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## On the second-order regularity of solutions to the parabolic $p$ -Laplace equation

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*Abstract.* In this paper, we study the second-order Sobolev regularity of solutions to the parabolic  $p$ -Laplace equation. For any  $p$ -parabolic function  $u$ , we show that  $D(|Du|^{\frac{p-2+s}{2}} Du)$  exists as a function and belongs to  $L^2_{loc}$  with  $s > -1$  and  $1 < p < \infty$ . The range of  $s$  is sharp.

### 1. Introduction

The elliptic  $p$ -Laplace equation has an extensive literature on the second-order regularity. In contrast, the second-order regularity for the parabolic  $p$ -Laplace equation

$$u_t = \operatorname{div}(|Du|^{p-2} Du) \quad (1.1)$$

is much less studied. Throughout the paper, we have  $1 < p < \infty$ . In the elliptic case, one of the known estimates shows  $W^{1,2}_{loc}$  regularity for the nonlinear expression of the gradient

$$|Du|^{\frac{p-2+s}{2}} Du$$

proven by Dong et al. [10] with  $s > 2 - \min\{p + \frac{n}{n-1}, 3 + \frac{p-1}{n-1}\}$ , and then extended to  $s > -1 - \frac{p-1}{n-1}$  by the third author [20]. The aim of this paper is to prove such a result to the parabolic  $p$ -Laplace equation. In other words, we prove in Theorem 2.2 that for any weak or viscosity solution  $u$  to (1.1),  $D(|Du|^{\frac{p-2+s}{2}} Du)$  exists, belongs to  $L^2_{loc}$  whenever  $s > -1$ , and this range is sharp.

In the parabolic case, Dong et al. [10] proved for  $p \in (1, 3)$  that the weak or viscosity solution  $u$  to (1.1) locally belongs to  $W^{2,2}$ . This result is obtained as a special case from ours by selecting  $s = 2 - p$ . Our result also contains as a special case Lindqvist's [16] result for  $|Du|^{p-2} Du$  and  $|Du|^{\frac{p-2}{2}} Du$  in the range  $p \geq 2$ . As a consequence, he also observed that the time derivative  $u_t$  exists and belongs to a Sobolev space. See also a recent paper by Cianchi and Maz'ya [5].

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The heuristic idea of the proof is to differentiate the equation (1.1), choose a test function  $\varphi = |Du|^s u_{x_k} \phi^2$  and use a fundamental inequality (the name stems from [10] for a related inequality)

$$|Du|^4 |D^2u|^2 \geq 2|Du|^2 |D^2u Du|^2 + \frac{(|Du|^2 \Delta u - \Delta_\infty u)^2}{n-1} - (\Delta_\infty u)^2$$

from [20], which holds for any smooth function. Here  $\Delta u := \sum_{i=1}^n u_{x_i x_i}$  denotes the Laplacian,  $\Delta_\infty u := \sum_{i,j=1}^n u_{x_i x_j} u_{x_i} u_{x_j}$  the infinity Laplacian, and  $|D^2u| := (\sum_{i,j=1}^n u_{x_i x_j}^2)^{1/2}$  the Hilbert–Schmidt norm for matrices. Surprisingly, it is sufficient for the sharp result to use the previous inequality in a rather simple form

$$|Du|^4 |D^2u|^2 \geq 2|Du|^2 |D^2u Du|^2 - (\Delta_\infty u)^2, \tag{1.2}$$

which we obtain by an elementary fact that the square  $(|Du|^2 \Delta u - \Delta_\infty u)^2$  is non-negative. Naturally the fact that the form (1.2) is sufficient, simplifies the proof. At the same time, the form of (1.2) makes the coefficient  $C = C(p, s)$  of estimate (2.3) in Theorem 2.2 independent of  $n$ . Note that in the elliptic case, the fundamental inequalities in [10, 20] include the parameter  $n$  so that both the range of  $s$  and  $C$  depend on  $n$ .

Unlike the second-order regularity, the lower-order regularity of the parabolic  $p$ -Laplace equation has been extensively studied since the 1980s, see DiBenedetto’s monograph [6] as well as, for example [2, 8, 9, 14, 15, 22, 23]. In the elliptic case, the second-order Sobolev regularity has been studied in addition to above mentioned [10], for example, in [1, 3, 18], and for a different parabolic equation in [11].

## 2. Preliminaries and main results

Let  $x_0 \in \mathbb{R}^n$ ,  $n \geq 1$  and  $r > 0$ . We denote by

$$B_r(x_0) = \{x \in \mathbb{R}^n : |x_0 - x| < r\}$$

the usual Euclidean ball in  $\mathbb{R}^n$ . For a space-time point  $(x_0, t_0) \in \mathbb{R}^{n+1}$  and a radius  $r > 0$ , we define the parabolic cylinder as

$$Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0 + r^2).$$

To ease the notation, we may write  $Q_r := Q_r(x_0, t_0)$ . Let  $\Omega \subset \mathbb{R}^n$  denote an open domain. For  $T > 0$ , we set

$$\Omega_T := \Omega \times (0, T).$$

If  $U$  is compactly contained in  $\Omega$ , i.e.,  $U \subset \Omega$  and the closure of  $U$  is a compact subset of  $\Omega$ , we write  $U \Subset \Omega$ . For  $0 < t_1 < t_2 < \infty$ , we set

$$U_{t_1, t_2} := U \times (t_1, t_2).$$

We denote the first partial derivatives of a function  $u : \Omega_T \rightarrow \mathbb{R}$  by  $u_{x_k}$  and  $u_t$ . The spatial gradient is denoted by  $Du$ , and the second derivatives by  $u_{x_i x_j}$ . Further,  $D^2u$  stands for matrix of second derivatives with respect to the space variables. As usual, the Sobolev space  $W^{1,p}(U)$  denotes the space of measurable functions  $u$  such that  $u \in L^p(U)$  and the distributional first partial derivatives  $u_{x_i}$  exist in  $U$  and belong to  $L^p(U)$ . We use the norm

