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86

Elena Gorshkova

A Posteriori Error Estimates and Adaptive Methods for Incompressible Viscous Flow Problems

JYVÄSKYLÄN YLIOPISTO

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ABSTRACT

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This thesis is focused on the development and numerical justification of a modern computational methodology that provides guaranteed upper bounds of the energy norms of an error. The methodology suggested is based on the so-called functional type a posteriori error estimates. Different linearizations of the Navier-Stokes equations are considered. Namely, estimates of the Stokes problem, the evolutionary Stokes problem and the system with rotation are proposed. For the system with rotation and semi-discrete approximations of the evolutionary Stokes problem, such type of estimates are presented for the first time.

For the Stokes problem and the system with rotation, different numerical strategies are implemented. Numerical tests are performed in Cartesian and Cylindrical coordinate system. For the Stokes problem, a posteriori error estimates on a certain subdomain of interest are also tested. It is shown that functional type a posteriori error estimation methods give reliable and robust upper bounds of the error and realistic error indication.

The approach suggested allows to construct efficient mesh-adaptive algorithms and provide a guaranteed accuracy for the approximate solutions.

Keywords: A posteriori error estimate, Mesh adaptation, Viscous incompressible flow, Stokes problem, evolutionary Stokes problem, Flow with rotation.

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Jyväskylä, December 2007
Elena Gorshkova

LIST OF SYMBOLS

$\mathbb{R}^n$	set of real numbers
$\mathbb{I}$	identity tensor
(a, b)	open interval of real numbers
$[a, b]$	closed interval of real numbers
$\mathcal{D}$	bounded domain with Lipschitz continuous boundary
$\partial\mathcal{D}$	boundary of domain $\mathcal{D}$
$\overline{\mathcal{D}}$	closure of domain $\mathcal{D}$
Q_T	space-time cylinder ($\mathcal{D} \times (0, T)$)
S_T	surface of the space-time cylinder ($\partial\mathcal{D} \times [0, T)$)
$\overline{Q_T}$	$Q_T \cup S_T$
$C^\infty(\mathcal{D})$	space of smooth functions on $\mathcal{D}$
$C_0^\infty(\mathcal{D})$	space of smooth functions with compact support on $\mathcal{D}$
$C^\infty(\mathcal{D}, \mathbb{R}^n)$	space of smooth vector-functions on $\mathcal{D}$
$C_0^\infty(\mathcal{D}, \mathbb{R}^n)$	space of smooth vector-functions with compact support on $\mathcal{D}$
$L_2(\mathcal{D})$	Lebesgue space of square integrable function over $\mathcal{D}$
$L_2(\mathcal{D}, \mathbb{R}^n)$	Lebesgue space of square integrable vector-valued function over $\mathcal{D}$
$\|\cdot\|_{\mathcal{D}}$ (or $\|\cdot\|$)	norm in the space $L_2(\mathcal{D})$, $L_2(\mathcal{D}, \mathbb{R}^n)$or $L_2(\mathcal{D}, \mathbb{M}^{n\times n})$
$H^1(\mathcal{D})$	Sobolev space $W^{1,2}(\mathcal{D})$
$\overset{o}{H}{}^1(\mathcal{D})$	subspace of $H^1(\mathcal{D})$of functions with zero traces on $\partial\mathcal{D}$
$H^1(\mathcal{D}, \mathbb{R}^n)$	Sobolev space of vector-valued function in $W^{1,2}(\mathcal{D})$
$\overset{o}{H}{}^1(\mathcal{D}, \mathbb{R}^n)$	subspace of $H^1(\mathcal{D}, \mathbb{R}^n)$of functions with zero traces on $\partial\mathcal{D}$
$L_2(\mathcal{D}, \mathbb{M}^{n\times n})$	Lebesgue space of square integrable tensor-valued function over $\mathcal{D}$
$L_2(\mathcal{D}, \mathbb{M}_s^{n\times n})$	Lebesgue space of square integrable symmetric tensor-function over $\mathcal{D}$
$\Sigma_{\mathrm{div}}(\mathcal{D})$	subspace of $L_2(\mathcal{D}, \mathbb{M}^{n\times n})$of tensor-functions which divergence belongs to the space $L_2(\mathcal{D}, \mathbb{R}^n)$
$J^\infty(\mathcal{D}, \mathbb{R}^n)$	subspace of $C^\infty(\mathcal{D}, \mathbb{R}^n)$of divergence-free functions
$\overset{o}{J}{}^\infty(\mathcal{D}, \mathbb{R}^n)$	subspace of $C_0^\infty(\mathcal{D}, \mathbb{R}^n)$of divergence-free functions
$J_2^1(\mathcal{D}, \mathbb{R}^n)$	closure of $J^\infty(\mathcal{D}, \mathbb{R}^n)$ in the norm of the space $H^1(\mathcal{D}, \mathbb{R}^n)$
$\overset{o}{J}{}_2^1(\mathcal{D}, \mathbb{R}^n)$	closure of $\overset{o}{J}{}^\infty(\mathcal{D}, \mathbb{R}^n)$ in the norm of the space $H^1(\mathcal{D}, \mathbb{R}^n)$
$L^\alpha([0, T], V)$	Bocher space of α-integrable mappings of the interval $[0, T]$ into Banach space V
$C^n([0, T], V)$	space of $n-$times continuously differentiable mappings from the interval $[0, T]$ into Banach space V

