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STABILITY OF SOLUTION FOR RAO–NAKRA SANDWICH BEAM MODEL WITH KELVIN–VOIGT DAMPING AND TIME DELAY

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Abstract. This paper deals with stability of solution for a one-dimensional model of Rao–Nakra sandwich beam with Kelvin–Voigt damping and time delay given by

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \kappa(-u + v + \alpha w_x) - a u_{xxt} - \mu u_{xxt}(\,\cdot\,, t - \tau) = 0,$$
$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \kappa(-u + v + \alpha w_x) - b v_{xxt} = 0,$$
$$\rho h w_{tt} + EI w_{xxxx} - \kappa\alpha(-u + v + \alpha w_x)_x - c w_{xxt} = 0.$$

A sandwich beam is an engineering model that consists of three layers: two stiff outer layers, bottom and top faces, and a more compliant inner layer called "core layer". Rao–Nakra system consists of three layers and the assumption is that there is no slip at the interface between contacts. The top and bottom layers are wave equations for the longitudinal displacements under Euler–Bernoulli beam assumptions. The core layer is one equation that describes the transverse displacement under Timoshenko beam assumptions. By using the semigroup theory, the well-posedness is given by applying the Lumer–Phillips Theorem. Exponential stability is proved by employing the Gearhart-Huang-Prüss' Theorem.

1. Introduction

Physical phenomena are usually modeled by equations involving differential operators of evolution type. A unique equation is not enough to describe, for instance, thermoelastic and viscoelastic processes of a given material, where the longitudinal and transverse displacement are the unknown parameters. These parameters follow different behavior depending on the material composition, and hence different differential equations govern them. In real life, transmission of the internal energy inherent to the system requires (needs) a short time to circulate from one place to another. In general, and for the sake of simplicity, time delays are usually neglected by the model. However, some experiments have shown that time

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delay could change the behavior of the original structure of physical phenomena. It could destabilize the system, and therefore some dissipative mechanism has to be introduced to thwart this effect.

In this manuscript, we deal with a Rao–Nakra system with viscoelastic damping and a time delay term in the first entry. More precisely, for $(x,t) \in (0,L) \times \mathbb{R}^+$, $L > 0$, we consider the system

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \kappa(-u + v + \alpha w_x) - a u_{xxt} - \mu u_{xxt}(\,\cdot\,, t-\tau) = 0, \tag{1.1}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \kappa(-u + v + \alpha w_x) - b v_{xxt} = 0, \tag{1.2}$$

$$\rho h w_{tt} + EI w_{xxxx} - \kappa\alpha(-u + v + \alpha w_x)_x - c w_{xxt} = 0, \tag{1.3}$$

subject to the Dirichlet–Neumann boundary conditions

$$\begin{aligned} ru(0,t) = u(L,t) = v(0,t) = v(L,t) = 0, \quad &\text{in} \quad \mathbb{R}^+, \\ w(0,t) = w_x(0,t) = w(L,t) = w_x(L,t) = 0, \quad &\text{in} \quad \mathbb{R}^+ \end{aligned} \tag{1.4}$$

and with corresponding initial data

$$\begin{aligned} r(u(x,0), v(x,0), w(x,0)) = (u_0(x), v_0(x), w_0(x)), \quad &\text{in} \quad (0,L), \\ (u_t(x,0), v_t(x,0), w_t(x,0)) = (u_1(x), v_1(x), w_1(x)), \quad &\text{in} \quad (0,L). \end{aligned} \tag{1.5}$$

Small vibrations of a beam are given by

$$\varrho_1 u_{tt} - k(u_x + \psi)_x = 0, \tag{1.6}$$

$$\varrho_2 \psi_{tt} - b\psi_{xx} + k(u_x + \psi) = 0. \tag{1.7}$$

This famous model has been introduced by S. P. Timoshenko [1] in 1921, where $u(x,t)$, $\psi(x,t)$ model the transverse displacement of the beam and the angular direction of the filament of the beam, respectively, and $\varrho_1, \varrho_2, k, b$ are positive real numbers. Since then, (1.6)–(1.7) have been widely studied by several authors in different contexts.

The Mead–Markus sandwich beam [2], of length $L > 0$ was introduced in 1969. The equations of motion based on the formulation given by Fabiano and Hansen [3] become

$$m u_{tt} + \Big(A + \frac{B^2}{C}\Big) u_{xxxx} - \frac{B}{C} s_{xxx} - \alpha u_{txx} = 0 \quad \text{in} \quad (0,L) \times (0,\infty), \tag{1.8}$$

$$\beta s_t + \gamma s - \frac{1}{C} s_{xx} + \frac{B}{C} u_{xxx} = 0 \;\text{ in }\; (0,L) \times (0,\infty). \tag{1.9}$$

For (1.8)–(1.9), $w(x,t)$ denotes the transverse displacement of the beam, $s(x,t)$ is proportional to the shear of the middle layer, $u(x,t)$ represents moment control, m is the mass of the beam, A, B and C are material constants, γ and β are the elastic and damping coefficients of the middle layer, respectively.

