tau(N, \hat{\rho}) \) that we will specify later, there exists a sequence of Lipschitz continuous functions \( p_j : B_1 \to \mathbb{R} \) such that

\[
p_{\min} \leq \inf_{B_1} p_j \leq \sup_{B_1} p_j \leq p_{\max} \quad \text{and} \quad (p_j)_{L} \leq p_L,
\]
functions \(f_j \in C(B_1)\) such that \(f_j \to 0\) uniformly in \(B_1\), vectors \(q_j \in \mathbb{R}^N\) and viscosity solutions \(w_j\) to

\[-\Delta w_j - (p_j(x) - 2) \frac{\langle D^2 w_j(Dw_j + q_j), Dw_j + q_j \rangle}{|Dw_j + q_j|^2 + \varepsilon^2} = f_j(x)\quad \text{in } B_1\]

such that \(w_j(0) = 0\), \(\text{osc}_{B_1} w_j \leq 1\) and

\[\text{osc}_{B_1}(w_j(x) - q' \cdot x) > \frac{\tau}{2}\quad \text{for all } q' \in \mathbb{R}^N. \tag{4.2}\]

By [13, Proposition 4.10], the functions \(w_j\) are uniformly Hölder continuous in \(B_r\) for any \(r \in (0, 1)\). Therefore by the Ascoli-Arzela theorem, we may extract a subsequence such that \(w_j \to w_\infty\) and \(p_j \to p_\infty\) uniformly in \(B_r\) for any \(r \in (0, 1)\). Moreover, \(p_\infty\) is \(p_L\)-Lipschitz continuous and \(p_{\text{min}} \leq p_\infty \leq p_{\text{max}}\). It then follows from (4.2) that

\[\text{osc}_{B_r}(w_\infty(x) - q' \cdot x) > \frac{\tau}{2}\quad \text{for all } q' \in \mathbb{R}^N. \tag{4.3}\]

We have two cases: either \(q_j\) is bounded or unbounded.

**Case \(q_j\) is bounded:** In this case \(q_j \to q_\infty \in \mathbb{R}^N\) up to a subsequence. It follows from the stability principle that \(w_\infty\) is a viscosity solution to

\[-\Delta w_\infty - (p_\infty(x) - 2) \frac{\langle D^2 w_\infty(Dw_\infty + q_\infty), Dw_\infty + q_\infty \rangle}{|Dw_\infty + q_\infty|^2 + \varepsilon^2} = 0\quad \text{in } B_1. \tag{4.4}\]

Hence by Theorem 3.9 we have \(\|Dw_\infty\|_{C^{1/2}(B_{1/2})} \leq C(N, \hat{\beta})\) for some \(\beta_1(N, \hat{\beta})\). The mean value theorem then implies the existence of \(q' \in \mathbb{R}^N\) such that

\[\text{osc}_{B_r}(u - q' \cdot x) \leq C_1(N, \hat{\beta})r^{1+\beta_1}\quad \text{for all } r \leq 1.\]

**Case \(q_j\) is unbounded:** In this case we take a subsequence such that \(|q_j| \to \infty\) and the sequence \(d_j := d_j/|d_j|\) converges to \(d_\infty \in \partial B_1\). Then \(w_j\) is a viscosity solution to

\[-\Delta w_j - (p_j(x) - 2) \frac{\langle D^2 w_j(|q_j|^{-1}Dw_j + d_j), |q_j|^{-1}Dw_j + d_j \rangle}{|q_j|^{-1}Dw_j + d_j|^2 + |q_j|^{-2}\varepsilon^2} = f_j(x)\quad \text{in } B_1.\]

It follows from the stability principle that \(w_\infty\) is a viscosity solution to

\[-\Delta w_j - (p_\infty(x) - 2) \langle D^2 w_\infty d_\infty, d_\infty \rangle = 0\quad \text{in } B_1.\]

By [13, Theorem 8.3] there exist positive constants \(\beta_2(N, \hat{\beta}), C_2(N, \hat{\beta}), r_2(N, \hat{\beta})\) and a vector \(q' \in \mathbb{R}^N\) such that

\[\text{osc}_{B_r}(w_\infty - q' \cdot x) \leq C_2r^{1+\beta_2}\quad \text{for all } r \leq r_2.\]

We set \(C_0 := \max(C_1, C_2)\) and \(\beta_0 := \min(\beta_1, \beta_2)\). Then by the two different cases there always exists a vector \(q' \in \mathbb{R}^N\) such that

\[\text{osc}_{B_r}(w_\infty - q' \cdot x) \leq C_0r^{1+\beta_0}\quad \text{for all } r \leq \min\left(\frac{1}{2}, r_2\right).\]
We take \( \tau \) so small that \( C_0 \tau^{\beta_0} \leq \frac{1}{4} \) and \( \tau \leq \min(\frac{1}{2}, r_2) \). Then, by substituting \( r = \tau \) in the above display, we obtain
\[
\operatorname{osc}_{B_r}(w_\infty - q' \cdot x) \leq C_0 \tau^{\beta_0} \tau \leq \frac{1}{4} \tau,
\] (4.5)
which contradicts (4.3).