∇	gradient by the spatial variables
$\nabla\cdot$	divergence
$\triangle$	Laplace operator
$\frac{\partial}{\partial t}$	partial derivative with respect to t
u	exact solution of the problem
$\tilde{v}$	approximate solution
v	divergence-free approximate solution
$c_{\mathcal{D}}$	constant from Friedrichs' inequality
C_{LBB}	constant from Ladyzhenskaya-Babuška-Brezzi inequality
RHS	right hand side of the equation or inequality
LHS	left hand side of the equation or inequality.
N	number of elements
$\mathbb{G}$	operator of averaging
E	square of the energy norm of the error
M	error majorant
M_p	primary term of an error majorant
M_r	reliability term of an error majorant
M_{div}	"div"-term of an error majorant
E_T	local contribution of the error on the element T
M_T	local contribution of the error majorant on the element T
e_T	normalized local contribution of the error on the element T
e_T	normalized local contribution of the error majorant on the element T
I_{eff}	efficiency index
p_{eff}	refinement effectivity
p_{eff}^{bulk}	refinement effectivity by the "bulk"-strategy
p_{eff}^{max}	refinement effectivity by the maximum strategy

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PI E. Gorshkova, S. Repin. On the functional type a posteriori error Estimates for the Stokes problem . *Proceeding of the ECCOMAS-2004, Jyväskylä, Finland, CD-ROM.*

PII E. Gorshkova, P. Neittaanmäki, S. Repin. Comparative study of the a posteriori error estimators for the Stokes Problem. *Numerical Mathematics and Advanced Application (ENUMATH 2005), Springer-Veglar, Berlin, Heidelberg, 2006, pp. 252-259 .*

PIII E. Gorshkova, P. Neittaanmäki, S. Repin. Mesh-adaptive methods for viscous flow problem with rotation. *Advances and Innovations in Systems, Computing and Software Engineering, Springer, pp. 105-107 .*

PIV E. Gorshkova, A. Mahalov, P. Neittaanmäki, S. Repin. A posteriori error estimates for viscous flow problems with rotation. *Journal of mathematical Sciences, Vol. 142, No. 1, 2007. pp. 1749-1762 .*

PV E. Gorshkova, P. Neittaanmäki, S. Repin. A posteriori error estimate for viscous flow problems with rotation. *Oberwolfach Report 29/2007 "Adaptive Numerical methods for PDE's", Mathematisches Forschungsinstitut Oberwolfach, pp. 18-20.*

PVI E. Gorshkova, S. Repin. A posteriori error estimates for semi-discrete approximations of the evolutionary Stokes problem. *To appear in Journal of mathematical Sciences, 2008.*

1 INTRODUCTION AND STRUCTURE OF THE STUDY

The use of adaptive methods for the numerical discretization of flow models is a subject of strong interest from both theoretical and practical points of view. They are often justified by a posteriori error estimates, which provide computable upper and lower error bounds and also serve as error indicators.

In this analysis, we pay major attention to two points: (a) error estimation in global (energy) norms and (b) local error estimation. The latter task is solved either by the error indicator that comes from the global error majorant or by local error estimation techniques. The latter information is used for the element marking and further mesh refinement.

The aim of this thesis is to present theoretically and study numerically functional type a posteriori error estimates for the different linearizations of the Navier-Stokes equations, namely the Stokes problem, the evolutionary Stokes problem and systems with rotation.

For the Stokes problem, we present a numerical investigation of the functional type a posteriori error estimates theoretically obtained in [1], [2], [3]. Practically efficient computational methods based on these estimates and their comparison with other known error indicators are presented. The first part of this thesis is devoted to the development of practically efficient computational technology of error control for the Stokes problem. This technology was verified on a large amount of tests, including different numerical methods and approximations of different types.

For the semi-discrete approximations of the evolutionary Stokes problem a new functional type error estimate is obtained. Its numerical investigation will be a subject of future research.

Also, a new a posterior error estimate is derived for viscous flow problems with rotation. It has been tested numerically and has demonstrated its robustness and efficiency.

The thesis is based on 6 publications and some unpublished results.

PI	E. Gorshkova, S. Repin, On the functional type a posteriori error Estimates for the Stokes problem, *In proceeding of the ECCOMAS-2004, Jyväskylä, Finland, CD-ROM*
PII	E. Gorshkova, P. Neittaanmäki, S. Repin, Comparative study of the a posteriori error estimators for the Stokes problem, *Numerical Mathematics and Advanced Application (ENUMATH 2005), Springer-Veglar, Berlin, Heidelberg, 2006, pp. 252-259.*
PIII	E. Gorshkova, P. Neittaanmäki, S. Repin, Mesh-adaptive methods for viscous flow problem with rotation. *Advances and Innovations in Systems, Computing and Software Engineering, Springer, 2007, pp. 105-107,*
PIV	E. Gorshkova, A. Mahalov, P. Neittaanmäki, S. Repin, A posteriori error estimates for viscous flow problems with rotation, *Journal of mathematical Sciences, Vol. 142, No. 1, 2007, pp. 1749-1762*
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PVI	E. Gorshkova, P. Neittaanmäki, S. Repin, A posteriori error estimates for semi-discrete approximations of the evolutionary Stokes problem. *To appear in Journal of mathematical Sciences, 2008*

The introductory part is organized as follows. In Chapter 2, we give an overview of a posteriori error estimation methods existed, paying major attention to the problem in the theory of viscous fluids.

In Chapter 3, we explain the main ideas of mesh-adaptive algorithms and comment on their practical implementation of them. Also, we discuss the Finite Element Methods for viscous flow problem and specific difficulties arising due to the construction of approximations, related to the well-known Ladyzhenskaya-Babuška-Brezzi condition.

Next, we discuss functional a posteriori error estimates and the respective adaptive strategies. For the sake of simplicity,we first explain the main principals on paradigm of the simple problem (Poisson's equation).