The following model for two-layer laminated beam was proposed by Hansen and Spies [4] in 1997 based on Timoshenko's theory

$$\varrho w_{tt} + G(\psi - w_x)_x = 0, \quad \text{in} \quad (0,L) \times \mathbb{R}^+, \tag{1.10}$$

$$I_\varrho(3s_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - u_x) = 0, \quad \text{in} \quad (0,L) \times \mathbb{R}^+, \tag{1.11}$$

$$3I_\varrho s_{tt} - 3D s_{xx} + 3G(\psi - w_x) + 4\mu s + 4\delta s_t = 0, \quad \text{in} \quad (0,L) \times \mathbb{R}^+, \tag{1.12}$$

where $\varrho, G, I_\varrho, D, \gamma$ and δ are positive constants and represent density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. The function $w(x,t)$ denotes the transversal displacement, $\psi(x,t)$ represents the rotational displacement, and $s(x,t)$ is proportional to the amount of slip along with the interface at time t and longitudinal spatial variable x. This model has received a lot of attention from several authors over the past several years. Please refer to [5] where the authors considered the dynamics of laminated Timoshenko beams.

The general three-layer laminated beam model was developed in 1999 by Liu–Trogdon–Yong [6]

$$\varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \tau = 0, \tag{1.13}$$

$$\varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \tau = 0, \tag{1.14}$$

$$\varrho h w_{tt} + EI w_{xxxx} - G_1 h_1 (w_x + \phi_1)_x - G_3 h_3 (w_x + \phi_3)_x - h_2 \tau_x = 0, \tag{1.15}$$

$$\varrho_1 I_1 \phi_{1,tt} - E_1 I_1 \phi_{1,xx} - \frac{h_1}{2}\tau + G_1 h_1 (w_x + \phi_1) = 0, \tag{1.16}$$

$$\varrho_3 I_3 \phi_{3,tt} - E_3 I_3 \phi_{3,xx} - \frac{h_3}{2}\tau + G_3 h_3 (w_x + \phi_3) = 0. \tag{1.17}$$

The physical parameters $h_i, \rho_i, E_i, G_i, I_i > 0$ are the thickness, density, Young's modulus, shear modulus, and moments of inertia of the i-th layer for $i = 1,2,3$, from the bottom to the top, respectively. In addition, $\varrho h = \varrho_1 h_1 + \varrho_2 h_2 + \varrho_3 h_3$ and $EI = E_1 I_1 + E_3 I_3$.

The Rao–Nakra system

$$\begin{aligned} &\varrho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \gamma w_x) = 0, \quad \text{in} \quad (0,L)\times\mathbb{R}^+,\\ &\varrho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \gamma w_x) = 0, \quad \text{in} \quad (0,L)\times\mathbb{R}^+,\\ &\varrho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \gamma w_x)_x = 0, \quad \text{in} \quad (0,L)\times\mathbb{R}^+, \end{aligned} \tag{1.18}$$

is obtained from (1.13)–(1.17) when the core material is considered to be linearly elastic i.e., $\tau = 2G_2\varsigma$ with the shear strain

$$\varsigma = \frac{1}{2h_2}(-u + v + \gamma w_x) \quad \text{and} \quad \gamma = h_2 + \frac{1}{2}(h_1 + h_3),$$

where $k := \frac{G_2}{h_2}$, the shear modulus $G_2 = \frac{E_2}{2(1+\nu)}$, and $-1 < \nu < \frac{1}{2}$ is the Poisson ratio.

When the extensional motion of the bottom and top layers is neglected, we obtain the two-layer laminated beam model proposed by Hansen–Spies. When $s(x,t) = 0$, system (1.10)–(1.12) reduces to the Timoshenko system. For more sandwich beam models found in the literature see for instance [7, 8] with references therein.

Systems with delay in time have been studied, among others, in several branches of Mathematics and Physics. Indeed, the control of Partial Differential Equations with delay has become an attractive area of research because time delays so often arise in many physical, chemical, biological, and economic phenomena, see [9] and the references therein. Whenever the energy is physically transmitted from one place to another, there is a delay associated with the transmission, see [10]. The central question is that the delays source can destabilize a system that is asymptotically stable in the absence of delays, see for instance [11–14] and the references therein.

Our motivation is the following Rao–Nakra model with internal damping and Kelvin-Voigt damping, considered in [15]

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) - a_1 u_{xxt} + a_2 u_t = 0, \tag{1.19}$$

$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) - b_1 u_{xxt} + b_2 u_t = 0, \tag{1.20}$$

$$\rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x - c_1 w_{xxxxt} + c_2 u_t = 0, \tag{1.21}$$

where $a_i, b_i, c_i \geqslant 0$, $i = 1, 2$. The authors in [15] showed that (1.19)–(1.21) is unstable if only one damping is imposed on the beam equation; beyond this, the exponential stability holds when all three displacements are damped while polynomial stability holds when just two of the three equations are damped. For the case $a_2 = b_2 = c_2 = 0$ we recover the system (1.1)–(1.3) without time delay and Kelvin–Voigt damping in the bottom layer. For $a_1 = b_1 = c_1 = 0$ in [16], the polynomial stability was proved when damping is just on one of the three wave equations and exponential stability was obtained by Özkan Özer-Hansen [17] when standard boundary damping is imposed on one end of the beam for all three displacements.