$$\|u\|_{W^{1,p}(U)} = \|u\|_{L^p(U)} + \|Du\|_{L^p(U)}.$$

By the *parabolic Sobolev space*  $L^p(t_1, t_2; W^{1,p}(U))$ , with  $0 < t_1 < t_2 < \infty$ , we mean the space of measurable functions  $u(x, t)$  such that the mapping  $x \mapsto u(x, t)$  belongs to  $W^{1,p}(U)$  for almost every  $t_1 < t < t_2$  and the norm

$$\|u\|_{L^p(t_1, t_2; W^{1,p}(U))} := \left( \int_{t_1}^{t_2} \|u(\cdot, t)\|_{W^{1,p}(U)}^p dt \right)^{1/p}$$

is finite. The space  $C(\Omega_T)$  denotes the space of continuous functions on  $\Omega_T$  and  $C_0^\infty(\Omega_T)$  denotes the space of smooth, compactly supported functions on  $\Omega_T$ . A function belongs to the local Sobolev space  $W_{loc}^{1,p}(\Omega)$  if it belongs to  $W^{1,p}(\Omega')$  for every open  $\Omega' \Subset \Omega$ . Other local spaces are defined analogously.

We study weak solutions to the parabolic  $p$ -Laplace equation

$$u_t - \Delta_p u = 0 \quad \text{in } \Omega_T, \tag{2.1}$$

where

$$\Delta_p u := \operatorname{div} (|Du|^{p-2} Du)$$

is the  $p$ -Laplace operator with  $1 < p < \infty$ .

**Definition 2.1.** A function  $u : \Omega_T \rightarrow [-\infty, \infty]$  is a *weak solution* to Eq. (2.1) if whenever  $U_{t_1, t_2} \Subset \Omega_T$  is an open cylinder, we have  $u \in C(U_{t_1, t_2}) \cap L^p(t_1, t_2; W^{1,p}(U))$ , and  $u$  satisfies the integral equality

$$\int_0^T \int_\Omega |Du|^{p-2} \langle Du, D\phi \rangle dx dt - \int_0^T \int_\Omega u \phi_t dx dt = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega_T).$$

Such solutions are called  *$p$ -parabolic functions*.

Under the above definition, weak solutions are equivalent to viscosity solutions to (2.1) for  $1 < p < \infty$ , see [13, 19, 21]. In this setting, gradients are bounded and Hölder continuous by a recent work of Imbert et al. [12]. Their result covers a more general class of equations containing the parabolic  $p$ -Laplace equation and the normalized  $p$ -parabolic equation arising from the game theory [17]. For earlier  $C^{1,\alpha}$ -regularity results based on the variational approach with various assumptions, see DiBenedetto and Friedman [7], Wiegner [24], Chen [4], and Chapter IX in [6].

2.1. Main results

For  $s \in \mathbb{R}$ , we define the vector field  $V_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$  as

$$V_s(z) := \begin{cases} |z|^{\frac{p-2+s}{2}} z & \text{for } z \in \mathbb{R}^n \setminus \{0\}; \\ 0 & \text{for } z = 0. \end{cases} \tag{2.2}$$

**Theorem 2.2.** *Let  $u : \Omega_T \rightarrow \mathbb{R}$  be a weak solution to the parabolic  $p$ -Laplace equation (2.1). If  $s > -1$ , then  $D(V_s(Du))$  exists and belongs to  $L^2_{\text{loc}}(\Omega_T)$ . Moreover, we have the estimate*

$$\int_{Q_r} |D(V_s(Du))|^2 dxdt \leq \frac{C}{r^2} \left( \int_{Q_{2r}} |V_s(Du)|^2 dxdt + \int_{Q_{2r}} |Du|^{s+2} dxdt \right) \tag{2.3}$$

where  $C = C(p, s) > 0$  and  $Q_r \subset Q_{2r} \Subset \Omega_T$  are concentric parabolic cylinders.

Note that here the range of  $s$  and the coefficient  $C$  do not depend on  $n$ . For the parabolic case, the range of  $s$  must satisfy the constraints of both the elliptic and parabolic terms. We get the elliptic restriction  $s > -1 - \frac{p-1}{n-1}$  by [20] and  $s > -1$  rising from the parabolic terms (see Remark 2.4). By combining them, we get the restriction  $s > \max\{-1 - \frac{p-1}{n-1}, -1\} = -1$ ; thus, the inequality (1.2) is sufficient and further the coefficient  $C = C(p, s)$  is independent of  $n$ .

*Remark 2.3.* In particular, we may set  $s = 0$ , and  $s = p - 2$  for any  $1 < p < \infty$  reproving Lindqvist’s result in [16] for  $|Du|^{\frac{p-2}{2}} Du$  and  $|Du|^{p-2} Du$ . If  $1 < p < 3$ , we may set  $s = 2 - p$  to reprove the second-order Sobolev regularity obtained in [10].

*Remark 2.4* (Counter example). The counterexample from [10] turns out to work also in our case, and shows that the range  $s > -1$  in Theorem 2.2 is sharp. By a direct calculation, the function

$$u(x_1, x_2) = \left(\frac{p}{p-1}\right)^{p-1} t + |x_1|^{1+\frac{1}{p-1}}$$

is a solution to (2.1) in  $\mathbb{R}^2 \times (0, \infty)$ , and

$$|D(|Du|^{\frac{p-2+s}{2}} Du)| = C(p, s)|x_1|^{\frac{-p+2+s}{2(p-1)}} \in L^2_{\text{loc}}(\mathbb{R}^2 \times (0, \infty))$$

if and only if  $s > -1$ .