The bound \( |q'| \leq C(N, \hat{p}) \) follows by observing that (4.5) together with the assumption \( \operatorname{osc}_{B_1} w \leq 1 \) yields \( |q'| \leq C \). Thus the contradiction is still there even if (4.3) is weakened by requiring additionally that \( |q'| \leq C \). \( \square \)

**Lemma 4.2.** Let \( \tau(N, \hat{p}) \) and \( \epsilon(N, \hat{p}) \) be as in Lemma 4.1. If \( \|f\|_{L^\infty(B_1)} \leq \epsilon \) and \( u \) is a viscosity solution to (4.1) in \( B_1 \) with \( q = 0 \), \( u(0) = 0 \) and \( \operatorname{osc}_{B_1} u \leq 1 \), then there exists \( \alpha \in (0, 1) \) and \( q_\infty \in \mathbb{R}^N \) such that
\[
\sup_{B_{\frac{1}{2}}} |u(x) - q_\infty \cdot x| \leq C(N, \hat{p}) \tau^{k(1+\alpha)} \quad \text{for all } k \in \mathbb{N}.
\]

**Proof. Step 1:** We show that there exists a sequence \( (q_k)_{k=0}^\infty \subset \mathbb{R}^N \) such that
\[
\operatorname{osc}_{B_{\frac{1}{2}}} (u(x) - q_k \cdot x) \leq \tau^{k(1+\alpha)}. \quad (4.6)
\]
When \( k = 0 \), this estimate holds by setting \( q_0 = 0 \) since \( u(0) = 0 \) and \( \operatorname{osc}_{B_1} u \leq 1 \). Next we take \( \alpha \in (0, 1) \) such that \( \tau^\alpha > \frac{1}{2} \). We assume that \( k \geq 0 \) and that we have already constructed \( q_k \) for which (4.6) holds. We set
\[
w_k(x) := \tau^{-k(1+\alpha)}(u(\tau^k x) - q_k \cdot (\tau^k x))
\]
and
\[
f_k(x) := \tau^{k(1-\alpha)}f(\tau^k x).
\]
Then by induction assumption \( \operatorname{osc}_{B_1}(w_k) \leq 1 \) and \( w_k \) is a viscosity solution to
\[
-\Delta w_k - \frac{(p(\tau^k x) - 2) \langle D^2w_k(Dw_k + \tau^{-k\alpha}q_k), Dw_k + \tau^{-k\alpha}q_k \rangle}{|Dw_k + \tau^{-k\alpha}q_k|^2 + (\tau^{-k\alpha}x)^2} = f_k(x) \quad \text{in } B_1.
\]
By Lemma 4.1 there exists \( q_k' \in \mathbb{R}^N \) with \( |q_k'| \leq C(N, \hat{p}) \) such that
\[
\operatorname{osc}_{B_1} (w_k(x) - q'_k \cdot x) \leq \frac{1}{2} \tau.
\]
Using the definition of \( w_k \), scaling to \( B_{\frac{1}{2}+1} \) and dividing by \( \tau^{-k(\alpha+1)} \), we obtain from the above
\[
\operatorname{osc}_{B_{\frac{1}{2}+1}} (u(x) - (q_k + \tau^{k\alpha} q'_k) \cdot x) \leq \frac{1}{2} \tau^{1+k(1+\alpha)} \leq \tau^{(k+1)(1+\alpha)}.
\]
Denoting \( q_{k+1} := q_k + \tau^{k\alpha} q'_k \), the above estimate is condition (4.6) for \( k + 1 \) and the induction step is complete.

**Step 2:** Observe that whenever \( m > k \), we have
\[
|q_m - q_k| \leq \sum_{i=k}^{m-1} \tau^{i\alpha} |q'_i| \leq C(N, \hat{p}) \sum_{i=k}^{m-1} \tau^{i\alpha}.
\]
Therefore $q_k$ is a Cauchy sequence and converges to some $q_\infty \in \mathbb{R}^N$. Thus

$$\sup_{x \in B_{r,k}} (q_k \cdot x - q_\infty \cdot x) \leq \tau^k |q_k - q_\infty| \leq \tau^k \sum_{i=k}^{\infty} \tau^\alpha q_i' \leq C(N, \hat{p}) \tau^{k(1+\alpha)}.$$ 

This with (4.6) implies that

$$\sup_{x \in B_{r,k}} |u(x) - q_\infty \cdot x| \leq C(N, \hat{p}) \tau^{k(1+\alpha)}.$$ 

\[ \square \]

**Theorem 4.3.** Suppose that $u$ is a viscosity solution to (4.1) in $B_1$ with $q = 0$ and $\text{osc}_{B_1} \leq 1$. Then there are constants $\alpha(N, \hat{p})$ and $C(N, \hat{p}, \|f\|_{L^\infty(B_1)})$ such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C.$$ 