In the literature, we find several studies on the effects of delay on beam systems. We will provide several examples to emphasize the importance of systems involving delays. For instance, Said-Houari and Larski [18] studied the following Timoshenko system with delay

$$\rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0,$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2 \psi(t - \tau) = 0,$$

and they proved that the associated energy decays exponentially, whenever $\mu_2 < \mu_1$. Raposo et al. [19] demonstrated the exponential stability of a thermoviscoelastic Timoshenko system with heat conduction modeled by the Cattaneo law. Nicaise and Pignotti [20] studied the abstract evolution delay model

$$U_t(t) = \mathcal{A}U(t) + F(U(t)) + k\mathcal{B}U(t - \tau), \quad U(0) = 0, \quad \mathcal{B}U(t - \tau) = f(t).$$

Under smallness assumption on the time delay feedback, and assuming that $\mathcal{B}$ is a bounded operator on adequate spaces, they showed that the system is exponentially stable.

The main purpose of this paper is to study the asymptotic behavior of the solution associated with (1.1)–(1.5) by showing that the system is exponentially stable, see Theorem 4.2. The paper is structured as follows. In Section 2, we introduce the new variable as in [21] to deal with the delay parameter and we obtain an equivalent system to (1.1)–(1.5). Then, we prove that the full energy of the equivalent system is not increasing. In Section 3, the well-posedness of the problem (1.1)–(1.5) is presented by using a semigroup approach. Finally, in Section 4 the exponential stability of the C_0-semigroup of contractions on an appropriated Hilbert space is proved by employing the Gearhart–Huang–Prüss' theorem [22–24].

2. Statement of the problem

To deal with the delay term, as in [21], we introduce an auxiliary function η defined by

$$\eta(x,y,t) = u_t(x,t-\tau y), \qquad (x,y,t) \in (0,L)\times(0,1)\times\mathbb{R}^+. \tag{2.1}$$

An immediate application of the chain rule yields

$$\tau\eta_t(x,y,t) + \eta_y(x,y,t) = 0, \qquad (x,y,t) \in (0,L)\times(0,1)\times\mathbb{R}^+.$$

Hereafter, we will use the notation $\eta(y)$ to refer to $\eta(x,y,t)$ and only when necessary, we will use $\eta(y,t)$, for example in case when dealing with the delay term. Hence, the system (1.1)–(1.3) becomes

$$\begin{aligned}\rho_1h_1u_{tt} - E_1h_1u_{xx} - \kappa(-u+v+\alpha w_x)&\\ - au_{xxt} - \mu\eta_{xx}(1) = 0 \text{ in } (0,L)\times\mathbb{R}^+&,\end{aligned} \tag{2.2}$$

$$\rho_3h_3v_{tt} - E_3h_3v_{xx} + \kappa(-u+v+\alpha w_x) - bv_{xxt} = 0 \text{ in } (0,L)\times\mathbb{R}^+, \tag{2.3}$$

$$\rho hw_{tt} + EIw_{xxxx} - \kappa\alpha(-u+v+\alpha w_x)_x - cw_{xxt} = 0 \text{ in } (0,L)\times\mathbb{R}^+, \tag{2.4}$$

$$\tau\eta_t(y) + \eta_y(y) = 0 \text{ in } (0,L)\times(0,1)\times\mathbb{R}^+. \tag{2.5}$$

In addition to the boundary and initial conditions (1.4)–(1.5), we add the following condition about η:

$$\eta(x,y,0) = f_0(x,-\tau y) \quad x\in(0,L),\ y\in(0,1), \tag{2.6}$$

where f_0 is a function defined in a suitable Sobolev space, see Section 3 for more details. By the very definition of η, we also have

$$\eta(x,0,t) = u_t(x,t), \quad \eta(x,1,t) = u_t(x,t-\tau) \qquad x\in(0,L),\ \tau\in(0,t). \tag{2.7}$$

Henceforth $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ will denote the usual inner product and norm in $L^2(0,L)$, that is

$$\langle u,v\rangle = \int_0^L u(x)\bar{v}(x)dx, \quad \|u\|^2 = \int_0^L |u(x)|^2dx.$$

In order to find the energy associated with the system (2.2)–(2.5), proceeding formally, we respectively multiply (2.2), (2.3) and (2.4) by u_t, v_t and w_t in $L^2(0,L)$ and (2.5) by $\frac{\delta}{\tau}\eta_{xx}$ in $L^2(0,1;L^2(0,L))$. Thus, adding the resulting identities, by previously using integration by parts, we obtain after taking the real part that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[&\rho_1h_1\|u_t\|^2 + \rho_3h_3\|v_t\|^2 + \rho h\|w_t\|^2 + E_1h_1\|u_x\|^2 + E_3h_3\|v_x\|^2\\ &+ EI\|w_{xx}\|^2 + \kappa\|-u+v+\alpha w_x\|^2 + \frac{\delta}{2}\int_0^1\|\eta_x(y)\|^2dy\Big]\\ &+ a\|u_{xt}\|^2 + b\|v_{xt}\|^2 + c\|w_{xt}\|^2 + \mu\langle\eta_x(1),\eta_x(0)\rangle\\ &+ \frac{\delta}{2\tau}\|\eta_x(1)\|^2 - \frac{\delta}{2\tau}\|\eta_x(0)\|^2 = 0,\end{aligned}$$

with δ being a constant whose value will be fixed later, see (2.9).