Indeed, we have

$$u_{x_1} = \frac{p}{p-1} |x_1|^{\frac{1}{p-1}-1} x_1, \quad u_{x_2} = 0,$$

and

$$u_{x_1 x_1} = \frac{p}{(p-1)^2} |x_1|^{\frac{1}{p-1}-1}, \quad u_{x_1 x_2} = u_{x_2 x_1} = u_{x_2 x_2} = 0.$$

Then,

$$\begin{aligned} |D(|Du|^{\frac{p-2+s}{2}} Du)| &= \left(\frac{p}{p-1}\right)^{\frac{p-2+s}{2}} |x_1|^{\frac{p-2+s}{2(p-1)}} \left| \frac{p}{(p-1)^2} |x_1|^{\frac{2-p}{p-1}} + \frac{p-2+s}{2} \frac{p}{(p-1)^2} |x_1|^{\frac{2-p}{p-1}} \right| \\ &= \left(\frac{p}{p-1}\right)^{\frac{p-2+s}{2}} \frac{p(p+s)}{2(p-1)^2} |x_1|^{\frac{2-p+s}{2(p-1)}} \\ &= C(p, s) |x_1|^{\frac{-p+2+s}{2(p-1)}}. \end{aligned}$$

Once we have proven the main result, Theorem 2.2, the existence and integrability of the time derivative easily follows as pointed out by Lindqvist [16] and Dong et al. [10]. We give the short proof for the convenience of the reader.

**Corollary 2.5** (Time derivative). *Let  $u : \Omega_T \rightarrow \mathbb{R}$  be a weak solution to the parabolic  $p$ -Laplace equation (2.1). Then, the time derivative  $u_t$  exists as a function and  $u_t \in L^2_{loc}(\Omega_T)$ .*

*Proof.* Let  $s = p - 2 > -1$ , then  $p + s = 2(p - 1) > 0$  and  $s + 2 = p > 1$ . By Theorem 2.2, for all  $Q_r \subset Q_{2r} \Subset \Omega_T$ , we have

$$\begin{aligned} &\int_{Q_r} |D(|Du|^{p-2} Du)|^2 dxdt \\ &\leq \frac{C(p)}{r^2} \left( \int_{Q_{2r}} |Du|^{2(p-1)} dxdt + \int_{Q_{2r}} |Du|^p dxdt \right), \end{aligned} \tag{2.4}$$

which implies

$$D(|Du|^{p-2} Du) \in L^2_{loc}(\Omega_T).$$

By the weak formulation

$$\int_{Q_r} u \phi_t dxdt = - \int_{Q_r} \operatorname{div}(|Du|^{p-2} Du) \phi dxdt \quad \text{for all } \phi \in C^\infty_0(Q_r),$$

we get that  $u_t$  exists, and  $u_t \in L^2_{loc}(\Omega_T)$ . □

### 3. Idea of the proof

In this section, for the convenience of the reader, we present the formal idea of the proof without excess details. In this setting, we assume that  $u \in C^\infty(\Omega_T)$  and  $Du \neq 0$ . The detailed proof is presented in Sect. 4.

Differentiating with respect to  $x_k$  in (2.1), we get

$$(u_{x_k})_t = \operatorname{div}(|Du|^{p-2} A Du_{x_k}) \tag{3.1}$$

where

$$A = I + (p - 2) \frac{Du \otimes Du}{|Du|^2}.$$

Here,  $I$  denotes the  $n \times n$  identity matrix and  $Du \otimes Du$  stands for the tensor product of two vectors in  $\mathbb{R}^n$ , resulting in a matrix in  $\mathbb{R}^{n \times n}$  with the entries  $u_{x_i} u_{x_j}$ .

We first study the term on the left hand side of (3.1) and choose a test function

$$\varphi = |Du|^s u_{x_k} \phi^2$$

with  $s > -1$  and  $\phi \in C_0^\infty(\Omega_T)$ . Summing over  $k$ , we get

$$\begin{aligned} \sum_{k=1}^n \int_{\Omega_T} (u_{x_k})_t \varphi dx dt &= \sum_{k=1}^n \int_{\Omega_T} (u_{x_k})_t |Du|^s u_{x_k} \phi^2 dx dt \\ &= \sum_{k=1}^n \int_{\Omega_T} \frac{1}{2} (u_{x_k}^2)_t |Du|^s \phi^2 dx dt \\ &= \int_{\Omega_T} \frac{1}{2} (|Du|^2)_t |Du|^s \phi^2 dx dt \\ &= \int_{\Omega_T} \frac{1}{s+2} (|Du|^{s+2})_t \phi^2 dx dt \\ &= - \int_{\Omega_T} \frac{1}{s+2} |Du|^{s+2} (\phi^2)_t dx dt. \end{aligned}$$

Recalling (3.1), we have

$$0 = \sum_{i=1}^n \int_{\Omega_T} \operatorname{div}(|Du|^{p-2} A Du_{x_i}) (|Du|^s u_{x_i} \phi^2) dx dt + \frac{2}{s+2} \int_{\Omega_T} |Du|^{s+2} \phi \phi_t dx dt.$$

Now the first integral on the right hand side is of the same form as in the elliptic case, and thus, the proof of [20, Lemma3.3] gives that for any  $\eta > 0$ ,

$$\begin{aligned} &\int_{\Omega_T} |Du|^{p-2+s} \left\{ |D^2u|^2 + (p-2+s-\eta) |D|Du|^2 + (s(p-2)-\eta) (\Delta_\infty^N u)^2 \right\} \phi^2 dx dt \\ &\leq \frac{C(p)}{\eta} \int_{\Omega_T} |Du|^{p+s} |D\phi|^2 dx dt + \frac{2}{s+2} \int_{\Omega_T} |Du|^{s+2} |\phi| |\phi_t| dx dt, \end{aligned} \tag{3.2}$$

where  $\Delta_\infty^N u := |Du|^{-2} \sum_{i,j=1}^n u_{x_i x_j} u_{x_i} u_{x_j}$  stands for the normalized or game theoretic infinity Laplacian. Observe that on the right hand side, we have bounded terms only. As a corollary, similarly as in [20, Corollary 3.4], we get

$$\begin{aligned} & \int_{\Omega_T} |Du|^{p-2+s} |D^2u|^2 \phi^2 dxdt \\ & \leq C(p, s, \eta) \int_{\Omega_T} |Du|^{p-2+s} |D|Du||^2 \phi^2 dxdt \\ & \quad + \frac{C(p)}{\eta} \int_{\Omega_T} |Du|^{p+s} |D\phi|^2 dxdt + \frac{2}{s+2} \int_{\Omega_T} |Du|^{s+2} |\phi||\phi_t| dxdt. \end{aligned} \tag{3.3}$$