In Chapter 4, we study the functional a posteriori error estimate for the

Stokes equation

$$\begin{aligned} -\nu\triangle u(x,t) &= f - \nabla p(x,t) && \text{in}\mathcal{D}, \\ \operatorname{div} u(x,t) &= 0 && \text{in } \mathcal{D}, \\ u(x,t) &= u_g && \text{on } \partial\mathcal{D}. \end{aligned}$$

The numerical investigation of the estimate is presented in papers **PI** and **PII**, where the estimate is compared with other methods. We present a series of numerical experiments and pay special attention on the sensitivity of the estimate with respect to the global constant involved. In Chapter 4, we also present some unpublished results concerning a posteriori error control on the estimate in the local norm.

Chapter 5 is based on the recent result for the semi-discrete approximation of the evolutionary Stokes equation (see **PVI**).

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu\triangle u(x,t) &= f - \nabla p(x,t) && \text{in}Q_T, \\ \operatorname{div} u(x,t) &= 0 && \text{in } Q_T, \\ u(x,t) &= 0 && \text{on } S_T, \\ u(x,0) &= \varphi(x) && \text{in } \mathcal{D}. \end{aligned}$$

In Chapter 6, we consider a model problem with rotation term

$$\begin{aligned} -\nu\triangle u(x,t) + \Omega \times u &= f - \nabla p(x,t) && \text{in}\mathcal{D}, \\ \operatorname{div} u(x,t) &= 0 && \text{in } \mathcal{D}, \\ u(x,t) &= u_g && \text{on } \partial\mathcal{D} \end{aligned}$$

and derive for this system a new guaranteed upper bound of the approximation error. The results were published in **PIII**, **PIV** and are presented in the Oberwolfach report (**PV**).

Also Chapter 6 contains certain generalizations of the proposed error estimate method. In particular, we obtain an error estimate for the systems with different vertical and horizontal viscosity.

1.1 Contribution of the author in joint publications

The author contributed by creating practically efficient computational methods based on the estimates proposed and by performing a comparative study of them versus other known error indicators. New error estimates for the

Stokes problem with rotation and semidiscrete approximations of the evolutionary Stokes problem are derived jointly with co-authors. In addition, the author significantly contributed to the writing and organization of all the papers.

2 A POSTERIORI ERROR ESTIMATE FOR FINITE ELEMENT METHODS

Reliable methods of numerical modelling are of high importance in the modern numerical analysis. Nowadays, it is necessary not only to solve some problem, but also estimate its accuracy and indicate the zones with excessively high errors.

Classical theory of approximation for differential equations provides so-called *a priori estimates.* They guarantee a convergence of the sequence of approximate solutions u_n (constructed on the finite dimensional spaces V_n, $dim\ V_n = n$), to true solutions u as $n \to \infty$. In addition, they qualify the rate of convergence with respect to n (see, e.g., [4]). Such estimates are unable to provide a guaranteed error bound for a particular approximation on a particular mesh. Typically, they require additional regularity of the solution, which may be difficult to guarantee in practice.

First works devoted to *a posteriori error estimation* for partial differential equations appeared in the middle of the past century (see, e.g., [5], [6]). In the 70s and 80s, adaptive algorithms based on a posteriori error indicators came into practice.

A posteriori error indicators for finite element approximations started receiving attention in the late 70s (see [7], [8]). First investigations were oriented towards error estimation for adaptive finite element methods for linear elliptic problems. Since then, a lot of work have been done for some other linear and nonlinear problems. We refer here to the monographs [9], [10], [11], [12] for surveys in the area.

For elliptic problems, the majority of estimators are based on various modifications of the residual method (originating from the papers [7], [8]) and methods using averaging (post-processing) techniques (see [13], [14], [15], [16],

[17] as well as from the references cited in these publications).

In the theory of fluids, methods of a posteriori error control are usually obtained in the framework of the residual methods. Typically, such estimates are derived for a particular numerical method and approximation type. For example, a modification of the estimates for a penalty method was constructed in [18]. For "mini-elements", the corresponding investigations were carried out in [19]. In that paper two methods of the error control are exposed. The first one is based on the estimation of residuals; the other one uses a solution of local Stokes problems. A simplification of the latter algorithm was suggested in [20]. In [21] method for conforming approximations (such as Taylor-Hood approximations) was suggested. In [22] a modification of the residual method for non-conforming finite elements (Crouzeix-Raviart) was obtained. A posteriori error control for the Discontinuous Galerkin method is presented in [23]. There exist many other modifications, which are described in numerous publications related to the topic. However, in this short overview we have no space to discuss all of them.

All these methods use specific features of the FEM solution and have certain restrictions in their applicability. First of all, they are valid only for Galerkin approximations, i.e., for the exact solution of the respective finite dimensional problem. Moreover, they depend on the discretization and the type of an approximation used. Theoretically they provide an upper bound of the error. However, they require sharp values of many local constants, that come from interpolation inequalities (therefore the latter are usually called *interpolation constants*). This problem is itself rather difficult. If these constants are defined approximately, then the guaranteed error bound is lost. On the other hand, an attempt to find guaranteed bounds for the constants may lead to a significant overestimation of the error(see [24] for elliptic equation). Nevertheless, these methods are widely used mainly as error indicators and have gained high popularity.

In the present work, we are focused on a posteriori estimates of a new type that are applicable for any conforming approximation of a boundary value problem considered. These estimates are derived by a pure functional analysis of the boundary value problem and contain no mesh-dependent constants. Therefore, they were called *functional a posteriori estimates*. Originally, the functional approach to a posteriori error control of boundary-value problem was presented and justified in [25]. A detailed explanation of this technique is exposed in the monograph [11]. For the Stokes problem, such type estimates in terms of energy norms have been derived in [1], [2] and numerically tested in [26], [27], [28].