The previous identity motivates us to define the energy associated with the system (2.2)–(2.5) as

$$E(t) = \frac{1}{2}\Big[\rho_1 h_1 \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h \|w_t\|^2 + E_1 h_1 \|u_x\|^2 + E_3 h_3 \|v_x\|^2 + EI\|w_{xx}\|^2 + \kappa\| - u + v + \alpha w_x\|^2 + \delta \int_0^1 \|\eta_x(y)\|^2 dy\Big], \tag{2.8}$$

and hence we obtain

$$\frac{d}{dt}E(t) = - a\|u_{xt}\|^2 - b\|v_{xt}\|^2 - c\|w_{xt}\|^2 - \mu\langle \eta_x(1), \eta_x(0)\rangle - \frac{\delta}{2\tau}\|\eta_x(1)\|^2 + \frac{\delta}{2\tau}\|\eta_x(0)\|^2.$$

Note that $\eta_x(0) = \eta_x(x, 0, t) = u_{xt}(x, t)$. Thus, applying Young's inequality, obtain

$$\frac{d}{dt}E(t) \leqslant -b\|v_{xt}\|^2 - c\|w_{xt}\|^2 + \Big(\frac{|\mu|}{2} - \frac{\delta}{2\tau}\Big)\|\eta_x(1)\|^2 + \Big(- a + \frac{|\mu|}{2} + \frac{\delta}{2\tau}\Big)\|\eta_x(0)\|^2.$$

Then, imposing the following condition on the constants a, δ and μ

$$a > \frac{\delta}{\tau} > |\mu| \tag{2.9}$$

we deduce that $\frac{d}{dt}E(t) \leqslant 0$. Hence, the energy of the system (2.2)–(2.5) is not increasing. The previous computations are only formal. The next step will make sense of those by defining the appropriate phase space and domain through a semigroup approach.

3. Well-posedness

We start this section by presenting the well-known Lumer–Phillips Theorem in its version for Hilbert spaces.

Theorem 3.1 (Lumer–Phillips, [25]). *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space X. If $\mathcal{A}$ is dissipative and there is a $\lambda_0 > 0$ such that the range $R(\lambda_0 I - \mathcal{A}) = X$, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X.*

For existence of solution, we use the following corollary of the Lumer–Phillips Theorem, see [26, Theorem 1.2.4].

Corollary 3.1. *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space X. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, then $\mathcal{A}$ is $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X.*

Now, by introducing the vector function $U = (u, \varphi, v, \xi, w, z, \eta)^T$, the system (2.2)–(2.5) can be rewritten as

$$\begin{cases} \frac{d}{dt}U(t) = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0 = (u_0, u_1, v_0, v_1, w_0, w_1, f_0)^T, \end{cases}$$

where the operator $\mathcal{A}\colon D(\mathcal{A}) \subset X \to X$ is defined for $U = (u, \varphi, v, \xi, w, z, \eta)^T$ by

$$\mathcal{A}U = \begin{pmatrix} \varphi \\ \frac{E_1}{\rho_1} u_{xx} + \frac{\kappa}{\rho_1 h_1}(-u + v + \alpha w_x) + \frac{a}{\rho_1 h_1}\varphi_{xx} + \frac{\mu}{\rho_1 h_1}\eta_{xx}(1) \\ \xi \\ \frac{E_3}{\rho_3} v_{xx} - \frac{\kappa}{\rho_3 h_3}(-u + v + \alpha w_x) + \frac{b}{\rho_3 h_3}\xi_{xx} \\ z \\ -\frac{EI}{\rho h} w_{xxxx} + \frac{\kappa\alpha}{\rho h}(-u + v + \alpha w_x)_x + \frac{c}{\rho h} z_{xx} \\ -\frac{1}{\tau}\eta_y(y) \end{pmatrix}, \tag{3.1}$$

and the phase space is

$$X = H_0^1 \times L^2 \times H_0^1 \times L^2 \times H^2 \cap H_0^1 \times L^2 \times L^2(0, 1; H_0^1)$$

which is a Hilbert space with respect to the inner product

$$\begin{aligned} \langle U, \tilde{U}\rangle_X = {} & \rho_1 h_1 \langle \varphi, \tilde{\varphi}\rangle + E_1 h_1 \langle u_x, \tilde{u}_x\rangle + \rho_3 h_3 \langle \xi, \tilde{\xi}\rangle \\ & + E_3 h_3 \langle v_x, \tilde{v}_x\rangle + \rho h \langle z, \tilde{z}\rangle + EI \langle w_{xx}, \tilde{w}_{xx}\rangle \\ & + \kappa\langle -u + v + \alpha w_x, -\tilde{u} + \tilde{v} + \alpha \tilde{w}_x\rangle + \delta \int_0^1 \langle \eta_x(y), \tilde{\eta}_x(y)\rangle dy \end{aligned} \tag{3.2}$$

and norm

$$\begin{aligned} \|U\|_X^2 = {} & \rho_1 h_1 \|\varphi\|^2 + E_1 h_1 \|u_x\|^2 + \rho_3 h_3 \|\xi\|^2 + E_3 h_3 \|v_x\|^2 + \rho h \|z\|^2 \\ & + EI \|w_{xx}\|^2 + \kappa \| - u + v + \alpha w_x\|^2 + \delta \int_0^1 \|\eta_x(y)\|^2 dy, \end{aligned}$$

where $U = (u, \varphi, v, \xi, w, z, \eta)^T$, $\tilde{U} = (\tilde{u}, \tilde{\varphi}, \tilde{v}, \tilde{\xi}, \tilde{w}, \tilde{z}, \tilde{\eta})^T \in X$. Recall that the domain of $\mathcal{A}$ consist of all $U \in X$ so that $\mathcal{A}U \in X$. Consequently, a straightforward computation shows that the domain of operator $\mathcal{A}$ can be defined by

$$D(\mathcal{A}) = (H^2 \cap H_0^1)^4 \times (H^4 \cap H_0^2) \times (H^2 \cap H_0^1) \times L^2(0, 1; H^2 \cap H_0^1). \tag{3.3}$$

In order to apply an efficient semigroup method, one needs to show the associated operator's dissipative property to the system. The following result shows that our operator $\mathcal{A}$ enjoys such a property.