**Proof.** Let $\epsilon(N, \hat{p})$ and $\tau(N, \hat{p})$ be as in Lemma 4.2. Set

$$v(x) := \kappa u(x/4)$$

where $\kappa := \epsilon(1 + \|f\|_{L^\infty(B_1)})^{-1}$. For $x_0 \in B_1$, set

$$w(x) := v(x + x_0) - v(x_0).$$

Then $\text{osc}_{B_1} w \leq 1$, $w(0) = 0$ and $w$ is a viscosity solution to

$$-\Delta w - \frac{(p(x/4 + x_0/4) - 2)(D^2wDw, Dw)}{|Dw|^2 + \epsilon^2 \kappa^2} = g(x) \quad \text{in } B_1,$$

where $g(x) := \kappa f(x/4 + x_0/4)/4^2$. Now $\|g\|_{L^\infty(B_1)} \leq \epsilon$ so by Lemma 4.2 there exists $q_\infty(x_0) \in \mathbb{R}^N$ such that

$$\sup_{x \in B_{r,k}} |w(x) - q_\infty(x_0) \cdot x| \leq C(N, \hat{p}) \tau^{k(1+\alpha)}$$

for all $k \in \mathbb{N}$.

Thus we have shown that for any $x_0 \in B_1$ there exists a vector $q_\infty(x_0)$ such that

$$\sup_{x \in B_{r,(x_0)}} |v(x) - v(x_0) - q_\infty(x_0) \cdot (x - x_0)| \leq C(N, \hat{p}) r^{1+\alpha}$$

for all $r \in (0, 1]$.

This together with a standard argument (see for example [4, Lemma A.1]) implies that $[Du]_{C^{1}(B_1)} \leq C(N, \hat{p})$ and so by definition of $v$, also $[Du]_{C^{1}(B_{1/4})} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)})$. The conclusion of the theorem then follows by a standard translation argument. \[ \square \]

5. **Proof of the main theorem**

In this section we finish the proof our main theorem.

**Proof of Theorem 1.1.** We may assume that $u \in C(\overline{B_1})$. By Comparison Principle (Lemma B.2 in the Appendix) $u$ is the unique viscosity solution to

$$\begin{cases}
-\Delta v - \frac{(p(x) - 2)(D^2vDv, Dv)}{|Dv|^2} = f(x) + u - v \quad \text{in } B_1, \\
v = u \quad \text{on } \partial B_1.
\end{cases} \quad (5.1)$$
By [21, Theorem 15.18] there exists a classical solution $u_\varepsilon$ to the approximate problem

$$
\begin{cases}
-\Delta u_\varepsilon - \frac{(p_\varepsilon(x) - 2)\langle D^2 u_\varepsilon, Du_\varepsilon \rangle}{|Du_\varepsilon|^2 + \varepsilon^2} = f_\varepsilon(x) + u_\varepsilon - u & \text{in } B_1, \\
u_\varepsilon = u & \text{on } \partial B_1,
\end{cases}
$$

where $p_\varepsilon, f_\varepsilon, u_\varepsilon \in C^\infty(B_1)$ are such that $p_\varepsilon \to p, f_\varepsilon \to f$ and $u_\varepsilon \to u_0$ uniformly in $B_1$ as $\varepsilon \to 0$ and $\|Dp_\varepsilon\|_{L^\infty(B_1)} \leq \|Dp\|_{L^\infty(B_1)}$. The maximum principle implies that $\|u_\varepsilon\|_{L^\infty(B_1)} \leq 2 \|f\|_{L^\infty(B_1)} + 2 \|u\|_{L^\infty(B_1)}$. By [13, Proposition 4.14] the solutions $u_\varepsilon$ are equicontinuous in $\overline{B}_1$ (their modulus of continuity depends only on $N, p, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}$, and modulus of continuity of $u$). Therefore by the Ascoli-Arzela theorem we have $u_\varepsilon \to v \in C(\overline{B}_1)$ uniformly in $\overline{B}_1$ up to a subsequence. By the stability principle, $v$ is a viscosity solution to (5.1) and thus by uniqueness $v \equiv u$.

By Corollary 4.3 we have $\alpha(N, \hat{p})$ such that

$$
\|Du_\varepsilon\|_{C^0(B_{1/2})} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}),
$$

and by the Lipschitz estimate A.2 also

$$
\|Du_\varepsilon\|_{L^\infty(B_{1/2})} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}).
$$

Therefore by the Ascoli-Arzela theorem there exists a subsequence such that $Du_\varepsilon \to \eta$ uniformly in $B_{1/2}$, where the function $\eta : B_{1/2} \to \mathbb{R}^N$ satisfies

$$
\|\eta\|_{C^0(B_{1/2})} \leq C(N, \hat{p}, \|f\|_{L^\infty(B_1)}, \|u\|_{L^\infty(B_1)}).
$$

Using the mean value theorem and the estimate (5.2), we deduce for all $x, y \in B_{1/2}$

$$
\begin{align*}
|u(y) - u(x) - (y - x) \cdot \eta(x)| & \leq |u_\varepsilon(x) - u_\varepsilon(y) - (y - x) \cdot Du_\varepsilon(x)| \\
& \quad + |u(y) - u_\varepsilon(y) - u(x) + u_\varepsilon(x)| + |x - y| |\eta(x) - Du_\varepsilon(x)| \\
& \leq C(N, \hat{p}, \|u\|_{L^\infty(B_1)}) |x - y|^{1 + \alpha} + o(\varepsilon)/\varepsilon.
\end{align*}
$$

Letting $\varepsilon \to 0$, this implies that $Du(x) = \eta(x)$ for all $x \in B_{1/2}$. □

**Appendix A. Lipschitz estimate**

In this section we apply the method of Ishii and Lions [23] to prove a Lipschitz estimate for solutions to the inhomogeneous normalized $p(x)$-Laplace equation and its regularized or perturbed versions. We need the following vector inequality.

