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Quasiconformal Uniformization of Metric Surfaces

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Toni Ikonen

List of included articles

This dissertation consists of an introductory part and the following four publications:

- **[A]** T. Ikonen, *Uniformization of metric surfaces using isothermal coordinates*, Annales Fennici Mathematici, 47(1), 155–180. https://doi.org/10.54330/afm.112781.
- **[B]** T. Ikonen and M. Romney, *Quasiconformal geometry and removable sets for conformal mappings*, Journal d'Analyse Mathématique, to appear.
- **[C]** T. Ikonen, *Quasiconformal Jordan domains*, Analysis and Geometry in Metric Spaces, 9(1), 167-185. https://doi.org/10.1515/agms-2020-0127.
- **[D]** T. Ikonen, *Two-dimensional metric spheres from gluing hemispheres*, arXiv:2106.01295.

The author of this dissertation has actively taken part in the research of the joint article [B].

Abstract

The main subject of this dissertation is the uniformization problem for nonsmooth surfaces. The foundational question is to find necessary and sufficient conditions for the existence of a homeomorphism taking a given nonsmooth surface into a smooth Riemannian surface while requiring minimal geometric distortion from the mapping. More specifically, we require the homeomorphism to be quasiconformal. Our approach is based on a recent work by Rajala. The dissertation consists of four articles.

In article [A], we prove a uniformization result for every nonsmooth surface satisfying mild geometric assumptions. In fact, we only assume that the surface can be covered by domains which can be quasiconformally mapped into the Euclidean plane. We prove that this is a sufficient (and necessary) condition for there to exist a quasiconformal map onto a smooth Riemannian surface.

In article [B], the author and Romney investigate weighted distances on the Euclidean plane. The main result of the article shows a surprising link between the nonsmooth uniformization problem and sets removable for conformal mappings, a notion of removability introduced by Ahlfors and Beurling in the 1950s.

In article [C], we examine the boundary structure of nonsmooth Euclidean disks which have finite two-dimensional Hausdorff measure and whose interiors can be quasiconformally mapped onto the Euclidean disk. We prove a generalized Carathéodory theorem in this setting and provide examples showing the sharpness of the result.

In article [D], we consider a metric version of the classical welding problem from complex analysis. We construct nonsmooth spheres by metrically welding the southern and northern hemispheres of the two-dimensional sphere along the equator using a homeomorphism from the equator onto itself. The goal is to understand when the resulting sphere can be quasiconformally mapped to the Euclidean sphere. A necessary condition we establish connects the metric welding problem to the classical one, while our sufficient conditions are related to measure-theoretic properties and modulus of continuity of the welding map.

Tiivistelmä

Väitöskirjan pääaihe on epäsileiden pintojen uniformisaatio. Perustavanlaatuinen kysymys on löytää riittäviä ja välttämättömiä ehtoja, jotta annetulta epäsileältä pinnalta on olemassa homeomorfismi sileälle pinnalle siten että kartta vääristää pinnan geometriaa mahdollisimman vähän. Tarkemmin sanottuna, vaadimme homeomorfismin olevan kvasikonformaalinen. Lähestymistapamme perustuu äskettäin julkaistuun Rajalan työhön. Väitöskirja koostuu neljästä artikkelista.

Artikkelin [A] päätulos todistaa uniformisaatiotuloksen kaikille epäsileille pinnoille, jotka toteuttavat heikon geometrisen oletuksen. Oletamme, että pinta voidaan peittää alueilla, joista jokainen voidaan kvasikonformaalisesti kuvata tasoon. Osoitamme tämän olevan riittävä (ja välttämätön) ehto sille, että annetulta pinnalta on olemassa kvasikonformikuvaus johonkin sileään Riemannin pintaan.

Artikkeli [B] on kirjoitettu yhdessä Matthew Romneyn kanssa. Tutkimme artikkelissa painotettuja etäisyyksiä Eukleideen tasossa. Artikkelin päätulos yhdistää yllättävällä tavalla epäsileän uniformisaatiokysymyksen erääseen Ahlforsin ja Beurlingin 1950-luvulla esittelemään poistuvien joukkojen käsitteeseen.

Artikkelissa [C] tutkitaan niiden epäsileiden Eukleideen kiekkojen reunan rakennetta, joilla on äärellinen kaksiulotteinen Hausdorffin mitta ja joiden sisuksesta on kvasikonformaalinen kuvaus Eukleideen kiekkoon. Artikkelin päätulos todistaa yleistetyn Carathéodoryn teoreeman ja esimerkit todistavat tuloksen olevan paras mahdollinen.

Artikkelin [D] aihe on metrinen versio klassisesta kompleksianalyysin hitsausongelmasta. Konstruoimme epäsileitä pintoja metrisesti hitsaamalla eteläinen ja pohjoinen pallonpuolisko päiväntasaajia pitkin. Tutkimme milloin saatu epäsileä pallo voidaan kvasikonformaalisesti kuvata Eukleideen pallolle. Välttämätön ehto yhdistää metrisen ongelman klassiseen hitsausongelmaan, kun taas riittävät ehdot liittyvät saumakuvauksen mittateoreettisiin sekä jatkuvuusominaisuuksiin.

INTRODUCTION

A *metric surface* X is a separable metric space which is homeomorphic to a topological surface without boundary and has a locally finite two-dimensional Hausdorff measure $\mathcal{H}^2_X$. We assume surfaces to be connected unless otherwise mentioned.

Typical examples of metric surfaces include domains V of the Euclidean plane $\mathbb{R}^2$, Riemannian surfaces M endowed with the length distance and the Riemannian area measure, and weighted distances in the plane. Under suitable regularity assumptions on a *weight* $\omega\colon \mathbb{R}^2 \to [0,\infty]$, the *weighted distance*

$$d_\omega(x,y) = \inf \int_\gamma \omega\, dt, \tag{1}$$

the infimum taken over absolutely continuous paths $\gamma\colon [a,b] \to \mathbb{R}^2$ joining x to y, defines a distance on $\mathbb{R}^2$ for which $X_\omega = (\mathbb{R}^2, d_\omega)$ is a metric surface. Similar constructions can be considered when the plane $\mathbb{R}^2$ is replaced by the sphere $\mathbb{S}^2$ or another smooth Riemannian surfaces. We elaborate on the significance of these examples in the coming sections.

Nonsmooth examples can be obtained by considering, for example, the graph of $u(x) = |x|^\alpha$, for $1 \geq \alpha > 0$, or much more involved constructions as in Figure 1.

1. Modulus and quasiconformality

An important tool in this dissertation is the so-called *conformal modulus*: Let X be a metric surface, and fix a subset $F \subset X$. A path family Γ (in F) is a collection of paths $\gamma\colon [a,b] \to F$. A Borel function $\rho\colon F \to [0,\infty]$ is *admissible* for Γ if

$$1 \leq \int_\gamma \rho\, ds \quad \text{for every } \gamma \in \Gamma. \tag{2}$$

The *modulus* of Γ is

$$\operatorname{mod}\Gamma = \inf \int_F \rho^2\, d\mathcal{H}^2_X, \tag{3}$$

where the infimum is taken over all Borel functions admissible for Γ.

Given two metric surfaces X and Y and two subsets $F \subset X$, $F' \subset Y$, a continuous map $\varphi\colon F \to F'$ has *bounded outer dilatation* if there exists a constant $K \geq 1$ such that

$$\operatorname{mod}\Gamma \leq K \operatorname{mod}\varphi\Gamma \quad \text{for all path families } \Gamma \text{ in } F. \tag{4}$$

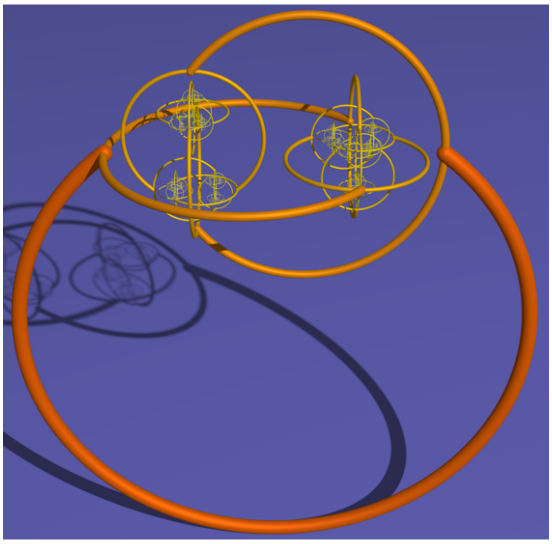


FIGURE 1. Alexander's horned sphere: A nonsmooth metric surface.
Source: https://upload.wikimedia.org/wikipedia/commons/0/0a/Alexander_horned_sphere.png

Here $\varphi\Gamma$ denotes the collection of all paths $\varphi \circ \gamma$ with $\gamma \in \Gamma$. The smallest constant K for which (4) holds is called the *outer dilatation* of φ, denoted by $K_O(\varphi)$. If the preimage of each point under φ is connected, we say that φ is *monotone*. Following [NR21b], we say that a monotone and surjective mapping with finite outer dilatation is *weakly quasiconformal*.

We say that φ is *quasiconformal* if φ is a homeomorphism and there exists a constant $K \geq 1$ such that $K_O(\varphi), K_O(\varphi^{-1}) \leq K$. Given such a K, we say that φ is *K-quasiconformal* and the smallest K with this property is called the *maximal dilatation* of φ.

A particular class of quasiconformal homeomorphisms is given by the class of *bi-Lipschitz homeomorphisms*: We say that $\varphi \colon X \to Y$ is *Lipschitz* if there exists a constant $L \geq 0$ for which

$$d_Y(\varphi(x), \varphi(y)) \leq L d_X(x, y) \quad \text{for every } x, y \in X. \tag{5}$$

Any mapping satisfying (5) is called *L-Lipschitz*. We say that a homeomorphism is (L-)bi-Lipschitz if φ and its inverse are L-Lipschitz for some $L \geq 0$. By directly working with (2) and (3), the L^4-quasiconformality of L-bi-Lipschitz homeomorphisms follows.

Lastly, we say that a homeomorphism $\varphi \colon X \to Y$ between metric spaces is a *quasisymmetry* if there exists a homeomorphism $\eta \colon [0, \infty) \to [0, \infty)$ such

that
$$\frac{d(\varphi(x),\varphi(y))}{d(\varphi(z),\varphi(y))} \leq \eta\left(\frac{d(x,y)}{d(z,y)}\right) \quad \text{for every } x,y,z \in X,\, z \neq y, \tag{6}$$
and whenever (6) holds, we refer to φ as an *η-quasisymmetry*. Observe that if φ is L-bi-Lipschitz, it is η-quasisymmetric for $\eta(t) = L^2 t$.

2. Uniformization problems

The Riemann mapping theorem [AS60, Ahl78] states that every simply connected domain $X \subset \mathbb{R}^2$, with $X \neq \mathbb{R}^2$, is the 1-quasiconformal image of the Euclidean disk $M = \mathbb{D}$. That is, there exists a 1-quasiconformal homeomorphism
$$u\colon M \to X \tag{7}$$
from the *model space* $M = \mathbb{D}$. More generally, a version of the *uniformization problem* asks the following: which Riemannian surfaces are quasiconformal images of some "model" Riemannian surfaces. The following fundamental theorem, see e.g. [AS60, Cou77, Hub06, dSG10], settles a version of this question.

Theorem 2.1 (Riemannian uniformization). *Suppose that X is a Riemannian surface. Then there exists a complete and constant curvature Riemannian surface M and a 1-quasiconformal homeomorphism $u\colon M \to X$.*

The curvature in Theorem 2.1 refers to the Gaussian curvature, and the curvature can be normalized to be exactly one of the numbers $-1, 0, 1$, uniquely determined by X. Theorem 2.1 has several different proofs, with approaches based on elliptic PDE's and isothermal coordinates, curvature flows, or an approach based on classification of universal covers of simply connected Riemannian surfaces, just to name a few. We refer the interested reader to the article [Abi81] for further reading.

We discuss three different approaches to generalizing Theorem 2.1 to the metric surface setting: the bi-Lipschitz, quasisymmetric, and quasiconformal variants. A basic motivation for these problems is to understand the geometry and/or analysis of a nonsmooth surface by reducing to the smooth setting.

A typical approach in such uniformization problems is to try to identify different *invariants* of the model spaces preserved by the *uniformization maps* of interest. In the following sections, we highlight some of the connections and distinctions between the three problems.

3. Bi-Lipschitz uniformization

Consider the sphere $\mathbb{S}^2$ endowed with the usual geodesic distance σ of Gaussian curvature one. We consider for a moment a metric space X that is a bi-Lipschitz image of $\mathbb{S}^2$.

It is straightforward to verify the following two geometric bi-Lipschitz invariants: First, whenever r is smaller than the diameter

$$\operatorname{diam}(X) := \sup_{x,y\in X} d_X(x,y),$$

the two-dimensional Hausdorff measure of a ball B_r of radius r in X is comparable to r^2. We refer to this property as *Ahlfors regularity* of X. Note, in particular, that there exists a constant $C > 0$ such that

$$\mathcal{H}^2_X(B_r) \leq Cr^2 \quad \text{for all } r > 0. \tag{8}$$

We define a second invariant, called *linear local connectivity* (or LLC, for short). This consists of the conditions defined below, involving a constant $\lambda \geq 1$.

- If $B_r(x_0)$ is a ball in X and $x, y \in B_r(x_0)$, there exists a compact and connected set $C \subset B_{\lambda r}(x_0)$ containing x and y.
- If $B_r(x_0)$ is a ball in X and $x, y \in X \setminus B_r(x_0)$, there exists a compact and connected set $C \subset X \setminus B_{r/\lambda}(x_0)$ containing x and y.

Since the LLC property is a bi-Lipschitz (and quasisymmetric) invariant, we ask the following.

Question 3.1 (Bi-Lipschitz uniformization)**.** *If X is Ahlfors regular, linearly locally connected and homeomorphic to $\mathbb{S}^2$, does there exists a bi-Lipschitz homeomorphism $u\colon \mathbb{S}^2 \to X$?*

The answer turns out to be negative, and we briefly discuss some counterexamples below. However, when *bi-Lipschitz* is replaced by *quasisymmetry*, Question 3.1 has a positive answer:

Theorem 3.2 (Bonk–Kleiner [BK02])**.** *If X satisfies the assumptions of Question 3.1, there exists a quasisymmetry $\varphi\colon \mathbb{S}^2 \to X$.*

The authors proved that whenever X is Ahlfors regular and homeomorphic to $\mathbb{S}^2$, the mapping φ exists if and only if X is linearly locally connected. Their proof uses an interesting discrete version of Theorem 2.1, usually formulated using circle packings, see [Ste05].

Theorem 3.2 is a particular example of the *quasisymmetric uniformization* problem. In fact, under the assumptions of Theorem 3.2, the mapping φ is also quasiconformal. Hence Theorem 3.2 answers positively also the quasiconformal uniformization problem in the class of metric surfaces defined in Question 3.1.

We elaborate on the connection between Question 3.1 and Theorem 3.2. To any given pair (X, φ) as in Theorem 3.2, we associate the *pullback measure*

$$\varphi^*\mathcal{H}^2_X(B) := \mathcal{H}^2_X(\varphi(B)) \quad \text{for every Borel } B \subset \mathbb{S}^2.$$

Then $\varphi^*\mathcal{H}^2_X$ is *doubling*, i.e., there exists a constant $C > 0$ such that for every ball $B_{2r}(x)$, the ball $B_r(x)$ satisfies

$$\varphi^*\mathcal{H}^2_X(B_{2r}(x)) \leq C\varphi^*\mathcal{H}^2_X(B_r(x)).$$

Moreover, up to enlarging C, we also have

$$C^{-1}D(x,y) \leq \left(\varphi^*\mathcal{H}^2_X(B_{2\sigma(x,y)}(x))\right)^{1/2} \leq CD(x,y) \quad \text{for every } x,y \in \mathbb{S}^2 \tag{9}$$

for the distance $D(x,y) = d(\varphi(x), \varphi(y))$ on the sphere. For a proof of these properties of $\varphi^*\mathcal{H}^2_X$, we refer the interested reader to [Laa02]. Laakso proved the result for the Euclidean plane $\mathbb{R}^2$ in place of the sphere $\mathbb{S}^2$, but a similar argument works in our case. See also [Sem93, Sem96b, BHS04].

If one replaces the measure $\mu = \varphi^*\mathcal{H}^2_X$ by some other doubling measure μ and the distance D by an arbitrary distance D_μ on the sphere, still satisfying

$$C^{-1}D_\mu(x,y) \leq \left(\mu(B_{2\sigma(x,y)}(x))\right)^{1/2} \leq CD(x,y) \quad \text{for every } x,y \in \mathbb{S}^2 \tag{10}$$

for some $C > 0$, cf. (9), we say that μ is a *metric doubling measure.* For such a measure, one can prove that $X_\mu := (\mathbb{S}^2, D_\mu)$ is a metric surface satisfying the assumptions of Question 3.1. In fact,

$$\text{the change of distance map } \varphi_\mu \colon \mathbb{S}^2 \to X_\mu \text{ is a quasisymmetry;} \tag{11}$$

here $\varphi_\mu(x) = x$. This observation follows by unwinding the definitions, see [Laa02].

Metric doubling measures were introduced by David and Semmes in [DS90], in the Euclidean setting. Having proved many interesting results about such measures, among them Poincaré inequalities and absolute continuity with respect to the Lebesgue measure, the authors asked in the Euclidean setting the following question: For which metric doubling measures μ does there exist a quasiconformal homeomorphism $f \colon \mathbb{S}^2 \to \mathbb{S}^2$ and a constant $C \geq 1$ such that

$$C^{-1}f^*\mathcal{H}^2_{\mathbb{S}^2}(B) \leq \mu(B) \leq Cf^*\mathcal{H}^2_{\mathbb{S}^2}(B) \quad \text{for every Borel set } B \subset \mathbb{S}^2? \tag{12}$$

As explained in [Laa02, BHS04, Bis07], the question of David and Semmes turns out to be equivalent to Question 3.1, in the following sense. First, if $u \colon \mathbb{S}^2 \to X_\mu$ is bi-Lipschitz and φ_μ as in (11), then $f = u^{-1} \circ \varphi_\mu$ satisfies (12). Conversely, if (12) is satisfied, then $u := \varphi_\mu \circ f^{-1}$ is a bi-Lipschitz parametrization of X_μ.

Laakso was the first to construct an example of a metric doubling measure which does *not* satisfy (12), thereby answering Question 3.1 in the negative. In [Laa02], he constructed an example of a compact and connected set $E \subset \mathbb{S}^2$ for which there are $s > 1$ and a continuous weight $\omega \colon \mathbb{S}^2 \to [0, \infty)$ satisfying

$$C^{-1}\sigma^{s-1}(x,E) \leq \omega(x) \leq C\sigma^{s-1}(x,E), \tag{13}$$

where $\sigma(x, E)$ is the spherical distance from x to the set E. The weight and E were constructed in such a way that the Borel measure

$$\mu(B) := \int_B \omega^2(x)\, d\mathcal{H}^2_{\mathbb{S}^2}, \quad B \subset \mathbb{S}^2 \tag{14}$$

is metrically doubling on $\mathbb{S}^2$. By a careful construction, Laakso proved that the weighted distance d_ω in (1) satisfies condition (9). An interesting consequence of his construction is that the X_μ cannot be bi-Lipschitz embedded into any finite-dimensional Banach space, or any uniformly convex Banach space. In particular, X_μ cannot be bi-Lipschitz equivalent to $\mathbb{S}^2$. Later on, Bishop [Bis07] used a different construction to find an example of a complete Ahlfors regular and LLC metric surface X, quasisymmetrically equivalent to $\mathbb{R}^2$, which can be bi-Lipschitz embedded into $\mathbb{R}^3$ but not into $\mathbb{R}^2$.

We see from the examples of Laakso and Bishop that the geometric variants of Ahlfors regularity and linear local connectivity are not enough to recognize the bi-Lipschitz images of $\mathbb{S}^2$ or $\mathbb{R}^2$. For metric surfaces $X \subset \mathbb{R}^N$, for some $N \geq 3$, satisfying local variants of Ahlfors regularity and linear local connectivity, there are known necessary and sufficient conditions for X to admit *local parametrizations* by bi-Lipschitz homeomorphisms. We refer the interested reader to [HK11], and to the related papers [Tor94, BL03].

4. Quasisymmetric uniformization

The definition of a quasisymmetric homeomorphism, recall (6), is sensible and useful even for homeomorphisms between fractal-type spaces. For this reason, quasisymmetries have found applications in many fields: complex dynamics [HP09, BM17], group theory [BK02, BK05], uniformization problems in higher dimensions [Sem93, Sem96b], these examples barely scratching the surface. See the mentioned articles and the monograph [MT10] for further information. For the sake of brevity, we focus our attention to the two-dimensional setting.

Quasisymmetric images of $\mathbb{R}^2$ and $\mathbb{S}^2$ satisfy the following properties: they have the LLC property mentioned above, *metric doubling* property, and 2-*rectifiability* [Tys00]. Note that the horned sphere in Figure 1 can be constructed in such a way that both the LLC and the 2-rectifiability properties fail, while the Ahlfors regularity upper bound (8) remains valid, cf. [Raj17, Proposition 17.1]. Such a horned sphere does not admit a parametrization by quasisymmetries or bi-Lipschitz mappings.

We highlight some of the results known in the metric surface setting, specifically in the setting of Ahlfors regular and LLC metric surfaces. When X is homeomorphic to a domain in $\mathbb{S}^2$, the question is subtle, and the *boundary structure* of X plays a delicate role. Nevertheless, there are known sufficient conditions [BK02, Wil08, MW13, RRR19, RR21]. Moreover, when X is homeomorphic to a compact metric surface, a full analog of Theorem 3.2 is known; see [GW18, FM22] for the case of orientable surfaces and article [A] for the case of non-orientable surfaces. In [Wil10], Wildrick constructed quasisymmetric structures — an atlas of quasisymmetric maps — on metric surfaces while only assuming local variants of Ahlfors regularity and linear local connectivity.

While every quasisymmetry from the Euclidean plane onto itself is quasiconformal (and vice versa), the two properties are typically unrelated when considering mappings between metric surfaces. We next consider an example illustrating this fact.

Fix an arbitrary quasisymmetric homeomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$. Whenever $t \in \mathbb{R}$, the plane $E_t = \{t\} \times \mathbb{R}^2$ is mapped onto a topological surface $X_t = \varphi(E_t)$. One can argue, using Fubini's theorem and the area formula for φ, that X_t is a metric surface for almost every t. For such a t, it follows from the work of Tyson [Tys00] that the restriction $f_t = \varphi|_{E_t}$ has finite outer dilatation, with an upper bound $K_O(f_t) \leq K$ depending only on φ. In other words, such f_t are weakly quasiconformal. This raises the following question.

Question 4.1. *Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a quasisymmetry and $E_t = \{t\} \times \mathbb{R}^2$, for each $t \in \mathbb{R}$. Is $f_t = \varphi|_{E_t}$ a quasiconformal homeomorphism onto its image $X_t = f_t(E_t)$, for almost every $t \in \mathbb{R}$?*

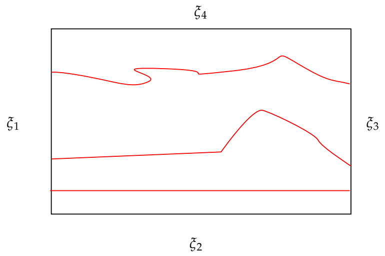
Question 4.1 is closely related to two of the thirty-three yes-or-no questions posed by Heinonen and Semmes in [HS97] (see also [Geh75, Geh76, Väi81, ABH02, Rom19b, NR21a] and references therein). More specifically, Question 15 asks if the inverse of an arbitrary quasisymmetry from the plane $\mathbb{R}^2$ into a metric surface is absolutely continuous with respect to the two-dimensional Hausdorff measure, while Question 16 asks if the same is true even when the target is not a metric surface.

Recently, Romney [Rom19b] answered Question 15 in the negative, thereby also answering Question 16. Later on, Ntalampekos and Romney constructed in [NR21a] (counter)examples even in $\mathbb{R}^3$. We quote the following special case of their result:

Theorem 4.2 ([NR21a])**.** *There exists a quasisymmetry $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $X = \varphi(\{0\} \times \mathbb{R}^2)$ is a metric surface, yet $\varphi^{-1}|_X$ sends a set of negligible two-dimensional Hausdorff measure to a set of positive two-dimensional Hausdorff measure.*

The following observation connects Question 15 to Question 4.1.

Theorem 4.3 (Section 17, [Raj17])**.** *If X is a metric surface and $f\colon \mathbb{R}^2 \to X$ a quasiconformal homeomorphism, then f^{-1} is absolutely continuous with respect to the two-dimensional Hausdorff measure.*



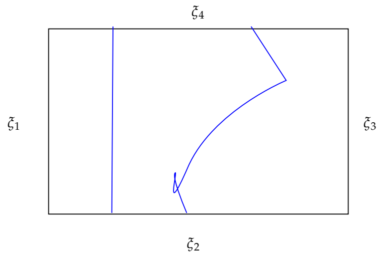


FIGURE 2. The modulus $\operatorname{mod}(\Gamma(\xi_1, \xi_3; R))$ of the horizontal paths joining the vertical sides is equal to the *aspect ratio* b/a of the rectangle, while $\operatorname{mod}(\Gamma(\xi_2, \xi_4; R)) = (b/a)^{-1}$

We observe from Theorem 4.3 that the homeomorphism $\varphi^{-1}|_X$ from Theorem 4.2 cannot be quasiconformal. Question 4.1 asks if the collection of all t where this property fails is a negligible set.

5. Quasiconformal uniformization

The goal of this section is to derive some quasiconformal invariants of the plane $\mathbb{R}^2$ and its open subsets, and outline the main results in articles [A], [B], [C] and [D]. For the former purpose, we introduce the following notation: Given three subsets A, B, C of X, $\Gamma(A, B; C)$ denotes the collection of paths starting at A, ending at B, with image lying in C.

5.1. **Rectangles and the Grötzsch problem.** Given $R = [0, a] \times [0, b]$, we denote by ξ_1, ξ_2, ξ_3, and ξ_4 its left, bottom, right and top sides.

Fix four numbers $a, a', b, b' > 0$ and consider the rectangles $R = [0, a] \times [0, b]$ and $R' = [0, a'] \times [0, b']$. The Grötzsch problem asks the following: What is the smallest K for which there exists a K-quasiconformal homeomorphism from R onto R', sending sides to sides? That is, sending ξ_i to ξ_i' for each $i = 1, 2, 3, 4$. It turns out that the unique linear map satisfying these properties has the minimal K.

The problem can be solved by considering the modulus of the path family $\Gamma = \Gamma(\xi_1, \xi_3; R)$ joining the left side ξ_1 of R to the right side ξ_3. The estimate $\operatorname{mod} \Gamma \leq b/a$ follows by considering the test function $\rho(z) = \chi_R(z)/a$, where χ_R is the *characteristic function* of the rectangle R. The lower bound is established by considering an arbitrary admissible ρ for Γ and the foliation of R by the horizontal arcs $\gamma_t = [0, a] \times \{t\}$, for $0 \leq t \leq b$, and by applying Fubini's theorem and Hölder's inequality. Similarly, for the *dual family* $\Gamma_* = \Gamma(\xi_2, \xi_4; R)$ joining the other two sides, we obtain $\operatorname{mod} \Gamma_* = (b/a)^{-1}$; see Figure 2.

Having fixed a K-quasiconformal homeomorphism $\varphi\colon R \to R'$ mapping sides to sides, we observe that

$$\frac{1}{K}\operatorname{mod}\Gamma(\xi_1,\xi_3;R) \leq \operatorname{mod}\Gamma(\xi_1',\xi_3';R') \leq K\operatorname{mod}\Gamma(\xi_1,\xi_3;R).$$

Since $\operatorname{mod}\Gamma(\xi_1',\xi_3';R') = b'/a'$, we conclude that

$$\begin{aligned} K &\geq \max\left\{\frac{\operatorname{mod}\Gamma(\xi_1,\xi_3;R)}{\operatorname{mod}\Gamma(\xi_1',\xi_3';R')}, \frac{\operatorname{mod}\Gamma(\xi_1',\xi_3';R')}{\operatorname{mod}\Gamma(\xi_1,\xi_3;R)}\right\} \\ &= \max\left\{\frac{a}{b}\left(\frac{a'}{b'}\right)^{-1}, \left(\frac{a}{b}\left(\frac{a'}{b'}\right)^{-1}\right)^{-1}\right\} \geq 1. \end{aligned} \tag{15}$$

Observe, in particular, that if φ is known to be 1-quasiconformal, then the inequalities (15) force the *aspect ratios* a/b and a'/b' of R and R' to coincide. If, on the other hand, the aspect ratios are *not* the same, every K-quasiconformal homeomorphism $R \to R'$, taking sides to sides, satisfies $K > 1$. In this manner, we have found a *conformal invariant* of rectangles. For general Riemannian surfaces, there are more subtle conformal invariants and several authors have contributed to the research of the invariants, see the monographs [AS60, Cou77, Ahl78, Hub06, Hub16] and references therein.

5.2. **Quadrilaterals and the duality principle.** While we were investigating the Grötzsch problem for a rectangle $R = [0,a] \times [0,b]$, we noticed that

$$\operatorname{mod}\Gamma(\xi_1,\xi_3;R)\operatorname{mod}\Gamma(\xi_2,\xi_4;R) = 1. \tag{16}$$

Next, fix a subset R' of a metric surface X and the rectangle R as above, and suppose the existence of a K-quasiconformal homeomorphism $\varphi\colon R \to R'$. We denote $\xi_i' := \varphi(\xi_i)$ for every $i = 1,2,3,4$ and $\kappa := K^2$. By applying (16), we observe the following:

$$\kappa^{-1} \leq \operatorname{mod}\Gamma(\xi_1',\xi_3';R')\operatorname{mod}\Gamma(\xi_2',\xi_4';R') \leq \kappa. \tag{17}$$

Hence the subset R' *almost* satisfies the *duality principle* (16). The Riemann mapping theorem implies that whenever $R' \subset \mathbb{R}^2$ we may take $\kappa = 1$ in (17).

In the sequel, any R' for which there exists a homeomorphism $f\colon [0,1] \times [0,1] \to R'$ is called a *quadrilateral*. The fixed homeomorphism determines a notion of a left, bottom, right and top sides of R', by considering the images of the corresponding sides of $[0,1] \times [0,1]$, and labeling the sides ξ_1', ξ_2', ξ_3' and ξ_4' as above.

5.3. **Annulus.** The Grötzsch problem has a version on Euclidean annuli. For this purpose, fix an arbitrary point $x \in \mathbb{R}^2$ and two pairs of radii $0 < r < R$

and $0 < r' < R'$. The modified problem asks to estimate the minimal K for which there exists a K-quasiconformal homeomorphism taking the *annulus*

$$A(x, r, R) := \{y : r < d(y, x) < R\}$$

onto the annulus $A(x', r', R') = \{y : r' < d(y, x') < R'\}$.

We are not going through the detailed argument, but one natural conformal invariant relates to the paths joining the complementary components of the annulus: let $\Gamma(x, r, R)$ denote the collection of all the paths joining the ball $\overline{B}_r(x)$ to the complement of $B_R(x)$. Once again, by finding an appropriate admissible function, and by arguing using Fubini's theorem, polar coordinates, and Hölder's inequality, it is possible to establish

$$\operatorname{mod} \Gamma(x, r, R) = \frac{2\pi}{\log \frac{R}{r}}.$$

The exact formula is not important for us, but in the sequel the following fact is:

$$\lim_{r \to 0^+} \operatorname{mod} \Gamma(x, r, R) = 0 \quad \text{for every } x \in \mathbb{R}^2 \text{ and all } R > 0. \tag{18}$$

Next, we fix an open set Ω in the plane and a metric surface X, and suppose the existence of a quasiconformal mapping $\phi \colon \Omega \to X$. We investigate the consequences of (18).

First, consider the path family $\Gamma(x, r, R)$ on X. We claim that

$$\lim_{r \to 0^+} \operatorname{mod} \Gamma(x, r, R) = 0 \quad \text{for every } x \in X \text{ and all } R > 0. \tag{19}$$

Whenever (19) fails at a given $x \in X$, we say that x has *positive capacity*. Otherwise, we say that x has *negligible capacity*.

Observe that there are more paths whenever R is lowered so

$$R \mapsto \operatorname{mod} \Gamma(x, r, R)$$

is *decreasing*. Hence (19) holds at a given $x \in X$ if it holds at x for all small radii R.

On the other hand, if we fix a small enough $r > 0$, every path in $\Gamma(x, r, R)$ must contain a subpath joining the boundary components of the image of an Euclidean annulus $A(x', s, S)$. Once again, basic monotonicity properties of modulus then yield

$$\operatorname{mod} \Gamma(x, r, R) \leq \operatorname{mod} \phi(\Gamma(x', s, S)) \leq K \operatorname{mod} \Gamma(x', s, S).$$

Observe that as $r \to 0^+$, we may also pass to the limit $s \to 0^+$. Hence (18) and our monotonicity considerations imply (19). Thus *every* point in X has negligible capacity.

5.4. **Reciprocality.** Based on the observations (17) and (19), Rajala [Raj17] posed the following definition.

Definition 5.1. *Let X be a metric surface. We say that X is κ-*reciprocal *if the conditions* (20)-(22) *hold: Every quadrilateral R' satisfies*

$$\kappa^{-1} \le \operatorname{mod}\Gamma(\xi_1', \xi_3'; R') \operatorname{mod}\Gamma(\xi_2', \xi_4'; R'), \tag{20}$$

$$\kappa \ge \operatorname{mod}\Gamma(\xi_1', \xi_3'; R') \operatorname{mod}\Gamma(\xi_2', \xi_4'; R'), \tag{21}$$

and

$$\lim_{r\to 0^+} \operatorname{mod}\Gamma(x, r, R) = 0 \quad \text{for every } x \in X \text{ and all } R > 0. \tag{22}$$

A metric surface is reciprocal *if it is κ-reciprocal for some $\kappa > 0$.*

It is now understood that (20) holds for some universal $\kappa > 0$ in *every* metric surface [RR19, EP21], with the *sharp* constant $(4/\pi)^2$ obtained in [EP21]. Hence only (21) and (22) can fail to hold in a given metric surface.

It turns out that whenever the upper Ahlfors regularity (8) holds on a metric surface X, then X is reciprocal [Raj17, Theorem 1.6]. In fact, it is enough to have the following:

$$\sup_{x\in X} \limsup_{r\to 0^+} \frac{\mathcal{H}^2(B_r(x))}{\pi r^2} \le C, \quad \text{see [RRR19]}. \tag{23}$$

Property (23) is readily verified, for example, for graphs of $|x|^\alpha$ over $\mathbb{R}^2$ for each $0 < \alpha \le 1$. The case $\alpha = 1$ corresponds to the case of a cone which is a bi-Lipschitz image of the plane, while in the case $0 < \alpha < 1$ the graph is not linearly locally connected.

Nonsmooth surfaces such as the Alexander's horned sphere in Figure 1 can be constructed in such a manner that (23) holds while linear local connectivity fails. For related constructions, see [Fed69, 4.2.25, pages 420-423], [HaZ16] and [Raj17, Proposition 17.1].

Example 5.2. *We consider the following example for which* (21) *and* (22) *fail. Let E denote the vertical slit $E = \{0\} \times [0,1]$ and as weight ω consider the characteristic function $\chi_{\mathbb{R}^2\setminus E}$ of $\mathbb{R}^2 \setminus E$. We define*

$$d_\omega(x,y) = \inf_\gamma \int_\gamma \omega \, ds,$$

where the infimum is taken over all absolutely continuous paths joining x to y, recall (1). *Observe that $d_\omega(x,y) = 0$ for every $x, y \in E$.*

We consider the quotient space X_ω, identifying x and y precisely when $d_\omega(x,y) = 0$. We endow X_ω with the associated quotient distance and observe that X_ω is a metric surface homeomorphic to $\mathbb{R}^2$.

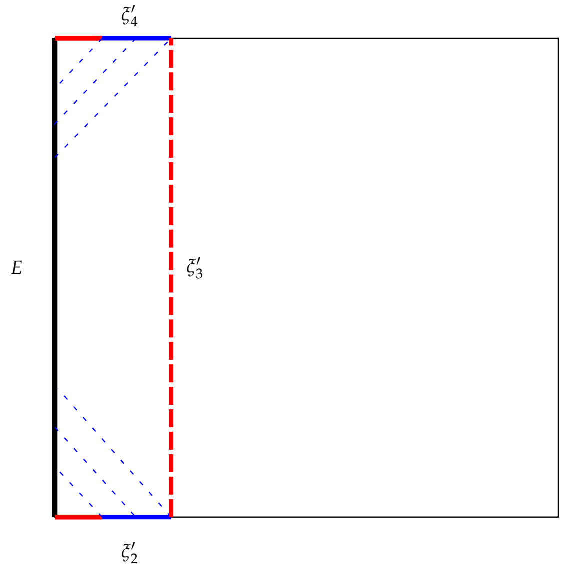


FIGURE 3. The red and blue horizontal arcs have lengths δ and 2δ, respectively. For each $0 < \delta < 1/3$, the modulus in X_ω of the concatenations of the dashed blue paths has a lower bound independent of δ. In particular, (21) and (22) fail on X_ω.

Fix $1 > \delta > 0$. *Consider in* X_ω *the quadrilateral* R' *corresponding to the rectangle* $[0, 3\delta] \times [0, 1]$, *endowed with the boundary decomposition* $\xi_1' = ([0, \delta] \times \{0, 1\}) \cup E$, $\xi_3' = \{3\delta\} \times [0, 1]$, *with* $\xi_2' = [\delta, 3\delta] \times \{0\}$ *and* $\xi_4' = [\delta, 3\delta] \times \{1\}$.

Observe that the modulus M_δ *of* $\Gamma(\xi_1', \xi_3'; R')$ *in the quotient space* X_ω *satisfies*

$$\frac{1}{3\delta} \leq M_\delta,$$

equaling the corresponding Euclidean modulus. In contrast, the modulus $M_{*,\delta}$ *of the dual family* $\Gamma(\xi_2', \xi_4'; R')$ *in* X_ω *satisfies*

$$\frac{1}{c} \leq M_{*,\delta} \quad \text{for every } \delta > 0 \tag{24}$$

for some constant $c > 0$. *The failure of* (21) *is seen by considering the product* $M_\delta M_{*,\delta}$ *and passing to the limit* $\delta \to 0^+$, *while the failure of* (22) *follows from the lower bound* (24).

5.5. **Metric Riemann mapping theorem.** We now state [Raj17, Theorem 1.4].

Theorem 5.3. *Let* X *be a metric surface homeomorphic to* $\mathbb{R}^2$. *Then* X *is reciprocal if and only if there exists a domain* $\Omega \subset \mathbb{R}^2$ *and a quasiconformal homeomorphism* $\varphi \colon \Omega \to X$.

The "if"-direction follows from the elementary computations we did in the previous sections, while the other direction requires an elaborate construction. The basic idea is to start with a quadrilateral $R' \subset X$ in a κ-reciprocal space and prove the existence of a K-quasiconformal homeomorphism onto some rectangle $R = [-a, a] \times [-b, b]$, with K depending only on κ. The next step involves exhausting the space X by an increasing sequence of such quadrilaterals R' and applying a normal family argument to suitably renormalized family of the "metric Riemann maps". This approach is successful because the obtained family consists of K-quasiconformal mappings, for uniform K.

It is simple to extend Theorem 5.3 to reciprocal X homeomorphic to $\mathbb{S}^2$. The key point is that each point in X has negligible capacity. Based on this fact, Rajala obtained a new proof of Theorem 3.2, see [Raj17, Corollary 1.7].

5.6. **Optimizing parametrizations.** Let X be a metric surface and $\Omega' \subset X$ homeomorphic to $\mathbb{R}^2$. Theorem 5.3 yields a necessary and sufficient condition for there to exist a quasiconformal homeomorphism $\varphi\colon \Omega \to \Omega'$ for some $\Omega \subset \mathbb{R}^2$.

If such a parametrization exists, we wish to make the maximal dilatation of the mapping as small as possible, in analogy to the 1-quasiconformality result in Theorem 2.1. This can be achieved as follows: it turns out that the quasiconformality of φ implies some Sobolev regularity for φ [Wil12], and the Sobolev regularity implies the existence of a measurable field of norms N_φ on Ω encoding some analytic information about the mapping φ. This is made precise in [Raj17, LW17, Rom19a, ILP21] and article [A], respectively.

Rajala associated in [Raj17] to the field of norms N_φ a measurable Riemannian norm field G, by associating to each of unit balls of N_φ its *John ellipse* [TJ89] and setting G to be the norms induced by the John ellipses. The precise definition is not important for the purposes of this introduction, but the key point is that the identity map from $(\mathbb{R}^2, \mathrm{N}_{\varphi,x})$ to $(\mathbb{R}^2, \mathrm{G}_x)$ is 2-quasiconformal, for almost every $x \in \Omega$. This implies that G is *uniformly elliptic*, in a suitable sense. This allowed him to apply the so-called *measurable Riemann mapping theorem* [AB60, AIM09] to find a quasiconformal homeomorphism $f\colon U \to \Omega$ such that the composition of G with the differential of f defines on U a measurable Riemannian norm field whose unit balls are Euclidean balls (possibly with varying radii). Now, by repeating the aforementioned computations for $\varphi \circ f$, it turns out that $\varphi \circ f$ is 2-quasiconformal. We refer the interested reader to [Raj17, Section 14] for further details.

Later on, Romney verified that when the John ellipses above are replaced by the so-called *distance ellipses* (see [Rom19a] or article [A]), the argument above yields the existence of a $(\pi/2)$-quasiconformal parametrization Ω'. More precisely, Romney established the following:

Theorem 5.4. *Let X be a metric surface homeomorphic to $\mathbb{R}^2$. Then X is reciprocal if and only if there exists an open set $\Omega \subset \mathbb{R}^2$ and a quasiconformal homeomorphism $\varphi\colon \Omega \to X$ satisfying*

$$\frac{\pi}{4} \operatorname{mod} \Gamma \leq \operatorname{mod} \varphi\Gamma \leq \frac{\pi}{2} \operatorname{mod} \Gamma \quad \textit{for all path families on } \Omega. \tag{25}$$

Theorem 5.4 answered in the positive a conjecture in [Raj17] about the optimality of (25). Indeed, in Example 2.2 of [Raj17], Rajala solves the Grötzsch problem for quasiconformal homeomorphisms mapping rectangles of $(\mathbb{R}^2, \| \cdot \|_\infty)$, endowed with the supremum norm $\| \cdot \|_\infty$, into the Euclidean plane $\mathbb{R}^2$ with the Euclidean norm. To obtain the claimed optimality of (25), two families of rectangles must be considered. The first family consists of rectangles with sides parallel to the coordinate axes and the second family of such rectangles rotated by $(\pi/4)$ radians.

When optimizing the quasiconformal parametrization using the approach by Romney, the composition $\varphi \circ f$ has the key property that the associated Riemannian norm field G, corresponding to the distance ellipses of the norm field $\mathrm{N}_{\varphi \circ f}$, is such that the unit ball of G_x is a Euclidean ball of some radius $r_x > 0$ for almost every x. Whenever this property holds for some quasiconformal homeomorphism $\varphi'\colon \Omega \to \Omega'$ for some open $\Omega \subset \mathbb{R}^2$ and $\Omega' \subset X$, we say that φ' is an *isothermal parametrization*.

5.7. **Quasiconformal surfaces.** A metric surface X is a *quasiconformal surface* if there exists a countable collection of open sets $\Omega_i' \subset X$ and $\Omega_i \subset \mathbb{R}^2$ together with quasiconformal homeomorphisms $\varphi_i\colon \Omega_i \to \Omega_i'$, with $X = \bigcup_{i=1}^\infty \Omega_i$. By arguing as in the previous section, we may assume that each φ_i is an isothermal parametrization. The main theorem of article [A] states the following:

Theorem 5.5. *For every quasiconformal surface X, there exists a Riemannian surface M and a quasiconformal homeomorphism $u\colon M \to X$ satisfying*

$$\frac{\pi}{4} \operatorname{mod} \Gamma \leq \operatorname{mod} u\Gamma \leq \frac{\pi}{2} \operatorname{mod} \Gamma \quad \textit{for all path families on } M.$$

The basic idea of article [A] is to consider the maximal collection $\mathcal{I}$ of all isothermal parametrizations mapping into X. The collection $\mathcal{I}$ defines on X a *conformal structure* since the transition maps between the elements of this collection are 1-quasiconformal *diffeomorphisms*. Then a variant of Theorem 2.1 implies the existence of a Riemannian norm field G such that $\mathcal{I}$ consists of all 1-quasiconformal *diffeomorphic* parametrizations of the Riemannian surface $X_G := (X, d_G)$. It turns out that the identity map $u\colon X_G \to X$ satisfies the properties claimed in Theorem 5.5. Note that the existence of u implies that every quasiconformal surface is reciprocal. This implies that reciprocality is a local property.

The X_G can be assumed to be complete and have constant curvature -1, 0, or 1, which basically follows from the construction of G, or from Theorem 2.1. With this further assumption, whenever X is a compact, Ahlfors regular and LLC metric surface, it is possible to prove that u is a quasisymmetry. This statement is made quantitative in article [A].

Whenever X is a metric surface homeomorphic to a domain in the sphere $\mathbb{S}^2$, the X_G admits a 1-quasiconformal embedding into $\mathbb{S}^2$ [AS60, Section III.4]. This fact was applied in [RR21] to study the quasisymmetric uniformization problem, for finitely connected X. It would be interesting to know if the approach of [RR21], or a close variant, works in the setting of [MW13] of infinitely connected planar domains.

5.8. **Weighted quasiconformal surfaces.** We considered in article [B] examples of metric surfaces constructed using the weighted distances as in (1).

If the weight has a uniform lower bound $c > 0$ and an upper bound $C < \infty$, it is not difficult to see that the corresponding $X_\omega := (\mathbb{S}^2, d_\omega)$ is bi-Lipschitz equivalent to the plane. When no such lower bound exists, it is not necessarily the case that d_ω is a distance, as the vertical slit Example 5.2 illustrates. For this reason, we need to consider the quotient mapping $\pi_\omega\colon \mathbb{S}^2 \to X_\omega$ identifying each point x with the set $\{y\colon d_\omega(x,y) = 0\}$. We say that a weight ω is *reciprocal* if the quotient space X_ω, endowed with the quotient distance, is a quasiconformal surface.

In article [B] we consider the following weights $\omega_p := \sigma(\cdot, E)^{p-1}$, for $p \geq 1$, where $\sigma(x, E)$ is the spherical distance from the point x to a given set E. The main results of article [B] establish the following:

Theorem 5.6. *Let $E \subset \mathbb{S}^2$ be a compact set with Hausdorff dimension s with $\mathbb{S}^2 \setminus E$ connected and nonempty.*

If there exists a $p > \max\{s, 1\}$ such that ω_p is a reciprocal weight, then the set E is removable for conformal mappings.

Conversely, if E is removable for conformal mappings, then ω_p is a reciprocal weight for every $p \geq 1$.

We say that a compact set $E \subset \mathbb{S}^2$ is *removable for conformal mappings* if every 1-quasiconformal embedding from $\mathbb{S}^2 \setminus E$ into $\mathbb{S}^2$ is the restriction of a Möbius transformation. This notion was studied by Ahlfors and Beurling in [AB50] and, rather curiously, Theorem 5.6 connects this purely Euclidean notion to the quasiconformal uniformization problem.

We note that Theorem 5.6 is sharp in the following sense: In [Sem96b], Semmes proved for a suitable self-similar arc $E \subset \mathbb{S}^2$ of Hausdorff dimension $s > 1$ that the weight $\omega_s(x)$ defines a metric doubling measure via the formula

$$\mu(B) := \int_B \omega_s^2(x)\, d\mathcal{H}^2_{\mathbb{S}^2}(x), \quad \text{recall (14).}$$

Since the set E is homeomorphic to $[0,1]$, it cannot be removable for conformal mappings [AB50], as a consequence of the Riemann mapping theorem. Note also the similarity of these weights to the example (13) by Laakso.

In article [B], we construct an example of a complete and geodesic metric surface X $(\pi/2)$-quasiconformally equivalent to $\mathbb{R}^2$ in such a way that X is *not* 1-quasiconformally equivalent to any metric surface Z, with Z bi-Lipschitz embeddable into $\mathbb{R}^2$. This implies that the *1-quasiconformal equivalence classes* of quasiconformal surfaces do not necessarily contain representatives that can be bi-Lipschitz parametrized by Riemannian surfaces. However, it is not clear if there always exists an element of the equivalence class that satisfies local versions of Ahlfors regularity and LLC.

5.9. **Boundary behaviour of quasiconformal mappings.** In article [C] of the dissertation, we are interested in the boundary behaviour of quasiconformal homeomorphisms. More precisely, we generalize *Carathéodory's theorem* from classical complex analysis. The classical result says that every quasiconformal homeomorphism between the interiors of quadrilaterals $R, R' \subset \mathbb{R}^2$ extends to a quasiconformal homeomorphism between the quadrilaterals.

In [C], we study quasiconformal Jordan domains: a metric surface is a *quasiconformal Jordan domain* if X is a metric surface for which there exists a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to X$, with X having the additional property that the completion $\overline{X}$ has finite two-dimensional Hausdorff measure and $\partial X := \overline{X} \setminus X$ is homeomorphic to the unit circle $\mathbb{S}^1$.

One of the main results of the article proves that given an arbitrary quasiconformal Jordan domain and a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to X$, there exists a weakly quasiconformal extension $\Phi\colon \overline{\mathbb{D}} \to \overline{X}$ of ϕ. It is natural to consider when this improves to quasiconformality. The main result of the article proves the following.

Theorem 5.7. *A given quasiconformal homeomorphism $\varphi\colon \mathbb{D} \to X$ admits a quasiconformal extension $\Phi\colon \overline{\mathbb{D}} \to \overline{X}$ if and only if every point of ∂X has negligible capacity in $\overline{X}$.*

Example 5.2 illustrates that some geometric conditions on ∂X, such as points having negligible capacity, are necessary to guarantee the quasiconformality of Φ. Interestingly, φ can admit a homeomorphic extension Φ even without the extension being quasiconformal. Such a situation arises by studying the metric surfaces X_ω as in the previous section, for weights $\omega = \chi_{\mathbb{R}^2 \setminus E}$ for suitable Cantor sets $E \subset \mathbb{R} \times \{0\}$.

We also consider some sufficient conditions which guarantee that the restriction of the quasiconformal homeomorphism Φ to $\mathbb{S}^1$ is a quasisymmetric parametrization of ∂X, and investigate sufficient conditions for the bi-Lipschitz embeddability of ∂X into plane. These questions are closely connected to the catalogue of quasisymmetric images of $\mathbb{S}^1$, see [Mey11, HM12].

5.10. **Gluing hemispheres.** In article [D] of the dissertation, we construct metric surfaces homeomorphic to $\mathbb{S}^2$ by starting with an orientation-preserving homeomorphism $g\colon \mathbb{S} \to \mathbb{S}$, where $\mathbb{S}$ is the equator of $\mathbb{S}^2$.

Let Z_1 and Z_2 denote the open southern and northern hemispheres of $\mathbb{S}^2$, respectively. We obtain the metric surface Z_g from g in such a way that there exists a 1-Lipschitz local isometric embedding $\iota_i\colon Z_i \to Z_g$, the 1-Lipschitz extension to $\overline{Z}_i$ mapping $\mathbb{S}$ monotonically onto the set *seam* $\mathbb{S}_g$, for $i = 1, 2$, with images of Z_1 and Z_2 being disjoint.

We are not going to outline the precise construction but mention some of our main results.

Theorem 5.8. *Let $g\colon \mathbb{S} \to \mathbb{S}$ be an orientation-preserving homeomorphism. Then the following are equivalent, quantitatively:*

- *g is L-bi-Lipschitz,*
- *there exists an L'-bi-Lipschitz homeomorphism $\Phi\colon \mathbb{S}^2 \to Z_g$, and*
- *there exists a constant $C > 0$ such that*

$$\sup_{z \in \mathbb{S}_g} \liminf_{r \to 0^+} \frac{\mathcal{H}^2_{Z_g}(B(z,r))}{\pi r^2} \leq C.$$

The equivalence between the second and third conditions is surprising since the pointwise information about the seam $\mathbb{S}_g$ improves to Ahlfors regularity and the LLC property, and even to the existence of a bi-Lipschitz parametrization of Z_g. In particular, if g is non-bi-Lipschitz, the conclusion that Z_g is a quasiconformal surface does not follow from the two-dimensional density upper bound (23), for example, and other methods need to be used.

Before stating one of the main results of article [D], Theorem 5.9 below, we state a definition. We say that an orientation-preserving homeomorphism $g\colon \mathbb{S} \to \mathbb{S}$ is a *welding homeomorphism* if there exist quadrilaterals $R_1, R_2 \subset \mathbb{S}^2$ and orientation-preserving 1-quasiconformal homeomorphisms $\phi_i\colon \overline{Z}_i \to R_i$, with $R_1 \cap R_2 = \Gamma$ homeomorphic to $\mathbb{S}$, and $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}}$. We call any such Γ a *welding curve* of g.

Theorem 5.9. *For every welding homeomorphism $g\colon \mathbb{S} \to \mathbb{S}$, there exists a weakly quasiconformal $f\colon \mathbb{S}^2 \to Z_g$, with $K_O(f) = 1$.*

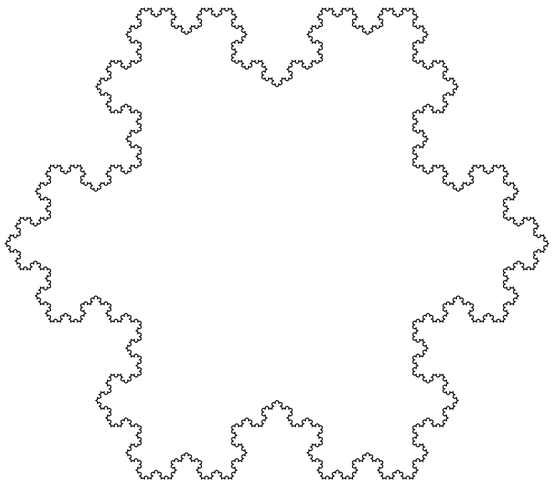


FIGURE 4. The von Koch snowflake. The welding homeomorphisms corresponding to the snowflake are quasisymmetries.

The proof of Theorem 5.9 is somewhat involved, requiring some careful harmonic analysis of the welding curves of g, and proving an intimate connection between the one-dimensional Hausdorff measures on the seam S_g and on the (tangents of the) welding curve.

Theorems 5.8 and 5.9 are connected to metric doubling measures. Indeed, we establish that the measure $\mu(E) := \mathcal{H}^2_{Z_g}(f(E))$, for f as in Theorem 5.9, is a metric doubling measure if and only if the welding homeomorphism g is bi-Lipschitz.

Examples of welding homeomorphisms include the class of orientation-preserving quasisymmetries. It would be natural to expect that whenever g is a quasisymmetry, the mapping f in Theorem 5.9 is a 1-quasiconformal homeomorphism. However, this turns out to be false. Indeed, when g and its inverse fail to be absolutely continuous with respect to length measures, points of positive capacity in Z_g can exist. For example, it is sufficient to consider the welding homeomorphisms corresponding to the von Koch snowflake; see Figure 4. Nevertheless, we are able to guarantee that Z_g is a quasiconformal surface in some situations.

Proposition 5.10. *Let $g\colon \mathbb{S} \to \mathbb{S}$ be an orientation-preserving quasisymmetry and g^{-1} absolutely continuous. Then Z_g is a quasiconformal surface.*

Some assumption similar to quasisymmetry is needed. In fact, based on [Oik61, Example 1] and [Vai89, Theorem 3], we constructed an example of a Lipschitz g^{-1} that is locally bi-Lipschitz outside a single point $z_0 \in \mathbb{S}$, yet the corresponding point $x_0 \in \mathbb{S}_g$ has positive capacity in Z_g.

It turns out that g being a quasisymmetry is not strictly needed. For example, if g admits an extension to a homeomorphism $H\colon \mathbb{S}^2 \to \mathbb{S}^2$ that is a mapping of *exponentially integrable distortion*, then the absolute continuity of g^{-1} is still enough to guarantee that Z_g is a quasiconformal surface. The interested reader is refered to article [D] for further details.

6. Open questions

6.1. **Weakly quasiconformal mappings.** In recent years, a lot of research has been done towards understanding metric surfaces which are not quasiconformal surfaces.

For a simple example of such a surface, we consider the metric surface X_ω constructed in Example 5.2. As outlined in Figure 3, there are obstructions for X_ω admitting a a quasiconformal parametrization. Nevertheless, the space X_ω admits a weakly quasiconformal parametrization: the quotient map

$$x \mapsto \{y\colon d_\omega(x,y) = 0\}$$

is such a mapping.

Using the recent results [MW21, NR21b], we now know the following: whenever X is metric surface with a length distance (*length metric surface*), and $R \subset X$ a quadrilateral with rectifiable boundary, there exists a weakly quasiconformal map $f\colon \overline{\mathbb{D}} \to R$.

Whenever such a mapping exists, f can always be assumed to satisfy $K_O(f) \leq 4/\pi$. This can be seen by employing the proof method from Section 5.6, or by applying the energy-minimization scheme developed by Lytchak and Wenger in [LW17, LW18, LW20] in the context of metric spaces admitting *quadratic isoperimetric inequalities*. This interesting result by [MW21, NR21b] is related to the following open question by Rajala and Wenger (see e.g. article [B], [MW21] or [NR21b]):

Question 6.1. *If X is a metric surface homeomorphic to $\mathbb{R}^2$, does there exists a domain $\Omega \subset \mathbb{R}^2$ and a weakly quasiconformal $f\colon \Omega \to X$?*

The question has a straight-forward generalization to the non-simply connected setting.

It is not difficult to see that whenever a metric surface is a quasiconformal surface, every weakly quasiconformal map is a quasiconformal homeomorphism, see Lemma 6.3. Moreover, if f is weakly quasiconformal and $f^{-1}(x_0)$ is a connected subset of Ω containing at least two points, then x_0 has positive capacity. It would be interesting to know the following:

Question 6.2. *Suppose that X is a metric surface homeomorphic to $\mathbb{R}^2$ and $x_0 \in X$ has positive capacity. Does there exists a weakly quasiconformal $f'\colon \Omega' \to X$ for some $\Omega' \subset \mathbb{R}^2$ such that $(f')^{-1}(x_0)$ contains more than a single point?*

This is closely related to [CR22], where Creutz and Romney investigate the structure of the fibers of weakly quasiconformal mappings in various contexts. We also observe the following:

Lemma 6.3. *Let X be a metric surface homeomorphic to $\mathbb{R}^2$. If $f\colon \Omega \to X$ is weakly quasiconformal and X is reciprocal, then f is a quasiconformal homeomorphism.*

The basic idea is the following: if there exists $x_0 \in X$ such that $f^{-1}(x_0)$ is not a point, then x_0 has positive capacity. Indeed, there exists a compact and connected set $E \subset \Omega \setminus f^{-1}(x_0)$ such that $\operatorname{mod} \Gamma(E, f^{-1}(x_0); \Omega) > 0$; this follows by slightly modifying the idea in the vertical slit example, see Figure 3 or from the 2-Loewner property of $\mathbb{R}^2$ [Hei01, Example 8.24.]. Then the weak quasiconformality implies $\operatorname{mod} \Gamma(f(E), x_0; X) > 0$ hence necessarily x_0 has positive capacity.

Next, if f is a weakly quasiconformal *homeomorphism* and $\varphi\colon \Omega' \to X$ is a quasiconformal homeomorphism for $\Omega' \subset \mathbb{R}^2$, then $h = \varphi^{-1} \circ f$ is a weakly quasiconformal homeomorphism between planar domains. Such mappings are always quasiconformal homeomorphisms, see [Res93, AIM09, Raj17]. These facts imply Lemma 6.3.

Theorem 4.2 gives an example of a metric surface X that admits a weakly quasiconformal (even quasisymmetric) parametrization $f\colon \mathbb{R}^2 \to X$ which is not quasiconformal (due to Theorem 4.3). Hence X is not reciprocal. It would be interesting to know if Question 6.2 has a positive answer in this special case.

6.2. **Duality principles.** Recently, there has been advances in understanding duality principles akin to (20) and (21) in the metric surface setting. In [RR19], it is proved that the duality lower bound (20) holds in every metric surface with a universal constant κ_0, with the sharp version $\kappa_0 = (4/\pi)^2$ proved in [EP21].

The authors of [EP21] also proved higher dimensional versions of their results, see also Section 6.3 below. In the context of complete metric measure spaces having a doubling measure and a local $(1,1)$-Poincaré inequality, a full analog of the duality principle holds; see [LR21, JL20]. See also [Zie67, AO99, Rom08, Loh20, Loh21, Zha21] for related duality results.

A metric space X is a *PI metric surface* if X is a complete metric surface, with $\mathcal{H}^2_X$ doubling, and supporting a local $(1,1)$-Poincaré inequality.

Question 6.4. *Let X be a PI metric surface. Does there exists a constant $\kappa > 0$ such that the duality upper bound (21) holds for every quadrilateral $R' \subset X$ with the constant κ?*

Question 6.4 does not immediately follow from the duality results in [LR21, JL20] and the reason is the following. Given the path family $\Gamma := \Gamma(\zeta_1', \zeta_3'; R')$

in a PI metric surface, the dual family Γ_* considered in [LR21, JL20] is not necessarily related to any path modulus, in particular to $\Gamma(\zeta_2', \zeta_4'; R')$. Indeed, the collection Γ_* consists of *sets of finite perimeter* separating the sets ζ_1 and ζ_3 within R, in a suitable sense. Moreover, instead of using the length measure to define admissible functions as in (2), the length measure is instead replaced by a suitable codimension one measure (or the perimeter measure). Therefore, being able to compare $\operatorname{mod}\Gamma(\zeta_2, \zeta_4; R)$ to the dual modulus considered in [LR21, JL20] requires careful analysis of the essential boundary of sets of finite perimeter and the codimension one measure in the PI metric surface context.

We also ask the following related question.

Question 6.5. *Are there PI metric surfaces which have points of positive capacity? What is the Hausdorff dimension of the set of points which have positive capacity?*

Let X be a PI metric surface and $x_0 \in X$. The point x_0 having positive capacity is closely related to the (non-essential) singularities of Green functions analyzed, for example, in [BBL20, BBL21]. Points of positive capacity also have interesting connections to the duality properties of Sobolev spaces and capacities investigated in [AS21], see [AS21, Theorem 5.7]. It is always the case on metric surfaces that the collection of points having positive capacity is negligible with respect to the two-dimensional Hausdorff measure. However, not much else is known.

We recall that every PI metric surface X is bi-Lipschitz equivalent to a geodesic PI metric surface [Che99, Kei03]. Hence we know that every quadrilateral with rectifiable boundary in X is the weakly quasiconformal image of the closed Euclidean disk, as a consequence of [MW21, NR21b]. Therefore it is reasonable to investigate Questions 6.4 and 6.5 using such a mapping directly.

6.3. **Generalized reciprocality.** The duality principle (16) actually holds in a much stronger form. To formulate this properly, we denote by mod_p the *p-modulus* where the number 2 in (3) is replaced by $1 < p < \infty$. Then, it turns out that whenever $1 < p < \infty$ and $q = p/(p-1)$, every quadrilateral $R \subset \mathbb{R}^2$ satisfies

$$\left(\operatorname{mod}_p \Gamma(\zeta_1, \zeta_3; R)\right)^{1/p} \left(\operatorname{mod}_q \Gamma(\zeta_2, \zeta_4; R)\right)^{1/q} = 1; \tag{26}$$

see, for example, [Zie67, Rom08]. In [EP21], the authors established in an arbitrary metric surface X and quadrilateral $R \subset X$ the sharp lower bound

$$\left(\operatorname{mod}_p \Gamma(\zeta_1, \zeta_3; R)\right)^{1/p} \left(\operatorname{mod}_q \Gamma(\zeta_2, \zeta_4; R)\right)^{1/q} \geq \frac{\pi}{4}. \tag{27}$$

Motivated by (26), (27), and Definition 5.1, we state the following definition.

Definition 6.6. *A metric surface X is κ-(p,q)*-reciprocal *if $1 < p \leq 2$, $q = p/(p-1)$, and the conditions (28) and (29) hold: Every quadrilateral $R \subset X$ satisfies*

$$\kappa \geq \left(\mathrm{mod}_p \Gamma(\xi_1, \xi_3; R)\right)^{1/p} \left(\mathrm{mod}_q \Gamma(\xi_2, \xi_4; R)\right)^{1/q}, \tag{28}$$

and

$$\lim_{r \to 0^+} \mathrm{mod}_p \Gamma(x, r, R) = 0 \quad \textit{for every } x \in X \textit{ and all } R > 0. \tag{29}$$

A metric surface is (p,q)-reciprocal *if it is κ-(p,q)-reciprocal for some $\kappa > 0$.*

We note that being (p,q)-reciprocal is a bi-Lipschitz invariant and every Riemannian surface is 1-(p,q)-reciprocal (notice $1 < p \leq 2$ in the definition). In fact, the density upper bound (23) is strong enough to imply (28), with only minor modifications needed to the proof of [RRR19, Proposition 3.9], the proof there written in the case $p = 2$. A consequence of Hölder's inequality is that if (29) holds for $p = 2$, then it also holds for all $1 \leq p < 2$. Thus, if X is a metric surface satisfying the density upper bound (23), then X is (p,q)-reciprocal.

Question 6.7. *Let X be a metric surface and $1 < p < 2$. If X is (p,q)-reciprocal, is X (p',q')-reciprocal for some $p < p' \leq 2$ and $q' = p'/(p'-1)$? What about $1 < p' < p$?*

We also ask the following:

Question 6.8. *Let $1 < p < 2$. If X is a (p,q)-reciprocal metric surface homeomorphic to $\mathbb{R}^2$, does there exists a domain $\Omega \subset \mathbb{S}^2$ and a homeomorphism $\varphi\colon \Omega \to X$ satisfying*

$$\mathrm{mod}_p \varphi\Gamma \leq K \,\mathrm{mod}_p \Gamma \quad \textit{for all } \Gamma \textit{ on } \Omega? \tag{30}$$

It might seem reasonable to expect that whenever X is as in Question 6.8, then there exists $\varphi\colon \Omega \to X$ that satisfies

$$K^{-1} \mathrm{mod}_p \Gamma \leq \mathrm{mod}_p \varphi\Gamma \leq K \,\mathrm{mod}_p \Gamma \quad \text{for all } \Gamma \text{ on } \Omega. \tag{31}$$

However, (31) is too strong of a condition in the generality of Question 6.8, as we argue next. It is known that every complete Ahlfors regular LLC metric surface X supports a local (1,1)-Poincaré inequality [Sem96a]. Therefore it is possible to prove for such X, see e.g. [BBL21, Theorem 1.1], that for every $\operatorname{diam} X/4 > R_0 > 0$,

$$c^{-1} r^{2-p} \leq \mathrm{mod}_p \Gamma(x, r, R) \leq c r^{2-p} \quad \text{whenever } 0 < 2r \leq R \leq R_0, \tag{32}$$

for a constant c depending on R_0 and X but not on the point $x \in X$. Based on (32), it is not difficult to see that if there exists a homeomorphism φ from $\mathbb{S}^2$ onto X satisfying (31), then φ is (locally) bi-Lipschitz. An analogous result is proved in [HKM92, Theorem 5.7] in the Euclidean setting.

It is now enough to recall the example by Laakso from Section 3. He constructed on $\mathbb{S}^2$ a metric doubling measure μ for which $X_\mu = (\mathbb{S}^2, D_\mu)$ cannot be bi-Lipschitz embedded into any finite-dimensional Banach space. In particular, no homeomorphism $\varphi\colon \mathbb{S}^2 \to X_\mu$ satisfies (31) for some $1 < p < 2$.

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Included articles

[A]

Uniformization of metric surfaces using isothermal coordinates

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Uniformization of metric surfaces using isothermal coordinates

Toni Ikonen

Abstract. We establish a uniformization result for metric surfaces—metric spaces that are topological surfaces with locally finite Hausdorff 2-measure. Using the geometric definition of quasiconformality, we show that a metric surface that can be covered by quasiconformal images of Euclidean domains is quasiconformally equivalent to a Riemannian surface. To prove this, we construct an atlas of suitable isothermal coordinates.

Metristen pintojen uniformisaatio isotermisillä koordinaateilla

Tiivistelmä. Todistamme metristen pintojen uniformisaatiolauseen. Metrinen pinta on topologinen pinta varustettuna etäisyysfunktiolla, jonka kaksiulotteinen Hausdorffin mitta on lokaalisti äärellinen. Tutkimme milloin metrinen pinta on riemannilaisen pinnan geometrisesti kvasikonformaalinen kuva. Osoitamme riittäväksi ehdoksi, että metrinen pinta voidaan peittää eukleideen avaruuden alueiden kvasikonformaalisilla kuvilla. Konstruoimme todistusta varten kartaston isotermisiä koordinaatteja.

1. Introduction

1.1. Overview. The Riemann mapping theorem states that given a simply connected proper subdomain U of $\mathbb{R}^2$, there exists a conformal map $\phi\colon \mathbb{D} \to U$, where $\mathbb{D}$ is the Euclidean disk. Recall that conformal maps preserve angles but they do not necessarily preserve lengths of paths or areas. We say that domains U and V are *conformally equivalent* if there exists a conformal map from U to V.

When the topological type of U is more complicated, so is the classification result. For example, if $U = A(1, r) \subset \mathbb{R}^2$ in an Euclidean annulus of inner radius 1 and outer radius $r > 1$, two such annuli $A(1, r)$ and $A(1, r')$ are conformally equivalent if and only if $r = r'$.

If we relax the definition of conformal map to allow for distortion of infinitesimal balls in a uniformly controlled manner, we obtain the class of quasiconformal maps. With this relaxation, it turns out that for every pair of outer radii $1 < r$ and $1 < r'$, there exists a quasiconformal map from $A(1, r)$ onto $A(1, r')$. Such a map takes the infinitesimal Euclidean balls in $A(1, r)$ to infinitesimal ellipses in $A(1, r')$, and the distortion is determined from the eccentricity of the ellipses.

Similar questions can be considered when the topology type of the surface is more complicated. This is the domain of Teichmüller theory of surfaces; see for example [Leh87, IT92, Hub06]. Roughly speaking, the Teichmüller theory classifies Riemann

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surfaces up to conformal maps, and quasiconformal maps measure how far apart two Riemann surfaces are from one another.

Quasiconformal maps also arise when we try to find isothermal coordinates in a given Riemannian surface, that is, a smooth surface with a smooth Riemannian metric. Indeed, given a Riemannian surface (Y, g) and a smooth chart $f\colon V \to U \subset \mathbb{R}^2$, by considering a smaller open set $V' \subset V$, we may assume without loss of generality that f is quasiconformal. We interpret the Riemannian metric g on V as a particular choice of an ellipse at each point of V. Then the chart f maps these ellipses to ellipses in U. We ask whether it is possible to find a diffeomorphism $\eta\colon U \to W \subset \mathbb{R}^2$ such that the particular ellipses in U are mapped to Euclidean balls by η. The existence of such a diffeomorphism η is guaranteed by the measurable Riemann mapping theorem; see, for example, [AB60, AIM09]. When we apply this theorem to the ellipse field of f, the composition $\eta \circ f$ maps the ellipses in V to Euclidean balls. Classically, the coordinates $\eta \circ f$ are called *isothermal coordinates.*

We are interested in two questions. Given a metric space (Y, d_Y) homeomorphic to a surface, what conditions guarantee that there exists a Riemannian surface Z and a quasiconformal map $f\colon Y \to Z$? Moreover, is it possible to find a good notion of isothermal coordinates on Y?

We use an approach based on [Raj17]. Let Y be a metric surface and $V \subset Y$ homeomorphic to $\mathbb{R}^2$. We say that V is a *reciprocal disk* if there exists a quasiconformal homeomorphism $f\colon V \to U \subset \mathbb{R}^2$. Given such an f, the inverse f^{-1} has an *approximate metric differential*, which defines a field of convex bodies on U. We obtain a field of ellipses on U by associating to each of the convex bodies its *distance ellipse* (see for example [Rom19, Section 2], [TJ89, Chapter 37] or Section 4). As before, there exists a quasiconformal homeomorphism $\eta\colon U \to W \subset \mathbb{R}^2$ mapping the field of distance ellipses to Euclidean balls. We call $(V, \eta \circ f)$ an *isothermal chart* of Y. The reason we define the charts in this manner is that every isothermal chart is $(\pi/2)$-quasiconformal; see [Rom19] or Section 4. We prove that whenever Y can be covered by reciprocal disks, the isothermal charts form an atlas $\mathcal{C}$ on Y with transition maps holomorphic or antiholomorphic. Using the atlas $\mathcal{C}$, we prove that Y is quasiconformally equivalent to a Riemannian surface.

Given a metric surface, a cover by reciprocal disks can be found if the 2-dimensional Hausdorff measure of any ball is bounded from above by a constant multiple of the radius squared [Raj17, Theorem 1.6]. In fact, it suffices to require a (locally) uniform upper bound for the 2-dimensional Hausdorff upper density [RRR21, Proposition 3.9]. Next, we give an example for which such a cover does not exist. To this end, we consider a Cantor set $E \subset \mathbb{R}^2$ of positive Lebesgue measure and any continuous function $\omega\colon \mathbb{R}^2 \to [0, \infty)$ with $E = \{x\colon \omega(x) = 0\}$. We define a distance d_ω by setting $d_\omega(x, y) = \inf \int_\gamma \omega\, ds$, the infimum taken over absolutely continuous paths joining x to y. The metric space $(\mathbb{R}^2, d_\omega)$ is homeomorphic to the plane but no Lebesgue density point of E can be covered by a reciprocal disk $V \subset (\mathbb{R}^2, d_\omega)$ [Raj17, Example 2.1].