3 MESH-ADAPTIVE ALGORITHMS

The aim of an adaptive strategy is to compute a numerical solution such that the error of it is less than the tolerance given. The error is defined to be the difference between the exact solution and the numerically computed solution measured in a suitable norm. In the framework of the modern results in adaptive methods, this aim is achieved with the help of the following principal algorithm (see, e.g., [29], [30]).

Repeat

$$\ldots \boxed{SOLVE} - \boxed{ESTIMATE} - \boxed{MARK} - \boxed{REFINE} \ldots$$

until a stopping criterion is satisfied.

More precisely, the algorithm can be described as follows:

step 1	Construct initial triangulation $\mathfrak{T}_h$.
step 2	Solve system.
step 3	Estimate the error of approximate solution. If the error is less then required tolerance, then exit.
step 4	Evaluate the error on each element.
step 5	Mark the elements with extensively large errors.
step 6	Refine the mesh. Go to step 2.

Below we comment on each step of this algorithm.

3.1 Solution of the problem

In what follows, use the standard Finite Element Method (as it is described, e.g., in [4]) and impose usual assumptions on the triangulation $\mathfrak{T}_h$ of a domain $\overline{\mathcal{D}}$:

- $\overline{\mathcal{D}} = \bigcup_{T\in\mathfrak{T}_h} T$, T is a simplex;

- for any $T \in \mathfrak{T}_h$ the set T is closed and its interior $\overset{o}{T}$ is nonempty.

- We assume, that for any $T_i, T_j \in \mathfrak{T}_h$ the intersection $\overset{o}{T_i} \bigcap \overset{o}{T_j}$ is empty.

The triangulations $\mathcal{F} = \{\mathfrak{T}_h\}$ are supposed to be regular (see, e.g., [4], [31]). In particular, we accent the "minimal angle condition", which means that $\alpha_T \geq \alpha_0 > 0 \quad \forall T \in \mathfrak{T}_h$, where α_T means the minimal angle of T.

In the thesis, we restrict ourselves to polygonal domains and assume that the triangulation is "exact", i.e.

$$\overline{\mathcal{D}} = \bigcup_h T_h$$

Let us now proceed to specific approximations for viscous flow problems. For these, there exist many finite element spaces(see, e.g., [32], [33]).

Flow problems have two basic variables: velocity and pressure. From the viewpoint of the approximation, the major difficulty consists in the fact that their approximations must be properly balanced in order to guarantee the stability condition of the discrete system. Mathematically, it means that a discrete analog of the inf-sup (LBB) condition

$$\inf_{\phi\in M_h;\,\phi\neq 0} \sup_{w\in V_h;\,w\neq 0} \frac{\int_{\mathcal{D}} \phi \operatorname{div}\, w\, dx}{\|\phi\| \, \|\nabla w\|} \geq \gamma_h \geq \gamma > 0$$

must be satisfied.

All approximations that satisfy such condition and are used in practice, can be divided into three groups:

- Approximations, which exactly fulfilled the incompressibility condition, (e.g., approximation based on the stream function w) Such approximation belongs to the space $J_2^1(\mathcal{D}, \mathbb{R}^n)$ (closure of the differentiable divergence-free functions in the norm of the space $H^1(\mathcal{D}, \mathbb{R}^n)$). We call them *conforming approximations*.

- *partly-conforming approximations* (approximations from the energy class, but without the divergence-free property, e.g., Taylor-Hood approximations, approximations constructed under Mini-elements, macro-elements)

- *non-conforming approximations* (approximations that do not belong to the energy space, e.g., Crouzeix-Raviart approximation).

Here we do not discus methods of solution of the respective system. Some of those can be found, e.g., in [32], [33]. In the present work we have used the relaxation method and the direct MATLAB solver.

3.2 Estimation of the error

Let us make first some general comments on a posteriori error estimation. Denote by u an exact solution of the problem, by v an approximate solution, and let $|||\cdot|||$ be an energy norm. Our aim is to establish the estimate

$$|||u - v||| \leq M(v, D), \tag{3.2.1}$$

where by D we denote problem data (domain, coefficients etc.). Such an estimate gives a guaranteed upper bound of the error and is explicitly computable. Any estimate of a practical interest must possess an additional property:

$$M(v_k, D) \to 0 \quad \text{when} \quad ||v_k - u|| \to 0. \tag{3.2.2}$$

Estimates with such a property are usually called "consistent". Not all a posteriori error estimates used nowadays satisfy such requirements (e.g., they may contain unknown high order terms, generic constants, etc.).

In the present research we investigate functional type a posteriori error estimates. To obtain these estimates we use purely functional analysis of a problem in question. They satisfy (3.2.1) and (3.2.2).Let us consider them more precisely.

3.2.1 Principal structure of functional a posteriori estimates

A profound explanation of the functional type a posteriori error estimate can be found in [11]. The estimates are constructed by the relations that jointly define the exact solution. Typically, they contain global constants that come from functional inequalities (e.g., Friedrichs, Poincare, Ladyzhenskaya-Babuška--Brezzi) or from inequalities for boundary traces. In other words, a majorant of $|||\ u\ -\ v\ |||$, where u is the exact solution of a boundary-value problem and v is an arbitrary approximation from the respective energy class, is the sum of the terms that can be thought of as penalties for unconformity in each of the basic relations. The respective multipliers are defined by the constants in the embedding inequalities for the spaces associated with a mathematical formulation of the problem.

Consider the Poisson's equation

$$\begin{aligned} -\triangle u = f \qquad & \text{in} \mathcal{D}, \\ u = 0 \qquad & \text{on } \partial\mathcal{D}. \end{aligned}$$

Let v be an approximate solution. We can estimate the energy norm of the difference between the exact and the approximate solutions as follows:

$$||\nabla(u - v)|| \leq ||\nabla v - \tau|| + c_{\mathcal{D}}||\operatorname{div} \tau + f||, \tag{3.2.4}$$

where $c_{\mathcal{D}}$ is a constant in the Friedrichs inequality. The estimate (3.2.4) was first obtained in [25], what gives start to the "functional" direction in the a posteriori control.