Proposition 3.1. *The operator $\mathcal{A}$ defined by (3.1) and (3.3) is dissipative and satisfies*

$$\operatorname{Re}\langle \mathcal{A}U, U\rangle_X \leqslant -\alpha\|\varphi_x\|^2 - b\|\xi_x\|^2 - c\|z_x\|^2 - \beta\|\eta_x(1)\|^2 \tag{3.4}$$

for all $U = (u, \varphi, v, \xi, w, z, \eta)^T \in D(\mathcal{A})$. Here $\alpha := a - \frac{\delta}{2\tau} - \frac{|\mu|}{2} > 0$ and $\beta := \frac{\delta}{2\tau} - \frac{|\mu|}{2} > 0$.

Proof. Let $U = (u, \varphi, v, \xi, w, z, \eta)^T \in D(\mathcal{A})$. According to the definition of the inner product in X given in (3.2), and applying several times integration by parts formula combined with Fundamental Theorem of Calculus, one deduces

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}U, U\rangle_X = & - a\|\varphi_x\|^2 - b\|\xi_x\|^2 - c\|z_x\|^2 \\ & - \mu \operatorname{Re}\langle \eta_x(1), \eta_x(0)\rangle - \frac{\delta}{2\tau}\|\eta_x(1)\|^2 + \frac{\delta}{2\tau}\|\eta_x(0)\|^2. \end{aligned}$$

Taking into account that $\eta_x(0) = \eta_x(x, 0, t) = \varphi_x(x, t)$, Young's inequality yields

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle_X \leqslant -\Big(a - \frac{|\mu|}{2} - \frac{\delta}{2\tau}\Big)\|\varphi_x\|^2 - b\|\xi_x\|^2 - c\|z_x\|^2 - \Big(\frac{\delta}{2\tau} - \frac{|\mu|}{2}\Big)\|\eta_x(1)\|^2.$$

From condition (2.9) imposed on a, δ and μ, it follows directly that $\mathcal{A}$ is dissipative and the estimate (3.4) is then satisfied. □

PROPOSITION 3.2. *Let $\rho(\mathcal{A})$ be the resolvent of the operator $\mathcal{A}$. Then $0 \in \rho(\mathcal{A})$.*

PROOF. By *reductio ad absurdum*, we assume that $0 \in \sigma(\mathcal{A})$, the spectrum of the operator $\mathcal{A}$. This implies that there exists a sequence

$$U_n = (u_n, \varphi_n, v_n, \xi_n, w_n, z_n, \eta_n)^T \in D(\mathcal{A}),$$

indexed by $n \in \mathbb{N}$, with $\|U_n\|_X = 1$ such that $\mathcal{A}U_n = o(1)$, that is, $\mathcal{A}U_n \to 0$ in X. Then by (3.4), one has

$$\|\varphi_{n,x}\|,\ \|\xi_{n,x}\|,\ \|z_{n,x}\|,\ \|\eta_{n,x}(1)\| \leqslant \|\mathcal{A}U_n\|_X = o(1). \tag{3.5}$$

The definition (2.1) and Poincaré inequality leads to

$$\|\varphi_n\|,\ \|\xi_n\|,\ \|z_n\|,\ \|\eta_n\|_{L^2(0,1;L^2(0,L))} = o(1). \tag{3.6}$$

Let $V_n = (0, -u_n, 0, -v_n, 0, -w_n, 0)$. Since (V_n) is bounded in X, we have that $\langle \mathcal{A}U_n, V_n\rangle = o(1)$. Then

$$\begin{aligned} &E_1h_1\|u_{n,x}\|^2 + E_3h_3\|v_{n,x}\|^2 + EI\|w_{n,xx}\|^2 + \kappa\| - u_n + v_n + \alpha w_{n,x}\|^2 \\ &\quad + a\langle \varphi_{n,x}, u_{n,x}\rangle + b\langle \xi_{n,x}, v_{n,x}\rangle + c\langle z_{n,x}, w_{n,x}\rangle + \mu\langle \eta_{n,x}(1), u_{n,x}\rangle = o(1). \end{aligned} \tag{3.7}$$

The estimates obtained in (3.5) and the boundedness of the sequences $(u_{n,x})$, $(v_{n,x})$ and $(w_{n,x})$ in L^2 lead us to

$$\|u_{n,x}\|,\ \|v_{n,x}\|,\ \|w_{n,xx}\|,\ \| - u_n + v_n + \alpha w_{n,x}\| = o(1). \tag{3.8}$$

From (3.6) and (3.8), we deduce that $\|U_n\|_X = o(1)$, which contradicts our assumption. Hence, $0 \in \rho(\mathcal{A})$, and the proof is now completed. □

THEOREM 3.2. *The operator $\mathcal{A}$ defined above is the infinitesimal generator of a C_0-semigroup $e^{t\mathcal{A}}$ of contractions in the Hilbert space X.*

PROOF. It is obvious that $D(\mathcal{A})$ is dense in X. By the previous propositions, the operator $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$. Then, by Corollary 3.1, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $e^{t\mathcal{A}}$ of contractions in the Hilbert space X. □

The well-posedeness is given by the following result.