1.2. Main results. A metric space (Y, d_Y) with a locally finite Hausdorff 2-measure is a *metric surface* if it is homeomorphic to a connected 2-manifold without boundary.

Definition 1.1. A metric surface (Y, d_Y) is a *quasiconformal surface* if every point of (Y, d_Y) is contained in a quasiconformal image of an open set $U \subset \mathbb{R}^2$.

A necessary and sufficient condition for Y to be a quasiconformal surface is given by [Raj17, Theorem 1.4]. Note that every Riemannian surface is a quasiconformal surface and being a quasiconformal surface is a quasiconformal invariant.

We now state the first of our main results.

Theorem 1.2. *Every quasiconformal surface is quasiconformally equivalent to a Riemannian surface.*

To prove Theorem 1.2 for a given quasiconformal surface (Y, d_Y), we construct in Section 4 an atlas of isothermal charts for (Y, d_Y). The atlas defines a conformal structure $\mathcal{C}$ on (Y, d_Y), uniquely determined from the distance d_Y. The classical uniformization theorem implies the existence of a Riemannian norm field G on $(Y, \mathcal{C})$ of Gaussian curvature -1, 0, or 1 in such a way that the associated length distance d_G on Y is complete and that every element of $\mathcal{C}$ is an isothermal chart for the Riemannian surface. The norm field G is not uniquely determined by $\mathcal{C}$ but different choices of G are conformally equivalent. Having fixed such a G, the identity map from (Y, d_G) to (Y, d_Y) is called the *uniformization map* and denoted by u. Theorem 1.2 follows from our next theorem.

Theorem 1.3. *For every quasiconformal surface (Y, d_Y), the uniformization map $u\colon (Y, d_G) \to (Y, d_Y)$ is $(\pi/2)$-quasiconformal. More precisely, it satisfies*

$$\frac{\pi}{4} \operatorname{mod} \Gamma \leq \operatorname{mod} u\Gamma \leq \frac{\pi}{2} \operatorname{mod} \Gamma \tag{1}$$

for all path families Γ in (Y, d_G).

In this generality, both the lower and upper bounds in (1) are best possible for any quasiconformal map from a Riemannian surface onto (Y, d_Y) [Raj17, Example 2.2].

As a particular application of Theorem 1.3, we consider a quasiconformal surface (Y, d_Y) homeomorphic to a domain in the sphere $\mathbb{S}^2$. Using the notation from Theorem 1.3, we recall the existence of a 1-quasiconformal embedding $\psi\colon (Y, d_G) \to \mathbb{S}^2$ [AS60, Section III.4]. Then the composition $f = \psi \circ u^{-1}$ is a $(\pi/2)$-quasiconformal embedding of (Y, d_Y) into the sphere $\mathbb{S}^2$, satisfying the bounds $(2/\pi) \operatorname{mod} \Gamma \leq \operatorname{mod} f\Gamma \leq (4/\pi) \operatorname{mod} \Gamma$ for all path families in (Y, d_Y). Romney proved in [Rom19] the existence of such an embedding for reciprocal disks.

Next, we refer the reader to Section 6.2 for the definitions of Ahlfors 2-regularity, linear local contractibility, and quasisymmetries.

Theorem 1.4. *If (Y, d_Y) is a compact, linearly locally contractible, and Ahlfors 2-regular metric surface, then (Y, d_Y) is a quasiconformal surface. Furthermore, a uniformization map $u\colon (Y, d_G) \to (Y, d_Y)$ is η-quasisymmetric with η depending only on the data of (Y, d_Y).*

In the statement, *the data* of (Y, d_Y) refers to the constants appearing in the definitions of linear local contractibility and Ahlfors 2-regularity. When (Y, d_Y) is homeomorphic to $\mathbb{S}^2$, we need to choose the uniformization map with care.

The main theorem from [BK02] proves that if (Y, d_Y) is as in the statement of Theorem 1.4 and homeomorphic to $\mathbb{S}^2$, then there exists an η'-quasisymmetry $\psi\colon \mathbb{S}^2 \to (Y, d_Y)$. We recover this result from Theorem 1.4, since (Y, d_G) is isometric to $\mathbb{S}^2$.

Theorem 1.2 of [GW18] proves that if (Y, d_Y) is as in the statement of Theorem 1.4, orientable and not homeomorphic to $\mathbb{S}^2$, there exists a complete Riemannian surface Z of constant curvature and an η'-quasisymmetric homeomorphism $\phi\colon Z \to (Y, d_Y)$ with η' depending only on the data of (Y, d_Y). Using Theorem 1.3, our isothermal coordinates, and a modified version of their proof, we prove that the uniformization map is η-quasisymmetric with η depending only on the data of (Y, d_Y). The modified proof also works for the non-orientable case.

We refer the interested reader to [BK02, Raj17, GW18], and references therein, for further reading about the quasisymmetric uniformization problem.

2. Outline of the paper

In Section 3, we introduce our notations and recall some prerequisite knowledge. In Section 4, we prove the existence of isothermal charts and the uniformization mapping. In Section 5, we analyze quasiconformal homeomorphisms between quasiconformal surfaces. These results are applied in Section 6, where we introduce isothermal parametrizations of quasiconformal surfaces by Riemannian surfaces. We prove that up to a conformal diffeomorphism, the isothermal parametrizations are uniquely determined by the uniformization mapping. We also prove Theorem 1.4 in this section. In Section 7, we have some concluding remarks.

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3. Preliminaries

Let (Y, d_Y) be a metric space. We drop the subscript from d_Y when convenient. We recall the definition of Hausdorff measure. For all $Q \geq 0$, the *Q-dimensional Hausdorff measure* is defined by

$$\mathcal{H}^Q_Y(B) = \frac{\alpha(Q)}{2^Q} \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^Q \colon B \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < \delta \right\}$$

for all sets $B \subset Y$, where the normalization constant is chosen in such a way that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers n.

A *path* is a continuous function from a compact interval into a metric space. A path in Y will typically be denoted by γ. The *length* of the path $\gamma\colon [a,b] \to Y$ is defined as

$$\ell_d(\gamma) = \sup \sum_{j=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all finite sequences $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. A path is *rectifiable* if it has finite length.

The *metric speed* of a path $\gamma\colon [a,b] \to Y$ at the point $t \in [a,b]$ is defined as

$$v_\gamma(t) = \lim_{t \neq s \to t} \frac{d(\gamma(s), \gamma(t))}{|t-s|}$$

whenever this limit exists. If γ is rectifiable, its metric speed exists at $\mathcal{L}^1$-almost every $t \in [a,b]$ [Dud07, Theorem 2.1].

A rectifiable path $\gamma\colon [a,b] \to Y$ is *absolutely continuous* if for all $a \leq s \leq t \leq b$,

$$d(\gamma(t), \gamma(s)) \leq \int_s^t v_\gamma(u)\, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a,b])$ where $\mathcal{L}^1$ is the Lebesgue measure on the real line. Equivalently, γ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $\mathcal{H}^1_Y$-measure zero in its image [Dud07, Section 3]. We refer to Chapter 5 of [HKST15] for further details about rectifiable paths.

If $\gamma\colon [a,b] \to Y$ is rectifiable, then there exist a 1-Lipschitz path $\widetilde{\gamma}\colon [0,\ell(\gamma)] \to Y$ whose metric speed equals one $\mathcal{L}^1$-almost everywhere on $[0,\ell(\gamma)]$, and for which there exists a non-decreasing surjective map $\psi\colon [a,b] \to [0,\ell(\gamma)]$ with $\widetilde{\gamma} \circ \psi = \gamma$.

Let $\rho\colon Y \to [0,\infty]$ be a Borel function. The *(path) integral* of ρ over γ is defined by

$$\int_\gamma \rho \, ds = \int_0^{\ell(\gamma)} \rho \circ \widetilde{\gamma} \, d\mathcal{L}^1. \tag{2}$$

A Borel function ρ is *integrable over* γ if (2) is finite. If γ is an absolutely continuous path, then

$$\int_\gamma \rho \, ds = \int_a^b (\rho \circ \gamma) \, v_\gamma \, d\mathcal{L}^1;$$

see [Dud07]. A path is *non-constant* if $\ell(\gamma) > 0$.

Let Γ be a family of paths in Y. A Borel function $\rho\colon Y \to [0,\infty]$ is *admissible* for Γ if for every rectifiable path in Γ,

$$\int_\gamma \rho \, ds \geq 1. \tag{3}$$

The *(conformal) modulus* of Γ is

$$\operatorname{mod} \Gamma = \inf \int_Y \rho^2 \, d\mathcal{H}^2_Y, \tag{4}$$

where the infimum is taken over all admissible functions ρ. A Borel function $\rho\colon Y \to [0,\infty]$ is *weakly admissible* for Γ if there exists a path family $\Gamma' \subset \Gamma$ such that $\operatorname{mod} \Gamma' = 0$ and for every $\gamma \in \Gamma \setminus \Gamma'$ (3) holds. We refer to [HKST15, Section 5.2] and [Wil12, Lemma 2.2] for basic properties of modulus. We recall that $\Gamma \mapsto \operatorname{mod} \Gamma$ is an outer measure on the collection of path families.

We say that a path family Γ is *negligible* if $\operatorname{mod} \Gamma = 0$. A property holds for *almost every* path if the path family along which it fails is negligible. We recall that a family Γ of non-constant paths is negligible if and only if there exists $\rho \in L^2(Y)$ such that the integral of ρ over every rectifiable $\gamma \in \Gamma$ is infinite [HKST15, Lemma 5.2.8]. The equivalence also holds for $\rho \in L^2_{\mathrm{loc}}(Y)$ by the countably subadditivity of modulus.

Let $\phi\colon (Y,d_Y) \to (Z,d_Z)$ be a homeomorphism between metric surfaces. The map ϕ is an element of the Sobolev space $N^{1,2}_{\mathrm{loc}}(Y,Z)$ if there exists a non-negative Borel function $\rho \in L^2_{\mathrm{loc}}(Y)$ such that for all non-constant rectifiable paths $\gamma\colon [a,b] \to Y$,

$$d_Z(\phi(\gamma(a)), \phi(\gamma(b))) \leq \int_\gamma \rho \, ds. \tag{5}$$

Such a function ρ is called an *upper gradient* of ϕ. A Borel function is a *weak upper gradient* of ϕ if (5) holds for almost all non-constant paths. A weak upper gradient ρ of $\phi \in N^{1,2}_{\mathrm{loc}}(Y,Z)$ is *minimal* if for every other weak upper gradient $\widetilde{\rho} \in L^2_{\mathrm{loc}}(Y)$, $\rho \leq \widetilde{\rho}$ $\mathcal{H}^2_Y$-almost everywhere. Every $\phi \in N^{1,2}_{\mathrm{loc}}(Y,Z)$ has a minimal weak upper gradient, uniquely defined $\mathcal{H}^2_Y$-almost everywhere, which we denote by ρ_ϕ. We refer the reader to [HKST15] and [Wil12] for details.

Let $C \subset Y$ be a Borel set. The length of γ in C, denoted by $\ell(\gamma \cap C)$, is the integral of χ_C over γ. Then Γ_C^+ denotes those rectifiable paths that have positive length in C.

Observe that if $\mathcal{H}_Y^2(C) = 0$, then Γ_C^+ is negligible; consider the admissible function $\infty \cdot \chi_C$. We prove in Lemma 3.2 a partial converse of this fact. We use the converse later on, since quasiconformal surfaces can have Borel subsets $C \subset Y$ of positive measure for which $\operatorname{mod}\Gamma_C^+ = 0$. See Remark 3.4 for further discussion.

Definition 3.1. For a metric surface (Y, d_Y) and for each Borel set $C \subset Y$, we denote $\nu_Y(C) = \int_C \rho_{\mathrm{id}_Y}\, d\mathcal{H}_Y^2$.

Lemma 3.2. *Let (Y, d_Y) be a metric surface. Then there exists a Borel set $C_0 \subset Y$ such that $\rho_{\mathrm{id}_Y} = \chi_{Y \setminus C_0}$. Moreover, for each Borel set $C \subset Y$, $\operatorname{mod}\Gamma_C^+ = 0$ if and only if $\nu_Y(C) = 0$.*

Proof. Fix a Borel representative ρ of the minimal weak upper gradient ρ_{id_Y}. Since ρ and χ_Y are weak upper gradients of id_Y, so is their pointwise minimum. Therefore, we may assume without loss of generality that $\rho \leq \chi_Y$ everywhere.

For $A = \{\rho < 1\}$, we have that $\operatorname{mod}\Gamma_A^+ = 0$, since otherwise ρ cannot be a weak upper gradient of id_Y [HKST15, Proposition 6.3.3]. Therefore, $\rho_0 = \rho\chi_{Y\setminus A} = \chi_{Y \setminus A}$ is a weak upper gradient of id_Y, and $\rho_0 \leq \rho$ implies that ρ_0 is a representative of ρ_{id_Y}. We denote $C_0 := A$.

Consider $\rho_0 = \chi_{Y \setminus C_0}$ as above. If $C \subset Y$ is a Borel set with $\operatorname{mod}\Gamma_C^+ = 0$, then $\rho_0\chi_{Y\setminus C}$ is a representative of ρ_{id_Y}, so $0 = \mathcal{H}_Y^2(C \setminus C_0) = \nu_Y(C)$. Conversely, if $0 = \nu_Y(C) = \mathcal{H}_Y^2(C \setminus C_0)$, then $\operatorname{mod}\Gamma_{C\setminus C_0}^+ = 0$. Also, $\operatorname{mod}\Gamma_{C\cap C_0}^+ \leq \operatorname{mod}\Gamma_{C_0}^+ = 0$. These facts imply that $\operatorname{mod}\Gamma_C^+ = 0$. The set C_0 has the claimed properties. $\square$

Consider a homeomorphism $\phi\colon (Y, d_Y) \to (Z, d_Z)$ between metric surfaces. We denote $\phi^*\mathcal{H}_Z^2(A) = \mathcal{H}_Z^2(\phi(A))$ for all sets $A \subset Y$. Then there exists a decomposition $\phi^*\mathcal{H}_Z^2 = J_\phi\mathcal{H}_Y^2 + \mu^\perp$ with $\mathcal{H}_Y^2$ and $\mu^\perp$ singular [Bog07, Sections 3.1–3.2, Volume I]. We refer to the density J_ϕ as the *Jacobian* of ϕ.

We say that ϕ satisfies Lusin's Condition (N) if $\phi^*\mathcal{H}_Z^2$ is absolutely continuous with respect to $\mathcal{H}_Y^2$. It satisfies Lusin's Condition (N^{-1}) if $\mathcal{H}_Y^2$ is absolutely continuous with respect to $\phi^*\mathcal{H}_Z^2$.

A homeomorphism $\phi\colon (Y, d_Y) \to (Z, d_Z)$ between metric surfaces is *quasiconformal* if there exist constants $K_O, K_I \geq 1$ such that $K_O^{-1}\operatorname{mod}\Gamma \leq \operatorname{mod}\phi\Gamma \leq K_I \operatorname{mod}\Gamma$ for every path family Γ in (Y, d_Y). Recalling [Wil12, Theorem 1.1], an equivalent definition is obtained by requiring

$$\phi \in N^{1,2}_{\mathrm{loc}}(Y, Z) \quad \text{and} \quad \rho_\phi^2 \leq K_O J_\phi \quad \mathcal{H}_Y^2\text{-a.e.} \quad \text{and} \tag{6}$$

$$\phi^{-1} \in N^{1,2}_{\mathrm{loc}}(Z, Y) \quad \text{and} \quad \rho_{\phi^{-1}}^2 \leq K_I J_{\phi^{-1}} \quad \mathcal{H}_Z^2\text{-a.e.} \tag{7}$$

with the same constants K_O and K_I. The smallest constant K_O (resp. K_I) for which (6) (resp. (7)) holds is called the *outer dilatation* of ϕ (resp. *inner dilatation*) and denoted by $K_O(\phi)$ (resp. $K_I(\phi)$). We say that a quasiconformal mapping is K-quasiconformal if $K_O(\phi) \leq K$ and $K_I(\phi) \leq K$. The smallest $K \geq 1$ for which ϕ is K-quasiconformal is called the *maximal dilatation* of ϕ.

Having defined quasiconformal mappings, we prove the following.

Lemma 3.3. *Let $\phi\colon (Y, d_Y) \to (Z, d_Z)$ be a quasiconformal homeomorphism between metric surfaces. Then for each Borel sets $C \subset Y$, the following four conditions* *are equivalent:*

$$\nu_Y(C)=0, \quad \operatorname{mod}\Gamma_C^+=0, \quad \operatorname{mod}\Gamma_{\phi(C)}^+=0 \quad \textit{and} \quad \nu_Z(\phi(C))=0.$$

Proof. Let K denote the maximal dilatation of ϕ. Fix Borel representatives of ρ_ϕ and $\rho_{\phi^{-1}}\circ\phi$. We denote $\rho=\rho_\phi\,(\rho_{\phi^{-1}}\circ\phi)$. We recall from (6) and (7) that

$$\rho_\phi^2 \le KJ_\phi \in L^1_{\mathrm{loc}}(Y) \quad \text{and} \quad \rho_{\phi^{-1}}^2 \le KJ_{\phi^{-1}} \in L^1_{\mathrm{loc}}(Z) \tag{8}$$

hold $\mathcal{H}^2_Y$- and $\mathcal{H}^2_Z$-almost everywhere, respectively.

Proposition 6.3.3 of [HKST15] implies that for almost every non-constant absolutely continuous path $\gamma\colon [0,1]\to Y$, the path $\phi\circ\gamma$ is absolutely continuous and for $\mathcal{L}^1$-almost every $0\le t\le 1$,

$$v_{\phi\circ\gamma}(t) \le (\rho_\phi\circ\gamma(t))\,v_\gamma(t) \in L^1([0,1]). \tag{9}$$

The right-hand side is interpreted to be zero in the set $\{v_\gamma\equiv 0\}$. Let Γ_1 denote the collection of those non-constant paths for which (9) fails.

As above, for almost every non-constant absolutely continuous path $\theta\colon [0,1]\to V$, the path $\phi^{-1}\circ\theta$ is absolutely continuous and for $\mathcal{L}^1$-almost every $0\le t\le 1$,

$$v_{\phi^{-1}\circ\theta}(t) \le (\rho_{\phi^{-1}}\circ\theta(t))\,v_\theta(t) \in L^1([0,1]). \tag{10}$$

Let Γ_2 denote the collection of those paths γ in Y for which $\theta=\phi\circ\gamma$ fails (10).

Since ϕ is quasiconformal, $\operatorname{mod}(\Gamma_1\cup\Gamma_2)=0$. Therefore, for almost every absolutely continuous $\gamma\colon [0,1]\to Y$ and $\theta=\phi\circ\gamma$ both (9) and (10) hold $\mathcal{L}^1$-almost everywhere. For such γ,

$$v_\gamma(t)\le(\rho\circ\gamma(t))\,v_\gamma(t)$$

for $\mathcal{L}^1$-almost every $0\le t\le 1$. This implies that ρ is a weak upper gradient of the identity map $\mathrm{id}_Y\colon Y\to Y$, and we conclude from (8) that

$$\rho\in L^2_{\mathrm{loc}}(Y). \tag{11}$$

Similar reasoning as above yields that

$$\rho\circ\phi^{-1}\in L^2_{\mathrm{loc}}(Z) \tag{12}$$

is a weak upper gradient of id_Z.

Let Γ_3 denote the collection of those absolutely continuous paths in U along which ρ fails to be integrable or those γ for which $\rho\circ\phi^{-1}$ fails to be integrable along $\phi\circ\gamma$. Then (11) and (12) imply that $\operatorname{mod}\Gamma_3=0$ as well.

Consider $\Gamma_0=\Gamma_1\cup\Gamma_2\cup\Gamma_3$. Observe that given a Borel set $C\subset U$, an absolutely continuous path $\gamma\colon [0,1]\to U\notin\Gamma_0$ has positive length in C, i.e.,

$$\int_0^1 (\chi_C\circ\gamma)\,v_\gamma\,d\mathcal{L}^1>0$$

if and only if the absolutely continuous path $\phi\circ\gamma$ has positive length in $\phi(C)$. Since Γ_0 and $\phi\Gamma_0$ are negligible, we deduce from this that $\operatorname{mod}\Gamma_C^+=0$ if and only if $\operatorname{mod}\Gamma_{\phi(C)}^+=0$. Then Lemma 3.2 proves the claim. □

Remark 3.4. As a consequence of Lemma 3.2, a quasiconformal homeomorphism ϕ from (Y,d_Y) into (Z,d_Z) satisfies Lusin's Conditions (N) and (N^{-1}) with respect to the measures ν_Y and ν_Z. That is, for all Borel subsets $B\subset Y$, $\nu_Y(B)=0$ if and only if $\nu_Z(\phi(B))=0$. We use this fact in Section 5.

As an application of Lemma 3.2, we fix a Borel set $B_0\subset Y$ such that $\nu_Y=\chi_{Y\setminus B_0}\mathcal{H}^2_Y$ and $\nu_Z=\chi_{Z\setminus\phi(B_0)}\mathcal{H}^2_Z$. The product $\rho_\phi(\rho_{\phi^{-1}}\circ\phi)$ is uniquely defined ν_Y-almost everywhere, since every representative of ρ_ϕ is zero $\mathcal{H}^2_Y$-almost everywhere in B_0 and $\rho_{\phi^{-1}}$ zero $\mathcal{H}^2_Z$-almost everywhere in $\phi(B_0)$. We apply this fact already in Section 4.

If Z is an open subset of $\mathbb{R}^2$ or a Riemannian surface, we have $\nu_Z \equiv \mathcal{H}^2_Z$ in Lemma 3.3. Therefore, for such Z, any quasiconformal mapping ϕ as above satisfies Lusin's Condition (N). For such a Z, if we have $\rho_{\mathrm{id}_Y} = \chi_{Y\setminus B_0}$ with $\mathcal{H}^2_Y(B_0) > 0$, ϕ fails Lusin's Condition (N^{-1}), with respect to the Hausdorff 2-measures, exactly at Borel subsets of B_0 of positive measure. We note that there are quasiconformal surfaces for which $\mathcal{H}^2_Y(B_0) > 0$; see [Raj17, Proposition 17.1]. Due to this fact, many results in Section 5 are only phrased in terms of ν_Y.

We sometimes write $TU = U \times \mathbb{R}^2$ when $U \subset \mathbb{R}^2$ is an open set. We refer to TU as the *tangent bundle* of U. For each $x \in U$, we refer to $\{x\} \times \mathbb{R}^2$ as a *fiber* of TU and denote it by T_xU.

At times, we consider quasiconformal maps $\psi\colon U \to \widetilde{U}$ between open subsets of $\mathbb{R}^2$. Such maps have a *differential* $D\psi$ $\mathcal{L}^2$-almost everywhere, which just means its classical derivative. The differential defines a map

$$D\psi\colon TU \to T\widetilde{U},$$

where the fiber T_xU is taken to $T_{\psi(x)}\widetilde{U}$ by the linear map $D_x\psi$.

Next, we consider a measurable seminorm field $\mathrm{N}\colon TU \to [0,\infty]$. This means that we have a measurable map from TU into $[0,\infty]$ such that for $\mathcal{L}^2$-almost every $x \in U$, the restriction of N to T_xU is a seminorm. If the restriction of N to $\mathcal{L}^2$-almost every fiber is a norm, we say that N is a *norm field*. In this case, the pair (TU, N) is called a *normed bundle*, where the fibers refer to $(TU,\mathrm{N})_x := (T_xU, \mathrm{N}\,|_{T_xU})$.

We sometimes consider the differential $D\psi$ between two normed bundles, i.e., the map

$$D\psi\colon (TU,\mathrm{N}) \to (T\widetilde{U}, \widetilde{\mathrm{N}}). \tag{13}$$

The *operator norm* $\|D\psi\|$ of (13) at $x \in U$ refers to the operator norm of the linear map $D_x\psi\colon (TU,\mathrm{N})_x \to (TU,\mathrm{N})_{\psi(x)}$. We denote the Jacobian of $D\psi$ at x by $J_2(D\psi)(x)$. The *outer dilatation* $K_O(D\psi)$ at $x \in U$ is defined as

$$K_O(D\psi)(x) = \frac{\|D\psi\|^2(x)}{J_2(D\psi)(x)}. \tag{14}$$

The *inner dilatation* $K_I(D\psi)$ at $x \in U$ is defined by the formula

$$K_I(D\psi)(x) = K_O(D(\psi^{-1}))(\psi(x)). \tag{15}$$

The *maximal dilatation* $K(D\psi)$ of $D\psi$ at $x \in U$ is the maximum of (14) and (15).

The objects (13), (14) and (15) are well-defined even if we consider norms $\{N_x\}_{x\in U}$ and $\{\widetilde{N}_x\}_{x\in U}$ together with linear maps $L_x\colon (TU,N)|_x \to (TU,\widetilde{N})|_x$. The objects above are defined similarly when U is an open subset of a smooth surface.

4. Proof of Theorem 1.3

We define isothermal parametrizations in Section 4.1 and state some of their properties. In Section 4.2, we analyze general quasiconformal maps from planar domains into metric surfaces. Using results from that subsection, we prove the claims from Section 4.1 in Section 4.3.

We construct the atlas of isothermal coordinates for (Y, d_Y) in Section 4.4. We define the uniformization map in Section 4.5 and prove Theorem 1.3 there.

4.1. Isothermal parametrizations.

Definition 4.1. A quasiconformal homeomorphism $\phi\colon U \to V \subset Y$, with $U \subset \mathbb{R}^2$ open, is an *isothermal parametrization* of V if for every other quasiconformal homeomorphism $\widetilde{\phi}\colon \widetilde{U} \to V$ with $\widetilde{U} \subset \mathbb{R}^2$,

$$\rho_\phi(x)\left(\rho_{\phi^{-1}} \circ \phi(x)\right) \le \rho_{\widetilde{\phi}}(\widetilde{x})\left(\rho_{\widetilde{\phi}^{-1}} \circ \widetilde{\phi}(\widetilde{x})\right) \tag{16}$$

for $\widetilde{x} = (\widetilde{\phi}^{-1} \circ \phi)(x)$ and $\mathcal{L}^2$-almost every $x \in U$. If the image of ϕ is clear, we say that ϕ is *isothermal*.

Here ρ_ϕ denotes a minimal weak upper gradient of ϕ and $\rho_{\phi^{-1}}$ a minimal weak upper gradient of ϕ^{-1}. Lemma 3.3 implies that both sides of (16) are independent of the representatives we use.

It turns out that the left-hand side of (16) is the geometric mean of the pointwise versions of the dilatations $K_O(\phi)$ and $K_I(\phi)$; this is made precise in (19) and the discussion following (19). This observation implies that isothermal parametrizations minimize the geometric mean of the pointwise dilatations; see Theorem 4.12 for the precise statement. We highlight two consequences of Theorem 4.12.

Proposition 4.2. *Let $\phi\colon U \to V$ be K-quasiconformal, $U \subset \mathbb{R}^2$ and $V \subset Y$ open. Then there exist a set $\widetilde{U} \subset \mathbb{R}^2$ and a $(4K/\pi)$-quasiconformal homeomorphism $\psi\colon \widetilde{U} \to U$ such that $\widetilde{\phi} = \phi \circ \psi$ is isothermal.*

Proposition 4.3. *Every isothermal parametrization $\phi\colon U \to V$ satisfies*

$$\frac{\pi}{4} \operatorname{mod} \Gamma \le \operatorname{mod} \phi\Gamma \le \frac{\pi}{2} \operatorname{mod} \Gamma \tag{17}$$

for all path families $\Gamma \subset U$. Moreover, if $V' \subset V$ is open and $\phi'\colon U' \to V'$ is quasiconformal with $U' \subset \mathbb{R}^2$, ϕ' is an isothermal parametrization of V' if and only if $\phi^{-1} \circ \phi'$ is holomorphic or antiholomorphic.

We see from Proposition 4.3 that isothermal parametrizations satisfy the same dilatation bounds as the parametrizations constructed in [Rom19]. In fact, our isothermal parametrizations coincide with the parametrizations considered by Romney for simply connected domains. This observation is not immediately apparent from our definition, but is a corollary of Theorem 4.12.

4.2. Quasiconformal parametrizations. Before proving the existence of isothermal parametrizations, we first analyze a given quasiconformal map $\phi\colon U \to V \subset Y$ with open $U \subset \mathbb{R}^2$ and Y a metric surface. Since $\phi \in N^{1,2}_{\mathrm{loc}}(U, V)$, there exists a measurable seminorm field

$$\mathrm{N}_\phi\colon TU \to \mathbb{R}$$

that encodes the following geometric properties of ϕ.

Lemma 4.4. *The following properties hold.*

(a) *The maximal stretching of N_ϕ,*

$$L(\mathrm{N}_\phi)(x) := \sup_{\|v\|_2 \le 1} \mathrm{N}_\phi(x, v) \quad \textit{for } x \in U,$$

defines a representative of the minimal weak upper gradient ρ_ϕ;

(b) *The Jacobian function*

$$x \mapsto J_2(\mathrm{N}_\phi)(x) := \frac{\pi}{\mathcal{L}^2\left(\{v \in \mathbb{R}^2 \colon \mathrm{N}_\phi(x, v) \le 1\}\right)}$$

is a representative of the Jacobian J_ϕ of ϕ;

(c) *For almost every non-constant absolutely continuous path $\gamma\colon [a,b] \to U$,*

$$v_{\phi\circ\gamma}(t) = \mathrm{N}_\phi \circ D\gamma(t)$$

for $\mathcal{L}^1$-almost every t, where $D\gamma(t) = (\gamma(t), \gamma'(t))$ is the derivative of γ at t.

See [LW18, Sections 3.3-3.4 and 3.6] for the proof of Lemma 4.4. The seminorm field N_ϕ is refered to as the *approximate metric differential* of ϕ. Lemma 3.3 implies that ϕ satisfies Lusin's Condition (N^{-1}) (see also Remark 3.4). Then the Sobolev regularity of ϕ implies the following; see, for example, [Raj17, Lemma 14.1].

Lemma 4.5. *The homeomorphism ϕ satisfies Lusin's Condition (N^{-1}) and there exists a Borel set $B_0 \subset U$ with $\mathcal{L}^2(B_0) = 0$ such that $\phi|_{U\setminus B_0}$ satisfies Lusin's Condition (N).*

Lemma 4.5 implies the following.

Corollary 4.6. *If B_0 is as in Lemma 4.5, then the Jacobian of ϕ^{-1} equals $1/(J_\phi \circ \phi^{-1})$ $\mathcal{H}^2_Y$-almost everywhere in $V \setminus \phi(B_0)$.*

Rajala's example [Raj17, Proposition 17.1] illustrates that the set $\phi(B_0)$ can have positive $\mathcal{H}^2_Y$-measure, so ϕ does not necessarily satisfy Lusin's Condition (N).

Since ϕ satisfies Lusin's Condition (N^{-1}), the Jacobian of ϕ is non-zero $\mathcal{L}^2$-almost everywhere in U. In other words, the approximate metric differential N_ϕ is a norm $\mathcal{L}^2$-almost everywhere in U. Consequently, $\omega(\mathrm{N}_\phi)(x) := \inf_{\|v\|_2 \geq 1} \mathrm{N}_\phi(x,v)$ is an element in $(0,\infty)$ for $\mathcal{L}^2$-almost every $x \in U$.

Lemma 4.7. *Let B_0 be as in Lemma 4.5 and*

$$\widetilde{\rho}(y) = \left(\frac{1}{\omega(\mathrm{N}_\phi)} \circ \phi^{-1}(y)\right) \chi_{V\setminus\phi(B_0)}(y) \quad \text{for each } y \in V.$$

Here $\widetilde{\rho} \equiv 0$ in $\phi(B_0)$. Then $\widetilde{\rho}$ is a representative of the minimal upper gradient $\rho_{\phi^{-1}}$.

Proof. The L^2_{loc}-integrability of $\widetilde{\rho}$ follows from the change of variables formula for ϕ. Lemma 4.5 and Lemma 3.3 imply that $\operatorname{mod}\Gamma^+_{\phi(B_0)} = 0$.

We conclude that almost every non-constant path has zero length in $\phi(B_0)$ and that $\widetilde{\rho}$ is integrable over the path. We may also assume that the image path γ in U is absolutely continuous and satisfies Lemma 4.4 (c). These facts imply that $\widetilde{\rho}$ is a weak upper gradient of ϕ^{-1}.

To see that $\widetilde{\rho}$ is a minimal upper gradient, it suffices to fix a upper gradient $\rho \in L^2_{\mathrm{loc}}(Y)$ of ϕ^{-1} and to prove $\widetilde{\rho} \leq \rho$ $\mathcal{H}^2_Y$-almost everywhere. This is clear everywhere in $\phi(B_0)$. Since $\phi|_{U\setminus B_0}$ satisfies Lusin's Condition (N) and (N^{-1}), it suffices to verify $\widetilde{\rho}(y_0) \leq \rho(y_0)$ for $y_0 = \phi(x_0)$ for $\mathcal{L}^2$-almost every $x_0 \in U \setminus B_0$. We fix $v_0, w_0 \in \mathbb{S}^1$ perpendicular to one another.

Consider now a rectangle $R \subset U$ with a foliation $\gamma_t(s) = x_0 + tv + sw$, for $-1 \leq s,t \leq 1$, $r = \|v\|_2 = \|w\|_2$ with $v = rv_0$ and $w = rw_0$. For $\mathcal{L}^1$-almost every t, Lemma 4.4 (c) holds for γ_t, and $\theta_t := \phi \circ \gamma_t$ is absolutely continuous. Then the upper gradient inequality and Fubini's theorem imply

$$\rho(\phi(x))\,\mathrm{N}_\phi((x,w)) \geq \|w\|_2 \quad \text{for } \mathcal{L}^2\text{-almost every } x \in R \setminus B_0.$$

Covering U by such rectangles implies

$$\rho(\phi(x)) \geq \frac{1}{\mathrm{N}_\phi((x,w_0))} \quad \text{for } \mathcal{L}^2\text{-almost every } x \in U \setminus B_0. \tag{18}$$

Since the inequality (18) holds for a countable dense set $\{w_i\}_{i=1}^{\infty} \subset \mathbb{S}^1$ for $\mathcal{L}^2$-almost every $x_0 \in U \setminus B_0$, taking the supremum over i yields $\rho(\phi(x)) \geq \widetilde{\rho}(\phi(x))$ for $\mathcal{L}^2$-almost every $x \in U \setminus B_0$. This was sufficient for the claim. □

Definition 4.8. Let $\phi\colon U \to V$ be quasiconformal. The *pointwise outer dilatation* of ϕ at $x \in U$ is

$$K_O(\phi)(x) = \frac{\rho_\phi^2(x)}{J_\phi(x)}$$

and the *pointwise inner dilatation* of ϕ at $x \in U$ is

$$K_I(\phi)(x) = \left(\rho_{\phi^{-1}}^2(\phi(x))\right) J_\phi(x)\chi_{U\setminus B_0}(x).$$

The *pointwise maximum dilatation* of ϕ at $x \in U$ is the maximum of the corresponding outer and inner dilatations.

We consider the differential

$$D\mathrm{id}\colon (TU, \|\cdot\|_2) \to (TU, \mathrm{N}_\phi) \tag{19}$$

as defined in (13). Then Lemma 4.4 (a) implies that the operator norm of Did from (19) is a representative of ρ_ϕ. Similarly, Lemma 4.4 (b) implies that the Jacobian $J_2(D\mathrm{id})$ is a representative of the Jacobian of ϕ. Lemma 4.7 and Corollary 4.6 yield similar identities for the inverse of the map in (19). Consequently, the pointwise outer (resp. inner) dilatation of ϕ and the differential in (19) coincide. These facts imply that the left-hand side of (16) equals $\sqrt{K_O(D\mathrm{id})K_I(D\mathrm{id})}$ $\mathcal{L}^2$-almost everywhere. Therefore, the left-hand side in (16) is the geometric mean of the outer and inner dilatations of the differential (19). This fact connects the definition of isothermal parametrizations to convex analysis.

4.3. Banach–Mazur distance and isothermal parametrizations. In this section, we associate a Beltrami differential to the approximate metric differential of any given quasiconformal parametrization. For this purpose, we introduce Banach–Mazur distance from convex analysis.

Definition 4.9. Let M and N be norms on $\mathbb{R}^2$. Then $GL_2[M, N]$ is the collection of all invertible linear maps $S\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, N)$. An invertible linear map $S \in GL_2[M, N]$ is a *Banach–Mazur minimizer from M to N* if S attains the infimum

$$\rho(M, N) = \inf_{T \in GL[M,N]} \sqrt{K_O(T)K_I(T)}.$$

If the domain and codomain of the linear map S are clear from the context, we say that S is a *Banach–Mazur minimizer*. The number $\rho(M, N)$ is the *Banach–Mazur distance* from M to N.

If N is induced by an inner product, $\rho(M, N) \leq \sqrt{2}$ [TJ89, Proposition 9.12], with $\rho(M, N) = \sqrt{2}$ if M is the supremum norm [TJ89, Proposition 37.6]. Therefore, $\rho(M, N) \leq 2$ for every pair of norms. Then a compactness argument implies that Banach–Mazur minimizers exist for each pair of norms, see e.g. [TJ89, Section 37].

We recall some notations. The group O_2 is the group of linear isometries of $\mathbb{R}^2$ and $\mathbb{R}_+ \cdot O_2$ denotes the group of invertible linear maps $L = \lambda \cdot S$, where $\lambda > 0$ and $S \in O_2$. The group SO_2 consists of the elements of O_2 with determinant equal to 1. The group $\mathbb{R}_+ \cdot O_2$ are the linear conformal automorphisms of $\mathbb{R}^2$, and $\mathbb{R}_+ \cdot SO_2$ the subgroup of $\mathbb{R}_+ \cdot O_2$ whose elements have positive determinant.

Lemma 4.10. *Let M be a norm on $\mathbb{R}^2$ and $L\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, \|\cdot\|_2)$ a Banach–Mazur minimizer. Then*

$$\frac{\pi}{4}\rho^2(M, \|\cdot\|_2) \leq K_O(L) \leq \frac{\pi}{2} \quad \text{and} \tag{20}$$

$$\frac{2}{\pi}\rho^2(M, \|\cdot\|_2) \leq K_I(L) \leq \frac{4}{\pi}. \tag{21}$$

Moreover, $L' \in GL_2[M, \|\cdot\|_2]$ is a Banach–Mazur minimizer if and only if $L' \circ L^{-1} \in \mathbb{R}_+ \cdot O_2$.

Proof. The inequalities (20) and (21) are slight reformulations of Lemma 2.1 of [Rom19]. Lemma 2.2 of [Rom19] proves that if L' is a Banach–Mazur minimizer, then $L' \circ L^{-1} \in \mathbb{R}_+ \cdot O_2$. Conversely, if $L' = S \circ L$ for some $S \in \mathbb{R}_+ \cdot O_2$, the outer and inner dilatations of L' and L coincide. Therefore, L' is a Banach–Mazur minimizer. □

If M is the supremum norm, we have that $\rho^2(M, \|\cdot\|_2) = 2$. Thus (20) and (21) are equalities in this case. In fact, $K_O(L) = \pi/2$ and $K_I(L) = 4/\pi$ for a Banach–Mazur minimizer from M to $\|\cdot\|_2$ if and only if M is isometric to the supremum norm [TJ89, Proposition 37.4].

We identify $\mathbb{R}^2$ with the complex plane in the following statement.

Corollary 4.11. *Suppose that M is a norm on $\mathbb{R}^2$. Then there exists a unique complex number μ_M in the Euclidean ball $\mathbb{D}$ such that*

$$T_M = \mathrm{id} + \mu_M \cdot \overline{\mathrm{id}}\colon \big(\mathbb{R}^2, M\big) \to \big(\mathbb{R}^2, \|\cdot\|_2\big)$$

is a Banach–Mazur minimizer from M to $\|\cdot\|_2$. Moreover, μ_M and T_M depend continuously on the norm M.

Here $\mu_M \cdot \overline{\mathrm{id}}$ refers to the complex multiplication and $\overline{\mathrm{id}}(w) = \overline{w_1 + iw_2} = w_1 - iw_2$ denotes the complex conjugation map.

Proof. Consider an orientation-preserving Banach–Mazur minimizer $L\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, \|\cdot\|_2)$, the existence of which follows from Lemma 4.10.

Fix an orientation-preserving $L' \in GL_2[M, \|\cdot\|_2]$. Lemma 4.10 implies that L' is a Banach–Mazur minimizer if and only if $L' = S \circ L$ for some $S \in \mathbb{R}_+ \cdot SO_2$. Such an S exists if and only if L' and L have the same Beltrami differential [AIM09, Section 2.4]. Moreover, for a given L, there exists $S \in \mathbb{R}_+ \cdot SO_2$ such that $L = S \circ T$ for some $T = \mathrm{id} + \mu \cdot \overline{\mathrm{id}}$ with $\mu \in \mathbb{D}$. So $T = T_M$ and $\mu = \mu_M$ are uniquely defined.

Next, we establish the continuity of $M \mapsto \mu_M$ and $M \mapsto T_M$. To this end, given a sequence of norms $(M_j)_{j=1}^{\infty}$ and a norm M, with $M_j \to M$ uniformly in compact subsets of $\mathbb{R}^2$, we claim that $T_{M_j} \to T_M$. First, we note that Banach–Mazur distances $\rho(M_j, \|\cdot\|_2)$ converge to $\rho(M, \|\cdot\|_2)$. Indeed, for every $\epsilon > 0$, there exists j_0 such that for every $j \geq j_0$, the identity mapping from $(\mathbb{R}^2, M_j)$ to $(\mathbb{R}^2, M)$ is $(1+\epsilon)$-bi-Lipschitz. This implies the claimed convergence. This convergence implies that $(T_{M_j})_{j=1}^{\infty}$ is a normal family.

Consider a convergent subsequence with $T_{M_{j_i}} \to T$. Then the sequence of outer (resp. inner) dilatations of $T_{M_{j_i}}\colon (\mathbb{R}^2, M_{j_i}) \to (\mathbb{R}^2, \|\cdot\|_2)$ converge to the outer (resp. inner) dilatation of $T\colon (\mathbb{R}^2, M) \to (\mathbb{R}^2, \|\cdot\|_2)$. Therefore,

$$\rho(M, \|\cdot\|_2) \leq \sqrt{K_O(T)K_I(T)} = \lim_{i\to\infty} \sqrt{K_O(T_{j_i})K_I(T_{j_i})} = \lim_{i\to\infty} \rho(M_{j_i}, \|\cdot\|_2).$$

The right-hand side equals $\rho(M, \|\cdot\|_2)$, so T must be a Banach–Mazur minimizer. Since every accumulation point of $(T_{M_j})_{j=1}^{\infty}$ is of the form $T = \mathrm{id} + \mu \cdot \overline{\mathrm{id}}$, we conclude that $T = T_M$ by the uniqueness of T_M. This implies $\mu = \mu_M$. Since $(T_{M_j})_{j=1}^{\infty}$ has a unique accumulation point, the sequence itself converges to T_M. This also implies $\mu_{M_j} \to \mu_M$. □

Let M and T_M be as in Corollary 4.11. We call the ellipse

$$\mathcal{E}_M := \left\{ v \in \mathbb{R}^2 \colon \left\| T_M^{-1} \right\| \left\| T_M(v) \right\|_2 \leq 1 \right\}$$

the *distance ellipse* of $\{M \leq 1\}$. We note that for every ellipse $\mathcal{E} \subset \{M \leq 1\}$ and every $\lambda > 0$ with $\{M \leq 1\} \subset \lambda\mathcal{E}$, we have $\lambda \geq \rho(M, \|\cdot\|_2)$. The equality $\lambda = \rho(M, \|\cdot\|_2)$ holds if and only if $\mathcal{E}$ is the distance ellipse. Observe that $\mathcal{E}_M$ is an Euclidean ball if and only if $\mu_M = 0$.

In the following statement, $\phi\colon U \to V \subset Y$ is a quasiconformal homeomorphism with $U \subset \mathbb{R}^2$ open. Furthermore, we denote $\mu_\phi := \mu_{\mathrm{N}_\phi}$ for the approximate metric differential N_ϕ. We refer to μ_ϕ as the Beltrami differential of ϕ.

Theorem 4.12. *Let $W \subset \mathbb{R}^2$ be open and $\psi\colon W \to U$ be a quasiconformal map, possibly orientation-reversing. Then the following are equivalent:*

(a) *The composition $\phi \circ \psi$ is isothermal;*
(b) *The equality $\mu_{\phi\circ\psi} = 0$ holds $\mathcal{L}^2$-almost everywhere.*

If either one of the conditions hold and ϕ is K-quasiconformal, then ψ is $(4K/\pi)$-quasiconformal. Moreover, the above conditions are equivalent to any one of the following.

(c) *Either ψ^{-1} or $\overline{\psi^{-1}}$ is an orientation-preserving solution of the Beltrami equation $\partial_{\overline{z}} f = \mu_\phi \partial_z f$;*
(d) *The map $D\mathrm{id}_W\colon (TW, \mathrm{N}_{\phi\circ\psi}) \to (TW, \|\cdot\|_2)$ is a Banach–Mazur minimizer pointwise $\mathcal{L}^2$-almost everywhere;*
(e) *The pointwise dilatations satisfy the equality*

$$K_O(\phi \circ \psi) K_I(\phi \circ \psi) = \rho^2(\|\cdot\|_2, \mathrm{N}_{\phi\circ\psi})$$

$\mathcal{L}^2$-almost everywhere in W.

We discussed normed bundles (TU, N_ϕ) in Section 3. We refer the reader to [AIM09, Chapter 5] for the basics of Beltrami equations and the measurable Riemann mapping theorem.

Proof of Theorem 4.12. Lemma 4.10 yields that

$$D(\psi^{-1})\colon (TU, \mathrm{N}_\phi) \to (TW, \|\cdot\|_2)$$

is a Banach–Mazur minimizer $\mathcal{L}^2$-almost everywhere if and only if there exists a measurable map $x \mapsto S(x) \in \mathbb{R}_+ \cdot O_2$ such that $D(\psi^{-1}) = S \circ T_{\mathrm{N}_\phi}$ pointwise $\mathcal{L}^2$-almost everywhere. The map ψ is orientation-preserving if and only if S is orientation-preserving $\mathcal{L}^2$-almost everywhere. In that case $\mu_{\psi^{-1}} = \mu_\phi$ holds $\mathcal{L}^2$-almost everywhere. Otherwise, $\overline{\psi^{-1}}$ is orientation-preserving and $\mu_{\overline{\psi^{-1}}} = \mu_\phi$ holds $\mathcal{L}^2$-almost everywhere. These facts and the chain rule $\mathrm{N}_{\phi\circ\psi} = \mathrm{N}_\phi \circ D\psi$ now imply that Properties (c) and (d) are equivalent.

We recall from (19) and the following discussion that the pointwise dilatations satisfy $K_O(\phi \circ \psi) = K_O(D\mathrm{id}_W)$ and $K_I(\phi \circ \psi) = K_I(D\mathrm{id}_W)$ $\mathcal{L}^2$-almost everywhere. Therefore, the dilatations also satisfy

$$K_O(\phi \circ \psi) K_I(\phi \circ \psi) \geq \rho^2(\|\cdot\|_2, \mathrm{N}_{\phi\circ\psi}) \quad \mathcal{L}^2\text{-almost everywhere.} \tag{22}$$

Moreover, the equality (22) holds $\mathcal{L}^2$-almost everywhere if and only if Property (d) holds.

Also, if ψ_1 and ψ_2 are two maps for which $\phi \circ \psi_1$ and $\phi \circ \psi_2$ are isothermal parametrizations, the pointwise dilatations satisfy

$$K_O(\phi \circ \psi_1)K_I(\phi \circ \psi_1) = [K_O(\phi \circ \psi_2)K_I(\phi \circ \psi_2)] \circ (\psi_2^{-1} \circ \psi_1) \tag{23}$$

$\mathcal{L}^2$-almost everywhere.

By applying (22) and (23), the equivalence of Properties (c) to (e) and Property (a) follows if it can be shown that there exists a quasiconformal map ψ such that the equality in (22) holds $\mathcal{L}^2$-almost everywhere. By Property (c), it suffices to solve the Beltrami equation $\mu_f = \mu_\phi$ induced by ϕ.

Suppose that we know that the L^∞-norm of μ_ϕ is bounded from above by some constant $C < 1$. Then we extend μ_ϕ as zero to the Euclidean plane and let f be the normalized solution to the corresponding Beltrami equation. The existence of f is guaranteed by the measurable Riemann mapping theorem; see for example [AIM09]. The restriction of f^{-1} to the appropriate open set is the desired map ψ.

Lemma 4.10 implies that

$$\|\mu_\phi\|_{L^\infty(U)} \leq \frac{\frac{4}{\pi}K - 1}{\frac{4}{\pi}K + 1} =: C,$$

where we use the fact that ϕ is K-quasiconformal. This inequality also implies that the maximal dilatation of ψ is bounded from above by $(4K/\pi)$.

By expressing $\phi \circ \psi$ as $(\phi \circ \psi) \circ \mathrm{id}_W$, we see that Property (b) is equivalent to the other properties. □

Proof of Proposition 4.2. Let ψ^{-1} solve the Beltrami equation $\partial_{\overline{z}} f = \mu_\phi \partial_z f$ induced by ϕ. Then Theorem 4.12 proves that ψ is $(4K/\pi)$-quasiconformal and $\widetilde{\phi} = \phi \circ \psi$ isothermal. □

Proof of Proposition 4.3. The outer and inner dilatation bounds follow from Theorem 4.12 (d) and the dilatation bounds in Lemma 4.10.

Next, consider an open set $U' \subset \mathbb{R}^2$, $V' \subset V$ and a quasiconformal homeomorphism $\phi' \colon U' \to V'$. Here $\phi' = \phi \circ \psi$ for $\psi = \phi^{-1} \circ \phi'$. Theorem 4.12 (d) proves that ϕ' is isothermal if and only if ψ^{-1}, or $\overline{\psi^{-1}}$, is orientation-preserving and its Beltrami differential equals $\mu_\phi = 0$. Thus, [AIM09, Weyl's lemma] yields that ϕ' is isothermal if and only if ψ is holomorphic or antiholomorphic. □

4.4. Conformal surfaces. We fix a quasiconformal surface (Y, d) for this section. Given an open set $V \subset (Y, d)$ and a quasiconformal homeomorphism $\phi' \colon U' \to V$ with $U' \subset \mathbb{R}^2$, Proposition 4.2 yields the existence of an isothermal parametrization $\phi \colon U \to V$ of V. Given such a ϕ, we denote $f := \phi^{-1}$ and call the pair (V, f) an *isothermal chart* of (Y, d).

Let $\mathcal{I}_d = \{(V_i, f_i)\}_{i \in I}$ denote the collection of all isothermal charts of (Y, d). Since a quasiconformal surface (Y, d) can be covered by quasiconformal images of planar domains, we conclude that $\bigcup_{i \in I} V_i = Y$. The subscript d refers to the dependence of the collection on the distance of Y.

Definition 4.13. A *conformal atlas* $\mathcal{D}$ is an atlas whose transition maps are holomorphic or antiholomorphic maps. A conformal atlas $\mathcal{D}$ is *maximal* if for every other conformal atlas $\mathcal{D}'$ with $\mathcal{D} \cap \mathcal{D}' \neq \emptyset$, we have $\mathcal{D}' \subset \mathcal{D}$. If $\mathcal{D}$ is a maximal conformal atlas, the pair $(Y, \mathcal{D})$ is a *conformal surface.* A smooth surface is defined analogously.

Proposition 4.14. *The pair $(Y, \mathcal{I}_d)$ is a conformal surface.*

Proof. Proposition 4.3 implies that restrictions of isothermal charts to open subsets of their domains are isothermal charts, and that the transition maps between isothermal charts are holomorphic or antiholomorphic. Consequently, $\mathcal{I}_d$ is a conformal atlas. The maximality of $\mathcal{I}_d$ also follows from Proposition 4.3. □

We define and recall some terminology from Riemannian geometry. A *Riemannian norm (field) G* on a conformal (or a smooth) surface $(Y, \mathcal{A})$ is a map $G\colon TY \to \mathbb{R}$ for which there exists a smooth Riemannian metric g such that $G(v) = [g(v,v)]^{1/2}$ for $v \in TY$. Here TY is the tangent bundle of Y.

The length distance induced by g is denoted by d_G. We say that d_G is the *Riemannian distance* induced by G. The metric space (Y, d_G) has *constant curvature* k if the corresponding Riemannian metric g has constant curvature k. The curvature refers to Gaussian curvature.

A *Riemannian surface* is a conformal (or smooth) surface with a Riemannian norm field. A map $\psi\colon (Y_1, G_1) \to (Y_2, G_2)$ between Riemannian surfaces is *conformal in the Riemannian sense* if ψ is a diffeomorphism and there exists a positive smooth function $h\colon Y_2 \to (0, \infty)$ such that the pushforward Riemannian norm field $\psi_* G_1$ equals $h \cdot G_2$. A Riemannian norm G is *compatible* with a conformal atlas $\mathcal{I}$ if every chart $(V, f) \in \mathcal{I}$ is conformal in the Riemannian sense.

Proposition 4.15. *The conformal surface $(Y, \mathcal{I}_d)$ has a Riemannian distance d_G such that G is compatible with the isothermal charts $\mathcal{I}_d$ of Y and (Y, d_G) is complete and has constant curvature -1, 0 or 1. Additionally, $\mathcal{I}_d = \mathcal{I}_{d_G}$ and the charts $(V, f) \in \mathcal{I}_{d_G}$ are conformal in the Riemannian sense.*

Proof. The existence of G follows from the classical uniformization theorem. Theorem 4.12 Property (e) and [AIM09, Weyl's lemma] imply that the elements of $\mathcal{I}_{d_G}$ are conformal in the Riemannian sense. The construction of G implies that when the elements of $\mathcal{I}_d$ are considered as maps from Euclidean domains into (Y, d_G), then they are conformal in the Riemannian sense. Thus $\mathcal{I}_d = \mathcal{I}_{d_G}$. □

4.5. Uniformization map. Let d_G denote the Riemannian distance obtained from Proposition 4.15. We define $Y_G = (Y, d_G)$ and let $Y = (Y, d)$. We denote the Hausdorff 2-measure of Y_G by $\mathcal{H}^2_G$.

We call the map $u = \mathrm{id}_Y\colon Y_G \to Y$ the *uniformization map*. Proposition 4.15 implies that every isothermal parametrization of $V \subset Y$ can be written in the form $u \circ \phi$ for an isothermal parametrization $\phi\colon U \to u^{-1}(V)$.

Let Y be a quasiconformal surface. If $u \circ \phi_1$ and $u \circ \phi_2$ are isothermal charts and $\psi = \phi_2^{-1} \circ \phi_1$, then $\mathrm{N}_{u\circ\phi_2} \circ D\psi = \mathrm{N}_{u\circ\phi_1}$ by the chain rule. Since $D\psi$ is a diffeomorphism, the equality actually holds everywhere whenever the left-hand side or the right-hand side are defined.

Remark 4.16. For a given quasiconformal surface Y, there is a norm field N on Y_G such that for every isothermal parametrization $u \circ \phi\colon U \to V$, its approximate metric differential $\mathrm{N}_{u\circ\phi}$ satisfies $\mathrm{N}_{u\circ\phi} = \mathrm{N} \circ D\phi$ everywhere.

Corollary 4.17. *Let u be the uniformization map. Then the pointwise dilatations of u satisfy*

$$\rho^2(G, \mathrm{N}) = K_O(u) K_I(u) \quad \mathcal{H}^2_G\text{-almost everywhere}, \tag{24}$$

where $\rho(G, \mathrm{N})$ is the Banach–Mazur distance between (TY, G) and (TY, N). In par-

ticular,

$$\frac{\pi}{4} \bmod \Gamma \leq \bmod u\Gamma \leq \frac{\pi}{2} \bmod \Gamma \tag{25}$$

for all path families Γ *in* Y_G.

Proof. It suffices to verify (24) and (25) in any given planar domain $V \subset Y_G$. Consider an isothermal parametrization $\phi\colon U \to V \subset Y_G$. Remark 4.16 yields $\mathrm{N} \circ D\phi = \mathrm{N}_{u\circ\phi}$ and Proposition 4.15 implies $G \circ D\phi = \omega \|\cdot\|_2$ for some smooth function ω. The equality (24) follows from the corresponding claim about $u \circ \phi$, see Theorem 4.12. The inequalities (25) follow the corresponding property of $u \circ \phi$, see Proposition 4.3. □

Proof of Theorem 1.3. The claim was that the uniformization map satisfies $K_O(u) \leq 4/\pi$ and $K_I(u) \leq \pi/2$. These inequalities follow from (25). □

Lemma 4.18. *The map* $u\colon (Y, \mathcal{H}^2_G) \to (Y, \nu_Y)$ *satisfies Lusin's Conditions* (N) *and* (N^{-1}).

Proof. This follows from Lemma 3.3 since $\nu_{Y_G} \equiv \mathcal{H}^2_G$. □

We sometimes consider the differential

$$Du\colon (TY, G) \to (TY, \mathrm{N}),$$

where the norm field N is understood to be well-defined ν_Y-almost everywhere in Y. This makes sense due to Lemma 4.18.

5. Quasiconformal maps between quasiconformal surfaces

Given two quasiconformal surfaces $Y_1 = (Y_1, d_1)$ and $Y_2 = (Y_2, d_2)$, we let $Y_{G_i} = (Y_i, d_{G_i})$ and $u_i\colon Y_{G_i} \to Y_i$ be as in Section 4.5 for $i = 1, 2$. For $i = 1, 2$, we denote $\nu_i = \nu_{Y_i}$ for the measures from Definition 3.1.

Our goal is to understand an analog of Corollary 4.17 for the quasiconformal surfaces Y_1 and Y_2 and for an arbitrary quasiconformal map

$$\Psi\colon Y_1 \to Y_2. \tag{26}$$

A technical difficulty is posed by the fact that Ψ can fail to satisfy Lusin's Condition (N) and (N^{-1}) with respect to Hausdorff measures. As a consequence, the pointwise results we prove hold only ν_1-almost everywhere.

We observe that the mapping

$$\widetilde{\Psi} = u_2^{-1} \circ \Psi \circ u_1\colon Y_{G_1} \to Y_{G_2}$$

is quasiconformal as a map between two Riemannian surfaces, it is classically differentiable $\mathcal{H}^2_{G_1}$-almost everywhere and it satisfies Lusin's Conditions (N) and (N^{-1}). Then Lemma 4.18 implies the following.

Lemma 5.1. *The differential*

$$D\Psi\colon (TY_1, \mathrm{N}_1) \to (TY_2, \mathrm{N}_2) \tag{27}$$

is well-defined ν_1*-almost everywhere. Moreover,*

$$D(\Psi^{-1}) \circ D\Psi = D\mathrm{id}_{Y_1}\colon (TY_1, \mathrm{N}_1) \to (TY_1, \mathrm{N}_1)$$

ν_1*-almost everywhere.*

Lemma 5.1 implies that we can compute the operator norm and the Jacobian of (27) ν_1-almost everywhere. These objects are defined as in Section 3. The chain rule implies that the inverse of (27) is well-defined ν_2-almost everywhere. We define pointwise outer and inner dilatations $K_O(\Psi) = \rho_\Psi^2/J_\Psi$ and $K_I(\Psi) = \rho_{\Psi^{-1}}^2 J_\Psi$, which are uniquely defined ν_1-almost everywhere.

Theorem 5.2. *The equalities $K_O(\Psi) = K_O(D\Psi)$ and $K_I(\Psi) = K_I(D\Psi)$ hold ν_1-almost everywhere. In particular, the pointwise dilatations satisfy*

$$K_O(D\Psi)K_I(D\Psi) \geq \rho^2(\mathrm{N}_1, \mathrm{N}_2 \circ D\Psi) \tag{28}$$

ν_1-almost everywhere. The equality (28) *holds ν_1-almost everywhere if and only if the differential*

$$D\Psi\colon (TY_1, \mathrm{N}_1) \to (TY_2, \mathrm{N}_2)$$

is a Banach–Mazur minimizer ν_1-almost everywhere.

Since Ψ is 1-quasiconformal if and only if the pointwise dilatations satisfy $K_O(\Psi) = \chi_{Y_1}$ and $K_I(\Psi) = \chi_{Y_1}$ ν_1-almost everywhere, Theorem 5.2 implies the following.

Corollary 5.3. *A quasiconformal homeomorphism $\Psi\colon Y_1 \to Y_2$ is 1-quasiconformal if and only if there exists a Borel function $\omega\colon Y_1 \to (0, \infty)$ such that $\mathrm{N}_2 \circ D\psi = \omega\, \mathrm{N}_1$ ν_1-almost everywhere.*

The rest of the section is spent on proving Theorem 5.2. To this end, let $B_0 \subset Y_{G_1}$ be a Borel set of $\mathcal{H}^2_{G_1}$-measure zero such that the restrictions of u_1 and $u_2 \circ \widetilde{\Psi}$ to $Y_{G_1} \setminus B_0$ satisfy Conditions (N) and (N^{-1}). The existence of such a set is guaranteed by Lemma 4.18 and by the fact that $\widetilde{\Psi}$ satisfies Conditions (N) and (N^{-1}). We fix such a set for the rest of this section.

Lemma 5.4. *The Jacobian J_Ψ of Ψ equals $J_2(D\Psi)$ $\mathcal{H}^2_{Y_1}$-almost everywhere in $Y_1 \setminus u_1(B_0)$. In particular, this identity holds ν_1-almost everywhere.*

Proof. The claim is local, so it suffices to consider the claim using isothermal charts of Y_1 and Y_2. The isothermal charts satisfy Conditions (N) and (N^{-1}) when restricted to the complement of $u_1(B_0)$ and $\Psi \circ u_1(B_0)$, respectively. Then the claim follows from the chain rule of Jacobians of linear maps between Banach spaces [AK00, Lemma 4.2] and the corresponding Euclidean results formulated in Lemma 4.4 and Corollary 4.6. □

We fix a Borel set $B_1 \supset B_0$ of zero $\mathcal{H}^2_{G_1}$-measure for which the following properties hold:

(a) The maps $Y_1 \setminus u_1(B_1) \ni y \mapsto \mathrm{N}_1(y)$ and $Y_2 \setminus \Psi(u_1(B_1)) \ni y \mapsto \mathrm{N}_2(y)$ are norms everywhere and also Borel measurable;
(b) The maps $Y_1 \setminus u_1(B_1) \ni y \mapsto D\Psi(y)$ and $Y_2 \setminus \Psi(u_1(B_1)) \ni y \mapsto D(\Psi^{-1})(y)$ are Borel measurable and the chain rule $D(\Psi^{-1}) \circ D\Psi = D\mathrm{id}_{Y_1}$ holds everywhere in $Y_1 \setminus u_1(B_1)$.

The set B_1 is defined to guarantee that the operator norms of $D\Psi$ and its inverse $D(\Psi^{-1})$ are well-defined everywhere in the complement of $u_1(B_1)$ and $\Psi(u_1(B_1))$, respectively. Also, the restriction of Ψ to the complement of $u_1(B_1)$ satisfies Conditions (N) and (N^{-1}).

Proposition 5.5. *The Borel functions $x \mapsto \|D\Psi\|(x)\left(\chi_{Y_1 \setminus u_1(B_1)}(x)\right) =: I_\Psi(x)$ and $x \mapsto \|D(\Psi^{-1})\|(x)\left(\chi_{Y_2 \setminus \Psi(u_1(B_1))}(x)\right) =: I_{\Psi^{-1}}(x)$ are minimal weak upper gradients of Ψ and Ψ^{-1}, respectively.*

Proof. First, for almost every non-constant absolutely continuous path $\theta\colon [0,1] \to Y_1$, the paths $u_1^{-1} \circ \theta$, $\Psi \circ \theta$, and $u_2^{-1} \circ \Psi \circ \theta$ are absolutely continuous, and the measures on $[0,1]$ induced by their metric speeds are absolutely continuous with respect to one another.

Second, Lemma 3.3 and Lemma 4.18 imply that the path families $\Gamma^+_{u_1(B_1)}$ and $\Gamma^+_{\Psi \circ u_1(B_1)}$ are negligible. The fact that I_Ψ is a minimal weak upper gradient of Ψ is a local property. For this reason, we fix isothermal parametrizations $u_i \circ \phi_i \colon U_i \to V_i$ for $i = 1, 2$ with $\Phi(V_1) = V_2$. Now for almost every path $\theta\colon [0,1] \to V_1$, Lemma 4.4 (c) holds for $(u_1 \circ \phi_1)^{-1} \circ \theta$ and for $(u_2 \circ \phi_2)^{-1} \circ (\Psi \circ \theta)$.

The previous two paragraphs imply that I_Ψ is a weak upper gradient of Ψ. To see the minimality of I_Ψ, we fix an upper gradient $\rho \in L^2_{\mathrm{loc}}(V_1)$ of Ψ. Fix a rectangle $R \subset U_1$ with a foliation $\gamma_t(s) = x_0 + tv + sw$, for $-1 \leq s, t \leq 1$, with v and w orthogonal. By arguing as in the proof of Lemma 4.7, Fubini's theorem implies for $\mathcal{L}^2$-almost every $x \in R \setminus \phi_1^{-1}(B_1)$ and $y = u_1(\phi_1(x))$,

$$\rho(y)\, \mathrm{N}_{u_1 \circ \phi_1}((x, w)) \geq \mathrm{N}_{\Psi \circ (u_1 \circ \phi_1)}((x, w)). \tag{29}$$

By slightly modifying the corresponding argument from the proof of Lemma 4.7, the inequality (29) implies $\rho(y) \geq I_\Psi(y)$ $\mathcal{H}^2_{Y_1}$-almost everywhere in $V_1 \setminus u_1(B_1)$. Therefore, I_Ψ is a representative of ρ_Ψ. The claim for $I_{\Psi^{-1}}$ follows from symmetry, given the fact from Lemma 3.3 that $\nu_1(B) = 0$ if and only if $\nu_2(\Psi(B)) = 0$. □

Proof of Theorem 5.2. Lemma 5.4 proves that the Jacobian of Ψ and the Jacobian $J_2(D\Psi)$ coincide ν_1-almost everywhere. Proposition 5.5 implies that the operator norm of $D\Psi$ determines the minimal weak upper gradient of Ψ ν_1-almost everywhere. This implies that the pointwise outer dilatation of Ψ is determined by the outer dilatation of $D\Psi$. Similar reasoning holds for the inner dilatation.

The inequality (28) follows from the fact that $D\Psi$ is a linear map between Banach spaces. The defining property of a Banach–Mazur minimizer yields that $D\Psi$ is a Banach–Mazur minimizer ν_1-almost everywhere if and only if the inequality (28) is an equality ν_1-almost everywhere. □

6. Applications

In Section 6.1, we establish the uniqueness of the uniformization map up to conformal diffeomorphisms. We prove Theorem 1.4 in Section 6.2.

6.1. Isothermal parametrizations using Riemannian surfaces. We start this section by considering global isothermal parametrizations of quasiconformal surfaces.

Definition 6.1. (Isothermal parametrizations) Let Z be a Riemannian surface and $\Psi\colon Z \to Y$ a quasiconformal map. The pair (Z, Ψ) is an *isothermal parametrization of* Y if for every other Riemannian surface $\widetilde{Z}$ and quasiconformal map $\widetilde{\Psi}\colon \widetilde{Z} \to Y$ we have that

$$(K_O(\Psi)K_I(\Psi))(z) \leq (K_O(\widetilde{\Psi})K_I(\widetilde{\Psi}))(\widetilde{z}) \tag{30}$$

for $\widetilde{z} = (\widetilde{\Psi}^{-1} \circ \Psi)(z)$ at $\mathcal{H}^2_Z$-almost every $z \in Z$. If the image of the map (Z, Ψ) is clear from the context, we say that (Z, Ψ) is *isothermal*. If also the domain is clear, we simply say that Ψ is *isothermal*.

The following theorem is a global version of Theorem 4.12.

Theorem 6.2. *The uniformization map u is isothermal. Moreover, the following are equivalent for every Riemannian surface Z and a quasiconformal homeomorphism $\Psi\colon Z \to Y$:*

(a) *The map Ψ is isothermal;*
(b) *The composition $u^{-1} \circ \Psi$ is conformal in the Riemannian sense;*
(c) *The pointwise dilatations satisfy*

$$(K_O(\Psi)K_I(\Psi)) \circ (\Psi^{-1} \circ u) = K_O(u)K_I(u) \tag{31}$$

$\mathcal{H}^2_G$-almost everywhere in Y_G.

(d) *The differential $D\Psi\colon (TZ, G) \to (TY, \mathrm{N})$ is a Banach–Mazur minimizer at $\mathcal{H}^2_Z$-almost every point $z \in Z$.*

Proof. Since $\Psi\colon Z \to Y$ is quasiconformal, Theorem 5.2 shows that

$$\begin{aligned} K_O(\Psi)K_I(\Psi) &\geq \rho^2(G_Z, \mathrm{N} \circ D\Psi) \\ &= \rho^2(G_Z \circ D(\Psi^{-1}) \circ Du, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi) \end{aligned} \tag{32}$$

$\mathcal{H}^2_Z$-almost everywhere in Z. The composition $G_Z \circ D(\Psi^{-1}) \circ Du$ is a norm induced by a Riemannian norm $\mathcal{H}^2_G$-almost everywhere in Y. Therefore the identity

$$\rho^2(G_Z \circ D(\Psi^{-1}) \circ Du, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi) = \rho^2(G, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi)$$

holds $\mathcal{H}^2_Z$-almost everywhere in Z. Applying Corollary 4.17 to the latter term shows that

$$\rho^2(G_Z \circ D(\Psi^{-1}) \circ Du, \mathrm{N} \circ Du) \circ (u^{-1} \circ \Psi) = (K_O(u)K_I(u)) \circ (u^{-1} \circ \Psi) \tag{33}$$

$\mathcal{H}^2_Z$-almost everywhere in Z. Now (32) and (33) show that

$$(K_O(\Psi)K_I(\Psi)) \circ (\Psi^{-1} \circ u) \geq K_O(u)K_I(u) \tag{34}$$

$\mathcal{H}^2_G$-almost everywhere in Y_G. We deduce from (34) that u is isothermal.

The map Ψ is isothermal if and only if the inequality in (34) is an equality $\mathcal{H}^2_G$-almost everywhere, and, by (32) and (33), this happens if and only if

$$D\Psi\colon (TZ, G_Z) \to (TY, \mathrm{N}) \tag{35}$$

is a Banach–Mazur minimizer $\mathcal{H}^2_Z$-almost everywhere. Hence, Properties (a), (c), and (d) are equivalent.

Having verified that Properties (a) and (d) are equivalent, we see that the property of being isothermal is a local property. Hence, the equivalence of Properties (a) and (b) follow after we verify the equivalence in the domain of an arbitrary isothermal chart of Z.

Let $\phi_1\colon U_1 \to V_1 \subset Z$ be an isothermal parametrization of a domain $V_1 \subset Z$. Then $\mathrm{N}_{\phi_1} = G_Z \circ D\phi_1 = \omega \left\|\cdot\right\|_2$ for some smooth function $\omega > 0$. Observe that $\Psi|_{V_1}$ is isothermal if and only if $\Psi \circ \phi_1$ is isothermal. Proposition 4.15 implies that the latter property holds if and only if $u^{-1} \circ (\Psi \circ \phi_1)$ is conformal in the Riemannian sense if and only if $u^{-1} \circ \Psi|_{V_1}$ is conformal in the Riemannian sense. This establishes the claim. □

Theorem 6.2 can be applied, for example, in the following manner. Given an isothermal map $\Phi\colon Z \to Y$ and a 1-quasiconformal homeomorphism $f\colon Y \to Y$, the mapping $\Phi^{-1} \circ f \circ \Phi\colon Z \to Z$ is conformal in the Riemannian sense. To see why, we first apply Corollary 5.3 to show that $f \circ \Phi$ is isothermal. Then Theorem 6.2 implies that $\Phi^{-1} \circ (f \circ \Phi)$ is conformal in the Riemannian sense. This fact imposes a structure and size restriction on the group generated by such f. A similar reasoning implies that for any given 1-quasiconformal homeomorphism $f\colon Y_1 \to Y_2$ and isothermal $\Phi_i\colon Z_i \to Y_i$, the homeomorphism $\Phi_2^{-1} \circ f \circ \Phi_1\colon Z_1 \to Z_2$ is conformal in the Riemannian sense.

6.2. Quasisymmetries. In this section, we investigate properties of isothermal charts of (Y, d_Y) under the assumption that (Y, d_Y) is compact, Ahlfors 2-regular, and linearly locally contractible.

6.2.1. Basic definitions. Let Y and Z be metric spaces. For a homeomorphism $\phi\colon Y \to Z$, $y \in Y$ and $r > 0$, let

$$L_\phi(y,r) = \sup\left\{d_Z(\phi(y), \phi(w)) \mid d_Y(y,w) \leq r\right\} \quad \text{and}$$
$$l_\phi(y,r) = \inf\left\{d_Z(\phi(y), \phi(w)) \mid d_Y(y,w) \geq r\right\}.$$

The map ϕ is *quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ for which for every $y \in Y$ and $0 < r_1, r_2 < \operatorname{diam} Y$,

$$L_\phi(y, r_1) \leq \eta\left(\frac{r_1}{r_2}\right) l_\phi(y, r_2). \tag{36}$$

Such a homeomorphism η is called a *(quasisymmetric) distortion function* of ϕ and we say that ϕ is η-quasisymmetric.

A metric surface Y is *Ahlfors* 2*-regular* if there exists a constant $C_A \geq 1$ such that for every $y \in Y$ and $\operatorname{diam} Y > r > 0$,

$$C_A^{-1} r^2 \leq \mathcal{H}_Y^2(\overline{B}(y,r)) \leq C_A r^2. \tag{37}$$

Here $\overline{B}(y,r) \subset Y$ is the closed ball of radius r centered at y.

