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Testing the Sobolev property with a single test plan

by

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Abstract. We prove that on an arbitrary metric measure space the following property holds: a single test plan can be used to recover the minimal weak upper gradient of any Sobolev function. This means that, in order to identify which are the exceptional curves in the weak upper gradient inequality, it suffices to consider the negligible sets of a suitable Borel measure on curves, rather than the ones of the p-modulus. Moreover, on RCD spaces we can improve our result, showing that the test plan can also be chosen to be concentrated on an equi-Lipschitz family of curves.

Introduction. Throughout the past two decades, the classical theory of first-order Sobolev spaces has been successfully generalised to the abstract setting of metric measure spaces. Two strategies played a central role in the development of this subject: the relaxation procedure based on the notion of upper gradient (introduced by J. Cheeger [7]) and the analysis of the behaviour along curves (proposed by N. Shanmugalingam [22]), later revisited by L. Ambrosio, N. Gigli, and G. Savaré [4, 5]. As eventually proven in [4], all these approaches are fully equivalent.

Let (X, d) be a (complete and separable) metric space endowed with a (boundedly finite) Borel measure $\mathfrak{m}$. Let $p \in (1, \infty)$ be fixed. Then the *p-Sobolev space* $W^{1,p}(\mathrm{X})$ is a Banach space whose elements f are associated with a minimal object $|Df|_p \in L^p(\mathfrak{m})$, which is called the *minimal generalised p-upper gradient* [7], the *minimal p-relaxed slope* [4], or the *minimal p-weak upper gradient* [22, 5], and is the smallest p-integrable function that bounds from above the (modulus of the) variation of f. For the purposes of this paper, it is convenient to begin with the notion of relaxed slope introduced by Ambrosio–Gigli–Savaré [4], which is a variant of the original Cheeger's approach: the function $|Df|_p$ can be characterised as the minimal possible strong $L^p(\mathfrak{m})$-limit of $\mathrm{lip}(f_n)$ among all sequences $(f_n)_n \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$ with

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$\lim_n \|f - f_n\|_{L^p(\mathfrak{m})} = 0$, where $\mathrm{lip}(f_n)$ stands for the slope of f_n (see (1.1)). In duality with this 'Eulerian' relaxation procedure, it is possible—from a more 'Lagrangian' viewpoint—to identify $|Df|_p$ by looking at the behaviour of f along rectifiable curves. Namely, $|Df|_p$ is the minimal function $G \in L^p(\mathfrak{m})$ such that for *almost every* absolutely continuous curve γ the function $f \circ \gamma$ is absolutely continuous and

$$(\star) \qquad \left|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\right| \leq G(\gamma_t)|\dot{\gamma}_t| \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1].$$

There are different ways to detect the negligible families of curves that are excluded from the weak upper gradient condition $(\star)$. In Shanmugalingam's approach, the exceptional curves are measured with respect to the *p-modulus* Mod_p, which is an outer measure on paths that plays a crucial role in function theory [18]. Ambrosio, Gigli, and Savaré proposed the alternative notion of test plan: writing $q \in (1,\infty)$ for the conjugate exponent of p, they define a *q-test plan* on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ as a Borel probability measure $\boldsymbol{\pi}$ on $C([0,1],\mathrm{X})$ that is concentrated on the set $AC([0,1],\mathrm{X})$ of absolutely continuous curves and satisfies

$$\exists C > 0: \quad (\mathrm{e}_t)_{\#}\boldsymbol{\pi} \leq C\mathfrak{m} \quad \forall t \in [0,1], \qquad \iint_0^1 |\dot{\gamma}_t|^q \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma) < +\infty,$$

where the evaluation map e_t is given by $\mathrm{e}_t(\gamma) \coloneqq \gamma_t$. The first condition is a compression estimate—which grants that the plan does not concentrate mass too much at any time—while the second one is an integral bound on the speed of the curves selected by the plan. It is then possible to express $|Df|_p$ as the minimal $G \in L^p(\mathfrak{m})$ such that for every q-test plan $\boldsymbol{\pi}$ the inequality $(\star)$ holds for $\boldsymbol{\pi}$-a.e. γ.

There are two main differences between the p-modulus and a q-test plan: firstly, the former is an outer measure, while the latter is a σ-additive Borel measure (but a priori one has to consider possibly uncountably many test plans to identify the minimal weak upper gradient); secondly, in the definition of test plan the parametrisation of the curves involved plays an essential role, while the modulus is parametrisation-invariant. The duality between modulus and plans has been studied in [3].

The aim of this paper is to show that we can find a single q-test plan $\boldsymbol{\pi}_q$—which we shall call the *master test plan*—that is sufficient to recover the minimal weak upper gradient of any given Sobolev function. More precisely, for every $f \in W^{1,p}(\mathrm{X})$, $|Df|_p$ is the minimal $G \in L^p(\mathfrak{m})$ such that $(\star)$ holds for $\boldsymbol{\pi}_q$-a.e. γ. This result will be achieved on arbitrary metric measure spaces. Let us briefly outline the ideas behind the proof:

(a) The main tool we use is the *plan representing the gradient* of a Sobolev function, a concept introduced by Gigli in [11]. This means, roughly speaking, that the 'derivative' at time $t = 0$ of the test plan coincides with the gradient of the given function.

(b) In lack of a linear structure underlying the ambient space X, we work within the framework of the abstract tensor calculus built by Gigli [12], which relies upon the theory of *normed modules*. This supplies the functional-analytic tools we will need.

(c) We will further investigate the plans representing a gradient and fit them in the setting of the normed modules calculus, which was still not available at the time of [11]. More precisely, we prove—in a suitable sense—that if a test plan $\boldsymbol{\pi}$ represents the gradient of $f \in W^{1,p}(\mathrm{X})$, then for every $g \in W^{1,p}(\mathrm{X})$ and $\boldsymbol{\pi}$-a.e. γ the derivative at time $t = 0$ of $g \circ \gamma$ coincides with $\mathrm{d}g(\nabla f)(\gamma_0)$. See Proposition 2.3 for the precise statement.

(d) Given a sequence $(f_n)_n$ dense 'in energy' in $W^{1,p}(\mathrm{X})$ and writing $\boldsymbol{\pi}^n$ for the plan representing the gradient of f_n, we show—by using the results we mentioned in item (c)—that the sequence $(\boldsymbol{\pi}^n)_n$ of q-test plans is sufficient to identify the minimal weak upper gradient of each Sobolev function. Finally, by suitably combining the measures $\boldsymbol{\pi}^n$ we obtain the desired master test plan $\boldsymbol{\pi}_q$. See Theorem 2.6 for the details.

The weak upper gradient condition ($\star$) can be additionally used (when considered with respect to the modulus, or to the totality of test plans) to detect which functions are Sobolev. Currently, it is not known whether the same holds for the master test plan; cf. Problem 2.7.

In the last part of the paper, we improve our existence result of master test plans in the case in which the metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ satisfies a lower Ricci curvature bound. More specifically, we consider the so-called RCD *spaces*, which are infinitesimally Hilbertian metric measure spaces (i.e., the associated 2-Sobolev space is Hilbert [11]) fulfilling the celebrated curvature-dimension condition introduced by Lott–Sturm–Villani [20, 23, 24]. In this framework, we show that it is possible to construct an ∞-test plan $\boldsymbol{\pi}_\infty$ (i.e., a test plan concentrated on an equi-Lipschitz family of curves) which acts as a master q-test plan for *every* exponent $q \in (1, \infty)$; cf. Theorem 3.4. This sort of property has to do with the dependence on the exponent p of minimal p-weak upper gradients; see Remark 3.6 for a more detailed discussion. To prove Theorem 3.4, instead of plans representing the gradient we employ the theory of *regular Lagrangian flows*, available on RCD spaces thanks to [6].

1. Preliminaries

1.1. Sobolev calculus on metric measure spaces. For the purposes of this article, by a *metric measure space* we mean a triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, where

$$\begin{aligned} &(\mathrm{X}, \mathsf{d}) \quad \text{is a complete and separable metric space,} \\ &\mathfrak{m} \geq 0 \quad \text{is a boundedly finite Borel measure on } (\mathrm{X}, \mathsf{d}). \end{aligned}$$

For the sake of brevity, we will use the shorthand notation $\mathcal{L}_1$ to indicate the restriction of the 1-dimensional Lebesgue measure $\mathcal{L}^1$ to the real interval $[0,1]$, that is,

$$\mathcal{L}_1 \coloneqq \mathcal{L}^1|_{[0,1]}.$$

The space $C([0,1],\mathrm{X})$ of continuous curves in X is a complete and separable metric space when equipped with the supremum distance $\mathsf{d}_\infty(\gamma,\sigma) \coloneqq \max\{\mathsf{d}(\gamma_t,\sigma_t) \mid t \in [0,1]\}$. The *evaluation map* $\mathrm{e}\colon C([0,1],\mathrm{X}) \times [0,1] \to \mathrm{X}$ is defined as $\mathrm{e}(\gamma,t) \coloneqq \gamma_t$ for every $\gamma \in C([0,1],\mathrm{X})$ and $t \in [0,1]$, while for any $t \in [0,1]$ we denote by $\mathrm{e}_t\colon C([0,1],\mathrm{X}) \to \mathrm{X}$ the *evaluation map at time t*, i.e., we set $\mathrm{e}_t(\gamma) \coloneqq \mathrm{e}(\gamma,t)$ for all $\gamma \in C([0,1],\mathrm{X})$. Given any $s,t \in [0,1]$ with $s<t$ we define the *restriction map* $\mathrm{restr}_s^t\colon C([0,1],\mathrm{X}) \to C([0,1],\mathrm{X})$ as $\mathrm{restr}_s^t(\gamma)_r \coloneqq \gamma_{rt+(1-r)s}$. Observe that e, e_t, and restr_s^t are continuous maps.

A curve $\gamma \in C([0,1],\mathrm{X})$ is said to be *absolutely continuous* provided there exists $g \in L^1(0,1)$ such that $\mathsf{d}(\gamma_t,\gamma_s) \le \int_s^t g(r)\,\mathrm{d}r$ for all $s,t \in [0,1]$ with $s<t$. In this case, the limit $|\dot\gamma_t| \coloneqq \lim_{h\to 0}\mathsf{d}(\gamma_{t+h},\gamma_t)/|h|$ exists at $\mathcal{L}_1$-a.e. $t \in [0,1]$ and defines a function in $L^1(0,1)$, which is the minimal one (in the a.e. sense) satisfying the inequality in the absolute continuity condition. The function $|\dot\gamma|$, which is declared to be 0 at those $t \in [0,1]$ where the above limit does not exist, is called the *metric speed* of γ. We denote by $AC([0,1],\mathrm{X})$ the family of all absolutely continuous curves on X. Given any $q \in (1,\infty)$, we define the family of *q-absolutely continuous curves* as

$$AC^q([0,1],\mathrm{X}) \coloneqq \{\gamma \in AC([0,1],\mathrm{X}) \mid |\dot\gamma| \in L^q(0,1)\}.$$

The space of all real-valued Lipschitz functions on (X,d) having bounded support is denoted by $\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$. Given any function $f \in \mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$, we define its *slope* $\mathrm{lip}(f)\colon \mathrm{X} \to [0,+\infty)$ as

$$\mathrm{lip}(f)(x) \coloneqq \varlimsup_{y\to x} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)} \quad \text{if } x \in \mathrm{X} \text{ is an accumulation point,} \tag{1.1}$$

and $\mathrm{lip}(f)(x) \coloneqq 0$ otherwise. Furthermore, for any $q \in (1,\infty)$ we denote by $\mathscr{P}_q(\mathrm{X})$ the set of all Borel probability measures μ on (X,d) having *finite qth moment*, i.e., satisfying

$$\int \mathsf{d}^q(\cdot,\bar x)\,\mathrm{d}\mu < +\infty \qquad \text{for some (thus any) point } \bar x \in \mathrm{X}.$$

In what follows, we will often consider the integral (in the sense of Bochner [9]) of maps of the form $[0,1] \ni t \mapsto \Phi_t \in \mathbb{B}$, where $\mathbb{B}$ is a separable Banach space; more precisely, $\mathbb{B}$ will always be an L^p-space, for some exponent $p \in [1,\infty)$. The fact that the maps $\Phi\colon [0,1] \to \mathbb{B}$ we will consider are strongly Borel follows by standard arguments, thus we will not insist further on measurability issues. Let us just recall that if a map $\Phi\colon [0,1] \to L^p(\mu)$ is Bochner integrable, then $(\int_0^1 \Phi_t\,\mathrm{d}t)(x) = \int_0^1 \Phi_t(x)\,\mathrm{d}t$ for μ-a.e. $x \in \mathrm{X}$.

Remark 1.1. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix any exponent $q\in(1,\infty)$. Then there exists a measure $\tilde{\mathfrak{m}}\in\mathscr{P}_q(\mathrm{X})$ such that $\mathfrak{m}\ll\tilde{\mathfrak{m}}\leq C\mathfrak{m}$ for some constant $C>0$.

In order to prove this, fix any point $\bar{x}\in\mathrm{X}$. Given that (X,d) is separable, we can find a sequence $(x_k)_k\subseteq\mathrm{X}$ such that $\mathrm{X}=\bigcup_{k\in\mathbb{N}}B_1(x_k)$. Recall that $\mathfrak{m}(B_1(x_k))<+\infty$ for all $k\in\mathbb{N}$. We define $A_1:=B_1(x_1)$ and $A_k:=B_1(x_k)\setminus(A_1\cup\cdots\cup A_{k-1})$ for every $k\geq 2$. Let us put

$$\mu:=\sum_{k=1}^{\infty}\frac{\mathfrak{m}|_{A_k}}{2^k(\mathsf{d}(x_k,\bar{x})+1)^q\max\{\mathfrak{m}(A_k),1\}},\qquad\tilde{\mathfrak{m}}:=\frac{\mu}{\mu(\mathrm{X})}.$$

Then μ is a Borel measure on X satisfying $\mu(\mathrm{X})\leq\sum_{k=1}^{\infty}2^{-k}=1$, whence $\tilde{\mathfrak{m}}$ is a (well-defined) Borel probability measure on X. If a Borel set $N\subseteq\mathrm{X}$ satisfies $\mu(N)=0$, then $\mathfrak{m}(N)=\sum_{k=1}^{\infty}\mathfrak{m}(N\cap A_k)=0$, thus showing that $\mathfrak{m}\ll\tilde{\mathfrak{m}}$. Moreover, observe that $\mu\leq\sum_{k=1}^{\infty}2^{-k}\mathfrak{m}|_{A_k}\leq\mathfrak{m}$ and accordingly $\tilde{\mathfrak{m}}\leq\mu(\mathrm{X})^{-1}\mathfrak{m}$. Finally, since $\mathsf{d}(\cdot,\bar{x})\leq\mathsf{d}(x_k,\bar{x})+1$ on A_k for any $k\in\mathbb{N}$, we conclude that

$$\int\mathsf{d}^q(\cdot,\bar{x})\,\mathrm{d}\tilde{\mathfrak{m}}=\frac{1}{\mu(\mathrm{X})}\sum_{k=1}^{\infty}\frac{1}{2^k\max\{\mathfrak{m}(A_k),1\}}\int_{A_k}\left(\frac{\mathsf{d}(\cdot,\bar{x})}{\mathsf{d}(x_k,\bar{x})+1}\right)^q\mathrm{d}\mathfrak{m}\leq\frac{1}{\mu(\mathrm{X})},$$

thus proving that the measure $\tilde{\mathfrak{m}}$ has finite qth moment.