We call the first term of (3.2.4) "primary" (it contains the main information on the error), another another term is that of "reliability" (due to the presence of it, we know that the upper bound is guaranteed). It is easy to observe that the right-hand side of (3.2.4) is nonnegative and vanishes if and only if $v = u$ and $\tau = \nabla u$. Moreover, it is exact in the sense that τ can be taken such that the right-hand side of (3.2.4) is equal to the left-hand one. In this case, the "reliability term" is equal to zero, while the "primary term" is equal to the error.

Several possible strategies of the implementation are shown below. First we note, that clearly τ should be in a sense close to the ∇u. Thus, we suggest to use the error majorant in one of the following ways.

If we already have some adaptive meshes obtained by any method, the following algorithms can be implemented.

- way 1 (the simplest)
  Initial guess ($\tau = \mathbb{G}\nabla v_h$),
  where ($\mathbb{G}$ is operator of averaging (see, e.g., [14])). This method is simple and cheap. However, it provides a coarse estimation.

- way 2 (more accurate)
  Some adaptive refinement process:
  Meshes $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, ...\mathfrak{T}_k, \mathfrak{T}_{k+1}, ...$
  To estimate error on mesh $\mathfrak{T}_k$:
  take $\tau_k = (\mathbb{G}\nabla v)_{k+1}$ from finer mesh. In this case the estimation is performed with one step retardation. This method is also simple, but gives much better results.

- way 3 (best estimate)
  Full minimization with respect to τ. This method is expensive, but provides the best error bounds.

For viscous flow problems, we have tested all the three above strategies and can confirm that the third one gives the best results.

For the theory of fluids, the functional type error majorant consists of three terms. In addition to "primary" and "reliability", it contains a term, which penalizes the violation of the divergence-free condition:

$$|||u - v||| \leq \sqrt{M_p} + \sqrt{M_r} + \sqrt{M_{div}}. \tag{3.2.5}$$

This estimate (3.2.5) can be rewritten in the quadratic form by introducing positive constants $\beta_i > 0$.

$$E := |||u - v|||^2 \leq (1 + \beta_1 + \beta_2)M_p + (1 + \frac{1}{\beta_1} + \beta_3)M_r +$$
$$+ (1 + \frac{1}{\beta_2} + \frac{1}{\beta_3})M_{div} =: M.$$

To asses the quality of the error estimation we define the efficiency index

$$I_{eff} = \sqrt{\frac{M}{E}},$$

that characterizes the quality of an error estimation. By definition, it is greater then 1, and equals 1 if and only if the estimator is equal to the error.

In the numerical experiments presented, one can observe that it is possible to achieve efficiency index quite close to 1.

If this estimate is applied to FEM approximations, then M_p, M_r and M_{div} can be presented as sums of element-wise quantities. Thus, they are used as error indicators to mark the zones with the extensively high errors and to make necessary refinement.

The local quantities

$$E_T := |||u - v|||_T^2 \leq (1 + \beta_1 + \beta_2)(M_p)_T + (1 + \frac{1}{\beta_1} + \beta_3)(M_r)_T +$$
$$+ (1 + \frac{1}{\beta_2} + \frac{1}{\beta_3})(M_{div})_T =: M_T$$

are used for the refinement procedures.

Certainly, the best possible adaptive algorithm can be constructed on the basis of the true error distribution obtained by comparing the true and approximate solutions. We denote by e_T the normalized local contribution of the error on the element, by m_T the normalized local contribution of error indicator. To compare them not only qualitatively but also quantitatively we introduce a special coefficient

$$p_{shape} = 1 - \frac{\Sigma|m_T - e_T|}{N},$$

where N is a number of elements. p_{shape} is equal to one only in the ideal case, when the normalized error indicator coincides with the normalized true error, what means that they may differ by a factor only.

3.3 Marking strategies

In our experiments, we use the simplest (two-color) marking strategy. In other words, we set "one" or "zero" to each element, and construct an element-wise boolean function

$$R(M_T) = \{0,\ 1\}.$$

"Zero" value of such a function means, that the element will not be refined and the value "one" means that the element is subject to further subdivision.

In our numerical tests we have used two refinement strategies: The first strategy follows the so-called "maximum criterion": within its framework we set

$$R_{max}(M_T) = 1 \quad \text{if} \quad M_T \geq \theta_{max} M_{max},$$

where θ_{max} is a given parameter. Typically, $\theta_{max} = 1/2$, In this case, an element is refined if the error is bigger than one half of the maximum error (see, e.g., [19]).

The second strategy is the so-called "bulk criterion" strategy. Here, the elements are ranked by the values of the local errors. For the refinement, we take the ones, that contain maximum errors and jointly give some certain part (θ_{bulk}) of the total error. In other words,

$$R^{bulk}(M_T) = 1, \quad \text{if} \quad \Sigma M_T \geq \theta_{bulk} \Sigma M_T.$$

In our tests we take $\theta_{bulk} = 60\%$.

To estimate the quality of error indication, we compare the indication of the error with the "etalon indicator". By "etalon" indicator we understand an indicator made with help of the true error distribution. If the exact solution of the model problem is unknown, then by the "true error" we assume a comparison with the function on a very fine mesh (reference solution).

Let us define p_{eff} which shows the percent of the elements marked in the same way as in the etalon marking, i.e.

$$p_{eff} = 1 - \frac{\Sigma |R_T - R_T^{etalon}|}{N}.$$

In general, any good error indicator should provide good results regardless of the marking strategy employed, because it should provide a correct representation of the true error. In fact, we observed this for the indicators that follow from the functional a posteriori estimate. However, for other indicators (as, e.g., gradient averaging) this is not always true (see, e.g., Figure 2 in Article **PII**). We demonstrate some examples, with smooth solution on simple domains, and also in more complicated problems. In the numerous numerical experiment we have implemented for the functional type error, we have been observing reliable error indication in all the cases.