Let $\lambda \geq 1$. A metric surface Y is λ*-linearly locally contractible* if for every $y \in Y$ and $0 < r < \frac{\operatorname{diam} Y}{\lambda}$, the metric ball $B(y,r)$ is contractible inside the ball $B(y, \lambda r)$. That is, there exists $y_0 \in B(y, \lambda r)$ and a continuous map $H\colon B(y,r) \times [0,1] \to B(y, \lambda r)$ such that $H(z,0) = z$ and $H(z,1) = y_0$ for every $z \in B(y,r)$.

6.2.2. Global parametrizations of compact surfaces. When we say that something in this section depends only on the *data of* Y, we mean that it depends only on C_A and λ, defined as above. Theorem 1.4 is an immediate consequence of Theorem 6.3 and Theorem 6.4.

Theorem 6.3. *Suppose that Y is an Ahlfors 2-regular metric surface that is linearly locally contractible and homeomorphic to $\mathbb{S}^2$. Then there exists a Riemannian distance $d_{G'}$ on Y of constant curvature 1 for which*

$$u' = \operatorname{id}_Y\colon Y_{G'} \to Y$$

is isothermal and η-quasisymmetric with η depending only on the data of Y.

Proof. Let $(Y, d_G) = Y_G$ denote the Riemannian surface obtained from Proposition 4.15. The surface has curvature equal to one. The uniformization map $u = \operatorname{id}_Y\colon Y_G \to Y$ is isothermal, and therefore $\frac{\pi}{2}$-quasiconformal.

We fix an isometry $I\colon \mathbb{S}^2 \to Y_G$, and choose three points $p_1, p_2, p_3 \in Y$ such that $d_Y(p_i, p_j) \geq \operatorname{diam} Y/2$ for each $i \neq j$. There exists a Möbius transformation $M\colon \mathbb{S}^2 \to \mathbb{S}^2$ so that $v' = u \circ I \circ M$ takes the north pole to p_1, the south pole to p_3, and a point from the equator to p_2. Since v' is $(\pi/2)$-quasiconformal, v' is η-quasisymmetric with η depending only on the data of Y; see [BK02, Proposition 9.1 and Section 3]. We denote $d_{G'}(x,y) := d_{\mathbb{S}^2}((I \circ M)^{-1}(x), (I \circ M)^{-1}(y))$ for all $x, y \in Y_G$ and set $Y_{G'} := (Y, d_{G'})$. Then the identity mapping $u'\colon (Y, d_{G'}) \to (Y, d_Y)$ is isothermal and η-quasisymmetric. □

Theorem 6.4. *Suppose that Y is a compact Ahlfors 2-regular and linearly locally contractible metric surface that is not homeomorphic to $\mathbb{S}^2$. Then the uniformization map*

$$u = \mathrm{id}_Y \colon Y_G \to Y \tag{38}$$

is η-quasisymmetric, where η depends only on the data of Y.

We postpone the proof of Theorem 6.4 until the end of this section.

Lemma 6.5. *Let Y be a quasiconformal surface and suppose that $\phi\colon \mathbb{D} \to V \subset Y$ is an η-quasisymmetric homeomorphism. Then ϕ is K-quasiconformal with K depending only on η. Moreover, there exists a $(4K/\pi)$-quasiconformal homeomorphism $\psi\colon \mathbb{D} \to \mathbb{D}$ such that $\psi(0) = 0$ and $\phi \circ \psi$ is an isothermal η'-quasisymmetric map with η' depending only on η.*

Proof. It follows from [Tys00, Theorem 3.13] that the outer dilatation of ϕ is bounded by some constant K_O depending only on η. Since V has a $(\pi/2)$-quasiconformal chart, the inner dilatation bound $(\pi/2)^2 K_O$ of ϕ follows from Euclidean regularity results [AIM09, Definition 3.1.1 and Theorem 3.7.7]. Therefore, ϕ is K-quasiconformal with $K = (\pi/2)^2 K_O$.

Proposition 4.2 and the Riemann mapping theorem, together with Proposition 4.3, imply the existence of a $(4K/\pi)$-quasiconformal mapping $\psi\colon \mathbb{D} \to \mathbb{D}$ with $\psi(0) = 0$ such that $\phi \circ \psi$ is isothermal. Corollary 3.10.4 of [AIM09] implies that ψ is $\widetilde{\eta}$-quasisymmetric with $\widetilde{\eta}$ depending only on the maximal dilatation of ψ. Hence, $\phi \circ \psi$ is $\eta \circ \widetilde{\eta}$-quasisymmetric. Since K and $\widetilde{\eta}$ depend only on η, the claim follows. □

Proposition 6.6. *Let Y_G be a complete Riemannian surface of curvature -1, 0, or 1 and*

$$\phi\colon \mathbb{D} \to Y_G$$

a conformal embedding. Suppose that Y_G is not homeomorphic to the sphere $\mathbb{S}^2$ or that

$$2 \operatorname{diam} \phi(\mathbb{D}) \le \operatorname{diam} Y_G.$$

Then there is a constant $2^{-1} > \beta > 0$ and a distortion function $\widetilde{\eta}$ for which

$$\phi(\beta\mathbb{D}) \subset B_G\left(\phi(0), \frac{l_\phi(0, \frac{1}{2})}{6}\right) \tag{39}$$

and the restriction of ϕ to $\beta\mathbb{D}$ is $\widetilde{\eta}$-quasisymmetric. The constant β and distortion function $\widetilde{\eta}$ are independent of ϕ and the surface Y_G.

Proof. First, suppose that Y_G is not homeomorphic to the sphere $\mathbb{S}^2$. The surface Y_G has a universal cover $\pi\colon \Omega \to Y_G$, where π is a local isometry and where Ω is either the hyperbolic disk $\mathbb{D}_{\mathrm{hyp}}$, the Euclidean plane $\mathbb{R}^2$, or the Riemann sphere $\mathbb{S}^2$. If $\Omega = \mathbb{S}^2$, the covering group of π is generated by the antipodal map.

Suppose that $\phi\colon \mathbb{D} \to Y_G$ is as in the claim. Then there exists a conformal embedding $\psi\colon \mathbb{D} \to \Omega$ for which $\phi = \pi \circ \psi$. Since ϕ is an embedding, so are ψ and the restriction of π to the image of ψ.

Claim (1): There exists a $2^{-1} > \beta' > 0$ and a distortion function η for which the restriction of ψ to $\beta'\mathbb{D}$ is η-quasisymmetric.

Proof of Claim (1): If Ω is the hyperbolic disk or the Euclidean plane, the existence of β' and η follows from Propositions 5 and 7 of [GW18] (which are stated for the case when ψ is orientable. However, the non-orientable case follows from the orientable one by applying the conjugate map $z \mapsto \overline{z}$ in the Euclidean unit disk $\mathbb{D}$).

Consider the case $\Omega = \mathbb{S}^2$. We rotate the sphere $\mathbb{S}^2$ in such a way that $\psi(0) = (0,0,-1)$. Moreover, we identify $\mathbb{S}^2$ with the extended plane $\mathbb{R}^2 \cup \{\infty\}$ using the stereographic projection $\tau\colon \mathbb{S}^2 \to \mathbb{R}^2 \cup \{\infty\}$ which fixes the *equator* $\mathbb{S}^1 = \mathbb{S}^1 \times \{0\} \subset \mathbb{R}^3$ and maps the *south pole* $(0,0,-1)$ to 0. With this identification, τ maps the southern hemisphere to the unit disk $\mathbb{D}$. Recall that τ is a conformal map.

By construction, the restriction of π to the image of ψ is injective. We claim that $\psi(10^{-1}\mathbb{D})$ is contained in the southern hemisphere. We prove this by employing the following growth estimate for conformal embeddings [Dur83, Theorem 2.6]: If $0 < r < 1$ and $\|x\|_2 = r$, then

$$\|D(\tau \circ \psi)\|(0)\frac{r}{(1+r)^2} \leq \|\tau \circ \psi\|_2(x) \leq \|D(\tau \circ \psi)\|(0)\frac{r}{(1-r)^2}. \tag{40}$$

If $\psi(10^{-1}\mathbb{D})$ is not contained in the southern hemisphere, then (40) implies that

$$\frac{81}{10} \leq \|D(\tau \circ \psi)\|(0). \tag{41}$$

Then (40) and (41) imply that $\tau \circ \psi(2^{-1}\mathbb{D})$ contains the closed unit disk $\overline{\mathbb{D}}$. This is a contradiction with the injectivity of π in the image of ψ.

The restriction of the stereographic projection τ to the southern hemisphere is a biLipschitz map. Also, the restriction of $\tau \circ \psi$ to the disk $10^{-1}\mathbb{D}$ is η'-quasisymmetric with η' independent of ψ [AIM09, Theorem 3.6.2]. The existence of β' and η follows.

Claim (2): Let $\beta' > 0$ be as in Claim (1). There exists a constant $\beta' > \beta'' > 0$ such that

$$\psi(\beta''\mathbb{D}) \subset B_{d_\Omega}\left(\psi(0), \frac{l_\psi(0,\frac{1}{2})}{6}\right). \tag{42}$$

Proof of Claim (2): Suppose that $\beta' > 0$ and η are as in Claim (1) and consider $\beta' > \beta'' > 0$. Since the restriction of ψ to the disk $\beta'\mathbb{D}$ is η-quasisymmetric,

$$L_\psi(0,\beta'') \leq \eta\left(\frac{\beta''}{\beta'}\right) l_\psi(0,\beta') \leq \eta\left(\frac{\beta''}{\beta'}\right) l_\psi\left(0,\frac{1}{2}\right).$$

Therefore, it suffices to pick $\beta'' > 0$ so small that $\eta\left(\frac{\beta''}{\beta'}\right) < \frac{1}{6}$. Claim (2) follows.

We complete the proof of the claim using Claims (1) and (2) (when Y_G is not homeomorphic to $\mathbb{S}^2$). Recall that the restriction of π to $\psi(\mathbb{D})$ is injective. Let $\beta'' > 0$ be as in Claim (2). Since

$$B_{d_\Omega}\left(\psi(0), l_\psi\left(0,\frac{1}{2}\right)\right) \subset \psi\left(2^{-1}\mathbb{D}\right),$$

the restriction of π to $B_{d_\Omega}\left(\psi(0), 6^{-1}l_\psi\left(0,\frac{1}{2}\right)\right)$ is an isometry onto its image. This is an immediate consequence of the fact that

$$d_G(x,y) = \inf\left\{d_\Omega(x',y') \mid x' \in \pi^{-1}(x) \text{ and } y' \in \pi^{-1}(y)\right\}.$$

In conclusion, the map ψ can be replaced with ϕ and Ω with Y_G everywhere in Claims (1) and (2). We define $\beta = \beta''$ as in Claim (2) and $\widetilde{\eta} = \eta$ as in Claim (1) to conclude the proof of Proposition 6.6 when Y_G is not homeomorphic to $\mathbb{S}^2$.

We are left to consider the case when Y_G is homeomorphic to $\mathbb{S}^2$. Then there exists an isometry $\pi\colon \mathbb{S}^2 \to Y_G$. Therefore, there exists a conformal embedding $\psi\colon \mathbb{D} \to \mathbb{S}^2$ for which $\phi = \pi \circ \psi$. By rotating the sphere, we can assume that $\psi(0)$ is the south pole. The diameter bound on the image of ϕ implies that $\psi(10^{-1}\mathbb{D})$ is contained in the southern hemisphere. The rest of the proof is argued as above. □

For the rest of the section, we assume that $\operatorname{diam} Y = 1$. This can be done without loss of generality since the properties we study are left unchanged by rescaling. The diameter normalization is needed for the results we use from [GW18]. We formulate the following corollary of [GW18, Theorem 9] and Lemma 6.5.

Proposition 6.7. *There is a quantity $A_0 \geq 1$ and a distortion function η, each depending only on the data of Y, such that for every $0 < R \leq \frac{1}{A_0}$ and $y \in Y$, there is a neighbourhood U of y for which*

(a) $B(y, \frac{R}{A_0}) \subset U \subset B(y, A_0 R)$;
(b) *there exists an η-quasisymmetric homeomorphism $f \colon U \to \mathbb{D}$ that is an isothermal chart of Y with $f(y) = 0$.*

The only difference between [GW18, Theorem 9] and Proposition 6.7 is the condition that f is an isothermal chart. We state next a modified version of [GW18, Lemma 10].

Lemma 6.8. *Suppose that $2^{-1} > \beta > 0$ is the constant in Proposition 6.6 and η is as in Proposition 6.7. Then there exist radii α and $r_0 > 0$ and a positive integer n such that the following statements hold.*

(a) *There exists an atlas $\mathcal{A}_\beta = \{(U_j, f_j)\}_{j=1}^n$, where $f_j(U_j) = \mathbb{D}$ and each f_j is an η-quasisymmetric isothermal chart of Y.*
(b) *Let $x_j = f_j^{-1}(0)$. The collection $\{B(x_j, r_0)\}_{j=1}^n$ is pairwise disjoint.*
(c) *The collection $\{B(x_j, 2r_0)\}_{j=1}^n$ covers Y.*
(d) *For each $j = 1, \dots, n$, we have $B(x_j, 10r_0) \subset U_j$ and*

$$\alpha\mathbb{D} \subset f_j(B(x_j, r_0)) \subset f_j(B(x_j, 10r_0)) \subset \beta\mathbb{D}.$$

The radii α and r_0, and the integer n depend only on the data of Y and β.

Lemma 6.8 is proved exactly as [GW18, Lemma 10], but instead of applying [GW18, Theorem 9] as in the proof of [GW18, Lemma 10], we apply Proposition 6.7.

Proof of Theorem 6.4. Let $(Y, d_G) = Y_G$ denote the Riemannian surface obtained from Proposition 4.15. The surface Y_G has curvature equal to 1, 0, or -1 and is not homeomorphic to $\mathbb{S}^2$. Let $u = \mathrm{id}_Y \colon Y_G \to Y$ denote the uniformization map.

Recall that the claim is that u is quasisymmetric with distortion depending only on the data of Y. It suffices to prove that $v = u^{-1} = \mathrm{id}_Y \colon Y \to Y_G$ is quasisymmetric with quasisymmetric distortion function depending only on the data of Y.

For the duration of the proof, we use the notations introduced in Lemma 6.8, and denote $\psi_j = v \circ f_j^{-1} \colon \mathbb{D} \to Y_G$. We first observe that for each $j = 1, 2, \dots, n$,

$$v\big|_{B(x_j, 10r_0)} = \psi_j \circ f_j\big|_{B(x_j, 10r_0)} \quad \text{is } \eta_1\text{-quasisymmetric} \tag{43}$$

with $\eta_1 = \widetilde{\eta} \circ \eta$, where η is from Lemma 6.8 and $\widetilde{\eta}$ from Proposition 6.6. Recall that $\widetilde{\eta}$ is independent of Y and the η depends only on the data of Y.

Next, we claim that for each $x, x' \in Y$ with $d_Y(x, x') = 4r_0$,

$$d_G(v(x), v(x')) \geq \delta = C^{-1} \operatorname{diam} Y_G, \tag{44}$$

where C depends only on the data of Y. To this end, since $\{B(x_j, 2r_0)\}_{j=1}^n$ covers Y, the union $\bigcup_{j=1}^n \psi_j(\beta\mathbb{D})$ covers Y_G. As Y_G is connected, we conclude

$$\max_j \{\operatorname{diam} \psi_j(\beta\mathbb{D})\} \geq \frac{\operatorname{diam} Y_G}{n}. \tag{45}$$

Consider a pair of indices $i,k = 1,2,\dots,n$ with $d_Y(x_i,x_k) < 4r_0$. Then $x_k \in B(x_i, 10r_0)$, so Lemma 6.8 implies $d_G(v(x_i), v(x_k)) \leq L_{\psi_i}(0,\beta)$. If i and k are distinct, $d_Y(x_i,x_k) > r_0$, so the same lemma implies $d_G(v(x_i),v(x_k)) \geq \ell_{\psi_i}(0,\alpha)$. Observe that

$$\ell_{\psi_i}(0,\alpha) \geq \frac{L_{\psi_i}(0,\beta)}{\widetilde{\eta}\left(\frac{\beta}{\alpha}\right)} \geq \frac{\operatorname{diam}\psi_i(\beta\mathbb{D})}{2\widetilde{\eta}\left(\frac{\beta}{\alpha}\right)}.$$

We have now verified that the quantities

$$\ell_{\psi_i}(0,\alpha), L_{\psi_i}(0,\beta), \operatorname{diam}\psi_i(\beta\mathbb{D}), d_G(v(x_i),v(x_k)) \tag{46}$$

are comparable with constants depending only on the data of Y.

Observe that for every pair $i,j = 1,2,\dots,n$ with $i \neq j$, there exists $m \leq n$ and a chain $\{x_{i_k}\}_{k=1}^m$ with $x_{i_1} = x_i$ and $x_{i_m} = x_j$, and $4r_0 > d_Y(x_{i_k}, x_{i_{k+1}}) > r_0$ for each $k = 1,2,\dots m-1$. Recall from Lemma 6.8 that n depends only on the data of Y. This fact and (46) imply that there exists $C_0 > 0$, depending only on the data of Y, such that for every pair $i,j = 1,2,\dots,n$,

$$\ell_{\psi_i}(0,\alpha) \geq \frac{\operatorname{diam}\psi_j(\beta\mathbb{D})}{C_0}. \tag{47}$$

Given the inequalities (45) and (47), we have

$$\ell_{\psi_i}(0,\alpha) \geq \frac{\operatorname{diam} Y_G}{nC_0} \quad \text{for every } i. \tag{48}$$

Suppose that $x, x' \in Y$ with $d_Y(x,x') = 4r_0$. Then there exist i and k such that $d_Y(x,x_i) < 2r_0$ and $d_Y(x',x_k) < 2r_0$. As $2r_0 \leq d_Y(x',x_i) \leq 6r_0$, we have $x, x', x_k \in B(x_i, 10r_0)$. Then (43) implies

$$d_G(v(x'),v(x)) \geq \frac{d_G(v(x'),v(x_i))}{\eta_1(3/2)}. \tag{49}$$

Since $x' \in Y \setminus B_Y(x_i, r_0)$, the inequality (48) yields that

$$d_G(v(x'),v(x_i)) \geq \ell_{\psi_i}(0,\alpha) \geq \frac{\operatorname{diam} Y_G}{nC_0}. \tag{50}$$

The inequality (44) follows from the inequalities (49) and (50).

Lastly, Lemma 6.8 implies that $L = 8r_0$ is a Lebesgue number of $\{B(x_j,10r_0)\}_{j=1}^n$. Then a theorem by Tukia and Väisälä, as formulated in [GW18, Theorem 4], states that v is η_2-quasisymmetric, where η_2 depends only on η_1 from (43) and the ratios $\frac{\operatorname{diam} Y}{L} = \frac{1}{L}$ and $\frac{\operatorname{diam} Y_G}{\delta}$, where δ is from (44). Hence η_2 depends only on the data of Y. This implied the claim. □

7. Concluding remarks

The classical uniformization theorem states that every smooth Riemannian surface Y is 1-quasiconformally equivalent to a complete Riemannian surface of curvature -1, 0, or 1. For such Y, our uniformization map $u\colon Y_G \to Y$ is 1-quasiconformal. Given this observation, we pose the following question.

Open Problem A. Let Y be a quasiconformal surface. Is Y 1-quasiconformally equivalent to a metric surface Z with desirable geometric properties?

One might ask if Open Problem A holds in such a way that Z is bi-Lipschitz equivalent to the space Y_G obtained from Proposition 4.15, or even if the space is $\sqrt{2}$-bi-Lipschitz equivalent to Y_G.

When (Y, d_Y) is constructed from a sufficiently regular norm field on a smooth surface, such a Z can be constructed using John's theorem and regularity results for Beltrami differential equations. However, we cannot always take in Open Problem A the surface Z to be bi-Lipschitz equivalent to Y_G, or to any other Riemannian surface.

Theorem 7.1. [IRar, Theorem 1.6] *There exists a distance d on $\mathbb{R}^2$ such that the identity map $\iota\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, d)$ is an isothermal parametrization, but ι does not factor as $\iota = \widehat{\iota} \circ P$, where (Z, d_Z) is a metric surface, $\widehat{\iota}\colon (Z, d_Z) \to (\mathbb{R}^2, d)$ is quasiconformal with distortion $H(\widehat{\iota}) < \sqrt{2}$ and $P\colon (\mathbb{R}^2, \|\cdot\|_2) \to (Z, d_Z)$ is bi-Lipschitz.*

Here $H(\widehat{\iota}) = \operatorname{ess\,sup} \sqrt{K_O(\widehat{\iota})(x) K_I(\widehat{\iota})(x)}$ for the pointwise dilatations of $\widehat{\iota}$. Theorem 7.1 shows that we cannot require in Open Problem A Z to be bi-Lipschitz equivalent to Y_G, even if we allow for non-conformal distortion $1 < H(\widehat{\iota}) < \sqrt{2}$. We note that the isothermal parametrization ι in Theorem 7.1 has distortion exactly $H(\iota) = \sqrt{2}$.

It is not clear whether Z in Open Problem A can be chosen in such a way that Z is locally quasisymmetrically equivalent to some Riemannian surface, or even what is the answer to the following problem.

Open Problem B. Is every quasiconformal surface 1-quasiconformally equivalent to a metric surface Z that is locally Ahlfors 2-regular and locally linearly locally contractible?

We note that Open Problem A is trivially true for each quasiconformal surface for which the uniformization map is 1-quasiconformal. This holds, for example, when (Y, d_Y) has bounded integral curvature [Res01] and [BL03], or $(Y, d_Y) \subset \mathbb{R}^N$ for $N \geq 2$.

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Quasiconformal geometry and removable sets for conformal mappings

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QUASICONFORMAL GEOMETRY AND REMOVABLE SETS FOR CONFORMAL MAPPINGS

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Abstract. We study metric spaces defined via a conformal weight, or more generally a measurable Finsler structure, on a domain $\Omega \subset \mathbb{R}^2$ that vanishes on a compact set $E \subset \Omega$ and satisfies mild assumptions. Our main question is to determine when such a space is quasiconformally equivalent to a planar domain. We give a characterization in terms of the notion of planar sets that are removable for conformal mappings. We also study the question of when a quasiconformal mapping can be factored as a 1-quasiconformal mapping precomposed with a bi-Lipschitz map.

Contents

1. Introduction

1.1. **Overview.** Let (X, d_X) and (Y, d_Y) be metric spaces with locally finite Hausdorff 2-measure. A homeomorphism $f\colon X \to Y$ is K-*quasiconformal* if there exists $K \geq 1$ such that

$$K^{-1} \operatorname{mod} \Gamma \leq \operatorname{mod} f\Gamma \leq K \operatorname{mod} \Gamma \tag{1}$$

for all path families Γ in X, where $\operatorname{mod} \Gamma$ denotes the conformal modulus of Γ. The map f is *quasiconformal* if it is K-quasiconformal for some $K \geq 1$. This definition is generally referred to as the *geometric definition* of quasiconformal mappings, and it is one of several possible generalizations of Euclidean quasiconformal maps to the setting of metric spaces. The definition of modulus, as well as other terms used in this introduction, is reviewed in Section 2.

The *quasiconformal uniformization problem* asks one to determine which metric spaces can be mapped onto a domain in the Euclidean plane or the 2-sphere by a mapping that is quasiconformal, according to one of the several definitions. This problem is based on the classical uniformization theorem, which states that every simply connected Riemannian 2-manifold is *conformally* equivalent to either the

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Euclidean plane, the 2-sphere, or the hyperbolic plane. Outside the 2-dimensional Riemannian setting, conformality is a very strong property, and it is natural to require only quasiconformality. Motivation comes from connections to neighboring fields such as complex dynamics [BM17] and geometric group theory [Bon06].

In the following, let (X, d) be a metric space homeomorphic to a 2-dimensional manifold and having locally finite Hausdorff 2-measure. Such a space is referred to in this paper as a *metric surface*. By *quasiconformal surface*, we mean a metric surface (X, d) that is quasiconformally equivalent to a smooth Riemannian 2-manifold.

The uniformization problem for metric surfaces has been studied recently using various axiomatic approaches. Rajala has proved that a metric surface X homeomorphic to $\mathbb{R}^2$ is a quasiconformal surface if and only if it satisfies a condition called *reciprocality* (Definition 2.8 below) [Raj17]. Roughly speaking, this condition says that X does not have too many more rectifiable paths, as quantified by conformal modulus, than Euclidean space. In this case, as shown in [Rom19], there exists a quasiconformal map $f\colon X \to \Omega \subset \mathbb{R}^2$ that satisfies the modulus inequality

$$\frac{2}{\pi} \operatorname{mod} \Gamma \leq \operatorname{mod} f\Gamma \leq \frac{4}{\pi} \operatorname{mod} \Gamma$$

for all path families Γ in X. This inequality is sharp, as can be shown by considering the plane equipped with either the $\|\cdot\|_1$- or $\|\cdot\|_\infty$-norm. These results are extended to arbitrary metric surfaces in [Iko21]. A different approach was taken in a series of papers of Lytchak and Wenger [LW17], [LW18], [LW20] based on the assumption that the space satisfies a *quadratic isoperimetric inequality*.

The goal of the present paper is to understand the uniformization results described above in the context of concrete constructions of metric surfaces. We study a general scheme for constructing surfaces based on specifying a measurable Finsler structure on a planar domain that vanishes on some subset of the plane. The natural problem is to decide when this construction yields a quasiconformal surface.

We provide an answer by linking the uniformization problem for metric surfaces to a separate topic in complex analysis: removable sets for classes of holomorphic functions. There are several notions of removability; see [You15] for a recent survey. For us, the relevant definition is the following. A compact set $E \subset \mathbb{R}^2$ is *removable for conformal mappings* if every conformal embedding $f\colon \mathbb{R}^2 \setminus E \to \widehat{\mathbb{R}}^2$ extends to a conformal mapping $\widetilde{f}\colon \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$, that is, to a Möbius transformation. Here, $\widehat{\mathbb{R}}^2$ denotes the extended plane, which can be identified with $\mathbb{S}^2$ via stereographic projection. There seems to be no standard terminology for sets satisfying this condition. This is referred to as *S-removability* in the survey [You15], while the terms *set of absolute area zero* and *neglible set for extremal distance* are also used. Note that this is different from the notion of *conformal removability*, which requires that every *homeomorphism* of $\widehat{\mathbb{R}}^2$ that is conformal on the set $\widehat{\mathbb{R}}^2 \setminus E$ be a Möbius transformation.

This connection to removable sets is natural in hindsight but does not appear to have been made before. On the other hand, removable sets are inherently connected to a different type of uniformization problem, namely of multiply connected planar domains onto some canonical class of domain, typically slit domains or circle domains. We recall that whether an arbitrary planar domain can be mapped conformally onto a circle domain is the well-known *Koebe Kreisnormierungsproblem* [HS93]. We hope the present paper will add a new perspective on these various topics.

1.2. **Motivating examples.** A basic observation, made in Example 2.1 in [Raj17], is that not every metric surface is a quasiconformal surface. A simple example is the following. Define a length pseudometric d_σ on $\mathbb{R}^2$ via the conformal weight

$\sigma = \chi_{\mathbb{R}^2 \setminus \mathbb{D}}$. More precisely, we define the σ-length of an absolutely continuous path γ to be $\ell_\sigma(\gamma) = \int_\gamma \sigma \, ds$, and let $d_\sigma(x,y) = \inf \ell_\sigma(\gamma)$, the infimum taken over all absolutely continuous paths γ connecting x and y. If we let X be the quotient space of $\mathbb{R}^2$ formed by collapsing the unit disk to a single point, then d_σ induces a metric on X, denoted by $\widetilde{d}_\sigma$, that is locally Euclidean outside the origin. The space $(X, \widetilde{d}_\sigma)$, while being homeomorphic to $\mathbb{R}^2$, is not quasiconformally equivalent to a planar domain. This is because the family of paths in X that intersect the collapsed point has positive modulus, while the modulus of the family of paths intersecting a single point in the Euclidean plane is zero. This example is included as Example 11.3 in [LW18].

A second example, and the one that comprises Example 2.1 in [Raj17], is a continuous conformal weight σ that vanishes on a Cantor set E of positive area. In this case, d_σ is a metric on $\mathbb{R}^2$, and the identity map $(\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, d_\sigma)$ is a homeomorphism. Nevertheless, the vanishing of the weight increases the conformal modulus of path families in $(\mathbb{R}^2, d_\sigma)$ in a way incompatible with admitting a quasiconformal parametrization by $\mathbb{R}^2$.

At the other extreme, it is not hard to show that if the analogous construction is carried out for a set E with Hausdorff dimension smaller than one, then the resulting space is quasiconformally equivalent to the plane. Indeed, the set E is then negligible for length and so has no effect on modulus. What happens in the intermediate situation—when the Hausdorff dimension satisfies $1 \leq \dim_{\mathcal{H}} E < 2$ or when $\mathcal{H}^2(E) = 0$—is not *a priori* clear and is one of the motivations of our work.

Similar constructions appear in a number of related contexts. One of these is the notion of *strong A_∞-weight* introduced by David and Semmes in [DS90]. Such a weight determines a metric on $\mathbb{R}^2$ that is Ahlfors 2-regular and quasisymmetrically equivalent to the plane. Conversely, the Jacobian of a quasisymmetric mapping from $\mathbb{R}^2$ to an Ahlfors 2-regular metric space induces a strong A_∞-weight on $\mathbb{R}^2$. We do not define this term here but refer the reader to [Sem96, Def. 1.5]. Such weights appear naturally when trying to recognize metric spaces that are bi-Lipschitz embeddable in some Euclidean space. See [DS90, Sem93, Sem96, Laa02, Bis07] for various contributions to this topic. A separate set of papers [BKR98, BHR01] studies metrics on the unit disk defined by conformal weights satisfying a Harnack-type inequality and an area growth condition, and shows that a number of results of classical complex analysis have natural analogues in this setting. All of the metric surfaces constructed in these two sets of papers are quasiconformally equivalent to a planar domain.

In the above examples, when a space fails to be a quasiconformal surface, this is due to the space "collapsing" on the set E where the weight vanishes. In fact, it may be the case that this is essentially the only way that a metric surface can fail to admit a quasiconformal parametrization. This is made precise by the following question of Rajala and Wenger [Raj20].

Question 1.1. Let (X,d) be a metric space homeomorphic to $\mathbb{R}^2$ with locally finite Hausdorff 2-measure. Does there exist a domain $\Omega \subset \mathbb{R}^2$ and a surjective continuous monotone mapping $f \colon \Omega \to X$ such that f is in the metric Sobolev space $N^{1,2}_{\mathrm{loc}}(\Omega, X)$ and satisfies the one-sided dilatation condition

$$g_f^2(x) \leq K J_f(x)$$

for some constant $K \geq 1$ and almost every $x \in \Omega$?

Here, g_f is the minimal weak upper gradient of f and J_f is the Jacobian of f; see Section 2.2. We say that $f \colon \Omega \to X$ is *monotone* if the preimage of every point $x \in X$ is a connected and compact subset of Ω.

1.3. **Setting and main results.** Let Ω be a planar domain and $E \subset \Omega$ be a compact set that does not separate Ω. We consider a measurable seminorm field $N\colon \Omega \times \mathbb{R}^2 \to [0,\infty)$ that vanishes exactly on the set E and satisfies certain mild assumptions, namely lower semicontinuity, local boundedness, and having locally bounded distortion. The seminorm at the point $x \in \Omega$ is denoted throughout this paper by N_x. We think of N as a Finsler structure on $\mathbb{R}^2$, determining a Finsler metric on $\mathbb{R}^2$, although requiring no regularity beyond the previous assumptions.

For conciseness, and since N_x is a norm for all $x \in \Omega \setminus E$, we use the term *norm field* and not *seminorm field* throughout this paper when referring to N. A norm field N satisfying the above hypotheses is said to be *admissible* (Definition 3.1). We define the *N-length* of an absolutely continuous path $\gamma\colon I \to \Omega$ by

$$\ell_N(\gamma) = \int_I N \circ D\gamma(t)\, dt. \tag{2}$$

In interpreting (2), note that the base point of N is understood to be $\gamma(t)$ even though this is omitted from the notation. One then obtains a pseudometric d_N on Ω by setting $d_N(x,y) = \inf \ell_N(\gamma)$, the infimum taken over all absolutely continuous paths γ from x to y contained in Ω. Let $\mathcal{E}_N$ denote the collection of equivalence classes of points in $\mathbb{R}^2$, declaring x to be equivalent to y if $d_N(x,y) = 0$. Then d_N determines a metric on the quotient space $\mathbb{R}^2/\mathcal{E}_N$ denoted by $\widetilde{d}_N$. In Section 3, we describe this construction in more detail.

We make the following definition.

Definition 1.2. The admissible norm field N is *reciprocal* if the corresponding space $(\Omega/\mathcal{E}_N, \widetilde{d}_N)$ is reciprocal (Definition 2.8).

The natural problem is to characterize as best as possible those norm fields N that are reciprocal. Our first result is the following.

Theorem 1.3. Let $\Omega \subset \mathbb{R}^2$ be a domain and $E \subset \Omega$ a compact set. If E is removable for conformal mappings, then every admissible norm field $N\colon \Omega \times \mathbb{R}^2 \to [0,\infty)$ that vanishes exactly on E is reciprocal.

Recall that our definition of admissibility includes the statement that N is locally bounded. It turns out that this assumption can be relaxed. In Proposition 4.5, we show that Theorem 1.3 still holds provided there exists some $p > 2$ such that the maximal stretching $L(N)$ is in $L^p_{\mathrm{loc}}(\Omega)$. This generalization follows from Theorem 1.3 by an approximation argument.

Next, we consider whether some converse to Theorem 1.3 holds. Observe first that the strongest possible converse to Theorem 1.3 is false: a reciprocal norm field N may vanish on a set E that is not removable for conformal mappings. As a simple example, take $E \subset \mathbb{R}^2$ to be a snowflake arc and let $N = \chi_{\mathbb{R}^2\setminus E}\|\cdot\|_2$. Since $\mathcal{H}^1_{\|\cdot\|_2}(|\gamma| \cap E) = 0$ for every absolutely continuous path γ, we see that d_N actually coincides with the Euclidean metric. However, it is a consequence of the Riemann mapping theorem that any compact set that is removable for conformal mappings is totally disconnected.

On the other hand, if one requires that the norm field N decays fast enough near E and N is reciprocal, then examples of the type just described are not possible. To illustrate this, consider two admissible norm fields N_1 and N_2 that satisfy $N_1 \leq N_2$. Every path that has finite N_2-length also has finite N_1-length, while the opposite may fail to be true for a large family of paths. In this sense, the space generated by the smaller norm field N_1 has more rectifiable paths and the reciprocality condition is harder to satisfy. This leads to the following partial converse to Theorem 1.4.

Theorem 1.4. Let $\Omega \subset \mathbb{R}^2$ be a domain and $E \subset \Omega$ a compact set that does not separate Ω, and let $N_p(x) = \min\left\{1, d_{\|\cdot\|_2}(x,E)^p\right\} \|\cdot\|_2$. If N_p is reciprocal for some $p > \max\{\dim_{\mathcal{H}} E - 1, 0\}$, then the set E is removable for conformal mappings.

Our method of proof actually yields a stronger conclusion. The relevant property of the norm field $N = N_p$, verified in Lemma 5.1 below, is that the quotient map π_N maps E onto a set of zero 1-dimensional Hausdorff measure with respect to the metric $\widetilde{d}_N$. Thus, for any reciprocal norm field N such that $\pi_N(E)$ has 1-dimensional Hausdorff measure zero, the corresponding set E on which N vanishes is removable for conformal mappings. For example, one can show that, if E is contained in a continuum F satisfying $\mathcal{H}^1_{\|\cdot\|_2}(F) < \infty$, $\pi_N(E)$ has 1-dimensional Hausdorff measure zero for any admissible norm field N vanishing exactly on E. For such compact sets, the strongest converse to Theorem 1.3 holds. That is, if any admissible norm field N vanishing exactly on E is reciprocal, then E is removable for conformal mappings.

The lower bound for p in Theorem 1.4 is sharp. Consider an arc $E \subset \mathbb{R}^2$ that is bi-Lipschitz equivalent to $([0,1], |\cdot|^{1/d})$ for some $d \in (1,2)$. Then E is a snowflake arc of Hausdorff dimension d. It follows from [Sem96, Theorem 6.3] that the square of the weight $\sigma_{d-1}(x) = \min\left\{1, d_{\|\cdot\|_2}(x,E)^{d-1}\right\}$ is a strong A_∞-weight, as defined in [Sem96, Definition 1.5], and hence the norm field N_{d-1} is reciprocal. However, the arc E is not removable for conformal mappings.

Theorems 1.3 and 1.4 show that reciprocal norm fields are almost characterized by whether the set on which they vanish is removable for conformal mappings. We now mention a few facts about removable sets for conformal mappings that are known, many of them coming from an influential paper of Ahlfors–Beurling [AB50]. First, every compact set of positive Hausdorff 2-measure is non-removable. Second, every compact set of zero Hausdorff 1-measure is removable. More intriguingly, for Cantor sets $E \subset \mathbb{R} \times \{0\}$ of positive Hausdorff 1-measure, both outcomes are possible. In [AB50], Ahlfors and Beurling give examples of Cantor sets in $\mathbb{R} \times \{0\}$ of positive $\mathcal{H}^1$-measure that are removable for conformal maps, as well as such Cantor sets that are non-removable. A similar example in the related context of circle domain uniformization can be found as Theorem 11.1 of an early version of a paper of Schramm [Sch95]. Next, by Theorem 10 in [AB50] and Proposition 3.3 in [KKR19], removable sets for conformal mappings are *metrically removable*: for every $\varepsilon > 0$, each pair of points $x, y \in \mathbb{R}^2$ can be connected by a curve disjoint from $E \setminus \{x, y\}$ that has length at most $\|x-y\|_2 + \varepsilon$. See [HH08] and [KKR19] for more on the topic of metric removability. Removable sets for conformal mappings are also examples of the *quasiextremal distance exceptional sets* considered in [GM85] and the related literature. Finally, an equivalent definition can be given by replacing the word "conformal" with "quasiconformal" in the definition [You15, Prop. 4.7]. Thus the property of removability is invariant under quasiconformal mappings of the complementary domain.

This should be compared with the notion of *removable sets for bounded analytic functions*. The problem of characterizing such sets is known as *Painlevé's problem* and has received considerable attention, with a satisfactory resolution obtained by Tolsa in [Tol03]. We note here that this is a stronger notion of removability: every set that is removable for bounded analytic functions is removable for conformal mappings. See Proposition 4.3 of [You15] for a proof. For example, a removable set for bounded analytic functions must have Hausdorff dimension at most 1. Moreover, according to David's resolution of Vitushkin's conjecture [Dav98], a compact set E with finite Hausdorff 1-measure is removable for bounded analytic functions if and only if it is purely 1-unrectifiable.

Finally, we remark that the notion of uniformly disconnected sets provides a further class of examples to which these results apply. In [Sem96], Semmes studies metrics of the form d_{N_p}, where N_p is as in Theorem 1.4, with the additional assumption that the set E is *uniformly disconnected*, meaning that there exists $\varepsilon > 0$ with the property that, for any two distinct points $x, y \in E$, there is no sequence of points $x = x_0, x_1, \ldots, x_m = y$ in E satisfying $\|x_{j-1} - x_j\|_2 < \varepsilon\|x - y\|_2$ for all $j \in \{1, \ldots, m\}$. He proves that for such an E and every $p > 0$, the square of the weight $\sigma_p(x) = \min\{1, d_{\|\cdot\|_2}(x, E)^p\}$ is a strong A_∞-weight and hence the norm field N_p in Theorem 1.4 is reciprocal. Therefore Theorem 1.4 implies that uniformly disconnected Cantor sets are removable for conformal mappings. This removability can alternatively be deduced in many ways from the existing literature. Note in particular that a uniformly disconnected set E can have Hausdorff dimension arbitrarily close to 2.

1.4. **Factorization of quasiconformal mappings.** This section is motivated by the following factorization problem. Consider a quasiconformal surface (X, d) and corresponding isothermal parametrization $f \colon \Omega \to X$, where Ω is a smooth Riemannian surface. Following [Iko21], a quasiconformal homeomorphism $f \colon \Omega \to X$ is *isothermal* if it is distortion-minimizing at almost every point in a suitable sense. Roughly speaking, the *pointwise distortion* of f at x is the aspect ratio of the image of a small ball centered at x. The existence of an isothermal parametrization for every quasiconformal surface is established in [Iko21, Theorem 6.2]. See Section 2.5 for the precise definition of distortion and Section 7.1 for the definition of isothermal map. We ask: can one find a metric surface $(\widehat{X}, \widehat{d})$ such that f factors as $f = \widehat{f} \circ P$, where $\widehat{f} \colon \widehat{X} \to X$ is 1-quasiconformal and $P \colon \Omega \to \widehat{X}$ is bi-Lipschitz? In other words, can one find a "conformal representative" for the space X within the class of bi-Lipschitz surfaces?

If the metric is defined by a continuous reciprocal norm field of bounded distortion, then such a factorization can always be found. Recall that, by the classical uniformization theorem, for every domain $\Omega \subset \mathbb{R}^2$ there exists a smooth Riemannian norm field $G = \sigma\|\cdot\|_2$ on Ω such that (Ω, d_G) is complete and has Gaussian curvature 0 or -1. We have the following result.

Proposition 1.5. Let $\Omega \subset \mathbb{R}^2$ be a domain and N a reciprocal norm field with distortion H. If N is continuous outside the set $E = \{x \in \Omega : N_x = 0\}$, then there exists a distance $\widehat{d}$ on Ω such that:

(i) The identity map $P \colon (\Omega, d_G) \to (\Omega, \widehat{d})$ satisfies

$$d_G(x, y) \leq \widehat{d}(P(x), P(y)) \leq H d_G(x, y) \tag{3}$$

for all $x, y \in \Omega$.

(ii) The identity map $\widehat{\iota} \colon (\Omega, \widehat{d}) \to (\Omega, d_N)$ is 1-quasiconformal.

If the identity map $\iota \colon \Omega \to (\Omega, d_N)$ is isothermal, then it has distortion at most $\sqrt{2}$ [Iko21, Lemma 4.10], and so (3) holds with $H = \sqrt{2}$. The example of the ℓ^∞-norm on $\mathbb{R}^2$ shows that the value $H = \sqrt{2}$ in (3) is sharp for the case of general isothermal maps. Since every quasiconformal surface has an isothermal parametrization, this raises the question of finding conditions on N that guarantee that the conclusion of Proposition 1.5 holds with $H = \sqrt{2}$. In turn, this question is related to the regularity of the Beltrami coefficient derived from distance ellipse field corresponding to N and does not appear to have a straightforward answer. We briefly address this issue in Section 7.3.

In general, the conclusion of Proposition 1.5 may fail if N_x is discontinuous outside of E. In the final part of the paper, we present a lengthy construction giving a negative answer to the above factorization question in general. In fact, we obtain the stronger conclusion that no quasiconformal map $\widehat{f}$ in such a factorization can have distortion smaller than that of f.

Theorem 1.6. There is a metric d on $\mathbb{R}^2$ such that the identity map $\iota\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, d)$ is an isothermal quasiconformal homeomorphism, but ι does not factor as $\iota = \widehat{\iota} \circ P$, where $(\widehat{X}, \widehat{d})$ is a metric surface, $\widehat{\iota}\colon (\widehat{X}, \widehat{d}) \to (\mathbb{R}^2, d)$ is quasiconformal with distortion $H(\widehat{\iota}) < \sqrt{2}$ and $P\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\widehat{X}, \widehat{d})$ is bi-Lipschitz.

The identity map ι in our construction has distortion $H(\iota) = \sqrt{2}$, so the inequality $H(\widehat{\iota}) < \sqrt{2}$ is sharp.

The metric d in Theorem 1.6 is defined via a lower semicontinuous norm field of the form

$$N_x = \begin{cases} c_x \|\cdot\|_1 & \text{if } x \in F \\ c_x \|\cdot\|_\infty & \text{if } x \notin F \end{cases}$$

for some measurable set $F \subset \mathbb{R}^2$ and measurable function $x \mapsto c_x$, where $0 \leq c_x \leq 1$ and c_x vanishes at a single point. Note that this fits exactly into the construction scheme of this paper, and therefore $(\mathbb{R}^2, d)$ is a quasiconformal surface.

One might initially expect that the metric $\widehat{d}$ on $\mathbb{R}^2$ defined by

$$\widehat{N}_x = \begin{cases} \|\cdot\|_1 & \text{if } x \in F \\ \sqrt{2}\|\cdot\|_\infty & \text{if } x \notin F \end{cases}$$

with $\widehat{\iota}$ and P the identity map on $\mathbb{R}^2$, or some variation on this, gives a factorization satisfying the properties given in Theorem 1.6. Observe that $\|\cdot\|_2 \leq \widehat{N} \leq \sqrt{2}\|\cdot\|_2$ everywhere, so the map P in this situation is bi-Lipschitz. However, the map $\widehat{\iota}$ may fail to be 1-quasiconformal. The reason for this is that the norm field $\widehat{N}$ corresponding to F is typically not lower semicontinuous, in which case the metric derivatives of P need not coincide with $\widehat{N}_x$ almost everywhere. Indeed, we prove Theorem 1.6 by specifying explicitly a set F and coefficients c_x for which this failure of 1-quasiconformality occurs for the norm field $\widehat{N}$ defined above, and in fact for any conformal rescaling of $\widehat{N}$ bi-Lipschitz equivalent to the Euclidean norm field.

The basic idea of our construction is to define a sequence of nested Cantor sets K_i as the intersection of a collection of squares in the plane. This is done so that the odd-indexed Cantor sets are formed from squares in the standard (i.e., non-rotated) alignment, while the even-indexed Cantor sets are formed from squares aligned diagonally. Next, the norm field on $K_i \setminus K_{i+1}$ for odd values of i is defined to be the supremum norm $\|\cdot\|_\infty$, scaled by a constant c_i satisfying $c_i \to 0$ as $i \to \infty$, while the norm field for even values of i is defined to be the $\|\cdot\|_1$-norm, also scaled by a constant c_i' satisfying $c_i' \to 0$ as $i \to \infty$. A consequence of the distortion inequality for $\widehat{\iota}$ is that the metric derivatives of P and ι cannot differ by more than a fixed amount, up to rescaling. With a suitable choice of constants c_i, c_i', the alternating arrangement of the Cantor sets K_i then forces the metric derivatives of P to be arbitrarily small at some points.

Lytchak–Wenger [LW18] and Creutz–Soultanis [CS20] study similar types of factorizations for *minimal disks* or *solutions to Plateau's problem* with metric space target, though without trying to optimize the properties of P in the way that we have proposed. Here, we simply remark that the map ι in our example is also an energy-minimizing map (for the Reshetnyak energy) in the sense of these papers on each closed disk. We refer the reader to the above papers for definitions of these terms.

1.5. **Outline.** Our paper is organized as follows. Section 2 gives an overview of basic results and notation related to metric Sobolev spaces, quasiconformal mappings, and removable sets. In Section 3, we give a detailed overview of the construction of metric spaces from a prescribed norm field under suitable assumptions. In Section 4, we prove the first of the main results, Theorem 1.3, stating that an admissible norm field is reciprocal if it vanishes exactly on a set that is removable for conformal mappings. In Section 5, we prove the partial converse, Theorem 1.4. Section 6 gives a pair of examples of spaces constructed from conformal weights that each vanish on a linear Cantor set of positive length, one of which is reciprocal and one of which is not. This can be viewed as the borderline case. Finally, Section 7 gives the proof of Proposition 1.5 as well as the construction for Theorem 1.6.

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2. Preliminaries

2.1. **Notation.** In this paper, we frequently consider several metrics in close proximity to one another. For this reason, we will consistently use subscripts to denote the metric being referred to. Let (X, d) be a metric space. The open ball centered at a point $x \in X$ of radius $r > 0$ with respect to the metric d is denoted by $B_d(x, r)$.

The Euclidean metric is denoted by $\|\cdot\|_2$. Thus, for example, we write $B_{\|\cdot\|_2}(x, r)$ for an open ball with respect to this metric, and $ds_{\|\cdot\|_2}$ for the Euclidean length element.

We recall the definition of Hausdorff measure. Let (X, d) be a metric space. For all $p \geq 0$, the *p-dimensional Hausdorff measure*, or *Hausdorff p-measure*, is defined by

$$\mathcal{H}^p_X(A) = \sup_{\delta>0} \inf \left\{ \frac{\alpha(p)}{2^p} \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^p : A \subset \bigcup_{i=1}^{\infty} A_i, \operatorname{diam} A_i < \delta \right\}$$

for all sets $A \subset X$, where $\alpha(p) = \pi^{\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-1}$. The constant $\alpha(p)$ is chosen so that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers.

If the space X is understood but not the metric d, then we use the notation $\mathcal{H}^p_d$ instead of $\mathcal{H}^p_X$. The Hausdorff dimension of a set $E \subset X$ is the infimal value of p for which $\mathcal{H}^p_X(E) = 0$ and is denoted by $\dim_{\mathcal{H}_d} E$. For the basics of Hausdorff measure, see for example [AT04, Chapter 2].

Unless otherwise noted, in this paper a metric surface (X, d) is always equipped with the Hausdorff 2-measure generated by the metric d. For example, the phrase *almost every* refers to the Hausdorff 2-measure. Similarly, an interval in $\mathbb{R}$ is equipped with the Lebesgue measure $\mathcal{L}^1$.

A *path* is a continuous function from an interval into a metric space. A path in X will typically be denoted by γ. The image of γ is denoted by $|\gamma|$. The *length* of a path $\gamma\colon [a, b] \to X$ is defined as

$$\ell_d(\gamma) = \sup \sum_{j=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

the supremum taken over all finite sequences $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. A path is *rectifiable* if it has finite length. A path is *locally rectifiable* if its restriction to any compact subinterval is rectifiable.

The *metric speed* of a path $\gamma\colon [a, b] \to X$ at the point $t \in [a, b]$ is defined as

$$v_\gamma(t) = \lim_{h\to 0} \frac{d(\gamma(t + h), \gamma(t))}{h}$$

whenever this limit exists. If γ is rectifiable, its metric speed exists at $\mathcal{L}^1$-almost every $t \in [a,b]$; see Theorem 2.1 of [Dud07].

A rectifiable path $\gamma\colon [a,b] \to X$ is *absolutely continuous* if for all $a \le s \le t \le b$,

$$d(\gamma(t), \gamma(s)) \le \int_s^t v_\gamma(u)\, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a,b])$ and $\mathcal{L}^1$ the Lebesgue measure on the real line. Equivalently, γ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $\mathcal{H}^1_X$-measure zero in its image; see Section 3 of [Dud07].

A path $\widetilde{\gamma}\colon [c,d] \to X$ is a *reparametrization* of γ if there exists a map $\psi\colon [a,b] \to [c,d]$ that is surjective, non-decreasing, and continuous such that $\gamma = \widetilde{\gamma} \circ \psi$. If ψ is absolutely continuous, we say that $\widetilde{\gamma}$ is an *absolutely continuous reparametrization* of γ. Note that this is different from $\widetilde{\gamma}$ itself being an absolutely continuous path.

Every rectifiable path γ has a reparametrization $\widetilde{\gamma}\colon [0, \ell_d(\gamma)] \to X$ such that the metric speed of $\widetilde{\gamma}$ equals one $\mathcal{L}^1$-almost everywhere. In this case, we write $\gamma_s = \widetilde{\gamma}$, and refer to γ_s as the *unit speed parametrization* of γ. See Chapter 5 of [HKST15] for details.

If γ is rectifiable, the unit speed parametrization γ_s is 1-Lipschitz and hence absolutely continuous [HKST15, Proposition 5.1.8].

Let γ be a rectifiable path. Then the *path integral* of a Borel function $\rho\colon X \to [0,\infty]$ *over* γ is

$$\int_\gamma \rho\, ds = \int_0^{\ell_d(\gamma)} \rho \circ \gamma_s \, d\mathcal{L}^1, \tag{4}$$

where $\mathcal{L}^1$ is the Lebesgue measure on the real line.

If γ is absolutely continuous and $\widetilde{\gamma}$ is an absolutely continuous reparametrization of γ, the chain rule for metric speeds [Dud07, Theorem 3.16 and Remark 3.4] states that

$$v_\gamma = (v_{\widetilde{\gamma}} \circ \psi) \cdot \psi' \in L^1([c,d]),$$

where the right-hand side is understood to be zero whenever the derivative $\psi' = 0$ (even if $v_{\widetilde{\gamma}} \circ \psi$ is not defined or is infinite at such a point).

Moreover, for absolutely continuous γ, the unit speed parametrization γ_s is an absolutely continuous reparametrization of γ. Therefore (4) can be restated for absolutely continuous $\gamma\colon [a,b] \to X$ as follows:

$$\int_\gamma \rho\, ds = \int_a^b (\rho \circ \gamma) \cdot v_\gamma \, d\mathcal{L}^1.$$

Given a Borel set $A \subset X$, the length of a path $\gamma\colon [a,b] \to X$ in A is defined as $\int_X \chi_A(x) \#(\gamma^{-1}(x))\, d\mathcal{H}^1_X(x)$, where $\#(\gamma^{-1}(x))$ is the counting measure of $\gamma^{-1}(x)$. This formula makes sense for paths that are not necessarily rectifiable; see Theorem 2.10.13 [Fed69]. If γ is rectifiable, the number coincides with the path integral of χ_A over γ.

2.2. **Metric Sobolev spaces.** In this section, we give an overview of the theory of Sobolev spaces in the metric space setting. We refer the reader to the book [HKST15] for a comprehensive introduction to this topic. Throughout this section, assume that (X, d_X) and (Y, d_Y) are metric surfaces.

The conformal modulus provides a basic way of measuring the size of a family of paths. It is a conformal invariant in the Euclidean case, which accounts for both its nomenclature and its usefulness. Let Γ be a family of paths in X. A Borel function $\rho\colon X \to [0,\infty]$ is *admissible* for Γ if the path integral $\int_\gamma \rho\, ds \ge 1$ for all locally rectifiable paths $\gamma \in \Gamma$. The *conformal modulus*, or simply *modulus*, of Γ is

$$\operatorname{mod} \Gamma = \inf \int_X \rho^2 \, d\mathcal{H}^2_X,$$

where the infimum is taken over all admissible functions ρ for Γ.

If ρ is admissible for a path family $\Gamma' \subset \Gamma$ such that $\Gamma \setminus \Gamma'$ has modulus zero, then ρ is said to be *weakly admissible* for Γ. If a property holds for every path $\gamma \in \Gamma$ except in a subfamily of modulus zero, then this property is said to hold *on almost every path* in Γ. If $\operatorname{mod} \Gamma < \infty$, then there exists a weakly admissible Borel function $\rho \in L^2(X)$ such that

$$\operatorname{mod} \Gamma = \int_X \rho^2 \, d\mathcal{H}^2_X.$$

Such a ρ is called a *minimizer* of Γ. Such a minimizer is unique $\mathcal{H}^2_X$-almost everywhere.

Let $f\colon (X, d_X) \to (Y, d_Y)$ be a mapping between metric surfaces X and Y. A function $g\colon X \to [0, \infty]$ is an *upper gradient of* f if

$$d_Y(f(x), f(y)) \leq \int_\gamma g \, ds$$

for every rectifiable path $\gamma\colon [0,1] \to X$ connecting x to y. The function g is a *weak upper gradient of* f if the same holds for almost every rectifiable path.

The weak upper gradient $g \in L^2_{\mathrm{loc}}(X)$ is *minimal* if it satisfies $g \leq \widetilde{g}$ almost everywhere for all weak upper gradients $\widetilde{g} \in L^2_{\mathrm{loc}}(X)$ of f. If f has a weak upper gradient $g \in L^2_{\mathrm{loc}}(X)$, then f has a minimal weak upper gradient, which we denote by g_f. The existence of g_f follows from the fact that the weak upper gradients of f form a lattice. This also implies that g_f is unique up to measure zero; see Section 6 of [HKST15] and Section 3 of [Wil12] for details. In general, g_f is only a weak upper gradient.

Proposition 6.3.3 of [HKST15] and countable subadditivity of modulus (see also Lemmas 3.2 and 3.3 of [Wil12]) establish that a Borel function $\rho\colon X \to [0, \infty]$ belonging to $L^2_{\mathrm{loc}}(X)$ is a weak upper gradient of f if and only if for almost every absolutely continuous path $\gamma\colon [a,b] \to X$, the composition $f \circ \gamma$ is an absolutely continuous path for which the metric speeds $v_{f\circ\gamma}$ and v_γ satisfy

$$v_{f \circ \gamma} \leq (\rho \circ \gamma) \cdot v_\gamma \tag{5}$$

$\mathcal{L}^1$-almost everywhere on $[a,b]$. Since $\rho \in L^2_{\mathrm{loc}}(X)$ the right-hand side of (5) is integrable on its domain for almost every γ.

Let Z be a metric space such that $\mathcal{H}^2_{d_Z}(Z) < \infty$. Choose a point $y \in Y$, and let $d_y = d_Y(\cdot, y)$. The space $L^2(Z, Y)$ is defined as the set of measurable mappings $f\colon Z \to Y$ such that $d_y \circ f$ is in $L^2(Z)$. One can check that this definition is independent of the choice of y.

We define $L^2_{\mathrm{loc}}(X, Y)$ to consist of those measurable mappings $f\colon X \to Y$ for which, for all $x \in X$, there is an open set $U \subset X$ containing x such that $f|_U$ is in $L^2(U, Y)$.

The metric Sobolev space $N^{1,2}_{\mathrm{loc}}(X, Y)$ consists of those mappings $f\colon X \to Y$ in $L^2_{\mathrm{loc}}(X, Y)$ that have a minimal weak upper gradient $g_f \in L^2_{\mathrm{loc}}(X)$.

For open $U \subset X$ with $\mathcal{H}^2_X(U) < \infty$, we say that $f \in N^{1,2}(U, Y)$ if $f|_U \in N^{1,2}_{\mathrm{loc}}(U, Y)$ in such a way that $g_f|_U \in L^2(U)$ and for some $y \in Y$, $f_y(x) = d_y \circ f|_U \in L^2(U)$.

Next we define the Jacobian of f for continuous $f\colon X \to Y$. The *pullback measure* $f^*\mathcal{H}^2_Y$ is defined for Borel sets $A \subset X$ by the formula

$$f^*\mathcal{H}^2_Y(A) = \int_Y \#(A \cap f^{-1}(y))\, d\mathcal{H}^2_Y,$$

where $\#(A \cap f^{-1}(y))$ is the multiplicity function of f relative to A. The measure $f^*\mathcal{H}^2_Y$ can be defined equivalently using a suitable Carathéodory construction; see [Fed69, 2.10.10]. In fact, $f^*\mathcal{H}^2_Y$ is a Borel regular outer measure.

If the pullback measure $f^*\mathcal{H}^2_Y$ is locally finite, the measure has a Lebesgue decomposition $f^*\mathcal{H}^2_Y = J_f\mathcal{H}^2_X + \mu^\perp$, where $\mu^\perp$ and $\mathcal{H}^2_X$ are singular [Bog07, Section 3.1-3.2, Volume I]. The density J_f is called the *Jacobian* of f. The local finiteness of $f^*\mathcal{H}^2_Y$ and $\mathcal{H}^2_X$ imply that J_f is locally integrable.

2.3. **Seminorms.** We introduce the terminology and notation we use for seminorms. Recall that a *seminorm* S on $\mathbb{R}^2$ is a function $S\colon \mathbb{R}^2 \to [0,\infty)$ satisfying the following conditions for all $v, w \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

(i) (absolute homogeneity) $S(\lambda v) = |\lambda| S(v)$ whenever $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^2$;
(ii) (triangle inequality) $S(v + w) \leq S(v) + S(w)$.

The seminorm S is a *norm* if it has the additional property that $S(v) = 0$ only if $v = 0$. The *maximal stretching* of S is

$$L(S) = \sup\left\{S(v)\colon\ \|v\|_2 \leq 1\right\}. \tag{6}$$

The *minimal stretching* of S is

$$\omega(S) = \inf\left\{S(v)\colon\ \|v\|_2 \geq 1\right\}. \tag{7}$$

The *Jacobian* of the seminorm S is

$$J_2(S) = \frac{\pi}{\mathcal{L}^2\left(\{v : S(v) \leq 1\}\right)}.$$

Observe that $J_2(S) = 0$ in the case that S is only a seminorm. The *distortion* of S is

$$H(S) = \frac{L(S)}{\omega(S)} \tag{8}$$

if $\omega(S) > 0$ and $H(S) = \infty$ otherwise. The latter case occurs if S is a non-zero seminorm that is not a norm. The *outer dilatation* and *inner dilatation* of S are defined by, respectively,

$$K_O(S) = \frac{L(S)^2}{J_2(S)}, \quad K_I(S) = \frac{J_2(S)}{\omega(S)^2}$$

if $J_2(S) > 0$, and $K_O(S) = K_I(S) = \infty$ otherwise. The *maximal dilatation* of S is $K(S) = \max\{K_O(S), K_I(S)\}$. Observe that $K_O(S) \geq 1$ and $K_I(S) \geq 1$.

The seminorm S induces a pseudometric d_S on $\mathbb{R}^2$ by the formula $d_S(x, y) = S(x - y)$. The identity map $\iota_S\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, d_S)$ has the constant function $L(S)$ as its minimal weak upper gradient and $J_2(S)$ as its Jacobian. Its inverse ι_S^{-1} has the constant function $\omega(S)^{-1}$ as its minimal weak upper gradient.

The following lemma gives a relationship between the maximal dilatation and distortion.

Lemma 2.1. The distortion $H(S)$ and maximal dilatation $K(S)$ of S satisfy $H(S) \leq K(S) \leq H(S)^2$.

Proof. If $\omega(S) = 0$, then $H(S) = K(S) = \infty$. Otherwise, $H(S)$ and $K(S)$ are both finite. Observe the relationship $H(S)^2 = K_O(S)K_I(S) \leq K(S)^2$. On the other hand, the relationships $K_O(S) \geq 1$ and $K_I(S) \geq 1$ imply respectively that $H(S)^2 \geq K_I(S)$ and $H(S)^2 \geq K_O(S)$. We conclude that $H(S)^2 \geq K(S)$. □

2.4. **Metric derivatives of Lipschitz mappings.** Throughout this section, we let Ω denote a domain in $\mathbb{R}^2$ and (X,d) denote a metric space. We refer to Section 2.3 for basic terminology about seminorms.

Definition 2.2. Let $f\colon (\Omega, \|\cdot\|_2) \to (X,d)$ be a Lipschitz map. For all $x \in \Omega$ and $v \in \mathbb{R}^2$, the *metric derivative of f at x in the direction v* is

$$\mathrm{N}_{f,x}(v) = \limsup_{t\to 0^+} \frac{d(f(x), f(x+tv))}{t}. \tag{9}$$

A result by Ivanov [Iva08] states the following. Similar results are proved in [Kir94, DCP90, DCP91, DCP95].

Theorem 2.3. Let $f\colon (\Omega, \|\cdot\|_2) \to (X,d)$ be a Lipschitz map. There exists a Borel set $N_0 \subset \Omega$ of zero Lebesgue measure such that, for all $x \in \Omega \setminus N_0$ and all $v \in \mathbb{R}^2$, the limit superior in (9) is an actual limit, and $v \mapsto \mathrm{N}_{f,x}(v)$ is a seminorm for every $x \in \mathbb{R}^2 \setminus N_0$.

As a consequence of Theorem 2.3, the metric derivative of a Lipschitz map defines a seminorm field on Ω.

Proposition 2.4. Let $f\colon \Omega \to X$ be a Lipschitz function and N_f its metric derivative. The maximal stretching $x \mapsto L(\mathrm{N}_f(x))$ is a minimal weak upper gradient of f, and f satisfies the change of variables formula

$$\int_\Omega \rho(z) J_2(\mathrm{N}_{f,z})\, d\mathcal{L}^2(z) = \int_X \int_{f^{-1}(x)} \rho(y) d\mathcal{H}^0(y)\, d\mathcal{H}^2_d(x) \tag{10}$$

for all Borel functions $\rho\colon \Omega \to [0,\infty]$.

Proof. Theorem 2.3 implies that the metric derivative, as defined in Definition 2.2, coincides with the metric derivative of Kirchheim [Kir94] $\mathcal{L}^2$-almost everywhere in Ω. Thus the change of variables formula (10) follows from [Kir94, Corollary 8]. The statement that $L(\mathrm{N}_f)$ is a minimal weak upper gradient of f is proved in [LW17, Section 4]. ☐

The metric differential can be used to compute the metric speed of an absolutely continuous path.

Lemma 2.5. If $\gamma\colon [a,b] \to \Omega$ is an absolutely continuous path, then for almost every $t \in [a,b]$, the metric speed $v_{f\circ\gamma}(t)$ of $f \circ \gamma$ exists and satisfies

$$v_{f\circ\gamma}(t) = \mathrm{N}_f \circ D\gamma(t),$$

where $D\gamma(t)$ is the derivative of γ at t.

Proof. It follows from [Iva08, Proposition 2.7] that $\ell_d(f\circ\gamma) = \ell_{\mathrm{N}_f}(\gamma)$ for every Lipschitz path $\gamma\colon [a,b] \to \mathbb{R}^2$. Since every absolutely continuous path has a Lipschitz parametrization, the same result holds for absolutely continuous paths $\gamma\colon [a,b] \to \mathbb{R}^2$. The lemma now follows from the Lebesgue differentiation theorem. ☐

2.5. **Quasiconformal mappings.** Recall the geometric definition of quasiconformal mapping given in (1). A result of Williams states that this geometric definition is equivalent to an analytic definition based on metric Sobolev spaces. We state the two-dimensional case of this result, or rather a generalization to the case of continuous monotone maps. Recall that a mapping $f\colon X \to Y$ is *monotone* if the preimage of every point $y \in Y$ is a connected and compact subset of X.

Theorem 2.6 (cf. [Wil12])**.** Let X and Y be metric surfaces with locally finite Hausdorff 2-measure. Let $f\colon X \to Y$ be continuous and monotone, and suppose that the pullback measure $f^*\mathcal{H}^2_Y$ is locally finite. The following are equivalent for the same constant $K \geq 1$:

(i) $\operatorname{mod}\Gamma \leq K \operatorname{mod} f\Gamma$ for all path families Γ in X.
(ii) $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ and satisfies

$$g_f^2(x) \leq K J_f(x)$$

for $\mathcal{H}^2_X$-almost every $x \in X$.

Theorem 2.6 can be established using the original proof in [Wil12] with slight modifications which deal with the multiplicity of f. This is omitted here. A similar result can be found as Proposition 3.5 of [LW20].

The *outer dilatation* of f is the smallest constant $K \geq 1$ for which the modulus inequality $\operatorname{mod}\Gamma \leq K \operatorname{mod} f\Gamma$ holds for all Γ in X. The *inner dilatation* of f is the smallest constant $K \geq 1$ for which $\operatorname{mod} f\Gamma \leq K \operatorname{mod}\Gamma$ holds for all Γ in X. These are denoted respectively by $K_O(f)$ and $K_I(f)$. Thus a quasiconformal map is a homeomorphism with finite outer and inner dilatation.

Definition 2.7. The *pointwise distortion* of a quasiconformal homeomorphism $f\colon X \to Y$ at $x \in X$ is

$$H_f(x) = \begin{cases} g_f(x) g_{f^{-1}}(f(x)) & \text{if } g_f(x), g_{f^{-1}}(f(x)) \in (0,\infty) \\ 1 & \text{otherwise} \end{cases}.$$

A quasiconformal homeomorphism f satisfies condition (ii) in Theorem 2.6, and its inverse f^{-1} satisfies an analogous condition. It follows that the map $x \mapsto H_f(x)$ is $\mathcal{H}^2_X$-a.e. independent of the representatives of g_f and $g_{f^{-1}}$. The smallest $H \geq 1$ for which $H_f(x) \leq H$ for $\mathcal{H}^2_X$-a.e. $x \in X$ is called the *distortion* of f. In particular, $H \leq \sqrt{K_O(f)K_I(f)}$.

Consider now a quasiconformal map $f\colon \Omega \subset \mathbb{R}^2 \to X$ that is also Lipschitz. Then the equalities $g_f(x) = L(\mathrm{N}_{f,x})$ and $g_{f^{-1}} \circ f(x) = (\omega(\mathrm{N}_{f,x}))^{-1}$ hold for $\mathcal{L}^2$-almost every $x \in \Omega$ [Iko21, Lemmas 4.4 and 4.7]. Consequently, we have the equality $H_f(x) = H(\mathrm{N}_{f,x})$ for $\mathcal{L}^2$-almost every $x \in \Omega$.

In general, a quasiconformal map $f\colon \Omega \subset \mathbb{R}^2 \to X$ must satisfy *Lusin's Condition (N^{-1})*: for every Borel set $E \subset \Omega$ of positive Lebesgue measure, $f(E)$ has positive Hausdorff 2-measure. This is essentially proved in Remark 8.3 or Section 17 of [Raj17]. On the other hand, f need not satisfy *Lusin's Condition (N)*: for every Borel set $E \subset \Omega$ of zero Lebesgue measure, $f(E)$ has zero Hausdorff 2-measure. An example of this is given as Proposition 17.1 of [Raj17].

A uniformization theorem for quasiconformal mappings was proved by Rajala based on the notion of *reciprocality* [Raj17]. Let X be a metric surface. For a set $G \subset X$ and disjoint sets $F_1, F_2 \subset G$, let $\Gamma(F_1, F_2; G)$ denote the family of paths whose images are contained in G that start from F_1 and end in F_2. A *quadrilateral* is a set Q homeomorphic to $[0,1]^2$ with boundary consisting of four nonoverlapping boundary arcs, labelled $\xi_1, \xi_2, \xi_3, \xi_4$ in cyclic order.

Definition 2.8. A metric surface X is *reciprocal* if there exists a constant $\kappa \geq 1$ such that

$$\kappa^{-1} \leq \operatorname{mod}\Gamma(\xi_1, \xi_3; Q) \operatorname{mod}\Gamma(\xi_2, \xi_4; Q) \leq \kappa \tag{11}$$

for every quadrilateral $Q \subset X$, and

$$\lim_{r\to 0^+} \operatorname{mod}\Gamma\left(\overline{B}(x,r), X \setminus B(x,R); \overline{B}(x,R)\right) = 0 \tag{12}$$

for all $x \in X$ and $R > 0$ such that $X \setminus B(x,R) \neq \emptyset$.

We say that a reciprocal surface is *κ-reciprocal* if (11) holds for the constant κ. Note that, for all metric surfaces, the left inequality in (11) is satisfied for a universal constant $\widetilde{\kappa} > 0$ [RR19].

Theorem 1.4 in [Raj17] states that a metric surface X homeomorphic to $\mathbb{R}^2$ is reciprocal if and only if there exists a quasiconformal homeomorphism onto the disk or the Euclidean plane. This result is extended to arbitrary metric surfaces in [Iko21]. More precisely, Theorem 1.2 in [Iko21] implies that a metric surface X is locally reciprocal (that is, every point in X has a neighborhood that is reciprocal) if and only if X is quasiconformally equivalent to a smooth Riemannian 2-manifold. In fact, Theorem 1.3 in [Iko21] proves that such an X is $(\pi/2)$-quasiconformal equivalent to a Riemannian surface. In particular, a metric surface that is locally reciprocal is also globally reciprocal.

2.6. **Removable sets for conformal mappings.** We collect some background on removable sets for conformal mappings. Recall from the introduction that the compact set $E \subset \mathbb{R}^2$ is *removable for conformal mappings* if every conformal embedding $f\colon \mathbb{R}^2 \setminus E \to \widehat{\mathbb{R}}^2$ extends to a conformal mapping $F\colon \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$. Thus f is the restriction of a Möbius transformation.

This notion exists under several names, including *sets of absolute area zero* and *negligible sets for extremal distance.* This nomenclature reflects the following characterization.

Proposition 2.9. Let $E \subset \mathbb{R}^2$ be compact. The following are equivalent.

(i) E is removable for conformal mappings.
(ii) E has absolute area zero: for every conformal embedding $f\colon \mathbb{R}^2 \setminus E \to \widehat{\mathbb{R}}^2$, the complementary set $\widehat{\mathbb{R}}^2 \setminus f(\mathbb{R}^2 \setminus E)$ has Lebesgue measure zero.
(iii) E is negligible for modulus: for every domain $\Omega \subset \mathbb{R}^2$ and pair of disjoint compact sets $F, G \subset \Omega \setminus E$, $\operatorname{mod}\Gamma(F, G; \Omega) = \operatorname{mod}\Gamma(F, G; \Omega \setminus E)$.
(iv) Any quasiconformal embedding $f\colon \mathbb{R}^2 \setminus E \to \widehat{\mathbb{R}}^2$ has an extension to a quasiconformal mapping $F\colon \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$.
(v) For any open set $U \subset \mathbb{R}^2$, every quasiconformal mapping on $U \setminus E$ extends quasiconformally to the whole open set U.

The equivalence of (i), (ii) and (iii) is proved in [AB50]. The equivalence of (i) and (iv) is a consequence of the measurable Riemann mapping theorem. See Proposition 4.7 in [You15] for a proof. The equivalence of (i) and (v) can also be found in [You15] as Proposition 4.6. We see from (iv) and (v) that removability for conformal mappings is a local property and a quasiconformal invariant. If E contains a nontrivial connected component E_0, then there is a non-Möbius conformal map $f\colon \mathbb{R}^2 \setminus E_0 \to \mathbb{R}^2$ such that $\mathbb{R}^2 \setminus f(\mathbb{R}^2 \setminus E_0)$ is the closed unit disk. Thus Property (ii) implies that a removable set for conformal mappings is totally disconnected.