**Lemma A.1.** Let $a, b \in \mathbb{R}^N \setminus \{0\}$ with $a \neq b$ and $\varepsilon \geq 0$. Then

$$
\left| \frac{a}{\sqrt{|a|^2 + \varepsilon^2}} - \frac{b}{\sqrt{|b|^2 + \varepsilon^2}} \right| \leq \frac{2}{\max \{|a|, |b|\}} |a - b|.
$$
Therefore, in Our and where 

Proof. We may suppose that \(|a| = \max(|a|, |b|)\). Let \(s_1 := \sqrt{|a|^2 + \varepsilon^2}\) and \(s_2 := \sqrt{|b|^2 + \varepsilon^2}\). Then

\[
\frac{|a - b|}{s_1} = \frac{1}{s_1} |a - b| \leq \frac{1}{s_1} (|a| + \varepsilon^2 + |b|) \\
\frac{|a - b|}{s_2} = \frac{1}{s_2} |a - b| \leq \frac{1}{s_2} (|a| + \varepsilon^2 + |b|).
\]

Moreover,

\[
|s_2 - s_1| = \left| \sqrt{|a|^2 + \varepsilon^2} - \sqrt{|b|^2 + \varepsilon^2} \right| \leq \frac{|a| - |b|}{|a|} \leq |a - b|. \quad \Box
\]

**Theorem A.2 (Lipschitz estimate).** Suppose that \(p : B_1 \to \mathbb{R}\) is Lipschitz continuous, \(p_{\min} > 1\) and that \(f \in C(B_1)\) is bounded. Let \(u\) be a viscosity solution to

\[
-\Delta u - (p(x) - 2) \frac{\langle D^2 u(Du + q), Du + q \rangle}{|Du + q|^2 + \varepsilon^2} = f(x) \quad \text{in } B_1,
\]

where \(\varepsilon \geq 0\) and \(q \in \mathbb{R}^N\). Then there are constants \(C_0(N, \hat{p}, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)})\) and \(v_0(N, \hat{p})\) such that if \(|q| > v_0\) or \(|q| = 0\), then we have

\[
|u(x) - u(y)| \leq C_0 |x - y| \quad \text{for all } x, y \in B_{1/2}.
\]

Proof. We let \(r(N, \hat{p}) \in (0, 1/2)\) denote a small constant that will be specified later. Let \(x_0, y_0 \in B_{r/2}\) and define the function

\[
\Psi(x, y) := u(x) - u(y) - L \varphi(|x - y|) - \frac{M}{2} |x - x_0|^2 - \frac{M}{2} |y - y_0|^2,
\]

where \(\varphi : [0, 2] \to \mathbb{R}\) is given by

\[
\varphi(s) := s - s^\gamma \kappa_0, \quad \kappa_0 := \frac{1}{\gamma 2^{-\gamma + 1}},
\]

and the constants \(L(N, \hat{p}, \|u\|_{L^\infty(B_1)}), M(N, \hat{p}, \|u\|_{L^\infty(B_1)}) > 0\) and \(\gamma(N, \hat{p}) \in (1, 2)\) are also specified later. Our objective is to show that for a suitable choice of these constants, the function \(\Psi\) is non-positive in \(\overline{B_r} \times \overline{B_r}\). By the definition of \(\varphi\), this yields \(u(x_0) - u(y_0) \leq L |x_0 - y_0|\) which implies that \(u\) is \(L\)-Lipschitz in \(B_r\). The claim of the theorem then follows by standard translation arguments.

Suppose on contrary that \(\Psi\) has a positive maximum at some point \((\hat{x}, \hat{y}) \in \overline{B_r} \times \overline{B_r}\). Then \(\hat{x} \neq \hat{y}\) since otherwise the maximum would be non-positive. We have

\[
0 < u(\hat{x}) - u(\hat{y}) - L \varphi(|\hat{x} - \hat{y}|) - \frac{M}{2} |\hat{x} - x_0|^2 - \frac{M}{2} |\hat{y} - y_0|^2
\]

\[
\leq |u(\hat{x}) - u(\hat{y})| - \frac{M}{2} |\hat{x} - x_0|^2. \quad (A.1)
\]

Therefore, by taking

\[
M := \frac{8 \text{osc}_{B_1} u}{r^2}, \quad (A.2)
\]
we get

\[ |\hat{x} - x_0| \leq \sqrt{\frac{2}{M}} |u(\hat{x}) - u(\hat{y})| \leq r/2 \]

and similarly \(|\hat{y} - y_0| \leq r/2\). Since \(x_0, y_0 \in B_{r/2}\), this implies that \(\hat{x}, \hat{y} \in B_r\).