Next we estimate the first term on the right hand side in (3.3). Using the inequality of [20, Corollary 2.2]:

$$|Du|^4 |D^2u|^2 \geq 2|Du|^2 |D^2uDu|^2 + \frac{(|Du|^2 \Delta u - \Delta_\infty u)^2}{n-1} - (\Delta_\infty u)^2,$$

dividing both sides by  $|Du|^4$ , we have

$$\begin{aligned} |D^2u|^2 & \geq 2|D|Du||^2 + \frac{(\Delta u - \Delta_\infty^N u)^2}{n-1} - (\Delta_\infty^N u)^2 \\ & \geq 2|D|Du||^2 - (\Delta_\infty^N u)^2. \end{aligned} \tag{3.4}$$

On the last line, we used  $(\Delta u - \Delta_\infty^N u)^2 \geq 0$ . Now we use the previous inequality in (3.2) for the term containing  $|D^2u|^2$ , set  $\eta = \min\{\frac{1}{4}(p+s), \frac{1}{6}(p-1)(s+1)\}$ , and obtain

$$\begin{aligned} & |D^2u|^2 + (p-2+s-\eta)|D|Du||^2 + (s(p-2)-\eta)(\Delta_\infty^N u)^2 \\ & \geq 2|D|Du||^2 - (\Delta_\infty^N u)^2 + (p-2+s-\eta)|D|Du||^2 + (s(p-2)-\eta)(\Delta_\infty^N u)^2 \\ & = (p+s-\eta)|D|Du||^2 + (s(p-2)-1-\eta)(\Delta_\infty^N u)^2 \\ & = \eta|D|Du||^2 + (p+s-2\eta)|D|Du||^2 + (s(p-2)-1-\eta)(\Delta_\infty^N u)^2 \\ & \geq \eta|D|Du||^2 + (p+s+s(p-2)-1-3\eta)(\Delta_\infty^N u)^2 \\ & = \eta|D|Du||^2 + ((p-1)(s+1)-3\eta)(\Delta_\infty^N u)^2 \\ & \geq \eta|D|Du||^2 = C(p, s)|D|Du||^2, \end{aligned} \tag{3.5}$$

whenever  $s > -1$ . We also used

$$|D|Du||^2 = \frac{|D^2uDu|^2}{|Du|^2} \geq \left( \frac{\langle D^2uDu, Du \rangle}{|Du|^2} \right)^2 = (\Delta_\infty^N u)^2. \tag{3.6}$$

Thus,

$$\begin{aligned} & \int_{\Omega_T} |Du|^{p-2+s} |D|Du||^2 \phi^2 dxdt \\ & \leq C(p, s) \left( \int_{\Omega_T} |Du|^{p+s} |D\phi|^2 dxdt + \int_{\Omega_T} |Du|^{s+2} |\phi||\phi_t| dxdt \right). \end{aligned}$$



Combining this with (3.3), we get

$$\begin{aligned} & \int_{\Omega_T} |Du|^{p-2+s} |D^2u|^2 \phi^2 dx dt \\ & \leq C(p, s) \left( \int_{\Omega_T} |Du|^{p+s} |D\phi|^2 dx dt + \int_{\Omega_T} |Du|^{s+2} |\phi| |\phi_t| dx dt \right). \end{aligned}$$

By a direct calculation

$$\int_{\Omega_T} |D(|Du|^{\frac{p-2+s}{2}} Du)|^2 \phi^2 dx dt \leq C(p, s) \int_{\Omega_T} |Du|^{p-2+s} |D^2u|^2 \phi^2 dx dt, \quad (3.7)$$

and combining this with the previous estimate, we finally get

$$\begin{aligned} & \int_{\Omega_T} |D(|Du|^{\frac{p-2+s}{2}} Du)|^2 \phi^2 dx dt \\ & \leq C(p, s) \left( \int_{\Omega_T} |Du|^{p+s} |D\phi|^2 dx dt + \int_{\Omega_T} |Du|^{s+2} |\phi| |\phi_t| dx dt \right). \quad (3.8) \end{aligned}$$

The estimate in Theorem 2.2 is obtained by choosing  $\phi$  as a standard cutoff function.

#### 4. Detailed proof

In this section, we present a detailed proof of Theorem 2.2 by regularizing Eq. (2.1). Solutions to the regularized equation will be smooth, and thus, the differentiation of this equation is justified. Since the obtained estimates will be uniform with respect to the regularization, we will be able to pass to the original equation at the end.

To start with the above plan, let  $u : \Omega_T \rightarrow \mathbb{R}$  be a  $p$ -parabolic function. Fix a smooth subdomain  $U \Subset \Omega$  and  $0 < t_1 < t_2 < \infty$  such that  $U_{t_1, t_2} \Subset \Omega_T$ . Let  $\epsilon > 0$  be small and  $u^\epsilon : U_{t_1, t_2} \rightarrow \mathbb{R}$  be a weak solution to

$$\begin{cases} u_t^\epsilon - \operatorname{div} (\mu^{p-2} Du^\epsilon) = 0 & \text{in } U_{t_1, t_2}; \\ u^\epsilon = u & \text{on } \partial_p U_{t_1, t_2}, \end{cases} \quad (4.1)$$

where

$$\mu := \sqrt{|Du^\epsilon|^2 + \epsilon}$$

and the *parabolic boundary* is defined as

$$\partial_p U_{t_1, t_2} = (\overline{U} \times \{t_1\}) \cup (\partial U \times (t_1, t_2]).$$

According to standard parabolic theory, we get  $u^\epsilon \in C^\infty(U_{t_1, t_2}) \cap C(\overline{U}_{t_1, t_2})$ , see [7, 24].