Property (iii) in Proposition 2.9 indicates the connection between quasiconformal uniformization and removable sets. Observe that for each triple F, G, and Ω, $\Gamma(F, G; \Omega \setminus E)$ is a subset of $\Gamma(F, G; \Omega)$ and thus satisfies $\operatorname{mod}\Gamma(F, G; \Omega \setminus E) \leq \operatorname{mod}\Gamma(F, G; \Omega)$. In contrast, the metric space constructions in our paper collapse a domain at the set E and hence *increase* the modulus of a path family, up to a factor related to the dilatation bound of the norm field. Thus Theorem 1.3 and Theorem 1.4 can be summarized roughly by saying that *removing the set E does not decrease the modulus of any path family if and only if collapsing the plane at E does not increase the modulus of any path family.*

3. Constructing a metric from a norm field

In this section, we give a description of the metric spaces considered in this paper and develop their basic properties. These spaces are constructed from measurable Finsler structures satisfying additional assumptions. The precise definition is given in Section 3.1.

There is a vast literature on Riemannian and Finsler geometry, typically requiring smoothness or at least continuity of the Finsler structure. The idea of constructing metrics from Finsler structures with less regularity has been considered by various previous authors, and so the material in this section is more-or-less standard. In Section 3.2, we include a brief comparison with the existing literature.

We consider here seminorm fields N such that either N_x is a norm or $N_x = 0$ for all $x \in \Omega$. Recall from the introduction that, slightly abusing terminology, we use the term *norm field* to refer to an object of this type. Since a vector $v \in \mathbb{R}^2$ often comes with an implicit basepoint x, we will sometimes write $N(v)$ in place of $N_x(v)$, such as in the expression $N \circ D\gamma$.

3.1. Definition of the metric.

Let $\Omega \subset \mathbb{R}^2$ be a domain.

Definition 3.1. A norm field $N\colon \Omega \times \mathbb{R}^2 \to [0, \infty)$ is *admissible* if it satisfies the following:

(i) (lower semicontinuous) For all vectors $v \in \mathbb{R}^2$ and points $x \in \Omega$, we have $N_x(v) \leq \liminf_{y \to x} N_y(v)$.

(ii) (locally bounded) For all $x \in \Omega$, there is a neighborhood U of x and $M > 0$ such that $L(N_y) \leq M$ for all $y \in U$.

(iii) (locally bounded distortion) For all $x \in \Omega$, there is a neighborhood U of x and $H > 0$ such that $L(N_y) \leq H\omega(N_y))$ for all $y \in U$.

(iv) (nonseparating) The set $E = \{x \in \Omega : N_x = 0\}$ is compact and $\Omega \setminus E$ is connected.

An immediate consequence of having locally bounded distortion is that $N_x(v) = 0$ for some $v \in \mathbb{R}^2 \setminus \{0\}$ if and only if N_x is identically zero.

We use the norm field N to measure the length of an absolutely continuous path $\gamma\colon [a, b] \to \Omega$ in the following way. We define the *N-length* of γ to be

$$\ell_N(\gamma) = \int_a^b N \circ D\gamma(t)\, dt,$$

where $D\gamma\colon [a, b] \to \Omega \times \mathbb{R}^2$ is a Borel representative of the differential of γ.

Definition 3.2. Let N be an admissible seminorm field and $x, y \in \Omega$. The *N-distance* between x and y is defined as

$$d_N(x, y) = \inf \ell_N(\gamma),$$

where the infimum is taken over absolutely continuous paths γ joining x to y in Ω.

The function d_N is locally finite and satisfies the triangle inequality, but it may happen that $d_N(x, y) = 0$ for distinct points $x, y \in \Omega$. Thus, in general, d_N is only a pseudodistance. Let $\mathcal{E}_N$ be the partition of Ω into equivalence classes of points, where $x, y \in \Omega$ belong to the same equivalence class if $d_N(x, y) = 0$. This yields the quotient space $\Omega/\mathcal{E}_N$ and the natural quotient map $\pi_N : \Omega \to \Omega/\mathcal{E}_N$. The space $\Omega/\mathcal{E}_N$ comes equipped with the metric that is the pushforward of d_N under π_N, which we denote by $\widetilde{d}_N$.

A consequence of the local boundedness of N is that the quotient map π_N is locally Lipschitz. In particular, the results described in Section 2.4 apply to the map π_N.

3.2. **Remarks on definition of admissible norm fields.** We offer a few remarks about Definition 3.1 and give a comparison to the previous literature.

The lower semicontinuity assumption guarantees that the metric tangents of $\Omega/\mathcal{E}_N$ coincide with N almost everywhere. This implies, for example, that two conformally equivalent norm fields generate metrics that are 1-quasiconformally equivalent. In general, the metric tangents are not so well-behaved. For example, let $F \subset [0,1]$ be a Cantor set of positive linear measure, and let $E = F \times F \subset \mathbb{R}^2$. The norm field N defined by

$$N_x = \begin{cases} 2\|\cdot\|_\infty & \text{if } x \in E \\ \|\cdot\|_1 & \text{if } x \notin E \end{cases}$$

generates the same metric as the norm field $\|\cdot\|_1$, despite the fact that they differ on a positive measure set. Indeed, the inequality $\|x-y\|_1 \leq d_N(x,y)$ is immediate for all $x, y \in \mathbb{R}^2$, since $\|\cdot\|_1 \leq 2\|\cdot\|_\infty$. On the other hand, for all $x, y \in \mathbb{R}^2 \setminus E$, there is an ℓ_1-geodesic from x to y lying in $\mathbb{R}^2 \setminus E$. Thus $d_N(x,y) \leq \|x-y\|_1$ for such x, y. Since E has empty interior, we obtain the inequality $d_N(x,y) \leq \|x-y\|_1$ for all $x, y \in \mathbb{R}^2$. The lower semicontinuity assumption allows us to avoid this type of behaviour; see Lemma 3.6 below.

The fact that $E = \{x \in \Omega \colon N_x = 0\}$ is non-separating guarantees that the quotient space is homeomorphic to Ω (Corollary 3.9). For example, if E is the Euclidean unit circle and $N = \chi_{\mathbb{R}^2 \setminus E} \|\cdot\|_2$, the resulting quotient space is not a 2-manifold.

Now we discuss some of the related literature on non-smooth Finsler metrics. Perhaps the first investigations into this topic were carried out by Busemann–Mayer in [BM41]. Beginning in the 1940s, the Russian school led by Alexandrov developed a theory of *surfaces of bounded curvature*, also now known as *Alexandrov surfaces*, as a generalization of two-dimensional Riemannian geometry. See [AZ67] and [Res93] for an overview.

Finsler metrics on Lipschitz manifolds were systematically studied by De Cecco–Palmieri in the series of papers [DCP88, DCP90, DCP91, DCP95]. Note that they take a different approach to defining the distance d_N from a norm field N. The idea is to make the distance more robust by making the definition insensitive to changes in N on a set of measure zero. In particular, the norm field N need only be defined on a full measure subset. This is achieved as follows. For a set $F \subset \mathbb{R}^2$ of measure zero, let Γ_F be the family of absolutely continuous paths that intersect F in a set of length zero. Then one defines the metric $d_{N,F}$ as in Definition 3.2 but restricting to paths in Γ_F. Next one defines $D_N(x,y) = \sup d_{N,F}(x,y)$, the supremum taken over all measure zero sets F. This is called the *intrinsic distance* in [DCP95, GPP06] and *essential metric* in [AHPCS18] and further investigated in [CS20]. Observe that if N is continuous, then the essential metric coincides with the metric considered in this paper. However, we do not take this approach, since the norm fields we have in mind typically vanish on a set of measure zero, and we prefer the additional flexibility of only requiring N to be lower semicontinuous.

3.3. **Properties of length.** In the remainder of this section, we establish properties of admissible norm fields and their corresponding metric. Our first lemma states that the property of lower semicontinuity of N in each direction v can be promoted to lower semicontinuity at a point in all directions uniformly.

Lemma 3.3. Let N be an admissible norm field and $x \in \Omega$. For every $\varepsilon > 0$, there exists $r > 0$ such that

$$N_y(v) \geq (1-\varepsilon)N_x(v).$$

for all $y \in B(x,r)$ and $v \in \mathbb{R}^2$.

Proof. If N_x is the zero seminorm, then the conclusion follows immediately. Thus we may assume that N_x is a norm. By the positive homogeneity of N, we need only consider vectors $v \in \mathbb{S}^1$. Let $\varepsilon > 0$ and let $\delta = \varepsilon\omega(N_x)$, so that $N_x(v) - \delta \geq (1-\varepsilon)N_x(v)$ for all $v \in \mathbb{R}^2$. Thus it suffices to show that there exists a radius $r > 0$ such that

$$N_y(v) \geq N_x(v) - \delta$$

for all $y \in B(x, r)$ and $v \in \mathbb{S}^1$.

Assume to the contrary that no such r exists. Then there exist sequences $(y_n) \subset \Omega$ and $(v_n) \subset \mathbb{S}^1$ for which

$$N_{y_n}(v_n) < N_x(v_n) - \delta \tag{13}$$

for all $n \in \mathbb{N}$. By passing to a subsequence, we have that v_n converges to some vector $v \in \mathbb{S}^1$.

Let $M > 0$ be such that $L(N_y) \leq M$ for all y in a neighborhood of x. Then for every sufficiently large $n \in \mathbb{N}$,

$$N_x(v_n) - M \|v - v_n\|_2 \leq N_x(v)$$

and

$$N_{y_n}(v) \leq N_{y_n}(v_n) + M \|v - v_n\|_2 .$$

Moreover, the lower semicontinuity of N implies that

$$N_x(v) - \frac{\delta}{2} \leq N_{y_n}(v)$$

for all sufficiently large $n \in \mathbb{N}$. Combining these inequalities yields

$$N_x(v_n) - \left(2M \|v - v_n\|_2 + \frac{\delta}{2}\right) \leq N_{y_n}(v_n).$$

Let n be sufficiently large so that $\|v - v_n\|_2 < \delta(4M)^{-1}$. Then the preceding inequality contradicts (13), and the result follows. □

The next lemma shows that the metric d_N is locally well-behaved outside of the set E.

Lemma 3.4. Let N be an admissible norm field. For all $x \in \Omega \setminus E$, there exists $r > 0$ such that $\overline{B}(x, r) \subset \Omega \setminus E$ and the quotient map π_N is bi-Lipschitz in the neighborhood $B(x, r)$.

Proof. We let $\sigma(x) = \omega(N_x)$ denote the minimal stretching of N. Lemma 3.3 implies that σ is lower semicontinuous. Also $\sigma(x) = 0$ if and only if N_x is not a norm.

Let $x \in \Omega \setminus E$. Let $R > 0$ be such that the closed ball $\overline{B}(x, R)$ is contained in $\Omega \setminus E$ and satisfies $\sigma(z) \geq \sigma(x)/2$ for all $z \in B(x, R)$. Such an $R > 0$ exists by the lower semicontinuity of the map $z \mapsto \sigma(z)$. Moreover, the local boundedness of N implies that there exists $M > 0$ such that the maximal stretching $L(N_z)$ is bounded from above by M for all $z \in B(x, R)$. We conclude that

$$\frac{\sigma(x)}{2} \|v\|_2 \leq N_z(v) \leq M \|v\|_2$$

for all $z \in B(x, R)$ and all $v \in \mathbb{R}^2$.

Let $r = R/2$. We claim that

$$\frac{\sigma(x) \|y - z\|_2}{4} \leq d_N(y, z) \leq M \|y - z\|_2$$

for all $y, z \in B(x, r)$. Clearly, the line segment from y to z has N-length at most $M \|y - z\|_2$. For the lower bound, consider an arbitrary absolutely continuous path γ from y to z. If $|\gamma| \subset B(x,R)$, then we have the lower bound $\ell_N(\gamma) \geq \sigma(x)\left\|y-z\right\|_2/2$. If $|\gamma|$ is not contained in $B(x,R)$, then its length is at least

$$\sigma(x)(R-r) = \frac{\sigma(x)R}{2} \geq \frac{\sigma(x)\left\|y-z\right\|_2}{4}.$$

Since our path is arbitrary, we obtain $d_N(y,z) \geq \sigma(x)\left\|y-z\right\|_2/4$. We conclude that d_N is bi-Lipschitz equivalent to the Euclidean distance on $B(x,r)$. □

Lemma 3.5. For $\mathcal{L}^2$-almost every $x \in \Omega$, the metric derivative N_{π_N} of π_N at x satisfies

$$\mathrm{N}_{\pi_N,x} \leq N_x.$$

Moreover, for every $x \in \Omega$,

$$N_x \leq \mathrm{N}_{\pi_N,x}.$$

In particular, the metric derivative N_{π_N} equals N $\mathcal{L}^2$-almost everywhere in $x \in \Omega$.

Proof. First, we show that the upper bound $\mathrm{N}_{\pi_N,x} \leq N_x$ holds $\mathcal{L}^2$-almost everywhere in Ω. Consider a fixed $v \in \mathbb{R}^2 \setminus \{0\}$. The local boundedness of N implies that the function $x \mapsto N_x(v)$ is locally integrable. Consider a rectangle $R \subset \Omega$ with one side parallel to v. There is a family of parallel line segments $\gamma_t \colon [0,h_0] \to R$, $\gamma_t(s) = x_t + vs$, that foliate R. Observe that for all t and s, $D\gamma_t(s) = v$. The definition of d_N implies that

$$\mathrm{N}_{\pi_N,\gamma_t(s)}(v) \leq \limsup_{h\to 0^+} \frac{1}{h}\int_{[s,s+h]} N_{\gamma_t(a)}(v)\, d\mathcal{L}^1(a).$$

According to Lebesgue's differentiation theorem, the lim sup on the right-hand side equals $N_{\gamma_t(s)}(v)$ for $\mathcal{L}^1$-almost every $s \in [0,h_0]$. Fubini's theorem implies that

$$\mathrm{N}_{\pi_N,x}(v) \leq N_x(v)$$

holds $\mathcal{L}^2$-almost everywhere in R. Since R is arbitrary, the same conclusion holds for almost every point in Ω. The first inequality follows.

Next, we show that the inequality $N_x \leq \mathrm{N}_{\pi_N,x}$ holds for all $x \in \Omega$. In the case that $x \in E$, the conclusion is immediate since then $N_x = 0$. We consider now the case that $x \in \Omega \setminus E$. Let $v \in \mathbb{R}^2 \setminus \{0\}$ and let $\varepsilon > 0$.

Let $r > 0$ be such that the conclusions of Lemma 3.3 and Lemma 3.4 hold for the point x and the given value of ε. In particular, Lemma 3.4 implies that there exists $\alpha \geq 1$ such that

$$\alpha^{-1} d_N(y,z) \leq \|y-z\|_2 \leq \alpha d_N(y,z)$$

for all $y,z \in B(x,r)$. Moreover, the local boundedness of N implies that there exists $M > 0$ such that the maximal stretching $L(N_y) \leq M$ for all $y \in B(x,r)$. Let

$$t_0 = \frac{1}{\left\|v\right\|_2}\frac{r}{2\alpha}\min\left\{\frac{1}{\alpha}, \frac{1}{\varepsilon M}\right\}.$$

For all $t \in (0,t_0)$, consider an absolutely continuous path $\gamma_t \colon [0,1] \to \Omega$ joining x to $x+tv$ that satisfies

$$\int_0^1 N \circ D\gamma_t \, d\mathcal{L}^1 \leq d_N(x,x+tv) + \varepsilon t N_x(v). \tag{14}$$

The right-hand side of (14) is bounded above by $\alpha t\left\|v\right\|_2 + \varepsilon M t\left\|v\right\|_2 < r/\alpha$. In particular, this implies that

$$|\gamma_t| \subset B_{\|\cdot\|}(x,r). \tag{15}$$

Next, observe that

$$tN_x(v) = N_x(tv) \leq M\|tv\|_2 \leq \alpha M d_N(x,x+tv). \tag{16}$$

Applying now the conclusion of Lemma 3.3 along γ_t, which is allowed due to (15), we have

$$(1-\varepsilon)N_x(D\gamma_t(s)) \le N \circ D\gamma_t(s), \tag{17}$$

for almost every $s \in [0,1]$. Note that the norm field N on the left-hand side has a fixed basepoint.

Since straight line segments are geodesics with respect to the norm N_x, by integrating both sides of (17) and applying (14) and (16), we obtain

$$(1-\varepsilon)N_x(tv) \le (1+\varepsilon\alpha M)d_N(x, x+tv). \tag{18}$$

We divide both sides of (18) by t and let $t \to 0$. We have

$$(1-\varepsilon)N_x(v) \le (1+\varepsilon\alpha M)\liminf_{t\to 0} \frac{d_N(x, x+tv)}{t}.$$

The lim inf on the right-hand side is bounded from above by the metric derivative $\mathrm{N}_{\pi_N,x}(v)$. The result follows by letting $\varepsilon \to 0$. □

Lemma 3.6. For every Borel function $\rho\colon \Omega/\mathcal{E}_N \to [0,\infty]$, we have the change of variables formula

$$\int_\Omega (\rho\circ\pi_N)\cdot J_2(N)\,d\mathcal{L}^2 = \int_{\Omega/\mathcal{E}_N} \rho\, d\mathcal{H}^2_{\widetilde{d}_N}.$$

Proof. It follows from Lemma 3.5 that the metric derivative of π_N equals N $\mathcal{L}^2$-almost everywhere. The change of variables formula Proposition 2.4 implies that $\mathcal{H}^2_{\widetilde{d}_N}(\pi_N(E)) = 0$. The fact that π_N is injective in the complement of E implies that the multiplicity term from Proposition 2.4 can be omitted. □

Lemma 3.7. For every absolutely continuous path γ in Ω, $\ell_N(\gamma) = \ell_{d_N}(\pi_N\circ\gamma)$. In particular, the equality

$$v_{\pi_N\circ\gamma} = N\circ D\gamma \tag{19}$$

holds almost everywhere in the domain of γ.

Proof. An immediate consequence of the definitions is that $\ell_{\widetilde{d}_N}(\pi_N\circ\gamma) \le \ell_N(\gamma)$ for every absolutely continuous γ in Ω. For the other direction, let L denote the N_{π_N}-length of γ:

$$L = \int_I \mathrm{N}_{\pi_N} \circ D\gamma(t)\, d\mathcal{L}^1(t).$$

Since $N(x) \le \mathrm{N}_{\pi_N}(x)$ for all $x\in\Omega$ by Lemma 3.5, we see that $\ell_N(\gamma) \le L$. By Lemma 2.5, the equality $L = \ell_{\widetilde{d}_N}(\pi_N\circ\gamma)$ holds for all absolutely continuous γ. The equality $\ell_N(\gamma) = \ell_{\widetilde{d}_N}(\pi_N\circ\gamma)$ now follows. The metric speed identity (19) follows from the Lebesgue differentiation theorem. □

As a consequence of the previous lemma, whenever $\gamma\colon I\to\Omega$ is an absolutely continuous path, we have the integral formula

$$\int_{\pi_N\circ\gamma} \rho\, ds_N = \int_I (\rho\circ\pi_N)(N\circ D\gamma)\, d\mathcal{L}^1$$

for all Borel measurable functions $\rho : \Omega/\mathcal{E}_N \to [0,\infty]$.

3.4. **The quotient map.**

Proposition 3.8. The quotient map $\pi_N\colon \Omega \to \Omega/\mathcal{E}_N$ is locally Lipschitz, locally bi-Lipschitz in the complement of E, and its restriction to $\Omega \setminus E$ is injective.

Moreover, the map π_N is closed and, for all $x \in \pi_N(E)$, the preimage $\pi_N^{-1}(x)$ is a connected and compact subset of E.

Proof. We already proved in Lemma 3.4 that π_N is locally bi-Lipschitz outside of E. Moreover, since N is locally bounded, π_N is locally Lipschitz at all points in Ω.

Next, let $x \in \Omega \setminus E$ and $U \subset \Omega \setminus E$ be a neighborhood of x such that $\pi_N|_U$ is bi-Lipschitz. The bi-Lipschitz property implies that $d_N(x, y) > 0$ for all $y \in U$. Next, let $r > 0$ be small enough so that $\overline{B}_{\|\cdot\|}(x, r) \subset U$, and let $c = \inf\{d_N(x, y)\colon y \in S_{\|\cdot\|}(x, r)\} > 0$. If $y \in \Omega \setminus U$, then any path from x to y must intersect $S_{\|\cdot\|}(x, r)$, which gives $d_N(x, y) \geq c > 0$. We conclude that π_N is injective in the complement of E.

Next, we prove that $\pi_N^{-1}(\widetilde{x})$ is a connected, compact subset of E for all $\widetilde{x} \in \pi_N(E)$. Let $x \in \pi_N^{-1}(\widetilde{x})$ and let K be the component of E containing x.

Let γ be a closed Jordan path in $\Omega \setminus E$ that separates K and the boundary of $\partial\Omega$. See [Why64, Section III.3] for the existence of such a path γ. Let U be the complementary component of $|\gamma|$ containing K and $c = \inf\left\{d_N(x, z)\colon z \in \Omega \setminus \overline{U}\right\}$. The image $|\gamma|$ has a small neighborhood V compactly contained in $\Omega \setminus E$. Every path joining the point x to $\Omega \setminus \overline{U}$ must pass through $\overline{V}$. The lower semicontinuity of N implies that $N \geq \alpha \|\cdot\|_2$ in $\overline{V}$ for some $\alpha > 0$ and hence that $c > 0$.

Let $y \in \pi_N^{-1}(\widetilde{x})$. Let (γ_n) be a sequence of Lipschitz paths joining x to y satisfying

$$\ell_N(\gamma_n) \leq 2^{-n}c$$

for all $n \in \mathbb{N}$. Observe that the image of each path γ_n is contained in $\overline{U}$. Moreover, for every $z_n \in |\gamma_n|$, we have that $d_N(x, z_n) \leq 2^{-n}c$. This implies that a subsequence of the sets $(|\gamma_n|)$ converges with respect to the Hausdorff distance to a connected subset C of $\pi_N^{-1}(\widetilde{x}) \cap \overline{U}$. This is a consequence of general properties of Hausdorff convergence in metric spaces; see Proposition 4.4.14 and Theorems 4.4.15 and 4.4.17 in [AT04]. Observe that, for a given point $z \in \Omega \setminus K$, the Jordan path γ above can be chosen so that $z \in \Omega \setminus \overline{U}$. Therefore, the limit continuum C does not contain z. It follows that C is a subset of K. Since $y \in \pi_N^{-1}(\widetilde{x})$ is arbitrary, we conclude that $\pi_N^{-1}(\widetilde{x})$ is a connected subset of K. Note that it is not necessarily the case that $K = \pi_N^{-1}(\widetilde{x})$.

The final step is to show that π_N is closed. Let $F \subset \Omega$ be a closed set and let $\widetilde{x}$ be a limit point of $\pi_N(F)$. Since $\pi_N^{-1}(\widetilde{x})$ is a singleton or contained in a component of E, there is a Jordan domain $U \subset \Omega$ such that ∂U is contained in $\Omega \setminus E$ and separates $\pi_N^{-1}(\widetilde{x})$ and $\partial\Omega$. Arguing as in the first part of the proof, we deduce that there is a constant $c > 0$ such that $d_N(\pi_N^{-1}(\widetilde{x}), z) \geq c$ for all $z \in \Omega \setminus U$. This implies that $\widetilde{x}$ is a limit point of $\pi_N(F \cap \overline{U})$. Let $(\widetilde{x}_j)$ be a sequence in $\pi_N(F \cap \overline{U})$ with limit $\widetilde{x}$. Let (y_j) be a sequence in $F \cap \overline{U}$ such that $\pi_N(y_j) = \widetilde{x}_j$. The compactness of $\overline{U}$ implies that there is a subsequence (y_{j_k}) that converges to a point $y \in F$. Since $\pi_N|_{\overline{U}}$ is Lipschitz, it follows that (x_{j_k}) converges to $\pi_N(y)$, and moreover that $\widetilde{x} = \pi_N(y)$. We conclude that $\widetilde{x} \in \pi_N(F)$, and hence that $\pi_N(F)$ is closed. □

Corollary 3.9. The space $\Omega/\mathcal{E}_N$ is homeomorphic to Ω.

Proof. By Proposition 3.8, π_N is a closed and monotone map. Thus each element of the decomposition $\mathcal{E}_N$ is a planar continuum. Since the components of E are non-separating, so are the elements of $\mathcal{E}_N$. It follows now from the classical theorem of Moore that $\Omega/\mathcal{E}_N$ is homeomorphic to Ω. See, for instance, Theorem 25.1 in [Dav86]. □

Next we study the analytic properties of π_N. A consequence of Proposition 2.4 and Lemma 3.5 is that $x \mapsto L(N_x)$ is a minimal weak upper gradient of π_N. The following lemma identifies the minimal weak upper gradient of the inverse of π_N.

Lemma 3.10. If $U \subset \Omega$ is an open set such that $\pi_N|_U$ is injective and its inverse h is an element of $N^{1,2}_{\mathrm{loc}}(\pi_N(U), \mathbb{R}^2)$, the function

$$g = \left(\frac{1}{\omega(N)} \chi_{U \setminus E} \right) \circ h$$

is a minimal weak upper gradient of h.

We use the convention $\frac{1}{0} \cdot 0 = 0$ in Lemma 3.10.

Proof. We show that the function g as in the claim is a weak upper gradient of h. First, the change of variables formula Lemma 3.6 implies that $\mathcal{H}^2_{\widetilde{d}_N}(\pi_N(E)) = 0$. Therefore the paths that have positive $\widetilde{d}_N$-length on $\pi_N(E)$ have zero modulus. Moreover, since h is an element of $N^{1,2}_{\mathrm{loc}}(\pi_N(U), \mathbb{R}^2)$, h maps almost every absolutely continuous path in $\pi_N(U)$ to an absolutely continuous path in U. Thus it suffices to check the upper gradient inequality for a path $\widetilde{\gamma}\colon [0,1] \to \pi_N(U)$ that intersects $\pi_N(E)$ in a set of $\widetilde{d}_N$-length zero and along which h is absolutely continuous.

Let $\widetilde{\gamma}\colon [0,1] \to \pi_N(U)$ be such a path, and let $x = \widetilde{\gamma}(0)$ and $y = \widetilde{\gamma}(1)$. Let $\gamma = h \circ \widetilde{\gamma}$. Note that the absolute continuity of h along $\widetilde{\gamma}$ implies that γ intersects E in a set of Euclidean length zero. Therefore, by reparametrizing, we can assume that the set $J = \gamma^{-1}(\Omega \setminus E)$ has full length in $[0,1]$.

By Lemma 2.5, the metric speed identity $v_{\widetilde{\gamma}} = N \circ D\gamma$ holds $\mathcal{L}^1$-almost everywhere for γ. Also, for almost every $t \in [0,1] \setminus \gamma^{-1}(E)$,

$$v_\gamma(t) = \|D\gamma(t)\|_2 \leq \frac{1}{\omega(N_{\gamma(t)})} N \circ D\gamma(t) = \frac{1}{\omega(N_{\gamma(t)})} v_{\widetilde{\gamma}}(t), \tag{20}$$

where $\omega(N_{\gamma(t)})$ is the minimal stretching of N at $\gamma(t)$. Since $\gamma^{-1}(E)$ has zero measure, we conclude from (20) that for almost every $t \in [0,1]$,

$$v_\gamma(t) \leq \left(\frac{\chi_{U \setminus E}}{\omega(N)} \circ \gamma(t) \right) \cdot v_{\widetilde{\gamma}}(t). \tag{21}$$

The right-hand side in (21) equals $g \circ (\pi_N \circ \gamma(t)) v_{\widetilde{\gamma}}(t)$. Therefore, integrating both sides of (21) implies that

$$\|h(x) - h(y)\|_2 \leq \int_{\widetilde{\gamma}} g \, ds_{\widetilde{d}_N}.$$

The local L^2-integrability of g follows from the fact that N has locally bounded distortion (Lemma 2.1) and the change of variables formula (Lemma 3.6).

We are left to check that g is a minimal weak upper gradient. Let $\rho \in L^2_{\mathrm{loc}}(\pi_N(U))$ be a weak upper gradient of h. We want to show that $g(x) \leq \rho(x)$ for $\mathcal{H}^2_{\widetilde{d}_N}$-almost every $x \in \pi_N(U)$. The set $\pi_N(E)$ is negligible, so it is sufficient to check this in the complement of $\pi_N(E)$. As h is locally bi-Lipschitz in the complement of $\pi_N(E)$, it suffices to check that

$$g \circ \pi_N(x) = \sup_{v \in \mathbb{S}^1} \frac{1}{N_x(v)} \leq \rho \circ \pi_N(x) \tag{22}$$

$\mathcal{L}^2$-almost every $x \in U \setminus E$.

Consider a square $R \subset U \setminus E$ with center point $x_0 \in U$ and the accompanying foliation given by

$$\gamma_t(s) = x_0 + sv + tw,$$

where $v, w \in \mathbb{R}^2$ are orthogonal vectors and $s, t \in [-1, 1]$.

The metric speed identity $v_{\pi_N \circ \gamma} = N \circ D\gamma$ implies that

$$v_{\gamma_t}(s) = \|v\|_2 \leq \rho \circ (\pi_N \circ \gamma_t(s)) \cdot N_{\gamma_t(s)}(v)$$

holds almost everywhere along the domain of γ_t for almost every t. Fubini's theorem implies that

$$\|v\|_2 \leq \rho \circ \pi_N(x) \cdot N_x(v)$$

holds for $\mathcal{L}^2$-almost every $x \in R$. Equivalently,

$$1 \leq \rho \circ \pi_N(x) \cdot N_x\left(\frac{v}{\|v\|_2}\right)$$

for $\mathcal{L}^2$-almost every $x \in R$. We can cover $U \setminus E$ by squares whose sides are parallel to v and w, so we deduce that

$$\frac{1}{N_x\left(\frac{v}{\|v\|_2}\right)} \leq \rho \circ \pi_N(x) \tag{23}$$

for $\mathcal{L}^2$-almost every $x \in U \setminus E$.

Let D be a countable dense subset of $\mathbb{S}^1$. We have shown that, for $\mathcal{L}^2$-almost every $x \in U \setminus E$, (23) holds for every $v \in D$. Consequently, (22) holds for $\mathcal{L}^2$-almost every $x \in U \setminus E$. □

3.5. **Local quasiconformality.** Let U be a subdomain of Ω such that $\overline{U} \subset \Omega$ is compact. Since the norm field N has locally bounded distortion, there exists $K(U) < \infty$ such that

$$L(N)^2 \leq K(U) J_2(N)$$

for the maximal stretching $L(N)$ and the Jacobian $J_2(N)$. Recall that $L(N)$ is a weak upper gradient of π_N, $J_2(N)$ is the Jacobian of π_N, and that the pullback measure $\pi_N^* \mathcal{H}^2_{\widetilde{d}_N}$ is locally finite. Thus Theorem 2.6 implies the following.

Proposition 3.11. For every path family Γ in U, we have that

$$\operatorname{mod} \Gamma \leq K(U) \operatorname{mod} \pi_N \Gamma.$$

If $(\Omega/\mathcal{E}_N, \widetilde{d}_N)$ is reciprocal, then it admits some quasiconformal parametrization from a domain in Euclidean space. We show here that the map π_N itself is a quasiconformal parametrization, at least locally.

Proposition 3.12. The metric surface $(\Omega/\mathcal{E}_N, \widetilde{d}_N)$ is reciprocal if and only if π_N is a homeomorphism that is locally quasiconformal.

Here, a map $\psi \colon X \to Y$ is *locally quasiconformal* if every point $x \in X$ has a neighborhood U such that the restriction of ψ to U is K-quasiconformal for some $K \geq 1$, where K is allowed to depend on x.

Proof. If π_N is a locally quasiconformal homeomorphism, every point in $Y = (\Omega/\mathcal{E}_N, \widetilde{d}_N)$ has a neighborhood that is reciprocal. By Theorem 1.2 of [Iko21], this implies that Y is reciprocal.

Conversely, suppose that Y is reciprocal. It suffices to fix an arbitrary quadrilateral $Q \subset \Omega$ with $E \cap \partial Q = \emptyset$ and check that $\pi_N|_{\text{int}(Q)}$ is quasiconformal.

The reciprocality of Y implies the existence of a homeomorphism $f \colon \pi_N(Q) \to \overline{\mathbb{D}}$ that is $\frac{\pi}{2}$-quasiconformal in $\pi_N(Q)$. Set $V = \text{int}(Q)$ and $h = f \circ \pi_N|_V$.

We claim that h is a homeomorphism. The mapping h satisfies the assumptions of Theorem 2.6 and condition (i) in this theorem. Let $y \in \mathbb{D}$ and $C = h^{-1}(y)$, and fix a non-trivial continuum $C' \subset \mathbb{D} \setminus \{y\}$. The set C is connected and compact. The modulus of the family of paths joining y to C' is zero, since planar domains satisfy (12). Hence condition (i) in Theorem 2.6 implies that the modulus of the family of paths joining C to $h^{-1}(C')$ is zero. Since $h^{-1}(C')$ is a non-trivial continuum, this happens only when C is a singleton; see for example [Raj17, Proposition 3.5]. Therefore h is injective. Consequently, h is a homeomorphism between planar domains satisfying condition (i) from Theorem 2.6. According to the theory of planar quasiconformal mappings, this suffices to show that h is a quasiconformal homeomorphism; see [Väi71, Theorem 34.3] or [AIM09, Section 3]. □

Remark 3.13. Proposition 3.12 gives two simple criteria for $(\Omega/\mathcal{E}_N, \widetilde{d}_N)$ to fail to be reciprocal. First, if $\mathcal{L}^2(E) > 0$, then π_N is not locally quasiconformal since Lusin's Condition (N^{-1}) is violated. Second, if π_N is not injective, then $(\Omega/\mathcal{E}_N, \widetilde{d}_N)$ is not reciprocal.

4. Removable implies reciprocal

The objective of this section is to prove Theorem 1.3. An outline of the proof is as follows. First, we give a pair of reductions, Lemmas 4.1 and 4.2, showing that it suffices to consider only the case of admissible norm fields of the form $N = \sigma\|\cdot\|_2$ defined on all of $\mathbb{R}^2$ for some bounded function $\sigma\colon \mathbb{R}^2 \to [0,\infty)$. Next, Proposition 4.3 gives a criterion for the mapping π_N in our situation to be quasiconformal: it suffices to show that π_N preserves the modulus of the path families $\Gamma(\xi_1,\xi_3;R)$ and $\Gamma(\xi_2,\xi_4;R)$ for a single rectangle R containing E with boundary edges ξ_1,ξ_2,ξ_3,ξ_4.

We complete the proof by verifying the modulus condition of Proposition 4.3. This part is an application of the classical theorem of uniformization onto slit domains. This argument is based on the proof of Theorem 9 in [AB50]. In Section 4.3, we extend Theorem 1.3 by relaxing the assumption that $L(N) \in L^\infty_{\mathrm{loc}}(\Omega)$ to the assumption that $L(N) \in L^p_{\mathrm{loc}}(\Omega)$ for some $p \in (2,\infty)$.

Lemma 4.1. An admissible norm field N on Ω is reciprocal if and only if the norm field $\widehat{N} = \omega(N)\|\cdot\|_2$ induced by the minimal stretching $\omega(N)$ is reciprocal.

Proof. It follows immediately from Lemma 3.3 that $\widehat{N}$ is admissible for any admissible norm field N. By Proposition 3.12, it suffices to show that the metrics generated by N and $\widehat{N}$ are locally quasiconformally equivalent. Observe first that it follows directly from the definition that $\widehat{N} \le N$. Since N has locally bounded distortion, every point has a neighborhood U such that $N_x \le H\widehat{N}_x$ for some $H > 0$ and every $x \in U$. These facts imply that the corresponding distances are locally bi-Lipschitz equivalent. □

For the following lemma, fix a subdomain $\Omega' \subset \Omega$ that contains E and is compactly contained in Ω. Let $K = \overline{\Omega'}$. Given an admissible norm field $N = \sigma\|\cdot\|_2$, there exists $\alpha > 0$ such that $\sigma < \alpha$ everywhere on K. We define

$$\widehat{N} = \left(\sigma\chi_K + \alpha\chi_{\mathbb{R}^2\setminus K}\right)\|\cdot\|_2 . \tag{24}$$

The choice of α implies that $\widehat{N}$ is admissible on $\mathbb{R}^2$ vanishing exactly on E. Also, $\widehat{N}$ coincides with N in Ω'.

Lemma 4.2. The norm field $N = \sigma\|\cdot\|_2$ is reciprocal in Ω if and only if the extension $\widehat{N}$ defined by (24) is reciprocal in $\mathbb{R}^2$. Moreover, in either one of these cases the quotient maps $\pi_{\widehat{N}}$ and π_N are 1-quasiconformal homeomorphisms.

Proof. First of all, since N and $\widehat{N}$ are equal in Ω', there exists a homeomorphism

$$f\colon \pi_N(\Omega') \to \pi_{\widehat{N}}(\Omega')$$

for which $\pi_{\widehat{N}} = \pi_N \circ f$ on Ω'. In fact, the map f is a local isometry and hence 1-quasiconformal.

Since the restrictions of π_N and $\pi_{\widehat{N}}$ to the complement of E are locally bi-Lipschitz, we deduce that they are locally quasiconformal if and only if their restrictions to Ω' are locally quasiconformal. These two conditions are equivalent for the maps since f is quasiconformal. We conclude from Proposition 3.12 that N is reciprocal if and only if $\widehat{N}$ is reciprocal.

We are left to check that if π_N is locally quasiconformal, then it is actually 1-quasiconformal. Combining Theorem 2.6 with the local quasiconformality of π_N, we conclude that $h = \pi_N^{-1}$ has the Sobolev regularity required for Lemma 3.10. Therefore,

$$\rho = \left(\frac{1}{\sigma}\chi_{\Omega\setminus E}\right) \circ h$$

is a minimal weak upper gradient of h. Lemma 3.6 implies that the Jacobian J_h equals $(\sigma^{-2}\chi_{\Omega\setminus E}) \circ h$ $\mathcal{H}^2_{\widetilde{d}_N}$-almost everywhere. Therefore, condition (ii) in Theorem 2.6 holds with $K = 1$. Consequently, $K_I(\pi) \leq 1$. The outer dilatation bound for π_N follows from Proposition 3.11. We conclude that π_N is 1-quasiconformal. The 1-quasiconformality of $\pi_{\widehat{N}}$ is argued in a similar manner. □

4.1. **A criterion for quasiconformality.** We consider an admissible norm field $N = \sigma \|\cdot\|_2$ defined on a domain $\Omega \subset \mathbb{R}^2$ vanishing exactly on a non-separating compact set $E \subset \Omega$.

We consider a quadrilateral $Q \subset \Omega$ whose boundary ∂Q does not intersect the set E. Let $(\xi_1, \xi_2, \xi_3, \xi_4)$ be a decomposition of ∂Q into four nonoverlapping arcs labelled in counterclockwise order.

Since ∂Q does not intersect E, $\pi_N|_{\partial Q}$ is a homeomorphism onto its image (Proposition 3.8). As a consequence of Corollary 3.9, the image $\pi_N Q$ is a Jordan domain with boundary $\pi_N \partial Q$ consisting of the arcs $(\pi_N\xi_1, \pi_N\xi_2, \pi_N\xi_3, \pi_N\xi_4)$.

We fix some notation for the following proof. Let

$$\Gamma_1 = \Gamma(\xi_1, \xi_3; Q) \quad \text{and} \quad \widetilde{\Gamma}_1 = \Gamma(\pi_N\xi_1, \pi_N\xi_3; \pi_N Q);$$
$$\Gamma_2 = \Gamma(\xi_2, \xi_4; Q) \quad \text{and} \quad \widetilde{\Gamma}_2 = \Gamma(\pi_N\xi_1, \pi_N\xi_3; \pi_N Q).$$

We defined $\Gamma(F_1, F_2; G)$ in Section 2.5. Observe that $\pi_N\Gamma_1 \subset \widetilde{\Gamma}_1$ and $\pi_N\Gamma_2 \subset \widetilde{\Gamma}_2$.

Proposition 4.3. Let $N = \sigma\|\cdot\|_2$ be admissible. If $\operatorname{mod}\Gamma_1 = \operatorname{mod}\widetilde{\Gamma}_1$ and $\operatorname{mod}\Gamma_2 = \operatorname{mod}\widetilde{\Gamma}_2$, then the restriction of π_N to Q is a homeomorphism and 1-quasiconformal.

Proof. Proposition 3.11 and the special form of N imply that $K_O(\pi_N) = 1$, so we only need to check that $\pi_N|_Q$ is injective and that its inverse has its outer dilatation bounded above by 1.

It was proved in [RR19] that there exists a continuous function

$$\widetilde{u}_1 \colon \pi_N Q \to [0, 1]$$

in the Sobolev space $N^{1,2}(\pi_N Q)$ whose minimal weak upper gradient $\widetilde{\rho}_1$ is a minimizer for $\operatorname{mod}\widetilde{\Gamma}_1$. The function $\widetilde{u}_1$ satisfies the boundary conditions $\widetilde{u}_1(\pi_N\xi_1) = 0$ and $\widetilde{u}_1(\pi_N\xi_3) = 1$.

Consider $u_1 = \widetilde{u}_1 \circ \pi_N$. Since $N = \sigma\|\cdot\|_2$ and π_N has bounded outer dilatation, it is readily verified that $\rho_1 = (\widetilde{\rho}_1 \circ \pi_N)\sigma \in L^2(Q)$ is a weak upper gradient of u_1 with L^2-norm $\operatorname{mod}\widetilde{\Gamma}_1 = \operatorname{mod}\Gamma_1$. Therefore $u_1 \in N^{1,2}(Q)$.

A consequence of Weyl's lemma [AIM09, A.6.10] and continuity of u_1 is that u_1 is harmonic in the interior of Q; it minimizes the Dirichlet energy among continuous Sobolev maps $u \colon Q \to [0, 1]$ with boundary values $u(\xi_1) = 0$ and $u(\xi_3) = 1$.

We repeat the above argument for the path families Γ_2 and $\widetilde{\Gamma}_2$. Let u_2 and $\widetilde{u}_2$ denote the corresponding functions, where $u_2(\xi_2) = 0$ and $u_2(\xi_4) = 1$.

Let $M = \operatorname{mod}\Gamma_1$. A consequence of the Riemann mapping theorem is that Mu_2 is a harmonic conjugate of u_1 and the restriction of $f = (u_1, Mu_2)$ to the interior of Q is conformal. The map extends as a homeomorphism to the boundary ∂Q.

Let $\widetilde{f} = (\widetilde{u}_1, M\widetilde{u}_2)$. Then $f = \widetilde{f} \circ \pi_N$ by construction. Since f is bijective, this implies that the restriction of π_N to Q is a homeomorphism.

Since ∂Q does not intersect E, there is a Jordan neighborhood $U \supset Q$ such that $\pi_N|_U$ is a homeomorphism and $U \cap E = Q \cap E$ (Proposition 3.8). Let h denote the inverse of $\pi_N|_U$.

We claim that $h \in N^{1,2}_{\text{loc}}(\pi_N(U), U)$. Since π_N is locally bi-Lipschitz in the complement of E and $E \cap U \subset \operatorname{int}(Q)$, it suffices to verify that $h|_V$ is an element of $N^{1,2}_{\text{loc}}(V, U)$, where $V = \pi_N(\operatorname{int}(Q))$. This regularity follows readily since the restriction of f to the interior of Q is locally bi-Lipschitz, $\widetilde{f}$ is an element of $N^{1,2}(\pi_N Q, [0,1] \times [0, M])$, and $h = f^{-1} \circ \widetilde{f}$ in V. Now the outer dilatation bound $K_O(h) \leq 1$ follows from Lemma 3.10 and the change of variables formula for π_N. □

Remark 4.4. Given an admissible norm field $N = \sigma \|\cdot\|_2$, the equalities $\operatorname{mod}\Gamma_1 = \operatorname{mod}\widetilde{\Gamma}_1$ and $\operatorname{mod}\Gamma_2 = \operatorname{mod}\widetilde{\Gamma}_2$ in Proposition 4.3 hold if and only if

$$\operatorname{mod}\widetilde{\Gamma}_1 \operatorname{mod}\widetilde{\Gamma}_2 \leq 1. \tag{25}$$

Furthermore, if (25) holds, Proposition 4.3 implies that a 1-quasiconformal homeomorphism $\varphi\colon \pi_N(Q) \to \overline{\mathbb{D}}$ exists. Conversely, if such a homeomorphism φ exists, the inequality (25) follows.

Proposition 4.3 is related to a question posed by Rajala in [Raj17]. Rajala asks whether the reciprocal upper bound (11) implies that points have zero modulus in the sense of (12). Proposition 4.3 verifies this for admissible norm fields $N = \sigma \|\cdot\|_2$ satisfying the sharp upper bound (25).

4.2. **Proof of Theorem 1.3.** Let $E \subset \mathbb{R}^2$ be removable for conformal mappings. We want to prove that for any domain $\Omega \supset E$ and admissible norm field $N\colon \Omega \times \mathbb{R}^2 \to [0, \infty)$ vanishing exactly on E, the quotient space $(\Omega/\mathcal{E}_N, \widetilde{d}_N)$ is reciprocal.

As shown in Lemma 4.1 and Lemma 4.2, we only need to consider the case where $\Omega = \mathbb{R}^2$ and $N = \sigma \|\cdot\|_2$.

Let $R = [a,b] \times [c,d]$ be a rectangle whose interior contains E. Let $\xi_1 = \{a\} \times [c,d]$, $\xi_2 = [a,b] \times \{c\}$, $\xi_3 = \{b\} \times [c,d]$, and $\xi_4 = [a,b] \times \{d\}$. Let $\Gamma_1 = \Gamma(\xi_1, \xi_3; R)$ and $\Gamma_2 = \Gamma(\xi_2, \xi_4; R)$.

Let $\widetilde{\Gamma}_1$ denote the family of paths joining $\pi_N \xi_1$ to $\pi_N \xi_3$ in $\pi_N R$ and $\widetilde{\Gamma}_2$ the family of paths joining $\pi_N \xi_2$ to $\pi_N \xi_4$ in $\pi_N R$. We claim that $\operatorname{mod}\widetilde{\Gamma}_1 = \operatorname{mod}\Gamma_1$ and $\operatorname{mod}\widetilde{\Gamma}_2 = \operatorname{mod}\Gamma_2$. Proposition 4.3 then implies that π_N is 1-quasiconformal in the interior of R. Since R is an arbitrary rectangle containing E, it then follows that π_N is globally 1-quasiconformal.

Observe that the inequalities $\operatorname{mod}\widetilde{\Gamma}_1 \geq \operatorname{mod}\Gamma_1$ and $\operatorname{mod}\widetilde{\Gamma}_2 \geq \operatorname{mod}\Gamma_2$ hold in general by Proposition 3.11. Thus we only need to verify the opposite inequalities.

A standard fact is that there is a sequence of finitely connected domains $\Omega_k \subset \mathbb{R}^2 \setminus E$ such that $\overline{\Omega}_k \subset \Omega_{k+1}$ for all $k \in \mathbb{N}$, each component of $\partial\Omega_k$ is a closed analytic Jordan path, and $\bigcup_{k=1}^{\infty} \Omega_k = \mathbb{R}^2 \setminus E$. We assume without loss of generality that $\partial R \subset \Omega_1$.

For each $n \in \mathbb{N}$, there exists a conformal embedding $\varphi_n\colon \Omega_n \to \mathbb{R}^2$ normalized as

$$\varphi_n(z) = z + \frac{a_{1,n}}{z} + \frac{a_{2,n}}{z^2} + \cdots$$

near ∞ such that the real part of $a_{1,n}$ is the smallest among all conformal embeddings $\psi\colon \Omega_n \to \mathbb{R}^2$ of the form

$$\psi(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots . \tag{26}$$

See for example Section V.2 of [Gol69].

For each $n \in \mathbb{N}$, the minimizer φ_n is unique and its image is a domain $U_n \subset \mathbb{R}^2$ whose complement consists of finitely many line segments parallel to the vertical axis.

For each $n \in \mathbb{N}$, the minimality of the real part of $a_{1,n}$ implies that $0 \geq \mathrm{Re}(a_{1,k}) \geq \mathrm{Re}(a_{1,n})$ for each $k \geq n$. Hence the mappings $\varphi_k|_{\Omega_n} \colon \Omega_n \to \mathbb{R}^2$, $k \geq n$, form a normal family. See the proof of Theorem 1 of [Gol69, Section V.2] for details. A diagonal argument then implies that $(\varphi_n)_{n=1}^{\infty}$ is a normal family. Thus every subsequence of $(\varphi_n)_{n=1}^{\infty}$ has a further subsequence converging uniformly on compact sets to a conformal map $f \colon \mathbb{R}^2 \setminus E \to \mathbb{R}^2$ satisfying the normalization (26) around ∞. By the removability of E, the map extends to a Möbius transformation, and thus (26) implies that $f(z) = z$ for all $z \in \mathbb{R}^2$. Hence the sequence $(\varphi_n)_{n=1}^{\infty}$ itself must converge to the identity map uniformly on compact sets in $\mathbb{R}^2 \setminus E$.

Let Q_n denote the quadrilateral bounded by the Jordan curve $\varphi_n(\partial R)$. The quadrilaterals Q_n converge to R with respect to Hausdorff distance as $n \to \infty$. Let π_1 and π_2 denote projection onto the x-axis and y-axis, respectively, and let $a_n = \sup \pi_1(\varphi_n(\xi_1))$, $b_n = \inf \pi_1(\varphi_n(\xi_3))$, $c_n = \inf \pi_2(\varphi_n(\xi_2))$ and $d_n = \sup \pi_2(\varphi_n(\xi_4))$.

Let $R_n = [a_n, b_n] \times [c_n, d_n]$ and $\widehat{E}_n = \mathbb{R}^2 \setminus \varphi_n(\Omega_n)$. Observe that $\widehat{E}_n$ consists of finitely many vertical slits. Moreover, the sets $\widehat{E}_n$ converge to E in the Hausdorff distance as $n \to \infty$.

There exists n_0 such that for all $n \geq n_0$, the slits $\widehat{E}_n$ are contained in the interior of R_n, and $0 < b_n - a_n$ and $0 < d_n - c_n$. Fix such an n. We claim that

$$\operatorname{mod} \widetilde{\Gamma}_1 \leq \frac{d_n - c_n}{b_n - a_n}. \tag{27}$$

Consider the function $\rho_n \colon \mathbb{R}^2/\mathcal{E}_N \to [0, \infty]$ defined as zero in the complement of $\pi_N(\Omega_n)$, and otherwise by

$$\rho_n = \left(\left(\frac{\chi_{R_n \setminus \widehat{E}_n}}{b_n - a_n} \circ \varphi_n \right) \cdot \frac{J_{\varphi_n}^{-1/2}}{\sigma} \right) \circ (\pi_N|_{\Omega_n})^{-1}.$$

We claim that ρ_n is admissible for $\widetilde{\Gamma}_1$. Let $\gamma \in \widetilde{\Gamma}_1$ be locally rectifiable with respect to $\widetilde{d}_N$.

We consider the restriction of γ to the set $I = \gamma^{-1}\left(\pi_N \varphi_n^{-1}(R_n \setminus \widehat{E}_n)\right)$. We have

$$\int_\gamma \rho_n \, ds_{\widetilde{d}_N} \geq \int_I (\rho_n \circ \gamma) \cdot v_\gamma \, d\mathcal{L}^1.$$

The function $\theta = \varphi_n \circ (\pi_N|_{\Omega_{n+1}})^{-1} \circ \gamma|_I$ is well-defined and satisfies

$$\int_I (\rho_n \circ \gamma) \cdot v_\gamma \, d\mathcal{L}^1 = \int_I \left(\frac{\chi_{R_n \setminus \widehat{E}_n}}{b_n - a_n} \circ \theta \right) \cdot v_\theta \, d\mathcal{L}^1.$$

Since $\widehat{E}_n$ consists of finitely many vertical slits, we conclude using the area formula for paths and the projection onto the x-axis that

$$\int_I \left(\frac{\chi_{R_n \setminus \widehat{E}_n}}{b_n - a_n} \circ \theta \right) \cdot v_\theta \, d\mathcal{L}^1 \geq \frac{1}{b_n - a_n} \mathcal{L}^1 \left(|\pi_1 \circ \theta| \right) \geq 1.$$

Therefore

$$\int_\gamma \rho_n \, ds_{\widetilde{d}_N} \geq 1,$$

and we conclude that ρ_n is admissible. The change of variables formulas for π_N and φ_n yield that

$$\int_{\mathbb{R}^2/\mathcal{E}_N} \rho_n^2 \, d\mathcal{H}^2_{\widetilde{d}_N} = \frac{d_n - c_n}{b_n - a_n}.$$

This verifies (27). Finally, observe that $d_n - c_n \to d - c$ and $b_n - a_n \to b - a$ as $n \to \infty$. This shows that

$$\operatorname{mod} \widetilde{\Gamma}_1 \leq \frac{d-c}{b-a} = \operatorname{mod} \Gamma_1.$$

A similar argument, using conformal mappings onto horizontal slit domains, shows that $\operatorname{mod} \widetilde{\Gamma}_2 \leq \operatorname{mod} \Gamma_2$. This completes the proof.

4.3. **An extension of Theorem 1.3 to integrable norm fields.** In this section, we extend Theorem 1.3 to the case of lower semicontinuous norm fields N with locally bounded distortion such that $L(N) \in L^p_{\mathrm{loc}}(\Omega)$ for some $p \in (2, \infty)$. We assume that N vanishes exactly on a compact set $E \subset \Omega$ that is removable for conformal mappings.

For this section, we allow the possibility for N_x to be infinite at some points $x \in \Omega$. To say this more precisely, in the definition of seminorm in Section 2.3, we consider a seminorm to be a function $S \colon \mathbb{R}^2 \to [0, \infty]$ satisfying the same assumptions listed there, following the convention that $0 \cdot \infty = 0$. An admissible norm field is now a function $N \colon \Omega \times \mathbb{R}^2 \to [0, \infty]$ satisfying the conditions of Definition 3.1, except that local boundedness of N is now replaced by the assumption that $L(N) \in L^p_{\mathrm{loc}}(\Omega)$. Observe that the local boundedness of the distortion then implies that if $N_x(v) = \infty$ for some $v \in \mathbb{R}^2 \setminus \{0\}$, then N_x must have the form

$$N_x(v) = \begin{cases} \infty & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases}.$$

In particular, $\omega(N_x) = L(N_x) = \infty$. Note also that the minimal stretching $\omega(N)$ is lower semicontinuous, and that Lemma 3.3 remains true for $x \in \Omega$ with $\omega(N_x) < \infty$.

We define the pseudodistance d_N exactly as in Definition 3.2. Then for every $x, y \in \Omega$ and any absolutely continuous path $\gamma \colon [0,1] \to \Omega$ joining x to y,

$$d_N(x,y) \leq \int_0^1 (N \circ D\gamma)\, d\mathcal{L}^1 \leq \int_\gamma L(N)\, ds_{\|\cdot\|_2}. \tag{28}$$

Given $z \in \Omega$, for each $x \in \Omega$ we let $u_z(x) = \inf \int_\gamma L(N)\, ds$, the infimum taken over all absolutely continuous paths joining z to x in Ω. Then $x \mapsto u_z(x)$ defines a locally Hölder continuous function [HKST15, Theorems 9.3.1, 9.2.14] having $L(N)$ as a locally L^p-integrable upper gradient. The Hölder exponent depends only on p, and the local Hölder constant depends only on the local L^p-norm of $L(N)$. Moreover, (28) yields that

$$d_N(x,y) \leq \sup_{z \in \Omega} |u_z(x) - u_z(y)|\,. \tag{29}$$

As before, we identify the two points $x, y \in \Omega$ if $d_N(x,y) = 0$ and let X denote the corresponding quotient space. Let $\pi \colon \Omega \to X$ denote the associated quotient map. The quotient distance d_X on X is defined as follows: for every $x, y \in X$, we set $d_X(x,y) = d_N(\pi^{-1}(x), \pi^{-1}(y))$, observing that this is independent of the choice of element in $\pi^{-1}(x)$ and $\pi^{-1}(y)$ and hence well-defined. The inequality (29) implies that π is locally Hölder continuous with $L(N)$ as its locally L^p-integrable upper gradient. In particular, $\pi \in N^{1,p}_{\mathrm{loc}}(\Omega, X)$.

We are now ready for the main result of this section. We recall that N is assumed to vanish on a compact set E removable for conformal mappings.

Proposition 4.5. The metric space X has locally finite Hausdorff 2-measure, and the quotient map π is a locally quasiconformal homeomorphism. In particular, X is a quasiconformal surface.

Proof. We first prove that π is a homeomorphism. To this end, let $\sigma(z) = \omega(N_z)$ and $\widehat{N}_z = \sigma(z)\|\cdot\|_2$ for every $z \in \Omega$. For each $k \in \mathbb{N}$, we define the function $\sigma_k \colon \Omega \to [0,\infty)$ by

$$\sigma_k(z) = \min\{\sigma(z), k\}.$$

Each function σ_k is bounded and lower semicontinuous in Ω, and $\sigma_k(z) = 0$ if and only if $\sigma(z) = 0$. For every $z \in \Omega$, the sequence $(\sigma_k(z))_{k=1}^{\infty}$ is non-decreasing and converges to $\sigma(z)$.

Let $N_k = \sigma_k\|\cdot\|_2$ and $d_k = d_{N_k}$. Since N_k is bounded and lower semicontinuous, Theorem 1.3 implies that $\pi_k \colon \Omega \to (\Omega, d_k)$ defined by $\pi_k(z) = z$ is a 1-quasiconformal homeomorphism. Since $N_k \leq \omega(N)\|\cdot\|_2 \leq N$ everywhere, we see that

$$d_k(\pi_k(x), \pi_k(y)) \leq d_X(\pi(x), \pi(y))$$

for all $x, y \in \Omega$. Since π_k is a homeomorphism, we see that π is injective. Now the map $\psi_k \colon X \to (\Omega, d_k)$ defined by $\psi_k = \pi_k \circ \pi^{-1}$ is 1-Lipschitz, hence $\pi^{-1} = \pi_k^{-1} \circ \psi_k$ is continuous. Therefore π is a homeomorphism.

Recall that N has locally bounded distortion. From this and the fact that, for every $x \in X$, $\pi^{-1}(\overline{B}_X(x,r))$ is compact for sufficiently small $r > 0$, we see that the induced distances d_N and $d_{\widehat{N}}$ are locally bi-Lipschitz equivalent. We assume from this point onwards, without loss of generality, that $N = \widehat{N} = \sigma\|\cdot\|_2$.

Let Γ_0 denote the family of paths along which $\sigma = L(N)$ fails to be integrable. Since $\sigma \in L^p_{\mathrm{loc}}(\Omega) \subset L^2_{\mathrm{loc}}(\Omega)$, the family Γ_0 has zero modulus. The inequality $d_X \geq d_k$ and Lemma 3.7 imply that for any absolutely continuous path θ in Ω

$$\ell_{d_X}(\pi \circ \theta) \geq \lim_{k\to\infty} \ell_{d_k}(\psi_k \circ \theta) = \lim_{k\to\infty} \ell_{N_k}(\theta) = \ell_N(\theta),$$

where the latter equality follows from monotone convergence. If $\theta \notin \Gamma_0$, we have $\ell_N(\theta) < \infty$ and the definition of d_X implies $\ell_{d_X}(\pi \circ \theta) \leq \ell_N(\theta)$. So $\ell_{d_X}(\pi \circ \theta) = \ell_N(\theta)$. Since the equality $\ell_{d_X}(\pi \circ \theta) = \ell_N(\theta)$ holds outside the negligible family Γ_0, the norm field N is the *approximate metric differential* of π_N; see [LW18, Sections 3.3 and 3.4]. Consequently, $L(N) = \sigma$ is a minimal weak upper gradient of π_N and $J_2(N) = \sigma^2$ the Jacobian of π_N. Since we also have that $\pi \in N^{1,p}_{\mathrm{loc}}(\Omega, X)$ for $p > 2$, it satisfies Lusin's Condition (N) [Vod00, Theorem 7.1]. Therefore, for each compact set $K \subset \Omega$,

$$\mathcal{H}^2_X(\pi(K)) = \int_K \sigma^2 \, d\mathcal{L}^2 < \infty.$$

We conclude that X has locally finite Hausdorff 2-measure. An application of Theorem 2.6 yields that $K_O(\pi) = 1$.

The proof is complete after we verify $K_O(\pi^{-1}) = 1$. Since π_k is 1-quasiconformal for every k, it suffices to verify $K_O(\psi_k) = 1$ for some k. To this end, we fix an arbitrary $k \in \mathbb{N}$ and recall that ψ_k is 1-Lipschitz.

Since π_k is a quasiconformal homeomorphism, it satisfies Lusin's Condition (N^{-1}). This implies that the map π^{-1} satisfies Lusin's Condition (N). As a consequence, the Jacobian of ψ_k coincides with ρ_k^2 for $\rho_k = ((\sigma_k/\sigma)\chi_{\Omega\setminus E}) \circ \pi^{-1}$.

Since ψ_k is Lipschitz, we have $\psi_k \in N^{1,2}_{\mathrm{loc}}(X, (\Omega, d_k))$. We claim that any minimal weak upper gradient of ψ_k coincides with ρ_k almost everywhere in X. If we verify this, then $K_O(\psi_k) = 1$ follows from Theorem 2.6.

Consider an absolutely continuous path $\gamma \colon [0,1] \to X$ with $|\gamma| \subset X \setminus \pi(E)$. Then $\psi_k \circ \gamma$ is absolutely continuous, and since d_k and $\|\cdot\|_2$ are locally bi-Lipschitz equivalent in a neighborhood of the image of $\theta = \pi_k^{-1} \circ \psi_k \circ \gamma$, the path θ is absolutely continuous with respect to $\|\cdot\|_2$. Then, by monotone convergence and Lemma 3.7 applied to each d_n,

$$\ell_N(\theta) = \lim_{n\to\infty} \ell_{N_n}(\theta) = \lim_{n\to\infty} \ell_{d_n}(\psi_n \circ \gamma).$$

Since every ψ_n is 1-Lipschitz,

$$\lim_{n\to\infty} \ell_{d_n}(\psi_n \circ \gamma) \leq \ell_{d_X}(\gamma).$$

Therefore $\ell_N(\theta) \leq \ell_{d_X}(\gamma) < \infty$, and, by the construction of d_X, $\ell_{d_X}(\gamma) \leq \ell_N(\theta)$.

Since $\ell_N(\theta) = \ell_{d_X}(\gamma)$ holds for every subpath of γ, we see that $v_\gamma = N \circ D\theta = (\sigma \circ \theta) \cdot v_\theta$ and $v_{\psi_k \circ \gamma} = (\sigma_k \circ \theta) \cdot v_\theta$ almost everywhere in the domain of γ. We conclude from this that

$$v_{\psi_k \circ \gamma} = (\rho_k \circ \gamma) \cdot v_\gamma \tag{30}$$

almost everywhere with respect to the length measure of γ. If $\widetilde{\Gamma}_0$ denotes the family of absolutely continuous paths in X that have positive length on the set $\pi(E)$, then $\mathcal{H}^2_X(\pi(E)) = 0$ implies $\operatorname{mod} \widetilde{\Gamma}_0 = 0$. The equality (30) remains valid for every absolutely continuous path $\gamma \notin \widetilde{\Gamma}_0$. Indeed, for any such path $\gamma\colon [a,b] \to X$, the set $\gamma^{-1}(\pi(E))$ is a compact set having $v_\gamma \mathcal{L}^1$-measure zero. This observation and the fact that (30) holds on compact intervals contained in $[a,b] \setminus \gamma^{-1}(\pi(E))$ yield the validity of (30). Since (30) is valid for every γ outside a negligible family, ρ_k is a minimal weak upper gradient of ψ_k. □

Remark 4.6. The norm field $N = \sigma \|\cdot\|_2$ defined by the weight $\sigma(x) = \|x\|_2^{-1}(1 - \log \|x\|_2)^{-1} \in L^2(\mathbb{D})$ induces a complete hyperbolic metric on the punctured disk of radius e. In particular, the origin is at infinite distance from any other point. Consequently, the assumption $p > 2$ in Proposition 4.5 cannot be relaxed to $p = 2$.

5. Reciprocal implies removable

This section is dedicated to a proof of Theorem 1.4. Recall that we consider a compact set $E \subset \Omega$ for which $\Omega \setminus E$ is connected, together with the norm field N defined by $N_x = \min\{1, d_{\|\cdot\|_2}(E,x)^p\} \|\cdot\|_2$ for some $p > \max\{\dim_{\mathcal{H}} E - 1, 0\}$. The norm field N induces a decomposition $\mathcal{E}_N$ of Ω, a metric $\widetilde{d}_N$ on $\Omega/\mathcal{E}_N$, and a quotient map $\pi\colon \Omega \to (\Omega/\mathcal{E}_N, \widetilde{d}_N)$, as described in Section 3.

5.1. Decay of the norm field near E.

The following lemma states that if N decays to zero sufficently fast near E, then each component of E collapses to a point under the quotient map π_N.

Lemma 5.1. Let $N_x = \min\{1, d_{\|\cdot\|_2}(x,E)^p\} \|\cdot\|_2$. For all $p > \max\{\dim_{\mathcal{H}} E - 1, 0\}$, $\mathcal{H}^1_{\widetilde{d}_N}(\pi_N(E)) = 0$. Consequently, the preimage of every $x \in \pi_N(E)$ is a connected component of E.

Proof. Let $p > \dim_{\mathcal{H}} E - 1$ and let $\varepsilon > 0$. By the definition of Hausdorff dimension, there exists $\delta > 0$ and a countable collection of sets $\mathcal{A} = \{A_j\}$ such that $E \subset \bigcup_j A_j$, $\operatorname{diam}_{\|\cdot\|_2} A_j \leq \delta$ for all j, and

$$\sum_j (\operatorname{diam}_{\|\cdot\|_2} A_j)^{p+1} < \varepsilon.$$

Without loss of generality, we may assume that $A_j \cap E \neq \emptyset$ for all j. Let $d_j = \operatorname{diam}_{\|\cdot\|_2} A_j$. Thus $A_j \subset \overline{B}_{\|\cdot\|_2}(y, d_j)$ for some $y \in E$. By integrating N over the straight-line path from y to a point $z \in A_j$, it follows that

$$d_N(y,z) \leq \int_0^{d_j} t^p \, dt = \frac{d_j^{p+1}}{p+1}.$$

Thus $\operatorname{diam}_{d_N} A_j \leq 2(p+1)^{-1} d_j^{p+1} < 2(p+1)^{-1}\delta^{p+1}$, and $\sum_j \operatorname{diam}_{d_N} A_j < 2(p+1)^{-1}\varepsilon$. This is sufficient to show that $\mathcal{H}^1_{\widetilde{d}_N}(\pi_N(E)) = 0$.

Next, let $x \in \pi_N(E)$. Proposition 3.8 implies that $\pi_N^{-1}(x)$ is a subset of a connected component F of E. Since $\pi_N(F)$ is a connected, compact subset of $\pi_N(E)$, we have that

$$\operatorname{diam} \pi_N(F) \leq \mathcal{H}^1_{\widetilde{d}_N}(\pi_N(F)) \leq \mathcal{H}^1_{\widetilde{d}_N}(\pi_N(E)) = 0.$$

Hence $\pi_N(F) = x$ and we must have $F = \pi_N^{-1}(x)$. □

5.2. **Proof of Theorem 1.4.** We first observe that if $(\Omega/\mathcal{E}_N, \widetilde{d}_N)$ is reciprocal, then the space formed by taking the same set E and the same definition for N, but applied to all points $x \in \mathbb{R}^2$, is also reciprocal. Thus the choice of domain Ω is not relevant for the proof, and we assume for the remainder of the section that $\Omega = \mathbb{R}^2$.

We prove the contrapositive: if E is not removable for conformal mappings, then $(\mathbb{R}^2/\mathcal{E}_N, \widetilde{d}_N)$ is not reciprocal.

Let $E \subset \mathbb{R}^2$ be a set that is not removable for conformal mappings. As a consequence of Proposition 2.9, there is a compact set $\widehat{E} \subset \mathbb{R}^2$ of positive Lebesgue measure and a conformal map $f\colon \mathbb{R}^2 \setminus \widehat{E} \to \mathbb{R}^2 \setminus E$ whose extension to $\widehat{\mathbb{R}}^2 \setminus \widehat{E}$ fixes ∞. Let $\widehat{N} = \chi_{\mathbb{R}^2 \setminus \widehat{E}} \|\cdot\|_2$ and let $\widehat{\pi}\colon \mathbb{R}^2 \to (\mathbb{R}^2/\mathcal{E}_{\widehat{N}}, \widetilde{d}_{\widehat{N}})$ be the associated quotient map. Observe that $\widehat{N}$ is an admissible norm field vanishing on the set $\widehat{E}$.

The following lemma states that f extends to a mapping of the respective quotient spaces. For brevity, let $\widehat{Y} = \mathbb{R}^2/\mathcal{E}_{\widehat{N}}$ and $Y = \mathbb{R}^2/\mathcal{E}_N$.

Lemma 5.2. The map $f\colon \mathbb{R}^2 \setminus \widehat{E} \to \mathbb{R}^2 \setminus E$ induces a continuous monotone map $\widehat{f}\colon \widehat{Y} \to Y$. That is, there is a monotone map $\widehat{f}\colon \widehat{Y} \to Y$ satisfying $\widehat{f} \circ \widehat{\pi}(x) = \pi_N \circ f(x)$ for all $x \in \mathbb{R}^2 \setminus \widehat{E}$.

Proof. Let $y \in \widehat{Y}$, and let $\widehat{F}$ denote its preimage under $\widehat{\pi}$. If $\widehat{F} = \{x\}$ for some point $x \notin \widehat{E}$, then we set $\widehat{f}(y) = \pi_N \circ f(x)$.

Otherwise, $\widehat{F}$ is a subset of some component $\widehat{A}$ of $\widehat{E}$. For all $m \in \mathbb{N}$, let $\widehat{\gamma}_m$ be a Jordan path with image contained in $B_{\|\cdot\|_2}(\widehat{A}, 1/m) \setminus \widehat{E}$ that separates $\widehat{A}$ and infinity. The curve $|\widehat{\gamma}_m|$ is the boundary of a closed region $\widehat{A}_m$ containing $\widehat{A}$. We assume without loss of generality that $|\widehat{\gamma}_{m+1}| \subset \widehat{A}_m$ for all m.

By assumption, $\gamma_m = f \circ \widehat{\gamma}_m$ is a Jordan loop whose image bounds a compactly contained domain A_m. Let $A = \bigcap_m \overline{A}_m$. It is immediate that A is nonempty and compact. The intersection is also connected; see for example Section 28 of [Wil70]. This implies that A is a connected component of E. Therefore $\pi_N(A)$ is a point by Lemma 5.1. We define $\widehat{f}(y) = \pi_N(A)$.

We now check that $\widehat{f}$ is continuous. Let $y \in \widehat{Y}$ and let (y_n) be a sequence in $\widehat{Y}$ converging to y. Let $\widehat{F}_n = \widehat{\pi}^{-1}(y_n)$. In the case that $\widehat{F} = \{x\}$ for some $x \notin \widehat{E}$, the continuity is obvious. Otherwise, we proceed as follows. For each fixed $m \in \mathbb{N}$, $F_n \subset \widehat{A}_m$ for sufficiently large n. This implies that $\widehat{f}(y_n) \subset \pi_N(A_m)$. Therefore the accumulation points of $\widehat{f}(y_n)$ are in the intersection of $\pi_N(A_m)$. Since the intersection equals $\pi_N(A)$, the sequence $\widehat{f}(y_n)$ converges to $\pi_N(A) = \widehat{f}(y)$. The continuity follows.

By construction, the preimage of a point in $\mathbb{R}^2/\mathcal{E}$ under $\widehat{\pi} \circ \widehat{f}$ is either a single-point set or a component of $\widehat{E}$. We conclude that $\widehat{f}$ is monotone. □

Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle whose interior contains $\widehat{E}$. Let Γ_1 denote the family of paths $\gamma_t\colon [a, b] \to \mathbb{R}^2$, where $t \in [c, d]$, defined by $\gamma_t(s) = (s, t)$. Thus Γ_1 is a foliation of R by horizontal paths. Let Γ_2 denote the corresponding foliation of R by vertical paths.

Next, let Q be the Jordan domain bounded by $f(\partial R)$, and let $\xi_1, \xi_2, \xi_3, \xi_4$ denote, respectively, the image of the left, bottom, right, and top side of R. Let $\widetilde{\Gamma}_1$ denote the family paths joining $\pi_N\xi_1$ to $\pi_N\xi_3$ in $\pi_N Q$ and $\widetilde{\Gamma}_2$ the family of paths joining $\pi_N\xi_2$ to $\pi_N\xi_4$.

By Lemma 4.2, it suffices to show that Y is not 1-reciprocal. Thus the proof is complete after we verify the inequalities

$$1 < \operatorname{mod} \widehat{\pi}\Gamma_1 \operatorname{mod} \widehat{\pi}\Gamma_2 \tag{31}$$

and

$$\operatorname{mod} \widehat{\pi}\Gamma_1 \operatorname{mod} \widehat{\pi}\Gamma_2 \leq \operatorname{mod} \widetilde{\Gamma}_1 \operatorname{mod} \widetilde{\Gamma}_2. \tag{32}$$

Define the function $P\colon \mathbb{R}^2 \to [0,\infty]$ by

$$P(x) = \begin{cases} L(N_{f(x)})\,\|D_x f\| & \text{if } x \notin \widehat{E} \\ 0 & \text{if } x \in \widehat{E} \end{cases}.$$

Since N is a weighted Euclidean norm and f is conformal in the complement of $\widehat{E}$, it follows that $N \circ D_x f(v) = P(x)\,\|v\|_2$ for all $v \in \mathbb{R}^2$ and all $x \in \mathbb{R}^2 \setminus \widehat{E}$.