By [13, Proposition 4.10] there exist constants \(C'(N, \hat{\mu}, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)})\) and \(\beta(N, \hat{\mu}) \in (0, 1)\) such that

\[ |u(x) - u(y)| \leq C'|x - y|^\beta \text{ for all } x, y \in B_r. \tag{A.3} \]

It follows from (A.1) and (A.3) that for \(C_0 := \sqrt{2C}M\) we have

\[ M |\hat{x} - x_0| \leq C_0 |\hat{x} - \hat{y}|^{\beta/2}, \]

\[ M |\hat{y} - y_0| \leq C_0 |\hat{x} - \hat{y}|^{\beta/2}. \tag{A.4} \]

Since \(\hat{x} \neq \hat{y}\), the function \((x, y) \mapsto \varphi(|x - y|)\) is \(C^2\) in a neighborhood of \((\hat{x}, \hat{y})\) and we may invoke the Theorem of sums [14, Theorem 3.2]. For any \(\mu > 0\) there exist matrices \(X, Y \in \mathcal{S}^N\) such that

\[
(D_x(L\varphi(|x - y|))(\hat{x}, \hat{y}), X) \in \mathcal{J}^{2,+}(u - \frac{M}{2}|x - x_0|^2)(\hat{x}),
\]

\[
(-D_y(L\varphi(|x - y|))(\hat{x}, \hat{y}), Y) \in \mathcal{J}^{2,-}(u + \frac{M}{2}|y - y_0|^2)(\hat{y}),
\]

which by denoting \(z := \hat{x} - \hat{y}\) and

\[
a := L\varphi'(|z|) \frac{z}{|z|} + M(\hat{x} - x_0),
\]

\[
b := L\varphi'(|z|) \frac{z}{|z|} - M(\hat{y} - y_0),
\]

can be written as

\[
(a, X + MI) \in \mathcal{J}^{2,+}u(\hat{x}), \quad (b, Y - MI) \in \mathcal{J}^{2,-}u(\hat{y}). \tag{A.5}
\]

By assuming that \(L\) is large enough depending on \(C_0\), we have by (A.4) and the fact \(\varphi' \in [\frac{q}{4}, 1]\)

\[
|a|, |b| \leq L |\varphi'(|\hat{x} - \hat{y}|)| + C_0 |\hat{x} - \hat{y}|^{\beta/2} \leq 2L, \tag{A.6}
\]

\[
|a|, |b| \geq L |\varphi'(|\hat{x} - \hat{y}|)| - C_0 |\hat{x} - \hat{y}|^{\beta/2} \geq \frac{1}{2}L. \tag{A.7}
\]

Moreover, we have

\[
-(\mu + 2 \|B\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \frac{2}{\mu} \begin{pmatrix} B^2 & -B^2 \\ -B^2 & B^2 \end{pmatrix}, \tag{A.8}
\]

where
\[ B = L\varphi''(|z|) \frac{z}{|z|} \otimes \frac{z}{|z|} + \frac{L\varphi'(|z|)}{|z|} \left( I - \frac{z}{|z|} \otimes \frac{z}{|z|} \right), \]
\[ B^2 = BB = L^2(\varphi''(|z|))^2 \frac{z}{|z|} \otimes \frac{z}{|z|} + \frac{L^2(\varphi'(|z|))^2}{|z|^2} \left( I - \frac{z}{|z|} \otimes \frac{z}{|z|} \right). \]

Using that \( \varphi''(|z|) < 0 < \varphi'(|z|) \) and \( |\varphi''(|z|)| \leq \varphi'(|z|)/|z| \), we deduce that
\[ \|B\| \leq \frac{L\varphi'(|z|)}{|z|} \quad \text{and} \quad \|B^2\| \leq \frac{L^2(\varphi'(|z|))^2}{|z|^2}. \]  

Moreover, choosing
\[ \mu := 4L \left( |\varphi''(|z|)| + \frac{|\varphi'(|z|)|}{|z|} \right), \]
and using that \( \varphi''(|z|) < 0 \), we have
\[ \left\langle B \frac{z}{|z|}, \frac{z}{|z|} \right\rangle + \frac{2}{\mu} \left\langle B^2 \frac{z}{|z|}, \frac{z}{|z|} \right\rangle = L\varphi''(|z|) + \frac{2}{\mu} L^2 |\varphi''(|z|)| \leq \frac{L}{2} \varphi''(|z|). \]  

We set \( \eta_1 := a + q \) and \( \eta_2 := b + q \). By (A.6) and (A.7) there is a constant \( \nu_0(L) \) such that if \( |q| = 0 \) or \( |q| > \nu_0 \), then
\[ |\eta_1|, |\eta_2| \geq \frac{L}{2}. \]