THEOREM 3.3. *Let $U_0 \in X$, then there exists a unique weak solution U of problem (1.1)–(1.5) satisfying*

$$U \in C([0, +\infty); X). \tag{3.9}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C([0, +\infty); D(\mathcal{A})) \cap C^1([0, +\infty); X). \tag{3.10}$$

PROOF. From semigroup theory, see e.g. [25], since $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $e^{t\mathcal{A}}$ of contractions in the Hilbert space X, we deduce that $U(t) = e^{t\mathcal{A}}U_0$ is the unique solution of problem (1.1)–(1.5) satisfying (3.9) and (3.10). □

4. Exponential stability

In this section we will prove that the C_0-semigroup $e^{\mathcal{A}t}$ of contractions is exponentially stable. For this purpose, we will use the following theorem that gives necessary and sufficient conditions for the exponential stability of a C_0-semigroup of contractions. This result was independently obtained by Gearhart [22] and Huang [23], and more recently by Pruss [24].

THEOREM 4.1. *Let $\rho(\mathcal{A})$ be the resolvent set of the operator $\mathcal{A}$ and $S(t) = e^{t\mathcal{A}}$ be the C_0-semigroup of contractions generated by $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{4.1}$$

and

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty. \tag{4.2}$$

In order to establish our main result we will prove that the operator $\mathcal{A}$ satisfies (4.1) and (4.2). Let's start with the first condition.

PROPOSITION 4.1. *Let $\rho(\mathcal{A})$ be the resolvent of the operator $\mathcal{A}$. Then*

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

PROOF. We prove the inclusion by using a contradiction argument. If the inclusion does not hold, then there exist $\omega \in \mathbb{R}$, $\omega \neq 0$ with $\|\mathcal{A}^{-1}\| \leqslant |\omega| < \infty$, and a couple of sequences $(\lambda_n) \subset \mathbb{R}$, $(U_n) \subset D(\mathcal{A})$ with

$$|\lambda_n| < |\omega|, \quad \lambda_n \to \omega, \qquad \text{as} \quad n \to \infty$$

and

$$U_n := (u_n, \varphi_n, v_n, \xi_n, w_n, z_n, \eta_n)^T, \quad \|U_n\|_X = 1, \quad \|(i\lambda_n I - \mathcal{A})U_n\|_X \to 0. \tag{4.3}$$

Setting

$$(i\lambda_n I - \mathcal{A})U_n = F_n, \quad F_n = (f_n^1, f_n^2, \ldots, f_n^6, f_n^7)^T$$

and by previous convergence in X, we have

$$\begin{aligned} f_n^1, f_n^3 \to 0 \ \text{ in } \ H_0^1(0,L), \quad &\text{as} \ \ n \to \infty, \\ f_n^2, f_n^4, f_n^6 \to 0 \ \text{ in } \ L^2(0,L), \quad &\text{as} \ \ n \to \infty, \\ f_n^5 \to 0 \ \text{ in } \ H^2(0,L) \cap H_0^1(0,L), \quad &\text{as} \ \ n \to \infty, \\ f_n^7 \to 0 \ \text{ in } \ L^2(0,1; H_0^1(0,L)), \quad &\text{as} \ \ n \to \infty. \end{aligned} \tag{4.4}$$

To take advantage of the dissipative property of $\mathcal{A}$, we take the inner product in X of $(i\lambda_n I - \mathcal{A})U_n$ against U_n, and after taking the real part of the resulting identity, we immediately deduce from Proposition 3.1 that

$$\varphi_{n,x}, \ \xi_{n,x}, \ z_{n,x}, \ \eta_{n,x}(1) \to 0 \ \text{ in } \ L^2(0,L) \quad \text{as} \ \ n \to \infty.$$

Moreover, thanks to Poincaré inequality, we also have the convergences

$$\varphi_n,\ \xi_n,\ z_n,\ \eta_n(1) \to 0 \quad \text{in} \quad L^2(0,L) \quad \text{as} \quad n \to \infty. \tag{4.5}$$

The next step consists in proving that the sequence (U_n) converges to zero in X, which would be a contradiction with the unitary property of U_n, see (4.3), and the proof shall be completed. To do so, the identity $(i\lambda_n I - \mathcal{A})U_n = F_n$ reads as follows

$$i\lambda_n u_n - \varphi_n = f_n^1, \quad \text{in} \quad H_0^1, \tag{4.6}$$

$$\begin{aligned} i\lambda_n \varphi_n - \frac{E_1}{\rho_1} u_{n,xx} - \frac{\kappa}{\rho_1 h_1}(-u_n + v_n + \alpha w_{n,x}) \\ - \frac{a}{\rho_1 h_1}\varphi_{n,xx} - \frac{\mu}{\rho_1 h_1}\eta_{n,xx}(1) = f_n^2, \quad \text{in} \quad L^2, \end{aligned} \tag{4.7}$$