We consider the function $\widehat{P}\colon \mathbb{R}^2/\widehat{\mathcal{E}} \to [0,\infty]$ defined by taking $\widehat{P}(x) = P(\widehat{\pi}^{-1}(x))$. Observe that $\widehat{P}$ is well-defined since $\widehat{\pi}$ is injective outside of $\widehat{E}$. Loosely speaking, $\widehat{P}$ is a weak upper gradient of $\widehat{f}$.

Let $\rho\colon \widehat{f}(\widehat{R}) \to [0,\infty]$ be an admissible function for $\widetilde{\Gamma}_1$, and let $\widehat{\rho} = (\rho \circ \widehat{f})\widehat{P}$. We first observe that

$$\int_{\widehat{R}} \widehat{\rho}^2 \, d\mathcal{H}^2_{\widehat{d}} = \int_{\widehat{f}(\widehat{R})} \rho^2 \, d\mathcal{H}^2_{\widetilde{d}_N}. \tag{33}$$

Indeed, the integrals are left unchanged by the removal of $\pi_N(E)$ and $\widehat{\pi}(\widehat{E})$ from both sides. With this reduction, the identity (33) follows from the Jacobian identities $J_f \equiv \|Df\|^2$, $J_{\widehat{\pi}} = \chi_{\mathbb{R}^2\setminus E}$, and $J_{\pi_N} = L^2(N)$.

Next, we claim that $\widehat{\rho}$ is weakly admissible for $\widehat{\pi}\Gamma_1$. Let $\widehat{\gamma}_t$ denote the image under $\widehat{\pi}$ of the horizontal path γ_t in the quotient space $\mathbb{R}^2/\widehat{\mathcal{E}}$. Lemma 3.7 implies that

$$v_{\widehat{f}\circ\widehat{\gamma}_t}(s) = N \circ Df \circ D\gamma_t(s) = \left(\widehat{P} \circ \widehat{\gamma}_t(s)\right) \cdot v_{\widehat{\gamma}_t}(s) \tag{34}$$

for $\mathcal{L}^1$-almost every $s \in [a,b] \setminus \gamma_t^{-1}(\widehat{E})$ and that the total variation of $\widehat{\gamma}_t$ in $\widehat{\pi}\widehat{E}$ is zero. Similarly, since $\mathcal{H}^1_{\widetilde{d}_N}(\pi_N(E)) = 0$ by Lemma 5.1, the area formula [Fed69, Theorem 2.10.13] for paths implies that the total variation of $\widehat{f} \circ \widehat{\gamma}_t$ in $\pi_N(E)$ is zero. We conclude that $\widehat{f} \circ \widehat{\gamma}_t$ is absolutely continuous as long as the right-hand side of (34) is integrable.

Observe that (33) holds with the characteristic function $\chi_{\widehat{f}\widehat{R}}$ in place of ρ and $\widehat{P}$ in place of $\widehat{\rho}$. Then an application of Fubini's theorem implies that the function in the right-hand side of (34) is integrable for $\mathcal{L}^1$-almost every t. For such t, we conclude from (34) that

$$1 \leq \int_{\widehat{f}\circ\widehat{\gamma}_t} \rho \, ds = \int_{\widehat{\gamma}_t} \widehat{\rho} \, ds.$$

Therefore $\widehat{\rho}$ is weakly admissible for $\widehat{\pi}\Gamma_1$, and the equality (33) implies that

$$\operatorname{mod} \widehat{\pi}\Gamma_1 \leq \operatorname{mod} \widetilde{\Gamma}_1.$$

A similar argument applied to the path family $\widehat{\pi}\Gamma_2$ gives $\operatorname{mod} \widehat{\pi}\Gamma_2 \leq \operatorname{mod} \widetilde{\Gamma}_2$. The inequality (32) now follows.

To conclude the proof, we prove (31). Let ρ be admissible for $\widehat{\pi}\Gamma_1$. Then for all $t \in [c,d]$, we have $1 \leq \int_c^d \rho \circ \widehat{\pi} \chi_{\mathbb{R}^2\setminus\widehat{E}}(s,t)\, dt$. Applying Fubini's theorem and

Hölder's inequality gives

$$d - c \leq \int_R \rho \circ \widehat{\pi} \chi_{\mathbb{R}^2 \setminus \widehat{E}} \, d\mathcal{L}^2 \leq \left(\int_R \rho \circ \widehat{\pi}^2 \chi_{\mathbb{R}^2 \setminus \widehat{E}} \, d\mathcal{L}^2 \right)^{1/2} \mathcal{L}^2(R \setminus \widehat{E})^{1/2}.$$

After rearranging and taking the infimum over admissible ρ, we find that

$$(d-c)^2 / \mathcal{L}^2(R \setminus \widehat{E}) \leq \operatorname{mod} \widehat{\pi} \Gamma_1.$$

The analogous argument gives $(b-a)^2/\mathcal{L}^2(R \setminus \widehat{E}) \leq \operatorname{mod} \widehat{\pi}\Gamma_2$. Thus

$$1 < \frac{(b-a)^2(d-c)^2}{\mathcal{L}^2(R \setminus \widehat{E})^2} \leq \operatorname{mod} \widehat{\pi}\Gamma_1 \operatorname{mod} \widehat{\pi}\Gamma_2.$$

This establishes (31) and completes the proof.

6. Linear Cantor sets: two examples

We call a Cantor set $E \subset \mathbb{R} \times \{0\}$ a *linear Cantor set*. As remarked in Section 1.3, a norm field vanishing on a linear Cantor set E of positive length may or may not be reciprocal. For completeness, we include here two explicit examples to illustrate both of these cases. Recall from the discussion following the statement of Theorem 1.4 that a compact set $E \subset [0,1] \times \{0\}$ is removable for conformal mappings if there exists an admissible norm field N vanishing on E that is reciprocal. Such a set E is necessarily a linear Cantor set by Proposition 3.12. Conversely, if there exists an admissible norm field vanishing on a linear Cantor set E that is not reciprocal, then E is not removable for conformal mappings. Versions of these examples are already present in [AB50, Sections 6-7]. A closely related construction, and the one that we directly based Example 6.1 on, is found in Section 11 of an early version of the paper [Sch95].

Example 6.1. We construct a lower semicontinuous weight $\sigma \colon \mathbb{R}^2 \to [0, \infty]$ that vanishes on a Cantor set $E \subset [0,1] \times \{0\}$ of positive length such that the space $(\mathbb{R}^2, d_\sigma)$ is not reciprocal. The idea is to make E sufficiently large so that the modulus of the path family joining $(0,0)$ to $(0,1)$ in $(\mathbb{R}^2, d_\sigma)$ is positive.

Identify $[0,1]$ with the set $[0,1] \times \{0\} \subset \mathbb{R}^2$. Let $a_1 = 1/2$, and now define inductively sequences (a_j), (b_j) by the rules $b_j = a_j / \exp(4^j)$ and $a_{j+1} = (a_j - b_j)/2$.

Let I_1 be an open interval centered at $t_1 = 1/2$ of length $2b_1$. Define next open intervals I_j inductively as follows. Assume that we have a collection of disjoint open intervals $I_1, \ldots, I_{j-1}$. From the complement $[0,1] \setminus \bigcup_{k=1}^{j-1} I_k$, choose any closed interval J_j of largest length. Let t_j be the midpoint of J_j, and let I_j be the open interval centered at t_j of length $2b_j$. We record the observation that $d_{\|\cdot\|_2}(t_j, \{0,1\}) = \min\{t_j, 1-t_j\} \geq a_j$. Let $E = [0,1] \setminus \bigcup_j I_j$, and let $\sigma = \chi_{\mathbb{R}^2 \setminus E}$. This yields a corresponding metric d_σ on $\mathbb{R}^2$. Note that the metric d_σ agrees with the Euclidean metric locally outside of E. Thus the Hausdorff 2-measure relative to d_σ coincides with Lebesgue 2-measure. Also, observe that the Lebesgue 1-measure of $[0,1] \setminus E$ is at most $\sum_{j=1}^\infty 2b_j < 1$.

Consider now an interval I_j. For all $t \in (t_j - a_j, t_j - b_j)$, let $\gamma_{j,t}$ be the path that connects t to $2t_j - t$ along the upper semicircle of the circle centered at t_j with radius $t_j - t$. Let Γ_j be the family of all such paths $\gamma_{j,t}$. Observe that Γ_j is a full-modulus subfamily of the family of paths in the upper half-plane H that separate the sets $\overline{B}_{\|\cdot\|_2}((t_j, 0), b_j)$ and $H \setminus B_{\|\cdot\|_2}((t_j,0), a_j)$.

Since the metric speed of $\gamma \in \Gamma_j$ with respect to Euclidean distance and with respect to d_σ coincide almost everywhere along γ, the modulus of Γ_j with respect to the metric d_σ equals the Euclidean modulus: $\operatorname{mod} \pi_\sigma \Gamma_j = \log(a_j/b_j)/\pi$. See for example [Hei01, Lemma 7.18].

We claim that the metric d_σ violates the reciprocality condition (12). Let $F_1 = \{(0,0)\}$ and $F_2 = \{(1,0)\}$ and let $\Gamma = \Gamma(F_1, F_2; \mathbb{R}^2)$. Recall the notation $\Gamma(F_1, F_2; G)$ defined in Section 2.5.

Observe that Γ is a subfamily of $\Gamma(F_1, \mathbb{R}^2 \setminus \mathbb{D}; \mathbb{R}^2)$, which is majorized by the annular path families $\Gamma(B_{\|\cdot\|_2}(0,\varepsilon), \mathbb{R}^2 \setminus \mathbb{D}; \mathbb{R}^2)$ for all $\varepsilon > 0$. In particular,

$$\operatorname{mod} \pi_\sigma \Gamma(B_{\|\cdot\|_2}(0,\varepsilon), \mathbb{R}^2 \setminus \mathbb{D}; \mathbb{R}^2) \geq \operatorname{mod} \pi_\sigma \Gamma$$

for all $\varepsilon > 0$. Thus it is sufficient to show that $\operatorname{mod} \pi_\sigma \Gamma > 0$.

Let ρ be an admissible function for Γ for the metric d_σ. For each $j \in \mathbb{N}$, let $m_j = \inf\{\int_\gamma \rho \, ds_\sigma : \gamma \in \Gamma_j\}$. If $m_j > 0$, this implies that ρ/m_j is admissible for the path family Γ_j, and thus that

$$\int_{\mathbb{R}^2} \frac{\rho^2}{m_j^2} \, d\mathcal{H}^2_\sigma \geq \operatorname{mod} \Gamma_j = \frac{\log(a_j/b_j)}{\pi}. \tag{35}$$

For each $j \in \mathbb{N}$, let γ_j be a path in Γ_j such that $\int_{\gamma_j} \rho \, ds_\sigma \leq \max\{2m_j, 2^{-j-1}\}$. For each $i \in \mathbb{N}$, we define $\eta_i \colon [0,1] \to \mathbb{R}^2$ by

$$\eta_i(t) = (t, \theta_i(t)),$$

where $\theta_i(t) = \sup_{1 \leq j \leq i} \pi_2(|\gamma_j| \cap (\{t\} \times \mathbb{R}))$. Here, π_2 denotes projection onto the vertical axis and the supremum over the empty set is meant to be zero. Observe that $\eta_i \in \Gamma$.

Let $\theta(t) = \lim_{i\to\infty} \theta_i(t)$, and observe for every $t \in (0,1) \setminus E$, $\theta(t) > 0$ since for each j the projection of $|\gamma_j|$ to the x-axis covers the interval I_j. We set $\eta(t) = (t, \theta(t))$ for $0 \leq t \leq 1$, and note that

$$\ell_\sigma(\eta) \leq \liminf_{i\to\infty} \ell_\sigma(\eta_i) \leq \sum_{j=1}^\infty \ell_{\|\cdot\|_2}(\gamma_j) \leq \sum_{j=1}^\infty \pi a_j \leq \pi.$$

Consequently, η is d_σ-rectifiable, and

$$1 \leq \int_\eta \rho \, ds_\sigma \leq \sum_{j=1}^\infty \int_{\gamma_j} \rho \, ds_\sigma \leq \sum_{j=1}^\infty \max\{2m_j, 2^{-j-1}\}.$$

From the identity $\sum_{j=1}^\infty 1/2^j = 1$, it follows that $m_j \geq 1/(2^{j+1})$ for some $j \in \mathbb{N}$. This together with (35) gives

$$\begin{aligned} \frac{1}{2^{j+1}} \leq m_j &\leq \left(\frac{\pi}{\log(a_j/b_j)}\right)^{1/2} \left(\int_{\mathbb{R}^2} \rho^2 \, d\mathcal{H}^2_{\|\cdot\|_2}\right)^{1/2} \\ &= \left(\frac{\pi}{4^j}\right)^{1/2} \left(\int_{\mathbb{R}^2} \rho^2 \, d\mathcal{H}^2_{\|\cdot\|_2}\right)^{1/2}. \end{aligned}$$

This yields the lower bound

$$\frac{1}{4\pi} \leq \int_{\mathbb{R}^2} \rho^2 \, d\mathcal{H}^2_{\|\cdot\|_2}.$$

We conclude that $(\mathbb{R}^2, d_\sigma)$ is not reciprocal.

Example 6.2. We construct a lower semicontinuous weight $\sigma \colon \mathbb{R}^2 \to [0,\infty]$ that vanishes on a Cantor set $E \subset [0,1] \times \{0\}$ of positive length such that the space $(\mathbb{R}^2, d_\sigma)$ is reciprocal.

Consider the quadrilateral $Q = [0,1] \times [-1,1]$. Let Γ be the family of paths in Q connecting the left and right edges of Q.

Fix for the time being a value $t \in (0,1/2)$. Let $I = [t, 1-t] \times \{0\} \subset (0,1) \times \{0\}$ and let $\sigma_1 = \chi_{\mathbb{R}^2 \setminus I}$, noting that σ_1 vanishes on the set I. Let $\mathcal{E}_1$ denote the decomposition of $\mathbb{R}^2$ corresponding to I. The weight σ_1 determines a metric $\widetilde{d}_{\sigma_1}$

on $\mathbb{R}^2/\mathcal{E}_1$ that is not reciprocal. Let π_{σ_1} denote the associated quotient map. The metric $\widetilde{d}_{\sigma_1}$, like all other metrics in this example, agrees with the Euclidean metric locally outside of $\pi_{\sigma_1}(I)$, and the Hausdorff 2-measure relative to the metric $\widetilde{d}_{\sigma_1}$ coincides with Lebesgue measure.

Let $\widetilde{\rho}$ be an admissible function for $\pi_{\sigma_1}\Gamma$ with respect to the metric $\widetilde{d}_{\sigma_1}$ satisfying $\int_{\mathbb{R}^2/\mathcal{E}_1} \widetilde{\rho}^2 \, d\mathcal{H}^2_{\widetilde{d}_{\sigma_1}} \leq 2 \operatorname{mod} \pi_{\sigma_1}\Gamma$. Since the function

$$\widetilde{g} = \frac{\chi_{[0,t)\times[-1,1]} + \chi_{(1-t,1]\times[-1,1]}}{2t}$$

is admissible for $\pi_{\sigma_1}\Gamma$, it follows that

$$\int_{\mathbb{R}^2/\mathcal{E}_1} \widetilde{\rho}^2 \, d\mathcal{H}^2_{\widetilde{d}_{\sigma_1}} \leq 2 \int_{\mathbb{R}^2/\mathcal{E}_1} \widetilde{g}^2 \, d\mathcal{H}^2_{\widetilde{d}_{\sigma_1}} = \frac{2}{t}. \tag{36}$$

Let $\rho = \chi_Q + \widetilde{\rho} \circ \pi_{\sigma_1}$.

For all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, let φ_i^n denote the similarity mapping of $\mathbb{R}^2$ taking I to the interval $[(i-1+t)/n, (i-t)/n]$. Explicitly,

$$\varphi_i^n(x) = x/n + ((i-1)/n, 0).$$

Let $I_n = \bigcup_{i=1}^n \varphi_i^n(I)$ and let $\mathcal{E}_n$ denote the corresponding decomposition of $\mathbb{R}^2$. Let $\sigma_n = \chi_{\mathbb{R}^2 \setminus I_n}$, and $\widetilde{d}_{\sigma_n}$ the resulting metric on $\mathbb{R}^2/\mathcal{E}_n$.

Let $\rho_i^n = \rho \circ (\varphi_i^n)^{-1}$. Define now the function $\rho_n \colon Q \to [0, \infty]$ by

$$\rho_n(x) = \begin{cases} \rho_i^n(x) & \text{if } x \in \varphi_i^n((0,1) \times [-1,1]) \text{ for some } i \in \{1, \ldots, n\} \\ 1 & \text{otherwise} \end{cases}.$$

For all $x \in \pi_{\sigma_n}(Q)$, we define $\widetilde{\rho}_n(x) = \rho_n(\pi_{\sigma_n}^{-1}(x))$. We claim that $\widetilde{\rho}_n$ is admissible for $\pi_{\sigma_n}\Gamma$ with respect to the metric $\widetilde{d}_{\sigma_n}$.

Consider an arbitrary path $\gamma \in \pi_{\sigma_n}\Gamma$. For each $i \in \{1, \ldots, n\}$, let

$$Q_i^n = [(i-1)/n, i/n] \times [-1, 1],$$

and let γ_i^n be a subpath of γ that traverses Q_i^n horizontally. It suffices to show that

$$\int_{\gamma_i^n} \rho_n \sigma_n \, ds_{\|\cdot\|_2} \geq 1/n.$$

If γ_i^n does not intersect I_n, then this is clear since $\rho_i^n \geq 1$ on $Q_i^n \setminus I_n$. If γ_i^n is contained in $\varphi_i^n(Q)$, then this is also immediate by the admissibility of $\widetilde{\rho}$. Finally, if γ_i^n intersects both I_n and $Q_i^n \setminus \varphi_i^n(Q)$, then γ_i^n must travel a vertical distance of $1/n$, and again the conclusion follows. We conclude that ρ_n is admissible for Γ with respect to the metric d_{σ_n}.

Next, we have the upper bound

$$\int_Q \rho_n^2 \, d\mathcal{L}^2 \leq \int_Q 1 \, d\mathcal{L}^2 + \sum_{i=1}^n \int_{Q_i^n} (\rho_i^n)^2 \, d\mathcal{L}^2 \leq 2 + \frac{\|\rho\|^2_{L^2(Q)}}{n}. \tag{37}$$

Observe that $2 = \operatorname{mod} \Gamma$. Thus, by taking n to be sufficiently large, the modulus of $\pi_{\sigma_n}\Gamma$ with respect to $\widetilde{d}_{\sigma_n}$ becomes arbitrarily close to the Euclidean modulus.

We can now define the Cantor set E as follows. For a given $t \in (0, 1/2)$ and $n \in \mathbb{N}$, let $I(t)$, $I_n(t)$ and $\sigma_n(t)$ denote respectively the sets I and I_n and the weight σ_n constructed above. For all $j \in \mathbb{N}$, let $t_j = 2^{-j-2}$, observing that $\mathcal{L}^1(I(t_j)) = 1 - 2t_j$. Let $\widetilde{\sigma}_j = \sigma_{n_j}(t_j)$. By choosing n_j sufficiently large, we can guarantee that

$$\mathcal{L}^1(I_{n_j}(t_j) \cap I_{n_{j-1}}(t_{j-1})) \geq (1 - 4t_j)\mathcal{L}^1(I_{n_{j-1}}(t_{j-1}))$$

and that $\operatorname{mod} \pi_{\widetilde{\sigma}_j}\Gamma \leq 2 + 1/j$ by applying (36) and (37). Inductively choosing n_j in this manner, we have

$$\mathcal{L}^1\left(\bigcap_{i=1}^{j} I_{n_i}(t_i)\right) \geq \prod_{i=1}^{j}(1-4t_i) = \prod_{i=1}^{j}(1-2^{-i}).$$

Let $E = \bigcap_{j=1}^{\infty} I_{n_j}(t_j)$ and let $\sigma = \chi_{\mathbb{R}^2\setminus E}$, yielding the metric d_σ on $\mathbb{R}^2$. Then $\mathcal{L}^1(E) = \prod_{j=1}^{\infty}(1-2^{-j}) > 0$. Moreover, $\sigma \geq \widetilde{\sigma}_j$ for all $j \in \mathbb{N}$. This fact, combined with Theorem 2.6, yields that $2 \leq \operatorname{mod} \pi_\sigma \Gamma \leq \operatorname{mod} \pi_{\widetilde{\sigma}_j}\Gamma$ for all $j \in \mathbb{N}$. We conclude that $\operatorname{mod} \pi_\sigma\Gamma = 2 = \operatorname{mod} \Gamma$.

Let Γ^* denote the family of paths connecting the bottom and top edges of Q. It is clear that the function $\rho^* = (1/2)\chi_{\pi_\sigma Q}$ is admissible for $\pi_\sigma\Gamma^*$ with respect to the metric $\widetilde{d}_\sigma$. Thus $\operatorname{mod} \pi_\sigma\Gamma^* = 1/2 = \operatorname{mod}\Gamma^*$. By Proposition 4.3, this suffices to show that $\widetilde{d}_\sigma$ is reciprocal.

7. Factoring quasiconformal mappings

The goal of this section is to prove Proposition 1.5 and Theorem 1.6. To prepare for this, we first give in Section 7.1 an overview of isothermal quasiconformal mappings. See [Iko21] for a more complete treatment. Section 7.2 gives the proof of Proposition 1.5. This is followed by a discussion in Section 7.3 of the problem of optimizing the distortion constant in Proposition 1.5. Finally, in Section 7.4, we prove Theorem 1.6.

7.1. **Isothermal Parametrizations.** Let X be a quasiconformal surface. By Theorem 6.2 in [Iko21], there exists a complete Riemannian surface Y of constant curvature and a quasiconformal map

$$\psi \colon Y \to X$$

with minimal pointwise distortion at almost every point: for every other Riemannian surface Z and quasiconformal map $\varphi\colon Z \to X$, the inequality

$$\big(g_\psi \cdot (g_{\psi^{-1}} \circ \psi)\big) \circ (\psi^{-1} \circ \varphi) \leq g_\varphi \cdot (g_{\varphi^{-1}} \circ \varphi) \tag{38}$$

holds $\mathcal{H}^2_Z$-almost everywhere on Z. Recall that g_ψ and $g_{\psi^{-1}}$ refer to the minimal weak upper gradients of ψ and ψ^{-1}, respectively. In this case, we say that (Y, ψ) is an *isothermal parametrization* of X. By Theorem 6.2 and Lemma 4.10 of [Iko21], any isothermal parametrization ψ is quasiconformal with outer dilatation $K_O(\psi)$ at most $4/\pi$ and inner dilatation $K_I(\psi)$ at most $\pi/2$. Also, the pointwise distortion of ψ is bounded from above by $\sqrt{2}$ $\mathcal{H}^2_Y$-almost everywhere.

We elaborate on the meaning of (38) in the case when $X = (\mathbb{R}^2, d_N)$ for some norm N. Then we can take $Y = \mathbb{R}^2$ and ψ to be a linear map

$$\psi\colon \mathbb{R}^2 \to (\mathbb{R}^2, d_N)$$

such that $g_\psi = L(N\circ\psi)$ and $g_{\psi^{-1}} = \omega(N\circ\psi)^{-1}$. Recall that L and ω denote, respectively, the maximal and minimal stretching, defined in (6) and (7).

The inequality (38) implies that, for all other linear maps $\varphi\colon \mathbb{R}^2 \to (\mathbb{R}^2, N)$, we have

$$\frac{L(N\circ\psi)}{\omega(N\circ\psi)} \leq \frac{L(N\circ\varphi)}{\omega(N\circ\varphi)}. \tag{39}$$

In terms of the *distortion* of a norm defined in (8), the inequality (39) implies that $N\circ\psi$ has the smallest possible distortion among such pairs ψ and φ. This can be phrased in terms of the *Banach–Mazur distance* in convex geometry; see [Rom19] and [Iko21, Section 4].

An isothermal parametrization of a quasiconformal surface is essentially unique. This is also part of the content of Theorem 6.2 of [Iko21], partially quoted here.

Theorem 7.1 ([Iko21])**.** *Let $\psi\colon Y \to X$ be an isothermal parametrization of X, and $\varphi\colon Z \to X$ a quasiconformal map from a Riemannian surface Z onto X. Then φ is isothermal if and only if $\psi^{-1} \circ \varphi$ is a conformal diffeomorphism.*

Let N be an admissible reciprocal norm field on $\mathbb{R}^2$ that vanishes on the compact set $E \subset \mathbb{R}^2$. The following lemma is a consequence of Theorem 4.12 of [Iko21].

Lemma 7.2. *The identity map $\iota\colon \mathbb{R}^2 \to (\mathbb{R}^2, d_N)$ is isothermal if and only if*

$$\frac{L(N_x)}{\omega(N_x)} \leq \frac{L(N_x \circ \varphi)}{\omega(N_x \circ \varphi)} \tag{40}$$

for all $\varphi \in \mathrm{GL}_2$, for $\mathcal{L}^2$-almost every $x \in \mathbb{R}^2$.

Observe that (40) is satisfied by the norm $N_x = \|\cdot\|_\infty$, and more generally by any norm N_x whose unit ball is a square [TJ89, Proposition 37.6]. Thus Lemma 7.2 has the following corollary.

Corollary 7.3. *Suppose that N is reciprocal and that the unit ball of N_x is a square for $\mathcal{L}^2$-almost every $x \in \mathbb{R}^2$. Then the identity map $\iota\colon \mathbb{R}^2 \to (\mathbb{R}^2, d_N)$ is isothermal.*

7.2. **Proof of Proposition 1.5.** Recall that we are assuming that N is a reciprocal norm field such that $\pi_N\colon \Omega \to (\Omega, d_N)$ is isothermal, and that N is continuous outside the set $E = \{x \in \Omega : N_x = 0\}$.

Let G be a complete Riemannian norm field on Ω of constant Gaussian curvature -1 or 0, which exists by the classical uniformization theorem. This norm field is of the form $G = \sigma \|\cdot\|_2$ for some smooth positive function σ. Consider the norm field

$$M = \chi_{\Omega \setminus E} \frac{\sigma}{\omega(N)} N + \chi_E G.$$

The function $1/\omega(N)$ is continuous in $\Omega \setminus E$ due to the continuity of N outside E. The distortion bound on N implies that M is a lower semicontinuous norm field satisfying $G \leq M \leq HG$ everywhere.

Let $\widehat{d} = d_M$ denote the distance induced by M. Then

$$d_G \leq \widehat{d} \leq H d_G,$$

so the identity map $P = \pi_M\colon (\Omega, d_G) \to (\Omega, \widehat{d})$ satisfies (3) and in particular is H-bi-Lipschitz. Lemma 3.5 states that the metric differential of P coincides with M $\mathcal{L}^2$-almost everywhere.

The proof is complete after we show that $\widehat{\iota} = \pi_N \circ P^{-1}$ is 1-quasiconformal. Recall that the metric derivatives of π_N and P coincide with N and M, respectively. The 1-quasiconformality is equivalent to proving that for $\mathcal{L}^2$-almost every $x \in \Omega$, the distortion of the identity map from $(\mathbb{R}^2, M_x)$ to $(\mathbb{R}^2, N_x)$ equals one $\mathcal{L}^2$-almost everywhere [Iko21, Corollary 5.3].

Observe that, by the change of variables formula Lemma 3.6 and the Lusin's Condition (N^{-1}) of π_N, the set E has zero $\mathcal{L}^2$-measure, so we only need to check the pointwise distortion in the complement of E. Here the claim is immediate, since $M_z = \sigma(z) N_z / \omega(N_z)$ for every $z \in \Omega \setminus E$. We conclude that $\widehat{\iota}$ is 1-quasiconformal.

7.3. **Remarks on optimal distortion.** We discuss the question of when the optimal constant $H = \sqrt{2}$ in (3) in Proposition 1.5 can be achieved. We recall that any planar quasiconformal mapping $f\colon \Omega \to \widehat{\Omega}$ is a solution of the *Beltrami equation* $f_{\overline{z}} = \mu f_z$, where $\mu\colon \Omega \to \mathbb{C}$ is a measurable function satisfying $\|\mu\|_\infty < 1$. Conversely, the *measurable Riemann mapping theorem* provides a homeomorphic solution to the Beltrami equation for any such μ. The function μ is called the *Beltrami coefficient*. Geometrically, the choice of a Beltrami coefficient corresponds to the choice of a measurable ellipse field on Ω modulo rescaling of the ellipses. See Chapter 5 of [AIM09] for an overview of the topic.

Given a reciprocal norm field $\widehat{N}$ on a domain $\widehat{\Omega} \subset \mathbb{R}^2$, one obtains an ellipse field on $\widehat{\Omega}$ by associating to each norm $\widehat{N}_x$ its distance ellipse, that is, the unique ellipse $\mathcal{E} \subset B_{\widehat{N}_x}(0,1)$ having minimal $\lambda \geq 1$ such that $B_{\widehat{N}_x}(0,1) \subset \lambda\mathcal{E}$. This in turn gives a Beltrami coefficient $\mu_{\widehat{N}}$ corresponding to $\widehat{N}$. We refer the reader to [Iko21, Section 4] for more details.

This choice of ellipse field also determines an underlying Riemannian structure on the metric space $(\widehat{\Omega}, d_{\widehat{N}})$. A consequence of the classical slit domain uniformization theorem [AS60, Section III.4] and [Iko21, Theorem 1.3] is the existence of a domain $\Omega \subset \mathbb{R}^2$ and a locally quasiconformal map $\psi\colon \Omega \to \widehat{\Omega}$ such that $\widehat{f} = \pi_{\widehat{N}} \circ \psi$ is isothermal. Consider the distance $d(x,y) = d_{\widehat{N}}(\widehat{f}(x), \widehat{f}(y))$ on Ω and the norm field $N = \widehat{N} \circ D\psi$. Then the identity map $\iota\colon \Omega \to (\Omega, d)$ is isothermal and the metric differential of ι exists and equals N $\mathcal{L}^2$-almost everywhere. If the norm field N obtained in this manner is continuous and non-zero outside $E = \psi^{-1}\left(\{x \in \widehat{\Omega} : \widehat{N}_x = 0\}\right)$, then Proposition 1.5 now holds with constant $H = \sqrt{2}$ for the space (Ω, d) and hence the original space $(\widehat{\Omega}, d_{\widehat{N}})$ as well.

The question of when the norm field N is continuous, in turn, depends upon the regularity of the map ψ. In fact, if ψ is $\mathcal{C}^1$-smooth in Ω and $\widehat{N}$ is continuous, then N is continuous and non-zero outside E. Since the map ψ arises as a solution to the Beltrami equation, this leads to the question of regularity of solutions to the Beltrami equation. Indeed, if we consider a domain U compactly contained in $\widehat{\Omega}$, the restriction of ψ^{-1} to U solves the Beltrami equation induced by $\mu_{\widehat{N}}|_U$. The $\mathcal{C}^1$-smoothness of ψ in U is known to hold, for example, when $\mu_{\widehat{N}}|_U$ is $\mathcal{C}^1$-smooth, locally Hölder continuous [AIM09, Theorem 15.0.7] or in $W^{1,p}_{\text{loc}}(U)$ for a large enough $p > 1$ depending on the L^∞-norm of $\mu_{\widehat{N}}|_U$ [BCO19, Proposition 4].

Solutions of the Beltrami equation for $\mu_{\widehat{N}}$, even when $\widehat{N}$ is a continuous Riemannian norm field, need not always be $\mathcal{C}^1$-smooth. In the following, we use complex notation $z = z_1 + iz_2$ to denote the point $(z_1, z_2) \in \mathbb{R}^2$ and $\overline{z} = z_1 - iz_2$ to denote the complex conjugate of z. See Section 2.4 of [AIM09] for a brief overview of complex notation. The following example is based on Section 15.1 of [AIM09]. Let

$$\mu(z) = \frac{z}{\overline{z}(1 + \log \|z\|_2^2)}$$

and consider the continuous Riemannian norm field $\widehat{N}$ on $\widehat{\Omega} = B_{\|\cdot\|_2}(0, e^{-1/2})$ defined by $\widehat{N}_z(v) = \|v + \mu(z)\overline{v}\|_2$. Then $\mu(z) = \mu_{\widehat{N}_z}$, where $\mu_{\widehat{N}_z}$ is the Beltrami coefficient corresponding to the $\widehat{N}$ as described earlier in this remark. Even though $\widehat{N}$ is continuous, every solution for the Beltrami equation for $\mu_{\widehat{N}_z} = \mu(z)$ has a discontinuous derivative at the origin. This is seen by considering the particular solution $g(z) = -z \log \|z\|_2^2$ and noticing that the differential Dg is discontinuous at the origin. It is enough to check this property for g since, by the Stoïlow factorization theorem [AIM09, Theorem 5.5.1], every other quasiconformal solution is of the form $\Psi \circ g$ for some conformal diffeomorphism Ψ.

For more general norm fields, we have the additional complexity that $\mu_{\widehat{N}}$ can be smooth even though $\widehat{N}$ is not. For example, consider the continuous norm field $\widehat{N}$ defined by $\widehat{N}_z(v) = \left\|e^{i\|z\|_2}v\right\|_\infty$. Since the supremum norm is not $\mathcal{C}^1$-smooth in $\mathbb{R}^2 \setminus \{0\}$, we see that $\widehat{N}$ is not $\mathcal{C}^1$-smooth, for example by considering the basepoint

$z = \pi/4$ and the vector $v = 1$, even though $\mu_{\widehat{N}} = 0$. The identity $\mu_{\widehat{N}} = 0$ follows from Corollary 7.3.

7.4. **Proof of Theorem 1.6.** In this section, we present the construction used to prove Theorem 1.6, namely of a quasiconformal surface whose isothermal parametrization cannot be factored as a bi-Lipschitz mapping postcomposed with a quasiconformal mapping of smaller distortion. We begin by introducing the notation and parameters involved in Section 7.4.1. We develop various properties of this construction in the following subsections, culminating with the proof of Theorem 1.6 in Section 7.4.6.

7.4.1. *Notation.* Let us introduce the notation used in our construction. Our first task is to construct a sequence of nested Cantor sets, denoted by $K_1, K_2, \ldots$ and satisfying $K_1 \supset K_2 \supset \cdots$. There are two intermediate steps used to obtain the sets K_i. First, we define sets E_i^j for all $i, j \in \mathbb{N}$, $j \geq i$, to serve as base collections of squares from which the Cantor sets are taken. Each set E_i^j is the union of a collection of congruent closed squares $Q_i^j(k,l)$ that covers almost all of $[0,1]^2$. The main feature of our construction is that the squares $Q_i^j(k,l)$ have the standard non-rotated alignment for odd values of i, while the square $Q_i^j(k,l)$ are aligned diagonally for even values of i.

In the second intermediate step, we define inductively

$$F_i^j = E_i^j \cap F_{i-1}^j \cap F_i^{j-1} \cap [0,1]^2$$

for all $i, j \in \mathbb{N}$, $j \geq i$, with the convention that $F_0^j = F_i^{i-1} = [0,1]^2$ for all i, j. By taking $F_i = \bigcap_j F_i^j$, we obtain a collection of nested Cantor sets. However, to obtain Theorem 1.6, we need the further property that the intersection of the Cantor sets is small. For this reason, we later define K_i to be a subset of F_i with the property that $\operatorname{diam} K_i \to 0$ as $i \to \infty$.

In the following, let $I = J = [0,1]$ and let $Q = I \times J = [0,1]^2$. We identify I with the set $[0,1] \times \{0\}$ and J with the set $\{0\} \times [0,1]$. Let π_1 denote the standard projection map from $\mathbb{R}^2$ onto the first coordinate axis, and let π_2 denote the standard projection map from $\mathbb{R}^2$ onto the second coordinate axis.

As mentioned above, the even-numbered Cantor sets are formed from squares that are rotated by $\pi/4$ from the standard alignment. Let Q^* denote the square with vertices $(1/2, -1/2)$, $(3/2, 1/2)$, $(1/2, 3/2)$, and $(-1/2, 1/2)$. Let $I^* = J^* = [0, \sqrt{2}]$. We also identify I^* with the set $[0, \sqrt{2}] \times \{0\}$ and J^* with the set $\{0\} \times [0, \sqrt{2}]$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the orientation-preserving isometry that maps $[0, \sqrt{2}]^2$ onto Q^* and satisfies $\varphi(0,0) = (1/2, -1/2)$. Explicitly,

$$\varphi(x,y) = (1/2, -1/2) + \frac{1}{\sqrt{2}}(x - y, x + y).$$

Thus $\varphi(I^* \times J^*) = Q^*$. Next, let π_1^* denote the projection map from Q^* onto $\varphi(I^*)$, and let π_2^* denote the projection map from Q^* onto $\varphi(J^*)$. Explicitly, $\pi_1^*(x,y) = \varphi(\pi_1(\varphi^{-1}(x,y)), 0)$ and $\pi_2^*(x,y) = \varphi(0, \pi_2(\varphi^{-1}(x,y)))$.

The definition of the sets E_i^j involves three sets of parameters: $\varepsilon_i^j > 0$, $N_i^j \in \mathbb{N}$, and $a_i^j \in \mathbb{N}$. A short explanation of these parameters is the following. The first parameter ε_i^j gives an upper bound on the proportion of area lost when passing from one step of the construction to the next. The second parameter N_i^j gives the number of subdivisions of the initial interval I or I^* that are made when forming the squares that comprise E_i^j. The final parameter a_i^j corresponds to the side length of these squares. The precise relation is that the side length of a square in E_i^j is $(1 - 2(a_i^j)^{-1})/N_i^j$ for i odd and $\sqrt{2}(1 - 2(a_i^j)^{-1})/N_i^j$ for i even.

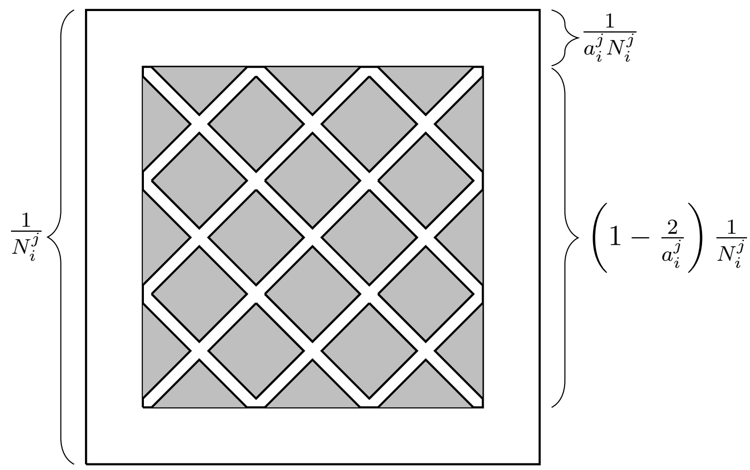

FIGURE 1. A square $Q_i^j(k,l) \subset E_i^j$ for i odd and the intersection $Q_i^j(k,l) \cap E_{i+1}^j$, shaded gray. The large outer square is $I_i^j(k) \times J_i^j(l)$.

7.4.2. *Constructing the sets* E_i^j. For two pairs of indices (i,j) and (i',j'), we say that $(i,j) \preceq (i',j')$ if $j < j'$ or if $j = j'$ and $i \le i'$. The relation $\preceq$ gives an ordering on the set of indices (i,j). We consider the sets E_i^j as being traversed in this order. We also write $(i,j) \prec (i',j')$ if $j < j'$ or if $j = j'$ and $i < i'$. Recall that here and throughout this proof we consider only those indices $i, j \in \mathbb{N}$ for which $j \ge i$. This ordering is illustrated in Figure 2.

We first choose the parameters $\varepsilon_i^j > 0$ so that they satisfy $\prod_{i,j}(1 - \varepsilon_i^j) \ge 1/2$. The factors in the product are traversed according to the ordering on $\{(i,j)\}$ just defined.

The sets E_i^j are defined for all $i, j \in \mathbb{N}$ satisfying $j \ge i$ in the following way. Assume for the moment that we have made suitable choices of $N_i^j, a_i^j \in \mathbb{N}$. In the case that i is odd, we divide I into N_i^j equal subintervals $I_i^j(k) = [(k-1)/N_i^j, k/N_i^j]$ and J into N_i^j equal subintervals $J_i^j(l) = [(l-1)/N_i^j, l/N_i^j]$. If i is even, we divide I^* into N_i^j equal subintervals $I_i^j(k) = [\sqrt{2}(k-1)/N_i^j, \sqrt{2}k/N_i^j]$ and J^* into N_i^j equal subintervals $J_i^j(l) = [\sqrt{2}(l-1)/N_i^j, \sqrt{2}l/N_i^j]$. This yields a collection of squares $I_i^j(k) \times J_i^j(l)$, where $k, l \in \{1, \dots, N_i^j\}$. If i is odd, let $Q_i^j(k,l)$ be the square of side length $(1 - 2(a_i^j)^{-1})/N_i^j$ with the same center and alignment as $I_i^j(k) \times J_i^j(l)$. If i is even, define $Q_i^j(k,l)$ to be the square of side length $\sqrt{2}(1-2(a_i^j)^{-1})/N_i^j$ with the same center and alignment as $\varphi(I_i^j(k) \times J_i^j(l))$. In this case, the square $Q_i^j(k,l)$ is contained in Q^* and is aligned diagonally. Let

$$E_i^j = \bigcup_{k,l} Q_i^j(k,l).$$

The square $Q_i^j(k,l)$, for i odd, is given explicitly by

$$\left[\left(k-1+\frac{1}{a_i^j}\right)\frac{1}{N_i^j}, \left(k-\frac{1}{a_i^j}\right)\frac{1}{N_i^j}\right] \times \left[\left(l-1+\frac{1}{a_i^j}\right)\frac{1}{N_i^j}, \left(l-\frac{1}{a_i^j}\right)\frac{1}{N_i^j}\right].$$

Let $v_i^j(k,l,1), v_i^j(k,l,2), v_i^j(k,l,3), v_i^j(k,l,4)$ denote the four vertices of $Q_i^j(k,l)$, traversed counterclockwise from the bottom left. Let $w_i^j(k,l)$ denote the center point of $Q_i^j(k,l)$.

$$\begin{array}{ccccccc} E_1^1 & \to & E_1^2 & & E_1^3 & \cdots \\ & & \downarrow & \nearrow & \downarrow & \\ & & E_2^2 & & E_2^3 & \cdots \\ & & & & \downarrow & \\ & & & & E_3^3 & \cdots \\ & & & & & \ddots \end{array}$$

FIGURE 2. The sets E_i^j as ordered by $\preceq$.

Similarly, the square $Q_i^j(k,l)$, for i even, is the image under φ of the square

$$\left[\left(k-1+\frac{1}{a_i^j}\right)\frac{\sqrt{2}}{N_i^j}, \left(k-\frac{1}{a_i^j}\right)\frac{\sqrt{2}}{N_i^j}\right] \times \left[\left(l-1+\frac{1}{a_i^j}\right)\frac{\sqrt{2}}{N_i^j}, \left(l-\frac{1}{a_i^j}\right)\frac{\sqrt{2}}{N_i^j}\right]. \tag{41}$$

Let $v_i^j(k,l,1), v_i^j(k,l,2), v_i^j(k,l,3), v_i^j(k,l,4)$ denote the four vertices of $Q_i^j(k,l)$, where $v_i^j(k,l,1)$ is the image under φ of the bottom left vertex of the square in (41) and the rest are labelled in counterclockwise order. Let $w_i^j(k,l)$ denote the center point of $Q_i^j(k,l)$.

The values of N_i^j and a_i^j are chosen inductively using the ordering $\preceq$. Let $N_1^1 = 2$ and choose $a_1^1 \in \mathbb{N}$ so that $\mathcal{L}^2(E_1^1) \geq 1 - \varepsilon_1^1$. For the inductive step, assume that we have chosen $N_{i'}^{j'}$ and $a_{i'}^{j'}$ for some pair (i',j'), and that

$$\mathcal{L}^2\left(\bigcap_{(i'',j'')\preceq(i',j')} E_{i''}^{j''}\right) \geq \prod_{(i'',j'')\preceq(i',j')} (1-\varepsilon_{i''}^{j''}).$$

Let (i,j) denote the pair immediately succeeding (i',j').

Define now $N_i^j = 2a_{i'}^{j'}N_{i'}^{j'}$. We then choose a_i^j so that

$$\mathcal{L}^2\left(\bigcap_{(i'',j'')\preceq(i,j)} E_{i''}^{j''}\right) \geq \prod_{(i'',j'')\preceq(i,j)} (1-\varepsilon_{i''}^{j''}).$$

This can be done because E_i^j can be made to have arbitrarily large area in Q or Q^*, respectively, by making a_i^j sufficiently large.

We make the following observation. Fix (i,j) and consider a square $Q_i^j(k,l)$. For all (i',j') such that $(i,j) \prec (i',j')$ and $m,n \in \{1,\ldots,N_{i'}^{j'}\}$, the square $I_{i'}^{j'}(m) \times J_{i'}^{j'}(n)$, if i' is odd, or $\varphi(I_{i'}^{j'}(m) \times J_{i'}^{j'}(n))$, if i' is even, is either entirely contained in $Q_i^j(k,l)$, has interior disjoint from $Q_i^j(k,l)$, or intersects $Q_i^j(k,l)$ in a triangle whose vertices are three of the vertices of $I_{i'}^{j'}(m) \times J_{i'}^{j'}(n)$.

We also observe a uniformity to how the squares are distributed. For each i,j,k,l, we divide the square $Q_i^j(k,l)$ into four triangles whose vertices are two adjacent vertices of $Q_i^j(k,l)$ and the midpoint of $Q_i^j(k,l)$. Denote these by $T_i^j(k,l,1)$, $T_i^j(k,l,2)$, $T_i^j(k,l,3)$, $T_i^j(k,l,4)$, where $T_i^j(k,l,m)$ contains the edge $[v_i^j(k,l,m), v_i^j(k,l,m+1)]$, taking $v_i^j(k,l,5) = v_i^j(k,l,1)$.

Lemma 7.4. Let $i,j,i',j' \in \mathbb{N}$, where $(i,j) \prec (i',j')$. For all $k,l \in \{1,\ldots,N_i^j\}$ and $m \in \{1,\ldots,4\}$ satisfying $T_i^j(k,l,m) \subset Q$, the sets $T_i^j(k,l,m) \cap E_{i'}^{j'}$ are all congruent.

Proof. This proof depends on the property that $2a_i^jN_i^j$ divides $N_{i'}^{j'}$. As a result, squares at different levels of the construction intersect nicely. We consider the case when i is odd.

First, suppose that i' is also odd. For each $m \in \{1, \dots, 4\}$, consider the edge $e_i^j(k,l,m)$ as defined above. We have $\pi_1(v_i^j(k,l,1)) = \pi_1(v_i^j(k,l,4)) = k_1(k,l)/N_{i'}^{j'}$ and $\pi_1(v_i^j(k,l,3)) = \pi_1(v_i^j(k,l,2)) = k_3(k,l)/N_{i'}^{j'}$, where

$$k_1(k,l) = \frac{(a_i^j k - a_i^j + 1)N_{i'}^{j'}}{a_i^j N_i^j} \quad \text{and} \quad k_3(k,l) = \frac{(a_i^j k - 1)N_{i'}^{j'}}{a_i^j N_i^j}.$$

Similarly, we have $\pi_2(v_i^j(k,l,2)) = \pi_2(v_i^j(k,l,1)) = k_2(k,l)/N_{i'}^{j'}$ and $\pi_4(v_i^j(k,l,3)) = \pi_4(v_i^j(k,l,4)) = k_4(k,l)/N_{i'}^{j'}$, where

$$k_2(k,l) = \frac{(a_i^j l - a_i^j + 1)N_{i'}^{j'}}{a_i^j N_i^j} \quad \text{and} \quad k_4(k,l) = \frac{(a_i^j l - 1)N_{i'}^{j'}}{a_i^j N_i^j}.$$

Observe that $k_i(k,l) \in \mathbb{N}$ for all $i \in \{1, \dots, 4\}$. We have then

$$Q_i^j(k,l) = [k_1(k,l)/N_{i'}^{j'}, k_3(k,l)/N_{i'}^{j'}] \times [k_2(k,l)/N_{i'}^{j'}, k_4(k,l)/N_{i'}^{j'}].$$

We conclude from this that the intersection $Q_i^j(k,l) \cap E_{i'}^{j'}$ is precisely the union of the squares

$$\{Q_{i'}^{j'}(k',l') : k_2(k,l) + 1 \le k' \le k_4(k,l), k_1(k,l) + 1 \le l' \le k_3(k,l)\}.$$

We also observe that

$$|k_3(k,l) - k_1(k,l)| = |k_4(k,l) - k_2(k,l)| = \frac{(a_i^j - 2)N_{i'}^{j'}}{a_i^j N_i^j}.$$

Thus the sets $Q_i^j(k,l) \cap E_{i'}^{j'}$ are congruent for all $k,l \in \{1, \dots, N_i^j\}$. Moreover, notice that each set $Q_i^j(k,l) \cap E_{i'}^{j'}$ is invariant under rotations by $\pi/4$ about the center point $w_i^j(k,l)$. We conclude from this that the sets $T_i^j(k,l,m) \cap E_{i'}^{j'}$ are all congruent.

Next, suppose that i' is even. Consider now a triangle $T_i^j(k,l,m)$. The two shorter edges of $T_i^j(k,l,m)$ are the edges of a rectangle $R_i^j(k,l,m)$ of side length $\sqrt{2}(a_i^j - 2)/(a_i^j N_i^j)$. To keep the exposition more manageable, we write out the argument only for $T_i^j(k,l,1)$. We compute

$$\varphi^{-1}(v_i^j(k,l,1)) = \left(\frac{(a_i^j k + a_i^j l - 2a_i^j + 2)\sqrt{2}}{2a_i^j N_i^j}, -\frac{(-a_i^j k + a_i^j l + a_i^j N_i^j)\sqrt{2}}{2a_i^j N_i^j} \right)$$

$$\varphi^{-1}(v_i^j(k,l,2)) = \left(\frac{(a_i^j k + a_i^j l - a_i^j)\sqrt{2}}{2a_i^j N_i^j}, \frac{(-a_i^j k + a_i^j l - a_i^j + 2 + a_i^j N_i^j)\sqrt{2}}{2a_i^j N_i^j} \right).$$

Comparing this with (41) and using the property that $2a_i^j N_i^j$ divides $N_{i'}^{j'}$, we have

$$\varphi^{-1}(R_i^j(k,l,m)) = \left[\frac{k_1(k,l)\sqrt{2}}{N_{i'}^{j'}}, \frac{k_3(k,l)\sqrt{2}}{N_{i'}^{j'}} \right] \times \left[\frac{k_2(k,l)\sqrt{2}}{N_{i'}^{j'}}, \frac{k_4(k,l)\sqrt{2}}{N_{i'}^{j'}} \right]$$

for some $k_1(k,l), \dots, k_4(k,l) \in \mathbb{N}$ satisfying

$$|k_3(k,l) - k_1(k,l)| = |k_4(k,l) - k_2(k,l)| = \frac{(a_i^j - 2)N_{i'}^{j'}}{2a_i^j N_i^j}.$$

The intersection $R_i^j(k,l,1) \cap E_{i'}^{j'}$ is the union of the squares

$$\{Q_{i'}^{j'}(k',l') : k_2(k,l) + 1 \le k' \le k_4(k,l), k_1(k,l) + 1 \le l' \le k_3(k,l)\}.$$

Thus the sets $R_i^j(k,l,1) \cap E_{i'}^{j'}$ are all congruent, and by symmetry it follows that the sets $T_i^j(k,l,1) \cap E_{i'}^{j'}$ are all congruent as well.

The case when i is even is similar, and its proof is omitted. $\square$

7.4.3. *Constructing the Cantor sets.* For all i, j, let $F_0^j = Q$ and $F_i^{i-1} = Q$. Define now

$$F_i^j = E_i^j \cap F_{i-1}^j \cap F_i^{j-1}$$

for all $j \geq i$. Observe that $\bigcap_{(i'',j'') \preceq (i,j)} E_{i''}^{j''} \subset F_i^j$, so we have $\mathcal{L}^2(F_i^j) \geq 1/2$ for all i, j. Next, let $F_i = \bigcap_{j \geq i} F_i^j$.

Let $K_0 = \mathbb{R}^2$. For each $i \geq 1$, pick inductively a square $Q_i = Q_i^i(j_i, k_i)$ with the property that $Q_i \subset Q_{i-1}$. Let

$$K_i = F_i \cap Q_i.$$

From Lemma 7.4, it follows that $\mathcal{L}^2(K_i) = \mathcal{L}^2(F_i)/(N_i^i)^2$. Moreover, we have that $\operatorname{diam} K_i \to 0$ as $i \to 0$, and in particular that $\bigcap_i F_i$ is a single point set.

7.4.4. *Dense networks of paths.* The following portion of the argument relates to having a "dense network of paths" at every stage.

We define the following subset of K_i. If i is even, let

$$H_i = K_i \cap \pi_2^{-1}(I \setminus \pi_2(K_{i+1})).$$

If i is odd, let

$$H_i = K_i \cap (\pi_2^*)^{-1}(I^* \setminus \pi_2^*(K_{i+1})).$$

For example, in Figure 1 representing the case where i is odd, a point $x \in K_i \cap Q_i^j(k,l)$ belongs to H_i if the line $t \mapsto x + (t,t)$ does not intersect any of the gray boxes.

Lemma 7.5. For every point $x \in K_{i+1}$ and $r > 0$, the set $H_i \cap B_{\|\cdot\|_2}(x,r)$ has positive $\mathcal{L}^2$-measure.

Proof. In the first case, we assume that i is even, and hence that $i+1$ is odd. Let $x \in K_{i+1}$ and $r > 0$. Consider a square $Q_{i+1}^j(k,l)$ containing x for some j sufficiently large so that $Q_{i+1}^j(k,l) \subset B(x, r/3)$ and such that $Q_{i+1}^j(k,l) \subset Q_{i+1}$. Pick a horizontal edge S of $Q_{i+1}^j(k,l)$ whose interior is contained in $\operatorname{int}(Q_{i+1})$.

Consider now the set E_i^{j+1}. Subdivide S into $(a_{i+1}^j - 2)N_i^{j+1}/(a_{i+1}^j N_{i+1}^j)$ congruent subintervals. Each subinterval is the diagonal of a square $\varphi(I_i^{j+1}(k') \times J_i^{j+1}(l'))$, with corresponding square $Q_i^{j+1}(k',l') \subset E_i^{j+1}$. From such a square $Q_i^{j+1}(k',l')$, we may extract a triangle $T_i^{j+1}(k',l',m')$, as defined prior to the statement of Lemma 7.4, whose interior does not intersect E_{i+1}^j. Observe further that $T_i^{j+1}(k',l',m') \subset Q_i$, so that

$$T_i^{j+1}(k',l',m') \cap F_i = T_i^{j+1}(k',l',m') \cap K_i.$$

As a consequence of Lemma 7.4, we have that

$$\mathcal{L}^2(T_i^{j+1}(k',l',m') \cap F_i) = \frac{\mathcal{L}^2(F_i)}{4(N_i^{j+1})^2}.$$

Moreover, $T_i^{j+1}(k',l',m')$ lies in the neighborhood of $Q_{i+1}^j(k,l)$ of radius

$$1/N_i^{j+1} \leq \operatorname{diam} Q_{i+1}^j(k,l) < 2r/3,$$

so $T_i^{j+1}(k',l',m') \subset B_{\|\cdot\|_2}(x,r)$. Also, we have that

$$\operatorname{int}(T_i^{j+1}(k',l',m')) \cap F_i \subset H_i.$$

This verifies the claim.

The case that i is even and $i+1$ is odd is similar, and we omit the details. $\square$

Lemma 7.6. Let $x \in \pi_1(K_i)$ be a Lebesgue density point of $\pi_1(K_i)$, where $i \in \mathbb{N}$ is odd. Let $\delta > 0$, and let $t_0 > 0$ be such that

$$\frac{\mathcal{L}^1(\pi_1(K_i) \cap (x, x+t))}{t} \geq 1 - \delta$$

for all $t \in (0, t_0)$. Then for all $y \in \pi_1(K_i)$ satisfying $|y - x| < 2\delta t$,

$$\frac{\mathcal{L}^1(\pi_1(K_i) \cap (y, y+t))}{t} \geq 1 - 2\delta.$$

The same result holds with π_2 instead of π_1. If i is even, the corresponding result holds for π_1^* and π_2^*, identifying I^* and J^* with the interval $[0, \sqrt{2}]$.

Proof. The first claim follows from the relationship

$$\mathcal{L}^1(\pi_1(K_i) \cap (y, y+t)) \geq \mathcal{L}^1(\pi_1(K_i) \cap (x, x+t)) - |x - y|.$$

The other claims follow from a similar inequality. □

7.4.5. *Defining the metric on* $\mathbb{R}^2$. Define a norm field N on $\mathbb{R}^2$ by the formula

$$N_x = \begin{cases} 2^{-i/2}\|\cdot\|_1 & \text{if } x \in K_i \setminus K_{i+1},\ i \text{ even}, \\ 2^{-(i-1)/2}\|\cdot\|_\infty & \text{if } x \in K_i \setminus K_{i+1},\ i \text{ odd}. \end{cases}$$

The norm field N is admissible in the sense of Definition 3.1, in particular being lower semicontinuous, and induces a metric d on $\mathbb{R}^2$ as described in Section 3. Observe that N vanishes at a single point. Theorem 1.3 and Corollary 7.3 imply the following.

Proposition 7.7. The identity map $\iota\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, d)$ is an isothermal quasiconformal mapping.

7.4.6. *Proof of Theorem 1.6.* We suppose to the contrary that there is a metric space $(\widehat{X}, \widehat{d})$ such that a factorization $\iota = \widehat{\iota} \circ P$ as in the statement of Theorem 1.6 exists, that is, that P is bi-Lipschitz and that $\widehat{\iota}$ has distortion $H(\widehat{\iota}) < \sqrt{2}$. Since ι is Lipschitz, it follows that $\widehat{\iota}$ is also Lipschitz.

By considering the metric $\widehat{d}(P(x), P(y))$ on $\mathbb{R}^2$, we assume without loss of generality that $\widehat{X} = \mathbb{R}^2$ and that $\widehat{\iota}$ and P are each the identity map on $\mathbb{R}^2$. Let $\widehat{N}$ denote the metric derivative of the map $P\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, \widehat{d})$, cf. Definition 2.2.

By assumption, the identity map $\widehat{\iota}\colon (\mathbb{R}^2, \widehat{d}) \to (\mathbb{R}^2, d)$ is quasiconformal with $H(\widehat{\iota}) < \sqrt{2}$. Moreover, since P is Lipschitz, there exists $C > 0$ such that $\widehat{N}_x \leq C\,\|\cdot\|_1$ for every $x \in \mathbb{R}^2$. Let $v = (1, 0)$ and $w = (1/\sqrt{2}, 1/\sqrt{2})$, and let $a = H(\widehat{\iota})/\sqrt{2} < 1$. It suffices to show that, for all $i \geq 0$ and almost every $x \in K_i \setminus K_{i+1}$,

$$\begin{aligned} \widehat{N}_x(w) &\leq Ca^i && \text{if } i \text{ is even}, \\ \widehat{N}_x(v) &\leq Ca^i && \text{if } i \text{ is odd}. \end{aligned} \tag{42}$$

This provides a contradiction. Indeed, given that P is bi-Lipschitz, $\widehat{N}_x(w)$ and $\widehat{N}_x(v)$ are bounded from below for all $x \in \mathbb{R}^2$ by some constant $C' > 0$.

Observe that, when i is even, $N_x(w) = \sqrt{2}N_x(v)$ for all $x \in K_i \setminus K_{i+1}$. Similarly, when i is odd, $N_x(v) = \sqrt{2}N_x(w)$. It follows from Theorem 5.2 of [Iko21] that the pointwise distortion of $\widehat{\iota}$ coincides with the distortion of the identity map from $(\mathbb{R}^2, \widehat{N}_x)$ to $(\mathbb{R}^2, N_x)$ for almost every $x \in \mathbb{R}^2$. As a consequence, for almost every $x \in K_i \setminus K_{i+1}$,

$$\begin{aligned} \widehat{N}_x(v) &\leq a\widehat{N}_x(w) && \text{if } i \text{ is even}, \\ \widehat{N}_x(w) &\leq a\widehat{N}_x(v) && \text{if } i \text{ is odd}. \end{aligned} \tag{43}$$

We verify (42) by induction on i. The claim is immediate for $i = 0$, recalling that $K_0 = \mathbb{R}^2$. For the inductive step, fix $i \geq 1$ and assume that (42) holds for almost every $x \in \mathbb{R}^2 \setminus K_i$. We show that (42) holds for almost every $x \in K_i \setminus K_{i+1}$. Let N_{i-1} denote the set of points in $K_{i-1} \setminus K_i$ for which (42) or (43) fails. We split into two cases based upon on whether i is odd or even. The idea is the same in each, but the bookkeeping requires separate statements.

Case 1. Assume that i is odd. By the inductive hypothesis, we have $\widehat{N}_x(v) \leq a\widehat{N}_x(w) \leq Ca^i$ for every $x \in (K_{i-1} \setminus K_i) \setminus N_{i-1}$, where N_{i-1} has $\mathcal{L}^2$-measure zero. We claim that $\widehat{N}_x(v) \leq Ca^i$ for almost every $x \in K_i \setminus K_{i+1}$. Assume to the contrary that there exists a set $G \subset K_i \setminus K_{i+1}$ of positive measure and a constant $b > 0$ such that $\widehat{N}_x(v) \geq (C+b)a^i$ for all $x \in G$.

For all $t \in [0,1]$, let $\gamma_t \colon I \to \mathbb{R}^2$ be the path defined by $\gamma_t(s) = (s,t)$. According to Lemma 2.5, for every path γ_t and every subinterval $I' \subset I$,

$$\ell_{\widehat{d}}(\gamma_t|_{I'}) = \int_{I'} \widehat{N}_{\gamma_t(s)}(v)\, d\mathcal{L}^1.$$

Consider now the interval $[s_0, s_0+h]$ for some $s_0 \in (0,1)$ and $h \in (0, 1-s_0)$. Differentiating, we have for $\mathcal{L}^2$-almost every $(s_0,t) \in G$ that

$$\lim_{h\to 0} \frac{\widehat{d}(\gamma_t(s_0), \gamma_t(s_0+h))}{h} = \lim_{h \to 0} \frac{\ell_{\widehat{d}}(\gamma_t|_{[s_0,s_0+h]})}{h} \geq (C+b)a^i.$$

In particular, for almost every $x \in G$, there exists $r_0 > 0$ such that

$$\widehat{d}(x, x+rv) \geq (C+b/2)a^i r \tag{44}$$

for all $r \in (0, r_0)$.

On the other hand, consider now a point $x \in G$ such that $\pi_1^*(x)$ is a Lebesgue density point of $\pi_1^*(K_{i-1})$ and $\pi_2^*(x)$ is a Lebesgue density point of $\pi_2^*(K_{i-1})$. Note that by Fubini's theorem, $\mathcal{L}^2$-almost every point in G has this property. Let $\delta > 0$ and let $t_0 = t_0(\delta)$ be such that the hypothesis in Lemma 7.6 is satisfied for both the point $\pi_1^*(x)$ and the point $\pi_2^*(x)$.

For all $\varepsilon > 0$, let $H(x,\varepsilon)$ be the set comprising those points $y \in H_{i-1} \cap \overline{B}_{\|\cdot\|_2}(x,\varepsilon)$ for which

$$\mathcal{H}^1_{\|\cdot\|_2}(N_{i-1} \cap \pi_2^{-1}(\pi_2(y))) = 0.$$

Recall that the set H_{i-1} is defined in Section 7.4.4. By Lemma 7.5, the set $H_{i-1} \cap \overline{B}_{\|\cdot\|_2}(x,\varepsilon)$ has positive $\mathcal{L}^2$-measure. Since N_{i-1} has $\mathcal{L}^2$-measure zero, an application of Fubini's theorem shows that $H(x,\varepsilon)$ is a full measure subset of $H_{i-1} \cap \overline{B}_{\|\cdot\|_2}(x,\varepsilon)$. Let $r \in (0,t_0)$ and $\varepsilon \in (0, 2\delta r)$.

Consider a point $y \in H(x,\varepsilon)$. Let $\gamma_y \colon [0,r] \to \mathbb{R}^2$ be the path defined by $\gamma_y(s) = y + sv$. Lemma 2.5 implies that

$$\ell_{\widehat{d}}(\gamma_y) = \int_{[0,r]} \widehat{N}_{\gamma_y(s)}(v)\, d\mathcal{L}^1(s),$$

and the definition of $H(x,\varepsilon)$ implies that

$$\widehat{N}_z(v) \leq Ca^i \tag{45}$$

for $\mathcal{H}^1_{\|\cdot\|_2}$-almost every $z \in K_{i-1} \cap |\gamma_y|$.

Next, we estimate the $\mathcal{H}^1_{\|\cdot\|_2}$-measure of $K_{i-1} \cap |\gamma_y|$. To this end, observe that the path $\gamma_y^1 \colon [0,r] \to \mathbb{R}$, $\gamma_y^1(s) = y + sw/\sqrt{2}$, intersects K_{i-1} in a set congruent to $\pi_1^*(K_{i-1}) \cap \pi_1^*(|\gamma_y^1|)$. Similarly, the path $\gamma_y^2 \colon [0,r] \to \mathbb{R}$, $\gamma_y^2(s) = y + s\overline{w}/\sqrt{2}$, where $\overline{w} = (1/\sqrt{2}, -1/\sqrt{2})$, intersects K_{i-1} in a set congruent to $\pi_2^*(K_{i-1}) \cap \pi_2^*(|\gamma_y^2|)$. Since $|\pi_m^*(y) - \pi_m^*(x)| < 2\delta r$, Lemma 7.6 gives, for $m \in \{1,2\}$,

$$\frac{\mathcal{H}^1_{\|\cdot\|_2}(K_{i-1} \cap |\gamma_y^m|)}{r/\sqrt{2}} \geq 1 - 2\delta. \tag{46}$$

We combine this with the following observation: for any measurable sets $E_1, E_2 \subset [0,r]$ satisfying $|E_j| \geq (1-\varepsilon_j)r$ for some $\varepsilon_j \in (0,1)$, $j \in \{1,2\}$, the diagonal path $\gamma\colon [0,r] \to [0,r]^2$ defined by $\gamma(s) = (s,s)$ intersects $E_1 \times E_2$ in a set of length at least $\sqrt{2}(1-\varepsilon_1-\varepsilon_2)r$. Since K_{i-1} is constructed as a product set relative to which γ_y is a diagonal path, we conclude from (46) that

$$\frac{\mathcal{H}^1_{\|\cdot\|_2}(K_{i-1} \cap |\gamma_y|)}{r} \geq 1 - 4\delta. \tag{47}$$

Using (45) and the fact that $\widehat{N}_z(v) \leq C$ for all $z \in \mathbb{R}^2$, the inequality (47) gives

$$\widehat{d}(y, y+rv) \leq (1-4\delta)Ca^i r + 4\delta C r.$$

Next, by making the initial choice of δ sufficiently small, we have

$$\widehat{d}(y, y+rv) \leq (1+\delta)Ca^i r.$$

From this and the relationship $\widehat{d} \leq C d_{\|\cdot\|_1} \leq \sqrt{2} C d_{\|\cdot\|_2}$, it follows that

$$\widehat{d}(x, x+rv) \leq 2\sqrt{2}C\varepsilon + (1+\delta)Ca^i r.$$

Since $\varepsilon \in (0, 2\delta r)$ is arbitrary, we obtain

$$\widehat{d}(x, x+rv) \leq (1+\delta)Ca^i r.$$

Since this estimate holds for $\mathcal{L}^2$-almost every $x \in G$, this contradicts our earlier statement (44) when δ is sufficiently small. We conclude that $\widehat{N}_x(v) \leq Ca^i$ for almost every $x \in K_i \setminus K_{i+1}$.

Case 2. We now consider the case that i is even. The idea is the same as in the first case, but now everything is rotated by $\pi/4$. By the inductive hypothesis, we have that $\widehat{N}_x(w) \leq a\widehat{N}_x(v) \leq Ca^i$ for every $x \in (K_{i-1} \setminus K_i) \setminus N_{i-1}$. We claim that $\widehat{N}_x(w) \leq Ca^i$ for almost every $x \in K_i \setminus K_{i+1}$. Assume to the contrary that there exists a set $G \subset K_i \setminus K_{i+1}$ of positive measure and a constant $b > 0$ such that $\widehat{N}_x(w) \geq (C+b)a^i$ for all $x \in G$.

For all $t \in J^*$, let $\gamma_t\colon I^* \to \mathbb{R}^2$ be the path defined by $\gamma_t(s) = \varphi(s,t)$. Consider as before the interval $[s_0, s_0+h]$ for some $s_0 \in (0, \sqrt{2})$ and $h \in (0, \sqrt{2} - s_0)$. Differentiating, we have that

$$\lim_{h \to 0} \frac{\ell_{\widehat{d}}(\gamma_t|_{[s_0, s_0+h]})}{h} \geq (C+b)a^i.$$

In particular, for $\mathcal{L}^2$-almost every $x \in G$, there exists $r_0 > 0$ such that

$$\widehat{d}(x, x+rw) \geq (C+b/2)a^i r \tag{48}$$

for all $r \in (0, r_0)$.

On the other hand, consider a point $x \in G$ such that $\pi_1(x)$ is a Lebesgue density point of $\pi_1(K_{i-1})$ and $\pi_2(x)$ is a Lebesgue density point of $\pi_2(K_{i-1})$. Let $\delta > 0$ and let $t_0 = t_0(\delta)$ be the corresponding value in Lemma 7.6. For all $\varepsilon > 0$, define the set $H(x,\varepsilon)$ as the set of points $y \in H_{i-1} \cap \overline{B}_{\|\cdot\|_2}(x,\varepsilon)$ for which

$$\mathcal{H}^1_{\|\cdot\|_2}(N_{i-1} \cap (\pi_2^*)^{-1}(\pi_2^*(y))) = 0.$$

As before, $H(x,\varepsilon)$ is a full measure subset of $H_{i-1} \cap \overline{B}_{\|\cdot\|_2}(x,\varepsilon)$. Let $r \in (0,t_0)$ and $\varepsilon \in (0, 2\delta r)$.

For all $y \in H(x,\varepsilon)$, define the path $\gamma_y\colon [0,r] \to \mathbb{R}^2$ by $\gamma_y(s) = y + sw$. Recall from Lemma 2.5 that

$$\ell_{\widehat{d}}(\gamma_y) = \int_{[0,r]} \widehat{N}_{\gamma_y(s)}(w)\, d\mathcal{L}^1.$$

Moreover, $\widehat{N}_z(w) \leq Ca^i$ for $\mathcal{H}^1_{\|\cdot\|_2}$-almost every $z \in K_{i-1} \cap |\gamma_y|$. Arguing as in the first case, we obtain the inequality

$$\widehat{d}(y, y + rw) \leq (1 - 4\delta)Ca^i r + 2\delta Cr.$$

Next, by taking δ sufficiently small, we then have $\widehat{d}(y, y + rw) \leq (1 + \delta)Ca^i r$. As before, since $\varepsilon \in (0, 2\delta r)$ is arbitrary,

$$\widehat{d}(x, x + rw) \leq (1 + \delta)Ca^i r,$$

which contradicts (48) for sufficiently small $\delta > 0$. We conclude that $\widehat{N}_x(w) \leq Ca^i$ for almost every $x \in K_i \setminus K_{i+1}$.

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Quasiconformal Jordan domains

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Quasiconformal Jordan Domains

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Abstract: We extend the classical Carathéodory extension theorem to quasiconformal Jordan domains (Y, d_Y). We say that a metric space (Y, d_Y) is a *quasiconformal Jordan domain* if the completion $\overline{Y}$ of (Y, d_Y) has finite Hausdorff 2-measure, the *boundary* $\partial Y = \overline{Y} \setminus Y$ is homeomorphic to $\mathbb{S}^1$, and there exists a homeomorphism $\phi \colon \mathbb{D} \to (Y, d_Y)$ that is quasiconformal in the geometric sense.
We show that ϕ has a continuous, monotone, and surjective extension $\Phi \colon \overline{\mathbb{D}} \to \overline{Y}$. This result is best possible in this generality. In addition, we find a necessary and sufficient condition for Φ to be a quasiconformal homeomorphism. We provide sufficient conditions for the restriction of Φ to $\mathbb{S}^1$ being a quasisymmetry and to ∂Y being bi-Lipschitz equivalent to a quasicircle in the plane.

Keywords: quasiconformal; metric surface; Carathéodory; Beurling–Ahlfors

MSC: Primary 30L10, Secondary 30C65, 28A75, 51F99.

1 Introduction

Let (X, d_X) be a metric space with locally finite Hausdorff 2-measure. If X is also homeomorphic to a 2-manifold, we say that (X, d_X) is a *metric surface*. A homeomorphism $\phi \colon (X, d_X) \to (Y, d_Y)$ between metric surfaces is *quasiconformal* if there exists $K \geq 1$ such that for all path families Γ,

$$K^{-1} \operatorname{mod} \Gamma \leq \operatorname{mod} \phi\Gamma \leq K \operatorname{mod} \Gamma, \tag{1.1}$$

where $\operatorname{mod} \Gamma$ is the *conformal modulus* of Γ, see Section 2.3.

We say that a metric surface (Y, d_Y) is a *metric Jordan domain* if the metric completion $\overline{Y}$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$, the *boundary* $\partial Y = \overline{Y} \setminus Y$ is homeomorphic to the unit circle $\mathbb{S}^1$, and the Hausdorff 2-measure of $\overline{Y}$ is finite.