**Lemma 4.1.** *Let  $u^\epsilon : U_{t_1, t_2} \rightarrow \mathbb{R}$  be a weak solution to (4.1). If  $s > -1$ , then for any  $\phi \in C_0^\infty(U_{t_1, t_2})$ , we have*

$$\begin{aligned} & \int_{U_{t_1, t_2}} \mu^{p-2+s} |D^2 u^\epsilon|^2 \phi^2 dx dt \\ & \leq C \left( \int_{U_{t_1, t_2}} \mu^{p-2+s} |Du^\epsilon|^2 |D\phi|^2 dx dt + \int_{U_{t_1, t_2}} \mu^{s+2} |\phi| |\phi_t| dx dt \right) \end{aligned}$$

where  $C = C(p, s) > 0$  is independent of  $\epsilon$ .

To prove Lemma 4.1, we use the inequality (1.2). In Sect. 3, under the assumption  $Du \neq 0$ , we can directly divide both sides of (1.2) by  $|Du|^4$  to get the inequality (3.4), which gives the lower bound of  $|D^2 u|^2$ . In order to get an inequality similar to (3.4), we also need to consider the case when  $Du = 0$ . Thus, we reformulate (1.2) here in a way that allows us to apply it in this context.

For the reformulation, we introduce some notations. Let  $v : U_{t_1, t_2} \rightarrow \mathbb{R}$  be a smooth function. In particular,  $|Dv|$  is locally Lipschitz continuous (by triangle inequality) and thus, by Rademacher’s theorem, differentiable almost everywhere on each time slice, hence also in  $U_{t_1, t_2}$ .

Note that if  $(x_0, t_0) \in U_{t_1, t_2}$  is a space-time point where  $|Dv|$  is differentiable and  $Dv(x_0, t_0) = 0$ , then  $D|Dv|(x_0, t_0) = 0$ . Indeed, if we had  $D|Dv|(x_0, t_0) \neq 0$ , then we could find a point  $\xi \in U \times \{t_0\}$  (close to  $(x_0, t_0)$ ) such that  $|Dv|(\xi) < 0$ , which is obviously impossible. On the other hand, if  $Dv(x_0, t_0) \neq 0$  for some  $(x_0, t_0) \in U_{t_1, t_2}$ , then  $|Dv|$  is differentiable at  $(x_0, t_0)$  and

$$D|Dv|(x_0, t_0) = \frac{D^2 v(x_0, t_0) Dv(x_0, t_0)}{|Dv(x_0, t_0)|}.$$

For each point in  $U_{t_1, t_2}$  where  $Dv \neq 0$ , we fix an orthonormal basis of  $\mathbb{R}^n$ ,  $\{e_1, \dots, e_n\}$ , such that  $e_n = \frac{Dv}{|Dv|}$ . Hence, we have, for those points where  $Dv \neq 0$ ,

$$\frac{D^2 v Dv}{|Dv|} = \langle e_1, D|Dv| \rangle e_1 + \dots + \langle e_{n-1}, D|Dv| \rangle e_{n-1} + \left\langle \frac{Dv}{|Dv|}, D|Dv| \right\rangle \frac{Dv}{|Dv|}.$$

For those points where  $|Dv|$  is differentiable, let us define the part of  $D|Dv|$  which is tangential to the spatial level sets of  $v$  as

$$D_T |Dv| := \begin{cases} \langle e_1, D|Dv| \rangle e_1 + \dots + \langle e_{n-1}, D|Dv| \rangle e_{n-1} & \text{if } Dv \neq 0, \\ 0 & \text{if } Dv = 0, \end{cases}$$

and its orthogonal counterpart, the normalized infinity Laplacian, as

$$\Delta_\infty^N v := \begin{cases} \left\langle \frac{Dv}{|Dv|}, D|Dv| \right\rangle = \frac{\Delta_\infty v}{|Dv|^2} & \text{if } Dv \neq 0, \\ 0 & \text{if } Dv = 0. \end{cases}$$

We employ these notations to write

$$|D|Dv||^2 = |D_T|Dv||^2 + (\Delta_\infty^N v)^2 \quad \text{a.e. in } U_{t_1, t_2}. \tag{4.2}$$

In Sect. 3, without dividing  $|D|Dv||^2$  into two parts, we use the inequality (3.4) and (3.6) to get the estimate (3.5). When using (3.6), we need to be careful and check that if the coefficient of  $|D|Dv||^2$  is nonnegative. For the regularization, the coefficients of each terms become more complicated, and thus, by using the equality (4.2), we can consider the coefficients together in the last step of the estimate. Now we can restate (1.2).

**Lemma 4.2.** *Let  $v : U_{t_1, t_2} \rightarrow \mathbb{R}$  be a smooth function. Then,*

$$|D^2v|^2 \geq 2|D_T|Dv||^2 + (\Delta_\infty^N v)^2 \quad \text{a.e. in } U_{t_1, t_2}. \tag{4.3}$$

*Proof.* Recall that  $|Dv|$  is differentiable a.e. in  $U_{t_1, t_2}$ . From now on, consider such points of  $U_{t_1, t_2}$  where  $|Dv|$  is differentiable. When  $n = 1$ , by the definition of  $D_T|Dv|$ , we have  $D_T|Dv| = 0$  and (4.3) is obviously an identity. Then, we consider the case  $n \geq 2$ . If  $Dv = 0$ , then (4.3) holds trivially by what we defined above. If  $Dv \neq 0$ , then by [20, Corollary 2.2], we have

$$|Dv|^4|D^2v|^2 \geq 2|Dv|^2|D^2vDv|^2 + \frac{(|Dv|^2\Delta v - \Delta_\infty v)^2}{n - 1} - (\Delta_\infty v)^2.$$