3.4 Refinement

In our computations, the refinement strategy that we use is the redgreen isotropic refinement strategy. The redgreen isotropic refinement strategy works in the following way:

(a) define the basic mesh that is contained in any further mesh; (b) once an element has been marked for the refinement, it is refined in a regular manner, whereby a triangle is divided up into four similar triangles by connecting the midpoints of the sides (this is termed red refinement, see Figure 1); (c) in doing so hanging nodes are immediately created on neighboring elements, and the triangulation is no longer admissible. Green refinement splits the neighboring elements into two as shown in Figure 1; in the case of an element acquiring two or more hanging nodes, red refinement is performed. Figure 1 indicates a procedure, where the bold line defines the initial unrefined element. Within this meshing strategy, elements may also be removed, providing that they do not lie in the original mesh.

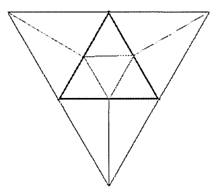

FIGURE 1 Refinement strategy.

In our work, we use the PDE MATLAB toolbox, where the proposed refinement algorithm is realized.

4 A POSTERIORI ERROR ESTIMATION FOR THE STOKES EQUATION

4.1 Formulation of the problem

Let $\mathcal{D}$ be an open bounded domain in $\mathbb{R}^n$, with Lipschitz continuous boundary $\partial\mathcal{D}$. Let $f \in L_2(\mathcal{D}, \mathbb{R}^n)$ be a given vector-valued function. The classical Stokes problem consists in determination a vector-valued function u (the velocity of the fluid), and a scalar-valued function p (the pressure), which are defined in $\mathcal{D}$ and satisfy the following equations and boundary conditions:

$$-\nu \triangle u = f - \nabla p \quad \text{in } \mathcal{D}, \tag{4.1.1}$$

$$\operatorname{div}\, u = 0 \quad \text{in } \mathcal{D}, \tag{4.1.2}$$

$$u = u_g \quad \text{on } \partial\mathcal{D}, \tag{4.1.3}$$

where ν is the kinematical viscosity coefficient and $u_g \in H^1(\mathcal{D}, \mathbb{R}^n)$ defines the Dirichlet boundary conditions on $\partial\mathcal{D}$. It is assumed that $\operatorname{div} u_g = 0$.

Two well known variational formulation of the Stokes problem (see, e.g., [34]) are as follows:

$$\int\limits_{\mathcal{D}} \nabla u : \nabla v = \int\limits_{\mathcal{D}} (f - \nabla p) \cdot v\, dx \quad \forall v \in \overset{o}{H}{}^1(\mathcal{D}, \mathbb{R}^n), \tag{4.1.4a}$$

$$-\int\limits_{\mathcal{D}} q \operatorname{div} u = 0 \quad \forall q \in \widetilde{L}_2(\mathcal{D}) = \{q \in \mathbf{L}_2(\mathcal{D}) | \int\limits_{\mathcal{D}} q = 0\} \tag{4.1.4b}$$

and

$$\nu \int_{\mathcal{D}} \nabla u : \nabla v dx = \int_{\mathcal{D}} f \cdot v\, dx \quad \forall v \in \overset{o}{J}{}^1_2(\mathcal{D}, \mathbb{R}^n), \tag{4.1.5}$$

where $J_2^1(\mathcal{D},\mathbb{R}^n)$ - is the closure of the set $\overset{o}{J}{}^{\infty}(\mathcal{D},\mathbb{R}^n)$ on the norm of space $H^1(\mathcal{D},\mathbb{R}^n)$

$$\overset{o}{J}{}^{\infty}(\mathcal{D},\mathbb{R}^n) = \{v \in C_0^{\infty}(\mathcal{D},\mathbb{R}^n) \mid \operatorname{div} v = 0, \operatorname{supp} v \subset\subset \mathcal{D}\}. \tag{4.1.6}$$

In the first variational formulation (4.1.4), test functions are taken from the space $\overset{o}{H}{}^1(\mathcal{D},\mathbb{R}^n)$, in the second one (4.1.5), they belong to the space of divergence-free function. Below we present estimates in terms of the energy norm of the difference between exact and approximate solution.

4.2 Functional type a posteriori error estimates

Functional type a posteriori error estimates for the Stokes problem were firstly obtained theoretically in [1] (see also [2]). Their numerical investigations were made in [26], [27], [28], (works [26] and [28] are included in the thesis).

The main results are formulated in the following theorems:

Theorem 4.2.1 (Estimate for divergence-free approximations). *For any $v \in \overset{o}{J}{}_2^1(\mathcal{D},\mathbb{R}^n) + u_g$, $\tau \in \Sigma_{\text{div}}(\mathcal{D})$, $q \in H^1(\mathcal{D})$ the following estimate holds:*

$$\|\nu\nabla(u - v)\| \leq \|\tau - \nu\nabla v\| + c_{\mathcal{D}}\|f + \operatorname{div}\tau - \nabla q\|. \tag{4.2.1}$$

Theorem 4.2.2 (Estimate for non divergence-free approximations). *For any $\tilde{v} \in \overset{o}{H}{}^1(\mathcal{D},\mathbb{R}^n) + u_g$, $\tau \in \Sigma_{\text{div}}(\mathcal{D})$, $q \in H^1(\mathcal{D})$ the following estimate holds:*

$$\|\nu\nabla(u - \tilde{v})\| \leq \|\tau - \nu\nabla\tilde{v}\| + c_{\mathcal{D}}\|f + \operatorname{div}\tau - \nabla q\| + \frac{2\nu}{C_{LBB}}\|\operatorname{div}\tilde{v}\|, \tag{4.2.2}$$

where the constant C_{LBB} is the constant from Ladyzhenskaya-Babuška-Brezzi inequality.