1.1.1. *Definition of Sobolev space.* Let us recall the notion of p-Sobolev space, based upon the relaxation of the slope, proposed by Ambrosio–Gigli–Savaré [4] as a variant of Cheeger's approach [7]. Other equivalent definitions will be discussed in Sections 1.2 and 1.3.

Definition 1.2 (Sobolev space [4]). Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $p\in(1,\infty)$. Then a function $f\in L^p(\mathfrak{m})$ belongs to the *p-Sobolev space* $W^{1,p}(\mathrm{X})$ provided there exists a sequence $(f_n)_n\subseteq\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$ such that $f_n\to f$ in $L^p(\mathfrak{m})$ and

$$\varliminf_{n\to\infty}\int\mathrm{lip}^p(f_n)\,\mathrm{d}\mathfrak{m}<+\infty.$$

The Sobolev space $W^{1,p}(\mathrm{X})$ is a Banach space if endowed with the norm

$$\|f\|_{W^{1,p}(\mathrm{X})}:=(\|f\|^p_{L^p(\mathfrak{m})}+p\,\mathrm{E}_{\mathrm{Ch},p}(f))^{1/p}\quad\text{for every }f\in W^{1,p}(\mathrm{X}),$$

where the *Cheeger p-energy* $\mathrm{E}_{\mathrm{Ch},p}$ is given by

$$\mathrm{E}_{\mathrm{Ch},p}(f):=\inf_{(f_n)_n}\varliminf_{n\to\infty}\frac{1}{p}\int\mathrm{lip}^p(f_n)\,\mathrm{d}\mathfrak{m},$$

with the infimum taken over all sequences $(f_n)_n\subseteq\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$ such that $f_n\to f$ in $L^p(\mathfrak{m})$. We observe that for every $f\in W^{1,p}(\mathrm{X})$ there exists a unique function $|Df|_p \in L^p(\mathfrak{m})$ such that

$$\mathrm{E}_{\mathrm{Ch},p}(f) = \frac{1}{p}\int |Df|_p^p \, \mathrm{d}\mathfrak{m}.$$

The function $|Df|_p$ is called the *minimal p-relaxed slope* of f.

Remark 1.3. The minimal p-relaxed slope might depend on the exponent p. More precisely, if $p, p' \in (1,\infty)$ and $f \in W^{1,p}(\mathrm{X}) \cap W^{1,p'}(\mathrm{X})$, then it might happen that $|Df|_p \neq |Df|_{p'}$. Some examples of spaces in which this phenomenon occurs can be found in [8].

Remark 1.4. The reflexivity properties of the Sobolev spaces are investigated in [2], where the authors proved, e.g., that $W^{1,p}(\mathrm{X})$ is reflexive as soon as the underlying space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is metrically doubling. Moreover, the reflexivity of $W^{1,p}(\mathrm{X})$ implies its separability. An example of non-reflexive (and non-separable) Sobolev space was also constructed in [2]. To the best of our knowledge, no example of separable non-reflexive Sobolev space is currently available.

1.1.2. *The theory of normed modules.* We need to recall a few basic notions in the theory of normed modules introduced in [12, 13]. Given a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and an exponent $p \in (1,\infty)$, we say that $\mathscr{M}$ is an *$L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module* if it is a module over the ring $L^\infty(\mathfrak{m})$ and it is equipped with a *pointwise norm* $|\cdot| \colon \mathscr{M} \to L^p(\mathfrak{m})$ satisfying

$$\begin{aligned} |v| \geq 0 &\quad \text{for every } v \in \mathscr{M}, \text{ with } |v| = 0 \text{ if and only if } v = 0, \\ |f \cdot v| = |f|\,|v| &\quad \text{for every } v \in \mathscr{M} \text{ and } f \in L^\infty(\mathfrak{m}), \\ |v + w| \leq |v| + |w| &\quad \text{for all } v, w \in \mathscr{M}, \end{aligned}$$

where equalities and inequalities are understood in the $\mathfrak{m}$-a.e. sense. Moreover, we require the norm $\|v\|_{\mathscr{M}} := \||v|\|_{L^p(\mathfrak{m})}$ to be complete, whence $\mathscr{M}$ has a Banach space structure.

The *dual* of $\mathscr{M}$ is given by the space $\mathscr{M}^*$ of $L^\infty(\mathfrak{m})$-linear continuous maps $T \colon \mathscr{M} \to L^1(\mathfrak{m})$. Choosing $q \in (1,\infty)$ so that $1/p + 1/q = 1$, we see that $\mathscr{M}^*$ is an $L^q(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module if endowed with the following pointwise norm operator:

$$|T| := \operatorname{ess\,sup}\{|T(v)| \mid v \in \mathscr{M},\ |v| \leq 1\ \mathfrak{m}\text{-a.e.}\} \in L^q(\mathfrak{m}) \qquad \text{for all } T \in \mathscr{M}^*.$$

The link between the Sobolev calculus and the theory of normed modules is represented by the *cotangent module* $L^p(T^*\mathrm{X})$. It is an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module that comes with a linear differential operator $\mathrm{d}_p \colon W^{1,p}(\mathrm{X}) \to$

$L^p(T^*\mathrm{X})$ and is characterised by these two properties:

$$|\mathrm{d}_p f| = |Df|_p \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } f \in W^{1,p}(\mathrm{X}),$$

$$\Big\{ \sum_{i=1}^n g_i \cdot \mathrm{d}_p f_i \,\Big|\, (g_i)_i \subseteq L^\infty(\mathfrak{m}),\ (f_i)_i \subseteq W^{1,p}(\mathrm{X}) \Big\} \text{ is dense in } L^p(T^*\mathrm{X}).$$

The existence of the cotangent module when $p = 2$ is proven in [12], while the case $p \neq 2$ is treated in [15]. The dual $L^q(T\mathrm{X})$ of the space $L^p(T^*\mathrm{X})$ is called the *tangent module.*

For any $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$, we define $\mathsf{Dual}\colon \mathscr{M} \to 2^{\mathscr{M}^*}$ as

$$\text{(1.2)} \quad \mathsf{Dual}(v) := \{\omega \in \mathscr{M}^* \mid \omega(v) = |v|^p = |\omega|^q \ \mathfrak{m}\text{-a.e.}\} \quad \text{for all } v \in \mathscr{M}.$$

We notice that $\mathsf{Dual}(v) \neq \emptyset$ for every $v \in \mathscr{M}$, as a consequence of the Hahn–Banach theorem.

Another important construction is that of *pullback module.* Consider metric measure spaces $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ and $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$. Let $\varphi\colon \mathrm{X} \to \mathrm{Y}$ be a map *of bounded compression*, that is, a Borel map satisfying $\varphi_\# \mathfrak{m}_\mathrm{X} \leq C\mathfrak{m}_\mathrm{Y}$ for some constant $C > 0$. Then for any $L^p(\mathfrak{m}_\mathrm{Y})$-normed $L^\infty(\mathfrak{m}_\mathrm{Y})$-module $\mathscr{M}$ there exist a unique $L^p(\mathfrak{m}_\mathrm{X})$-normed $L^\infty(\mathfrak{m}_\mathrm{X})$-module $\varphi^*\mathscr{M}$ and a unique linear map $\varphi^*\colon \mathscr{M} \to \varphi^*\mathscr{M}$ such that

$$|\varphi^* v| = |v| \circ \varphi \quad \mathfrak{m}_\mathrm{X}\text{-a.e.} \quad \text{for every } v \in \mathscr{M},$$

$$\Big\{ \sum_{i=1}^n f_i \cdot \varphi^* v_i \,\Big|\, (f_i)_{i=1}^n \subseteq L^\infty(\mathfrak{m}_\mathrm{X}),\ (v_i)_{i=1}^n \subseteq \mathscr{M} \Big\} \text{ is dense in } \varphi^*\mathscr{M}.$$

1.1.3. *Infinitesimal strict convexity.* Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Let $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$. Following [11], we say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is *q-infinitesimally strictly convex* provided for any functions $f, g \in W^{1,p}(\mathrm{X})$ we have

$$\text{(1.3)} \quad \mathbb{1}_{\{|Df|_p > 0\}} \operatorname*{ess\,sup}_{\varepsilon < 0} \frac{|D(f + \varepsilon g)|_p^p - |Df|_p^p}{p\varepsilon |Df|_p^{p-2}} = \mathbb{1}_{\{|Df|_p > 0\}} \operatorname*{ess\,inf}_{\varepsilon > 0} \frac{|D(f + \varepsilon g)|_p^p - |Df|_p^p}{p\varepsilon |Df|_p^{p-2}} \quad \mathfrak{m}\text{-a.e.}$$

By arguing as in [12, Proposition 2.3.8] or [16, Proposition 4.3.1], one can check that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is q-infinitesimally strictly convex if and only if $\mathsf{Dual}(\mathrm{d}_p f)$ is a singleton for every $f \in W^{1,p}(\mathrm{X})$. In this case, writing $\nabla_p f \in L^q(T\mathrm{X})$ for the unique element of $\mathsf{Dual}(\mathrm{d}_p f)$, we see that $\mathrm{d}_p g(\nabla_p f)$ coincides $\mathfrak{m}$-a.e. with the function appearing in (1.3) for any choice of $f, g \in W^{1,p}(\mathrm{X})$. We say that $\nabla_p f$ is the *p-gradient* of the function f. Observe that 'p' refers to the fact that f belongs to the p-Sobolev space, but the function $|\nabla_p f|$ is actually q-integrable.

1.1.4. *Infinitesimal Hilbertianity.* Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. An $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ is said to be a *Hilbert module* provided the parallelogram rule holds:

$$|v+w|^2 + |v-w|^2 = 2|v|^2 + 2|w|^2 \quad \mathfrak{m}\text{-a.e.} \quad \text{for all } v, w \in \mathscr{M}. \tag{1.4}$$

The condition in (1.4) is equivalent to requiring that $\mathscr{M}$ is Hilbert when viewed as a Banach space. The *pointwise scalar product* $\langle \cdot, \cdot \rangle \colon \mathscr{M} \times \mathscr{M} \to L^1(\mathfrak{m})$ is then defined as follows:

$$\langle v, w \rangle := \frac{|v+w|^2 - |v|^2 - |w|^2}{2} \quad \mathfrak{m}\text{-a.e.} \quad \text{for all } v, w \in \mathscr{M}.$$

It can be straightforwardly checked that the map $\langle \cdot, \cdot \rangle$ is $L^\infty(\mathfrak{m})$-bilinear and continuous.

Remark 1.5. Consider a $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ and the map $\mathsf{Dual} \colon \mathscr{M} \to 2^{\mathscr{M}^*}$ as in (1.2). Then $\mathscr{M}$ is a Hilbert module if and only if Dual is single-valued and the unique element of $\mathsf{Dual}(v)$ linearly depends on $v \in \mathscr{M}$. The map associating to every $v \in \mathscr{M}$ the unique element $\mathsf{R}_{\mathscr{M}}(v) \in \mathscr{M}^*$ of $\mathsf{Dual}(v)$ is called the *Riesz isomorphism* of $\mathscr{M}$. Moreover, $\mathsf{R}_{\mathscr{M}} \colon \mathscr{M} \to \mathscr{M}^*$ is a linear isomorphism that preserves the pointwise norm. The above claims can be proven by arguing as in [16, Exercise 4.2.11].

A metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is said to be *infinitesimally Hilbertian* [11] provided the 2-Sobolev space $W^{1,2}(\mathrm{X})$ is a Hilbert space, or equivalently the cotangent module $L^2(T^*\mathrm{X})$ is a Hilbert module. Moreover, as proven in [12, Proposition 2.3.17], $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if and only if it is 2-infinitesimally strictly convex and the 2-gradient operator $\nabla_2 \colon W^{1,2}(\mathrm{X}) \to L^2(T\mathrm{X})$ is linear. Let us also observe that

$$\nabla_2 f = \mathsf{R}_{L^2(T^*\mathrm{X})}(\mathrm{d}_2 f) \quad \text{for every } f \in W^{1,2}(\mathrm{X}).$$

1.2. Modulus and Newtonian space. The notion of Sobolev space that we described in Section 1.1 corresponds, in the smooth framework, to the approach via approximation by smooth functions. Another viewpoint on weakly differentiable functions in the Euclidean space is the one introduced by B. Levi [19], which consists in checking the behaviour of functions along curves. This approach was further refined by B. Fuglede [10], who made it frame-independent by using the potential-theoretic notion of *modulus*. Later on, the theory was extended by N. Shanmugalingam [22] to the setting of metric measure spaces, by introducing the so-called *Newtonian space*, whose definition builds upon the notion of *upper gradient* introduced by J. Heinonen and P. Koskela [17].

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Given an exponent $p \in (1, \infty)$ and any family $\Gamma \subseteq AC([0,1], \mathrm{X})$ of non-constant curves, we define the *p-modulus* of Γ as

$$\mathrm{Mod}_p(\Gamma) := \inf_\rho \int \rho^p \, \mathrm{d}\mathfrak{m},$$

where the infimum is taken over all Borel functions $\rho\colon \mathrm{X} \to [0,+\infty]$ such that $\int_0^1 \rho(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t \geq 1$ for every $\gamma \in \Gamma$. Note that Mod_p is an outer measure. Typically, it is defined on all (non-parametric) curves, but here we prefer the above formulation since it better fits our approach. A property is said to hold Mod_p-*almost everywhere* provided it is satisfied by every γ in some set Γ of curves whose complement is Mod_p-negligible. Given two Borel functions $\bar{f}\colon \mathrm{X} \to \mathbb{R}$ and $G\colon \mathrm{X} \to [0,+\infty]$ with $G \in L^p(\mathfrak{m})$, we say that G is a *p-weak upper gradient* of $\bar{f}$ if for Mod_p-a.e. γ the function $\bar{f} \circ \gamma$ is absolutely continuous and $\left|\frac{\mathrm{d}}{\mathrm{d}t}\bar{f}(\gamma_t)\right| \leq G(\gamma_t)|\dot{\gamma}_t|$ for $\mathcal{L}_1$-a.e. $t \in [0,1]$.

Definition 1.6 (Newtonian space [22]). Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix any exponent $p \in (1,\infty)$. Then the *Newtonian space* $N^{1,p}(\mathrm{X})$ is the family of all $f \in L^p(\mathfrak{m})$ that admit a Borel representative $\bar{f}\colon \mathrm{X} \to \mathbb{R}$ having a p-weak upper gradient $G \in L^p(\mathfrak{m})$.

The Newtonian space can be made into a Banach space: given any $f \in N^{1,p}(\mathrm{X})$, we define

$$\|f\|_{N^{1,p}(\mathrm{X})} := \Big(\|f\|^p_{L^p(\mathfrak{m})} + \inf_{G \in D_p[f]} \|G\|^p_{L^p(\mathfrak{m})}\Big)^{1/p},$$

where $D_p[f]$ stands for the family of all Borel functions $G\colon \mathrm{X} \to [0,+\infty]$ that are p-weak upper gradients of some Borel version of f. It turns out that $\|\cdot\|_{N^{1,p}(\mathrm{X})}$ is a complete norm on $N^{1,p}(\mathrm{X})$. There exists a unique function $G_{f,p} \in D_p[f]$ having minimal $L^p(\mathfrak{m})$-norm among all elements of $D_p[f]$, and it is minimal also in the $\mathfrak{m}$-a.e. sense. We have:

Proposition 1.7. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p \in (1,\infty)$. Then $W^{1,p}(\mathrm{X}) \subseteq N^{1,p}(\mathrm{X})$, and $G_{f,p} \leq |Df|_p$ holds $\mathfrak{m}$-a.e. for every $f \in W^{1,p}(\mathrm{X})$.*

We refer the reader to the monograph [18] for a thorough discussion of this topic.