Dividing both sides by  $|Dv|^4$ , using the definitions of  $D_T|Dv|$  and  $\Delta_\infty^N v$ , we get the desired inequality by following:

$$\begin{aligned} |D^2v|^2 &\geq 2|D|Dv||^2 + \frac{(\Delta v - \Delta_\infty^N v)^2}{n - 1} - (\Delta_\infty^N v)^2 \\ &\geq 2|D|Dv||^2 - (\Delta_\infty^N v)^2 \\ &= 2|D_T|Dv||^2 + (\Delta_\infty^N v)^2. \end{aligned}$$

□

*Proof of Lemma 4.1.* The spatial partial derivatives  $u_{x_k}^\epsilon, k = 1, \dots, n$ , solve

$$(u_{x_k}^\epsilon)_t - \operatorname{div}(\mu^{p-2}ADu_{x_k}^\epsilon) = 0 \tag{4.4}$$

where

$$A = I + (p - 2)\frac{Du^\epsilon \otimes Du^\epsilon}{\mu^2}.$$

Note that

$$\min\{1, p - 1\}I \leq A \leq \max\{1, p - 1\}I \tag{4.5}$$

uniformly in  $U_{t_1, t_2}$  and for  $\epsilon$ .

We multiply Eq. (4.4) with  $\mu^s u_{x_k}^\epsilon$ , where  $s > -1$ , and obtain

$$\mu^s u_{x_k}^\epsilon (u_{x_k}^\epsilon)_t - \mu^s u_{x_k}^\epsilon \operatorname{div} (\mu^{p-2} A Du_{x_k}^\epsilon) = 0. \tag{4.6}$$

For the first item in the above display, we note that

$$u_{x_k}^\epsilon (u_{x_k}^\epsilon)_t = \frac{1}{2} ((u_{x_k}^\epsilon)^2 + \frac{\epsilon}{n})_t. \tag{4.7}$$

Summing (4.6) over  $k = 1, \dots, n$  and taking (4.7) into account gives that

$$\frac{1}{s+2} (\mu^{s+2})_t - \mu^s \sum_{k=1}^n u_{x_k}^\epsilon \operatorname{div} (\mu^{p-2} A Du_{x_k}^\epsilon) = 0. \tag{4.8}$$

Observe that

$$\begin{aligned} \operatorname{div} (\mu^{p-2+s} A D^2 u^\epsilon Du^\epsilon) &= \sum_{k=1}^n \operatorname{div} ((\mu^s u_{x_k}^\epsilon) (\mu^{p-2} A Du_{x_k}^\epsilon)) \\ &= \mu^s \sum_{k=1}^n u_{x_k}^\epsilon \operatorname{div} (\mu^{p-2} A Du_{x_k}^\epsilon) + \mu^{p-2+s} (|D^2 u^\epsilon|^2 \\ &\quad + (p-2+s) \frac{|D^2 u^\epsilon Du^\epsilon|^2}{\mu^2} + s(p-2) \frac{(\Delta_\infty u^\epsilon)^2}{\mu^4}). \end{aligned} \tag{4.9}$$

Above we used

$$\begin{aligned} \langle A Du_{x_k}^\epsilon, Du_{x_k}^\epsilon \rangle &= |Du_{x_k}^\epsilon|^2 + (p-2) \frac{\langle Du^\epsilon, Du_{x_k}^\epsilon \rangle^2}{\mu^2}, \\ \langle A Du_{x_k}^\epsilon, D^2 u^\epsilon Du^\epsilon \rangle &= \langle Du_{x_k}^\epsilon, D^2 u^\epsilon Du^\epsilon \rangle + (p-2) \frac{\langle Du^\epsilon, Du_{x_k}^\epsilon \rangle \Delta_\infty u^\epsilon}{\mu^2}, \end{aligned}$$

and a straightforward computation. In other words,

$$\mu^s \sum_{k=1}^n u_{x_k}^\epsilon \operatorname{div} (\mu^{p-2} A Du_{x_k}^\epsilon) = \operatorname{div} (\mu^{p-2+s} A D^2 u^\epsilon Du^\epsilon) - \mu^{p-2+s} \sigma, \tag{4.10}$$

where

$$\sigma := |D^2 u^\epsilon|^2 + (p-2+s) \frac{|D^2 u^\epsilon Du^\epsilon|^2}{\mu^2} + s(p-2) \frac{(\Delta_\infty u^\epsilon)^2}{\mu^4}.$$

By (4.8) and (4.10), we have

$$\mu^{p-2+s} \sigma = \operatorname{div} (\mu^{p-2+s} A D^2 u^\epsilon Du^\epsilon) - \frac{1}{s+2} (\mu^{s+2})_t. \tag{4.11}$$

We claim that for  $s > -1$ , we can find a small number  $\lambda = \lambda(p, s) > 0$  such that

$$\lambda |D^2 u^\epsilon|^2 \leq \sigma \quad \text{a.e. in } U_{t_1, t_2}. \tag{4.12}$$

Observe that this is not a trivial inequality since not all the coefficients are positive in  $\sigma$ .