4.3 On the C_{LBB} constant

We observe, that a posteriori error estimate in the form (4.2.2) requires the value of C_{LBB}.

The constants C_{LBB} play an important role in the numerical analysis of the Stokes problem as well as in the theoretical one. They affect the stability of mixed-type formulations and the efficiency of iteration methods (see, e.g., [35], [36]). Therefore, it is very desirable to have a unified numerical technology able to compute values of C_{LBB} for arbitrary Lipschitz domains. To the best of our knowledge, at present such a unified technology does not exist. An upper bound of C_{LBB}can be expressed throughout the constants in the Friedrichs and Poincare inequalities for the domain (see [37], [2]). However, in practice we are more interested in the lower bound. For rectangular domains, two-sided bounds for the constant were derived in [38], [39] and [40]. In [41], it was shown that for a unitary disc the constant equals $1/\sqrt{2}$. Some ideas numerical evaluation of the C_{LBB} constant are contained in the above cited publications and in the paper by M. [42], where the case of stretched domains is considered.

It can be shown, that the C_{LBB} constant is directly connected to the constant from the Nešas inequality and the constant in the closed range lemma. However, estimation of that constants presents the a difficult problem in modern numerical analysis.

In order to avoid estimation of C_{LBB}, it is possible to project an approximate solution to the space of divergence-free function (for example, by means of a stream function) and consider this projection as an estimated function. The corresponding algorithm is described in the following subsection.

4.4 Projection on solenoidal vector fields

The following algorithm constructs a new approximation $\widehat{v}^h \in \overset{o}{J}{}^1_2(\mathcal{D}, \mathbb{R}^n) + u_g$ from a given function $v^h \notin J^1_2(\mathcal{D}, \mathbb{R}^n) + u_g$. In the case of $\mathbb{R}^2$, the construction is realized through the stream function w with the help of the following formulas:

$$\widehat{v}^h_1 = \frac{\partial w}{\partial y}; \quad \widehat{v}^h_2 = -\frac{\partial w}{\partial x}. \tag{4.4.1}$$

A suitable divergence-free function $\widehat{v}^h \in J^1_2(\mathcal{D}, \mathbb{R}^n) + u_g$ should be found from the natural minimality condition

$$\|\nabla(\widehat{v}^h - v^h)\| \to min. \tag{4.4.2}$$

For approximations of the function w we should use C^1 element. In our experiments, we have used Cie-Clouh-Tocher elements. Let us briefly recall their structure. Cie-Clouh-Tocher element is a macroelement (triangle) T divided by the center of the mass to the tree triangles T_i. On the each of the triangles function is presented as a polynom of the degree 3. Since $\dim P_3(T_i) = 10$, then it is necessary to obtain 30 equations in order to find three polynoms

p_{iT_i} $1 \leq i \leq 3$. First of all, 21 can be obtained from the degrees of freedom related to the the T element. They are: nodal values of the function, nodal values of the derivatives and the values of the normal derivatives at the midpoints of the edges. Another 9 equations can be obtained from the fact, that the element belongs to the class of C^1-elements. It is necessary to satisfy the continuity condition at the central point for the function and its derivatives and the continuity condition of the normal derivatives at the midpoints of the edges.

We note that the projection procedure is not very expensive from the computational point of view. In it, the minimization is performed only with respect to three parameters (nodal values of the stream function). Value of the derivatives of the stream function should be taken with the help of known nodal values v_1^h, v_2^h. For example, for the Taylor-Hood elements the corresponding values can be calculate as follows:

$$\left.\frac{\partial w}{\partial x}\right|_i = -\left.v_2^h\right|_i, \left.\frac{\partial w}{\partial x}\right|_j = -\left.v_2^h\right|_j, \left.\frac{\partial w}{\partial x}\right|_k = -\left.v_2^h\right|_k, \tag{4.4.3}$$

$$\left.\frac{\partial w}{\partial y}\right|_i = \left.v_1^h\right|_i, \left.\frac{\partial w}{\partial y}\right|_j = \left.v_1^h\right|_j, \left.\frac{\partial w}{\partial y}\right|_k = \left.v_1^h\right|_k, \tag{4.4.4}$$

$$\left.\frac{\partial w}{\partial n}\right|_{ij} = \left.\left(-v_2^h \cdot n_x + v_1^h \cdot n_y\right)\right|_{ij}, \tag{4.4.5}$$

$$\left.\frac{\partial w}{\partial n}\right|_{ik} = \left.\left(-v_2^h \cdot n_x + v_1^h \cdot n_y\right)\right|_{ik}, \tag{4.4.6}$$

$$\left.\frac{\partial w}{\partial n}\right|_{jk} = \left.\left(-v_2^h \cdot n_x + v_1^h \cdot n_y\right)\right|_{jk}, \tag{4.4.7}$$

where w_i, w_j, w_k is defined by (4.4.2).