1.3. Test plans. To prove the equivalence between $W^{1,p}(\mathrm{X})$ and $N^{1,p}(\mathrm{X})$, L. Ambrosio, N. Gigli, and G. Savaré introduced in [5, 4] the notion of *test plan*, which furnishes a more 'probabilistic' way to measure the exceptional curves in the weak upper gradient condition.

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Given any $q \in (1,\infty)$ and $t \in (0,1]$, following [11] we define the *q-energy functional* $\mathrm{E}_{q,t}\colon C([0,1],\mathrm{X}) \to [0,+\infty]$ as

$$\mathrm{E}_{q,t}(\gamma) := t\Big(⨍_0^t |\dot{\gamma}_s|^q \, \mathrm{d}s\Big)^{1/q} \qquad \text{if } \gamma \in AC^q([0,1],\mathrm{X}),$$

and $\mathrm{E}_{q,t}(\gamma) := +\infty$ otherwise. It can be readily checked that $\mathrm{E}_{q,t}$ is a Borel mapping.

Definition 1.8 (Test plan [4]). Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $q \in (1, \infty)$. Then a Borel probability measure $\boldsymbol{\pi}$ on $C([0,1], \mathrm{X})$ is a *q-test plan* on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ provided:

(i) There is a constant $C > 0$ such that $(\mathrm{e}_t)_\# \boldsymbol{\pi} \leq C\mathfrak{m}$ for every $t \in [0,1]$. The minimal such C is denoted by $\mathrm{Comp}(\boldsymbol{\pi}) > 0$ and called the *compression constant*.

(ii) The measure $\boldsymbol{\pi}$ has *finite kinetic q-energy*, which means that

$$\mathrm{KE}_q(\boldsymbol{\pi}) := \int \mathrm{E}_{q,1}(\gamma)^q \, \mathrm{d}\boldsymbol{\pi}(\gamma) < +\infty.$$

In particular, $\boldsymbol{\pi}$ is concentrated on $AC^q([0,1], \mathrm{X})$.

Also, we say that a Borel probability measure $\boldsymbol{\pi}$ on $C([0,1], \mathrm{X})$ is an *∞-test plan* on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ provided it satisfies (i) and is concentrated on an equi-Lipschitz family of curves.

Observe that if $q, q' \in (1, \infty]$ satisfy $q' \leq q$, then every q-test plan is a q'-test plan. Moreover, if $\boldsymbol{\pi}$ is a q-test plan and $s, t \in [0,1]$ with $s < t$, then $(\mathrm{restr}_s^t)_\# \boldsymbol{\pi}$ is a q-test plan.

The relation between test plans and modulus has been deeply investigated in [3]. The following result (whose proof can be found, e.g., in [16, Lemma 2.2.26]) is sufficient for the purposes of this paper. As the formulation is slightly different, we report here also its proof.

Lemma 1.9. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Let $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$. Fix a q-test plan $\boldsymbol{\pi}$ and a family $\Gamma \subseteq AC([0,1], \mathrm{X})$ of non-constant curves with $\mathrm{Mod}_p(\Gamma) = 0$. Then there exists a Borel set $N \subseteq C([0,1], \mathrm{X})$ such that $\Gamma \subseteq N$ and $\boldsymbol{\pi}(N) = 0$.*

Proof. For any $n \in \mathbb{N}$, there is a Borel function $\rho_n \colon \mathrm{X} \to [0, +\infty]$ such that $\int_0^1 \rho_n(\gamma_t) |\dot\gamma_t| \, \mathrm{d}t \geq 1$ for every $\gamma \in \Gamma$ and $\int \rho_n^p \, \mathrm{d}\mathfrak{m} \leq 1/n$. Since $(\gamma, t) \mapsto \rho_n(\gamma_t)|\dot\gamma_t|$ is a Borel function, the set $N_n := \{\gamma \mid \int_0^1 \rho_n(\gamma_t)|\dot\gamma_t| \, \mathrm{d}t \geq 1\}$ is Borel. Therefore, the Borel set $N := \bigcap_n N_n$ contains Γ and satisfies

$$\begin{aligned}
\boldsymbol{\pi}(N) &\leq \inf_{n \in \mathbb{N}} \boldsymbol{\pi}(N_n) = \inf_{n \in \mathbb{N}} \int \mathbb{1}_{N_n}(\gamma) \, \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \inf_{n \in \mathbb{N}} \int \int_0^1 \rho_n(\gamma_t)|\dot\gamma_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&\leq \inf_{n \in \mathbb{N}} \Big(\int \int_0^1 \rho_n^p \circ \mathrm{e}_t \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi} \Big)^{1/p} \Big(\int \int_0^1 |\dot\gamma_t|^q \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma) \Big)^{1/q} \\
&\leq \mathrm{Comp}(\boldsymbol{\pi})^{1/p} \, \mathrm{KE}_q(\boldsymbol{\pi})^{1/q} \inf_{n \in \mathbb{N}} \frac{1}{n^{1/p}} = 0,
\end{aligned}$$

thus proving the statement. ∎

A proof of the following continuity result can be found, e.g., in [16, Proposition 2.1.4].

PROPOSITION 1.10. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Let* $q \in (1,\infty)$ *and* $r \in [1,\infty)$. *Let* $\boldsymbol{\pi}$ *be a* q*-test plan on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$. *Then for any function* $f \in L^r(\mathfrak{m})$,

$$[0,1] \ni t \mapsto f \circ \mathrm{e}_t \in L^r(\mathfrak{m}) \text{ is a strongly continuous map.}$$

1.3.1. $(\mathit{\Pi},p)$*-weak upper gradients.* We now focus on the role that test plans play in the Sobolev theory. The key point is that they can be used to select the 'negligible families of curves' in the weak upper gradient condition, as we are going to explain in the next definition.

DEFINITION 1.11 ($(\mathit{\Pi},p)$-weak upper gradient). Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $p,q \in (1,\infty)$ satisfy $1/p+1/q=1$. Let $\mathit{\Pi}$ be a family of q-test plans on X. Let $f \in L^p(\mathfrak{m})$. Then a function $G \in L^p(\mathfrak{m})$ is a $(\mathit{\Pi},p)$-*weak upper gradient* of f provided for any $\boldsymbol{\pi} \in \mathit{\Pi}$ we have $f \circ \gamma \in W^{1,1}(0,1)$ for $\boldsymbol{\pi}$-a.e. $\gamma \in AC^q([0,1],\mathrm{X})$ and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\right| \leq G(\gamma_t)|\dot\gamma_t| \quad \text{for } (\boldsymbol{\pi}\otimes\mathcal{L}_1)\text{-a.e. } (\gamma,t) \in AC^q([0,1],\mathrm{X}) \times [0,1].$$

We denote by $\mathrm{G}_{\mathit{\Pi},p}(f)$ the collection of all $(\mathit{\Pi},p)$-weak upper gradients of f. Also, we define

$$W^{1,p}_{\mathit{\Pi}}(\mathrm{X}) := \{f \in L^p(\mathfrak{m}) \mid \mathrm{G}_{\mathit{\Pi},p}(f) \neq \emptyset\}.$$

Observe that $N^{1,p}(\mathrm{X}) \subseteq W^{1,p}_{\mathit{\Pi}}(\mathrm{X})$ and $D_p[f] \subseteq \mathrm{G}_{\mathit{\Pi},p}(f)$ for every $f \in N^{1,p}(\mathrm{X})$ by Lemma 1.9.

LEMMA 1.12. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space and* $p,q \in (1,\infty)$ *satisfy* $1/p+1/q=1$. *Let* $\mathit{\Pi}$ *be a family of* q*-test plans on* X. *Then the set* $\mathrm{G}_{\mathit{\Pi},p}(f)$ *is a closed convex lattice of* $L^p(\mathfrak{m})$ *for every* $f \in W^{1,p}_{\mathit{\Pi}}(\mathrm{X})$.

Proof. Fix $f \in W^{1,p}_{\mathit{\Pi}}(\mathrm{X})$. Clearly, if $G_1, G_2 \in \mathrm{G}_{\mathit{\Pi},p}(f)$, then

$$\min\{G_1,G_2\} \in \mathrm{G}_{\mathit{\Pi},p}(f)$$

as well. Now fix a sequence $(G_n)_n \subseteq \mathrm{G}_{\mathit{\Pi},p}(f)$ such that $G_n \to G \in L^p(\mathfrak{m})$ strongly in $L^p(\mathfrak{m})$. Up to taking a subsequence (not relabelled), we have $G_n \to G$ pointwise $\mathfrak{m}$-a.e. Given any $t \in [0,1]$, it follows from the assumption $(\mathrm{e}_t)_\#\boldsymbol{\pi} \ll \mathfrak{m}$ that $G_n \circ \mathrm{e}_t \to G \circ \mathrm{e}_t$ pointwise $\boldsymbol{\pi}$-a.e. as $n \to \infty$. Also, by the Fubini theorem we see that for $\mathcal{L}_1$-a.e. $t \in [0,1]$ we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\right| \leq G_n(\gamma_t)|\dot\gamma_t| \quad \text{for every } n \in \mathbb{N},\ \boldsymbol{\pi}\text{-a.e. } \gamma \in AC^q([0,1],\mathrm{X}). \tag{1.5}$$

By letting $n \to \infty$ in (1.5), we find for $\mathcal{L}_1$-a.e. $t \in [0,1]$ that $\left|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\right| \leq G(\gamma_t)|\dot\gamma_t|$ is satisfied for $\boldsymbol{\pi}$-a.e. γ. By using the Fubini theorem again, we conclude that $G \in \mathrm{G}_{\Pi,p}(f)$. This shows that the set $\mathrm{G}_{\Pi,p}(f)$ is strongly closed in $L^p(\mathfrak{m})$, thus completing the proof of the statement. ■

Definition 1.13 (Minimal (Π,p)-weak upper gradient). Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $p,q \in (1,\infty)$ satisfy $1/p+1/q=1$. Let Π be a family of q-test plans on X. Fix any function $f \in W^{1,p}_{\Pi}(\mathrm{X})$. Then the (unique) minimal element of $\mathrm{G}_{\Pi,p}(f)$ is denoted by $|Df|_{\Pi,p}$ and called the *minimal (Π,p)-weak upper gradient* of f.

Whenever $\Pi=\{\boldsymbol{\pi}\}$ is a singleton, we use the shorthand notation $W^{1,p}_{\boldsymbol{\pi}}(\mathrm{X})$ and $|Df|_{\boldsymbol{\pi},p}$.

Remark 1.14. Observe that $|Df|_{\Pi,p} \leq G_{f,p}$ holds $\mathfrak{m}$-a.e. for every $f \in N^{1,p}(\mathrm{X})$.

As already mentioned above, by considering the totality of test plans it is possible to recover both the Sobolev space and the minimal relaxed slope of each Sobolev function:

Theorem 1.15 (Sobolev space via test plans [4]). *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p,q \in (1,\infty)$ be such that $1/p+1/q=1$. Denote by Π_q the family of all q-test plans on X. Then $W^{1,p}_{\Pi_q}(\mathrm{X})=W^{1,p}(\mathrm{X})$ and*

$$|Df|_{\Pi_q,p}=|Df|_p \quad \text{for every } f \in W^{1,p}(\mathrm{X}).$$

In particular, $N^{1,p}(\mathrm{X})=W^{1,p}(\mathrm{X})$ and $G_{f,p}=|Df|_p$ for every $f \in W^{1,p}(\mathrm{X})$.

Proposition 1.16. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $p,q \in (1,\infty)$ satisfy $1/p+1/q=1$. Let $\Pi \subseteq \Pi'$ be two given families of q-test plans on X. Then $W^{1,p}_{\Pi'}(\mathrm{X}) \subseteq W^{1,p}_{\Pi}(\mathrm{X})$ and the inequality $|Df|_{\Pi,p} \leq |Df|_{\Pi',p}$ is satisfied $\mathfrak{m}$-a.e. for every $f \in W^{1,p}_{\Pi'}(\mathrm{X})$. In particular, $W^{1,p}(\mathrm{X}) \subseteq W^{1,p}_{\Pi}(\mathrm{X})$ and*

$$|Df|_{\Pi,p} \leq |Df|_p \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } f \in W^{1,p}(\mathrm{X}).$$

Proof. To prove the first part of the claim, it suffices to observe that any (Π',p)-weak upper gradient is a (Π,p)-weak upper gradient, thus $W^{1,p}_{\Pi'}(\mathrm{X}) \subseteq W^{1,p}_{\Pi}(\mathrm{X})$ and for any $f \in W^{1,p}_{\Pi'}(\mathrm{X})$ the function $|Df|_{\Pi',p}$ is a (Π,p)-weak upper gradient of f. Consequently, the last part of the statement follows from the first one by recalling Theorem 1.15. ■

1.3.2. *Plans representing a gradient.* A special class of test plans is that of *plans representing a gradient*, which have been introduced by N. Gigli in [11]. Roughly speaking, they are test plans whose derivative at time 0 coincides with the gradient of a given Sobolev function, in some generalised sense. These objects will play a fundamental role in this paper.

Definition 1.17 (Test plan representing a gradient [11]). Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p,q \in (1,\infty)$ satisfy $1/p+1/q=1$. Let

$f \in W^{1,p}(\mathrm{X})$. Then a q-test plan $\boldsymbol{\pi}$ is said to *q-represent the gradient of f* provided the following properties hold:

$$\frac{f \circ \mathrm{e}_t - f \circ \mathrm{e}_0}{\mathrm{E}_{q,t}} \to |Df|_p \circ \mathrm{e}_0 \quad \text{strongly in } L^p(\boldsymbol{\pi}) \text{ as } t \searrow 0, \tag{1.6a}$$

$$\left(\frac{\mathrm{E}_{q,t}}{t}\right)^{q/p} \to |Df|_p \circ \mathrm{e}_0 \quad \text{strongly in } L^p(\boldsymbol{\pi}) \text{ as } t \searrow 0. \tag{1.6b}$$

REMARK 1.18. The above definition of test plan representing a gradient is slightly different from the one introduced in [11]. First of all, a plan $\boldsymbol{\pi}$ representing a gradient in the sense of [11] is not necessarily a test plan; however, for some $t \in (0,1)$, $(\mathrm{restr}_0^t)_{\#}\boldsymbol{\pi}$ is a test plan on X. Also, the approach we chose is not the original one proposed in [11, Definition 3.7], but is rather its equivalent reformulation provided in [11, Proposition 3.11].

LEMMA 1.19. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Let $p, q \in (1,\infty)$ satisfy $1/p + 1/q = 1$. Let $\boldsymbol{\pi}$ be a q-test plan that q-represents the gradient of some function $f \in W^{1,p}(\mathrm{X})$. Then*

$$\frac{f \circ \mathrm{e}_t - f \circ \mathrm{e}_0}{t} \to |Df|_p^p \circ \mathrm{e}_0 \qquad \textit{strongly in } L^1(\boldsymbol{\pi}) \textit{ as } t \searrow 0, \tag{1.7a}$$

$$\frac{\mathrm{E}_{q,t}}{t} \to |Df|_p^{p/q} \circ \mathrm{e}_0 \quad \textit{strongly in } L^q(\boldsymbol{\pi}) \textit{ as } t \searrow 0. \tag{1.7b}$$

Proof. First, let us prove (1.7b). Let $t_i \searrow 0$ be fixed. Since

$$(\mathrm{E}_{q,t_i}/t_i)^{q/p} \to |Df|_p \circ \mathrm{e}_0 \quad \text{strongly in } L^p(\boldsymbol{\pi}) \text{ as } i \to \infty$$

by (1.6b), we can assume (possibly passing to a subsequence) that $\mathrm{E}_{q,t_i}/t_i \to |Df|_p^{p/q} \circ \mathrm{e}_0$ pointwise $\boldsymbol{\pi}$-a.e. as $i \to \infty$ and that there exists $H \in L^p(\boldsymbol{\pi})$ such that $(\mathrm{E}_{q,t_i}/t_i)^{q/p} \leq H$ holds $\boldsymbol{\pi}$-a.e. for every $i \in \mathbb{N}$. In particular, for any $i \in \mathbb{N}$ we have the $\boldsymbol{\pi}$-a.e. inequalities

$$\left|\frac{\mathrm{E}_{q,t_i}}{t_i} - |Df|_p^{p/q} \circ \mathrm{e}_0\right|^q \leq 2^{q-1}\left(\frac{\mathrm{E}_{q,t_i}}{t_i}\right)^q + 2^{q-1}|Df|_p^p \circ \mathrm{e}_0 \leq 2^{q-1}(H^p + |Df|_p^p \circ \mathrm{e}_0).$$

Therefore, by the dominated convergence theorem we get

$$\int |\,\mathrm{E}_{q,t_i}/t_i - |Df|_p^{p/q} \circ \mathrm{e}_0|^q \, \mathrm{d}\boldsymbol{\pi} \to 0 \qquad \text{as } i \to \infty,$$

whence (1.7b) follows (thanks to the arbitrariness of $t_i \searrow 0$).