We denote \( A(x, \eta) := I + (p(x) - 2)\eta \otimes \eta \) and \( \bar{\eta} := \frac{\eta}{\sqrt{|\eta|^2 + x_2^2}} \). Since \( u \) is a viscosity solution, we obtain from (A.5)
\[ 0 \leq \text{tr}(A(\hat{x}, \eta_1)(X + MI)) - \text{tr}(A(\hat{y}, \eta_2)(Y - MI)) + f(\hat{x}) - f(\hat{y}) \]
\[ = \text{tr}(A(\hat{y}, \bar{\eta}_2)(X - Y)) + \text{tr}((A(\hat{x}, \bar{\eta}_2) - A(\hat{y}, \bar{\eta}_2))X) \]
\[ + \text{tr}((A(\hat{x}, \eta_1) - A(\hat{x}, \bar{\eta}_2))X) + M\text{tr}(A(\hat{x}, \eta_1) + A(\hat{y}, \bar{\eta}_2)) \]
\[ + f(\hat{x}) - f(\hat{y}) \]
\[ =: T_1 + T_2 + T_3 + T_4 + T_5. \]  

We will now proceed to estimate these terms. The plan is to obtain a contradiction by absorbing the other terms into \( T_1 \) which is negative by concavity of \( \varphi \).

**Estimate of** \( T_1 \): Multiplying (A.8) by the vector \( \left( \frac{z}{|z|}, -\frac{z}{|z|} \right) \) and using (A.10), we obtain an estimate for the smallest eigenvalue of \( X - Y \)
\[ \lambda_{\min}(X - Y) \leq \left\langle (X - Y) \frac{z}{|z|}, \frac{z}{|z|} \right\rangle \]
\[ \leq 4 \left\langle B \frac{z}{|z|}, \frac{z}{|z|} \right\rangle + \frac{8}{\mu} \left\langle B^2 \frac{z}{|z|}, \frac{z}{|z|} \right\rangle \leq 2L\varphi''(|z|). \]

The eigenvalues of \( A(\hat{y}, \bar{\eta}_2) \) are between \( \min(1, p_{\min} - 1) \) and \( \max(1, p_{\max} - 1) \). Therefore by [36]
\[ T_1 = \text{tr}(A(\hat{y}, \bar{\eta}_2)(X - Y)) \leq \sum_i \lambda_i(A(\hat{y}, \bar{\eta}_2))\lambda_i(X - Y) \]
\[ \leq \min(1, p_{\min} - 1)\lambda_{\min}(X - Y) \]
\[ \leq C(\hat{p})L\varphi''(|z|). \]

**Estimate of** \(T_2\): We have

\[ T_2 = \text{tr}((A(\hat{x}, \hat{\eta}_2) - A(\hat{y}, \hat{\eta}_2))X) \leq |p(\hat{x}) - p(\hat{y})| |X\hat{\eta}_2, \hat{\eta}_2| \leq C(\hat{p})|z| \|X\|, \]

where by (A.8) and (A.9)

\[ \|X\| \leq \|B\| + \frac{2}{\mu} \|B\|^2 \leq \frac{L|\varphi'(|z|)|}{|z|} + \frac{2L^2(\varphi'(|z|))^2}{4L(|\varphi''(|z|)| + |\varphi'(|z|)| |z|^2)} \]

\[ \leq \frac{2L\varphi'(|z|)}{|z|}. \quad (A.13) \]

**Estimate of** \(T_3\): From Lemma A.1 and the estimate (A.11) it follows that

\[ |\eta_1 - \eta_2| \leq \frac{2|\eta_1 - \eta_2|}{\max(|\eta_1|, |\eta_2|)} \leq \frac{4}{L} |\eta_1 - \eta_2| = \frac{4L}{L} |a - b| \]

\[ \leq \frac{4}{L} (M|\hat{x} - x_0| + M|\hat{y} - y_0|) \leq \frac{8C_0}{L} |z|^\beta/2, \quad (A.14) \]

where in the last inequality we used (A.4). Observe that

\[ \|\eta_1 \otimes \eta_1 - \eta_2 \otimes \eta_2\| = \|\overline{\eta}_1 - \overline{\eta}_2\| \leq \|\eta_1 + |\eta_2\| \leq (|\eta_1| + |\eta_2|) |\eta_1 - \eta_2|. \]

Using the last two displays, we obtain by [36] and (A.13)

\[ T_3 = \text{tr}((A(\hat{x}, \eta_1) - A(\hat{x}, \eta_2))X) \leq N \|A(x_1, \overline{\eta}_1) - A(x_1, \overline{\eta}_2)\| \|X\| \]

\[ \leq N |p(x_1) - 2|(|\eta_1| + |\eta_2|)|\eta_1 - \eta_2| \|X\| \]

\[ \leq \frac{C(N, \hat{p})C_0}{L} |z|^\beta/2 \|X\| \]

\[ \leq C(N, \hat{p}, \|u\|_{L^\infty}, \|f\|_{L^\infty}) \sqrt{M\varphi'(|z|)} |z|^\beta/2 - 1. \]

**Estimate of** \(T_4\) and \(T_5\): By Lipschitz continuity of \(p\) we have

\[ T_4 = M\text{tr}(A(\hat{x}, \eta_1) + A(\hat{y}, \overline{\eta}_2)) \leq 2MC(N, \hat{p}). \]

We have also

\[ T_5 = f(\hat{x}) - f(\hat{y}) \leq 2 \|f\|_{L^\infty(B_1)}. \]