A metric Jordan domain is a *quasiconformal Jordan domain* if there exists a quasiconformal homeomorphism $\phi \colon \mathbb{D} \to (Y, d_Y)$. A metric Jordan domain is a quasiconformal one if and only if (Y, d_Y) is *reciprocal* as introduced in [18, Theorem 1.4]; see Definition 2.5. This uses the facts that $\mathcal{H}^2_{\overline{Y}}(\overline{Y}) < \infty$ and that ∂Y is a non-trivial continuum.

In general, it is not true that the completion $\overline{Y}$ of a quasiconformal Jordan domain is a quasiconformal image of the closed unit disk $\overline{\mathbb{D}}$. We illustrate this with an example after Theorem 1.1. Contrast this with the classical case when Y is a Jordan domain in the plane $\mathbb{R}^2$. Then any 1-quasiconformal homeomorphism $\phi \colon \mathbb{D} \to Y$, i.e., any Riemann map from the unit disk onto Y extends to a homeomorphism $\Phi \colon \overline{\mathbb{D}} \to \overline{Y}$ by a result known as the Carathéodory extension theorem [9, Chapter I, Theorem 3.1]. In fact, the extension still satisfies (1.1) with $K = 1$. Additionally, if $\phi \colon \mathbb{D} \to Y$ is K-quasiconformal for some $K \geq 1$, it still has a homeomorphic extension to the boundary.

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1.1 Carathéodory's theorem

We prove the following generalization of the classical Carathéodory extension theorem of quasiconformal maps.

Theorem 1.1. Let $\phi\colon \mathbb{D} \to Y$ be a quasiconformal map onto a quasiconformal Jordan domain. Then there exists an extension $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ of ϕ that is surjective, monotone and $\Phi(\mathbb{S}^1) = \partial Y$.

Here we say that a map is *monotone* if it is continuous and the preimage of every point is a *continuum*, i.e., a compact and connected set.

The map Φ might fail to be a homeomorphism. As an example, consider the length space X homeomorphic to $\mathbb{R}^2$ obtained by collapsing the Euclidean square $[0,1]^2$ in $\mathbb{R}^2$ to a point. Let $\pi\colon \mathbb{R}^2 \to X$ denote the associated 1-Lipschitz quotient map. We define $Y = \pi((1,2)\times(0,1))$. Then $\partial Y = \pi(\partial\,[1,2]\times[0,1])$. The restriction of π to $(1,2)\times(0,1)$ is a 1-quasiconformal map, but its extension collapses the arc segment $\{1\}\times[0,1]$ to the singleton $\pi([0,1]^2)$. By considering a Riemann map $f\colon \mathbb{D} \to (0,1)^2$, the claim follows by setting $\phi = \pi\circ f$.

Next, we investigate when the extension in Theorem 1.1 is a quasiconformal homeomorphism. To this end, for every $y \in \overline{Y}$ and $\operatorname{diam}\overline{Y} \geq R > r > 0$, we let $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y}\setminus B_{\overline{Y}}(y,R); \overline{Y})$ denote the family of paths joining $\overline{B}_{\overline{Y}}(y,r)$ to $\overline{Y}\setminus B_{\overline{Y}}(y,R)$.

Proposition 1.2. The extension Φ in Theorem 1.1 is quasiconformal if and only if for every $y \in \partial Y$ and $R > 0$ for which $\overline{Y}\setminus B_{\overline{Y}}(y,R) \neq \emptyset$,

$$\lim_{r\to 0^+} \operatorname{mod}\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y}\setminus B_{\overline{Y}}(y,R); \overline{Y}) = 0. \tag{1.2}$$

Moreover, if (1.2) holds at each $y \in \partial Y$ and ϕ is K-quasiconformal, then Φ is K-quasiconformal.

A well-known fact is that if there exists $C_U > 0$ such that for all $y \in \partial Y$ and $0 < r < \operatorname{diam}\partial Y$,

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y,r)) \leq C_U r^2, \tag{1.3}$$

then (1.2) holds; see Lemma 2.8. The condition (1.2) has a close link to the reciprocality condition introduced in [18]; see Definition 2.5. The aforementioned example of the collapsed disk $[0,1]^2$ fails (1.2) at exactly one point.

It can happen that the extension Φ in Theorem 1.1 is a homeomorphism, but not quasiconformal; see [14, Example 6.1]. There we have a metric space X for which there exists a 1-Lipschitz homeomorphism $\pi\colon \mathbb{R}^2 \to X$ which is 1-quasiconformal outside a Cantor set $K \subset [0,1]\times\{0\}$, but $\pi|_{(0,1)^2}$ does not extend to a 1-quasiconformal homeomorphism on $[0,1]^2$. The claim follows by setting $Y = \pi((0,1)^2)$ and setting $\phi = \pi\circ f$ for any Riemann map $f\colon \mathbb{D} \to (0,1)^2$.

1.2 Quasicircles

Consider a quasiconformal Jordan domain Y whose boundary points satisfy the area growth inequality (1.3). We know from Proposition 1.2 that the extension $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ of any quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$ is a quasiconformal homeomorphism. In particular, the *boundary map* $g_\phi = \Phi|_{\mathbb{S}^1}\colon \mathbb{S}^1 \to \partial Y$ is a homeomorphism.

We are especially interested when we can deduce that ∂Y is a *quasicircle*, i.e., a quasisymmetric image of $\mathbb{S}^1$. We refer the reader to Section 2 for definitions.

Theorem 1.3 (Beurling–Ahlfors extension)**.** Suppose that Y is a quasiconformal Jordan domain whose boundary points satisfy the area growth (1.3).

If $\phi\colon \mathbb{D} \to Y$ is a quasiconformal homeomorphism, then the boundary map g_ϕ is a quasisymmetry if and only if ∂Y has bounded turning. If ∂Y has bounded turning, then any quasisymmetry $g\colon \mathbb{S}^1 \to \partial Y$ is the boundary map of some quasiconformal map $\phi\colon \mathbb{D} \to Y$.

Theorem 1.3 has a parallel in the classical literature. Ahlfors and Beurling proved in [2] that every quasisymmetry $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the boundary homeomorphism of some quasiconformal map $\phi\colon \mathbb{D} \to \mathbb{D}$. In fact, we apply their result in proving that quasisymmetries $g\colon \mathbb{S}^1 \to \partial Y$ extend like claimed. It is also known that the boundary homeomorphisms of quasiconformal maps $\phi\colon \mathbb{D} \to \mathbb{D}$ are quasisymmetries. So we also recover this result with the assumptions of Theorem 1.3.

We now know from Theorem 1.3 that ∂Y is a quasicircle in some situations. We are interested whether or not ∂Y can be bi-Lipschitz embedded into the plane. We say that a quasicircle Z is *planar*, or a *planar quasicircle*, if there exists a bi-Lipschitz embedding $h\colon Z \to \mathbb{R}^2$.

One of the main results obtained in [13] states that a quasicircle is planar if and only if its *Assouad dimension* is strictly less than two, see Definition 2.10. There are quasicircles for every Assouad dimension between 1 and ∞ since $Z = (\mathbb{S}^1, \|\cdot\|_2^{\alpha})$ for $0 < \alpha \leq 1$ has Assouad dimension α^{-1}.

Proposition 1.4. Let Y be a quasiconformal Jordan domain. If ∂Y is a quasicircle and the boundary points satisfy the area growth (1.3), then the Assouad dimension of ∂Y is at most two.

It is not clear if ∂Y in the above statement must be planar. However, if Y is *annularly linearly locally connected (ALLC)* and *Ahlfors* 2*-regular*, then ∂Y is a planar quasicircle [17, Theorems 8.1 and 8.2]; see [17] for the proofs and terminology. Quasiconformal Jordan domains satisfying these stronger assumptions appear in [24] and [3].

We localize these assumptions in the following statement and obtain the same conclusion.

Theorem 1.5. Let Y be a quasiconformal Jordan domain such that ∂Y is a quasicircle and its boundary points satisfy the area growth (1.3). Then the Assouad dimension of ∂Y is strictly less than two if the following two conditions are satisfied for some $r_0 > 0$, $C > 0$ and $\lambda > 1$:

(a) For every $y \in \partial Y$ and $0 < 2r < R < r_0$ and any pair $a, b \in \overline{B}_{\overline{Y}}(y, R) \setminus B_{\overline{Y}}(y, r)$, there exists a path $|\alpha| \subset \overline{B}_{\overline{Y}}(y, \lambda R) \setminus B_{\overline{Y}}(y, \lambda^{-1} r)$ containing a and b.
(b) For every $z \in Y$ with $0 < r < d(z, \partial Y) \leq r_0$, $\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(z, r)) \geq C^{-1} r^2$.

In particular, ∂Y is planar.

If (a) holds we say that ∂Y is *relatively ALLC* and if (b) holds, we say that Y satisfies the *Ahlfors lower bound* near ∂Y. The main point of Theorem 1.5 is to only restrict the geometry of $\overline{Y}$ near the boundary ∂Y.

The relative ALLC guarantees that ∂Y is *porous* in $\overline{Y}$ below the given scale r_0 (Lemma 5.2). The porosity allows us to pack many balls in Y, well-disjoint from ∂Y, near all points of ∂Y at all scales below r_0. Now the Ahlfors lower bound, valid for such balls, combined with the upper bound (1.3) allow us to control quantitatively the total amount of such non-overlapping balls in a given interval of scales. This quantification allows us to prove planarity for ∂Y. This idea appears in [3, Lemma 3.12], where the authors prove that a compact set in an Ahlfors regular space is porous if and only if its Assouad dimension is strictly smaller than the homogeneous dimension of the space. A similar argument also works in the setting of Theorem 1.5.

1.3 Outline

In Section 2, we introduce the notations we use and some preliminary results. In Section 3, we prove Theorem 1.1. Theorem 1.3 and Proposition 1.4 are proved in Section 4. Theorem 1.5 is proved in Section 5. Section 6 contains some concluding remarks.

2 Preliminaries

2.1 Notation

Let (Y, d_Y) be a metric space. The open ball centered at a point $y \in Y$ of radius $r > 0$ with respect to the metric d is denoted by $B_Y(y, r)$. The closed ball is denoted by $\overline{B}_Y(y, r)$. We sometimes omit the subscript from d_Y, from B_Y, and from $\overline{B}_Y$, respectively.

We recall the definition of Hausdorff measure. Let (Y, d) be a metric space. For all $Q \geq 0$, the *Q-dimensional Hausdorff measure (Hausdorff Q-measure)* is defined by

$$\mathcal{H}^Q_Y(B) = \frac{\alpha(Q)}{2^Q} \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^Q : B \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < \delta \right\}$$

for all sets $B \subset Y$, where $\alpha(Q) = \pi^{\frac{Q}{2}} \left(\Gamma\left(Q/2 + 1\right)\right)^{-1}$. The constant $\alpha(Q)$ is chosen in such a way that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers.

The *length* of a path $\gamma \colon [a, b] \to Y$ is defined as

$$\ell_d(\gamma) = \sup \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)),$$

the supremum taken over all finite partitions $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. A path is *rectifiable* if it has finite length.

The *metric speed* of a path $\gamma \colon [a, b] \to Y$ at the point $t \in [a, b]$ is defined as

$$v_\gamma(t) = \lim_{h \to 0^+} \frac{d(\gamma(t+h), \gamma(t))}{h}$$

whenever this limit exists. If γ is rectifiable, its metric speed exists at $\mathcal{L}^1$-almost every $t \in [a, b]$ [7, Theorem 2.1].

A rectifiable path $\gamma \colon [a, b] \to Y$ is *absolutely continuous* if for all $a \leq s \leq t \leq b$,

$$d(\gamma(t), \gamma(s)) \leq \int_s^t v_\gamma(u)\, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a, b])$ and $\mathcal{L}^1$ the Lebesgue measure on the real line. Equivalently, γ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $\mathcal{H}^1_Y$-measure zero in its image [7, Section 3].

Let $\gamma \colon [a, b] \to X$ be an absolutely continuous path. Then the *path integral* of a Borel function $\rho \colon X \to [0, \infty]$ *over* γ is

$$\int_\gamma \rho\, ds = \int_a^b (\rho \circ \gamma) v_\gamma\, d\mathcal{L}^1. \tag{2.1}$$

If γ is rectifiable, then the *path integral* of ρ over γ is defined to be the path integral of ρ over the arc length parametrization γ_s of γ; see for example Chapter 5 of [12].

2.2 Quasiconformal Jordan domains

We assume that Y is a quasiconformal Jordan domain. In particular, its completion $\overline{Y}$ is homeomorphic to $[0, 1]^2$ and has finite Hausdorff 2-measure.

Given a Borel set $A \subset Y$, the *length* of a path $\gamma \colon [a, b] \to Y$ *in* A is defined as $\int_Y \chi_A(y) \#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y)$, where $\#(\gamma^{-1}(x))$ is the counting measure of $\gamma^{-1}(x)$. This formula makes sense for paths that are not necessarily rectifiable [8, Theorem 2.10.13]. When γ is rectifiable, for every Borel function $\rho \colon \overline{Y} \to [0, \infty]$,

$$\int_\gamma \rho\, ds = \int_{\overline{Y}} \rho(x) \#(\gamma^{-1}(x))\, d\mathcal{H}^1_{\overline{Y}}(x). \tag{2.2}$$

The equality (2.2) follows from [8, Theorem 2.10.13] via a standard approximation argument using simple functions.

We recall a special case of the coarea inequality [8, 2.10.25]. Let $\alpha \in \{0, 1\}$, $B \subset \overline{Y}$ Borel, and $f\colon \overline{Y} \to \mathbb{R}$ 1-Lipschitz. Then

$$\int_{\mathbb{R}}^{\star} \mathcal{H}^{\alpha}_{\overline{Y}}(B \cap f^{-1}(t))\, d\mathcal{L}^1(t) \le C_\alpha \mathcal{H}^{\alpha+1}_{\overline{Y}}(B), \tag{2.3}$$

where $C_0 = 1$ and $C_1 = 4/\pi$. Here $\int^\star$ refers to the upper integral [8, 2.4.2]. If $\mathcal{H}^{\alpha+1}_{\overline{Y}}(B) < \infty$, the upper integral can be replaced with the usual one [8, 2.10.26].

Via a standard approximation argument using simple functions, we obtain the following.

Theorem 2.1. Let $f\colon \overline{Y} \to \mathbb{R}$ be 1-Lipschitz, $\alpha \in \{0, 1\}$ and C_α as in (2.3). Then, for every Borel function $g\colon \overline{Y} \to [0, \infty]$,

$$\int_{\mathbb{R}}^{\star} \int_{f^{-1}(t)} g(y)\, d\mathcal{H}^{\alpha}_{\overline{Y}}(y)\, d\mathcal{L}^1(t) \le C_\alpha \int_{\overline{Y}} g(y)\, d\mathcal{H}^{\alpha+1}_{\overline{Y}}(y).$$

When g is $\mathcal{H}^{\alpha+1}_{\overline{Y}}$-integrable, the upper integral can be replaced with the usual one.

In the following, we say that $C \subset \overline{Y}$ is a *continuum* if C is compact and connected. A compact set $F \subset \overline{Y}$ *separates* $x, y \in \overline{Y}$ if $x, y \in \overline{Y} \setminus F$ and the points are in different connected components of $\overline{Y} \setminus F$.

Lemma 2.2. Let $F \subset \overline{Y}$ be compact and $x, y \in \overline{Y}$ separated by F. Then there exists a continuum $C \subset F$ that separates x and y.

Proof. Let x, y and F be as in the claim. We consider a homeomorphism $h\colon \overline{Y} \to Z \subset \mathbb{S}^2$, where Z is the union of the equator and the southern hemisphere of $\mathbb{S}^2$. Then there exists a nested sequence of quadrilaterals $Z_n \supset Z_{n+1} \supset Z$ such that $h(x), h(y) \in Z_n$ is an interior point of Z_n for each $n \in \mathbb{N}$ and $Z = \bigcap_{n=1}^\infty Z_n$.

Since $F_n = \partial Z_n \cup h(F) \subset Z_n$ separates $h(x)$ and $h(y)$ in $\mathbb{S}^2$, there exists a continuum $C_n \subset F_n$ separating $h(x)$ and $h(y)$ in $\mathbb{S}^2$ [23, Chapter 2, Lemma 5.20]. In particular, for every path $\gamma\colon [0, 1] \to Z$ joining $h(x)$ to $h(y)$, there exists $z_n \in C_n \cap |\gamma|$ for every $n \in \mathbb{N}$. Up to passing to a subsequence and relabeling, the continua $(C_n)_{n=1}^\infty$ converge to a continuum $C' \subset \bigcap_{n=1}^\infty h(F) \cap Z_n = h(F)$ in the Hausdorff convergence [1, Theorems 4.4.15 and 4.4.17]. If γ and $(z_n)_{n=1}^\infty$ are as above, the accumulation points of $(z_n)_{n=1}^\infty$ are contained in $C' \cap |\gamma|$ [1, Proposition 4.4.14]. Consequently, $C' \cap |\gamma| \neq \emptyset$ for every such γ. Hence $C = h^{-1}(C') \subset F$ is a continuum separating x and y. □

2.3 Metric Sobolev spaces

In this section we give an overview of Sobolev theory in the metric surface setting, and refer to [12] for a comprehensive introduction.

Let $\Gamma \subset \mathcal{C}([0, 1]\,; Y)$ be a family of rectifiable paths in Y. A Borel function $\rho\colon Y \to [0, \infty]$ is *admissible* for Γ if the path integral $\int_\gamma \rho\, ds \ge 1$ for all rectifiable paths $\gamma \in \Gamma$. The *modulus* of Γ is

$$\operatorname{mod} \Gamma = \inf \int_Y \rho^2\, d\mathcal{H}^2_Y,$$

where the infimum is taken over all admissible functions ρ. Observe that if Γ_1 and Γ_2 are path families and every path $\gamma_1 \in \Gamma_1$ contains a subpath $\gamma_2 \in \Gamma_2$, then $\operatorname{mod} \Gamma_1 \le \operatorname{mod} \Gamma_2$. In particular, this holds if $\Gamma_1 \subset \Gamma_2$. A property holds for *almost every* path if the family of paths for which the property fails has zero modulus.

Let $\psi\colon (Y, d_Y) \to (Z, d_Z)$ be a mapping between metric spaces Y and Z. A Borel function $\rho\colon Y \to [0, \infty]$ is an *upper gradient of* ψ if

$$d_Y(\psi(x), \psi(y)) \le \int_\gamma \rho \, ds$$

for every rectifiable path $\gamma\colon [0, 1] \to Y$ connecting x to y. The function ρ is a *weak upper gradient of* ψ if the same holds for almost every rectifiable path.

A weak upper gradient $\rho \in L^2_{\mathrm{loc}}(Y)$ of ψ is *minimal* if it satisfies $\rho \le \widetilde{\rho}$ almost everywhere for all weak upper gradients $\widetilde{\rho} \in L^2_{\mathrm{loc}}(Y)$ of ψ. If ψ has a weak upper gradient $\rho \in L^2_{\mathrm{loc}}(Y)$, then ψ has a minimal weak upper gradient, which we denote by ρ_ψ. We refer to Section 6 of [12] and Section 3 of [26] for details.

Fix a point $z \in Z$, and let $d_z = d_Z(\cdot, z)$. The space $L^2(Y, Z)$ is defined as the collection of measurable maps $\psi\colon Y \to Z$ such that $d_z \circ \psi$ is in $L^2(Y)$.

Moreover, $L^2_{\mathrm{loc}}(Y, Z)$ is defined as those measurable maps $\psi\colon Y \to Z$ for which, for all $y \in Y$, there is an open set $U \subset Y$ containing y such that $\psi|_U$ is in $L^2(U, Z)$.

The metric Sobolev space $N^{1,2}_{\mathrm{loc}}(Y, Z)$ consists of those maps $\psi\colon Y \to Z$ in $L^2_{\mathrm{loc}}(Y, Z)$ that have a minimal weak upper gradient $\rho_\psi \in L^2_{\mathrm{loc}}(Y)$.

For open $\emptyset \ne U \subset Y$, we say that $\psi \in N^{1,2}(U, Z)$ if $\psi|_U \in N^{1,2}_{\mathrm{loc}}(U, Z)$, $\rho_{\psi|_U} \in L^2(U)$ and $\psi|_U \in L^2(U, Z)$.

Given a homeomorphism $\psi\colon Y \to Z$, the pullback measure $\psi^*\mathcal{H}^2_Z$ is defined by $\psi^*\mathcal{H}^2_Z(B) = \mathcal{H}^2_Z(\psi(B))$ for each Borel set $B \subset Y$. The pullback measure has a decomposition $\psi^*\mathcal{H}^2_Z = J_\psi \mathcal{H}^2_Y + \mu^\perp$, where J_ψ is locally integrable with respect to $\mathcal{H}^2_Y$, and the measures $\mathcal{H}^2_Y$ and $\mu^\perp$ are singular [5, Sections 3.1-3.2 in Volume I]. We call the density J_ψ the *Jacobian* of ψ.

2.4 Quasiconformal mappings

We define quasiconformal maps and recall some basics.

Definition 2.3. Let (Y, d_Y) and (Z, d_Z) be metric spaces with locally finite Hausdorff 2-measures. We say that a homeomorphism $\psi\colon (Y, d_Y) \to (Z, d_Z)$ is *quasiconformal* if there exists $K \ge 1$ such that for all path families Γ in Y

$$K^{-1} \operatorname{mod} \Gamma \le \operatorname{mod} \psi\Gamma \le K \operatorname{mod} \Gamma, \tag{2.4}$$

where $\psi\Gamma = \{\psi \circ \gamma\colon \gamma \in \Gamma\}$. If (2.4) holds with a constant $K \ge 1$, we say that ψ is *K-quasiconformal*.

Definition 2.3 is sometimes called the *geometric* definition of quasiconformality. A special case of [26, Theorem 1.1] yields the following.

Theorem 2.4. Let Y and Z be metric spaces with locally finite Hausdorff 2-measure and $\psi\colon Y \to Z$ a homeomorphism. The following are equivalent for the same constant $K > 0$:

(i) $\operatorname{mod} \Gamma \le K \operatorname{mod} \psi\Gamma$ for all path families Γ in Y.
(ii) $\psi \in N^{1,2}_{\mathrm{loc}}(Y, Z)$ and satisfies $\rho^2_\psi(y) \le K J_\psi(y)$ for $\mathcal{H}^2_Y$-almost every $y \in Y$.

The *outer dilatation* of ψ is the smallest constant $K_O \ge 0$ for which the modulus inequality $\operatorname{mod} \Gamma \le K_O \operatorname{mod} \psi\Gamma$ holds for all Γ in Y. The *inner dilatation* of ψ is the smallest constant $K_I \ge 0$ for which $\operatorname{mod} \psi\Gamma \le K \operatorname{mod} \Gamma$ holds for all Γ in Y. The number $K(\psi) = \max\left\{K_I(\psi), K_O(\psi)\right\}$ is the *maximal dilatation* of ψ.

For a set $G \subset Y$ and disjoint sets $F_1, F_2 \subset G$, let $\Gamma(F_1, F_2; G)$ denote the family of paths that start from F_1, end in F_2 and whose images are contained in G. A *quadrilateral* is a set Q homeomorphic to $[0, 1]^2$ with boundary ∂Q consisting of four boundary arcs, overlapping only at the end points, labelled $\xi_1, \xi_2, \xi_3, \xi_4$ in cyclic order.

Definition 2.5. A metric surface Y is *reciprocal* if there exists a constant $\kappa \ge 1$ such that

$$\kappa^{-1} \le \operatorname{mod} \Gamma(\xi_1, \xi_3; Q) \operatorname{mod} \Gamma(\xi_2, \xi_4; Q) \le \kappa \tag{2.5}$$

for every quadrilateral $Q \subset Y$, and

$$\lim_{r\to 0^+} \operatorname{mod} \Gamma\left(\overline{B}_Y(y,r), Y \setminus B_Y(y,R); \overline{B}_Y(y,R)\right) = 0 \tag{2.6}$$

for all $y \in Y$ and $R > 0$ such that $Y \setminus B_Y(y,R) \neq \emptyset$.

We note that the product in (2.5) is always bounded from below by a universal constant $\kappa_0 > 0$ [19]. We also have the following.

Proposition 2.6 (Corollary 12.3 of [18])**.** Let Y be a metric surface, $U \subset Y$ a domain, and $\psi\colon U \to \Omega \subset \mathbb{R}^2$ a homeomorphism. If $K_O(\psi) < \infty$, then $K_I(\psi) \leq (2 \cdot \kappa_0) K_O(\psi) < \infty$.

Recall the definition of quasiconformal Jordan domain from the introduction.

Proposition 2.7. Let Y be a quasiconformal Jordan domain and $\Psi\colon \overline{Y} \to \overline{\mathbb{D}}$ a homeomorphism. Then $K_O(\Psi) \leq K < \infty$ if and only if there exists a constant $C > 0$ such that

$$\liminf_{r\to 0^+} \operatorname{mod} \Psi^{-1}\Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x,r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x,2r); \overline{B}_{\overline{\mathbb{D}}}(x,2r)\right) \leq C \tag{2.7}$$

for every $x \in \overline{\mathbb{D}}$. The constants K and C depend on each other quantitatively. Moreover, $K_I(\Psi) \leq (2 \cdot \kappa_0) \cdot K$.

Proof. Since the Lebesgue 2-measure on $\overline{\mathbb{D}}$ is doubling, Theorem 1.2 of [26] states that an upper bound $K_O(\Psi) \leq K$ is quantitatively equivalent to the following statement: there exists $C' \geq 1$ such that for every $x \in \overline{\mathbb{D}}$,

$$\liminf_{r\to 0^+} \frac{r^2 \operatorname{mod} \Psi^{-1}\Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x,r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x,2r); \overline{B}_{\overline{\mathbb{D}}}(x,2r)\right)}{\mathcal{L}^2(\overline{B}_{\overline{\mathbb{D}}}(x,r))} \leq C'.$$

Since $\mathcal{L}^2(B_{\overline{\mathbb{D}}}(x,r))$ is comparable to r^2, $K_O(\Psi) \leq K$ if and only if (2.7) holds for some C, with K and C depending on one another quantitatively.

It remains to prove that $K_I(\Psi) \leq C_0 K$. To this end, consider $\psi = \Psi|_Y$ and $\phi = \psi^{-1}$. Proposition 2.6 implies that $K_I(\psi) = K_O(\phi) \leq C_0 K$. Then Proposition 2.4 implies that $\phi \in N^{1,2}(\mathbb{D}; Y)$ since the Jacobian J_ϕ of ϕ is integrable. Observe that the extension $\Phi = \Psi^{-1}$ of ϕ is an element of $N^{1,2}(\overline{\mathbb{D}}; \overline{Y})$. This can be seen by extending Φ to a neighbourhood of $\overline{\mathbb{D}}$ via reflection over $\mathbb{S}^1$ and by applying [15, Theorem 1.12.3]. The minimal weak upper gradient of Φ has a representative that vanishes in $\mathbb{S}^1$ since $\mathcal{L}^2(\mathbb{S}^1) = 0$. Therefore $\rho_\Phi^2 \leq C_0 K J_\Phi$ holds $\mathcal{L}^2$-almost everywhere in $\overline{\mathbb{D}}$. This implies that Φ satisfies the second condition in Proposition 2.4 with the constant $C_0 K$. □

We recall a sufficient condition for (1.2) for later use.

Lemma 2.8. Suppose that there exists $C_U > 0$ such that for all $y \in \partial Y$ and $0 < r < \operatorname{diam} \partial Y$,

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y,r)) \leq C_U r^2. \tag{2.8}$$

Then for $\widetilde{C}_U = 8C_U/\log 2$ and for every $y \in \partial Y$ and $0 < 2r < R < 2^{-1} \operatorname{diam} \partial Y$,

$$\operatorname{mod} \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y}) \leq \frac{\widetilde{C}_U}{\log \frac{R}{r}}. \tag{2.9}$$

In particular, (1.2) in Proposition 1.2 holds under the assumption (2.8).

Proof. The inequality (2.9) follows from (2.8) by considering the admissible function $\rho(x) = \frac{1}{\log \frac{R}{r}} \frac{1}{d(y,x)} \chi_{\{r \leq d(y,x) \leq R\}}$. We claim that ρ is admissible for the family $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$. To this end, fix a rectifiable $\gamma \in \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$.

We denote $f(x) = d(y,x)$. Whenever $x \in f^{-1}(t)$, we have $\rho(x)\#(\gamma^{-1}(x)) \geq \frac{1}{\log \frac{R}{r}} \frac{1}{t} \chi_{\{r \leq t \leq R\}}$. Therefore

$$\int_{\mathbb{R}}^{\star} \int_{f^{-1}(t)} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^0_{\overline{Y}}(x)\, d\mathcal{L}^1(t) \geq \int_r^R \frac{1}{\log \frac{R}{r}} \frac{1}{t}\, d\mathcal{L}^1(t) = 1.$$

Then Theorem 2.1 implies

$$\int_{\overline{Y}} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^1_{\overline{Y}}(x) \geq \int_{\mathbb{R}}^{\star} \int_{f^{-1}(t)} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^0_{\overline{Y}}(x)\, d\mathcal{L}^1(t).$$

The equality (2.2) yields $\int_\gamma \rho\, ds = \int_{\overline{Y}} \rho(x)\#(\gamma^{-1}(x))\, d\mathcal{H}^1_{\overline{Y}}(x)$. Hence ρ is admissible for $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$.

The L^2-norm of ρ is estimated from above by applying the area growth (2.8) on the annuli $A_l = \left\{2^l r \leq d(y,x) < 2^{l+1} r\right\}$ for $l = 0, 1, 2, \dots, k$ for $2^k r < R \leq 2^{k+1} r$, $k \in \mathbb{N}$. That is,

$$\int_{\overline{Y}} \rho^2(x)\, d\mathcal{H}^2_{\overline{Y}}(x) \leq \sum_{l=0}^{k} \int_{A_l} \rho^2(x)\, d\mathcal{H}^2_{\overline{Y}}(x) \leq \frac{1}{\log^2(\frac{R}{r})} \sum_{l=0}^{k} \frac{\mathcal{H}^2_{\overline{Y}}(\overline{B}(y, 2^{l+1} r))}{2^{2l} r^2}$$
$$\leq \frac{1}{\log^2(\frac{R}{r})} \sum_{l=0}^{k} \frac{C_U 2^{2l+2} r^2}{2^{2l} r^2} = 4C_U \frac{k+1}{\log^2 \frac{R}{r}} \leq \frac{8C_U/\log 2}{\log \frac{R}{r}}$$

since $k + 1 \leq (2/\log 2) \log \frac{R}{r}$. The inequality (2.9) follows.

We claim now that (1.2) in Proposition 1.2 holds. Let $y \in \partial Y$ and $R' > R > 0$ such that $\overline{Y} \setminus B_{\overline{Y}}(y, R') \neq \emptyset$ and $2^{-1} \operatorname{diam} \partial Y > R$. Then for every $0 < 2r < R$, every path in $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R'); \overline{Y})$ has a subpath in $\Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y})$. Hence

$$\operatorname{mod} \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R'); \overline{Y}) \leq \operatorname{mod} \Gamma(\overline{B}_{\overline{Y}}(y,r), \overline{Y} \setminus B_{\overline{Y}}(y,R); \overline{Y}).$$

The right-hand side converges to zero as $r \to 0^+$, given (2.9). This establishes (1.2).

□

2.5 Quasicircles

In this section we recall some basic properties of quasisymmetries and quasicircles. If $g\colon (Y, d_Y) \to (Z, d_Z)$, we denote

$$L_g(y,r) = \sup_{w \in \overline{B}_Y(y,r)} d_Z(g(y), g(w)) \quad \text{and} \quad \ell_g(y,r) = \inf_{w \in Y \setminus B_Y(y,r)} d_Z(g(y), g(w)).$$

Definition 2.9. Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism $g\colon (Y, d_Y) \to (Z, d_Z)$ between metric spaces is *η-quasisymmetric* if for every $y \in Y$ and $0 < r_1, r_2 < \operatorname{diam} Y$,

$$L_g(y, r_1) \leq \eta\left(\frac{r_1}{r_2}\right) \ell_g(y, r_2). \tag{2.10}$$

A homeomorphism g is *quasisymmetric* if it is η-quasisymmetric for some homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$.

A set $S \subset Y$ is *r-separated* if for every $x, y \in S$ with $x \neq y$, $d_Y(x,y) \geq r$, and an *r-net* if for every $y \in Y$, there exists $x \in S$ for which $d_Y(x,y) < r$. An r-separated set is *maximal* if it is also an r-net.

Definition 2.10. A metric space (Y, d_Y) has its *Assouad dimension* bounded from above by $Q > 0$ if for every $0 < \epsilon < 1$ and every $(y, r) \in Y \times (0, \operatorname{diam} Y)$, any ϵr-separated set $S \subset B_Y(y,r)$ satisfies

$$\#S \leq C\epsilon^{-Q}, \tag{2.11}$$

where C is a constant independent of ϵ, y, r and S. Here $\#S$ refers to the counting measure of S. The *Assouad dimension* of Y is the infimum of such Q.

A metric space (Y, d_Y) is said to be *doubling* if its Assouad dimension is finite.

Definition 2.11. Let $\lambda \geq 1$. A metric space (Y, d_Y) has λ-*bounded turning* if for every $y, z \in Y$ there exists a compact and connected set $E \subset Y$ containing y and z such that $\operatorname{diam} E \leq \lambda d_Y(y, z)$.

We recall that a metric space $\mathcal{C}$ homeomorphic to $\mathbb{S}^1$ is a quasisymmetric image of $\mathbb{S}^1$ if and only if $\mathcal{C}$ has bounded turning and is doubling [21]. We refer to any quasisymmetric image of $\mathbb{S}^1$ as a *quasicircle*.

3 Carathéodory's theorem

3.1 Proof of Theorem 1.1

We fix a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$ and claim that it has a monotone and surjective extension $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$. We are assuming that $\overline{Y}$ is homeomorphic to $[0, 1]^2$ and has finite Hausdorff 2-measure.

Fix $x_0 \in \mathbb{S}^1$. For each $0 < r < 2^{-1}$, denote $E_r = \mathbb{S}^1(x_0, r) \cap \overline{\mathbb{D}}$. Let V_r be the component of $\mathbb{D} \setminus E_r$ whose closure contains $\{x_0\}$.

Let $U_r = \overline{\phi(V_r)}$. Since V_r is connected, so is U_r. Moreover, as $V_{r'} \subset V_r$ whenever $r' < r$, we have $U_{r'} \subset U_r$. Therefore

$$\emptyset \neq \widetilde{C} = \bigcap_{0<r<2^{-1}} U_r \quad \text{is compact and connected [22, Theorem 28.2].} \tag{3.1}$$

Notice that $\widetilde{C} \subset \partial Y$.

Lemma 3.1 implies that $\widetilde{C}$ is a singleton. Let y_0 denote the unique element. We define $\Phi(x_0) := y_0$. We repeat the argument for every $x_0 \in \mathbb{S}^1$. By setting $\Phi(x) = \phi(x)$ for every $x \in \mathbb{D}$, we obtain a mapping

$$\Phi\colon \overline{\mathbb{D}} \to \overline{Y}.$$

We prove in Lemma 3.4 that Φ is continuous and surjective. Lemma 3.5 shows the monotonicity of Φ. Hence Theorem 1.1 follows after we verify these lemmas.

Lemma 3.1. Let d_r denote the diameter of U_r. Then $d_r \to \operatorname{diam} \widetilde{C} = 0$ for every $x_0 \in \mathbb{S}^1$.

Before proving Lemma 3.1, we prove a couple of technical lemmas. In the following, an *arc* refers to a set homeomorphic to $[0, 1]$.

Lemma 3.2. Let $C' \subset Y$ be an arc and $C \subset \partial Y$ a compact and connected set. Then

$$\operatorname{diam} C > 0 \quad \text{implies} \quad \operatorname{mod} \Gamma(C, C'; Y \cup C) > 0. \tag{3.2}$$

Proof of Lemma 3.2. Since C and C' are disjoint, there are Borel functions $\rho \in L^2(\overline{Y})$ admissible for $\Gamma(C, C'; Y \cup C)$. We fix such a function ρ and find a lower bound for the L^2-norm of ρ, depending only on C and C'. The claim (3.2) follows from this.

We argue as in the proof of [18, Proposition 3.5]. First, we join C and C' with an arc $\gamma\colon [0, 1] \to Y \cup C$ for which $r_0 = d(|\gamma|, \partial Y \setminus C) > 0$, and consider the Lipschitz function $f(z) = d(|\gamma|, z)$. Since C and C' are arcs, we can choose γ in such a way that $\gamma(0)$ separates C into two arcs J_1 and J_2, $\gamma(1)$ separates C' into two arcs J_3 and J_4, and $\gamma(t) \in Y \setminus (C \cup C')$ for every $0 < t < 1$.

Fix $0 < r_1 < r_0$ such that every J_i intersects $f^{-1}(r)$ for every $0 < r < r_1$. For every such $r > 0$, the level set $f^{-1}(r)$ separates $|\gamma|$ from the arc $\overline{\partial Y \setminus C}$. Then Theorem 2.2 provides us with a continuum $\Gamma_r \subset f^{-1}(r)$ that separates $|\gamma|$ from $\overline{\partial Y \setminus C}$. The continuum Γ_r must intersect every J_i, since otherwise we find a path joining $|\gamma|$ to $\overline{\partial Y \setminus C}$ that does not intersect Γ_r.

By applying Theorem 2.1 to the function $g(y) = \chi_{\overline{Y}}(y)$, we conclude that the level set $f^{-1}(r)$ has finite Hausdorff 1-measure for $\mathcal{L}^1$-almost every $0 < r < r_1$. In particular, the continuum Γ_r has finite Hausdorff

1-measure. Then every pair of points from Γ_r can be joined with a rectifiable path within Γ_r [20, Proposition 15.1]. Consequently, there exists a rectifiable arc $\theta\colon [0,1] \to \Gamma_r$ joining $J_1 \subset C$ to $J_3 \subset C'$. Since $0 < r < r_1 < r_0$, we have $\theta \in \Gamma(C, C'; Y \cup C)$. Hence

$$1 \leq \int_\theta \rho \, ds \leq \int_{f^{-1}(r)} \rho \, d\mathcal{H}^1_{\overline{Y}}.$$

Then Theorem 2.1 and Hölder's inequality imply

$$r_1 \leq \frac{4}{\pi}\left(\mathcal{H}^2_{\overline{Y}}(\overline{Y})\right)^{1/2} \|\rho\|_{L^2(\overline{Y})}.$$

Rearranging this inequality establishes the claim. □

Lemma 3.3. Let $\theta\colon (0,1) \to G \subset \overline{Y}$ be a homeomorphism and suppose that for every $0 < s < t < 1$,

$$\ell(\theta|_{[s,t]}) \leq \int_s^t h(a)\, d\mathcal{L}^1(a)$$

for some $h \in L^1([0,1])$. Then there exists an absolutely continuous extension $\overline{\theta}\colon [0,1] \to \overline{G}$ of θ that is surjective.

Proof of Lemma 3.3. Let $0 < s < t < 1$. Then

$$d(\theta(s), \theta(t)) \leq \int_0^1 \left(\chi_{[0,s]}(a) + \chi_{[0,t]}(a)\right) h(a)\, d\mathcal{L}^1(a). \tag{3.3}$$

By the absolute continuity of the integral, given $\epsilon > 0$, there exists $\delta > 0$ for which

$$|s|, |t| < \delta \quad \text{implies} \quad \int_0^1 \left(\chi_{[0,s]} + \chi_{[0,t]}\right)(a) h(a)\, d\mathcal{L}^1(a) < \epsilon. \tag{3.4}$$

This fact and (3.3) imply that for any given $(s_j)_{j=1}^\infty \subset (0,1)$ converging to zero, the sequence $(\theta(s_j))_{j=1}^\infty$ is Cauchy. Since $\overline{Y}$ is complete, the sequence converges to some $y_0 \in \overline{G}$. We define $\overline{\theta}(0) := y_0$. The inequalities (3.3) and (3.4) imply that y_0 is independent of the sequence $(s_j)_{j=1}^\infty$, and setting $\overline{\theta}(s) = \theta(s)$ for $0 < s$ defines a continuous extension of θ to $[0,1)$.

By arguing similarly for $t = 1$, we find a continuous extension $\overline{\theta}\colon [0,1] \to \overline{G}$ of θ. The inequality (3.3) extends to every $0 \leq s < t \leq 1$, implying the absolute continuity of $\overline{\theta}$. Notice that for every $y \in \overline{G}$, there exists a sequence $(t_j)_{j=1}^\infty \subset (0,1)$ such that $\overline{\theta}(t_j) \to y$. By passing to a subsequence and relabeling, we may assume that $(t_j)_{j=1}^\infty$ has a limit in $[0,1]$. This implies that $\overline{\theta}$ is surjective. □

Proof of Lemma 3.1. Since $\widetilde{C}$ is the intersection of the U_r and U_r are nested, we have $d_r \to \operatorname{diam} \widetilde{C}$. Hence the difficulty lies in proving $\operatorname{diam} \widetilde{C} = 0$.

Fix an arc $C' \subset Y$ for which $\phi^{-1}(C') \subset \mathbb{D} \setminus (E_r \cup V_r)$ for every $0 < r < 2^{-1}$. We assume $\operatorname{diam} \widetilde{C} > 0$ and derive a contradiction. Since $\operatorname{diam} \widetilde{C} > 0$, there exist a subarc $C \subset \widetilde{C}$ such that $r_0 = d(C, C' \cup (\partial Y \setminus \widetilde{C})) > 0$. We claim that

$$\operatorname{mod} \Gamma(C, C'; Y \cup C) = 0. \tag{3.5}$$

If (3.5) holds for C, we obtain a contradiction with Lemma 3.2.

So it suffices to prove (3.5). We claim that there exists a sequence $r_n \to 0^+$ such that every $\theta \in \Gamma(C, C'; Y \cup C)$ has a subpath in $\Gamma(\phi(E_{r_n} \cap \mathbb{D}), C'; Y)$. If this can be proved, then the K-quasiconformality of ϕ yields

$$\operatorname{mod} \Gamma(C, C'; Y \cup C) \leq K \operatorname{mod} \Gamma(E_{r_n} \cap \mathbb{D}, \phi^{-1}(C'); \mathbb{D}).$$

Given Lemma 2.8, the right-hand side converges to zero as $n \to \infty$, and we conclude (3.5). The rest of the proof is spent on finding the sequence of radii $(r_n)_{n=1}^\infty$.

The K-quasiconformality of ϕ yields that the minimal weak upper gradient ρ_ϕ satisfies $\rho_\phi^2 \le KJ_\phi \in L^1(\mathbb{D})$. The integrability of J_ϕ follows from the fact that Y has finite Hausdorff 2-measure. This implies that ϕ has an $L^2(\mathbb{D})$-integrable upper gradient g [12, Lemma 6.2.2].

We consider $g_0 = g\chi_{\mathbb{D}} \in L^2(\overline{\mathbb{D}})$. Polar coordinates centered at x_0 yield

$$\infty > \|g_0\|_{L^1(\overline{\mathbb{D}})} \ge \int_0^{2^{-1}} \int_{E_r} g_0 \, d\mathcal{H}^1 \, d\mathcal{L}^1(r). \tag{3.6}$$

In particular, g_0 has a finite path integral over E_r for almost every $0 < r < 2^{-1}$. Let I denote those $0 < r < 2^{-1}$ for which this holds.

Let Γ_0 be the family of non-constant rectifiable paths in $\overline{\mathbb{D}}$ along which g_0 is not integrable. Consider an absolutely continuous non-constant path $\gamma\colon [a, b] \to \overline{Y}$ with image in Y and which is not an element of Γ_0. Since g is an upper gradient of ϕ, we have that

$$\ell(\phi \circ \gamma) \le \int_\gamma g \, ds = \int_\gamma g_0 \, ds; \text{ see [12, Proposition 6.3.2].} \tag{3.7}$$

Consider a surjective Lipschitz $\gamma_r\colon [0, 1] \to E_r$ for $r \in I$. Then, for every $0 < s < t < 1$, (3.7) implies

$$\ell((\phi \circ \gamma_r)|_{[s,t]}) \le \int_{\gamma_r|_{[s,t]}} g_0 \, ds \le \int_{E_r} g_0 \, d\mathcal{H}^1,$$

where the last term on the right is finite. Therefore $\theta_r = \phi \circ \gamma_r\colon (0, 1) \to \phi(E_r \cap \mathbb{D})$ satisfies the assumptions of Lemma 3.3. Hence there exists a continuous extension $\overline{\theta_r}\colon [0, 1] \to F_r$ onto $F_r := \overline{\phi(E_r \cap \mathbb{D})}$.

Since ϕ is a homeomorphism, $\overline{\theta_r}(s) \notin Y$ for both $s = 0, 1$. Hence F_r is homeomorphic to a circle or an arc. We note that $\phi(V_r)$ is one of the connected components of $Y \setminus F_r$. In particular, $U_r = \overline{\phi(V_r)}$ is homeomorphic to $[0, 1]^2$, $U_r \cap \partial Y$ is a point or an arc, and $C \subset \widetilde{C} \subset U_r \cap \partial Y$. As the ends of $U_r \cap \partial Y$ and F_r coincide, $d(C, \partial Y \setminus \widetilde{C}) > 0$ implies that $U_r \cap \partial Y$ is an arc and $C \cap F_r = \emptyset$. This means that every path $\theta \in \Gamma(C, C'; Y \cup C)$ has a subpath in $\Gamma(F_r \cap Y, C'; Y)$. Then (3.5) follows by taking a sequence $(r_n)_{n=1}^\infty \subset I$ converging to zero. □

Lemma 3.4. The mapping $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ is continuous and surjective.

Proof. Let $x_0 \in \mathbb{S}^1$. If $\mathbb{D} \ni x_n \to x_0$, the accumulation points of $(\Phi(x_n))_{n=1}^\infty$ are contained in the intersection of U_r, where U_r are as in the definition of $\widetilde{C}$ in (3.1). Lemma 3.1 shows that the intersection is a singleton, which we defined to be $\Phi(x_0)$. This implies $\Phi(x_n) \to \Phi(x_0)$.

More generally, if $\overline{\mathbb{D}} \ni x_n \to x_0$, we find for every x_n an element $z_n \in \mathbb{D}$ such that $d_{\overline{Y}}(\Phi(z_n), \Phi(x_n)) \le 2^{-n}$ and $\|z_n - x_n\| \le 2^{-n}$. Then $\mathbb{D} \ni z_n \to x_0$ and $\Phi(z_n) \to \Phi(x_0)$. Since $d_Y(\Phi(z_n), \Phi(x_n)) \to 0$, we have $\Phi(x_n) \to \Phi(x_0)$. This implies that Φ is continuous.

Consider now $y_0 \in \overline{Y}$. Then there exists a sequence $Y \ni y_n \to y_0$. Up to passing to a subsequence and relabeling, $(\Phi^{-1}(y_n))_{n=1}^\infty$ converges to some $x_0 \in \overline{\mathbb{D}}$. The continuity of Φ implies $\Phi(x_0) = y_0$. □

Lemma 3.5. The mapping $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ is monotone.

Proof. Let $y \in \partial Y$ and suppose that there are two distinct points x_1 and x_2 from $\mathbb{S}^1$ such that $\Phi(x_1) = y = \Phi(x_2)$. Let I_i be radial lines from x_i to 0 for $i = 1, 2$ and define $I = I_0 \cup I_1$. Here $\Phi(I)$ is a Jordan loop intersecting ∂Y exactly at y. Let U denote the component of $Y \setminus \Phi(I)$ whose closure intersects ∂Y only at y. Then $V = \Phi^{-1}(U)$ is one of the components of $\mathbb{D} \setminus I$. By construction, $\overline{V} \cap \mathbb{S}^1$ is connected and is mapped to the singleton y. Therefore x_1 and x_2 can be joined with a path in $\Phi^{-1}(y)$. The monotonicity of Φ follows. □

3.2 Proof of Proposition 1.2

Recall that we fix a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$ and study its extension Φ under the assumption (1.2). This is done in several parts. First, Lemma 3.6 proves that the extension is a homeomorphism.

Lemma 3.8 implies that Φ is quasiconformal. After this is verified, Lemma 3.9 proves that $K_O(\Phi) = K_O(\phi)$ and $K_I(\Phi) = K_I(\phi)$. In particular, the maximal dilatations of Φ and ϕ coincide.

To finish up the proof of Proposition 1.2, we also need to verify that if Φ is a quasiconformal homeomorphism, then the assumption (1.2) holds. But this follows from the corresponding Euclidean result. So Proposition 1.2 follows after we prove the lemmas mentioned above.

Lemma 3.6. The mapping Φ is a homeomorphism.

Proof. Let $y \in \partial Y$ and suppose that there exists a non-trivial continuum $E \subset \Phi^{-1}(y)$. Let $F \subset Y \setminus \overline{B}_{\overline{Y}}(y, R)$ be a non-trivial arc for some $R > 0$. For $0 < r < R$, let $\Gamma(y, r, R) = \Gamma(\overline{B}_{\overline{Y}}(y, r), \overline{Y} \setminus B_{\overline{Y}}(y, R); \overline{Y})$. Then

$$\operatorname{mod} \Gamma(y, r, R) \geq \operatorname{mod}(\Gamma(y, r, R) \cap \mathrm{AC}(Y)),$$

where $\mathrm{AC}(Y)$ refers to those absolutely continuous paths in $\overline{Y}$ whose images lie in Y. Since $\phi = \Phi|_{\mathbb{D}}$ is K-quasiconformal, we have that

$$\operatorname{mod}(\Gamma(y, r, R) \cap \mathrm{AC}(Y)) \geq K^{-1} \operatorname{mod}(\Gamma(E, \Phi^{-1}(F); \mathbb{D} \cup E \cup \Phi^{-1}(F))).$$

The right-hand side is a strictly positive lower bound; recall Lemma 3.2. Therefore $\operatorname{mod} \Gamma(y, r, R) \geq C_0 > 0$ for a constant independent of r. By passing to the limit $r \to 0^+$, we find a contradiction with (1.2). The injectivity of Φ follows. Since Φ is continuous, surjective, and monotone, we conclude that Φ is a homeomorphism. □

Next we claim that Φ is quasiconformal. To this end, we let $\Psi = \Phi^{-1}$. Due to Proposition 2.7, it is sufficient to find a constant C_0 such that for every $x \in \mathbb{S}^1$,

$$\liminf_{r \to 0^+} \operatorname{mod} \Psi^{-1} \Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x, r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x, 2r); \overline{B}_{\overline{\mathbb{D}}}(x, 2r)\right) \leq C_0. \tag{3.8}$$

We denote $\Gamma(x, r, 2r) = \Gamma\left(\overline{B}_{\overline{\mathbb{D}}}(x, r), \overline{\mathbb{D}} \setminus B_{\overline{\mathbb{D}}}(x, 2r); \overline{B}_{\overline{\mathbb{D}}}(x, 2r)\right)$ for the rest of the section.

Fix $0 < r < 1/4$. Let $\xi_1 = \Psi^{-1}\left(\mathbb{S}^1(x, r) \cap \overline{\mathbb{D}}\right)$ and $\xi_3 = \Psi^{-1}\left(\mathbb{S}^1(x, 2r) \cap \overline{\mathbb{D}}\right)$. Let ξ_2 and ξ_4 denote the subarcs of ∂Y joining ξ_1 and ξ_3 in such a way that the arcs $\xi_1, \xi_2, \xi_3, \xi_4$ form the boundary decomposition of a quadrilateral Q in $\overline{Y}$. Then

$$\operatorname{mod} \Psi^{-1} \Gamma(x, r, 2r) = \operatorname{mod} \Gamma(\xi_1, \xi_3; Q) =: M.$$

Lemma 3.7. There exists a homeomorphism $f\colon Q \to [0, 1] \times [0, M]$ with the following properties: First, $f(\xi_1) = \{0\} \times [0, M]$ and $f(\xi_3) = \{1\} \times [0, M]$. Whenever $0 < a < b < M$ and $I = [a, b]$, let $Q^0 = f^{-1}([0, 1] \times I)$,

$$\xi_1^0 = f^{-1}(\{0\} \times I), \quad \xi_2^0 = f^{-1}([0, 1] \times \{a\}),$$
$$\xi_3^0 = f^{-1}(\{1\} \times I), \quad \xi_4^0 = f^{-1}([0, 1] \times \{b\}).$$

Then $b - a = \operatorname{mod} \Gamma(\xi_1^0, \xi_3^0; Q^0)$.

Proof. Proposition 9.1 [18] and [18, equation (57), Lemma 10.2] provide us with f having the stated properties. Notice that [18, Proposition 9.1] is applicable since (2.6) holds for every $x_0 \in \overline{Y}$ and the product in (2.5) is always bounded from below by a universal constant $\kappa_0 > 0$ [19]. These facts allow us to apply [18, equation (57), Lemma 10.2] as well. □

Lemma 3.8. The inequality (3.8) holds for a constant $C_0 = 2KC_1$, where C_1 depends only on $\mathbb{D}$ and K is the maximal dilatation of ϕ.

Proof. We let $b = 3M/4$ and $a = M/4$ in Lemma 3.7. Since the restriction of Ψ to Y is K-quasiconformal,

$$\operatorname{mod} \Gamma(\xi_1^0, \xi_3^0; Q^0) \leq K \operatorname{mod} \Gamma(\Psi(\xi_1^0), \Psi(\xi_3^0); \Psi(Q^0)).$$

Observe that $\Gamma(\Psi(\xi_1^0), \Psi(\xi_3^0); \Psi(Q^0)) \subset \Gamma(x, r, 2r)$. Therefore

$$\operatorname{mod} \Gamma(\Psi(\xi_1^0), \Psi(\xi_3^0); \Psi(Q^0)) \leq \operatorname{mod} \Gamma(x, r, 2r).$$

Here $\operatorname{mod}\Gamma(x, r, 2r) \leq C_1$ for a constant depending only on $\overline{\mathbb{D}}$. Then Lemma 3.7 yields $M \leq 2KC_1$. The claim follows by passing to the limit $r \to 0^+$. □

Lemma 3.9. The outer (resp. inner) dilatation of Φ coincides with the outer (resp. inner) dilatation of ϕ.

Proof. Lemmas 3.6 and 3.8 prove that Φ is a quasiconformal homeomorphism. In $\mathbb{D}$, the minimal weak upper gradients of Φ and ϕ coincide. This is also true for their Jacobians. Therefore they satisfy (ii) in Proposition 2.4 with the same constant. Hence $K_O(\Phi) = K_O(\phi)$.

Consider the Borel functions $g = \chi_{\mathbb{S}^1}$ and $\widetilde{g} = (g \circ \Phi^{-1})\rho_{\Phi^{-1}}$. The property (ii) in Proposition 2.4 implies

$$\|\widetilde{g}\|^2_{L^2(\overline{Y})} \leq K_O(\Phi^{-1})\, \|g\|^2_{L^2(\overline{\mathbb{D}})} = 0.$$

Hence $\widetilde{g} = 0$ $\mathcal{H}^2_{\overline{Y}}$-almost everywhere in $\overline{Y}$. We conclude that $\rho_{\Phi^{-1}}(y) = 0$ for $\mathcal{H}^2_{\overline{Y}}$-almost every $y \in \partial Y$. Hence $\rho_{\Phi^{-1}} = \rho_{\phi^{-1}}\chi_Y$ $\mathcal{H}^2_{\overline{Y}}$-almost everywhere in $\overline{Y}$. We conclude that ϕ and Φ satisfy (ii) in Proposition 2.4 with the same constant. In other words, $K_I(\Phi) = K_I(\phi)$. □

4 Beurling–Ahlfors extension

For this section we fix a quasiconformal Jordan domain Y satisfying (1.3) and a quasiconformal homeomorphism $\phi\colon \mathbb{D} \to Y$. Let $\Phi\colon \overline{\mathbb{D}} \to \overline{Y}$ denote the quasiconformal homeomorphic extension of ϕ, obtained from Proposition 1.2. We refer to $g_\phi = \Phi|_{\mathbb{S}^1}$ as the boundary map of ϕ. The goal of this section is to prove Theorem 1.3. We reduce the proof to Proposition 4.1.

Observe that if g_ϕ is a quasisymmetry, then ∂Y has bounded turning as this property is preserved by quasisymmetries [21]. Moreover, if we fix an arbitrary quasisymmetry $g\colon \mathbb{S}^1 \to \partial Y$, then $h = g_\phi^{-1} \circ g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetry. The Beurling–Ahlfors extension theorem [2] yields the existence of a quasiconformal map $H\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ whose boundary map equals h. Then $G = \Phi \circ H\colon \overline{\mathbb{D}} \to \overline{Y}$ is the quasiconformal extension of g whose existence we wanted to establish. So Theorem 1.3 is a consequence of the following result.

Proposition 4.1. If ∂Y has bounded turning and satisfies the mass upper bound (1.3), then the boundary map $g_\phi = \Phi|_{\mathbb{S}^1}$ is a quasisymmetry.

We start the proof of Proposition 4.1 by first establishing Proposition 1.4. There we claim that ∂Y has Assouad dimension at most two.

Lemma 4.2. Suppose that ∂Y has bounded turning with constant $\lambda > 1$. Let $C = (4\lambda)^3 \frac{2}{\pi}$. Then for all $y \in \partial Y$ and all $0 < r < \operatorname{diam} \partial Y$, $\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, r)) \geq C^{-1} r^2$.

Proof. Consider $y \in \partial Y$ and the 1-Lipschitz function $f(z) = d(y, z)$. For every $0 < r < (4\lambda)^{-1} \operatorname{diam} \partial Y$, we have that $f^{-1}(r) \cap \partial Y \neq \emptyset$. Let $y_0 \in \partial Y \setminus \overline{B}_{\overline{Y}}(y, 2\lambda r)$. We obtain from Theorem 2.2 a continuum $C_r \subset f^{-1}(r)$ separating y and y_0.

Let $E \subset \partial Y$ be the (closure of the) component of $\partial Y \setminus C_r$ that contains y and let a and b denote the ends of E. Here $a, b \in E \cap C_r \subset f^{-1}(r)$ so $d(a, y) = r = d(b, y)$.

Let $E_a \subset \partial Y$ be the arc ending at a and y with $\operatorname{diam} E_a \leq \lambda r$, and let E_b denote the corresponding arc for b and y. Then $E_a \cup E_b = E$. Indeed, otherwise $\max\{\operatorname{diam} E_a, \operatorname{diam} E_b\} \geq d(y, y_0) > 2\lambda r$.

Let $E' \subset \partial Y$ be the arc ending at a and b with smaller diameter. Then $\operatorname{diam} E' \leq 2^{-1} \operatorname{diam} \partial Y$. Since $\operatorname{diam} E \leq \operatorname{diam} E_a + \operatorname{diam} E_b \leq 2\lambda r < 2^{-1} \operatorname{diam} \partial Y$, we have $E' = E$.

We conclude that $\operatorname{diam} C_r \geq d(a, b) \geq \lambda^{-1} \operatorname{diam} E \geq \lambda^{-1} r$. Since C_r is a continuum, $\mathcal{H}^1_{\overline{Y}}(f^{-1}(r)) \geq \mathcal{H}^1_{\overline{Y}}(C_r) \geq \operatorname{diam} C_r \geq \lambda^{-1} r$ for each $0 < r < (4\lambda)^{-1} \operatorname{diam} \partial Y$ [8, 2.10.12].

By integrating over the interval $(0, r)$ and by applying Theorem 2.1, we conclude that

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, r)) \geq (4\lambda)^{-1} \frac{\pi}{2} r^2$$

whenever $0 < r < (4\lambda)^{-1} \operatorname{diam} \partial Y$. If $(4\lambda)^{-1} \operatorname{diam} \partial Y \le r < \operatorname{diam} \partial Y$, we have

$$\mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, r)) \ge \mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, (4\lambda)^{-1} r)) \ge (4\lambda)^{-1} \frac{\pi}{2} \left((4\lambda)^{-1} r\right)^2 .$$

The claim follows for $C = (4\lambda)^3 \frac{2}{\pi}$. □

Proof of Proposition 1.4. Let $0 < r < \operatorname{diam} \partial Y$ and $0 < \epsilon < 1$. We consider a point $y \in \partial Y$ and an ϵr-separated set $S \subset B_{\overline{Y}}(y, r) \cap \partial Y$. We conclude from Lemma 4.2 and (1.3) that for some $C \ge 1$

$$C(2r)^2 \ge \mathcal{H}^2_{\overline{Y}}(\overline{B}_{\overline{Y}}(y, 2r)) \ge \mathcal{H}^2_{\overline{Y}}\left(\bigcup_{x \in S} \overline{B}_{\overline{Y}}(x, \epsilon r)\right) \ge C^{-1}\left(\frac{\epsilon r}{2}\right)^2 \#S.$$

Therefore $\#S \le C\epsilon^{-2}$ for some constant C independent of r, y and ϵ. Hence the Assouad dimension of ∂Y is at most 2. □

Proof of Proposition 4.1. Let $0 < r_0$ be such that for every $x \in \mathbb{S}^1$ and every $0 < r \le r_0$, we have that $4L_g(x, 2r) < \operatorname{diam} \partial Y$.

Fix $x \in \mathbb{S}^1$ and $0 < r < r_0$. Let $z, a, b \in \mathbb{S}^1 \cap \mathbb{D}(x, r)$ be such that $0 < \|a - z\|_2 \le \|b - z\|_2$. We proved in Proposition 1.4 that ∂Y is doubling, so by a result of Tukia–Väisälä [21, Theorems 2.15 and 2.23], the quasisymmetry of $g = g_\phi$ follows if there exists a constant $H > 0$ depending only on the constants in (1.3), Lemma 4.2, the bounded turning constant λ of ∂Y, and the maximal dilatation K of ϕ such that $d(g(a), g(z)) \le H d(g(b), g(z))$.

Let $\ell = d(g(a), g(z))$ and let $M > 0$ be such that $\ell > M d(g(b), g(z))$. If we find H_0 such that $M \le H_0$ independently of a, b and z, we may set $H = H_0$. If $M \le (2\lambda)^2$, any choice $H_0 \ge (2\lambda)^2$ suffices. So we may assume $M > (2\lambda)^2$.

Fix $z' \in \mathbb{S}^1$ with $d(g(z'), g(x)) > 2L_g(x, 2r)$. Then

$$2L_g(x, 2r) < d(g(z'), g(z)) + d(g(z), g(x)) \le d(g(z'), g(z)) + L_g(x, 2r) \quad \text{and}$$
$$2^{-1}\ell \le 2^{-1}(d(g(a), g(x)) + d(g(x), g(z))) \le L_g(x, 2r).$$

Therefore $2^{-1}\ell < d(g(z'), g(z))$. We conclude that

$$g(a), g(z') \in \partial Y \setminus \overline{B}_{\overline{Y}}\left(g(z), \frac{\ell}{2}\right) \quad \text{and} \quad g(b) \in \partial Y \cap \overline{B}_{\overline{Y}}\left(g(z), \frac{\ell}{M}\right).$$

Let A' be the subarc of ∂Y joining $g(a)$ to $g(z')$ that does not contain $g(z)$. Then any arc joining A' to $g(z)$ within ∂Y must pass through either $g(a)$ or $g(z')$. Using this fact, the bounded turning of ∂Y yields

$$A' \subset \partial Y \setminus \overline{B}_{\overline{Y}}\left(g(z), \lambda^{-1}\frac{\ell}{2}\right). \tag{4.1}$$

Let B' be the subarc of ∂Y with smallest diameter which ends at $g(b)$ and $g(z)$. The bounded turning of ∂Y implies that

$$B' \subset \overline{B}_{\overline{Y}}\left(g(z), \lambda\frac{\ell}{M}\right). \tag{4.2}$$

The inclusions (4.1) and (4.2) imply that every path $\gamma \in \Gamma(A', B'; \overline{Y})$ has a subpath joining $\overline{Y} \setminus \overline{B}_{\overline{Y}}(g(z), \lambda^{-1}\frac{\ell}{2})$ to $\overline{B}_{\overline{Y}}\left(g(z), \lambda\frac{\ell}{M}\right)$ within $\overline{B}_{\overline{Y}}(g(z), \lambda^{-1}\frac{\ell}{2})$. Then Lemma 2.8 yields

$$\frac{\widetilde{C}_U}{\log \frac{M}{2\lambda^2}} \ge \operatorname{mod} \Gamma(A', B'; \overline{Y}). \tag{4.3}$$

Let $A = g^{-1}(A')$ and $B = g^{-1}(B')$. The relative distance $\Delta(A, B)$ satisfies

$$\Delta(A, B) := \frac{d(A, B)}{\min\{\operatorname{diam} A, \operatorname{diam} B\}} \le 2. \tag{4.4}$$

First, $d(A, B) \leq \|a - z\|_2$ since $a \in A$ and $z \in B$. Second, diam $A \geq \|a - z'\|_2 \geq r$, so $\|a - x\|_2 \leq r$ and $\|x - z\|_2 \leq r$ imply $2^{-1} \|a - z\|_2 \leq r$. Lastly, diam $B \geq \|b - z\|_2 \geq \|a - z\|_2$. These imply (4.4).

The 2-Loewner property of $\overline{\mathbb{D}}$ [11, Example 8.24] states that there exists a constant $C_2 > 0$ for which

$$\operatorname{mod} \Gamma(A, B; \overline{\mathbb{D}}) \geq C_2 \tag{4.5}$$

depending only on the upper bound (4.4). The K-quasiconformality of the extension Φ implies that $\operatorname{mod} \Gamma(A', B'; \overline{Y}) \geq K^{-1} \operatorname{mod} \Gamma(A, B; \overline{\mathbb{D}})$. Combining this inequality with (4.3) and (4.5) yields an upper bound on M in terms of C_2, K, $\widetilde{C}_U$, and λ. Setting H_0 to be the maximum of this bound and $(2\lambda)^2$ establishes the claim. □

5 Planar quasicircles

We prove Theorem 1.5 in this section. The main result of this section is the following.

Proposition 5.1. Under the assumptions of Theorem 1.5, the Assouad dimension of ∂Y is strictly less than 2.

Proof of Theorem 1.5 assuming Eq. (5.1).. Equation (5.1) states that ∂Y has Assouad dimension strictly less than 2. Having verified this, [13] yields the existence of a bi-Lipschitz embedding $h \colon \partial Y \to \mathbb{R}^2$, i.e., ∂Y is planar. □

So Theorem 1.5 follows from Proposition 5.1. We split the proof of Proposition 5.1 into a couple of sublemmas.

Lemma 5.2. Suppose that ∂Y has λ-bounded turning and satisfies (a) in Theorem 1.5. Then there exists a constant $C_p \geq 1$ depending only on λ such that for every $x \in \partial Y$ and $0 < r < \min\{r_0, \operatorname{diam} \partial Y\}$, there exists $y \in Y$ with $\overline{B}_{\overline{Y}}(y, C_p^{-1} 2r) \subset B_{\overline{Y}}(x, r) \setminus \partial Y$.

Proof. This is only a small modification of the proof of Theorem 8.2 of [17] but we include the details here for the convenience of the reader. Let $s = 8\lambda^2(2\lambda + 1)$ and $C_p = 8\lambda s$. Let $x \in \partial Y$ and $0 < r < \min\{r_0, \operatorname{diam} \partial Y\}$. We claim that there exists a point $v \in Y$ such that

$$B_{\overline{Y}}\left(v, C_p^{-1} 2r\right) \subset B_{\overline{Y}}(x, r) \setminus \partial Y. \tag{5.1}$$

Suppose for now that $r < (4\lambda)^{-1} \min\{r_0, \operatorname{diam} \partial Y\}$. Then there exists a point $z \in \partial Y$ for which $d(x, z) \geq 2\lambda r$. We fix such a z.

Let $|\gamma|$ denote the (closure of the) subarc of $\partial Y \setminus \left\{w \in \partial Y \colon d(x, w) = \frac{r}{4\lambda}\right\}$ that contains x. Let a and b denote the end points of $|\gamma|$ labelled in such a way that $\{x, a, z, b\}$ is cyclically ordered on ∂Y. We have that $d(x, a) = d(x, b) = \frac{r}{4\lambda}$ and $|\gamma| \subset \overline{B}_{\overline{Y}}\left(x, \frac{r}{4\lambda}\right)$.

The relative ALLC condition of ∂Y implies that there exists a path α joining a to b in $\overline{B}_{\overline{Y}}\left(x, \frac{r}{4}\right) \setminus B_{\overline{Y}}\left(x, \frac{r}{8\lambda^2}\right)$. We assume without loss of generality that α is an arc.

Let $|\gamma_a|$ denote the (closure of the) component of $\partial Y \setminus \{x, z\}$ joining x and z that contains a and let $|\gamma_b|$ be the other component. Observe that $d(a, |\gamma_b|) \geq (8\lambda^2)^{-1} r$ since otherwise we would find an arc $|\gamma'|$ joining $|\gamma_b|$ to a within ∂Y for which $(8\lambda)^{-1} r \geq \operatorname{diam} |\gamma'|$. This would imply the contradiction $(8\lambda)^{-1} r \geq d(a, \{x, z\})$.

The lower bound on $d(a, |\gamma_b|)$ and connectedness of $|\alpha|$ imply the existence of $v \in |\alpha|$ such that $d(v, |\gamma_b|) = \frac{r}{s}$. Fix such a v. Suppose that there exists $w \in |\gamma_a|$ for which $d(v, w) < \frac{r}{s}$. Then $d(w, |\gamma_b|) < 2\frac{r}{s}$ and there exists a path β' joining w to $|\gamma_b|$ within ∂Y for which diam $|\beta'| < 2\lambda \frac{r}{s}$. Since $|\beta'|$ contains either x or z, we have

$$d(v, \{x, z\}) \leq d(w, \{x, z\}) + d(v, w) < 2\lambda \frac{r}{s} + \frac{r}{s} = \frac{r}{8\lambda^2}.$$

This is a contradiction with the facts $d(x, z) > 2\lambda r$ and $v \in |\alpha| \subset \overline{B}_{\overline{Y}}\left(x, \frac{r}{4}\right) \setminus B_{\overline{Y}}\left(x, \frac{r}{8\lambda^2}\right)$. Since no such w exists, $d(v, |\gamma_a|) \geq \frac{r}{s}$. Consequently, $B_{\overline{Y}}\left(v, \frac{r}{s}\right) \subset B_{\overline{Y}}(x, r) \setminus \partial Y$.

If $(4\lambda)^{-1}\min\{r_0, \operatorname{diam}\partial Y\} \le r < \min\{r_0, \operatorname{diam}\partial Y\}$, then there exists a point $v \in Y$ such that

$$B_{\overline{Y}}\left(v, \frac{r}{4\lambda s}\right) \subset B_{\overline{Y}}\left(x, \frac{r}{4\lambda}\right) \setminus \partial Y \subset B_{\overline{Y}}(x, r) \setminus \partial Y.$$

In either case, (5.1) holds. □

Let $r_1 = \min\{r_0, \operatorname{diam}\partial Y\}$, where r_0 is the parameter from the assumptions of Theorem 1.5. There exists a constant $C \ge 1$ with the following properties:

(i) For every $y \in \partial Y$ and every $0 < r$, $\mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(y, r)) \le Cr^2$. Moreover, for every $y \in \partial Y$ and every $0 < r < r_1$, $\mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(y, r)) \ge C^{-1}r^2$.
(ii) For every $y \in \partial Y$ and $0 < r < r_1$, there exists $z \in Y$ such that $\overline{B}_{\overline{Y}}(z, C^{-1}2r) \subset B_{\overline{Y}}(y, r) \setminus \partial Y$.
(iii) For every $z \in Y$ with $d(z, \partial Y) \le r_1$ and every $0 < r < d(z, \partial Y)$, $C^{-1}r^2 \le \mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(z, r))$.

Remark 5.3. We have assumed that for all radii $0 < r < \operatorname{diam}\partial Y$ and every $y \in \partial Y$, $\mathcal{H}^2_{\overline{Y}}(B_{\overline{Y}}(y, r)) \le C'r^2$ for some $C' > 0$. Then the upper bound in (i) holds for every $r > 0$ if we replace C' with $C'' = \max\left\{C', \mathcal{H}^2_{\overline{Y}}(\overline{Y})/(\operatorname{diam}\partial Y)^2\right\}$. The lower bound for such balls follows from Lemma 4.2.

The property (ii) follows from Lemma 5.2 for some constant. Recall that, under the assumptions of Theorem 1.5, the lower bound in (iii) holds for each $0 < r < d(z, \partial Y) \le r_0$ for some constant C'. Hence (iii) follows.

Our claim is qualitative, so we use a uniform constant C for these various conditions.

In the following, if B is a ball and $\xi > 0$, ξB refers to the ball with the same center and ξ times the radius.

Lemma 5.4. There exists a collection $\mathcal{B}$ of pairwise disjoint balls in $Y = \overline{Y} \setminus \partial Y$ such that for every $(y, r) \in \partial Y \times (0, r_1)$ there exists a ball $B \in \mathcal{B}$ with

$$r_1/2 > \max\left\{d(\partial Y, B), \operatorname{diam} B\right\} \quad \text{and} \quad \operatorname{diam} B \simeq r \simeq d(y, B) \simeq d(\partial Y, B), \tag{5.2}$$

where $A_1 \simeq A_2$ means that there exists a *constant of comparability* $D > 0$ for which $D^{-1}A_1 \le A_2 \le DA_1$. Here the constants of comparability depend only on the constant C.

Proof of Lemma 5.4. For each $x \in \partial Y$ and each integer m such that $0 < 2^m < r_1$, consider a point $v_{x,m} \in Y = \overline{Y} \setminus \partial Y$ with $\overline{B}_{\overline{Y}}(v_{x,m}, C^{-1}2^{m+1}) \subset B_{\overline{Y}}(x, 2^m) \setminus \partial Y$. Then the ball $B_{x,m} := \overline{B}_{\overline{Y}}(v_{x,m}, C^{-1}2^m)$ satisfies

$$\operatorname{diam} B_{x,m} \simeq 2^m \simeq d(x, B_{x,m}) \simeq d(\partial Y, B_{x,m}) \tag{5.3}$$

with the constants of comparison depending only on C.