In order to prove (1.7a), observe that (1.6a), (1.7b), and the Hölder inequality yield

$$\frac{f \circ \mathrm{e}_t - f \circ \mathrm{e}_0}{t} = \frac{f \circ \mathrm{e}_t - f \circ \mathrm{e}_0}{\mathrm{E}_{q,t}} \frac{\mathrm{E}_{q,t}}{t} \to |Df|_p^{1+p/q} \circ \mathrm{e}_0 = |Df|_p^p \circ \mathrm{e}_0$$

strongly in $L^1(\boldsymbol{\pi})$ as $t \searrow 0$. ∎

The existence of plans representing a gradient has been proven in [11, Theorem 3.14]:

THEOREM 1.20 (Existence of test plans representing a gradient [11]). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Let* $p, q \in (1, \infty)$ *satisfy* $1/p + 1/q = 1$. *Fix any* $\mu \in \mathscr{P}_q(\mathrm{X})$ *such that* $\mu \leq C\mathfrak{m}$ *for some constant* $C > 0$. *Then for any* $f \in W^{1,p}(\mathrm{X})$ *there exists a q-test plan* $\boldsymbol{\pi}$ *that q-represents the gradient of* f *and satisfies* $(\mathrm{e}_0)_{\#}\boldsymbol{\pi} = \mu$.

1.3.3. *Velocity of a test plan.* Another useful tool is the *velocity of a test plan* $\boldsymbol{\pi}$, which consists in an abstract way to define—in a suitable sense—the velocity γ'_t at time t of $\boldsymbol{\pi}$-a.e. curve γ. Here, the concept of pullback of a normed module comes into play. The notion of velocity of a test plan was introduced in [12, Theorem 2.3.18] in the case where the tangent module is separable. Below, we adapt the original definition and construction to the case of arbitrary metric measure spaces, thus dropping the separability assumption. In Remark 1.22, we comment on the relation between this generalisation and the previous approach.

THEOREM 1.21 (Velocity of a test plan [12]). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and fix exponents* $p, q \in (1, \infty)$ *such that* $1/p + 1/q = 1$. *Let* $\boldsymbol{\pi}$ *be a given q-test plan on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. *Then there exists a unique element* $\boldsymbol{\pi}' \in (\mathrm{e}^* L^p(T^*\mathrm{X}))^*$ *such that, for any function* $f \in W^{1,p}(\mathrm{X})$ *and* $\mathcal{L}_1$*-a.e.* $t \in [0, 1]$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \mathrm{e}_t := \lim_{h \to 0} \frac{f \circ \mathrm{e}_{t+h} - f \circ \mathrm{e}_t}{h} = \boldsymbol{\pi}'(\mathrm{e}^* \mathrm{d}_p f)(\cdot, t), \tag{1.8}$$

where the derivative is taken with respect to the strong topology of $L^1(\boldsymbol{\pi})$. *Moreover,*

$$|\boldsymbol{\pi}'|(\gamma, t) = |\dot\gamma_t| \quad \textit{for } (\boldsymbol{\pi} \otimes \mathcal{L}_1)\textit{-a.e. } (\gamma, t) \in AC^q([0,1], \mathrm{X}) \times [0, 1]. \tag{1.9}$$

We call $\boldsymbol{\pi}'$ *the* p-velocity *of the* q*-test plan* $\boldsymbol{\pi}$.

Proof. By [16, Theorem 2.1.21], there exists $\mathsf{Der}\colon W^{1,p}(\mathrm{X}) \to L^1(\boldsymbol{\pi} \otimes \mathcal{L}_1)$ linear such that for every $f \in W^{1,p}(\mathrm{X})$,

$$\lim_{h \to 0} \left\| \frac{1}{h}(f \circ \mathrm{e}_{t+h} - f \circ \mathrm{e}_t) - \mathsf{Der}(f)(\cdot, t) \right\|_{L^1(\boldsymbol{\pi})} = 0 \quad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0, 1].$$

Moreover, the operator Der satisfies the estimate

$$|\mathsf{Der}(f)|(\gamma, t) \leq |Df|_p(\gamma_t)|\dot\gamma_t| \quad \text{for } (\boldsymbol{\pi} \otimes \mathcal{L}_1)\text{-a.e. } (\gamma, t). \tag{1.10}$$

Actually, [16, Theorem 2.1.21] was proven in the case where $p = 2$, but the arguments can be easily adapted to treat general exponents $p \in (1, \infty)$. Consider $\mathcal{V} := \{\mathrm{e}^* \mathrm{d}_p f \mid f \in W^{1,p}(\mathrm{X})\}$, which is a generating linear subspace of $\mathrm{e}^* L^p(T^*\mathrm{X})$. We define the operator $L\colon \mathcal{V} \to L^1(\boldsymbol{\pi} \otimes \mathcal{L}_1)$ as $L(\mathrm{e}^* \mathrm{d}_p f) := \mathsf{Der}(f)$. It follows from (1.10) that

$$|L(\mathrm{e}^* \mathrm{d}_p f)|(\gamma, t) \leq |\mathrm{e}^* \mathrm{d}_p f|(\gamma, t)|\dot\gamma_t| \quad \text{for } (\boldsymbol{\pi} \otimes \mathcal{L}_1)\text{-a.e. } (\gamma, t),$$

thus the map L is well-defined, linear, and continuous. Hence, [16, Proposition 3.2.9] grants the existence of a unique element $\boldsymbol{\pi}' \in (\mathrm{e}^* L^p(T^*\mathrm{X}))^*$ such that $\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p f) = L(\mathrm{e}^*\mathrm{d}_p f)$ for every $f \in W^{1,p}(\mathrm{X})$, so that (1.8) is verified. Moreover, $\boldsymbol{\pi}'$ satisfies $|\boldsymbol{\pi}'|(\gamma,t) \leq |\dot\gamma_t|$ for $(\boldsymbol{\pi}\otimes\mathcal{L}_1)$-a.e. (γ,t). Only the converse inequality is left to prove. To this end, fix any dense sequence $(x_i)_i$ in X and define the 1-Lipschitz functions $f_{ij}\colon \mathrm{X}\to\mathbb{R}$ as $f_{ij} := \max\{j-\mathsf{d}(\cdot,x_i),0\}$ for every $i,j\in\mathbb{N}$. Notice that the identity $\mathsf{d}(x,y) = \sup_{i,j}(f_{ij}(x)-f_{ij}(y))$ holds for every $x,y\in\mathrm{X}$. For every $i,j\in\mathbb{N}$ and $\boldsymbol{\pi}$-a.e. γ, we see that $f_{ij}\circ\gamma$ is absolutely continuous and thus $\frac{\mathrm{d}}{\mathrm{d}t} f_{ij}(\gamma_t) = \boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p f_{ij})(\gamma,t)$ for $\mathcal{L}_1$-a.e. $t\in[0,1]$. Therefore, for $(\boldsymbol{\pi}\otimes\mathcal{L}_1)$-a.e. $(\gamma,t)\in AC^q([0,1],\mathrm{X})\times[0,1]$ we may estimate

$$
\begin{aligned}
|\dot\gamma_t| &= \lim_{h\searrow 0}\frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{h} = \lim_{h\searrow 0}\sup_{i,j\in\mathbb{N}}\frac{f_{ij}(\gamma_{t+h})-f_{ij}(\gamma_t)}{h}\\
&= \lim_{h\searrow 0}\sup_{i,j\in\mathbb{N}} \fint_t^{t+h}\frac{\mathrm{d}}{\mathrm{d}s} f_{ij}(\gamma_s)\,\mathrm{d}s = \lim_{h\searrow 0}\sup_{i,j\in\mathbb{N}}\fint_t^{t+h}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p f_{ij})(\gamma,s)\,\mathrm{d}s\\
&\leq \varliminf_{h\searrow 0}\sup_{i,j\in\mathbb{N}}\fint_t^{t+h}|\boldsymbol{\pi}'|(\gamma,s)|Df_{ij}|_p(\gamma_s)\,\mathrm{d}s \leq \lim_{h\searrow 0}\fint_t^{t+h}|\boldsymbol{\pi}'|(\gamma,s)\,\mathrm{d}s\\
&= |\boldsymbol{\pi}'|(\gamma,t),
\end{aligned}
$$

where the last passage is obtained by applying the Lebesgue differentiation theorem to $|\boldsymbol{\pi}'|(\gamma,\cdot)$. This yields the identity in (1.9), whence accordingly the statement follows. ∎

Remark 1.22. In [12, Theorem 2.3.18]—where $L^q(T\mathrm{X})$ is assumed to be separable—the velocity of a q-test plan $\boldsymbol{\pi}$ is given by a family $\{\boldsymbol{\pi}'_t\}_{t\in[0,1]}$ of elements $\boldsymbol{\pi}'_t\in\mathrm{e}_t^*L^q(T\mathrm{X})$ such that for any $f\in W^{1,p}(\mathrm{X})$ we have

$$
\lim_{h\to 0}\left\|\frac{f\circ\mathrm{e}_{t+h}-f\circ\mathrm{e}_t}{h}-(\mathrm{e}_t^*\mathrm{d}_p f)(\boldsymbol{\pi}'_t)\right\|_{L^1(\boldsymbol{\pi})} = 0 \quad \text{for } \mathcal{L}_1\text{-a.e. } t\in[0,1],
$$

and $|\boldsymbol{\pi}'_t|(\gamma) = |\dot\gamma_t|$ for $(\boldsymbol{\pi}\otimes\mathcal{L}_1)$-a.e. (γ,t). Actually, in [12] just the case $p=2$ is considered, but all arguments can be easily carried over to the case of an arbitrary $p\in(1,\infty)$. This kind of statement can be recovered from Theorem 1.21 as follows. First, the separability of $L^q(T\mathrm{X})$ grants that $(\mathrm{e}^*L^p(T^*\mathrm{X}))^*$ can be identified with $\mathrm{e}^*L^q(T\mathrm{X})$ (according to [12, Theorem 1.6.7]). Moreover, write $\mathscr{M}$ for the set of all families $V=\{V_t\}_{t\in[0,1]}\in\prod_{t\in[0,1]}\mathrm{e}_t^*L^q(T\mathrm{X})$ such that

$$
C([0,1],\mathrm{X})\times[0,1]\ni(\gamma,t)\mapsto\mathrm{e}_t^*\omega(V_t)(\gamma)\in\mathbb{R}\text{ is Borel measurable,}
$$

for every $\omega\in L^p(T^*\mathrm{X})$. It can be readily checked that the resulting space $\mathscr{M}$ can be made into an $L^q(\boldsymbol{\pi}\otimes\mathcal{L}_1)$-normed $L^\infty(\boldsymbol{\pi}\otimes\mathcal{L}_1)$-module by defining the following pointwise operations:

$$\begin{aligned}(V+W)_t &:= V_t + W_t \qquad \text{for } \mathcal{L}_1\text{-a.e. } t\in[0,1],\\ (f\cdot V)_t &:= f(\cdot,t)\cdot V_t \quad \text{for } \mathcal{L}_1\text{-a.e. } t\in[0,1],\\ |V|(\gamma,t) &:= \operatorname{ess\,sup}\{\mathrm{e}_t^*\omega(V_t)(\gamma) \mid \omega\in L^p(T^*\mathrm{X}),\ |\omega|\le 1\ \mathfrak{m}\text{-a.e.}\},\end{aligned}$$

for all $V,W\in\mathscr{M}$ and $f\in L^\infty(\boldsymbol{\pi}\otimes\mathcal{L}_1)$. Under suitable functional-analytic assumptions—for instance, the separability of $L^q(T\mathrm{X})$—it turns out that $\mathscr{M}$ can be identified with $\mathrm{e}^*L^q(T\mathrm{X})$; we omit the details. Hence, the element $\boldsymbol{\pi}'\in\mathrm{e}^*L^q(T\mathrm{X})$ provided by Theorem 1.21 can be viewed as a family $\{\boldsymbol{\pi}'_t\}_{t\in[0,1]}\in\mathscr{M}$ satisfying the conclusions of [12, Theorem 2.3.18].

Proposition 1.23. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space and* $p,q\in(1,\infty)$ *satisfy* $1/p+1/q=1$. *Let* $\boldsymbol{\pi}$ *be a* q*-test plan on* X. *Then for every function* $f\in W^{1,p}(\mathrm{X})$ *the mapping* $t\mapsto f\circ\mathrm{e}_t$ *belongs to* $AC^q([0,1],L^1(\boldsymbol{\pi}))$ *and satisfies*

$$f\circ\mathrm{e}_t - f\circ\mathrm{e}_s = \int_s^t \boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p f)(\cdot,r)\,\mathrm{d}r \qquad \textit{for all } s,t\in[0,1] \textit{ with } s<t. \tag{1.11}$$

Proof. Define

$$\phi(r) := \Big(\int |\dot\gamma_r|^q\,\mathrm{d}\boldsymbol{\pi}(\gamma)\Big)^{1/q} \qquad \text{for } \mathcal{L}_1\text{-a.e. } r\in[0,1].$$

Given that $\int_0^1\phi(r)^q\,\mathrm{d}r = \int\int_0^1|\dot\gamma_r|^q\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\pi}(\gamma)<+\infty$, we have $\phi\in L^q(0,1)$. Fix $f\in W^{1,p}(\mathrm{X})$ and $s,t\in[0,1]$ with $s<t$. Then

$$\begin{aligned}\|f\circ\mathrm{e}_t - f\circ\mathrm{e}_s\|_{L^1(\boldsymbol{\pi})} &= \int |f(\gamma_t)-f(\gamma_s)|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\le \int\int_s^t |Df|_p(\gamma_r)|\dot\gamma_r|\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\le \int_s^t \Big(\int |Df|_p^p\circ\mathrm{e}_r\,\mathrm{d}\boldsymbol{\pi}\Big)^{1/p}\Big(\int|\dot\gamma_r|^q\,\mathrm{d}\boldsymbol{\pi}(\gamma)\Big)^{1/q}\mathrm{d}r\\ &\le \mathrm{Comp}(\boldsymbol{\pi})^{1/p}\||Df|_p\|_{L^p(\mathfrak{m})}\int_s^t\phi(r)\,\mathrm{d}r,\end{aligned}$$

which shows that the curve $[0,1]\ni t\mapsto f\circ\mathrm{e}_t\in L^1(\boldsymbol{\pi})$ is q-absolutely continuous. Moreover, we know from Theorem 1.21 that the $L^1(\boldsymbol{\pi})$-derivative $\frac{\mathrm{d}}{\mathrm{d}t}f\circ\mathrm{e}_t$ exists and equals $\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p f)(\cdot,t)$ at $\mathcal{L}_1$-a.e. $t\in[0,1]$. Therefore, the identity in (1.11) follows from [16, Proposition 1.3.16]. ∎

2. Master test plans on metric measure spaces

2.1. Properties of plans representing a gradient. In order to prove our main theorem, we first need to study some properties of plans representing a gradient. Roughly speaking, we aim to show that if $\boldsymbol{\pi}$ represents the gradient of f, then for any Sobolev function g the derivative of $t \mapsto g \circ \mathrm{e}_t$ at $t = 0$ coincides with $\mathrm{d}g(\nabla f) \circ \mathrm{e}_0$, in a sense; see Proposition 2.3.