Combining the estimates, we deduce the existence of positive constants \(C_1(N, \hat{p})\) and \(C_2(N, \hat{p}, \|u\|_{L^\infty(B_1)}), \|f\|_{L^\infty(B_1)}\) such that

\[ 0 \leq C_1L\varphi''(|z|) + C_2(L\varphi'(|z|) + \sqrt{M\varphi'}(|z|)) |z|^\beta - 1 + M + 1 \]

\[ \leq C_1L\varphi''(|z|) + C_2(L + \sqrt{M}) |z|^\beta - 1 + M + 1 \quad (A.15) \]

where we used that \(\varphi'(|z|) \in [\frac{3}{4}, 1]\). We take \(\gamma := \frac{\beta}{2} + 1\) so that we have
\[
\varphi''(|z|) = \frac{1 - \gamma}{2^{\gamma+1}} |z|^{-2} = \frac{-\beta}{2^{\frac{\beta}{2}+3}} |z|^\frac{\beta-1}{2} = -C_3 |z|^\frac{\beta-1}{2}.
\]

We apply this to (A.15) and obtain
\[
0 \leq (C_2 \sqrt{M} - C_1 C_3 L) |z|^\frac{\beta}{2} - C_2 (L + M + 1)
\] (A.16)

We fix \( r := \frac{1}{2} \left( \frac{6C_2}{C_1 C_3} \right)^{\frac{2}{\beta-1}} \). By (A.2) this will also fix \( M = (N, \hat{p}, \|u\|_{L^\infty(B_1)}) \). We take \( L \) so large that

\[
L > \max \left( \frac{2C_2 \sqrt{M}}{C_1 C_3}, M + 1 \right).
\]

Then by (A.16) we have

\[
0 < -\frac{1}{2} C_1 C_3 L |z|^\frac{\beta}{2} - 2C_2 L \leq L (-\frac{1}{2} C_1 C_3 (2r)^\frac{\beta-1}{2} + 2C_2)
\]

\[
= -LC_2 \leq 0,
\]

which is a contradiction. \( \square \)

**Appendix B. Stability and comparison principles**

**Lemma B.1.** Suppose that \( p \in C(B_1) \), \( p_{\min} > 1 \) and that \( f : B_1 \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous. Let \( u_\varepsilon \) be a viscosity solution to

\[
-\Delta u_\varepsilon - (p_\varepsilon(x) - 2) \frac{\langle D^2 u_\varepsilon D u_\varepsilon, D u_\varepsilon \rangle}{|D u_\varepsilon|^2 + \varepsilon^2} = f_\varepsilon(x, u(x)) \quad \text{in } B_1
\]

and assume that \( u_\varepsilon \rightarrow u \in C(B_1) \), \( p_\varepsilon \rightarrow p \) and \( f_\varepsilon \rightarrow f \) locally uniformly as \( \varepsilon \rightarrow 0 \). Then \( u \) is a viscosity solution to

\[
-\Delta u - (p(x) - 2) \frac{\langle D^2 u D u, D u \rangle}{|D u|^2} = f(x, u(x)) \quad \text{in } B_1.
\]

**Proof.** It is enough to consider supersolutions. Suppose that \( \varphi \in C^2 \) touches \( u \) from below at \( x \). Since \( u_\varepsilon \rightarrow u \) locally uniformly, there exists a sequence \( x_\varepsilon \rightarrow x \) such that \( u_\varepsilon - \varphi \) has a local minimum at \( x_\varepsilon \). We denote \( \eta_\varepsilon := D\varphi(x_\varepsilon) / \sqrt{|D\varphi(x_\varepsilon)|^2 + \varepsilon^2} \). Then \( \eta_\varepsilon \rightarrow \eta \in \overline{B}_1 \) up to a subsequence. Therefore we have

\[
0 \leq -\Delta \varphi(x_\varepsilon) - (p_\varepsilon(x_\varepsilon) - 2) \langle D^2 \varphi(x_\varepsilon) \eta_\varepsilon, \eta_\varepsilon \rangle - f_\varepsilon(x_\varepsilon, u_\varepsilon(x_\varepsilon))
\]

\[
\rightarrow -\Delta \varphi(x) - (p(x) - 2) \langle D^2 \varphi(x) \eta, \eta \rangle - f(x, u(x)),
\]

(B.1)

which is what is required in Definition 2.1 in the case \( D\varphi(x) = 0 \). If \( D\varphi(x) \neq 0 \), then \( D\varphi(x_\varepsilon) \neq 0 \) when \( \varepsilon \) is small and thus \( \eta = D\varphi(x) / |D\varphi(x)| \). Therefore B.1 again implies the desired inequality. \( \square \)

**Lemma B.2.** Suppose that \( p : B_1 \rightarrow \mathbb{R} \) is Lipschitz continuous, \( p_{\min} > 1 \) and that \( f \in C(B_1) \) is bounded. Assume that \( u \in C(\overline{B}_1) \) is a viscosity subsolution to \( -\Delta_{p(x)}^N u \leq f - u \) in \( B_1 \) and that \( v \in C(\overline{B}_1) \) is a viscosity supersolution to \( -\Delta_{p(x)}^N v \geq f - v \) in \( B_1 \). Then

\[
u \leq u \quad \text{on } \partial B_1
\]
implies
\[ u \leq v \quad \text{in } B_1. \]