If (4.12) holds, then the desired estimate follows easily. Indeed, we plug the estimate (4.12) into Eq. (4.11) to obtain

$$\lambda \mu^{p-2+s} |D^2 u^\epsilon|^2 \leq \operatorname{div} (\mu^{p-2+s} A D^2 u^\epsilon D u^\epsilon) - \frac{1}{s+2} (\mu^{s+2})_t. \tag{4.13}$$

Let  $\phi \in C_0^\infty(U_{t_1, t_2})$ . Multiplying (4.13) by  $\phi^2$  and then integrating over  $U_{t_1, t_2}$  yields

$$\begin{aligned} & \lambda \int_{U_{t_1, t_2}} \mu^{p-2+s} |D^2 u^\epsilon|^2 \phi^2 dx dt \\ & \leq \int_{U_{t_1, t_2}} \left( \operatorname{div} (\mu^{p-2+s} A D^2 u^\epsilon D u^\epsilon) - \frac{1}{s+2} (\mu^{s+2})_t \right) \phi^2 dx dt. \end{aligned}$$

We employ integration by parts, (4.5) and Young’s inequality to obtain the upper bound of the right hand side term in above inequality,

$$\begin{aligned} & \int_{U_{t_1, t_2}} \left( \operatorname{div} (\mu^{p-2+s} A D^2 u^\epsilon D u^\epsilon) - \frac{1}{s+2} (\mu^{s+2})_t \right) \phi^2 dx dt \\ & = - \int_{U_{t_1, t_2}} \mu^{p-2+s} \langle A D^2 u^\epsilon D u^\epsilon, D \phi^2 \rangle dx dt + \frac{1}{s+2} \int_{U_{t_1, t_2}} \mu^{s+2} (\phi^2)_t dx dt \\ & \leq \eta \int_{U_{t_1, t_2}} \mu^{p-2+s} |D^2 u^\epsilon|^2 \phi^2 dx dt + \frac{C}{\eta} \int_{U_{t_1, t_2}} \mu^{p-2+s} |D u^\epsilon|^2 |D \phi|^2 dx dt \\ & \quad + \frac{2}{s+2} \int_{U_{t_1, t_2}} \mu^{s+2} |\phi| |\phi_t| dx dt \end{aligned}$$

for any  $\eta > 0$  and some constant  $C = C(p) > 0$ . The desired estimate follows by choosing  $\eta = \frac{\lambda}{2}$ .

It remains to prove (4.12). As explained above in this section, we can write

$$\begin{aligned} \sigma & = |D^2 u^\epsilon|^2 + (p-2+s) \frac{|D u^\epsilon|^2}{\mu^2} |D |D u^\epsilon||^2 + s(p-2) \frac{|D u^\epsilon|^4}{\mu^4} (\Delta_\infty^N u^\epsilon)^2 \\ & = |D^2 u^\epsilon|^2 + (p-2+s) \frac{|D u^\epsilon|^2}{\mu^2} |D_T |D u^\epsilon||^2 \\ & \quad + \left( (p-2+s) \frac{|D u^\epsilon|^2}{\mu^2} + s(p-2) \frac{|D u^\epsilon|^4}{\mu^4} \right) (\Delta_\infty^N u^\epsilon)^2 \end{aligned}$$

almost everywhere in  $U_{t_1, t_2}$ . For  $\lambda \in (0, 1)$ , we write

$$\sigma = \lambda \sigma + (1-\lambda) \sigma. \tag{4.14}$$

For the latter part of  $\sigma$  on the right hand side of (4.14), we utilize the nonnegativity of the  $|D^2 u^\epsilon|^2$ -term via the inequality of Lemma 4.2:

$$|D^2 u^\epsilon|^2 \geq 2 |D_T |D u^\epsilon||^2 + (\Delta_\infty^N u^\epsilon)^2.$$

We obtain a lower bound

$$\begin{aligned} \sigma &\geq \left(2 + (p - 2 + s) \frac{|Du^\epsilon|^2}{\mu^2}\right) |D_T|Du^\epsilon||^2 \\ &\quad + \left(1 + (p - 2 + s) \frac{|Du^\epsilon|^2}{\mu^2} + s(p - 2) \frac{|Du^\epsilon|^4}{\mu^4}\right) (\Delta_\infty^N u^\epsilon)^2 =: \tau. \end{aligned}$$

Now we have

$$\sigma \geq \lambda(\sigma - \tau) + \tau. \tag{4.15}$$

Writing

$$1 = \frac{|Du^\epsilon|^2}{\mu^2} + \frac{\epsilon}{\mu^2}$$

allows us to divide the terms in  $\sigma$  and  $\tau$  according to the degree of  $\epsilon$ . This kind of regrouping is useful, because it separates the main terms that appear also in the formal calculation of Sect. 3 from those terms that appear as a result of the regularization.

Indeed, we write

$$\begin{aligned} \tau &= \left(2 \left(\frac{|Du^\epsilon|^2}{\mu^2} + \frac{\epsilon}{\mu^2}\right)^2 + (p - 2 + s) \frac{|Du^\epsilon|^2}{\mu^2} \left(\frac{|Du^\epsilon|^2}{\mu^2} + \frac{\epsilon}{\mu^2}\right)\right) |D_T|Du^\epsilon||^2 \\ &\quad + \left(\left(\frac{|Du^\epsilon|^2}{\mu^2} + \frac{\epsilon}{\mu^2}\right)^2 + (p - 2 + s) \frac{|Du^\epsilon|^2}{\mu^2} \left(\frac{|Du^\epsilon|^2}{\mu^2} + \frac{\epsilon}{\mu^2}\right)\right. \\ &\quad \left.+ s(p - 2) \frac{|Du^\epsilon|^4}{\mu^4}\right) (\Delta_\infty^N u^\epsilon)^2 \\ &= \left((p + s) \frac{|Du^\epsilon|^4}{\mu^4} + (p + s + 2) \frac{\epsilon|Du^\epsilon|^2}{\mu^4} + \frac{2\epsilon^2}{\mu^4}\right) |D_T|Du^\epsilon||^2 \\ &\quad + \left((p - 1)(s + 1) \frac{|Du^\epsilon|^4}{\mu^4} + (p + s) \frac{\epsilon|Du^\epsilon|^2}{\mu^4} + \frac{\epsilon^2}{\mu^4}\right) (\Delta_\infty^N u^\epsilon)^2 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \sigma - \tau &= |D^2 u^\epsilon|^2 - 2|D_T|Du^\epsilon||^2 - (\Delta_\infty^N u^\epsilon)^2 \\ &= |D^2 u^\epsilon|^2 - 2 \left(\frac{|Du^\epsilon|^4}{\mu^4} + \frac{2\epsilon|Du^\epsilon|^2}{\mu^2} + \frac{\epsilon^2}{\mu^4}\right) |D_T|Du^\epsilon||^2 \\ &\quad - \left(\frac{|Du^\epsilon|^4}{\mu^4} + \frac{2\epsilon|Du^\epsilon|^2}{\mu^2} + \frac{\epsilon^2}{\mu^4}\right) (\Delta_\infty^N u^\epsilon)^2. \end{aligned} \tag{4.17}$$