LEMMA 2.1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Let* $p, q \in (1, \infty)$ *satisfy* $1/p + 1/q = 1$. *Let* $f \in W^{1,p}(\mathrm{X})$. *Let* $\boldsymbol{\pi}$ *be a* q*-test plan that* q*-represents the gradient of* f. *Then for every function* $G \in L^p(\mathfrak{m})$ *with* $G \geq 0$ *there exists a family* $\{\Phi_t\}_{t \in (0,1)} \subseteq L^1(\boldsymbol{\pi})$ *such that*

$$\fint_0^t G \circ \mathrm{e}_s |\boldsymbol{\pi}'|(\cdot, s)\,\mathrm{d}s \leq \Phi_t \quad \boldsymbol{\pi}\text{-a.e.} \quad \text{for every } t \in (0,1) \tag{2.1}$$

and $\Phi_t \to G \circ \mathrm{e}_0 |Df|_p^{p/q} \circ \mathrm{e}_0$ *strongly in* $L^1(\boldsymbol{\pi})$ *as* $t \searrow 0$. *In particular, if for some* $g \in W^{1,p}(\mathrm{X})$ *and* $\ell \in L^1(\boldsymbol{\pi})$ *and* $t_i \searrow 0$ *we have*

$$\fint_0^{t_i} \boldsymbol{\pi}'(\mathrm{e}^* \mathrm{d}_p g)(\cdot, s)\,\mathrm{d}s \rightharpoonup \ell \quad \text{weakly in } L^1(\boldsymbol{\pi}) \text{ as } i \to \infty,$$

then

$$|\ell| \leq |Df|_p^{p/q} \circ \mathrm{e}_0 |Dg|_p \circ \mathrm{e}_0 \quad \boldsymbol{\pi}\text{-a.e.} \tag{2.2}$$

Proof. Let $G \in L^p(\mathfrak{m})$, $G \geq 0$ be fixed. If we define

$$R_t \coloneqq \fint_0^t |G \circ \mathrm{e}_s - G \circ \mathrm{e}_0|\, |\boldsymbol{\pi}'|(\cdot, s)\,\mathrm{d}s,$$

then

$$\begin{aligned}\fint_0^t G \circ \mathrm{e}_s |\boldsymbol{\pi}'|(\cdot, s)\,\mathrm{d}s &\leq R_t + G \circ \mathrm{e}_0 \fint_0^t |\boldsymbol{\pi}'|(\cdot, s)\,\mathrm{d}s \\ &\leq R_t + G \circ \mathrm{e}_0 \Big(\fint_0^t |\boldsymbol{\pi}'|^q(\cdot, s)\,\mathrm{d}s \Big)^{1/q} =: \Phi_t\end{aligned}$$

in the $\boldsymbol{\pi}$-a.e. sense. Observe that

$$\begin{aligned}\int R_t\,\mathrm{d}\boldsymbol{\pi} &= \int\fint_0^t |G \circ \mathrm{e}_s - G \circ \mathrm{e}_0| |\boldsymbol{\pi}'|(\cdot, s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} \\ &\leq \Big(\int\fint_0^t |G \circ \mathrm{e}_s - G \circ \mathrm{e}_0|^p\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} \Big)^{1/p} \Big(\int\fint_0^t |\boldsymbol{\pi}'|^q(\cdot, s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} \Big)^{1/q} \\ &= \Big(\fint_0^t \|G \circ \mathrm{e}_s - G \circ \mathrm{e}_0\|_{L^p(\boldsymbol{\pi})}^p\,\mathrm{d}s \Big)^{1/p} \Big(\int \frac{\mathrm{E}_{q,t}^q}{t^q}\,\mathrm{d}\boldsymbol{\pi} \Big)^{1/q} \to 0\end{aligned}$$

as $t \searrow 0$, where we used the fact that $\int \mathrm{E}_{q,t}^q / t^q\,\mathrm{d}\boldsymbol{\pi} \to \int |Df|_p^p \circ \mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi}$ as $t \searrow 0$ and the continuity of $[0,1] \ni s \mapsto G \circ \mathrm{e}_s \in L^p(\boldsymbol{\pi})$. Also, we see that $(\fint_0^t |\boldsymbol{\pi}'|^q(\cdot, s)\,\mathrm{d}s)^{1/q} = \mathrm{E}_{q,t}/t \to |Df|_p^{p/q} \circ \mathrm{e}_0$ strongly in $L^q(\boldsymbol{\pi})$ as $t \searrow 0$, whence accordingly $G\circ\mathrm{e}_0(\fint_0^t|\boldsymbol{\pi}'|^q(\cdot,s)\,\mathrm{d}s)^{1/q}\to G\circ\mathrm{e}_0|Df|_p^{p/q}\circ\mathrm{e}_0$ strongly in $L^1(\boldsymbol{\pi})$. All in all, we have proved that $\Phi_t\to G\circ\mathrm{e}_0|Df|_p^{p/q}\circ\mathrm{e}_0$ in $L^1(\boldsymbol{\pi})$.

Let us now prove the last claim. Thanks to the first part of the statement applied to the function $G:=|Dg|_p$, we can find a sequence $(\Phi_i)_i\subseteq L^1(\boldsymbol{\pi})$ such that $\Phi_i\to|Dg|_p\circ\mathrm{e}_0|Df|_p^{p/q}\circ\mathrm{e}_0$ strongly in $L^1(\boldsymbol{\pi})$ as $i\to\infty$ and

$$\fint_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_pg)(\cdot,s)\,\mathrm{d}s\leq\fint_0^{t_i}|Dg|_p\circ\mathrm{e}_s|\boldsymbol{\pi}'|(\cdot,s)\,\mathrm{d}s\leq\Phi_i\quad\boldsymbol{\pi}\text{-a.e.}\tag{2.3}$$

for every $i\in\mathbb{N}$. In order to prove the inequality in (2.2), we can argue by contradiction: suppose there exists a Borel set $P\subseteq C([0,1],\mathrm{X})$ with $\boldsymbol{\pi}(P)>0$ and $\ell(\gamma)>|Df|_p^{p/q}(\gamma_0)|Dg|_p(\gamma_0)$ for $\boldsymbol{\pi}$-a.e. $\gamma\in P$. Since $\fint_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_pg)(\cdot,s)\,\mathrm{d}s\rightharpoonup\ell$ and $\Phi_i\rightharpoonup|Df|_p^{p/q}\circ\mathrm{e}_0|Dg|_p\circ\mathrm{e}_0$ weakly in $L^1(\boldsymbol{\pi})$ as $i\to\infty$, and $\mathbb{1}_P\in L^\infty(\boldsymbol{\pi})$, we deduce that

$$\begin{aligned}\int_P|Df|_p^{p/q}\circ\mathrm{e}_0|Dg|_p\circ\mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi}\ &<\ \int_P\ell\,\mathrm{d}\boldsymbol{\pi}\\&=\lim_{i\to\infty}\int\mathbb{1}_P\fint_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_pg)(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}\\&\overset{(2.3)}{\leq}\lim_{i\to\infty}\int\mathbb{1}_P\Phi_i\,\mathrm{d}\boldsymbol{\pi}\\&=\int_P|Df|_p^{p/q}\circ\mathrm{e}_0|Dg|_p\circ\mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi},\end{aligned}$$

which leads to a contradiction. Therefore, (2.2) follows. ∎

Corollary 2.2. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Let* $p,q\in(1,\infty)$ *satisfy* $1/p+1/q=1$. *Let* $f\in W^{1,p}(\mathrm{X})$. *Let* $\boldsymbol{\pi}$ *be a* q*-test plan that* q*-represents the gradient of* f. *Fix* $g\in W^{1,p}(\mathrm{X})$ *and* $t_i\searrow 0$. *Then there exist a subsequence* $(t_{i_j})_j$ *and a function* $\ell\in L^1(\boldsymbol{\pi})$ *such that*

$$\fint_0^{t_{i_j}}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_pg)(\cdot,s)\,\mathrm{d}s\rightharpoonup\ell\qquad\textit{weakly in } L^1(\boldsymbol{\pi})\textit{ as } j\to\infty.\tag{2.4}$$

Proof. Pick functions $\{\Phi_t\}_{t\in(0,1)}\subseteq L^1(\boldsymbol{\pi})$ associated with $G:=|Dg|_p$ as in Lemma 2.1. Given that the sequence $(\Phi_{t_i})_i$ is strongly convergent in $L^1(\boldsymbol{\pi})$, we can find a subsequence $(t_{i_j})_j$ and a non-negative function $H\in L^1(\boldsymbol{\pi})$ such that $\Phi_{t_{i_j}}\leq H$ holds $\boldsymbol{\pi}$-a.e. for every $j\in\mathbb{N}$. Then

$$\Big|\fint_0^{t_{i_j}}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_pg)(\cdot,s)\,\mathrm{d}s\Big|\leq\fint_0^{t_{i_j}}|Dg|_p\circ\mathrm{e}_s|\boldsymbol{\pi}'|(\cdot,s)\,\mathrm{d}s\overset{(2.1)}{\leq}\Phi_{t_{i_j}}\leq H\quad\boldsymbol{\pi}\text{-a.e.}$$

for every $j\in\mathbb{N}$. Therefore, thanks to [16, Lemma 1.3.22] we know that there exists a function $\ell \in L^1(\boldsymbol{\pi})$ such that (possibly passing to a subsequence, not relabelled) the property in (2.4) holds. ∎

Proposition 2.3. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Let* $p,q \in (1,\infty)$ *satisfy* $1/p+1/q=1$. *Let* $f \in W^{1,p}(\mathrm{X})$. *Let* $\boldsymbol{\pi}$ *be a* q*-test plan that* q*-represents the gradient of* f. *Fix any two sequences* $(g_n)_n \subseteq W^{1,p}(\mathrm{X})$ *and* $t_i \searrow 0$. *Then there exist a subsequence* $(t_{i_j})_j$ *and an element* $\eta \in \mathsf{Dual}(\mathrm{e}_0^*\mathrm{d}_p f)$*—where the mapping* Dual *is defined as in* (1.2)*—such that*

$$\frac{g_n\circ \mathrm{e}_{t_{i_j}} - g_n\circ \mathrm{e}_0}{t_{i_j}} \rightharpoonup \eta(\mathrm{e}_0^*\mathrm{d}_p g_n) \quad \textit{weakly in } L^1(\boldsymbol{\pi}) \textit{ as } j\to\infty \tag{2.5}$$

for every $n\in\mathbb{N}$.

Proof. Fix $\mathcal{C}\subseteq\{f\}\cup\{g_n \mid n\in\mathbb{N}\}$ with the property that $\{\mathrm{e}_0^*\mathrm{d}_p g \mid g\in\mathcal{C}\}$ is a maximal linearly independent subset of $\{\mathrm{e}_0^*\mathrm{d}_p g \mid g\in\{f\}\cup\{g_n\}_n\}$. By Corollary 2.2 and a diagonalisation argument, the sequence $t_i\searrow 0$ admits a subsequence (not relabelled) such that $⨍_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s \rightharpoonup \ell_g$ weakly in $L^1(\boldsymbol{\pi})$ as $i\to\infty$ for every $g\in\mathcal{C}$, for some limit functions $\ell_g\in L^1(\boldsymbol{\pi})$. Denote by $\mathcal{V}\subseteq \mathrm{e}_0^*L^p(T^*\mathrm{X})$ the linear span of $\{\mathrm{e}_0^*\mathrm{d}_p g : g\in\mathcal{C}\}$. Then we define $L\colon\mathcal{V}\to L^1(\boldsymbol{\pi})$ as the unique linear operator satisfying $L(\mathrm{e}_0^*\mathrm{d}_p g)=\ell_g$ for every $g\in\mathcal{C}$. Given that $⨍_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s \rightharpoonup L(\mathrm{e}_0^*\mathrm{d}_p g)$ weakly in $L^1(\boldsymbol{\pi})$ as $i\to\infty$ for every $g\in\mathcal{V}$, it follows from the last part of the statement of Lemma 2.1 that

$$|L(\mathrm{e}_0^*\mathrm{d}_p g)| \le |Df|_p^{p/q}\circ\mathrm{e}_0|Dg|_p\circ\mathrm{e}_0 = |\mathrm{e}_0^*\mathrm{d}_p f|^{p/q}|\mathrm{e}_0^*\mathrm{d}_p g| \quad \boldsymbol{\pi}\text{-a.e.}$$

for every $g\in\mathcal{V}$. Hence, writing $\mathscr{M}$ for the $L^p(\boldsymbol{\pi})$-normed $L^\infty(\boldsymbol{\pi})$-submodule of $\mathrm{e}_0^*L^p(T^*\mathrm{X})$ generated by $\mathcal{V}$, we know from [16, Proposition 3.2.9] that there exists a unique $L^\infty(\boldsymbol{\pi})$-linear map $\tilde\eta\colon\mathscr{M}\to L^1(\boldsymbol{\pi})$ such that $\tilde\eta|_{\mathcal{V}}=L$ and $|\tilde\eta(\omega)|\le|\mathrm{e}_0^*\mathrm{d}_p f|^{p/q}|\omega|$ in the $\boldsymbol{\pi}$-a.e. sense for every $\omega\in\mathscr{M}$. By using the Hahn–Banach theorem, we can find a (not necessarily unique) element $\eta\in(\mathrm{e}_0^*L^p(T^*\mathrm{X}))^*$ which extends $\tilde\eta$ and satisfies the inequality $|\eta|\le|\mathrm{e}_0^*\mathrm{d}_p f|^{p/q}$ in the $\boldsymbol{\pi}$-a.e. sense. Observe that the property (2.5) is verified by construction. It only remains to show that $\eta\in\mathsf{Dual}(\mathrm{e}_0^*\mathrm{d}_p f)$. Since $\boldsymbol{\pi}$ represents the gradient of f, one sees that $(f\circ\mathrm{e}_{t_i}-f\circ\mathrm{e}_0)/t_i\to|Df|_p^p\circ\mathrm{e}_0=|\mathrm{e}_0^*\mathrm{d}_p f|^p$ strongly in $L^1(\boldsymbol{\pi})$ as $i\to\infty$. We also have $(f\circ\mathrm{e}_{t_i}-f\circ\mathrm{e}_0)/t_i\rightharpoonup\eta(\mathrm{e}_0^*\mathrm{d}_p f)$ weakly in $L^1(\boldsymbol{\pi})$ as $i\to\infty$ by definition of η, whence it follows that $\eta(\mathrm{e}_0^*\mathrm{d}_p f)=|\mathrm{e}_0^*\mathrm{d}_p f|^p$ holds $\boldsymbol{\pi}$-a.e. Then

$$|\mathrm{e}_0^*\mathrm{d}_p f|^p=\eta(\mathrm{e}_0^*\mathrm{d}_p f)\le|\eta|\,|\mathrm{e}_0^*\mathrm{d}_p f|\le|\mathrm{e}_0^*\mathrm{d}_p f|^{p/q+1}=|\mathrm{e}_0^*\mathrm{d}_p f|^p \quad \boldsymbol{\pi}\text{-a.e.},$$

whence $|\eta|=|\mathrm{e}_0^*\mathrm{d}_p f|^{p/q}$ holds $\boldsymbol{\pi}$-a.e. and accordingly $\eta\in\mathsf{Dual}(\mathrm{e}_0^*\mathrm{d}_p f)$, as required. ∎

Albeit not strictly needed for the purposes of this article, let us illustrate a reinforcement of Proposition 2.3 under some additional assumptions on the metric measure space.