$$i\lambda_n v_n - \xi_n = f_n^3, \quad \text{in} \quad H_0^1, \tag{4.8}$$

$$i\lambda_n \xi_n - \frac{E_3}{\rho_3} v_{n,xx} + \frac{\kappa}{\rho_3 h_3}(-u_n + v_n + \alpha w_{n,x}) - \frac{b}{\rho_3 h_3}\xi_{n,xx} = f_n^4, \quad \text{in} \quad L^2, \tag{4.9}$$

$$i\lambda_n w_n - z_n = f_n^5, \quad \text{in} \quad H^2 \cap H_0^1, \tag{4.10}$$

$$i\lambda_n z_n + \frac{EI}{\rho h} w_{n,xxxx} - \frac{\kappa\alpha}{\rho h}(-u_n + v_n + \alpha w_{n,x})_x - \frac{c}{\rho h} z_{n,xx} = f_n^6, \quad \text{in} \quad L^2, \tag{4.11}$$

$$i\lambda_n \eta_n(y) + \frac{1}{\tau}\eta_{n,y}(y) = f_n^7(y), \quad \text{in} \quad L^2(0,1;H_0^1). \tag{4.12}$$

Combining (4.4)–(4.5) with (4.6), (4.8) and (4.10), we easily get the convergences

$$u_n,\ v_n,\ w_n, \to 0 \quad \text{in} \quad H_0^1(0,L), \qquad -u_n + v_n + \alpha w_{n,x} \to 0 \quad \text{in} \quad L^2(0,L), \qquad \text{as} \quad n \to \infty.$$

It remains to prove that $w_{n,xx} \to 0$ in L^2 and $\eta_n \to 0$ in $L^2(0,1;H_0^1)$. Taking the inner product of (4.11) with by $\rho h w_n$ in $L^2(0,L)$, and integrating by parts, we obtain

$$\begin{aligned} EI\|w_{n,xx}\|^2 = & -\kappa\alpha\langle -u_n + v_n + \alpha w_{n,x}, w_{n,x}\rangle \\ & - c\langle z_{n,x}, w_{n,x}\rangle - i\lambda_n \rho h\langle z_n, w_n\rangle + \rho h\langle f_n^6, w_n\rangle. \end{aligned}$$

By previous convergences, one can easily check that each term on the right-hand side goes to zero in $\mathbb{C}$ when $n \to \infty$. Hence $w_{n,xx} \to 0$ in L^2 when $n \to \infty$ as desired. On the other hand, equation 4.12 can be explicitly solved. Indeed, using that $\eta_n(x,0) = \varphi_n$ combined with a variation of parameters method, we easily get

$$\eta_n(x,y) = e^{-i\tau\lambda_n y}\varphi_n + \tau\int_0^y e^{i\tau\lambda_n(s-y)} f_n^7(x,s)\,ds, \quad \text{a.e.} \quad (x,y) \in (0,L)\times(0,1).$$

Since $|e^{i\omega}| = 1$ for all $\omega \in \mathbb{R}$ and the norm in the space $L^2(0,1;H_0^1)$ only involves derivatives with respect to the spatial variable x, it follows that

$$\|\eta_n\|_{L^2(0,1;H_0^1)} \leqslant C(\|\varphi_{n,x}\| + \|f_n^7\|_{L^2(0,1;H_0^1)}) \to 0 \quad \text{as} \quad n \to \infty$$

for some constant $C > 0$. This completes the proof. □

We now move to verify the second condition.

Proposition 4.2. *The operator $\mathcal{A}$ satisfies the following resolvent estimate*

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I-\mathcal{A})^{-1}\| < +\infty.$$

Proof. We again use a contradiction argument. If the above condition does not hold, then there exist a couple of sequences $(\lambda_n)\subset\mathbb{R}$, $(U_n)\subset D(\mathcal{A})$ with

$$\lambda_n\to\infty, \quad \text{as} \quad n\to\infty$$

and

$$U_n := (u_n,\varphi_n,v_n,\xi_n,w_n,z_n,\eta_n)^T, \quad \|U_n\|_X=1, \qquad \|(i\lambda_n I-\mathcal{A})U_n\|_X\to 0. \tag{4.13}$$

Let us set

$$(i\lambda_n I-\mathcal{A})U_n = F_n, \quad F_n=(f_n^1,f_n^2,\dots,f_n^6,f_n^7)^T \tag{4.14}$$

By previous convergence in X, we deduce the same convergences as in (4.4). Taking the inner product of (4.14) with U_n in X, considering the real part and using again the dissipativity property of $\mathcal{A}$ given in Proposition 3.1, we obtain

$$\varphi_{n,x},\ \xi_{n,x},\ z_{n,x},\ \eta_{n,x}(1)\to 0 \ \text{ in } \ L^2(0,L) \quad \text{as} \quad n\to\infty, \tag{4.15}$$

and by Poincaré inequality, we also have

$$\varphi_n,\ \xi_n,\ z_n,\ \eta_n(1)\to 0 \ \text{ in } \ L^2(0,L) \quad \text{as} \quad n\to\infty. \tag{4.16}$$