As we plug (4.16) and (4.17) into (4.15), we can easily choose  $\lambda = \lambda(p, s) > 0$  so small that

$$\begin{aligned} \sigma &\geq \lambda |D^2 u^\epsilon|^2 + \left((p + s - 2\lambda) \frac{|Du^\epsilon|^4}{\mu^4} + (p + s + 2 - 4\lambda) \frac{\epsilon|Du^\epsilon|^2}{\mu^4}\right. \\ &\quad \left.+ (2 - 2\lambda) \frac{\epsilon^2}{\mu^4}\right) |D_T|Du^\epsilon||^2 + \left((p - 1)(s + 1) - \lambda\right) \frac{|Du^\epsilon|^4}{\mu^4} \end{aligned}$$

$$\begin{aligned}
 &+ (p + s - 2\lambda) \frac{\epsilon |Du^\epsilon|^2}{\mu^4} + (1 - \lambda) \frac{\epsilon^2}{\mu^4} \Big) (\Delta_\infty^N u^\epsilon)^2 \\
 &\geq \lambda |D^2 u^\epsilon|^2.
 \end{aligned}$$

This is indeed possible because  $s > -1$ . Now that we have shown (4.12), the proof is finished.  $\square$

*Proof of Theorem 2.2.* To prove Theorem 2.2, we need to justify letting  $\epsilon \rightarrow 0$  in Lemma 4.1. For notational convenience, we introduce the regularized version of the vector field  $V_s$  which corresponds to Lemma 4.1. Let us define  $V_s^\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$  as

$$V_s^\epsilon(z) := (|z|^2 + \epsilon)^{\frac{p-2+s}{4}} z \quad \text{for } z \in \mathbb{R}^n.$$

Similarly to (3.7), by Lemma 4.1, there exists a constant  $C = C(p, s) > 0$  such that

$$\begin{aligned}
 \int_{U_{t_1, t_2}} |D(V_s^\epsilon(Du^\epsilon))|^2 \phi^2 dx dt &\leq C \left( \int_{U_{t_1, t_2}} |V_s^\epsilon(Du^\epsilon)|^2 |D\phi|^2 dx dt \right. \\
 &\left. + \int_{U_{t_1, t_2}} (|Du^\epsilon|^2 + \epsilon)^{\frac{s+2}{2}} |\phi| |\phi_t| dx dt \right) \tag{4.18}
 \end{aligned}$$

for any  $\phi \in C_0^\infty(U_{t_1, t_2})$ .

The estimate (2.3) can be derived from (4.18) as follows. Let us fix a space-time point  $(x_0, t_0) \in U_{t_1, t_2}$ . Let  $r > 0$  be small enough such that the parabolic cylinder with center  $(x_0, t_0)$  and radius  $2r$  fits inside  $U_{t_1, t_2}$ , that is  $Q_{2r} \Subset U_{t_1, t_2}$ . Let  $\phi \in C_0^\infty(U_{t_1, t_2})$  be a cutoff function such that

$$\phi \equiv 1 \text{ in } Q_r, \quad |\phi| \leq 1, \quad \text{spt } \phi \subset Q_{2r}, \quad |D\phi| \leq \frac{10}{r} \quad \text{and} \quad |\phi_t| \leq \frac{10}{r^2}. \tag{4.19}$$

The estimate (4.18) implies that

$$\begin{aligned}
 &\int_{Q_r} |D(V_s^\epsilon(Du^\epsilon))|^2 dx dt \\
 &\leq \frac{C}{r^2} \left( \int_{Q_{2r}} |V_s^\epsilon(Du^\epsilon)|^2 dx dt + \int_{Q_{2r}} (|Du^\epsilon|^2 + \epsilon)^{\frac{s+2}{2}} dx dt \right) \tag{4.20}
 \end{aligned}$$

for  $C = C(p, s) > 0$ .

Since  $s > -1$ , we can apply, for example, [12] to conclude for the gradient

$$\|Du^\epsilon\|_{C^\alpha(Q_{2r})} \leq C.$$

Thus  $Du^\epsilon$  converge uniformly (and strongly in  $L^p$ ) by Arzelà–Ascoli theorem. It follows that the limit  $u$  is a solution to (2.1). Moreover, the right hand side of (4.20) is thus bounded from above by a constant independent of  $\epsilon$ . Thus  $\{D(V_s^\epsilon(Du^\epsilon))\}_\epsilon$  is bounded in  $L^2(Q_r)$ , and consequently we may extract a subsequence that converges weakly in  $L^2(Q_r)$ . Further, using integration by parts, we see that the limit is  $D(V_s(Du))$ , and thus

$$\begin{aligned} \int_{Q_r} |D(V_s(Du))|^2 dx dt &\leq \liminf_{\epsilon \rightarrow 0} \int_{Q_r} |D(V_s^\epsilon(Du^\epsilon))|^2 dx dt \\ &\leq \lim_{\epsilon \rightarrow 0} \frac{C}{r^2} \left( \int_{Q_{2r}} |V_s^\epsilon(Du^\epsilon)|^2 dx dt + \int_{Q_{2r}} (|Du^\epsilon|^2 + \epsilon)^{\frac{s+2}{2}} dx dt \right) \\ &\leq \frac{C}{r^2} \left( \int_{Q_{2r}} |V_s(Du)|^2 dx dt + \int_{Q_{2r}} |Du|^{s+2} dx dt \right), \end{aligned}$$

which is the desired estimate. □

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