For each $m \in \mathbb{Z}$ with $2^m < r_1$, let $\mathcal{B}_m$ denote the collection of the $B_{x,m}$ as $x \in \partial Y$ varies. The $5r$-covering theorem [8, 2.8.4] states that there exists subcollection $\mathcal{B}'_m \subset \mathcal{B}_m$ whose elements are pairwise disjoint and

$$\bigcup_{B \in \mathcal{B}_m} B \subset \bigcup_{B \in \mathcal{B}'_m} 5B. \tag{5.4}$$

Let $1 \le N \in \mathbb{N}$ and $m_1 \in \mathbb{Z}$ with $2^{m_1} < r_1 \le 2^{m_1+1}$. Consider the collection $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}'_{m_1-kN}$.

By choosing a sufficiently large N, the elements of $\mathcal{B}$ are pairwise disjoint and each element satisfies $\max\left\{d(\partial Y, B), \operatorname{diam} B\right\} < r_1/2$. The choice of N depends only on the constants in (5.3). The inclusion (5.4) implies that for each $0 < r < r_1$ and $x \in \partial Y$, there exists $B \in \mathcal{B}$ such that (5.3) holds with $B_{x,m}$ replaced by B and 2^m by r, with constants of comparison depending only on C. Then (5.2) follows. □

We fix arbitrary $0 < \epsilon < 1$, $0 < r < r_1$ and $y \in \partial Y$. Let $N = B_{\overline{Y}}(y, 2r) \cap \left\{x \in \overline{Y} \colon d(\partial Y, x) < 2^{-1}\epsilon r\right\}$. Let $\mathcal{B}_0 \subset \mathcal{B}$ denote the subcollection consisting of all $B \in \mathcal{B}$ with $B \subset B_{\overline{Y}}(y, 4r)$ with diameter at least $a\epsilon r$ for $a > 0$.

Lemma 5.5. There exist $1 > a > 0$, $b > 0$ and $\xi > 1$, with the choices depending only on the constants of comparability in (5.2), such that

$$T(x) := \sum_{B \in \mathcal{B}_0} \chi_{\xi B}(x) \ge -b\log(\epsilon) \quad \text{for all } x \in N, \tag{5.5}$$

where $\chi_{\xi B}$ is the characteristic function of the ball ξB.

Proof of Lemma 5.5. For each $(w, s) \in B(y, 3r) \cap \partial Y \times (0, r_1)$, let $B_s(w) \in \mathcal{B}$ be the ball obtained from Lemma 5.4. Lemma 5.4 implies the existence of $A > 1$ for which $A^{-1}s \le \operatorname{diam} B_s(w) < As$ and for the center $c_s(w)$ of $B_s(w)$, $A^{-1}s \le d(w, c_s(w)) < As$.

Let $0 < a < A^{-2}$. Then for every $Aa\epsilon r \le s < r/A$, we have $B_s(w) \in \mathcal{B}_0$. Moreover, if $\xi \ge 3a^{-1}$, the radius of $\xi B_s(w)$ is bounded from below by $3a^{-1} \operatorname{diam} B_s(w)/2$. Given $z \in B(w, 2^{-1}\epsilon r)$,

$$d(c_s(w), z) < s/(2Aa) + d(c_s(w), w) \le (a^{-1} + 2A^2)\frac{\operatorname{diam} B_s(w)}{2}.$$

Thus $\xi B_s(w) \supset B(w, 2^{-1}\epsilon r)$.

Let $k \in \mathbb{Z}$ be the largest integer for which $Aa\epsilon r < r/A^{2k+1}$. Let $\{\widetilde{s}_i\}_{i=1}^k$ be a strictly increasing sequence in the interval $\left(Aa\epsilon r, r/A^{2k+1}\right)$. Denote $s_i = A^{2(i-1)}\widetilde{s}_i$ for each $i = 1, 2, \ldots k$. Here $s_k < r/A$ and $As_{i-1} < A^{-1}s_i$ for each i. Hence the collection $\{B_{s_i}(w)\}_{i=1}^k$ contains k different balls. This implies

$$T(z) \ge k \quad \text{for every } z \in B(w, 2^{-1}\epsilon r). \tag{5.6}$$

We set now $a = A^{-4}$, $\xi = 3a^{-1}$, and $b = 1/(2\log(A))$. The maximality of k implies $k \ge -b\log(\epsilon)$.

If $z \in N$, there exists $w_z \in \partial Y$ such that $d(w_z, z) = d(\partial Y, z) < 2^{-1}\epsilon r$. In particular, $w_z \in B(y, 3r) \cap \partial Y$ and $z \in B(w_z, 2^{-1}\epsilon r)$. Then (5.6) implies (5.5) for the constants a, ξ, and b. □

Lemma 5.6. Let a, b, ξ, and $\mathcal{B}_0$ be as in Lemma 5.5. There exists a constant $d \ge 1$, depending only on the constants of comparability in (5.2) and C, such that $\mathcal{H}^2_{\overline{Y}}(5\xi B) \le d\mathcal{H}^2_{\overline{Y}}(B)$ for every $B \in \mathcal{B}_0$.

Proof of Lemma 5.6. Consider $B \in \mathcal{B}_0$. Then $5\xi B \subset B_{\overline{Y}}(y, \rho)$ for some $y \in \partial Y$ such that $5\xi \operatorname{diam} B \simeq 5\xi d(y, B) \simeq 5\xi d(\partial Y, B) \simeq \rho$, depending only on the constants of comparability in (5.2). The mass upper bound (i) yields $\mathcal{H}^2_{\overline{Y}}(5\xi B) \le C\rho^2 \simeq C25\xi^2(\operatorname{diam} B)^2$. Given (5.2), we have $\max\{\operatorname{diam} B, d(B, \partial Y)\} < r_1/2$. Hence the lower bound from (iii) implies $(\operatorname{diam} B)^2 \le C\mathcal{H}^2_{\overline{Y}}(B)$. The existence of d follows. □

Proof of Proposition 5.1. The claim is that there exist $0 < \delta < 2$ and $\widetilde{C} > 0$ such that for every $0 < \epsilon < 1$, every $0 < r < \operatorname{diam} \partial Y$, and every $y \in \partial Y$, any ϵr-separated set $E \subset B(y, r) \cap \partial Y$ satisfies $\#E \le \widetilde{C}\epsilon^{-\delta}$.

Suppose that the claim holds whenever $0 < r < r_1$. Consider $r_1 \le r < \operatorname{diam} \partial Y$. Let $E_0 \subset \partial Y$ be a maximal $r_1/2$-separated net. For every $f \in E_0$, the set $E_f = B(f, r_1/2) \cap E$ is $\epsilon r_1/2$-separated. Since $E = \bigcup_{f \in E_0} E_f$, we have $\#E \le \sum_{f \in E_0} \#E_f \le \#E_0 \widetilde{C}\epsilon^{-\delta}$. So the general case follows from the special one.

Now we prove the claim for each $0 < r < r_1$, $0 < \epsilon < 1$, and $y \in \partial Y$. We choose a, b and ξ as in Lemma 5.5 and let $\mathcal{B}_0 \subset \mathcal{B}$ be the collection defined before the statement Lemma 5.5. The collection $\mathcal{B}_0$ has the following properties:

A_1. $\sup_{B \in \mathcal{B}_0} \operatorname{diam} B < \infty$;
A_2. $0 < \mathcal{H}^2_{\overline{Y}}(5\xi B) \le d\mathcal{H}^2_{\overline{Y}}(B)$ for each $B \in \mathcal{B}_0$;
A_3. the balls in $\mathcal{B}_0$ are pairwise disjoint;
A_4. the measure of $S := \bigcup_{B \in \mathcal{B}_0} B \subset B_{\overline{Y}}(y, 4r)$ is finite.

Lemma 5.6 implies that the constant d in A_2 can be chosen to be independent of y, r, and ϵ.

Having verified properties A_1-A_4, [4, Theorem 9.6 (b)] explicitly states for $\mu = 1/12d^2$ the following:

$$\mathcal{H}^2_{\overline{Y}}\left(\{x \in \overline{B}_{\overline{Y}}(y, 2r) \colon T(x) \ge -b\log(\epsilon)\}\right) \le (1 + d)\mathcal{H}^2_{\overline{Y}}(S)e^{-\mu(-b\log(\epsilon))}.$$

Given that $S \subset B_{\overline{Y}}(y, 4r)$, the upper bound in (i) implies

$$\mathcal{H}^2_{\overline{Y}}\left(\{x \in \overline{B}_{\overline{Y}}(y, 2r) \colon T(x) \ge -b\log(\epsilon)\}\right) \le (1 + d)(C(4r)^2)\epsilon^{\mu b}. \tag{5.7}$$

Let $E \subset B_{\overline{Y}}(y, r) \cap \partial Y$ be an ϵr-separated set. We see from (5.5) and the lower bound in (i) that

$$\mathcal{H}^2_{\overline{Y}}\left(\{x \in \overline{B}_{\overline{Y}}(y, 2r) \colon T(x) \ge -b\log(\epsilon)\}\right) \tag{5.8}$$

$$\geq \mathcal{H}^2_{\overline{Y}}(N) \geq \mathcal{H}^2_{\overline{Y}}\left(\bigcup_{z\in E} B(z, 2^{-1}\epsilon r)\right) \geq (\#E)C^{-1}(2^{-1}\epsilon r)^2.$$

Now (5.7) and (5.8) yield $\#E \leq \widetilde{C}\epsilon^{-(2-\mu b)}$, where $\widetilde{C}$ is independent of y, r, and ϵ. We denote $\delta := 2 - \mu b < 2$. The claim follows. □

6 Concluding remarks

Consider any quasiconformal Jordan domain Y and ϕ as in Theorem 1.1. Since ϕ is a homeomorphism, the Jacobian J_ϕ of ϕ satisfies

$$\text{Area}(\phi) := \int_{\mathbb{D}} J_\phi \, d\mathcal{L}^2 \leq \mathcal{H}^2_{\overline{Y}}(Y) < \infty. \tag{6.1}$$

The number Area(ϕ) is called the *parametrized area* of ϕ.

Lytchak and Wenger consider in [16, Section 1.2] the class $\Lambda(\partial Y, \overline{Y})$ of those $u \in N^{1,2}(\mathbb{D}; \overline{Y})$ whose trace $u' : \mathbb{S}^1 \to \overline{Y}$ is a (weakly) monotone parametrization of ∂Y. Associated to such maps, one defines Area(u) by integrating a Jacobian of u [16, Section 1.2]. Also, $E(u) = \int_{\mathbb{D}} \rho_u^2 \, d\mathcal{L}^2$ is the corresponding energy.

Theorem 1.1 implies that every quasiconformal homeomorphism $\phi: \mathbb{D} \to Y$ defines an element of $\Lambda(\partial Y, \overline{Y})$. If ϕ is K-quasiconformal, then $E(\phi) \leq K\text{Area}(\phi) < \infty$ due to (6.1). Then [16, Theorem 7.6] yields the existence of $u_e \in \Lambda(\partial Y, \overline{Y})$ with minimal $E(u)$ among all $u \in \Lambda(\partial Y, \overline{Y})$, referred to as an energy minimizer. Similarly, [16, Theorem 1.1] yields the existence of $u_a \in \Lambda(\partial Y, \overline{Y})$ of minimal parametrized area.

For a general quasiconformal Jordan domain Y, it is not clear whether or not u_e (or u_a) is a quasiconformal homeomorphism. However, if we also assume that $\overline{Y}$ is geodesic, ∂Y is rectifiable, and $\overline{Y}$ satisfies a quadratic isoperimetric inequality and (1.2) at the boundary points of Y, the energy minimizer u_e is a quasiconformal homeomorphism [6, Theorem 1.3]. We refer the interested reader to [16] and [6] for further reading.

There are some ways to construct quasiconformal Jordan domains. For example, if X is a metric surface satisfying local versions of annular linear local connectivity and Ahlfors 2-regularity, Theorem 4.17 of [25] yields the existence of many quasiconformal Jordan domains $Y \subset X$ satisfying the assumptions of Theorem 1.5. Some examples of quasiconformal Jordan domains can also be obtained from [10, Theorem 2].

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Two-dimensional metric spheres from gluing hemispheres

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TWO-DIMENSIONAL METRIC SPHERES FROM GLUING HEMISPHERES

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Abstract. We study metric spheres (Z, d_Z) obtained by gluing two hemispheres of $\mathbb{S}^2$ along an orientation-preserving homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$, where d_Z is the canonical distance that is locally isometric to $\mathbb{S}^2$ off the seam.

We show that if (Z, d_Z) is quasiconformally equivalent to $\mathbb{S}^2$, in the geometric sense, then g is a welding homeomorphism with conformally removable welding curves. We also show that g is bi-Lipschitz if and only if (Z, d_Z) has a 1-quasiconformal parametrization whose Jacobian is comparable to the Jacobian of a quasiconformal mapping $h\colon \mathbb{S}^2 \to \mathbb{S}^2$. Furthermore, we show that if g^{-1} is absolutely continuous and g admits a homeomorphic extension with exponentially integrable distortion, then (Z, d_Z) is quasiconformally equivalent to $\mathbb{S}^2$.

1. Introduction

In this paper, we work in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. We denote the equator $\mathbb{S}^2 \cap (\mathbb{R}^2 \times \{0\})$ by $\mathbb{S}^1$ and endow $\mathbb{S}^2$ with the length distance σ induced by the Euclidean distance of $\mathbb{R}^3$. The open southern and northern hemispheres are denoted by Z_1 and Z_2, respectively. Here $(0,0,1) \in Z_2$.

Consider an orientation-preserving homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$, mapping the boundary of $\overline{Z}_1$ to the boundary of $\overline{Z}_2$. We identify each $z \in \mathbb{S}^1$ with its image $g(z) \in \mathbb{S}^1$. With this identification, we obtain a set Z and inclusion maps $\iota_1\colon \overline{Z}_1 \to Z$ and $\iota_2\colon \overline{Z}_2 \to Z$. We call $S_Z = \iota_1(\mathbb{S}^1) = \iota_2(\mathbb{S}^1)$ the *seam* of Z.

We construct a pseudodistance d_Z on Z, see Section 3, making the inclusion maps local isometries off the seam and 1-Lipschitz everywhere. We consider the quotient map $Q\colon Z \to \widetilde{Z}$ identifying points $x, y \in Z$ whenever $d_Z(x,y) = 0$, and endow $\widetilde{Z}$ with the associated quotient distance.

We are interested in this construction for the following reason: whenever the metric space $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$, there exist Riemann maps $\phi_1\colon Z_1 \to \Omega_1$, $\phi_2\colon Z_2 \to \Omega_2$ onto the complementary components of a Jordan curve $\mathcal{C}$ with $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$; with the Carathéodory theorem we can make sense of the composition $\phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$ [GM05]. Any such g is called a *welding homeomorphism* and $\mathcal{C}$ a *welding curve*. A long-standing problem is to understand which homeomorphisms g satisfy $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$ for some Riemann maps. We refer to the survey articles [Ham02], [You15] for further background information.

We also investigate the properties of $\widetilde{Z}$, given an arbitrary welding homeomorphism g. We show in Section 4 that the 1-dimensional Hausdorff measures on the seam $Q(S_Z)$ and on (the tangents of) $\mathcal{C}$ are closely connected, using results from classical complex analysis [GM05]. For example, our results show that a given subarc of the welding curve has tangents only in a set negligible to the 1-dimensional Hausdorff measure if and only if the quotient map Q collapses the corresponding part of the seam to a point.

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We present in Sections 7.1 and 7.2 examples illustrating that for some homeomorphisms g, after removing a portion E' of the seam $Q(S_Z)$, one can find a 1-quasiconformal embedding $\psi\colon \widetilde{Z} \setminus E' \to \mathbb{S}^2$, but not necessarily a quasiconformal homeomorphism $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$. A similar phenomenom was investigated in [Ham91] and [Bis07b] in more detail.

We now state our first result.

Theorem 1.1. *Let $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism. The following are quantitatively equivalent.*

(1) g is L-bi-Lipschitz;
(2) there exists an L'-bi-Lipschitz homeomorphism $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$;
(3) there exists $C' \geq 0$ such that for every $y \in Q(S_Z)$,

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{B}_{\widetilde{Z}}(y,r))}{\pi r^2} \leq C'.$$

In the implications "(1) $\Rightarrow$ (2)" we may take $L' = L$, in "(2) $\Rightarrow$ (3)" $C' = (L')^4$, and in "(3) $\Rightarrow$ (1)" $L = \pi C'$.

We prove "(1) $\Rightarrow$ (2)" by observing that if $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ admits an L'-bi-Lipschitz extension $\phi\colon \overline{Z}_2 \to \overline{Z}_2$, the space $\widetilde{Z}$ has an L'-bi-Lipschitz parametrization. That we may take $L' = L$ in "(1) $\Rightarrow$ (2)", follows by applying a known planar extension result [Kal14] and stereographic projection.

The claim "(2) $\Rightarrow$ (3)" is a straightforward consequence of the properties of Hausdorff measures. The implication "(3) $\Rightarrow$ (1)" is proved by carefully analyzing the behaviour of the inclusion mappings $\iota_i\colon \overline{Z}_i \to \widetilde{Z}$ at the equator $\mathbb{S}^1$. Notice that the ι_i are 1-Lipschitz everywhere and local isometries outside the equator. This implies $C' \geq 1$ in (3). Remark 5.9 shows two ways to improve the bi-Lipschitz constant $\pi C'$. The improvements imply that as $C' \to 1^+$ in (3), the bi-Lipschitz constant of g converges to one. In particular, (3) holds with $C' = 1$ if and only if g is an isometry.

Theorem 1.1 is closely related to the following result.

Theorem 1.2. *If an orientation-preserving homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is L-bi-Lipschitz, there exists a 1-quasiconformal homeomorphism $\varphi\colon \mathbb{S}^2 \to \widetilde{Z}$ and a K-quasiconformal homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the Jacobians satisfy*

$$C^{-1} J_h(x) \leq J_\varphi(x) \leq C J_h(x) \quad \text{for } \mathcal{H}^2_{\mathbb{S}^2}\text{-a.e. } x \in \mathbb{S}^2 \tag{1}$$

for $K = L^4$ and $C = L^2$. Conversely, if there exists K, C, and h for which (1) holds, then g is $\pi(KC)^2$-bi-Lipschitz.

The Jacobians are defined in Section 2.3. We note that if $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-preserving quasiconformal homeomorphism, the J_h coincides with the usual distributional Jacobian; see for example [AIM09, Section 3.8].

If g is L-bi-Lipschitz, the existence of φ and h is a straightforward consequence of the implication "(1) $\Rightarrow$ (2)" in Theorem 1.1. If h and φ exist, we first check that $\Psi = h \circ \varphi^{-1}$ is bi-Lipschitz, the study of the seam requiring a careful argument, and use the implications "(2) $\Rightarrow$ (3) $\Rightarrow$ (1)" from Theorem 1.1 to verify that g is bi-Lipschitz.

Theorem 1.2 is a special case of the *quasiconformal Jacobian problem*: which weights $\omega\colon \mathbb{S}^2 \to [0,\infty]$ are comparable to the Jacobians of quasiconformal homeomorphisms $h\colon \mathbb{S}^2 \to \mathbb{S}^2$; see [BHS04], [Bis07a], and references therein for further reading.

Given that (1) and (3) are equivalent in Theorem 1.1, it is not entirely clear for which classes of homeomorphisms one can expect $\widetilde{Z}$ to be quasiconformally equivalent to $\mathbb{S}^2$, or what kind of geometric properties one can expect from such a $\widetilde{Z}$.

Question 1.3. *Let $\widetilde{Z}$ be the metric space obtained from a homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$. When can we find a quasiconformal homeomorphism $\psi\colon \widetilde{Z} \to \mathbb{S}^2$? What kind of restrictions does this impose on g?*

As an example, if g is a welding homeomorphism corresponding to the von Koch snowflake, then $d_Z(x,y) = 0$ for every pair of points in the seam, see Remark 4.2. Hence $\widetilde{Z}$ can fail to be quasiconformally equivalent, or homeomorphic, to $\mathbb{S}^2$ when g is a quasisymmetry. We show that a simple measure-theoretic assumption removes this obstruction.

Proposition 1.4. *Let $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a quasisymmetry whose inverse is absolutely continuous. Then $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$.*

The absolute continuity of g^{-1} is used in two ways. First, it guarantees that $\widetilde{Z} = (Z, d_Z)$. Second, if $\psi\colon \overline{Z}_2 \to \overline{Z}_2$ is a quasisymmetric extension of g, we show that the homeomorphism $H\colon \mathbb{S}^2 \to \widetilde{Z}$ satisfying $H|_{Z_1} = \iota_1$ and $H|_{Z_2} = \iota_2 \circ \psi|_{Z_2}$ is quasiconformal. A key step in the proof is showing the Sobolev regularity $H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$; the absolute continuity of g^{-1} is applied here.

Proposition 1.4 is a special case of the following stronger result.

Theorem 1.5. *Let $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism whose inverse is absolutely continuous. If g extends to a homeomorphism $\psi\colon \overline{Z}_2 \to \overline{Z}_2$ for which $\psi|_{Z_2}$ has exponentially integrable distortion, then $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$.*

We now explain the main steps of the proof of Theorem 1.5. We first show that there exists a homeomorphism $H\colon \mathbb{S}^2 \to \widetilde{Z}$ with exponentially integrable distortion. We also have $H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$; see Remark 6.8. The exponential integrability of distortion of H is used to verify the reciprocality condition of $\widetilde{Z}$, see Definition 2.5. Then [Raj17, Theorem 1.4] shows that $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$. The key ingredients in the proof are the condenser estimates for mappings of exponentially distortion [KO06], applicable because $H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$, and the Stoilow factorization theorem [AIM09, Chapter 20]. There are some known criteria which guarantee that g admits an extension as in Theorem 1.5; see [Zak08] [KN21].

In Section 7.1, we present an example of $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ that is locally bi-Lipschitz outside a single point, but for which $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$. This illustrates that the absolute continuity of g^{-1} is not enough to guarantee that $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$. This fact is a consequence of the following result, partially answering Question 1.3.

Theorem 1.6. *Suppose that $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation-preserving homeomorphism for which there exists a quasiconformal homeomorphism $h\colon \mathbb{S}^2 \to \widetilde{Z}$. Then $\widetilde{Z} = (Z, d_Z)$ and there exists a 1-quasiconformal homeomorphism $\pi\colon \mathbb{S}^2 \to \widetilde{Z}$. Furthermore, g is a welding homeomorphism whose welding curves are conformally removable.*

The first step in the proof of Theorem 1.6 is showing that h can be assumed to be 1-quasiconformal. Then, up to an orientation-reversing Möbius transformation, $\phi_i = h^{-1} \circ \iota_i\colon Z_i \to \mathbb{S}^2$ are Riemann maps with welding curve $\mathcal{C} = h^{-1}(Q(S_Z))$ and welding homeomorphism $\phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$. The equality $\widetilde{Z} = (Z, d_Z)$ and the conformal removability of $\mathcal{C}$ follow from a connection we show between the tangents of the welding curve $\mathcal{C}$ and the Hausdorff 1-measure on the seam $Q(S_Z)$; see Section 4. The equality $\widetilde{Z} = (Z, d_Z)$ implies $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$.

We recall that a compact proper subset $K \subset \mathbb{S}^2$ is *conformally removable* if every homeomorphism $M\colon \mathbb{S}^2 \to \mathbb{S}^2$ conformal in $\mathbb{S}^2 \setminus K$ is Möbius. The von Koch snowflake example illustrates that conformal removability of a welding curve $\mathcal{C}$ is not enough to guarantee even that $\widetilde{Z}$ is homeomorphic to $\mathbb{S}^2$. We refer the reader to [You15] and [You18] for further reading on conformal weldings and the connections to conformal removability. See [HK03] for some results in the context of Theorem 1.5.

The paper is structured as follows. In Section 2, we introduce our notations and some preliminary results. In Section 3, we analyze the distance d_Z induced by any given homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$. When g is a welding homeomorphim, we establish in Section 4 a connection between the geometry of the seam S_Z and the tangents of the corresponding welding curves $\mathcal{C}$. We also prove Theorem 1.6 in this section. In Section 5, we prove Theorems 1.1 and 1.2. Proposition 1.4 and Theorem 1.5 are proved in Section 6. In Section 7, we give some concluding remarks.

2. Preliminaries

2.1. **Notation.** Let (Y, d_Y) be a metric space. We sometimes drop the subscript from d_Y when there is no chance for confusion. For all $Q \geq 0$, the *Q-dimensional Hausdorff measure*, or a *Hausdorff Q-measure*, is defined by

$$\mathcal{H}^Q_Y(B) = \frac{\alpha(Q)}{2^Q} \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^Q \colon B \subset \bigcup_{i=1}^{\infty} B_i, \operatorname{diam} B_i < \delta \right\}$$

for all sets $B \subset Y$, where $\alpha(Q)$ is chosen so that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers.

The *length* of a path $\gamma\colon [a,b] \to Y$ is defined as

$$\ell_d(\gamma) = \sup \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i+1})),$$

the supremum taken over all finite partitions $a = t_1 \leq t_2 \leq \cdots \leq t_{n+1} = b$. A path is *rectifiable* if it has finite length.

The *metric speed* of a path $\gamma\colon [a,b] \to Y$ at the point $t \in [a,b]$ is defined as

$$v_\gamma(t) = \lim_{t \neq s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|}$$

whenever this limit exists. The limit exists $\mathcal{L}^1$-almost everywhere for every rectifiable path [Dud07, Theorem 2.1].

A rectifiable path $\gamma\colon [a,b] \to Y$ is *absolutely continuous* if for all $a \leq s \leq t \leq b$,

$$d(\gamma(t), \gamma(s)) \leq \int_s^t v_\gamma(u)\, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a,b])$ and $\mathcal{L}^1$ the Lebesgue measure on the real line. Equivalently, the rectifiable path γ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $\mathcal{H}^1_Y$-measure zero [Dud07, Section 3].

Let $\gamma\colon [a,b] \to X$ be an absolutely continuous path. Then the *(path) integral* of a Borel function $\rho\colon X \to [0,\infty]$ *over* γ is

$$\int_\gamma \rho\, ds = \int_a^b (\rho \circ \gamma) v_\gamma \, d\mathcal{L}^1. \tag{2}$$

If γ is rectifiable, then the path integral of ρ over γ is defined to be the path integral of ρ over the arc length parametrization γ_s of γ; see [HKST15, Chapter 5] for further details.

Given a Borel set $A \subset Y$, the *length* of a path $\gamma\colon [a,b] \to Y$ *in* A is defined as $\int_Y \chi_A(y)\#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y)$, where $\#(\gamma^{-1}(x))$ is the counting measure of $\gamma^{-1}(x)$. For $A = Y$, [Fed69, Theorem 2.10.13] states

$$\ell(\gamma) = \int_Y \#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y). \tag{3}$$

When γ is rectifiable, for every Borel function $\rho\colon Y \to [0,\infty]$,

$$\int_\gamma \rho\, ds = \int_Y \rho(y)\#(\gamma^{-1}(y))\, d\mathcal{H}^1_Y(y). \tag{4}$$

The equality (4) follows from [Fed69, Theorem 2.10.13] via a standard approximation argument using simple functions.

2.2. **Metric Sobolev spaces.** In this section we give an overview of Sobolev theory in the metric surface setting, and refer to [HKST15] for a comprehensive introduction.

Let Γ be a family of paths in Y. A Borel function $\rho\colon Y \to [0,\infty]$ is *admissible* for Γ if the path integral $\int_\gamma \rho\, ds \geq 1$ for all rectifiable paths $\gamma \in \Gamma$. Given $1 \leq p < \infty$, the *p-modulus* of Γ is

$$\operatorname{mod}_p \Gamma = \inf \int_Y \rho^p\, d\mathcal{H}^2_Y,$$

where the infimum is taken over all admissible functions ρ. Observe that if Γ_1 and Γ_2 are path families and every path $\gamma_1 \in \Gamma_1$ contains a subpath $\gamma_2 \in \Gamma_2$, then $\operatorname{mod}_p \Gamma_1 \leq \operatorname{mod}_p \Gamma_2$. In particular, this holds if $\Gamma_1 \subset \Gamma_2$. When $p = 2$, and there is no chance for confusion, we omit the subscript from mod_2.

If ρ is admissible for a path family $\Gamma \setminus \Gamma_0$, where $\operatorname{mod}_p \Gamma_0 = 0$, we say that ρ is *p-weakly admissible* for Γ. If a property holds for every path $\gamma \in \Gamma$ except in a subfamily of p-modulus zero, the property is said to hold *on p-almost every path* in Γ. We also refer to 2-almost every path as *almost every* path.

We recall the following lemma [HKST15, Lemma 5.2.8].

Lemma 2.1. *Let $1 \leq p < \infty$. A family of nonconstant paths Γ satisfies $\operatorname{mod}_p \Gamma = 0$ if and only if there exists $\rho\colon Y \to [0,\infty]$, $\rho \in L^p(Y)$ with*

$$\infty = \int_\gamma \rho\, ds \quad \textit{for every } \gamma \in \Gamma.$$

Let $\psi\colon (Y, d_Y) \to (Z, d_Z)$ be a mapping between metric spaces Y and Z. A Borel function $\rho\colon Y \to [0,\infty]$ is an *upper gradient of* ψ if

$$d_Y(\psi(x), \psi(y)) \leq \int_\gamma \rho\, ds$$

for every rectifiable path $\gamma\colon [a,b] \to Y$ connecting x to y. The function ρ is a *p-weak upper gradient of* ψ if the same holds for p-almost every rectifiable path.

A p-weak upper gradient $\rho \in L^p_{\mathrm{loc}}(Y)$ of ψ is *minimal* if it satisfies $\rho \leq \widetilde{\rho}$ almost everywhere for all p-weak upper gradients $\widetilde{\rho} \in L^p_{\mathrm{loc}}(Y)$ of ψ. If ψ has a p-weak upper gradient $\rho \in L^p_{\mathrm{loc}}(Y)$, then ψ has a minimal p-weak upper gradient, which we denote by ρ_ψ. We refer to Section 6 of [HKST15] and Section 3 of [Wil12] for further details. Minimal 2-weak upper gradients are also refered to as *minimal weak upper gradients*.

Fix a point $z \in Z$, and let $d_z = d_Z(\cdot, z)$. The space $L^p(Y, Z)$ is defined as the collection of measurable maps $\psi\colon Y \to Z$ such that $d_z \circ \psi$ is in $L^p(Y)$. Moreover,

$L^p_{\mathrm{loc}}(Y,Z)$ is defined as those measurable maps $\psi\colon Y \to Z$ for which, for all $y \in Y$, there is an open set $U \subset Y$ containing y such that $\psi|_U$ is in $L^p(U,Z)$.

The metric Sobolev space $N^{1,p}_{\mathrm{loc}}(Y,Z)$ consists of those maps $\psi\colon Y \to Z$ in $L^p_{\mathrm{loc}}(Y,Z)$ that have a minimal p-weak upper gradient $\rho_\psi \in L^p_{\mathrm{loc}}(Y)$.

For subsets $\emptyset \neq U \subset Y$, we say that $\psi \in N^{1,p}(U,Z)$ if $\psi|_U \in N^{1,p}_{\mathrm{loc}}(U,Z)$, $\rho_{\psi|U} \in L^p(U)$ and $\psi|_U \in L^p(U,Z)$. If $Z = \mathbb{R}$, we denote $N^{1,p}(U,Z) = N^{1,p}(U)$, and in the case $p = 2$,

$$E(\psi) := 2^{-1} \left\|\rho_\psi\right\|^2_{L^2(U)}.$$

We refer to $E(\psi)$ as the *Dirichlet energy* of ψ.

We repeatedly use the following technical lemma in later sections.

Lemma 2.2. *Let $\psi\colon Y \to Z$ be continuous, $\rho\colon Y \to [0,\infty]$ a Borel function and $\gamma\colon [0,1] \to Y$ absolutely continuous with $\int_\gamma \rho\, ds < \infty$.*

Suppose that $E \subset Y$ is compact, $\mathcal{H}^1_Z(\psi(E)) = 0$, and $\ell(\psi \circ \gamma|_I) \leq \int_{\gamma|_I} \rho\, ds$ for each closed interval $I \subset [0,1] \setminus \gamma^{-1}(E)$. Then $\ell(\psi\circ\gamma) \leq \int_\gamma \rho\, ds$.

Proof. First, for every closed interval $J \subset [0,1] \setminus \gamma^{-1}(E)$, $\psi \circ \gamma|_J$ is absolutely continuous with $v_{\psi\circ\gamma}(s) \leq (\rho\circ\gamma)(s) v_\gamma(s)$ for $\mathcal{L}^1$-almost every $s \in J$. This follows from [HKST15, Proposition 6.3.2].

Second, consider the connected components $\{I_i\}_{i=1}^\infty$ of $[0,1] \setminus (\psi\circ\gamma)^{-1}(\psi(E))$. Notice that $I_i \subset [0,1] \setminus \gamma^{-1}(E)$ for every i.

Let $J_i = \overline{I_i}$. Then $v_{\psi\circ\gamma}(s) \leq (\rho \circ \gamma|_{J_i})(s)$ $\mathcal{L}^1$-almost everywhere on J_i (on I_i). This fact, the continuity of $\psi\circ\gamma$ and $\int_\gamma \rho\, ds < \infty$ imply

$$\ell(\psi\circ\gamma|_{J_i}) \leq \int_{I_i} (\rho\circ\gamma) v_\gamma \, ds < \infty.$$

By summing over i, we conclude

$$\sum_{i=1}^\infty \ell(\psi\circ\gamma|_{J_i}) \leq \int_{\bigcup_{i=1}^\infty I_i} (\rho\circ\gamma) v_\gamma\, ds \leq \int_\gamma \rho\, ds.$$

Given $\mathcal{H}^1_Z(\psi(E)) = 0$, (3) and (4) imply

$$\begin{aligned}
\ell(\psi\circ\gamma) &= \int_{Z\setminus\psi(E)} \#((\psi\circ\gamma)^{-1}(x))\, d\mathcal{H}^1_Z(x) \\
&\leq \sum_{i=1}^\infty \int_{Z\setminus\psi(E)} \#((\psi\circ\gamma|_{J_i})^{-1}(x))\, d\mathcal{H}^1_Z(x) \\
&= \sum_{i=1}^\infty \ell(\psi\circ\gamma|_{J_i}) \leq \int_\gamma \rho\, ds.
\end{aligned}$$

Hence $\ell(\psi\circ\gamma) \leq \int_\gamma \rho\, ds$. □

2.3. **Measure theory.** Let Y be a Borel subset of a complete and separable metric space. A Borel measure μ on Y is *σ-finite* if there exists a Borel decomposition $\{B_i\}_{i=1}^\infty$ of Y for which $\mu(B_i) < \infty$ for every i.

A pair of σ-finite Borel measures μ and ν on Y are said to be *mutually singular* if there exists a Borel set $B \subset Y$ such that $\mu(B) = 0$ and $\nu(Y \setminus B) = 0$. The measure μ admits a *Lebesgue decomposition* (with respect to ν), where $\mu = f\cdot\nu + \mu^\perp$, with $\mu^\perp$ and ν mutually singular and f Borel measurable [Bog07, Sections 3.1-3.2 in Volume I]. We say that μ and ν are *mutually absolutely continuous* if $\mu = f \cdot \nu$ with density $f > 0$ ν-almost everywhere.

Given a homeomorphism $\psi\colon Y \to Z$ and measures ν on Y and μ on Z, the measure $\psi^*\mu(B) = \mu(\psi(B))$ is called the *pullback measure*. Such a measure admits a decomposition $\psi^*\mu = f \cdot \nu + \mu^\perp$ with ν and $\mu^\perp$ mutually singular. If $\nu = \mathcal{H}^2_Y$ and $\mu = \mathcal{H}^2_Z$, the density f is called the *Jacobian* of ψ and denoted by J_ψ.

2.4. **Quasiconformal mappings.** Here we define quasiconformal maps and recall some basic facts.

Definition 2.3. *Let (Y, d_Y) and (Z, d_Z) be metric spaces with locally finite Hausdorff 2-measures. A homeomorphism $\psi\colon (Y, d_Y) \to (Z, d_Z)$ is* quasiconformal *if there exists $K \geq 1$ such that for all path families Γ in Y*

$$K^{-1} \operatorname{mod} \Gamma \leq \operatorname{mod} \psi\Gamma \leq K \operatorname{mod} \Gamma, \tag{5}$$

*where $\psi\Gamma = \{\psi \circ \gamma\colon \gamma \in \Gamma\}$. If (5) holds with a constant $K \geq 1$, we say that ψ is K-*quasiconformal.

A special case of [Wil12, Theorem 1.1] yields the following.

Theorem 2.4. *Let Y and Z be locally compact separable metric spaces with locally finite Hausdorff 2-measure and $\psi\colon Y \to Z$ a homeomorphism. The following are equivalent for the same constant $K > 0$:*

(i) $\operatorname{mod} \Gamma \leq K \operatorname{mod} \psi\Gamma$ *for all path families Γ in Y.*

(ii) $\psi \in N^{1,2}_{\mathrm{loc}}(Y, Z)$ *and satisfies*

$$\rho^2_\psi(y) \leq K J_\psi(y)$$

for $\mathcal{H}^2_Y$-almost every $y \in Y$.

The *outer dilatation* of ψ is the smallest constant $K_O \geq 0$ for which the modulus inequality $\operatorname{mod} \Gamma \leq K_O \operatorname{mod} \psi\Gamma$ holds for all Γ in Y. The *inner dilatation* of ψ is the smallest constant $K_I \geq 0$ for which $\operatorname{mod} \psi\Gamma \leq K_I \operatorname{mod} \Gamma$ holds for all Γ in Y. The number $K(\psi) = \max\{K_I(\psi), K_O(\psi)\}$ is the *maximal dilatation* of ψ.

For a set $G \subset Y$ and disjoint sets $F_1, F_2 \subset G$, let $\Gamma(F_1, F_2; G)$ denote the family of paths with each path starting at F_1, ending at F_2 and whose images are contained in G. A *quadrilateral* is a set Q homeomorphic to $[0,1]^2$ with boundary ∂Q consisting of four boundary arcs, overlapping only at the end points, labelled $\xi_1, \xi_2, \xi_3, \xi_4$ in cyclic order.

A *metric surface* is a separable metric space Y with locally finite Hausdorff 2-measure that is homeomorphic to a (connected) 2-manifold without boundary.

Definition 2.5. *A metric surface Y is* reciprocal *if there exists a constant $\kappa \geq 1$ such that*

$$\kappa^{-1} \leq \operatorname{mod} \Gamma(\xi_1, \xi_3; Q) \operatorname{mod} \Gamma(\xi_2, \xi_4; Q) \leq \kappa \tag{6}$$

for every quadrilateral $Q \subset Y$, and

$$\lim_{r \to 0^+} \operatorname{mod} \Gamma\left(\overline{B}_Y(y, r), Y \setminus B_Y(y, R); \overline{B}_Y(y, R)\right) = 0 \tag{7}$$

for all $y \in Y$ and $R > 0$ such that $Y \setminus B_Y(y, R) \neq \emptyset$.

We note that for every metric surface,

$$\kappa_0^{-1} \leq \operatorname{mod} \Gamma(\xi_1, \xi_3; Q) \operatorname{mod} \Gamma(\xi_2, \xi_4; Q), \tag{8}$$

with $\kappa_0 = (4/\pi)^2$ [EP21] [RR19].

We recall [Raj17, Theorem 1.4] stating the following.

Theorem 2.6. *Let (Y, d_Y) be a metric surface homeomorphic to $\mathbb{R}^2$ or to $\mathbb{S}^2$. Then there exists a quasiconformal embedding $\psi\colon (Y, d_Y) \to \mathbb{S}^2$ if and only if Y is reciprocal.*

Similarly, Theorem 1.3 of [Iko21b] shows that if a metric surface (Y, d_Y) can be covered by quasiconformal images of domains $V \subset \mathbb{R}^2$, then (Y, d_Y) is quasiconformally equivalent to a Riemannian surface. In particular, we have the following.

Theorem 2.7. *Let (Y, d_Y) be a metric surface homeomorphic to $\mathbb{S}^2$. Then there exists a quasiconformal homeomorphism $\psi\colon (Y, d_Y) \to \mathbb{S}^2$ if and only if each point $y \in Y$ is contained in an open set U from which there exists a quasiconformal homeomorphism $\phi\colon U \to V \subset \mathbb{R}^2$.*

Since (8) holds, Corollary 12.3 of [Raj17] shows the following.

Proposition 2.8. *Let Y be a metric surface, $U \subset Y$ a domain, and $\psi\colon U \to \Omega \subset \mathbb{R}^2$ a homeomorphism. If $K_O(\psi) < \infty$, then ψ is K-quasiconformal for $K = (2 \cdot \kappa_0) \cdot K_O(\psi)$.*

3. Hemispheres

We construct a (pseudo)distance d_Z on Z using a *predistance* $D\colon Z \times Z \to [0, \infty]$ defined in the following way, with the identification $S_Z \subset \overline{Z}_1$ for the seam,

$$D(x, y) = \begin{cases} \infty, & \text{if } (x, y) \in Z_1 \times Z_2 \cup Z_2 \times Z_1, \\ \min\{\sigma(x, y), \sigma(g(x), g(y))\}, & \text{if } x, y \in S_Z, \\ \sigma(x, y), & \text{otherwise.} \end{cases}$$

Then we denote $d_Z(x, y) = \inf \sum_{i=1}^{n} D(x_i, x_{i+1})$, the infimum taken over finite chains $(x_i)_{i=1}^{n+1}$ for which $x_1 = x$ and $x_{n+1} = y$. We obtain a metric space $\widetilde{Z}$ and a quotient map $Q\colon Z \to \widetilde{Z}$ by identifying $(x, y) \in Z \times Z$ whenever $d_Z(x, y) = 0$, and setting $d_{\widetilde{Z}}(x, y) = d_Z(Q^{-1}(x), Q^{-1}(y))$ for each $x, y \in \widetilde{Z}$.

In this section, we focus on analyzing the distance d_Z on the seam S_Z. The main results of this section are Lemmas 3.2 and 3.3 and Proposition 3.6.

In the following two lemmas we abuse notation and identify $\iota_i(Z_i)$ with Z_i when convenient.

Lemma 3.1. *The following hold:*

1. *Let $x, y \in \mathbb{S}^1 \subset \overline{Z}_1$ and $(x_i)_{i=1}^{n+1}$ a chain with $x_1 = x$, $x_{n+1} = y$, and $x_i \in Z_1$ otherwise. Then $\sum_{i=1}^{n} D(x_i, x_{i+1}) \geq D(x, y)$.*
2. *Let $x, y \in \mathbb{S}^1 \subset \overline{Z}_1$ and $(x_i)_{i=1}^{n+1}$ a chain with $g(x_1) = g(x)$, $g(x_{n+1}) = g(y)$, and $x_i \in Z_2$ otherwise. Then $\sum_{i=1}^{n} D(x_i, x_{i+1}) \geq D(x, y)$.*

Proof. Given the chain from the claim (1), for every i, $D(x_i, x_{i+1}) = \sigma(x_i, x_{i+1})$. Thus, $\sum_{i=1}^{n} D(x_i, x_{i+1}) \geq \sigma(x_1, x_{n+1}) \geq D(x_1, x_{n+1})$. The corresponding inequalities hold for the chain from (2). □

Lemma 3.1 implies that when computing $d_Z(\iota_1(x), \iota_1(y))$ for $x, y \in \mathbb{S}^1$, it is sufficient to consider chains with intermediate points staying within the seam.

Lemma 3.2. *If $x, y \in Z_1$, then*

$$d_Z(\iota_1(x), \iota_1(y)) = \begin{cases} \sigma(x, y), & \text{or there exist } w, w' \in \mathbb{S}^1 \text{ with} \\ \sigma(x, w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(w', y) \leq \sigma(x, y). \end{cases} \tag{9}$$

The corresponding identity holds for points $x, y \in Z_2$.

Furthermore, if $x \in Z_1$ and $y \in Z_2$, there exist $w, w' \in \mathbb{S}^1$ such that

$$d_Z(\iota_1(x), \iota_2(y)) = \sigma(x, w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(g(w'), y). \tag{10}$$

Proof. We show (9). Suppose that there exists a sequence $\epsilon_j \to 0^+$ and a sequence of chains $(x_{i,j})_{i=1}^{n_j+1}$ joining x to y with $d_Z(\iota_1(x), \iota_1(y)) \geq -\epsilon_j + \sum_{i=1}^{n_j} D(x_{i,j}, x_{i+1,j})$

so that every chain has an element in $\mathbb{S}^1$. If i_1 is the first index for which $x_{i,j} \in \mathbb{S}^1$ and i_2 the last one, then

$$\sum_{i=1}^{n_j} D(x_{i,j}, x_{i+1,j}) \geq \sigma(x, x_{i_1,j}) + d_Z(\iota_1(x_{i_1,j}), \iota_1(x_{i_2,j})) + \sigma(x_{i_2,j}, y)$$
$$\geq \inf \left\{ \sigma(x,w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(w', y) \right\},$$

the infimum taken over every $w, w' \in \mathbb{S}^1$. Observe that the infimum is realized by some $w, w' \in \mathbb{S}^1$. Given such $w, w' \in \mathbb{S}^1$, we pass to the limit $j \to \infty$ and conclude

$$d_Z(\iota_1(x), \iota_1(y)) \geq \sigma(x, w) + d_Z(\iota_1(w), \iota_1(w')) + \sigma(w', y).$$

Since "$\leq$" holds for every pair $w, w' \in \mathbb{S}^1$, the lower equality in (9) follows.

If no such sequence of $\epsilon_j \to 0^+$ exists, then there exists $\epsilon_0 > 0$ such that for every $\epsilon_0 > \epsilon > 0$, any chain joining x to y with $d_Z(\iota_1(x), \iota_2(y)) \geq -\epsilon + \sum_{i=1}^{n} D(x_i, x_{i+1})$ does not intersect $\mathbb{S}^1$. Hence $\sum_{i=1}^{n} D(x_i, x_{i+1}) \geq \sigma(x, y)$. So, either way, we obtain (9). The claims for each $x, y \in Z_2$ and $(x, y) \in Z_1 \times Z_2$ are proved in a similar manner. □

For $i = 1, 2$, we denote $\widetilde{\iota}_i := Q \circ \iota_i \colon \overline{Z}_i \to \widetilde{Z}$. Lemma 3.2 implies that $\widetilde{\iota}_i$ is 1-Lipschitz everywhere and a local isometry in Z_i. We also establish that $\widetilde{\iota}_i$ is *monotone*, i.e, the preimage of a point is a compact and connected set.

Lemma 3.3. *For $i = 1, 2$, the inclusion map $\widetilde{\iota}_i \colon \overline{Z}_i \to \widetilde{Z}$ is 1-Lipschitz everywhere and a local isometry on Z_i. Moreover, for every $z \in \widetilde{Z}$, the preimage $\widetilde{\iota}_i^{-1}(z)$ is compact and connected. It contains two or more points only if $\widetilde{\iota}_i^{-1}(z) \subset \mathbb{S}^1$.*

Before proving Lemma 3.3, we show two auxiliary results.

Lemma 3.4. *Let $x, y \in \mathbb{S}^1$ be distinct. Then there exists an arc $\gamma \colon [0,1] \to \mathbb{S}^1$ joining x to y with $D(\iota_1(x), \iota_1(y)) = \min \{\ell(\gamma), \ell(g \circ \gamma)\}$. The arc satisfies*

$$D(\iota_1(x), \iota_1(y)) \geq \sup_{\{t_i\}_{i=1}^{n+1}} \sum_{i=1}^{n} D(\iota_1(\gamma(t_i)), \iota_1(\gamma(t_{i+1}))),$$

the supremum taken over finite partitions of $[0,1]$. In particular, $D(\iota_1(x), \iota_1(y)) \geq \ell(\widetilde{\iota}_1(\gamma))$.

Proof. The existence of γ with $D(\iota_1(x), \iota_1(y)) = \min \{\ell(\gamma), \ell(g \circ \gamma)\}$ follows from the fact that σ is geodesic on $\mathbb{S}^1$. We identify $\iota_1(x)$ with x for every $x \in \mathbb{S}^1$ in the following computations.

The claim about the partitions is a consequence of the following observation and induction: If $0 \leq a < s < b \leq 1$, then

$$D(\gamma(a), \gamma(b)) \geq D(\gamma(a), \gamma(s)) + D(\gamma(s), \gamma(b)). \tag{11}$$

We first assume that $D(\gamma(a), \gamma(b)) = \sigma(\gamma(a), \gamma(b))$. Then γ is a length-minimizing geodesic joining $\gamma(a)$ and $\gamma(b)$. Consequently,

$$\sigma(\gamma(a), \gamma(b)) = \sigma(\gamma(a), \gamma(s)) + \sigma(\gamma(s), \gamma(b)).$$

Since $\sigma(c, d) \geq D(c, d)$ holds for every $c, d \in \mathbb{S}^1$, the inequality (11) holds in this case. In the remaining case, $g \circ \gamma$ is a length-minimizing geodesic joining $g(\gamma(a))$ and $g(\gamma(b))$ and

$$\sigma(g(\gamma(a)), g(\gamma(b))) = \sigma(g(\gamma(a)), g(\gamma(s))) + \sigma(g(\gamma(s)), g(\gamma(b))).$$

Since $\sigma(g(c), g(d)) \geq D(c, d)$ for every $c, d \in \mathbb{S}^1$, the inequality (11) holds also in this case.

The partition claim implies $D(x,y) \geq \sum_{i=1}^{n} d_{\widetilde{Z}}(\widetilde{\iota}_1(\gamma(t_i)), \widetilde{\iota}_1(\gamma(t_{i+1})))$ for every partition $\{t_i\}_{i=1}^{n+1}$ of $[0,1]$. The inequality $D(x,y) \geq \ell(\widetilde{\iota}_1(\gamma))$ follows by taking the supremum over such partitions. □

Lemma 3.5. *Let $x, y \in \mathbb{S}^1$ be distinct. Then there exists an arc $\gamma\colon [0,1] \to \mathbb{S}^1$ joining x to y such that $d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma))$.*

Proof. Let $\epsilon > 0$. The defining property of d_Z and Lemma 3.1 imply the existence of a chain $\{x_i\}_{i=1}^{n+1} \subset \mathbb{S}^1$ joining x to y for which

$$d_Z(\iota_1(x), \iota_1(y)) \geq -\epsilon + \sum_{i=1}^{n} D(\iota_1(x_i), \iota_1(x_{i+1})).$$

For each i, Lemma 3.4 yields the existence of an arc $\theta_i\colon [0,1] \to \mathbb{S}^1$ joining x_i to x_{i+1} with $D(\iota_1(x_i), \iota_1(x_{i+1})) \geq \ell(\widetilde{\iota}_1(\theta_i))$. Let θ denote the concatenation of these paths. Then $d_Z(\iota_1(x), \iota_1(y)) \geq -\epsilon + \ell(\widetilde{\iota}_1(\theta))$.

Let $\theta'\colon [0,1] \to \mathbb{S}^1$ be an arc joining x to y within the image of θ. Applying (4) on $\widetilde{Z}$ with $\rho \equiv \chi_{\widetilde{Z}}$ implies that $\ell(\widetilde{\iota}_1(\theta)) \geq \ell(\widetilde{\iota}_1(\theta'))$. Such a θ' is one of the arcs joining x to y within $\mathbb{S}^1$.

Let $\epsilon_j \to 0^+$ and consider θ'_j as above for every such ϵ_j. Up to passing to a subsequence and relabeling, we may assume that every such θ'_j is the same arc θ'. Passing to the limit $j \to \infty$ establishes $d_Z(\iota_1(x), \iota_1(y)) \geq \ell(\widetilde{\iota}_1(\theta')) \geq d_Z(\iota_1(x), \iota_1(y))$. We set $\gamma = \theta'$ to conclude the proof. □

Proof of Lemma 3.3. The claimed 1-Lipschitz and local isometry properties of $\widetilde{\iota}_1$ follow from Lemma 3.2. The local isometry property implies that given $z \in \widetilde{Z}$, the preimage $\widetilde{\iota}_1^{-1}(z)$ has more than two points only if the preimage is a subset of $\mathbb{S}^1$.

Suppose the existence of a distinct pair $x, y \in \widetilde{\iota}_1^{-1}(z)$. Then $x, y \in \mathbb{S}^1$. Lemma 3.5 shows that there exists an arc γ joining x to y within $\mathbb{S}^1$ satisfying

$$0 = d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma)).$$

This implies $|\gamma| \subset \widetilde{\iota}_1^{-1}(z)$. Since x and y were arbitrary, we conclude that $\widetilde{\iota}_1^{-1}(z)$ is path connected. Consequently, $\widetilde{\iota}_1^{-1}(z)$ is a connected and compact subset of $\mathbb{S}^1$.

The properties of $\widetilde{\iota}_2$ follow from a symmetry in the argument. Hence the claim follows. □

Proposition 3.6. *Let $g\colon (\mathbb{S}^1, \mathcal{H}^1_{\mathbb{S}^1}) \to (\mathbb{S}^1, \mathcal{H}^1_{\mathbb{S}^1})$ be a homeomorphism with $g^*\mathcal{H}^1_{\mathbb{S}^1} = v_g \mathcal{H}^1_{\mathbb{S}^1} + \mu^\perp$ with $\mathcal{H}^1_{\mathbb{S}^1}$ and $\mu^\perp$ mutually singular. Then, for every Borel set $B \subset \mathbb{S}^1$,*

$$\mathcal{H}^1_{d_{\widetilde{Z}}}(\widetilde{\iota}_1(B)) = \int_B \min\{1, v_g\}\, d\mathcal{H}^1_{\mathbb{S}^1} = \int_{\widetilde{\iota}_1(B)} \#(\widetilde{\iota}_1^{-1}(z))\, d\mathcal{H}^1_{\widetilde{Z}}(z). \tag{12}$$

Moreover, for every $x, y \in \mathbb{S}^1$, there exists an arc $|\gamma| \subset \mathbb{S}^1$ joining x to y for which

$$d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma)). \tag{13}$$

Before proving Proposition 3.6, we first consider a Carathéodory construction on $\mathbb{S}^1$. First, fix a Borel set $B_0 \subset \mathbb{S}^1$ for which $\mathcal{H}^1_{\mathbb{S}^1}(B_0) = 0$ and $\mu^\perp(\mathbb{S}^1 \setminus B_0) = 0$. Set $\nu^{ABS}(B) := \int_B \min\{1, v_g\}\, \chi_{\mathbb{S}^1 \setminus B_0}\, d\mathcal{H}^1_{\mathbb{S}^1}$ for all Borel sets $B \subset \mathbb{S}^1$.

For every arc $\gamma\colon [0,1] \to \mathbb{S}^1$, we denote $\xi^{ABS}(|\gamma|) := \nu^{ABS}(|\gamma|)$ and $\xi(|\gamma|) := D(\gamma(0), \gamma(1))$. The set function ξ^{ABS} and the family of arcs $|\gamma| \subset \mathbb{S}^1$ yields Carathéodory premeasures ν^{ABS}_δ for each $\delta > 0$.

Lemma 3.7. *For every Borel set $B \subset \mathbb{S}^1$, we have $\nu^{ABS}(B) = \sup_{\delta>0} \nu_\delta^{ABS}(B) \geq \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B))$.*

Proof. The equality $\nu^{ABS}(B) = \sup_{\delta>0} \nu_\delta^{ABS}(B)$ follows from the fact that ν^{ABS} is a finite Borel regular Borel measure.

We denote $B_1 = \{v_g \geq 1\} \cup B_0$ and $B_2 = \mathbb{S}^1 \setminus B_1$. If $B \subset \mathbb{S}^1$ is Borel, we have

$$\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B)) = \sum_{i=1}^{2} \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B \cap B_i)) \leq \mathcal{H}^1_{\mathbb{S}^1}(B \cap B_1) + \mathcal{H}^1_{\mathbb{S}^1}(g(B) \cap g(B_2))$$

since $\widetilde{\iota}_i$ is 1-Lipschitz for $i = 1, 2$. The right-hand side equals $\nu^{ABS}(B)$. Therefore $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(B)) \leq \nu^{ABS}(B)$ holds for all Borel sets. □

Lemma 3.8. *Let $x, y \in \mathbb{S}^1$ be distinct and $\gamma\colon [0,1] \to \mathbb{S}^1$ an arc joining x to y such that $d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \ell(\widetilde{\iota}_1(\gamma))$. Then $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma|)) = d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \nu^{ABS}(|\gamma|)$.*

Proof. Let $\pi/2 > \delta_0 > 0$ be such that

$$D(\iota_1(a), \iota_1(b)) < \delta_0 \quad \text{implies} \quad \max\{\sigma(a,b), \sigma(g(a), g(b))\} < \pi/2.$$

Given such a pair $a, b \in \mathbb{S}^1$, the length-minimizing geodesic $\theta\colon [0,1] \to \mathbb{S}^1$ joining a to b satisfies $\xi(|\theta|) = \min\{\ell(\theta), \ell(g \circ \theta)\}$. Then $\xi(|\theta|) \geq \xi^{ABS}(|\theta|)$.

Let γ be as in the claim. Let $0 < \delta < \delta_0$ and $0 < \epsilon < \delta/2$. We consider a partition $\{t_i\}_{i=1}^{n+1}$ of $[0,1]$ such that $\sigma(\gamma(t_i), \gamma(t_{i+1})) < \delta/2$ for every i. Then there exists a chain $\{x_{i,j}\}_{j=1}^{n_i+1} \subset \mathbb{S}^1$ joining the ends of $\gamma|_{[t_i, t_{i+1}]}$ so that

$$d_Z(\iota_1 \circ \gamma(t_i), \iota_1 \circ \gamma(t_{i+1})) \geq -\frac{\epsilon}{n} + \sum_{j=1}^{n_i} D(\iota_1(x_{i,j}), \iota_1(x_{i,j+1})).$$

In particular, $D(\iota_1(x_{i,j}), \iota_1(x_{i,j+1})) < \delta < \delta_0$ for every j. Hence the length-minimizing geodesic $\gamma_{i,j}$ joining $x_{i,j}$ to $x_{i,j+1}$ satisfies the assumptions of Lemma 3.4. For every i, Lemma 3.4 implies that, up to further partitioning the paths $\gamma_{i,j}$ and relabeling, we may assume $\sigma(x_{i,j}, x_{i,j}) < \delta$ for every j. Given this property, we conclude $D(\iota_1(x_{i,j}), \iota_1(x_{i,j+1})) = \xi(|\gamma_{i,j}|) \geq \xi^{ABS}(|\gamma_{i,j}|)$ and

$$\ell(\widetilde{\iota}_1(\gamma)) = \sum_{i=1}^{n} d_Z(\iota_1 \circ \gamma(t_i), \iota_1 \circ \gamma(t_{i+1})) \geq -\epsilon + \nu_\delta^{ABS}\left(\bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} |\gamma_{i,j}|\right).$$

Since the concatenation θ_i of $\{\gamma_{i,j}\}_{j=1}^{n_i}$ is a path joining $\gamma(t_i)$ to $\gamma(t_{i+1})$, the concatenation θ of $\{\theta_i\}_{i=1}^{n}$ is a path joining x to y. Hence $\bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} |\gamma_{i,j}| = |\theta|$ contains $|\gamma|$ or $\mathbb{S}^1 \setminus |\gamma|$, and

$$\ell(\widetilde{\iota}_1(\gamma)) \geq -\epsilon + \min\left\{\nu_\delta^{ABS}(|\gamma|), \nu_\delta^{ABS}(\mathbb{S}^1 \setminus |\gamma|)\right\}.$$

After passing to $\epsilon \to 0^+$ and then to $\delta \to 0^+$, we conclude

$$\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(\gamma)) = \ell(\widetilde{\iota}_1(\gamma)) \geq \min\left\{\nu^{ABS}(|\gamma|), \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma|)\right\}.$$

If we had $\nu^{ABS}(|\gamma|) > \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma|)$, this would contradict Lemma 3.7 and the length-minimizing property of $\widetilde{\iota}_1(\gamma)$. Hence $\nu^{ABS}(|\gamma|) \leq \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma|)$, and $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma|)) = d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) = \nu^{ABS}(|\gamma|)$ follows from Lemma 3.7. □

Proof of Proposition 3.6. The existence of γ and equality in (13) already follows from Lemma 3.4.

We claim that (12) holds. To this end, we consider three arcs $\gamma_i \colon [0,1] \to \mathbb{S}^1$ overlapping only at their end points, whose images cover $\mathbb{S}^1$, with the arcs satisfying $\nu^{ABS}(|\gamma_i|) \leq \nu^{ABS}(\mathbb{S}^1 \setminus |\gamma_i|)$.

Lemmas 3.7 and 3.8 imply that $\widetilde{\iota}_1 \circ \gamma_i$ is a length-minimizing geodesic joining its end points and $\nu^{ABS}(|\gamma_i|) = \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma_i|))$. Lemma 3.7 implies that the metric speed of $\widetilde{\iota}_1|_{|\gamma_i|}$ is bounded from above by $\min\{1, v_g\}$. Hence the equality $\nu^{ABS}(|\gamma_i|) = \mathcal{H}^1_{\widetilde{Z}}(\widetilde{\iota}_1(|\gamma_i|))$ forces the metric speed of $\widetilde{\iota}_1$ to equal $\min\{1, v_g\}$ $\mathcal{H}^1_{\mathbb{S}^1}$-almost everywhere on $|\gamma_i|$ for $i = 1,2,3$. The equality (12) follows from the area formula (4) and the fact that $\#(\widetilde{\iota}_1^{-1}(x)) = 1$ $\mathcal{H}^1_{\widetilde{Z}}$-almost everywhere. The fact $\#(\widetilde{\iota}_1^{-1}(x)) = 1$ $\mathcal{H}^1_{\widetilde{Z}}$-almost everywhere follows from the monotonicity of $\widetilde{\iota}_1$ and the integrability of the multiplicity. The integrability of the multiplicity follows from (3). □

Remark 3.9. *We consider a 2π-periodic doubling measure μ on $\mathbb{R}$ with $2\pi = \mu([0,2\pi])$ such that for some Borel set $B \subset [0,2\pi]$, $\mathcal{L}^1(B) = 0 = \mu([0,2\pi] \setminus B)$, the existence of which is established by Ahlfors–Beurling* [BA56, Section 7]. *Then $\psi(x) = \int_0^x d\mu$ is a homeomorphism and there exists a quasisymmetry $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ with $\theta \circ \psi = g \circ \theta$, where $\theta(t) = (\cos(t), \sin(t), 0)$. Then v_g in (12) is identically zero. Consequently, $d_Z \equiv 0$ on the seam S_Z.*

4. Harmonic measure and welding homeomorphisms

We consider a welding homeomorphism $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ and a welding circle $\mathcal{C}$ with complementary components Ω_1 and Ω_2, Riemann maps $\phi_i \colon Z_i \to \Omega_i$ for $i = 1,2$, and $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$. In this section, we consider the *harmonic measures* $\omega_i(E) = \phi_i^* \mathcal{H}^1_{\mathbb{S}^1}(E)/(2\pi)$ for all Borel sets $E \subset \mathbb{S}^2$.

We define a homeomorphism $\pi \colon \mathbb{S}^2 \to (Z, d_Z)$ and a quotient map $\widetilde{\pi} \colon \mathbb{S}^2 \to \widetilde{Z}$ via the formulas

$$\pi(x) = \begin{cases} \iota_1 \circ \phi_1^{-1}(x), & \text{when } x \in \overline{\Omega_1}, \\ \iota_2 \circ \phi_2^{-1}(x), & \text{when } x \in \Omega_2 \end{cases} \quad \text{and} \quad \widetilde{\pi} = Q \circ \pi. \tag{14}$$

Recall that $Q \colon Z \to \widetilde{Z}$ is the quotient map identifying $x, y \in Z$ whenever $d_Z(x,y) = 0$. Lemma 3.3 implies that $\widetilde{\pi}$ is monotone and $\widetilde{\pi}^{-1}(x)$ contains two or more points only if x is a point of the seam $Q(S_Z)$, and in such a case $\widetilde{\pi}^{-1}(x) \subset \mathcal{C}$.

For $\alpha = 1,2$, we denote, for every Borel set $B \subset \mathbb{S}^2$,

$$\widetilde{\pi}^* \mathcal{H}^\alpha_{\widetilde{Z}}(B) := \int_{\widetilde{Z}} \#(B \cap \widetilde{\pi}^{-1}(x)) \, d\mathcal{H}^\alpha_{\widetilde{Z}}(x) = \mathcal{H}^\alpha_{\widetilde{Z}}(\widetilde{\pi}(B)), \tag{15}$$

where the multiplicity can be ignored in the case $\alpha = 2$ since it equals one outside the negligible set $Q(S_Z)$. For $\alpha = 1$, the multiplicity is two or more only when it is ∞ and this happens in a set of negligible $\mathcal{H}^1_{\widetilde{Z}}$-measure. Either way, the multiplicity is negligible in (15), so the second equality is justified.

Proposition 4.1. *Let g be a welding homeomorphism with a welding circle $\mathcal{C}$ and $I \subset \mathcal{C}$ a subarc. Then $d_{\widetilde{Z}}(\widetilde{\pi}(x), \widetilde{\pi}(y)) = 0$ for all $x, y \in I$ if and only if $\omega_1|_I$ and $\omega_2|_I$ are mutually singular. If such an interval exists, then $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$.*

Remark 4.2. *If g is a welding homeomorphism obtained from Remark 3.9 or any welding g corresponding to the von Koch snowflake* [GM05, Example 4.3], *Proposition 4.1 implies that $Q(S_Z)$ is a singleton. In particular, $\widetilde{Z}$ is not even homeomorphic to the sphere. For a given g, this happens if and only if $g^* \mathcal{H}^1_{\mathbb{S}^1}$ and $\mathcal{H}^1_{\mathbb{S}^1}$ are mutually singular.*

A key step in the proof of the conformal removability in Theorem 1.6 is the following.

Proposition 4.3. *Let g be a welding homeomorphism and $\widetilde{\pi}$ as in (14). Then $\widetilde{\pi}$ is continuous, monotone, and surjective. Moreover, for all path families Γ on $\mathbb{S}^2$, $\operatorname{mod}\Gamma \leq \operatorname{mod}\widetilde{\pi}\Gamma$. The metric space $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$ if and only if $\widetilde{\pi}$ is a homeomorphism for which $\operatorname{mod}\Gamma = \operatorname{mod}\widetilde{\pi}\Gamma$ for all path families.*

The proof of Proposition 4.3 requires some preparatory work. Given the curve $\mathcal{C}$, we say that $x_0 \in \mathcal{C}$ is a *tangent point* if there exists a homeomorphism $\gamma\colon (-\epsilon,\epsilon) \to \mathcal{C}' \subset \mathcal{C}$ with $\gamma(0) = x_0$, and a tangent vector $v_0 \in T_{x_0}\mathbb{S}^2$ with unit length such that for every smooth $f\colon \mathbb{S}^2 \to \mathbb{R}$, its differential df satisfies

$$df(v_0) = \lim_{t\to 0^+} \frac{f(\gamma(t)) - f(x_0)}{\sigma(\gamma(t),x_0)} \quad \text{and} \quad df(-v_0) = \lim_{t\to 0^-} \frac{f(\gamma(t)) - f(x_0)}{\sigma(\gamma(t),x_0)}.$$

If v_0 exists, the tangent vector v_0 is independent of the parametrization γ and $\mathcal{C}'$ up to multiplication by -1; see [GM05, Chapter II, Section 4]. The collection of *tangents points* of $\mathcal{C}$ is denoted by $\operatorname{Tn}(\mathcal{C})$. The key properties of $\operatorname{Tn}(\mathcal{C})$ are self-contained in the following statement.

Lemma 4.4. *The Borel set $\operatorname{Tn}(\mathcal{C})$ has σ-finite Hausdorff 1-measure. Moreover, on the set $\operatorname{Tn}(\mathcal{C})$, the measures ω_1, ω_2, and $\mathcal{H}^1_{\mathcal{C}}$ are mutually absolutely continuous.*

Given any Borel set $E \subset \mathcal{C}$ with $\omega_1(E)\cdot\omega_2(E) > 0$, the restrictions $\omega_1|_E$ and $\omega_2|_E$ are mutually singular on E if and only if $\mathcal{H}^1_{\mathcal{C}}(\operatorname{Tn}(\mathcal{C})\cap E) = 0$.

Proof. The Borel measurability of $\operatorname{Tn}(\mathcal{C})$ follows from [GM05, Chapter II, Theorem 4.2] which connects the tangents of $\mathcal{C}$ and the angular derivatives of any given Riemann map $\phi_1'\colon Z_1 \to \Omega_1$, where $\partial\Omega_1 = \mathcal{C}$. The fact that $\operatorname{Tn}(\mathcal{C})$ has σ-finite Hausdorff 1-measure follows from [GM05, Chapter VI, Theorem 4.2].

Theorem 6.3 of [GM05, Chapter VI] states that if a Borel set $E \subset \mathcal{C}$ is such that $\omega_1(E)\cdot\omega_2(E) > 0$, then $\omega_1|_E$ and $\omega_2|_E$ are mutually singular on E if and only if $\mathcal{H}^1_{\mathcal{C}}(\operatorname{Tn}(\mathcal{C})\cap E) = 0$.

The fact that on the set $\operatorname{Tn}(\mathcal{C})$ the measures ω_1, ω_2, and $\mathcal{H}^1_{\mathcal{C}}$ are mutually absolutely continuous follows from [GM05, Chapter VI, Theorem 4.2 and the following discussion on p. 211]. □

Lemma 4.5. *The measures $\chi_{\mathcal{C}}\widetilde{\pi}^*\mathcal{H}^1_{\widetilde{Z}}$, $\chi_{\operatorname{Tn}(\mathcal{C})}\omega_1$, $\chi_{\operatorname{Tn}(\mathcal{C})}\omega_2$ and $\chi_{\operatorname{Tn}(\mathcal{C})}\mathcal{H}^1_{\mathcal{C}}$ are mutually absolutely continuous.*

More precisely, a given Borel set $B \subset \operatorname{Tn}(\mathcal{C})$ has positive 1-dimensional Hausdorff measure if and only if $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) > 0$. Furthermore, if $B \subset \mathcal{C}\setminus\operatorname{Tn}(\mathcal{C})$, then $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) = 0$.

Proof. We write $g^*\mathcal{H}^1_{\mathbb{S}^1} = v_g\mathcal{H}^1_{\mathbb{S}^1} + 2\pi\cdot\mu^\perp$ with $\mathcal{H}^1_{\mathbb{S}^1}$ and $\mu^\perp$ are mutually singular. We recall from Proposition 3.6 that for every Borel set $B \subset \mathcal{C}$,

$$\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) = \int_{\phi_1^{-1}(B)} \min\{1, v_g\}\, d\mathcal{H}^1_{\mathbb{S}^1}. \tag{16}$$

We denote $h = v_g \circ \phi_1^{-1}$ and observe the equality $\omega_2 = h\omega_1 + (\phi_1)_*\mu^\perp$. Then (16) is equivalent to

$$(2\pi)^{-1}\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(B)) = \int_B \min\{1, h\}\, d\omega_1. \tag{17}$$

Lemma 4.4 implies that the measures $\chi_{\mathcal{C}\setminus\operatorname{Tn}(\mathcal{C})}\omega_1$ and $\chi_{\mathcal{C}\setminus\operatorname{Tn}(\mathcal{C})}\omega_2$ are mutually singular. Consequently, $h = 0$ ω_1-almost everywhere in $\mathcal{C}\setminus\operatorname{Tn}(\mathcal{C})$. In particular, if $B = \mathcal{C}\setminus\operatorname{Tn}(\mathcal{C})$, the left-hand side equals zero in (17).

Lemma 4.4 yields that the measures $\chi_{\mathrm{Tn}(\mathcal{C})}\omega_1$, $\chi_{\mathrm{Tn}(\mathcal{C})}\omega_2$ and $\chi_{\mathrm{Tn}(\mathcal{C})}\mathcal{H}^1_{\mathcal{C}}$ are mutually absolutely continuous. Hence $\infty > h > 0$ ω_1-almost everywhere in $\mathrm{Tn}(\mathcal{C})$. This implies that the measure in (17) is mutually absolutely continuous with the measures $\chi_{\mathrm{Tn}(\mathcal{C})}\omega_1$, $\chi_{\mathrm{Tn}(\mathcal{C})}\omega_2$ and $\chi_{\mathrm{Tn}(\mathcal{C})}\mathcal{H}^1_{\mathcal{C}}$. The claim follows from the equalities (15) for $\alpha = 1$. □

Proof of Proposition 4.1. Fix a subarc $I \subset \mathcal{C}$. Proposition 3.6 implies that $\widetilde{\pi}(I)$ has zero $\mathcal{H}^1_{\widetilde{Z}}$-measure if and only if for every $x, y \in I$, $d_{\widetilde{Z}}(\widetilde{\pi}(x), \widetilde{\pi}(y)) = 0$ if and only if $v_g = 0$ $\mathcal{H}^1_{\mathbb{S}^1}$-almost everywhere on $\phi_1^{-1}(I)$. Equivalently, $\omega_1|_I$ and $\omega_2|_I$ are mutually singular.

Lemma 3.3 shows that $\widetilde{Z} \neq (Z, d_Z)$ if and only if there exists a closed arc $I \subset \mathbb{S}^1$ such that $y = \widetilde{\iota}_1(I)$. Assume that such an I exists. Having fixed $x_0 \in Z_1$ and $0 < s < \sigma(x_0, \mathbb{S}^1)$, there exists $c = c(x_0, I, s)$ for which

$$\operatorname{mod}\Gamma(I, \overline{B}_{\mathbb{S}^2}(x_0, s); I \cup Z_1) \geq c > 0;$$

a positive lower bound can be shown, for example, by estimating the modulus of all geodesics joining I to $\overline{B}_{\mathbb{S}^2}(x_0, s)$ in $I \cup Z_1$.