The idea of the proof is, as in Proposition 4.1, to prove that the sequence (U_n) goes to zero in X. However, the analysis of the convergences is more delicate because of in this case the sequence of real numbers (λ_n) goes to infinity. As $F_n=(f_n^1,\dots,f_n^6,f_n^7)$, we can rewrite the spectral equation (4.13) in terms of its components, and we will get the system (4.6)–(4.12) again. From (4.6), (4.8) and (4.10), we easily get

$$u_n=-\frac{i}{\lambda_n}(\varphi_n+f_n^1), \quad v_n=-\frac{i}{\lambda_n}(\xi_n+f_n^3), \quad w_n=-\frac{i}{\lambda_n}(z_n+f_n^5).$$

By (4.4), (4.15) and (4.16), all the sequences on the right-hand side are bounded in $H_0^1(0,L)$, and since $\lambda_n\to\infty$, we deduce

$$u_n,v_n,w_n\to 0 \ \text{ in } \ H_0^1(0,L) \quad \text{as} \quad n\to\infty$$

and therefore

$$-u_n+v_n+\alpha w_{n,x}\to 0 \ \text{ in } \ L^2(0,L), \quad \text{as} \quad n\to\infty. \tag{4.17}$$

Taking the inner product of (4.11) with $\rho h w_n$ in $L^2(0,L)$, and integrating by parts, we get

$$\begin{aligned} EI\|w_{n,xx}\|^2 = &-\kappa\alpha\langle -u_n+v_n+\alpha w_{n,x}, w_{n,x}\rangle \\ &- c\langle z_{n,x},w_{n,x}\rangle - i\lambda_n\rho h\langle z_n,w_n\rangle + \rho h\langle f_n^6,w_n\rangle. \end{aligned}$$

By previous convergences, namely (4.4) and (4.15)–(4.17), one can easily check that the first three terms on the right-hand side go to zero in $\mathbb{C}$ when $n\to\infty$. The last term can be written as

$$-i\lambda_n\rho h\langle z_n,w_n\rangle = \rho h\langle z_n, i\lambda_n w_n\rangle = \rho h\langle z_n, z_n+f_n^5\rangle.$$

Then, the convergences (4.4) and (4.16) imply that $i\lambda_n \rho h\langle z_n, w_n\rangle \to 0$. Hence $w_{n,xx} \to 0$ in L^2 when $n \to \infty$ as desired. On the other hand, once more as in the Proposition 4.1, by recalling that $\eta_n(x,0) = \varphi_n$, the variation of parameters method allows us to obtain the explicit solution of (4.12)

$$\eta_n(x,y) = e^{-i\tau\lambda_n y}\varphi_n + \tau\int_0^y e^{i\tau\lambda_n(s-y)} f_n^7(x,s)ds, \quad \text{a.e.} \quad (x,y) \in (0,L)\times(0,1).$$

Note that the norm in the space $L^2(0,1;H_0^1)$ only involves derivatives with respect to the spatial variable x, thus it follows that

$$\|\eta_n\|_{L^2(0,1;H_0^1)} \leqslant C(\|\varphi_{n,x}\| + \|f_n^7\|_{L^2(0,1;H_0^1)}),$$

where the constant $C > 0$ is independent of λ_n thanks to the fact $|e^{i\omega}| = 1$ for all $\omega \in \mathbb{R}$. Hence $\|\eta_n\|_{L^2(0,1;H_0^1)} \to 0$ as $n \to \infty$. This completes the proof. □

Finally, we establish our main result.

THEOREM 4.2. *The semigroup $S(t) = e^{t\mathcal{A}}$ generated by $\mathcal{A}$ is exponentially stable.*

PROOF. From Proposition 4.1 and Proposition 4.2, it follows that the conditions of Theorem 4.1 are satisfied and then our semigroup $S(t) = e^{t\mathcal{A}}$ generated by $\mathcal{A}$ is exponentially stable. □

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СТАБИЛНОСТ РЕШЕЊА РАО–НАКРА МОДЕЛА СЕНДВИЧ ГРЕДЕ СА КЕЛВИН–ВОИГТОВИМ ПРИГУШЕЊЕМ И ВРЕМЕНСКИМ КАШЊЕЊЕМ

Резиме. Овај рад се бави стабилношћу решења једнодимензионалног Рао-Накра модела вишеслојне (сендвич) греде са Келвин-Воигтовим пригушењем и временским кашњењем датим са

$$\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \kappa(-u + v + \alpha w_x) - a u_{xxt} - \mu u_{xxt}(\,\cdot\,, t - \tau) = 0,$$
$$\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \kappa(-u + v + \alpha w_x) - b v_{xxt} = 0,$$
$$\rho h w_{tt} + EI w_{xxxx} - \kappa\alpha(-u + v + \alpha w_x)_x - c w_{xxt} = 0.$$

Сендвич греда је инжењерски модел који се састоји од три слоја: два тврда спољна слоја, доње и горње стране, и више усклађеног унутрашњег слоја – језгра. Рао-Накра систем се састоји од три слоја и претпоставка је да нема клизања на интерфејсу између контаката. Горњи и доњи слој су описани таласном једначином за уздужна померања према претпоставкама Ојлер-Берноулијеве греде. Слој језгра је дат једном једначином која описује попречно померање према претпоставкама Тимошенкове греде. Добра постављеност модела је показана применом теорије полугрупа и Лумер-Филипсове теореме. Експоненцијална стабилност је доказана коришћењем Геархарт-Хуанг-Прусове теореме.

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