COROLLARY 2.4. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Let* $p,q\in(1,\infty)$ *satisfy* $1/p+1/q=1$. *Let* $f\in W^{1,p}(\mathrm{X})$. *Let* $\boldsymbol{\pi}$ *be a* q*-test plan on* X *that* q*-represents the gradient of* f.

(i) *If* $W^{1,p}(\mathrm{X})$ *is separable, then for any sequence* $t_i\searrow 0$ *there exist a subsequence* $(t_{i_j})_j$ *and an element* $\eta\in\mathsf{Dual}(\mathrm{e}_0^*\mathrm{d}_p f)$ *such that for any* $g\in W^{1,p}(\mathrm{X})$ *we have*

$$\frac{g\circ\mathrm{e}_{t_{i_j}}-g\circ\mathrm{e}_0}{t_{i_j}}\rightharpoonup\eta(\mathrm{e}_0^*\mathrm{d}_p g)\quad\textit{weakly in } L^1(\boldsymbol{\pi})\textit{ as } j\to\infty. \tag{2.6}$$

(ii) *If* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is infinitesimally Hilbertian and* $p=2$, *then for any* $g\in W^{1,2}(\mathrm{X})$ *we have*

$$\frac{g\circ\mathrm{e}_t-g\circ\mathrm{e}_0}{t}\rightharpoonup\langle\nabla_2 g,\nabla_2 f\rangle\circ\mathrm{e}_0\quad\textit{weakly in } L^1(\boldsymbol{\pi})\textit{ as } t\searrow 0. \tag{2.7}$$

Proof. (i) Fix a sequence $t_i\searrow 0$ and a countable, strongly dense subset $\mathcal{C}$ of $W^{1,p}(\mathrm{X})$. By virtue of Proposition 2.3, there exists $\eta\in\mathsf{Dual}(\mathrm{e}_0^*\mathrm{d}_p f)$ such that (up to a subsequence, not relabelled)

$$⨍_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s\rightharpoonup\eta(\mathrm{e}_0^*\mathrm{d}_p g)\quad\text{weakly in } L^1(\boldsymbol{\pi})\text{ as } i\to\infty \tag{2.8}$$

for every $g\in\mathcal{C}$. Now fix $g\in W^{1,p}(\mathrm{X})$. Choose any sequence $(g_n)_n\subseteq\mathcal{C}$ such that $g_n\to g$ with respect to the strong topology of $W^{1,p}(\mathrm{X})$. Fix any $h\in L^\infty(\boldsymbol{\pi})$ and some constant $M>0$ satisfying the inequality

$$\int\frac{\mathrm{E}_{q,t_i}^q}{t_i^q}\,\mathrm{d}\boldsymbol{\pi}\le M^q\quad\text{for every } i\in\mathbb{N}.$$

Given any $i,n\in\mathbb{N}$, we can estimate

$$\bigg|\int h⨍_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}-\int h\eta(\mathrm{e}_0^*\mathrm{d}_p g)\,\mathrm{d}\boldsymbol{\pi}\bigg|\le A_{i,n}+B_{i,n}+C_n, \tag{2.9}$$

where we set

$$\begin{aligned}
A_{i,n}&:=\bigg|\int h⨍_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p(g-g_n))(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}\bigg|,\\
B_{i,n}&:=\bigg|\int h⨍_0^{t_i}\boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g_n)(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}-\int h\eta(\mathrm{e}_0^*\mathrm{d}_p g_n)\,\mathrm{d}\boldsymbol{\pi}\bigg|,\\
C_n&:=\bigg|\int h\eta(\mathrm{e}_0^*\mathrm{d}_p(g_n-g))\,\mathrm{d}\boldsymbol{\pi}\bigg|.
\end{aligned}$$

Observe that

$$\begin{aligned}
A_{i,n} &\leq \|h\|_{L^\infty(\boldsymbol{\pi})} \int \fint_0^{t_i} |D(g-g_n)|_p \circ \mathrm{e}_s |\boldsymbol{\pi}'|(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} \\
&\leq \|h\|_{L^\infty(\boldsymbol{\pi})} \bigg(\int \fint_0^{t_i} |D(g-g_n)|_p^p \circ \mathrm{e}_s\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} \bigg)^{1/p} \bigg(\int \fint_0^{t_i} |\boldsymbol{\pi}'|^q(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} \bigg)^{1/q} \\
&\leq \mathrm{Comp}(\boldsymbol{\pi})^{1/p} \|h\|_{L^\infty(\boldsymbol{\pi})} \bigg(\int |D(g-g_n)|_p^p \,\mathrm{d}\mathfrak{m} \bigg)^{1/p} \bigg(\int \frac{\mathrm{E}_{q,t_i}^q}{t_i^q}\,\mathrm{d}\boldsymbol{\pi} \bigg)^{1/q} \\
&\leq M\,\mathrm{Comp}(\boldsymbol{\pi})^{1/p} \|h\|_{L^\infty(\boldsymbol{\pi})} \|g-g_n\|_{W^{1,p}(\mathrm{X})}.
\end{aligned}$$

Moreover, it follows from (2.8) that $\lim_{i\to\infty} B_{i,n} = 0$ for any given $n \in \mathbb{N}$. Finally, we estimate

$$\begin{aligned}
C_n &\leq \|h\|_{L^\infty(\boldsymbol{\pi})} \int |D(g_n-g)|_p \circ \mathrm{e}_0 |\eta|\,\mathrm{d}\boldsymbol{\pi} \\
&\leq \|h\|_{L^\infty(\boldsymbol{\pi})} \bigg(\int |D(g_n-g)|_p^p \circ \mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi} \bigg)^{1/p} \bigg(\int |\eta|^q\,\mathrm{d}\boldsymbol{\pi} \bigg)^{1/q} \\
&\leq \mathrm{Comp}(\boldsymbol{\pi})^{1/p} \|h\|_{L^\infty(\boldsymbol{\pi})} \bigg(\int |D(g_n-g)|_p^p\,\mathrm{d}\mathfrak{m} \bigg)^{1/p} \bigg(\int |Df|_p^p \circ \mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi} \bigg)^{1/q} \\
&\leq \mathrm{Comp}(\boldsymbol{\pi}) \|h\|_{L^\infty(\boldsymbol{\pi})} \|g_n-g\|_{W^{1,p}(\mathrm{X})} \|f\|_{W^{1,p}(\mathrm{X})}^{p/q}.
\end{aligned}$$

Hence, given any $\varepsilon > 0$, we can find $n \in \mathbb{N}$ such that $A_{i,n} + C_n \leq \varepsilon$ for every $i \in \mathbb{N}$. Then

$$\varlimsup_{i\to\infty} \bigg| \int h \fint_0^{t_i} \boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} - \int h\eta(\mathrm{e}_0^*\mathrm{d}_p g)\,\mathrm{d}\boldsymbol{\pi} \bigg| \overset{(2.9)}{\leq} \varepsilon + \lim_{i\to\infty} B_{i,n} = \varepsilon.$$

By letting $\varepsilon \searrow 0$, we conclude that

$$\lim_{i\to\infty} \int h \fint_0^{t_i} \boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi} = \int h\eta(\mathrm{e}_0^*\mathrm{d}_p g)\,\mathrm{d}\boldsymbol{\pi}$$

for every $h \in L^\infty(\boldsymbol{\pi})$, whence $\fint_0^{t_i} \boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s \rightharpoonup \eta(\mathrm{e}_0^*\mathrm{d}_p g)$ weakly in $L^1(\boldsymbol{\pi})$ as $i \to \infty$. Given that $(g \circ \mathrm{e}_{t_i} - g \circ \mathrm{e}_0)/t_i = \fint_0^{t_i} \boldsymbol{\pi}'(\mathrm{e}^*\mathrm{d}_p g)(\cdot,s)\,\mathrm{d}s$ by Proposition 1.23, we have proven (i).

(ii) The infinitesimal Hilbertianity assumption grants that both $W^{1,2}(\mathrm{X})$ and $L^2(T\mathrm{X})$ are separable; see, e.g., [16, Proposition 4.3.5]. In particular, we know from [12, Theorem 1.6.7] that the space $(\mathrm{e}_0^* L^2(T^*\mathrm{X}))^*$ is isometrically isomorphic to $\mathrm{e}_0^* L^2(T\mathrm{X})$. Thanks to this fact, we can identify any element η satisfying (2.6) (for some $t_{i_j} \searrow 0$) with an element v of the pullback module $\mathrm{e}_0^* L^2(T\mathrm{X})$. Since $(\mathrm{e}_0^*\mathrm{d}_2 f)(v) = |\mathrm{e}_0^*\mathrm{d}_2 f|^2 = |v|^2$ holds $\boldsymbol{\pi}$-a.e., we get

$$\begin{aligned}
|v - \mathrm{e}_0^*\nabla_2 f|^2 &= |v|^2 - 2\langle v, \mathrm{e}_0^*\nabla_2 f\rangle + |\mathrm{e}_0^*\nabla_2 f|^2 \\
&= |v|^2 - 2(\mathrm{e}_0^*\mathrm{d}_2 f)(v) + |\mathrm{e}_0^*\mathrm{d}_2 f|^2 = 0
\end{aligned}$$

in the $\boldsymbol{\pi}$-a.e. sense, whence $v = \mathrm{e}_0^*\nabla_2 f$. In particular, the limit v does not depend on $(t_{i_j})_j$, thus accordingly, as $t \searrow 0$, we have

$$\frac{g\circ \mathrm{e}_t - g\circ \mathrm{e}_0}{t} \rightharpoonup (\mathrm{e}_0^*\mathrm{d}_2 g)(\mathrm{e}_0^*\nabla_2 f) = \langle \nabla_2 g, \nabla_2 f\rangle \circ \mathrm{e}_0 \quad \text{weakly in } L^1(\boldsymbol{\pi})$$

for every $g \in W^{1,2}(\mathrm{X})$. Therefore, the sought conclusion (2.7) is reached. ∎

2.2. Existence of master test plans on metric measure spaces. We now have at our disposal all the ingredients that we need to prove our main theorem, which says that a single test plan is sufficient to identify the minimal relaxed slope of every Sobolev function. In this regard, the relevant notion is that of master test plan:

Definition 2.5 (Master test plan). Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix $p,q\in(1,\infty)$ such that $1/p+1/q=1$. Then a q-test plan $\boldsymbol{\pi}_q$ on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is said to be a *master q-test plan* provided that

$$|Df|_{\boldsymbol{\pi}_q,p} = |Df|_p \quad \text{for every } f\in W^{1,p}(\mathrm{X}).$$

Here, we are using the fact that $W^{1,p}(\mathrm{X}) \subseteq W^{1,p}_{\boldsymbol{\pi}_q}(\mathrm{X})$, which is granted by Proposition 1.16.

Hence, our main result about the identification of the minimal relaxed slope reads as follows:

Theorem 2.6 (Existence of master test plans). *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix any $p,q\in(1,\infty)$ such that $1/p+1/q=1$. Then there exists a master q-test plan $\boldsymbol{\pi}_q$ on $(\mathrm{X},\mathsf{d},\mathfrak{m})$.*

Proof. We subdivide the proof into several steps:

Step 1. First of all, fix a countable family $\mathcal{C}\subseteq W^{1,p}(\mathrm{X})$ having the following property: given any $f\in W^{1,p}(\mathrm{X})$, there exists a sequence $(f_n)_n\subseteq\mathcal{C}$ such that $f_n\to f$ and $|Df_n|_p\to|Df|_p$ strongly in $L^p(\mathfrak{m})$ as $n\to\infty$. The existence of $\mathcal{C}$ is granted by the separability of the product space $L^p(\mathfrak{m})\times L^p(\mathfrak{m})$ and thus, a fortiori, of its subset $\{(f,|Df|_p)\mid f\in W^{1,p}(\mathrm{X})\}$. Fix any measure $\tilde{\mathfrak{m}}\in\mathscr{P}_q(\mathrm{X})$ such that $\mathfrak{m}\ll\tilde{\mathfrak{m}}\leq C\mathfrak{m}$ for some $C>0$, whose existence is shown in Remark 1.1. Given any $f\in\mathcal{C}$, there exists a q-test plan $\boldsymbol{\pi}^f$ on X that q-represents the gradient of f and satisfies $(\mathrm{e}_0)_\#\boldsymbol{\pi}^f=\tilde{\mathfrak{m}}$ (by Theorem 1.20). Let us define $\Pi:=\{\boldsymbol{\pi}^f : f\in\mathcal{C}\}$. We aim to prove that

$$|Df|_{\Pi,p} = |Df|_p \quad \text{for every } f\in W^{1,p}(\mathrm{X}). \tag{2.10}$$

Since $|Df|_{\Pi,p}\leq|Df|_p$ $\mathfrak{m}$-a.e. by Proposition 1.16, to prove (2.10) it suffices to show that

$$\int|Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}} \leq \int|Df|_{\Pi,p}^p\,\mathrm{d}\tilde{\mathfrak{m}} \quad \text{for every } f\in W^{1,p}(\mathrm{X}). \tag{2.11}$$

Step 2. In order to show (2.11), let $f\in W^{1,p}(\mathrm{X})$ be fixed. Choose any sequence $(f_n)_n\subseteq\mathcal{C}$ such that $f_n\to f$ and $|Df_n|_p\to|Df|_p$ in $L^p(\mathfrak{m})$.