When $R > 0$ is small enough, for every $R > r > 0$ and every path in $\Gamma(I, \overline{B}_{\mathbb{S}^2}(x_0, s)); I \cup Z_1)$, we find a subpath $\gamma' \colon [0,1] \to Z_1$ so that $\widetilde{\iota}_1 \circ \gamma$ joins $\overline{B}_{\widetilde{Z}}(y, r)$ to $\widetilde{Z} \setminus B_{\widetilde{Z}}(y, R)$ within $\overline{B}_{\widetilde{Z}}(y, R)$. Since $\widetilde{\iota}_1$ is a local isometry off the seam, this implies

$$\liminf_{r \to 0^+} \operatorname{mod}\Gamma\left(\overline{B}_{\widetilde{Z}}(y, r), \widetilde{Z} \setminus B_{\widetilde{Z}}(y, R); \overline{B}_{\widetilde{Z}}(y, R)\right) \geq c.$$

Recalling Theorem 2.6, we see that $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$. □

Lemma 4.6. *For $i = 1, 2$, let $\rho_i \colon \Omega_i \to [0, \infty]$ denote the operator norm of the differential of $D(\phi_i^{-1})$. Then*

$$G = \chi_{\Omega_1}\rho_1 + \chi_{\Omega_2}\rho_2 + \infty \cdot \chi_{\mathrm{Tn}(\mathcal{C})} \in L^2(\mathbb{S}^2) \tag{18}$$

is a weak upper gradient of $\widetilde{\pi}$.

Proof. The L^2-integrability of G follows from the change of variables formulas of the Riemann maps ϕ_1 and ϕ_2 and the fact that $\mathrm{Tn}(\mathcal{C})$ has negligible area. Hence, as a consequence of Lemma 2.1, G is integrable along almost every absolutely continuous path $\gamma \colon [0,1] \to \mathbb{S}^2$. Given such a γ, we claim that

$$d_{\widetilde{Z}}(\widetilde{\pi}(\gamma(0)), \widetilde{\pi}(\gamma(1))) \leq \int_\gamma G \, ds, \tag{19}$$

implying that G is a weak upper gradient of $\widetilde{\pi}$.

Since G is integrable along γ, γ has negligible length in $\mathrm{Tn}(\mathcal{C})$. Then (4) implies $\mathcal{H}^1_{\mathbb{S}^2}(\mathrm{Tn}(\mathcal{C}) \cap |\gamma|) = 0$. We conclude $\mathcal{H}^1_{\widetilde{Z}}(\widetilde{\pi}(\mathcal{C}) \cap |\widetilde{\pi} \circ \gamma|) = 0$ from Lemma 4.5. The assumptions of Lemma 2.2 are satisfied and the conclusion $\ell(\widetilde{\pi} \circ \gamma) \leq \int_\gamma G \, ds$ follows. The inequality (19) is a consequence. □

We define the *Jacobian* of $\widetilde{\pi}$ to be the density of $\widetilde{\pi}^*\mathcal{H}^2_{\widetilde{Z}}$, defined in (15), with respect to $\mathcal{H}^2_{\mathbb{S}^2}$.

Lemma 4.7. *The mapping $\widetilde{\pi}$ satisfies Lusin's Condition (N) and the Jacobian $J_{\widetilde{\pi}}$ coincides with G^2 $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere, with G being from (18).*

Proof. The Lusin's Condition (N) of $\widetilde{\pi}$ follows from the fact that $\widetilde{\pi}(\mathcal{C})$ has negligible $\mathcal{H}^2_{\widetilde{Z}}$-measure, the fact that $\iota_i \colon Z_i \to \widetilde{Z}_i$ is a local isometry, and as $\phi_i^{-1} \colon \Omega_i \to Z_i$

satisfies Condition (N). Here $J_{\widetilde{\pi}} = 0$ $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere on $\mathcal{C}$, so the equality $J_{\widetilde{\pi}} = G^2$ follows from the fact that ϕ_1 and ϕ_2 are Riemann maps. □

Proof of Proposition 4.3. The claimed topological properties of $\widetilde{\pi}$ were already verified at the beginning of this section. Lemmas 4.6 and 4.7 prove that $J_{\widetilde{\pi}} = G^2 \in L^1(\mathbb{S}^2)$ with G being a weak upper gradient of $\widetilde{\pi}$. This fact and the fact that the multiplicity of $\widetilde{\pi}$ is negligible for $\widetilde{\pi}^*\mathcal{H}^2_{\widetilde{Z}}$ imply $\operatorname{mod}\Gamma \leq \operatorname{mod}\widetilde{\pi}\Gamma$ for all path families Γ.

Lastly, we argue that a K-quasiconformal map $\psi\colon \widetilde{Z} \to \mathbb{S}^2$ exists (for some $K \geq 1$) if and only if $\widetilde{\pi}$ is a 1-quasiconformal homeomorphism. The "if"-direction is obvious.

In the "only if"-direction, the fact that $\widetilde{\pi}$ is a homeomorphism follows from Proposition 4.1. So $h = \psi \circ \widetilde{\pi}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is a homeomorphism satisfying $\operatorname{mod}\Gamma \leq K \operatorname{mod} h\Gamma$ for all path families Γ. Theorem 2.4 and [AIM09, Definition 3.1.1 and Theorem 3.7.7] prove that h is K-quasiconformal. Consequently, $\widetilde{\pi}$ is K'-quasiconformal for some $K' \leq K^2$. This self-improves to $K' = 1$ due to Lemma 4.8 below. This yields $\operatorname{mod}\widetilde{\pi}\Gamma = \operatorname{mod}\Gamma$ for all path families. □

Lemma 4.8. *Suppose that $\widetilde{\pi}\colon \mathbb{S}^2 \to \widetilde{Z}$ from (14) is a homeomorphism. Then $\widetilde{\pi}\colon \mathbb{S}^2 \to \widetilde{Z}$ is 1-quasiconformal if and only if for every 1-Lipschitz $h\colon \mathbb{S}^2 \to \mathbb{R}$, $h \circ \widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z})$.*

Proof. The "only if"-claim is clear, given Theorem 2.4 (ii). In the "if"-direction, fix a 1-Lipschitz $h\colon \mathbb{S}^2 \to \mathbb{R}$ for now.

Consider the Borel function $G\colon \mathbb{S}^2 \to [0, \infty]$ defined on Lemma 4.6. Then $\rho = 1/G \circ \widetilde{\pi}^{-1}$ is such that ρ^2 is the Jacobian of $\widetilde{\pi}^{-1}$, as a consequence of Lemma 4.7. Hence $\rho \in L^2(\widetilde{Z})$.

Given that $h \circ \widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z})$ and $\mathcal{H}^2_{\widetilde{Z}}(Q(S_Z)) = 0$, for almost every $\gamma\colon [0,1] \to \widetilde{Z}$, the composition $(h \circ \widetilde{\pi}^{-1}) \circ \gamma$ is absolutely continuous, γ has negligible length on the seam $Q(S_Z)$, and $\int_\gamma \rho\, ds < \infty$. Indeed, the absolute continuity of $(h \circ \widetilde{\pi}^{-1}) \circ \gamma$ for almost every path follows from [HKST15, Proposition 6.3.2]. The fact that almost every path has negligible length on $Q(S_Z)$ follows from Lemma 2.1 and the L^2-integrability of $\infty \cdot \chi_{Q(S_Z)}$. Similarly, the conclusion $\int_\gamma \rho\, ds < \infty$ follows from Lemma 2.1 and the L^2-integrability of ρ.

If we denote $E = (h \circ \widetilde{\pi}^{-1})(|\gamma| \cap Q(S_Z))$, the absolute continuity of $(h \circ \widetilde{\pi}^{-1}) \circ \gamma$ implies $\mathcal{H}^1_{\mathbb{R}}(E) = 0$. Then Lemma 2.2 yields $\ell((h \circ \widetilde{\pi}^{-1}) \circ \gamma) \leq \int_\gamma \rho\, ds$. We conclude that ρ is a weak upper gradient of $h \circ \widetilde{\pi}^{-1}$.

Since ρ is independent of h and h is an arbitrary 1-Lipschitz function, Theorem 7.1.20 [HKST15] shows that ρ is a weak upper gradient of $\widetilde{\pi}^{-1}$. Since ρ^2 is the Jacobian of $\widetilde{\pi}^{-1}$, we conclude $K_O(\widetilde{\pi}^{-1}) = 1$. Recall $K_O(\widetilde{\pi}) = 1$ from Proposition 4.3. □

Remark 4.9. *If the welding curve $\mathcal{C}$ happens to be rectifiable, the Hausdorff 1-measure on $\mathcal{C}$ and $\chi_{\mathrm{Tn}(\mathcal{C})}\mathcal{H}^1_{\mathcal{C}}$ are mutually absolutely continuous* [GM05, Chapter VI, Theorem 1.2 (F. and M. Riesz)]. *With this fact at hand, Lemma 4.5 implies that $\widetilde{\pi}$ is a homeomorphism. Moreover, one can show that $h \circ \widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z})$ for every 1-Lipschitz $h\colon \mathbb{S}^2 \to \mathbb{R}$. Hence $\widetilde{\pi}$ is 1-quasiconformal.*

Proof of Theorem 1.6. Suppose the existence of a quasiconformal homeomorphism $\psi\colon \widetilde{Z} \to \mathbb{S}^2$. Up to postcomposing ψ by an orientation-reversing Möbius transformation of $\mathbb{S}^2$, we may assume that $\widetilde{\phi}_i := \psi \circ \widetilde{\iota}_i|_{Z_i}\colon Z_i \to \mathbb{S}^2$ is orientation-preserving for $i = 1, 2$. Let $\mathcal{C} = \psi(Q(S_Z))$.

The set $\mathbb{S}^2 \setminus \mathcal{C}$ is the disjoint union of Jordan domains Ω_1 and Ω_2, where Ω_i is the image of $\widetilde{\phi}_i$ for $i = 1, 2$.

Next, since $\psi \colon \widetilde{Z} \to \mathbb{S}^2$ is a quasiconformal homeomorphism, ψ satisfies Lusin's Condition (N) [Raj17, Section 17]. Consequently, $\mathcal{C}$ has zero 2-dimensional Hausdorff measure.

We consider the Beltrami differential $\mu = \chi_{\Omega_1}\mu_1 + \chi_{\Omega_2}\mu_2$, where μ_i is the Beltrami differential of $\widetilde{\phi}_i^{-1}$. If $h \colon \mathbb{S}^2 \to \mathbb{S}^2$ is a normalized solution to the Beltrami equation induced by μ [AIM09, Measurable Riemann mapping theorem], the mapping $\widetilde{\psi} = h \circ \psi$ is 1-quasiconformal. Since $\mathcal{C}$ has zero measure, this is readily verified by hand or by applying [Iko21b, Theorem 4.12].

We have verified that $(Z, d_Z) = \widetilde{Z}$ and we may assume that $\psi \colon (Z, d_Z) \to \mathbb{S}^2$ is 1-quasiconformal with $\phi_i = \psi \circ \widetilde{\iota}_i|_{Z_i}$ being Riemann maps [AIM09, Weyl's lemma]. Proposition 4.1 implies $(Z, d_Z) = \widetilde{Z}$. The definition of Z implies that $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$. Consequently, g is a welding homeomorphism.

In order to show the removability of $\mathcal{C} := \psi(S_Z)$, we are given an orientation-preserving homeomorphism $M \colon \mathbb{S}^2 \to \mathbb{S}^2$ conformal in the complement of $\mathcal{C}$. Then $\pi' := \psi^{-1} \circ M^{-1}$ defines a mapping as in (14) for the curve $\mathcal{C}' = M(\mathcal{C})$. Proposition 4.3 implies that π' is 1-quasiconformal. Consequently, $M^{-1} = \psi \circ \pi'$ is 1-quasiconformal, i.e., a Möbius transformation. □

5. Mass upper bound

In this section, we prove Theorems 1.1 and 1.2. We first consider the implication "(3) ⇒ (1)". Recall that we are given an orientation-preserving homeomorphism $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ and the canonical quotient map $Q \colon Z \to \widetilde{Z}$. We are assuming the existence of a constant $C > 0$ for which

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{B}_{\widetilde{Z}}(y, r))}{\pi r^2} \leq C \quad \text{for every } y \in Q(S_Z). \tag{20}$$

In order to make transparent how the Lipschitz constant of g (resp. g^{-1}) is related to C in (20), we define $C_1, C_2 \geq 0$ to be the smallest constants for which

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(y, r))}{\pi r^2} \leq C_1 \quad \text{for every } y \in Q(S_Z) \tag{21}$$

$$\liminf_{r \to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_2(Z_2)} \cap \overline{B}_{\widetilde{Z}}(y, r))}{\pi r^2} \leq C_2 \quad \text{for every } y \in Q(S_Z). \tag{22}$$

Recalling from Lemma 3.3 the fact that the inclusion maps are 1-Lipschitz and local isometries outside the seam, the limit infimums in (21) and (22) are bounded from below by $1/2$. Hence, $C_1, C_2 \geq 1/2$. Since the seam is negligible, we have $1 \leq C_1 + C_2 \leq C$.

We show that the constant C_1 in (21) and the Lipschitz constant L_1 of g^{-1} are connected via the following function

$$f(\epsilon) := \frac{\left(\sin|_{(0,\pi/2]}\right)^{-1}(\epsilon)}{\pi} + \frac{\sqrt{1-\epsilon^2}}{\pi\epsilon} \quad \text{for } 0 < \epsilon \leq 1. \tag{23}$$

Definition 5.1. *For every $C \geq 1/2$, $L = L(C) \geq 1$ denotes the unique positive number such that for every $0 < \epsilon \leq L^{-1}$, $f(\epsilon) \geq C$. Equivalently, $L = 1/f^{-1}(C)$.*

Remark 5.2. *We note that for every $0 < \epsilon \leq 1$, we have $f(\epsilon) \geq (\pi\epsilon)^{-1}$. We use this fact during the proof of Theorem 1.1.*

Proposition 5.3. *If* (21) *holds with constant* C_1 *and* $L_1 = L(C_1)$ *is as in Definition 5.1, then* g^{-1} *is* L_1*-Lipschitz and* $\widetilde{\iota}_1 \colon \overline{Z}_1 \to \widetilde{Z}$ *satisfies for every* $x, y \in \overline{Z}_1$, $\sigma(x,y) \geq d_Z(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) \geq \sigma(x,y)/L_1$.

The symmetry in the argument yields the following result.

Proposition 5.4. *If* (22) *holds with constant* C_2 *and* $L_2 = L(C_2)$ *is as in Definition 5.1, then* g *is* L_2*-Lipschitz and* $\widetilde{\iota}_2 \colon Z_2 \to \widetilde{Z}$ *satisfies for every* $x, y \in \overline{Z}_2$, $\sigma(x,y) \geq d_Z(\widetilde{\iota}_2(x), \widetilde{\iota}_2(y)) \geq \sigma(x,y)/L_2$.

We start the proof of Proposition 5.3. We consider the decomposition $g^*\mathcal{H}^1_{\mathbb{S}^1} = v_g \mathcal{H}^1_{\mathbb{S}^1} + \mu^\perp$ with $\mu^\perp$ and $\mathcal{H}^1_{\mathbb{S}^1}$ being singular. We fix a Borel representative of v_g. Let f be as in (23). The following statement holds for every $\widetilde{Z}$.

Proposition 5.5. *Given* $1 > \epsilon > 0$ *and a* $\mathcal{H}^1_{\mathbb{S}^1}$*-density point* $x_0 \in \mathbb{S}^1$ *of* $E := \{v_g \leq \epsilon\}$*, we have*

$$f(\epsilon) \leq \liminf_{r \to 0^+} \frac{\mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(x_0, r))}{\pi r^2}. \tag{24}$$

Proof. For the duration of the proof, we fix normal coordinates $F \colon B(0, \pi/2) \to \mathbb{S}^2$ centered at x_0 in such a way that the preimage of $\mathbb{S}^1 \cap B(x_0, \pi/2)$ is $(-\pi/2, \pi/2) \times \{0\}$ [Lee18, Section 5]. Recall that this means that F is an isometry along radial geodesics and the metric has the expansion $g_{ij}(x) = \delta_{ij} + O(\|x\|_2^2)$ in these coordinates. In particular, as $r \to 0^+$, the bi-Lipschitz constant of $F|_{B(0,r)}$ is of the form $1 + O(r^2)$. We denote $\Gamma(s) := F(s, 0)$ for $|s| \leq \pi/2$.

We fix $0 < \eta < 1/\epsilon - 1$. Since x_0 is a density point of E, there exists $s_0 < \pi/2$ such that for every $0 < s \leq s_0$,

$$\mathcal{H}^1_{\mathbb{S}^1}\left(\Gamma\left([-s, s]\right) \setminus E\right) \leq \epsilon \eta s. \tag{25}$$

We fix $0 < r \leq \epsilon s_0$. Then, for every $0 < s < r/\epsilon$, Proposition 3.6 yields, for both $I = [0, s]$ and $I = [-s, 0]$,

$$\ell(E \cap (\widetilde{\iota}_1 \circ \Gamma|_I)) \leq \epsilon s. \tag{26}$$

Since $\widetilde{\iota}_1$ is 1-Lipschitz, according to Lemma 3.3, (25) and (26) imply

$$\ell(\widetilde{\iota}_1 \circ \Gamma|_I) \leq s\epsilon + \epsilon \eta s = \epsilon(1 + \eta)s < s. \tag{27}$$

We denote for every $|s| < r/((1+\eta)\epsilon)$, $\rho_s := r - \epsilon(1+\eta)|s|$. For each $z \in Z_1 \cap B_{\mathbb{S}^2}(F(s,0), \rho_s)$, the inequality (27) implies $\widetilde{\iota}_1(z) \in \overline{B}_{\widetilde{Z}}(\widetilde{\iota}_1(x_0), r)$.

We estimate $A_r := \mathcal{H}^2_{\widetilde{Z}}(\overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(\widetilde{\iota}_1(x_0), r))$ as $r \to 0^+$. In estimating A_r, we use the fact that the seam $Q(S_Z)$ has negligible $\mathcal{H}^2_{\widetilde{Z}}$-measure and that $\widetilde{\iota}_1$ is a local isometry outside the seam. We claim that for each $0 < \theta < \pi/2$ the following holds:

$$A_r \geq (1 + O((r/\epsilon)^2))^{-2} \left(\theta r^2 + \cos(\theta) \frac{r^2}{(1 + O((r/\epsilon)^2))\epsilon(1+\eta)} \right). \tag{28}$$

The term $(1 + O((r/\epsilon)^2))^{-2}$ comes from estimating the Jacobian of $\widetilde{\iota}_1 \circ F$. The first term in the brackets comes from the fact that F preserves the speed of radial geodesics, so

$$(\widetilde{\iota}_1 \circ F)\left(\left\{(s,t) \colon \sqrt{s^2 + t^2} < r, 0 < t\right\}\right) \subset \overline{\widetilde{\iota}_1(Z_1)} \cap \overline{B}_{\widetilde{Z}}(\widetilde{\iota}_1(x_0), r).$$

We use this inclusion in a circular sector $C_\theta(r)$ which has a total angle 2θ and an angle bisector $\{0\} \times \mathbb{R}$.

The second term in the brackets is twice the area of a suitable triangle. The factor of two comes from the symmetry of the estimate (27) with respect to the parameter $s = 0$. We consider a triangle $T_\theta(r) \subset \mathbb{R}^2$ foliated by line segments $\ell(s)$, where $0 \leq s < r/((1+\eta)\epsilon)$, with $\ell(s)$ having the start point $(s, 0)$, tangent in the direction $(\sin(\theta), \cos(\theta))$, and has length $\rho_a/(1 + O((r/\epsilon)^2))$. The $\widetilde{\iota}_1 \circ F$ image of such a triangle $T_\theta(r)$ contributes to A_r. The inequality (28) follows.

We choose the angle θ to satisfy $\sin(\theta) = \epsilon(1+\eta)$. We divide (28) by πr^2, pass to the limit $r \to 0^+$, and then to $\eta \to 0^+$, and conclude

$$\liminf_{r \to 0^+} \frac{A_r}{\pi r^2} \geq \frac{\left(\sin|_{(0,\pi/2]}\right)^{-1}(\epsilon)}{\pi} + \frac{\sqrt{1-\epsilon^2}}{\pi \epsilon} = f(\epsilon). \tag{29}$$

The inequality (24) is the same as (29). □

Remark 5.6. *Given $0 < \epsilon < 1$, the lower bound in (29) is sharp. This can be shown by considering a bi-Lipschitz $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ with metric speed $v_g \equiv \epsilon$ everywhere in an open neighbourhood of $x_0 \in \mathbb{S}^1$.*

If the circular sector $C_\theta(r)$ and triangle $T_\theta(r)$ are defined as in the proof of the lower bound (29), with $\eta = 0$, and $\theta = \left(\sin|_{(0,\pi/2]}\right)^{-1}(\epsilon)$, we have

$$\liminf_{r \to 0^+} \frac{A_r}{\pi r^2} = \frac{\mathcal{H}^2_{\mathbb{R}^2}(C_\theta(1)) + 2\mathcal{H}^2_{\mathbb{R}^2}(T_\theta(1))}{\pi} = f(\epsilon).$$

This can be showed using Lemma 3.2 and Proposition 3.6. The key property of the angle θ is that the line on $\mathbb{R}^2$ containing $(r/\epsilon, 0)$ with tangent vector $(-\cos(\theta), \sin(\theta))$ intersects every ball $\overline{B}_{\mathbb{R}^2}((s,0), \rho_s)$ tangentially when $0 \leq s < 1/\epsilon$ and $\rho_s = 1 - \epsilon s$.

Proof of Proposition 5.3. Given (21) and Proposition 5.5, we have $v_g(x) \geq L_1^{-1}$ for $\mathcal{H}^1_{\mathbb{S}^1}$-almost every $x \in \mathbb{S}^1$. This implies that g^{-1} is absolutely continuous and $v_{g^{-1}}(x) \leq L_1$ for $\mathcal{H}^1_{\mathbb{S}^1}$-almost every $x \in \mathbb{S}^1$. Therefore g^{-1} is L_1-Lipschitz.

The fact that $\widetilde{\iota}_1$ 1-Lipschitz follows from Lemma 3.3. Proposition 3.6 implies that

$$d_Z(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) \geq \sigma(x,y)/L_1 \quad \text{for every } x, y \in \mathbb{S}^1.$$

Given this inequality, the equality (9) in Lemma 3.2 implies the corresponding inequality for every pair $x, y \in \overline{Z}_1$. Hence $\widetilde{\iota}_1^{-1}$ is L_1-Lipschitz. □

Next, we verify a lemma about radial extensions of bi-Lipschitz maps, which we need during the proof of Theorem 1.1.

For the south pole $P_1 \in Z_1$, we consider the stereographic projection $P \colon \mathbb{S}^2 \setminus \{P_1\} \to \mathbb{R}^2 \times \{0\}$ fixing the equator and mapping the north pole $P_2 = (0,0,1)$ to the origin. We identify $\mathbb{R}^2 \times \{0\}$ with $\mathbb{R}^2$. We note that P^{-1} has the explicit definition

$$P^{-1}(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2+y^2}{1+x^2+y^2} \right).$$

The Riemannian tensor of $\mathbb{S}^2$ in these coordinates is $I = (4/(1+r^2)^2) g_E$, where r is the distance to the origin and g_E the Euclidean inner product. In polar coordinates, $g_E = dr^2 + r^2 d\theta^2$. We see from the form of I that the bi-Lipschitz constant of $\widetilde{g} = P \circ g \circ (P|_{\mathbb{S}^1})^{-1}$ and $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ coincide.

We represent the polar coordinates using the complex notation $re^{i\theta}$. We note that there exists a homeomorphism $\widetilde{G} \colon \mathbb{R} \to \mathbb{R}$ with $\widetilde{g}(e^{i\theta}) = e^{i\widetilde{G}(\theta)}$ for every $\theta \in \mathbb{R}$. For every $0 \leq r \leq 1$ and $\theta \in \mathbb{R}$, we set $\widetilde{\psi}(re^{i\theta}) := re^{i\widetilde{G}(\theta)}$ and refer to $\widetilde{\psi}$ as the *radial extension* of $\widetilde{g}$. We recall from [Kal14, Theorem 2.2] that the bi-Lipschitz constants of $\widetilde{g}$ and $\widetilde{\psi}$ coincide. Let $\psi = P^{-1} \circ \widetilde{\psi} \circ P|_{Z_2} : Z_2 \to Z_2$.

We use the following fact during the proof of Lemma 5.8; see for example [DCJS16], [CS20].

Lemma 5.7. *For every $x, y \in \mathbb{S}^2$, $0 < \epsilon < 1$, and $0 < 4r < \sigma(x, y)$, the modulus of the family of paths joining $B_{\mathbb{S}^2}(x, r)$ to $B_{\mathbb{S}^2}(y, r)$ with length $(1 + \epsilon)\sigma(x, y)$ is positive.*

Lemma 5.8. *The map $\psi : Z_2 \to Z_2$ is L-bi-Lipschitz if g is L-bi-Lipschitz.*

Proof. We refer the interested reader to [Kal14, Section 2] for the proof of the fact that $\widetilde{\psi}$ is bi-Lipschitz if $\widetilde{g}$ (equivalently g) is bi-Lipschitz. We take this as a given.

Since $\widetilde{\psi}$ is bi-Lipschitz, it has a differential at $\mathcal{L}^2$-almost every point in $\mathbb{D}$. Given this fact, the following computations are understood to hold at $\mathcal{L}^2$-almost every $(x, y) = re^{i\theta}$ in the unit disk.

The pullback $\widetilde{\psi}^* I$ is a diagonal matrix with respect to the basis $(dr, d\theta)$, with diagonal $4/(1 + r^2)^2$ and $4|\widetilde{G}'(\theta)|^2 r^2/(1 + r^2)^2$. Hence the maximum of the operator norms of $D\widetilde{\psi} : (T\mathbb{D}, I) \to (T\mathbb{D}, I)$ and its inverse is equal to $L(re^{i\theta}) = \max\left\{|\widetilde{G}'(\theta)|, |\widetilde{G}'(\theta)|^{-1}\right\}$. Then, if L' denotes the essential supremum of $L(re^{i\theta})$, Lemma 5.7 implies that ψ is L'-bi-Lipschitz. On the other hand, L' is the bi-Lipschitz constant of g. □

Proof of Theorem 1.1. We first claim that "(1) ⇒ (2)". Lemma 5.8 provides us with an L-bi-Lipschitz $\psi : Z_2 \to Z_2$ extension of the given L-bi-Lipschitz g. We define $H(x) = \widetilde{\iota}_1(x)$ for each $x \in \overline{Z}_1$ and $H(x) = \widetilde{\iota}_2 \circ \psi(x)$ otherwise. Proposition 3.6 implies that H is L-bi-Lipschitz at the seam, and Lemma 3.2 implies that H is L-bi-Lipschitz everywhere.

Notice that if $H : \mathbb{S}^2 \to \widetilde{Z}$ is L'-bi-Lipschitz, we may choose $C = (L')^4$ as an upper bound for the 2-dimensional Hausdorff lower density. Hence "(2) ⇒ (3)" follows, quantitatively. Lastly, "(3) ⇒ (1)" follows from Propositions 5.3 and 5.4. In fact, given $C \geq 1$ for which the lower density bound (20) holds, g is L'-bi-Lipschitz for L' solving $C = f(1/L')$. Since $f(\epsilon) \geq 1/\pi\epsilon$ for every $0 < \epsilon \leq 1$, we have $C\pi \geq L'$. Hence g is $C\pi$-bi-Lipschitz. □

Remark 5.9. *The estimates between the constants in "(3) ⇒ (1)" in Theorem 1.1 can be improved in two ways. First, the constants C_1 and C_2 in (21) and (22) satisfy $\max\{C_1, C_2\} \leq C - 1/2$, so g is $(C - 1/2)\pi$-bi-Lipschitz.*

The second improvement is obtained by using the constant $L' = L(C - 1/2)$ from Definition 5.1. Then g is L'-bi-Lipschitz, where $L' \leq (C - 1/2)\pi$.

These improvements imply that the bi-Lipschitz constant of g converges to 1 as $C \to 1^+$. These facts also improve Theorem 1.2 and the following result, Proposition 5.10.

Before proving Theorem 1.2, we investigate a related problem. To this end, suppose that we are given Riemann maps $\phi_i : Z_i \to \Omega_i$ with Ω_1 and Ω_2 denoting the complementary components of a welding curve $\mathcal{C}$, and set $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$.

Proposition 5.10. *Let $K, C \geq 1$. The welding homeomorphism g is $\pi(KC)^2$-bi-Lipschitz if there exists a K-quasiconformal homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that for both $i = 1, 2$,*

$$C^{-1} J_h(x) \leq J_{\phi_i^{-1}}(x) \leq C J_h(x) \quad \text{for } \mathcal{H}^2_{\mathbb{S}^2}\text{-a.e. } x \in \Omega_i. \tag{30}$$

Conversely, if g is L-bi-Lipschitz, then there exists L^4-quasiconformal homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that (30) holds for $C = L^2$.

Proof. We first assume that $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is L-bi-Lipschitz. Then Theorem 1.1 provides us with an L-bi-Lipschitz homeomorphism $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$. Proposition 4.3 and (14) imply that $\widetilde{\pi}\colon \mathbb{S}^2 \to \widetilde{Z}$ defined via the formula

$$\widetilde{\pi}(x) = \begin{cases} \widetilde{\iota}_1 \circ \phi_1^{-1}(x), & x \in \overline{\Omega_1}, \\ \widetilde{\iota}_2 \circ \phi_2^{-1}(x), & x \in \Omega_2 \end{cases} \tag{31}$$

is a 1-quasiconformal homeomorphism. Therefore, $h := \Psi \circ \widetilde{\pi}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is K-quasiconformal for $K = L^4$, and as Ψ is L-bi-Lipschitz, the Jacobians of h and $\widetilde{\pi}$ are comparable with comparison constant $C = L^2$.

Next, we are given a Jordan curve $\mathcal{C} \subset \mathbb{S}^2$ corresponding to a welding homeomorphism $g = \phi_2^{-1} \circ \phi_1|_{\mathbb{S}^1}$, a K-quasiconformal homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{S}^2$, and a constant $C \geq 1$ such that

$$C^{-1} J_h(x) \leq J_{\widetilde{\pi}}(x) \leq C J_h(x) \quad \mathcal{H}^2_{\mathbb{S}^2}\text{-a.e. } x \in \mathbb{S}^2 \setminus \mathcal{C}. \tag{32}$$

For $i = 1, 2$, the composition $h \circ \phi_i$ is K-quasiconformal with Jacobian bounded from above C and below by C^{-1}, respectively; here we apply (32). Theorem 2.4 (ii) and Hadamard's inequality imply that $C^{-1} \leq \rho^2_{h \circ \phi_i} \leq KC$ $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in Z_i. Lemma 5.7 implies that the homeomorphism $h \circ \phi_i$ is locally L'-bi-Lipschitz for $L' = \sqrt{KC}$.

Since, for both $i = 1, 2$, $\overline{Z}_i$ is geodesic, it is immediate that $h \circ \phi_i\colon \overline{Z}_i \to \mathbb{S}^2$ is L'-Lipschitz. Since this holds for both $i = 1, 2$, the construction of d_Z implies that whenever $x, y \in \mathbb{S}^1$, $\sigma(h \circ \phi_1(x), h \circ \phi_1(y)) \leq L' d_{\widetilde{Z}}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y))$. Lemma 3.2 (9) establishes the same inequality for each $x, y \in \overline{Z}_1$. Hence the mapping $\widetilde{\pi}$ defined by the expression (31) is a homeomorphism and $\Psi := h \circ \widetilde{\pi}^{-1}$ is L'-Lipschitz on the southern hemisphere. A similar argument shows that Ψ is L'-Lipschitz on both of the hemispheres. Then Lemma 3.2 (10) implies that Ψ is L'-Lipschitz everywhere.

Since $\operatorname{mod} \Gamma \leq K \operatorname{mod} \Psi^{-1}\Gamma$ for all path families (recall Proposition 4.3), we have $\Psi^{-1} \in N^{1,2}(\mathbb{S}^2, \widetilde{Z})$. On the other hand, $\Psi(Q(S_Z))$ has negligible $\mathcal{H}^2_{\mathbb{S}^2}$-measure and Ψ^{-1} is locally L'-Lipschitz in the complement of that set. In particular, almost every absolutely continuous $\gamma\colon [0,1] \to \mathbb{S}^2$ has zero length in $\Psi(Q(S_Z))$ and $\Psi^{-1} \circ \gamma$ is absolutely continuous. As a consequence, $\mathcal{H}^1_{\widetilde{Z}}(Q(S_Z) \cap |\Psi^{-1} \circ \gamma|) = 0$.

Denoting $E = Q(S_Z) \cap |\Psi^{-1} \circ \gamma|$ and $\rho = L' \chi_{\mathbb{S}^2}$, we conclude from Lemma 2.2 that $\ell(\Psi^{-1} \circ \gamma) \leq \int_\gamma \rho \, ds \leq L' \ell(\gamma)$. Lemma 5.7 implies that Ψ^{-1} is L'-Lipschitz.

We have verified that Ψ is L'-bi-Lipschitz. By applying the implications "(2) $\Rightarrow$ (3) $\Rightarrow$ (1)" in Theorem 1.1, we conclude that g is L-bi-Lipschitz for $L = \pi (L')^4 = \pi (KC)^2$. ☐

Next, we prove Theorem 1.2. This essentially follows from Proposition 5.10.

Proof of Theorem 1.2. We claim that $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ is bi-Lipschitz if and only if there exists a quasiconformal homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ and a 1-quasiconformal homeomorphism $\varphi\colon \mathbb{S}^2 \to \widetilde{Z}$ such that J_φ and J_h are comparable.

If such φ and h exist, we may assume that $\phi_i = \varphi^{-1} \circ \widetilde{\iota}_i|_{Z_i}$ is a Riemann map for both $i = 1, 2$. Then Proposition 5.10 shows that g is bi-Lipschitz.

Conversely, if g is bi-Lipschitz, Theorem 1.1 provides a bi-Lipschitz homeomorphism $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$. Then Theorem 1.6 implies the existence of a 1-quasiconformal homeomorphism $\pi\colon \mathbb{S}^2 \to \widetilde{Z}$ such that $\phi_i = \pi^{-1} \circ \widetilde{\iota}_i|_{Z_i}$ is a Riemann map for $i = 1, 2$. We may also assume that $\Psi \circ \widetilde{\iota}_i|_{Z_i}$ is orientation-preserving for $i = 1, 2$, by post-composing Ψ with a suitable reflection, if need be. Defining $h = \Psi \circ \pi$ implies that the assumptions of Proposition 5.10 hold for g.

Since Theorem 1.1 and Proposition 5.10 are quantitative, so is Theorem 1.2. □

6. Mappings of finite distortion

In this section, we establish Proposition 1.4 and Theorem 1.5.

Definition 6.1. *Let $\Omega, \Omega' \subset \mathbb{S}^2$ be open. A homeomorphism $\psi \colon \Omega \to \Omega'$ is a* mapping of finite distortion *if $\psi \in N^{1,1}(\Omega, \mathbb{S}^2)$; second, the determinant $J(D\psi)$ of the differential $D\psi$ is nonnegative and integrable; lastly, there exists a function $1 \leq K'_\psi < \infty$ for which*

$$|D\psi|_g^2 \leq K'_\psi J(D\psi) \quad \mathcal{H}^2_{\mathbb{S}^2}\text{-a.e. in } \Omega. \tag{33}$$

Here $|D\psi|_g$ refers to the operator norm of the differential $D\psi$. We let K_ψ denote a smallest Borel function which is bounded from below by χ_Ω and for which (33) holds.

Definition 6.2. *A smooth strictly increasing function $\mathcal{A} \colon [1, \infty) \to [0, \infty)$ is* admissible *if*

(1) $\mathcal{A}(1) = 0$,
(2) $\int_1^\infty t^{-2} \mathcal{A}(t) \, d\mathcal{L}^1(t) = \infty$, and
(3) $t \mapsto t\mathcal{A}'(t)$ is increasing for large values t, and converges to ∞ as $t \to \infty$.

We obtain the same class of admissible $\mathcal{A}$ if we replace (2) with the condition

$$\int_1^\infty t^{-1} \mathcal{A}'(t) \, d\mathcal{L}^1(t) = \infty.$$

This follows from the fact that $\mathcal{A}(s)/s \leq 4 \int_s^{2s} t^{-2} \mathcal{A}(t) \, d\mathcal{L}^1(t)$ whenever $s \geq 1$ and the integration by parts formula.

Definition 6.3. *Let $\Omega, \Omega' \subset \mathbb{S}^2$ be open, and $\psi \colon \Omega \to \Omega'$ a homeomorphism. We say that ψ is* admissible *if ψ is a mapping of finite distortion and there exists an admissible $\mathcal{A}$ with*

$$\int_\Omega e^{\mathcal{A}(K_\psi)} \, d\mathcal{H}^2_{\mathbb{S}^2} < \infty. \tag{34}$$

If $\mathcal{A}(t) = pt - p$ for some $p > 0$, we say that ψ has exponentially integrable distortion.

We recall some properties of such ψ. First, ψ satisfies Lusin's Condition (N) [KKM+03, Theorem 1.1]. Second, $\psi^{-1} \in N^{1,2}(\Omega', \Omega)$ [KO06, Corollary 1.2]; this implies that ψ^{-1} satisfies Lusin's Condition (N) [AIM09, Theorem 3.3.7]. Third, the Jacobian $J(D\psi)$ appearing on the right-hand side of (33) coincides with the Jacobian J_ψ we defined in Section 2.2 [KKM+03].

In this section, we show the following theorem.

Theorem 6.4. *Suppose that $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism, g^{-1} absolutely continuous, and there exists a homeomorphism $\psi \colon \overline{Z}_2 \to \overline{Z}_2$ extending g with $\psi|_{Z_2}$ admissible. Then $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$.*

Note that Theorem 1.5 is a consequence of Theorem 6.4 so it suffices to verify Theorem 6.4.

Definition 6.5. *Given $x_0 \in \mathbb{S}^1$ and $\pi > R_0 > 0$, set $\widetilde{Q} := \overline{B}_{\mathbb{S}^2}(x_0, R_0) \subset \mathbb{S}^2$. We define $H(x) = \widetilde{\iota}_1(x)$ if $x \in \widetilde{Q} \cap \overline{Z}_1$ and $\widetilde{\iota}_2 \circ \psi(x)$ if $x \in \widetilde{Q} \cap Z_2$, and denote $\widetilde{R} = H(\widetilde{Q}) \subset \widetilde{Z}$.*

Proposition 6.6. *If $\widetilde{R}$ and H are as in Definition 6.5, then H is a homeomorphism and there exists a 1-quasiconformal homeomorphism $f = (u, v) \colon \widetilde{R} \to [0,1] \times [0, M]$ for some $M > 0$.*

Proof of Theorem 6.4 assuming Proposition 6.6. We cover the seam in $\widetilde{Z}$ by the interiors of $\widetilde{R}$ as in Definition 6.5. This implies that $\widetilde{Z}$ can be covered by quasiconformal images of planar domains, and the quasiconformal equivalence of $\widetilde{Z}$ and $\mathbb{S}^2$ follows from Theorem 2.7. □

The following lemma is a key step in proving Proposition 6.6.

Lemma 6.7. *The H from Definition 6.5 is a homeomorphism, $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$ and $H^{-1} \in N^{1,2}(\widetilde{R}, \widetilde{Q})$. Furthermore, H satisfies Lusin's Conditions (N) and (N^{-1}).*

Proof. The absolute continuity of g^{-1} implies for the Lebesgue decomposition $g^*\mathcal{H}^1 = v_g\mathcal{H}^1 + \mu^\perp$ that $\{v_g = 0\}$ has negligible $\mathcal{H}^1_{\mathbb{S}^1}$-measure in an open neighbourhood of $\mathbb{S}^1 \cap \widetilde{Q}$. Then Proposition 3.6 and Lemma 3.2 imply that H is a homeomorphism.

We recall from Lemma 3.3 the fact that the inclusion maps $\widetilde{\iota}_1|_{Z_1} \colon Z_1 \to \widetilde{Z}$ and $\widetilde{\iota}_2|_{Z_2} \colon Z_2 \to \widetilde{Z}$ are 1-Lipschitz local isometries. This implies that H and its inverse are absolutely continuous in measure; the seam has negligible Hausdorff 2-measure.

In the following proof, we write $\widetilde{\rho}_i$ for functions defined on $\widetilde{Q} \cap Z_i \subset \mathbb{S}^2$ and $\rho_i = (\widetilde{\rho}_i \circ \widetilde{\iota}_i^{-1})$ on $\widetilde{R} \cap \widetilde{\iota}_i(Z_i) \subset \widetilde{Z}$ for $i = 1, 2$.

Since $\psi^{-1} \in N^{1,2}(\widetilde{Q} \cap Z_2, \mathbb{S}^2)$, for $i = 1, 2$, there exists an upper gradient $\widetilde{\rho}_i \in L^2(\widetilde{Q} \cap Z_i)$ of $H^{-1} \circ \widetilde{\iota}_i|_{Z_i \cap \widetilde{Q}}$ for $i = 1, 2$. We fix such functions and denote $\rho := \chi_{\widetilde{R} \cap \widetilde{\iota}_1(Z_1)}\rho_1 + \chi_{\widetilde{R} \cap \iota_2(Z_2)}\rho_2 \in L^2(\widetilde{R})$.

Let Γ_0 denote the collection of non-constant paths on $\widetilde{R} \subset \widetilde{Z}$ which have positive length in the seam $Q(S_Z)$ or along which ρ fails to be integrable. Since $\rho + \infty \cdot \chi_{Q(S_Z)}$ is L^2-integrable, Lemma 2.1 yields $\operatorname{mod} \Gamma_0 = 0$.

Consider next an absolutely continuous path $\gamma \colon [0, 1] \to \widetilde{R}$ in the complement of Γ_0. Then $\theta = H^{-1} \circ \gamma$ is such that $\mathcal{H}^1_{\mathbb{S}^2}(|\theta| \cap \mathbb{S}^1) = 0$. Indeed, since γ has zero length in the seam, the area formula (4) implies $\mathcal{H}^1_{\widetilde{Z}}(|\gamma| \cap Q(S_Z)) = 0$. This implies $\mathcal{H}^1_{\mathbb{S}^1}(|\theta| \cap \mathbb{S}^1) = 0$ due to Proposition 3.6 and the absolute continuity of $g|^{-1}_{\mathbb{S}^1 \cap \widetilde{Q}}$. Since $\mathcal{H}^1_{\mathbb{S}^1}(|\theta| \cap \mathbb{S}^1) = 0$, the assumptions of Lemma 2.2 are satisfied. Hence

$$\ell(\theta) \leq \int_\gamma \rho \, ds < \infty.$$

This implies that H^{-1} has an L^2-integrable weak gradient, so $H^{-1} \in N^{1,2}(\widetilde{R}, \widetilde{Q})$.

Lastly, we claim that $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$. To this end, we observe that $H|_{\widetilde{Q} \cap Z_i}$ has an upper gradient $\widetilde{\rho}_i \in L^1(\widetilde{Q} \cap Z_i)$, and denote $\widetilde{\rho} = \sum_{i=1}^2 \chi_{\widetilde{Q} \cap Z_i}\widetilde{\rho}_i \in L^1(\widetilde{Q})$. Now $\widetilde{\rho}$ is integrable along 1-almost every absolutely continuous path $\gamma \colon [0, 1] \to \widetilde{Q}$ and 1-almost every such path has zero length in $\mathbb{S}^1$. Having fixed a path γ with these properties, Proposition 3.6 implies that $\theta = H \circ \gamma$ has zero length in the seam. The inequality $\ell(\theta) \leq \int_\gamma \rho \, ds$ follows from Lemma 2.2. This yields that $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$. □

Remark 6.8. *The Sobolev regularity $H^{-1} \in N^{1,2}(\widetilde{Q}, \widetilde{R})$ is crucial in the following. Typically, the Sobolev regularity of the inverse of a Sobolev homeomorphism is a subtle issue in the metric surface setting.*

To highlight the issue, we recall [IRar, Example 6.1]. *There an example of a metric surface X was constructed for which there exists a 1-Lipschitz homeomorphism $H \colon \mathbb{R}^2 \to X$ with $\operatorname{mod} \Gamma \leq \operatorname{mod} H\Gamma$ for all path families, but $H^{-1} \notin N^{1,2}(X, \mathbb{R}^2)$. In fact, H is a* *local isometry outside a Cantor set $E \subset \mathbb{R} \times \{0\}$ of positive $\mathcal{L}^1$-measure and $H(E)$ has negligible $\mathcal{H}^1_X$-measure. The key point is that X is not reciprocal; recall Definition 2.5.*

We define the following auxiliary function for later use:

$$P(t) := \begin{cases} t^2, & 0 \le t < 1, \\ \dfrac{t^2}{\mathcal{A}^{-1}(\log t^2)}, & t \ge 1. \end{cases}$$

We note that for every $a \in [0, \infty)$,

$$P(a) \le e^{\mathcal{A}(K_H)} + \frac{a^2}{K_H} \quad \text{for } \mathcal{H}^2_{\mathbb{S}^2}\text{-a.e. in } \widetilde{Q}. \tag{35}$$

This follows by first observing that $a^2 < e^{\mathcal{A}(K_H)}$ implies $P(a) \le e^{\mathcal{A}(K_H)}$ and otherwise $P(a) \le \frac{a^2}{K_H}$.

Also, for any measurable function $\widetilde{\rho} \colon \widetilde{Q} \to [0, \infty]$,

$$\int_{\widetilde{Q}} P(\widetilde{\rho})\, d\mathcal{H}^2_{\mathbb{S}^2} < \infty \quad \text{implies} \quad \int_{\widetilde{Q}} \widetilde{\rho}\, d\mathcal{H}^2_{\mathbb{S}^2} < \infty. \tag{36}$$

The implication (36) follows since $\mathcal{A}'(t)t$ is increasing for large t and converges to infinity as $t \to \infty$. Consequently, there exists $t_1 \ge 1$ for which the derivative of $h(t) = e^{\mathcal{A}(t)}/t^2$ is bounded from below by $h(t)/t$ for every $t \ge t_1$. This implies the existence of $t_0 \ge 1$ such that $h(t) \ge 1$ for every $t \ge t_0$. This is equivalent to saying that $P(t) \ge t$ for every $t \ge t_0$. This yields (36).

We set $K_\psi(x) = |D\psi|^2_g / J(D(\psi))(x)$ and $K_{\psi^{-1}}(x) = \left|D(\psi^{-1})\right|^2_g / J(D(\psi^{-1}))$. Observe that $K_\psi = K_{\psi^{-1}} \circ \psi$ $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere.

We set $K_H(x) = 1$ if $x \in \widetilde{Q} \cap \overline{Z}_1$ and $K_H(x) = K_\psi(x)$ in $x \in \widetilde{Q} \cap Z_2$. Then

$$\int_{\widetilde{Q}} e^{\mathcal{A}(K_H)}\, d\mathcal{H}^2_{\mathbb{S}^2} < \infty. \tag{37}$$

Also, $K_{H^{-1}} := \rho^2_{H^{-1}} / J_{H^{-1}}$ satisfies $K_H = K_{H^{-1}} \circ H$ $\mathcal{H}^2_{\widetilde{Z}}$-almost everywhere, since, outside a $\mathcal{H}^2_{\widetilde{Z}}$-negligible set, either the number is one, or $\rho^2_{H^{-1}} \circ \widetilde{\iota}_2 = \left|D(\psi^{-1})\right|^2_g$, $J_{H^{-1}} \circ \widetilde{\iota}_2 = J(D(\psi^{-1}))$, and $K_\psi = K_{\psi^{-1}} \circ \psi$.

For every $z \in \widetilde{Q}$ and every pair $0 < r < r_0$, we denote $\Gamma(z, r, r_0) := \Gamma(\overline{B}_{\mathbb{S}^2}(z, r), \widetilde{Q} \setminus B_{\mathbb{S}^2}(z, r_0); \widetilde{Q})$.

Lemma 6.9. *For every $z \in \widetilde{Q}$ and $0 < r < r_0$ with $\widetilde{Q} \setminus B_{\mathbb{S}^2}(z, r_0) \neq \emptyset$,*

$$\operatorname{mod} H\Gamma(z, r, r_0) \le \inf \left\{ \int_{\widetilde{Q}} \widetilde{\rho}^2 K_H\, d\mathcal{H}^2_{\mathbb{S}^2} \colon \ \widetilde{\rho} \text{ is admissible for } \Gamma(z, r, r_0) \right\}. \tag{38}$$

Proof. Fix an admissible function $\widetilde{\rho}$ for $\Gamma(z, r, r_0)$. Then for almost every $\gamma \in H\Gamma(z, r, r_0)$, $H^{-1} \circ \gamma$ is absolutely continuous, and

$$1 \le \int_{H^{-1} \circ \gamma} \widetilde{\rho}\, ds \le \int_\gamma (\widetilde{\rho} \circ H^{-1}) \rho_{H^{-1}}\, ds.$$

In particular, $\rho = (\widetilde{\rho} \circ H^{-1}) \rho_{H^{-1}}$ is weakly admissible for $H\Gamma(z, r, r_0)$. Consequently,

$$\operatorname{mod} H\Gamma(z, r, r_0) \le \int_{\widetilde{R}} \rho^2\, d\mathcal{H}^2_{\widetilde{Z}}.$$

The change of variables formula for H and the fact that the seam $Q(S_Z)$ is $\mathcal{H}^2_{\widetilde{Z}}$-negligible establish the claim, after taking the infimum over such $\widetilde{\rho}$. □

Having observed Lemma 6.9 and (37), the capacitary estimate [KO06, Theorem 5.3] implies that keeping r_0 fixed in (38), we obtain $\operatorname{mod} H\Gamma(z, r, r_0) \to 0$ as $r \to 0^+$. A key point is that $\mathcal{A}$ in (37) is admissible. Since H is a homeomorphism, this implies that (7) holds for every $y \in \operatorname{int}(\widetilde{R}) \subset \widetilde{Z}$. By repeating the argument with a slightly larger $\widetilde{Q}$, we conclude the following.

Lemma 6.10. *The identity (7) holds for every $y \in \widetilde{R} \subset \widetilde{Z}$.*

Fix a decomposition $\widetilde{\xi}_1, \widetilde{\xi}_2, \widetilde{\xi}_3, \widetilde{\xi}_4$ of $\partial\widetilde{Q}$ of four arcs overlapping only at their end points, labelled in cyclic order consistently with the orientation of $\mathbb{S}^2$. For each i, we denote $\xi_i = H(\widetilde{\xi}_i)$.

Given the validity of (7) for each $y \in \widetilde{R}$ and the universal lower bound (8), [Raj17, Proposition 9.1] yields the existence of a homeomorphism $f = (u, v) \colon \widetilde{R} \to [0,1] \times [0, M]$ with the following properties:

- $u \in N^{1,2}(\widetilde{R})$ with $2E(u) =: M$ [Raj17, Section 4];
- $u^{-1}(0) = \xi_1$, $u^{-1}(1) = \xi_3$, $v^{-1}(0) = \xi_2$, and $v^{-1}(M) = \xi_4$ [Raj17, Theorem 5.1 and Proposition 7.3];
- The minimal weak upper gradient ρ_u is weakly admissible for the path family $\Gamma(\xi_1, \xi_3; \widetilde{R})$ and is a minimizer, i.e., $M = \operatorname{mod}\Gamma(\xi_1, \xi_3; \widetilde{R})$ [Raj17, Section 4-5];
- For every Borel set $E \subset \widetilde{R}$, $\mathcal{L}^2(f(E)) = \int_E \rho_u^2 \, d\mathcal{H}^2_{\widetilde{Z}}$. In particular, the Jacobian of f coincides with ρ_u^2 [Raj17, Proposition 8.2].

The third point implies that if $u' \in N^{1,2}(\widetilde{R})$ has the same boundary values as u in $\xi_1 \cup \xi_3$, the Dirichlet energies satisfy $E(u) \leq E(u')$. Given this, we say that u is an *energy minimizer* for $\Gamma(\xi_1, \xi_3; \widetilde{R})$.

During the proof of Proposition 6.11, the Beltrami differential of H is defined to be zero in $\operatorname{int}(\widetilde{Q}) \cap \overline{Z_1}$, and coincide with the one of ψ in $\operatorname{int}(\widetilde{Q}) \cap Z_2$.

Proposition 6.11. *The map $f = (u, v) \colon \widetilde{R} \to [0,1] \times [0, M]$ is a 1-quasiconformal homeomorphism.*

The proof of Proposition 6.11 is split into several lemmas.

Lemma 6.12. *Let $0 < a < b < 1$ and $0 < c < d < M$ for which*

$$Q^0 = \left\{ x \in \widetilde{R} \colon f(x) \in [a,b] \times [c,d] \right\} \subset \operatorname{int}(\widetilde{R}) \setminus Q(S_Z).$$

Then $f|_{\operatorname{int}(Q^0)}$ is a 1-quasiconformal homeomorphism.

Proof. For the duration of the proof, we denote

$$\xi_1^0 = f^{-1}(\{a\} \times [c,d]), \quad \xi_2^0 = f^{-1}([0,1] \times \{c\}),$$
$$\xi_3^0 = f^{-1}(\{b\} \times [c,d]), \quad \xi_4^0 = f^{-1}([0,1] \times \{d\}).$$

There exists a Jordan domain $V \subset \operatorname{int}(Q) \cap Z_i$, for some $i = 1, 2$, such that $\widetilde{\iota}_i(\overline{V}) = Q^0$. Equation (57) [Raj17, Lemma 10.2] states that

$$\operatorname{mod}\Gamma(\xi_1^0, \xi_3^0; Q^0) = \frac{d-c}{b-a}.$$

Since $\widetilde{\iota}_i$ is 1-Lipschitz and a local isometry in $\overline{V}$, we have for every quadrilateral $Q' \subset Q^0$,

$$\operatorname{mod}\Gamma(\xi_1', \xi_3'; Q') \operatorname{mod}\Gamma(\xi_2', \xi_4'; Q') = 1. \tag{39}$$

In particular, we have

$$\operatorname{mod}\Gamma(\xi_1^0, \xi_3^0; Q^0) \operatorname{mod}\Gamma(\xi_2^0, \xi_4^0; Q^0) = 1. \tag{40}$$

We wish to apply [Raj17, Proposition 11.1]. There Rajala assumes that (6) holds for some $\kappa \geq 1$ and concludes that $2000 \cdot \sqrt{\kappa}\rho_u$ is a weak upper gradient of f. We do not assume this. However, a quick inspection of the proof shows that given any open set $\Omega \subset \text{int}(Q^0)$, the property (39) implies that $2000 \cdot \chi_{\text{int}(Q^0)} \cdot \rho_u$ is a weak upper gradient of $f|_{\text{int}(Q^0)}$ in Ω. By exhausting $\text{int}(Q^0)$ by such open sets, we conclude that $f|_{\text{int}(Q^0)} \in N^{1,2}(\text{int}(Q^0); \mathbb{R}^2)$.

Since $u \in N^{1,2}(\widetilde{R})$ is a continuous energy minimizer, the composition $u \circ \iota_i|_V$ is harmonic [AIM09, Weyl's lemma]. The Riemann mapping theorem, the Sobolev regularity of $f|_{\text{int}(Q^0)}$, the boundary values of the components of $f|_{Q^0}$, and (40) imply that $f \circ \iota_i|_V$ is a Riemann map. In particular, $f|_{\text{int}(Q^0)}$ is a 1-quasiconformal homeomorphism. □

Lemma 6.13. *The composition $\widetilde{f} = f \circ H \colon \widetilde{Q} \to [0,1] \times [0,M]$ is an element of $N^{1,1}(\text{int}(\widetilde{Q}), \mathbb{R}^2)$. Moreover, the Beltrami differential of $\widetilde{f}$ coincides with the one of H and (37) holds for $K_{\widetilde{f}}$ in place of K_H.*

Proof. Given Lemma 6.12, the Beltrami differential of $\widetilde{f}$ and H coincide $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in $\text{int}(\widetilde{Q}) \setminus \mathbb{S}^1$, i.e., $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in $\text{int}(\widetilde{Q})$. The result also implies that the pointwise distortions of $\widetilde{f}$ and H coincide $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere in $\text{int}(\widetilde{Q})$.

Next, we show that $\widetilde{u} = u \circ H \in N^{1,1}(\widetilde{Q})$. We recall that $H \in N^{1,1}(\widetilde{Q}, \widetilde{R})$. Moreover, if $\rho_0 \in L^2(\widetilde{R})$ is an upper gradient of u, the function $\rho = (\rho_0 \circ H)\rho_H$ is a 1-weak upper gradient of $\widetilde{u}$ with

$$\int_{\widetilde{Q}} P(\rho)\, d\mathcal{H}^2_Z \leq \int_{\widetilde{Q}} e^{\mathcal{A}(K_H)}\, d\mathcal{H}^2_Z + \|\rho_0\|^2_{L^2(Q)} < \infty,$$

where we apply (35) and the distortion inequality $\rho_H^2 \leq K_H J_H$. The $L^1(\widetilde{Q})$-integrability of ρ follows from (36), so $\widetilde{u} \in N^{1,1}(\widetilde{Q})$.

Let $\widetilde{v} = v \circ H$. Lemma 6.12 implies that $\rho = (\rho_0 \circ H)\rho_H \in L^1(\widetilde{Q})$ is a 1-weak upper gradient of $\widetilde{v}$ in every open $U \subset \text{int}(\widetilde{Q}) \setminus \mathbb{S}^1$. Therefore, $\widetilde{v} \in N^{1,1}(\text{int}(\widetilde{Q}) \setminus \mathbb{S}^1)$. Given the continuity of $\widetilde{v}$, we actually have $\widetilde{v} \in N^{1,1}(\text{int}(\widetilde{Q}))$. This is seen by verifying the ACL (absolute continuity on lines) property for $\widetilde{v}|_{\text{int}(\widetilde{Q})}$ on charts covering $\mathbb{S}^1 \cap \text{int}(\widetilde{Q})$. The ACL property on charts follows from a minor modification of the proof in [Väi71, Theorem 35.1] showing that closed sets with σ-finite Hausdorff 1-measure are quasiconformally removable. This implies that ρ is a 1-weak upper gradient of $\widetilde{v}$ on $\text{int}(\widetilde{Q})$. The claim follows from this. □

Lemma 6.14. *Let u' denote the energy minimizer for $\Gamma(\xi_2, \xi_4; Q)$. Then $v = Mu'$.*

Proof. Similarly to f and $\widetilde{f}$, let $f' = (u', v')$ and $\widetilde{f}'$ denote the homeomorphisms obtained from the energy minimizer u' for $\Gamma(\xi_2, \xi_4; \widetilde{R})$. Let R' denote the image of f' and R the image of f.

Lemma 6.13 shows that the Beltrami differentials of $\widetilde{f}$ and $\widetilde{f}'$ coincide with one another $\mathcal{H}^2_{\mathbb{S}^2}$-almost everywhere and their distortion satisfies (37) for an admissible $\mathcal{A}$. Then the Stoilow factorization theorem [AIM09, Theorems 20.5.1, 20.5.2] implies that $\varphi = \widetilde{f}' \circ \widetilde{f}^{-1}$ is conformal; note also that $\varphi = f' \circ f^{-1}$.

Since φ is conformal, the energy minimizer π_1 for $\Gamma(f'(\xi_2), f'(\xi_4); R')$ is such that $\pi_1 \circ \varphi$ is the energy minimizer for $\Gamma(f(\xi_2), f(\xi_4); R)$. On the other hand, here π_1 is the projection to the x-axis and $\pi_1 \circ \varphi$ is M^{-1} times the projection to the y-axis. Since $\varphi = f' \circ f^{-1}$, the equality $u' = \pi_1 \circ \varphi \circ f = M^{-1} v$ follows. □

Proof of Proposition 6.11. Lemma 6.14 implies that $f = (u, v) \in N^{1,2}(\widetilde{R}, \mathbb{R}^2)$. Furthermore, Lemma 6.12 implies $\rho_f^2 = J_f \in L^1(\widetilde{R})$. Hence $\operatorname{mod}\Gamma \leq \operatorname{mod} f\Gamma$ for every path family in $\widetilde{R}$. This improves to K-quasiconformality for some $K \geq 1$ due to Proposition 2.8. As $f(Q(S_Z) \cap \widetilde{R})$ is negligible due to the change of variables formula for f, and as f^{-1} is 1-quasiconformal outside $f(S_Z \cap \widetilde{R})$, we immediately obtain $\operatorname{mod}\Gamma \leq \operatorname{mod} f^{-1}\Gamma$ for every path family in $f(\widetilde{R})$. Thus f is 1-quasiconformal. □

Proof of Proposition 6.6. This is proved by Proposition 6.11. □

Remark 6.15. *Notice that if Lemma 6.10 holds for a given homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ having an admissible extension, even without assuming the absolutely continuity of g^{-1}, the rest of the proof of Proposition 6.11 (and Proposition 6.6) go through the same way.*

Proof of Proposition 1.4. Given a quasisymmetry $g\colon \mathbb{S}^1 \to \mathbb{S}^1$, its *Beurling–Ahlfors* extension $\psi\colon \overline{Z}_2 \to \overline{Z}_2$ is a quasisymmetry and $\psi|_{Z_2}$ is K-quasiconformal for some $K \geq 1$ [BA56]. Thus, if g^{-1} is absolutely continuous, g satisfies the assumptions of Theorem 1.5. Alternatively, if H is as in Definition 6.5, Lemma 6.9 implies that H^{-1} has outer dilatation $K_O(H^{-1}) \leq K$. Proposition 2.8 implies that H is quasiconformal; this self-improves to K-quasiconformality. Clearly H extends to a K-quasiconformal homeomorphism $H\colon \mathbb{S}^2 \to \widetilde{Z}$. □

7. Concluding remarks

7.1. **A point of positive capacity.** For a general orientation-preserving homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$, the $\widetilde{Z}$ can have points of positive capacity (in the sense that (7) can fail) even if g is locally bi-Lipschitz in the complement of a single point. For example, having fixed arbitrary $1 < \alpha < \beta$, we consider the homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x^{\alpha}, & x \geq 0, \\ -(-x)^{\beta}, & x < 0. \end{cases} \tag{41}$$

We construct a homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ by restricting h to the interval $[-1, 1]$, extending the restriction to $\mathbb{R}$ periodically, and by considering the covering map $\theta(t) = (\cos(\pi t), \sin(\pi t), 0)$, and a homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $g \circ \theta = \theta \circ h^{-1}$. Then g^{-1} is an L-Lipschitz homeomorphism for some $L \geq 1$, and one can check directly from the definition of d_Z that the inclusion map $\widetilde{\iota}_1\colon \overline{Z}_1 \to \widetilde{Z}$ is L-bi-Lipschitz onto its image.

Let $x_0 \in \widetilde{Z}$ denote the point corresponding to $(1, 0, 0)$. By using the techniques from Section 5, we can show that $\widetilde{Z} \setminus \{x_0\}$ can be covered by bi-Lipschitz images of planar domains. Then [Iko21b, Theorem 1.3] implies that $\widetilde{Z} \setminus \{x_0\}$ is 1-quasiconformally equivalent to a Riemannian surface (that is homeomorphic to a planar domain). Such a Riemannian surface can be conformally embedded into $\mathbb{S}^2$ [AS60, Section III.4]. Hence there exists a 1-quasiconformal embedding $\psi\colon \widetilde{Z} \setminus \{x_0\} \to \mathbb{S}^2$.

We claim that the complement of the image of ψ is a non-trivial continuum (which is equivalent to the failure of (7) at x_0). Indeed, otherwise ψ would extend to a 1-quasiconformal homeomorphism and g would be a welding homeomorphism, as a consequence of Theorem 1.6. This would contradict both [Oik61, Example 1] and [Vai89, Theorem 3], where both of these result show that g is not a welding homeomorphism.

In contrast, if we set $\alpha = \beta \geq 1$ in (41), the homeomorphism g is a quasisymmetry, so $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$, as a consequence of Proposition 1.4.

7.2. **Points of positive capacity.** We construct another example for which points of positive capacity occur. To this end, consider a Cantor set $E \subset [0,1]$ and

$$h(x) = \begin{cases} (\mathcal{L}^1([0,1] \setminus E))^{-1} \int_0^x \chi_{\mathbb{R} \setminus E}(y) \, d\mathcal{L}^1(y), & 0 \leq x \leq 1, \\ x, & \text{otherwise.} \end{cases} \tag{42}$$

Then $h\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz homeomorphism coinciding with the identity map outside $(0,1)$.

Next, consider the Möbius transformation $\theta_1(z) = (z-i)/(z+i)$ from the upper half-space $\overline{\mathbb{H}}$ onto the Euclidean unit disk $\overline{\mathbb{D}}$. Let $\theta_2(x,y) = (2x/(1+x^2+y^2), 2y/(1+x^2+y^2), (1-x^2-y^2)/(1+x^2+y^2))$. Then $\theta := \theta_2 \circ \theta_1 \colon \overline{\mathbb{H}} \to \overline{Z}_2$ defines a 1-quasiconformal homeomorphism, given that θ_2^{-1} is a(n orientation-reversing) stereographic projection.

There exists a unique homeomorphism $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $g \circ \theta = \theta \circ h^{-1}$. We see from (42) that g^{-1} is L-Lipschitz and $\widetilde{\iota}_1$ is L-bi-Lipschitz with a constant L depending only on $\mathcal{L}^1(E)$. In particular, $\widetilde{Z} = (Z, d_Z)$.

We denote $E' = \widetilde{\iota}_2(\theta(E)) \subset \widetilde{Z}$, and apply [Iko21b, Theorem 1.3] as in Section 7.1, and find a 1-quasiconformal embedding $\psi\colon \widetilde{Z} \setminus E' \to \mathbb{S}^2$.

Consider on $\mathbb{R}^2$ the distance d_E obtained as follows: For each absolutely continuous $\gamma\colon [0,1] \to \mathbb{R}^2$, denote $\ell_E(\gamma) := \int_\gamma \chi_{\mathbb{R}^2 \setminus E} \, ds$. We set $d_E(x,y) = \inf \ell_E(\gamma)$, the infimum taken over absolutely continuous paths joining x to y.

We denote $X = (\mathbb{R}^2, d_E)$. The change of distance map $H\colon \mathbb{R}^2 \to X$ is a 1-Lipschitz homeomorphism that is a local isometry on $\mathbb{R}^2 \setminus E$. Moreover, if $\theta\colon [0,1] \to \mathbb{R}^2$ is absolutely continuous, the metric speeds satisfy

$$v_{H\circ\theta} = \left(\chi_{\mathbb{R}^2 \setminus E} \circ \theta\right) \cdot v_\theta \quad \mathcal{L}^1\text{-almost everywhere.} \tag{43}$$

The composition $G = \widetilde{\iota}_2 \circ \theta \circ \left(H|_{[-1,2]\times[0,1]}\right)^{-1}$ is a 1-quasiconformal homeomorphism. This follows from Lemma 2.2, the equalities $\mathcal{H}^1_{\widetilde{Z}}(E') = 0 = \mathcal{H}^1_X(H(E))$, together with Proposition 3.6 and (43).

We consider a Cantor set E obtained from [IRar, Example 6.1]. The key property of E is the following: there exists a path family Γ on $[0,1]^2$, each path joining $(0,0)$ to $(1,0)$, such that $\operatorname{mod} H\Gamma \geq (4\pi)^{-1}$ and $\operatorname{mod} \Gamma = 0$. Given that G is 1-quasiconformal, the points $\widetilde{\iota}_2(\theta(x))$, where $x = (0,0), (1,0)$, fail (7). Consequently, $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$, and the embedding ψ does not have a quasiconformal extension $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$.

Question 7.1. *Are there Cantor sets E with $\mathcal{L}^1(E) > 0$ such that a quasiconformal embedding $\psi\colon \widetilde{Z} \setminus E' \to \mathbb{S}^2$ extends to a quasiconformal homeomorphism $\Psi\colon \widetilde{Z} \to \mathbb{S}^2$?*

Given a compact set $F \subset Y$ with $Y = \mathbb{R}^2$ or $Y = \mathbb{S}^2$, we say that F has *zero absolute area* if every 1-quasiconformal embedding $f\colon Y \setminus F \to \mathbb{S}^2$ satisfies $\mathcal{H}^2_{\mathbb{S}^2}(\mathbb{S}^2 \setminus f(Y \setminus F)) = 0$.

We expect that the quasiconformal extension Ψ exists if and only if the set $F = \mathbb{S}^2 \setminus \psi(\widetilde{Z} \setminus E')$ has zero absolute area; the "only if"-direction follows by applying the techniques used in Section 4, by noting that the composition $(f \circ \psi)^{-1}$ has a continuous, monotone, and surjective extension $\widetilde{\pi}$ with $\operatorname{mod} \Gamma \leq \operatorname{mod} \widetilde{\pi}\Gamma$ for all path families. We expect that the "if"-direction follows from [IRar, Theorems 1.3 and 1.4, together with Lemma 5.1].

If E in Question 7.1 has zero absolute area, [IRar, Theorem 1.3] implies that the change of distance map H is a 1-quasiconformal homeomorphism. Given that the G above is 1-quasiconformal, one readily verifies that $\widetilde{\iota}_2$ is a 1-quasiconformal homeomorphism onto its image. We ask the following.

Question 7.2. *Let E, g, and ψ be as in Question 7.1. If $\widetilde{\iota}_2 \colon \overline{Z}_2 \to \widetilde{Z}$ is a 1-quasiconformal parametrization of its image, does $\psi \colon \widetilde{Z} \setminus E' \to \mathbb{S}^2$ extend to a quasiconformal homeomorphism $\Psi \colon \widetilde{Z} \to \mathbb{S}^2$? In particular, if E has zero absolute area, does $F = \mathbb{S}^2 \setminus \psi(\widetilde{Z} \setminus E')$ have zero absolute area?*

It follows from [Iko21a, Theorem 1.1 and Proposition 1.2] that the inclusion map $\widetilde{\iota}_2$ is a 1-quasiconformal homeomorphism if and only if there exists a quasiconformal homeomorphism $h \colon \widetilde{\iota}_2(\overline{Z}_2) \to \overline{\mathbb{D}}$ where $\overline{\mathbb{D}}$ is the closed Euclidean unit disk.

7.3. **Welding homeomorphisms.** We consider a welding homeomorphism $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ with welding curve $\mathcal{C} \subset \mathbb{S}^2$. Consider the monotone mapping $\widetilde{\pi} \colon \mathbb{S}^2 \to \widetilde{Z}$ obtained from (14).

Question 7.3. *If $\widetilde{\pi}$ is a homeomorphism, is it a 1-quasiconformal homeomorphism?*

We showed in Proposition 4.1 that if $\widetilde{\pi}$ is not a homeomorphism, then $\widetilde{Z}$ is not quasiconformally equivalent to $\mathbb{S}^2$; the collapsing creates points of positive capacity — by which we mean that (7) fails — in $\widetilde{Z}$. Question 7.3 asks if the collapsing is the only obstruction for quasiconformal uniformization. Lemma 4.8 reduces the question to understanding when $\widetilde{\pi}^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$.

7.4. **Quasisymmetries.** Observe that the assumptions of Proposition 1.4 are satisfied by every quasisymmetry $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ that is *strongly quasisymmetric* [Sem86] [Bis88] [AZ91] [BJ94]: for every $\epsilon > 0$ there exists $\delta > 0$ such that for every subarc $I \subset \mathbb{S}^1$ and Borel set $E \subset I$,

$$\mathcal{H}^1_{\mathbb{S}^1}(E) \leq \delta \mathcal{H}^1_{\mathbb{S}^1}(I) \quad \text{implies} \quad \mathcal{H}^1_{\mathbb{S}^1}(g(E)) \leq \epsilon \mathcal{H}^1_{\mathbb{S}^1}(g(I)).$$

The welding curves corresponding to strongly quasisymmetric homeomorphisms are special cases of the *asymptotically conformal* quasicircles; see [Pom78]. One might ask whether or not $\widetilde{Z}$ is quasiconformally equivalent to $\mathbb{S}^2$ whenever g is a welding homeomorphism corresponding to such a curve. Corollary 4 of [Pom78] provides us with an example of asymptotically conformal quasicircle $\mathcal{C}$ which has an uncountable number of tangent points, with the tangent points dense in $\mathcal{C}$, but they also have zero 1-dimensional Hausdorff measure.

Lemma 7.4. *There exists a quasisymmetric $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ with asymptotically conformal welding curve $\mathcal{C}$ such that $\widetilde{Z}$ is not homeomorphic to $\mathbb{S}^2$.*

Lemma 7.4 follows from Proposition 4.1, Lemma 4.5, and the cited example.

Question 7.5. *Is the answer to Question 7.3 yes if we also assume that $g \colon \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetry?*

To answer Question 7.5 negatively, one needs to construct a quasisymmetry $\psi \colon \overline{Z}_2 \to \overline{Z}_2$, with $g = \psi|_{\mathbb{S}^1}$, for which the measures $g^*\mathcal{H}^1_{\mathbb{S}^1}$ and $\mathcal{H}^1_{\mathbb{S}^1}$ are not mutually singular in any subarc $I \subset \mathbb{S}^1$, yet the corresponding $\widetilde{Z}$ is not quasiconformally equivalent to $\widetilde{Z}$. Equivalently, one only needs to show that the homeomorphism $H \colon \mathbb{S}^2 \to \widetilde{Z}$, coinciding with $\widetilde{\iota}_1$ in Z_1 and with $\widetilde{\iota}_2 \circ \psi$ in Z_2, is not quasiconformal. By arguing as in the proof of Lemma 4.8, one sees that H is quasiconformal if and only if $H^{-1} \in N^{1,2}(\widetilde{Z}, \mathbb{S}^2)$.

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