**Proof. Step 1:** Assume on the contrary that the maximum of \( u - v \) in \( B_1 \) is positive. For \( x, y \in \overline{B}_1 \), set

\[ \Psi_j(x, y) := u(x) - v(y) - \varphi_j(x, y), \]

where \( \varphi_j(x, y) := \frac{j}{4} |x - y|^4 \). Let \( (x_j, y_j) \) be a global maximum point of \( \Psi_j \) in \( \overline{B}_1 \times \overline{B}_1 \). Then

\[ u(x_j) - v(y_j) - \frac{j}{4} |x_j - y_j|^4 \geq u(0) - v(0) \]

so that

\[ \frac{j}{4} |x_j - y_j|^4 \leq 2 \| u \|_{L^\infty(B_1)} + 2 \| v \|_{L^\infty(B_1)} < \infty. \]

By compactness and the assumption \( u \leq v \) on \( \partial B_1 \) there exists a subsequence such that \( x_j, y_j \to \hat{x} \in B_1 \) and \( u(\hat{x}) - v(\hat{x}) > 0 \). Finally, since \( (x_j, y_j) \) is a maximum point of \( \Psi_j \), we have

\[ u(x_j) - v(x_j) \leq u(x_j) - v(y_j) - \frac{j}{4} |x_j - y_j|^4, \]

and hence by continuity

\[ \frac{j}{4} |x_j - y_j|^4 \leq v(x_j) - v(y_j) \to 0 \quad \text{as } j \to \infty. \]

**Step 2:** If \( x_j = y_j \), then \( D_x^2 \varphi_j(x_j, y_j) = D_y^2 \varphi_j(x_j, y_j) = 0 \). Therefore, since the function \( x \mapsto u(x) - \varphi_j(x, y_j) \) reaches its maximum at \( x_j \) and \( y \mapsto v(y) - (-\varphi_j(x, y)) \) reaches its minimum at \( y_j \), we obtain from the definition of viscosity sub- and supersolutions that

\[ 0 \leq f(x_j) - u(x_j) \quad \text{and} \quad 0 \geq f(y_j) - v(y_j). \]

That is \( 0 \leq f(x_j) - f(y_j) + v(y_j) - u(x_j) \), which leads to a contradiction since \( x_j, y_j \to \hat{x} \) and \( v(\hat{x}) - u(\hat{x}) < 0 \). We conclude that \( x_j \neq y_j \) for all large \( j \). Next we apply the Theorem of sums [14, Theorem 3.2] to obtain matrices \( X, Y \in S^N \) such that

\[ (D_x \varphi(x_j, y_j), X) \in T^{2,+} u(x_j), \quad (-D_y \varphi(x_j, y_j), Y) \in T^{2,-} v(y_j) \]

and

\[ \left( \begin{array}{cc} X & 0 \\ 0 & -Y \end{array} \right) \leq D^2 \varphi(x_j, y_j) + \frac{1}{2} (D^2(x_j, y_j))^2, \quad \text{(B.3)} \]

where

\[ D^2(x_j, y_j) = \left( \begin{array}{cc} M & -M \\ -M & M \end{array} \right) \]

with \( M = j(2(x_j - y_j) \odot (x_j - y_j) + |x_j - y_j|^2 I) \). Multiplying the matrix inequality (B.3) by the \( \mathbb{R}^{2N} \) vector \((\xi_1, \xi_2)\) yields
\[
\langle X\xi_1,\xi_1 \rangle - \langle Y\xi_2,\xi_2 \rangle \leq \langle (M + 2j^{-1}M^2)(\xi_1 - \xi_2),\xi_1 - \xi_2 \rangle \\
\leq (\|M\| + 2j^{-1}\|M\|^2)\|\xi_1 - \xi_2\|^2.
\]

Observe also that \( \eta := D_x\varphi(x_j, y_j) = -D_y(x_j, y_j) = j|x_j - y_j|^2(x_j - y_j) \neq 0 \) for all large \( j \). Since \( u \) is a subsolution and \( v \) is a supersolution, we thus obtain

\[
f(y_j) - f(x_j) + u(x_j) - v(y_j) \\
\leq \text{tr}(X - Y) + (p(x_j) - 2)\left\langle X\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle - (p(y_j) - 2)\left\langle Y\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle \\
\leq (p(x_j) - 1)\left\langle X\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle - (p(y_j) - 1)\left\langle Y\frac{\eta}{|\eta|},\frac{\eta}{|\eta|}\right\rangle \\
\leq (\|M\| + 2j^{-1}\|M\|^2)\sqrt{p(x_j) - 1 - \sqrt{p(y_j) - 1}}^2 \\
\leq Cj|x_j - y_j|^2 \frac{|p(x_j) - p(y_j)|^2}{(\sqrt{p(x_j)} - 1 + \sqrt{p(y_j)} - 1)^2} \\
\leq C(\tilde{p})j|x_j - y_j|^4.
\]

This leads to a contradiction since the left-hand side tends to \( u(\tilde{x}) - v(\tilde{y}) > 0 \) and the right-hand side tends to zero by (B.2). □

References