Possibly passing to a subsequence (not relabelled), we may assume that $|Df_n|_p \to |Df|_p$ pointwise $\mathfrak{m}$-a.e. and that there exists a function $G \in L^p(\mathfrak{m})$ such that $|Df_n|_p \leq G$ holds $\mathfrak{m}$-a.e. for every $n \in \mathbb{N}$. For brevity, let us put $\boldsymbol{\pi}^n := \boldsymbol{\pi}^{f_n}$ for every $n \in \mathbb{N}$. Given any $n \in \mathbb{N}$, thanks to Proposition 2.3 there exist an element $\eta_n \in \mathsf{Dual}(\mathrm{e}_0^*\mathrm{d}_p f_n)$ and a sequence $(t_i^n)_i \subseteq (0,1)$ with $\lim_{i\to\infty} t_i^n = 0$ such that

$$\frac{f\circ \mathrm{e}_{t_i^n} - f\circ \mathrm{e}_0}{t_i^n} \rightharpoonup \eta_n(\mathrm{e}_0^*\mathrm{d}_p f) \quad \text{weakly in } L^1(\boldsymbol{\pi}^n) \text{ as } i\to\infty. \tag{2.12}$$

Therefore, by applying (2.12) we deduce that

$$\begin{aligned}
&\int \eta_n(\mathrm{e}_0^*\mathrm{d}_p f)\,\mathrm{d}\boldsymbol{\pi}^n \\
&\quad = \lim_{i\to\infty} \int \frac{f\circ \mathrm{e}_{t_i^n} - f\circ \mathrm{e}_0}{t_i^n}\,\mathrm{d}\boldsymbol{\pi}^n \leq \varliminf_{i\to\infty} \frac{1}{t_i^n}\int |f(\gamma_{t_i^n}) - f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}^n(\gamma) \\
&\quad \leq \varliminf_{i\to\infty} \int \fint_0^{t_i^n} \left|\frac{\mathrm{d}}{\mathrm{d}s} f(\gamma_s)\right| \mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}^n(\gamma) \leq \varliminf_{i\to\infty} \int \fint_0^{t_i^n} |Df|_{\Pi,p}(\gamma_s)|\dot\gamma_s|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}^n(\gamma) \\
&\quad \leq \varliminf_{i\to\infty} \left(\int \fint_0^{t_i^n} |Df|_{\Pi,p}^p \circ \mathrm{e}_s\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}^n\right)^{1/p} \left(\int \fint_0^{t_i^n} |\dot\gamma_s|^q\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}^n(\gamma)\right)^{1/q} \\
&\quad = \lim_{i\to\infty} \left(\fint_0^{t_i^n} \big\||Df|_{\Pi,p}\circ \mathrm{e}_s\big\|_{L^p(\boldsymbol{\pi}^n)}^p\,\mathrm{d}s\right)^{1/p} \left(\int \frac{\mathrm{E}_{q,t_i^n}^q}{(t_i^n)^q}\,\mathrm{d}\boldsymbol{\pi}^n\right)^{1/q} \\
&\quad = \left(\int |Df|_{\Pi,p}^p \circ \mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi}^n\right)^{1/p} \left(\int |Df_n|_p^p \circ \mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi}^n\right)^{1/q} \\
&\quad = \big\||Df|_{\Pi,p}\big\|_{L^p(\tilde{\mathfrak{m}})} \big\||Df_n|_p\big\|_{L^p(\tilde{\mathfrak{m}})}^{p/q}.
\end{aligned}$$

Furthermore, observe that for any $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\left|\int |Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}} - \int \eta_n(\mathrm{e}_0^*\mathrm{d}_p f)\,\mathrm{d}\boldsymbol{\pi}^n\right| \\
&\quad \leq \left|\int |Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}} - \int \eta_n(\mathrm{e}_0^*\mathrm{d}_p f_n)\,\mathrm{d}\boldsymbol{\pi}^n\right| + \left|\int \eta_n(\mathrm{e}_0^*\mathrm{d}_p (f_n - f))\,\mathrm{d}\boldsymbol{\pi}^n\right| \\
&\quad \leq \left|\int |Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}} - \int |\mathrm{e}_0^*\mathrm{d}_p f_n|^p\,\mathrm{d}\boldsymbol{\pi}^n\right| + \int |\eta_n|\,|D(f_n - f)|_p \circ \mathrm{e}_0\,\mathrm{d}\boldsymbol{\pi}^n \\
&\quad \leq \left|\int |Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}} - \int |Df_n|_p^p\,\mathrm{d}\tilde{\mathfrak{m}}\right| + \left(\int |\eta_n|^q\,\mathrm{d}\boldsymbol{\pi}^n\right)^{1/q} \left(\int |D(f_n - f)|_p^p\,\mathrm{d}\tilde{\mathfrak{m}}\right)^{1/p} \\
&\quad \leq \left|\int |Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}} - \int |Df_n|_p^p\,\mathrm{d}\tilde{\mathfrak{m}}\right| + C^{1/p} \left(\int |Df_n|_p^p\,\mathrm{d}\tilde{\mathfrak{m}}\right)^{1/q} \|f_n - f\|_{W^{1,p}(\mathrm{X})}.
\end{aligned}$$

Given that $|Df_n|_p^p \to |Df|_p^p$ pointwise $\tilde{\mathfrak{m}}$-a.e. and $|Df_n|_p^p \leq G^p \in L^1(\tilde{\mathfrak{m}})$ holds $\tilde{\mathfrak{m}}$-a.e. for all $n \in \mathbb{N}$, by using the dominated convergence theorem we deduce that $\int |Df_n|_p^p\,\mathrm{d}\tilde{\mathfrak{m}} \to \int |Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}}$. Consequently, by letting $n \to \infty$ in the above estimates we get $\int \eta_n(\mathrm{e}_0^*\mathrm{d}_p f)\,\mathrm{d}\boldsymbol{\pi}^n \to \int |Df|_p^p\,\mathrm{d}\tilde{\mathfrak{m}}$ as $n \to \infty$. All in all, we can conclude that

$$\begin{aligned}\int |Df|_p^p \,\mathrm{d}\tilde{\mathfrak{m}} &= \lim_{n\to\infty}\int \eta_n(\mathrm{e}_0^*\mathrm{d}_p f)\,\mathrm{d}\boldsymbol{\pi}^n \le \big\||Df|_{\Pi,p}\big\|_{L^p(\tilde{\mathfrak{m}})}\lim_{n\to\infty}\big\||Df_n|_p\big\|_{L^p(\tilde{\mathfrak{m}})}^{p/q}\\ &\le \big\||Df|_{\Pi,p}\big\|_{L^p(\tilde{\mathfrak{m}})}\big\||Df|_p\big\|_{L^p(\tilde{\mathfrak{m}})}^{p/q}.\end{aligned}$$

This proves the validity of (2.11) and accordingly of (2.10).

Step 3. It remains to deduce the claim from (2.10). Define $\Pi := (\boldsymbol{\pi}^k)_k$ and

$$\boldsymbol{\eta} := \sum_{k=1}^{\infty} \frac{\boldsymbol{\pi}^k}{2^k \max\{\mathrm{Comp}(\boldsymbol{\pi}^k), \mathrm{KE}_q(\boldsymbol{\pi}^k), 1\}}, \qquad \boldsymbol{\pi}_q := \frac{\boldsymbol{\eta}}{\boldsymbol{\eta}(C([0,1],\mathrm{X}))}.$$

Since all measures $\boldsymbol{\pi}^k$ are Borel measures concentrated on $AC^q([0,1],\mathrm{X})$, we see that $\boldsymbol{\eta}$ is a Borel measure concentrated on $AC^q([0,1],\mathrm{X})$ as well. Also, $\boldsymbol{\eta}(C([0,1],\mathrm{X})) \le \sum_{k=1}^{\infty} 1/2^k = 1$, so that $\boldsymbol{\pi}_q$ is well-defined and is thus a Borel probability measure concentrated on $AC^q([0,1],\mathrm{X})$. Given any $t \in [0,1]$ and a Borel set $E \subseteq \mathrm{X}$, we have

$$(\mathrm{e}_t)_\#\boldsymbol{\eta}(E) = \boldsymbol{\eta}(\mathrm{e}_t^{-1}(E)) \le \sum_{k=1}^{\infty} \frac{\boldsymbol{\pi}^k(\mathrm{e}_t^{-1}(E))}{2^k\,\mathrm{Comp}(\boldsymbol{\pi}^k)} \le \mathfrak{m}(E)\sum_{k=1}^{\infty}\frac{1}{2^k} = \mathfrak{m}(E),$$

whence $\boldsymbol{\pi}_q$ satisfies item (i) of Definition 1.8. Moreover, observe that

$$\int\int_0^1 |\dot\gamma_t|^q \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\gamma) \le \sum_{\substack{k\in\mathbb{N}:\\ \mathrm{KE}_q(\boldsymbol{\pi}^k)>0}} \frac{1}{2^k\,\mathrm{KE}_q(\boldsymbol{\pi}^k)}\int\int_0^1 |\dot\gamma_t|^q\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}^k(\gamma) \le \sum_{k=1}^{\infty}\frac{1}{2^k} = 1,$$

thus accordingly $\boldsymbol{\pi}_q$ has finite kinetic q-energy. All in all, $\boldsymbol{\pi}_q$ is a q-test plan on $(\mathrm{X},\mathsf{d},\mathfrak{m})$.

Finally, a given Borel subset of $C([0,1],\mathrm{X})$ is $\boldsymbol{\pi}_q$-negligible if and only if it is $\boldsymbol{\pi}^k$-negligible for all $k \in \mathbb{N}$, thus $W^{1,p}_{\boldsymbol{\pi}_q}(\mathrm{X}) = W^{1,p}_{\Pi}(\mathrm{X})$ and $|Df|_{\boldsymbol{\pi}_q,p} = |Df|_{\Pi,p}$ for every $f \in W^{1,p}_{\boldsymbol{\pi}_q}(\mathrm{X})$. Consequently, the statement follows from (2.10). ∎

Problem 2.7. Under the assumption of Theorem 2.6, does it hold that $W^{1,p}_{\boldsymbol{\pi}_q}(\mathrm{X}) = W^{1,p}(\mathrm{X})$? In other words, is the q-test plan $\boldsymbol{\pi}_q$ sufficient to detect which functions are Sobolev, and not only to identify the minimal p-relaxed slope of those functions that are known to be Sobolev?

A positive answer to the above question is known, for instance, in the Euclidean space (and, similarly, on Riemannian manifolds). Indeed, in this case the original approach to weakly differentiable functions pioneered by B. Levi [19] shows that to look at the behaviour along coordinate directions is sufficient to distinguish the Sobolev functions; by building upon this result, one can find a master q-test plan on $\mathbb{R}^n$ for which $W^{1,p}_{\boldsymbol{\pi}_q}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

3. Master test plans on RCD spaces. The aim of this section is to improve Theorem 2.6 in the case in which the space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ under consideration is an $\mathsf{RCD}(K,\infty)$ space for some $K\in\mathbb{R}$. An $\mathsf{RCD}(K,\infty)$ *space* is an infinitesimally Hilbertian space whose Ricci curvature is bounded from below by K, in a synthetic sense. For an account of this theory, we refer to [1] and the references therein.

An important feature of $\mathsf{RCD}(K,\infty)$ spaces is the presence of a vast class of 'highly regular' functions, which are referred to as the *test functions*. In order to introduce them, we first need to recall the notion of *Laplacian*: we declare that $f\in W^{1,2}(\mathrm{X})$ belongs to $D(\Delta)$ provided there exists a (uniquely determined) function $\Delta f\in L^2(\mathfrak{m})$ such that

$$\int g\Delta f\,\mathrm{d}\mathfrak{m}=-\int\langle\nabla_2 g,\nabla_2 f\rangle\,\mathrm{d}\mathfrak{m}\quad\text{ for every } g\in W^{1,2}(\mathrm{X}).$$

With this said, we are in a position to define

$$\mathrm{Test}^\infty(\mathrm{X})\coloneqq\{f\in D(\Delta)\cap L^\infty(\mathfrak{m})\mid |Df|_2,\ \Delta f\in L^\infty(\mathfrak{m}),\ \Delta f\in W^{1,2}(\mathrm{X})\}.$$

As proven in [21, 12], the family $\mathrm{Test}^\infty(\mathrm{X})$ is strongly dense in the Sobolev space $W^{1,2}(\mathrm{X})$.

We also point out that for any $q\in(1,\infty)$ we have

$$\mathsf{RCD}(K,\infty)\text{ spaces are } q\text{-infinitesimally strictly convex.}\tag{3.1}$$

Albeit expected, this property is far from being trivial. The reason is that, as we recalled in Remark 1.3, minimal p-relaxed slopes might depend on p. However, this issue cannot occur in the class of $\mathsf{RCD}(K,N)$ spaces, where for any $p,p'\in(1,\infty)$ we can see that

$$|Df|_p=|Df|_{p'}\quad\mathfrak{m}\text{-a.e.}\quad\text{ for every } f\in W^{1,p}(\mathrm{X})\cap W^{1,p'}(\mathrm{X}).\tag{3.2}$$

If N is finite, then the space X is (locally uniformly) doubling and satisfies a (weak, local) Poincaré inequality, thus (3.2) follows from the results of [7]. In the infinite-dimensional case, it is proven in [14]. With (3.2) at our disposal, we can argue in the following way: By combining the 2-infinitesimal strict convexity of RCD spaces with (3.2), we deduce that for any $p\in(1,\infty)$ the identity in (1.3) is satisfied whenever $f,g\in W^{1,p}(\mathrm{X})\cap W^{1,2}(\mathrm{X})$; cf. the discussion at the beginning of [11, Section 3.1]. By an approximation argument, we can thus conclude that (1.3) holds for all $f,g\in W^{1,p}(\mathrm{X})$, whence the claimed property (3.1) follows.

Additionally, the same reasoning shows, for any $p\in(1,\infty)$, that

$$\mathrm{d}_p f(\nabla_p g)=\mathrm{d}_p g(\nabla_p f)\quad\mathfrak{m}\text{-a.e.}\quad\text{ for all } f,g\in W^{1,p}(\mathrm{X}).\tag{3.3}$$

3.1. Regular Lagrangian flow. Another important ingredient that we will need to prove Theorem 3.4 is the notion of regular Lagrangian flow, which (in the metric setting) has been introduced by L. Ambrosio and D. Trevisan [6]. The following result is only a very special case of a much more general statement, but still it is sufficient for our purposes.

THEOREM 3.1 (Regular Lagrangian flow [6]). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space for some constant* $K\in\mathbb{R}$. *Let* $f\in\mathrm{Test}^\infty(\mathrm{X})$. *Then there exists a* ($\mathfrak{m}$*-a.e. uniquely determined*) regular Lagrangian flow $F_\cdot\colon\mathrm{X}\to C([0,1],\mathrm{X})$ *associated with* ∇f, *which means that:*

(i) *The map* $F_\cdot\colon\mathrm{X}\to C([0,1],\mathrm{X})$ *is Borel and satisfies* $F_0(x)=x$ *for* $\mathfrak{m}$*-a.e.* $x\in\mathrm{X}$.
(ii) *There exists a constant* $L>0$ *such that* $(F_t)_\#\mathfrak{m}\leq L\mathfrak{m}$ *for every* $t\in[0,1]$.
(iii) *Given any* $p\in(1,\infty)$ *and* $g\in W^{1,p}(\mathrm{X})$, *the function* $[0,1]\ni t\mapsto g(F_t(x))$ *belongs to* $W^{1,1}(0,1)$ *for* $\mathfrak{m}$*-a.e.* $x\in\mathrm{X}$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}g(F_t(x))=\mathrm{d}_pg(\nabla_pf)(F_t(x))\quad\text{for }(\mathfrak{m}\otimes\mathcal{L}_1)\text{-a.e. }(x,t).\tag{3.4}$$

Let us spend a few words about both the statement and the proof of Theorem 3.1:

REMARK 3.2. Observe that item (ii) is meaningful since the map $[0,1]\times\mathrm{X}\ni(t,x)\mapsto F_t(x)\in\mathrm{X}$ is Borel (as it is a Carathéodory function), thus in particular $\mathrm{X}\ni x\mapsto F_t(x)\in\mathrm{X}$ is Borel for every $t\in[0,1]$. Moreover, item (iii) is well-posed thanks to item (ii): given that $\mathrm{d}_pg(\nabla_pf)$ is defined $\mathfrak{m}$-a.e. and $(F_t)_\#\mathfrak{m}\ll\mathfrak{m}$, we see that $\mathrm{d}_pg(\nabla_pf)\circ F_t$ is defined $\mathfrak{m}$-a.e. as well.

Moreover, we point out that the formulation presented above is taken from [13], where only the case $p=2$ is considered. The case of an arbitrary exponent $p\in(1,\infty)$ can be deduced as follows. Fix any $g\in W^{1,p}(\mathrm{X})$. A standard cut-off argument shows that $W^{1,p}(\mathrm{X})\cap W^{1,2}(\mathrm{X})$ is dense in $W^{1,p}(\mathrm{X})$, thus we can find a sequence $(g_n)_n\subseteq W^{1,p}(\mathrm{X})\cap W^{1,2}(\mathrm{X})$ such that $g_n\to g$ in $W^{1,p}(\mathrm{X})$. In particular, $g_n\to g$ in $L^p(\mathfrak{m})$ and $\mathrm{d}_pg_n\to\mathrm{d}_pg$ in $L^p(T^*\mathrm{X})$, so that (by taking item (ii) of Theorem 3.1 into account) we obtain

$$\begin{aligned}\lim_{n\to\infty}\int|g_n(F_t(x))-g(F_t(x))|^p\,\mathrm{d}(\mathfrak{m}\otimes\mathcal{L}_1)(x,t)&=0,\\ \lim_{n\to\infty}\int|\mathrm{d}_pg_n(\nabla_pf)(F_t(x))-\mathrm{d}_pg(\nabla_pf)(F_t(x))|\,\mathrm{d}(\mathfrak{m}\otimes\mathcal{L}_1)(x,t)&=0.\end{aligned}$$

Therefore, up to a subsequence (not relabelled) in n, for $\mathfrak{m}$-a.e. $x\in\mathrm{X}$,

$$\begin{aligned}\lim_{n\to\infty}\int_0^1|g_n(F_t(x))-g(F_t(x))|^p\,\mathrm{d}t&=0,\\ \lim_{n\to\infty}\int_0^1|\mathrm{d}_pg_n(\nabla_pf)(F_t(x))-\mathrm{d}_pg(\nabla_pf)(F_t(x))|\,\mathrm{d}t&=0.\end{aligned}\tag{3.5}$$

Given that $\mathrm{d}_pg_n(\nabla_pf)=\langle\nabla_2g_n,\nabla_2f\rangle$ holds $\mathfrak{m}$-a.e. for every $n\in\mathbb{N}$ by (3.2), we see that each function g_n satisfies item (iii) of Theorem 3.1. By virtue of (3.5), so does g. Hence, Theorem 3.1 is proved for every $p\in(1,\infty)$.

Given any measure $\mu \in \mathscr{P}(\mathrm{X})$ such that $\mu \leq C\mathfrak{m}$ for some constant $C > 0$, we see that

$$\boldsymbol{\pi} := (F_\cdot)_\# \mu \text{ is an } \infty\text{-test plan on X.} \tag{3.6}$$

In particular, given any $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, we deduce that $\boldsymbol{\pi}$ is a q-test plan on X, thus we can consider its p-velocity $\boldsymbol{\pi}' \in (\mathrm{e}^* L^p(T^*\mathrm{X}))^* \cong \mathrm{e}^* L^q(T\mathrm{X})$. Therefore $\boldsymbol{\pi}' = \mathrm{e}^* \nabla_p f$; we refer to [13] for more details.

3.2. Existence of master test plans on RCD spaces. To begin with, we show that the regularity result in Proposition 1.23 can be sharpened when the test plan is induced by a regular Lagrangian flow (in the sense of (3.6) above):

LEMMA 3.3. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space for some* $K \in \mathbb{R}$. *Let* $f \in \mathrm{Test}^\infty(\mathrm{X})$. *Denote by* $F_\cdot$ *the regular Lagrangian flow associated with* ∇f. *Let* $\mu \in \mathscr{P}(\mathrm{X})$ *be such that* $\mu \leq C\mathfrak{m}$ *for some* $C > 0$ *and define* $\boldsymbol{\pi} := (F_\cdot)_\# \mu$. *Then for any exponent* $p \in (1, \infty)$ *and any* $g \in W^{1,p}(\mathrm{X})$ *the map* $[0,1] \ni t \mapsto g \circ \mathrm{e}_t \in L^1(\boldsymbol{\pi})$ *is of class* C^1 *and*

$$\frac{\mathrm{d}}{\mathrm{d}t} g \circ \mathrm{e}_t = \mathrm{d}_p g(\nabla_p f) \circ \mathrm{e}_t \quad \textit{for every } t \in [0,1].$$

Proof. We know from Proposition 1.23 that the curve $[0,1] \ni t \mapsto g \circ \mathrm{e}_t \in L^1(\boldsymbol{\pi})$ is absolutely continuous and its $L^1(\boldsymbol{\pi})$-strong derivative coincides at $\mathcal{L}_1$-a.e. $t \in [0,1]$ with

$$D_t := \boldsymbol{\pi}'(\mathrm{e}^* \mathrm{d}_p g)(\cdot, t) = (\mathrm{e}^* \mathrm{d}_p g)(\mathrm{e}^* \nabla_p f)(\cdot, t) = \mathrm{d}_p g(\nabla_p f) \circ \mathrm{e}_t.$$

Since $[0,1] \ni t \mapsto D_t \in L^1(\boldsymbol{\pi})$ is continuous by Proposition 1.10, the statement follows. ∎

We are now in a position to prove our existence result. Even though the ideas are very similar to those employed in the proof of Theorem 2.6, we still prefer to write down the whole argument since it presents many technical simplifications.

THEOREM 3.4 (Master test plans on RCD spaces). *Consider an* $\mathsf{RCD}(K, \infty)$ *space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, *for some* $K \in \mathbb{R}$. *Then there exists an* ∞*-test plan* $\boldsymbol{\pi}_\infty$ *on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *that is a master* q*-test plan for every exponent* $q \in (1, \infty)$.

Proof. First of all, fix a countable family $\mathcal{C} \subseteq \mathrm{Test}^\infty(\mathrm{X})$ that is strongly dense in $W^{1,p}(\mathrm{X})$ for every $p \in (1, \infty)$, whose existence can be proven by combining the properties of the heat flow on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ with a cut-off argument. Choose any $\tilde{\mathfrak{m}} \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m} \ll \tilde{\mathfrak{m}} \leq C\mathfrak{m}$ for some $C > 0$ (recall Remark 1.1). Given any $f \in \mathcal{C}$, we write $F^f_\cdot$ for the regular Lagrangian flow associated with ∇f and we set $\boldsymbol{\pi}^f := (F^f_\cdot)_\# \tilde{\mathfrak{m}}$. Define $\Pi := \{\boldsymbol{\pi}^f \mid f \in \mathcal{C}\}$. We claim that

$$|Df|_{\Pi,p} = |Df|_p \quad \text{for every } p \in (1, \infty) \text{ and } f \in W^{1,p}(\mathrm{X}). \tag{3.7}$$

Given that $|Df|_{\Pi,p} \le |Df|_p$ holds $\mathfrak{m}$-a.e. by Proposition 1.16, it is just sufficient to show the inequality $\int |Df|_p^p \,\mathrm{d}\tilde{\mathfrak{m}} \le \int |Df|_{\Pi,p}^p \,\mathrm{d}\tilde{\mathfrak{m}}$. To this end, fix a sequence $(f_n)_n \subseteq \mathcal{C}$ with $f_n \to f$ strongly in $W^{1,p}(\mathrm{X})$. In particular, up to taking a subsequence (not relabelled) in n, we may assume that $\mathrm{d}_p f_n \to \mathrm{d}_p f$ strongly in $L^p(T^*\mathrm{X})$ and $|Df_n|_p \to |Df|_p$ strongly in $L^p(\tilde{\mathfrak{m}})$. For brevity, let us denote $\boldsymbol{\pi}^n := \boldsymbol{\pi}^{f_n}$ for every $n \in \mathbb{N}$. Notice that $(\mathrm{e}_0)_\# \boldsymbol{\pi}^n = (F_0^{f_n})_\# \tilde{\mathfrak{m}} = \tilde{\mathfrak{m}}$, so Lemma 3.3 and the dominated convergence theorem yield

$$
\begin{aligned}
\int |Df|_p^p \,\mathrm{d}\tilde{\mathfrak{m}} &= \int \mathrm{d}_p f(\nabla_p f)\,\mathrm{d}\tilde{\mathfrak{m}} \\
&= \lim_{n\to\infty} \int \mathrm{d}_p f_n(\nabla_p f)\,\mathrm{d}\tilde{\mathfrak{m}} \overset{(3.3)}{=} \lim_{n\to\infty} \int \mathrm{d}_p f(\nabla_p f_n)\,\mathrm{d}\tilde{\mathfrak{m}} \\
&= \lim_{n\to\infty} \int \mathrm{d}_p f(\nabla_p f_n)\circ \mathrm{e}_0 \,\mathrm{d}\boldsymbol{\pi}^n = \lim_{n\to\infty} \lim_{t\searrow 0} \int \frac{f\circ \mathrm{e}_t - f\circ \mathrm{e}_0}{t}\,\mathrm{d}\boldsymbol{\pi}^n \\
&\le \varliminf_{n\to\infty} \varliminf_{t\searrow 0} \int \frac{|f(\gamma_t) - f(\gamma_0)|}{t}\,\mathrm{d}\boldsymbol{\pi}^n(\gamma) \\
&\le \varliminf_{n\to\infty} \varliminf_{t\searrow 0} \int \fint_0^t |Df|_{\Pi,p}(\gamma_s)|\dot{\gamma}_s|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}^n(\gamma) \\
&\le \lim_{n\to\infty} \lim_{t\searrow 0} \Big(\int \fint_0^t |Df|_{\Pi,p}^p \circ \mathrm{e}_s \,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}^n\Big)^{1/p} \Big(\int \fint_0^t |(\boldsymbol{\pi}^n)'|^q(\cdot,s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}^n\Big)^{1/q} \\
&= \lim_{n\to\infty} \lim_{t\searrow 0} \Big(\fint_0^t \big\| |Df|_{\Pi,p}\circ \mathrm{e}_s \big\|_{L^p(\boldsymbol{\pi}^n)}^p \,\mathrm{d}s\Big)^{1/p} \Big(\fint_0^t \big\| |\nabla_p f_n|\circ \mathrm{e}_s \big\|_{L^q(\boldsymbol{\pi}^n)}^q \,\mathrm{d}s\Big)^{1/q} \\
&= \lim_{n\to\infty} \Big(\int |Df|_{\Pi,p}^p \circ \mathrm{e}_0 \,\mathrm{d}\boldsymbol{\pi}^n\Big)^{1/p} \Big(\int |\nabla_p f_n|^q \circ \mathrm{e}_0 \,\mathrm{d}\boldsymbol{\pi}^n\Big)^{1/q} \\
&= \lim_{n\to\infty} \Big(\int |Df|_{\Pi,p}^p \circ \mathrm{e}_0 \,\mathrm{d}\boldsymbol{\pi}^n\Big)^{1/p} \Big(\int |Df_n|_p^p \circ \mathrm{e}_0 \,\mathrm{d}\boldsymbol{\pi}^n\Big)^{1/q} \\
&= \Big(\int |Df|_{\Pi,p}^p \,\mathrm{d}\tilde{\mathfrak{m}}\Big)^{1/p} \lim_{n\to\infty} \Big(\int |Df_n|_p^p \,\mathrm{d}\tilde{\mathfrak{m}}\Big)^{1/q} \\
&= \Big(\int |Df|_{\Pi,p}^p \,\mathrm{d}\tilde{\mathfrak{m}}\Big)^{1/p} \Big(\int |Df|_p^p \,\mathrm{d}\tilde{\mathfrak{m}}\Big)^{1/q},
\end{aligned}
$$

where $(\boldsymbol{\pi}^n)'$ stands for the p-velocity of the q-test plan $\boldsymbol{\pi}^n$. Therefore, the claimed identity (3.7) is satisfied.

In order to conclude, it remains to pass from the countable family Π to a single ∞-test plan $\boldsymbol{\pi}_\infty$. We proceed as follows: Define $\Pi := (\boldsymbol{\pi}^k)_k$. Given any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\boldsymbol{\pi}^k$ is concentrated on n_k-Lipschitz curves. Then let

$$
\boldsymbol{\pi}^{k,i} := (\mathrm{restr}_{(i-1)/n_k}^{i/n_k})_\# \boldsymbol{\pi}^k \qquad \text{for every } i = 1, \dots, n_k.
$$

Therefore, $\boldsymbol{\pi}^{k,1}, \dots, \boldsymbol{\pi}^{k,n_k}$ are ∞-test plans concentrated on 1-Lipschitz curves. Observe also that the family

$$\Pi' := \{\boldsymbol{\pi}^{k,i} \mid k \in \mathbb{N},\, i = 1, \ldots, n_k\}$$

satisfies $W^{1,p}_{\Pi'}(\mathrm{X}) = W^{1,p}_{\Pi}(\mathrm{X})$ and $|Df|_{\Pi',p} = |Df|_{\Pi,p}$ for every $p \in (1,\infty)$ and $f \in W^{1,p}_{\Pi'}(\mathrm{X})$. Finally, let

$$\boldsymbol{\eta} := \sum_{k=1}^{\infty}\sum_{i=1}^{n_k} \frac{\boldsymbol{\pi}^{k,i}}{2^{k+i}\max\{\mathrm{Comp}(\boldsymbol{\pi}^{k,i}), 1\}}, \qquad \boldsymbol{\pi}_\infty := \frac{\boldsymbol{\eta}}{\boldsymbol{\eta}(C([0,1],\mathrm{X}))}.$$

By arguing as we did in Step 3 of the proof of Theorem 2.6, we can see that $\boldsymbol{\pi}_\infty$ is an ∞-test plan (concentrated on 1-Lipschitz curves). Given that $W^{1,p}_{\boldsymbol{\pi}_\infty}(\mathrm{X}) = W^{1,p}_{\Pi'}(\mathrm{X})$ and $|Df|_{\boldsymbol{\pi}_\infty,p} = |Df|_{\Pi',p}$ for every $p \in (1,\infty)$ and $f \in W^{1,p}_{\boldsymbol{\pi}_\infty}(\mathrm{X})$, the statement follows from the identity (3.7). ∎

Remark 3.5. We point out that every ∞-test plan $\boldsymbol{\pi}$ induced by the regular Lagrangian flow associated with ∇f, as in (3.6), q-represents the gradient of f for every $q \in (1,\infty)$.

Indeed, for $(\boldsymbol{\pi} \otimes \mathcal{L}_1)$-a.e. (γ, t) we have

$$|\dot\gamma_t| = |\boldsymbol{\pi}'|(\gamma,t) = |\mathrm{e}^*\nabla_p f|(\gamma,t) = |Df|_p^{p/q}(\gamma_t)$$

and $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t) = |Df|_p^p(\gamma_t)$, whence

$$\frac{\mathrm{E}_{q,t}(\gamma)}{t} = \Big(\fint_0^t |\dot\gamma_s|^q \,\mathrm{d}s\Big)^{1/q} = \Big(\fint_0^t |Df|_p^p \circ \mathrm{e}_s \,\mathrm{d}s\Big)^{1/q}(\gamma),$$

$$\begin{aligned}\Big(\frac{f\circ \mathrm{e}_t - f\circ \mathrm{e}_0}{\mathrm{E}_{q,t}}\Big)(\gamma) &= \frac{t}{\mathrm{E}_{q,t}(\gamma)}\fint_0^t \frac{\mathrm{d}}{\mathrm{d}s} f(\gamma_s)\,\mathrm{d}s = \frac{t}{\mathrm{E}_{q,t}(\gamma)}\fint_0^t |Df|_p^p(\gamma_s)\,\mathrm{d}s\\ &= \Big(\fint_0^t |Df|_p^p \circ \mathrm{e}_s\,\mathrm{d}s\Big)^{1/p}(\gamma)\end{aligned}$$

for every $t \in (0,1)$ and $\boldsymbol{\pi}$-a.e. γ. By recalling Proposition 1.10, we conclude that the plan $\boldsymbol{\pi}$ q-represents the gradient of f, as claimed above. This means that Theorem 3.4 could have been alternatively proven by directly using the proof of Theorem 2.6.

Remark 3.6. A necessary condition for a statement as the one of Theorem 3.4 to hold is the fact that minimal p-weak upper gradients are independent of p. Hence, by recalling Remark 1.3, we see that Theorem 3.4 cannot be generalised to arbitrary metric measure spaces.

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