For the Creuzeix-Reaviar elements the nodal values are not defined, because the approximate velocity is not continuous. However, it is possible to modify the proposed algorithm if we replace (4.4.3)-(4.4.4) with the (4.4.8)-(4.4.9), i.e. take the averaging over the patch instead of velocity $\mathbb{G}v^h$:

$$\left.\frac{\partial w}{\partial x}\right|_i = -\left.\mathbb{G}v_2^h\right|_i, \left.\frac{\partial w}{\partial x}\right|_j = -\left.\mathbb{G}v_2^h\right|_j, \left.\frac{\partial w}{\partial x}\right|_k = -\left.\mathbb{G}v_2^h\right|_k, \tag{4.4.8}$$

$$\left.\frac{\partial w}{\partial y}\right|_i = \left.\mathbb{G}v_1^h\right|_i, \left.\frac{\partial w}{\partial y}\right|_j = \left.\mathbb{G}v_1^h\right|_j, \left.\frac{\partial w}{\partial y}\right|_k = \left.\mathbb{G}v_1^h\right|_k. \tag{4.4.9}$$

It is also possible to propose a similar algorithm in $\mathbb{R}^3$ and for the cylindrical coordinate system. However, it will be too complicated to present it here.

4.5 Functional type a posteriori error estimate in the local norms

The estimates presented (4.2.2) and (4.2.1) yield the overall accuracy of an approximate solution computed. As it was shown in [26], they also give quite reliable information about the distribution of the error over the domain. But sometimes this information is not enough and more detailed information is required. Let us show how to provide a guaranteed upper bound in a local norm. Functional type estimates in the local norms were first obtained in [43]. For the Stokes problem they were obtained in [3] (see also [44]).

Introduce a local norm

$$\|\nu\nabla(u-v)\|_\omega = \Big(\int_\omega |\nu\nabla(u-v)|^2\,dx\Big)^{1/2}, \tag{4.5.1}$$

where ω (subdomain of $\mathcal{D}$ with Lipschitz continuous boundary $\partial\omega$) is a "domain of interest".

Let ϕ be a divergence-free function, such that $\nabla\phi = 0$ a.e. in ω, i.e. $\phi = const$ a.e. in ω.

As is easy to see

$$\begin{aligned} \nu^2\|\nabla(u-v)\|_\omega^2 &= \nu^2\|\nabla(u-v-\phi)\|_\omega^2 \leq \nu^2\|\nabla(u-v-\phi)\|_{\mathcal{D}}^2 = \\ &= \nu^2\|\nabla(u-v)\|_\omega^2 + \nu^2\|\nabla(u-v-\phi)\|_{\mathcal{D}\setminus\omega}^2. \end{aligned} \tag{4.5.2}$$

As a matter of fact, a local error related to a subdomain ω is caused by the following two reasons: possible violation of the differential equations in ω and infringement of the boundary condition on $\partial\omega$ (see (4.5.2)). Thus, in case $u = v$ on $\partial\omega$, only the first reason form the error. Via taking $\phi = u - v$ in $\mathcal{D}\setminus\omega$ and $\phi \equiv 0$ in ω it is easy to observe, that the error estimator is "exact", i.e. the error majorant is equal to the true local error.

Taking into account, that (4.5.2) can be minimized with respect to ϕ, we rewrite (4.5.2) as follows

$$\nu^2\|\nabla(u-v)\|_\omega^2 \leq \inf_\phi \nu^2\|\nabla(u-v+\phi)\|_{\mathcal{D}}^2. \tag{4.5.3}$$

Consider $\widetilde{v} = v - \phi$ as an approximate solution. Its accuracy can be estimated via the (4.2.1):

$$\begin{aligned} \nu^2\|\nabla(u-v)\|_\omega^2 &= \nu^2\|\nabla(u-\widetilde{v})\|_\omega^2 \leq \inf_\phi \nu^2\|\nabla(u-v+\phi)\|_{\mathcal{D}}^2 \leq \\ &\leq \inf_\phi\,(\|\ \nu\nabla\widetilde{v} - \tau\ \| + c_{\mathcal{D}}\ \|\ \mathrm{div}\ \tau + f - \nabla q\ \|)^2. \end{aligned} \tag{4.5.4}$$

For a non-solenoidal approximations v a similar estimator can be obtained by using (4.2.2) instead of (4.2.1).

A more profound analysis of this type of error estimates can be found in [3].

4.6 Practical implementation

In this section, we discus practical implementation of the proposed local estimators. We present general algorithm and explain some moments, which help to improve estimators and economize computational time.

First of all, let us rewrite the majorant in a quadratic form, which is more convenient for practical implementation. For this purpose, we introduce introducing positive scalar parameters β_1, β_2, β_3 and represent (4.2.2) in the form

$$\nu^2\|\nabla(u-v)\|^2 \leq (1+\beta_1+\beta_2)\|\nu\nabla v-\tau(x)\|^2+ \\ +(1+\frac{1}{\beta_1}+\beta_3)c_{\mathcal{D}}^2\|\operatorname{div}\tau+f-\nabla q\|^2+ \\ +(1+\frac{1}{\beta_2}+\frac{1}{\beta_3})\frac{4}{C_{LBB}^2}\nu^2\|\operatorname{div} v\|^2. \quad (4.6.1)$$

Analogously, (4.2.1) can be rewritten as follows:

$$\nu^2 \parallel \nabla(u-v) \parallel^2 \leq (1+\beta) \parallel \nu\nabla v-\tau \parallel^2 \\ +(1+\frac{1}{\beta})c_{\mathcal{D}}^2 \parallel \operatorname{div}\ \tau+f-\nabla q \parallel^2 \quad (4.6.2)$$

Optimal value for scalar parameters β_i can be stated in the framework of a numerical procedure or analytically.

To compute local errors we use (4.5.4) and the following algorithm:

Algorithm for error estimation in subdomain ω

- Step 1. Construct the initial mesh "h"
- Step 2. Solve the problem on current mesh and find v_h
- Step 3. Make an averaging, find $G_h v_h$ and compute a coarse error bound
- Step 4. Improve theestimate by minimization over τ and q This gives a more accurate estimation (especially if second term of the (4.2.2) is closed to zero)

- Step 5. Local estimation: add a function ϕ such as $\nabla\phi = 0$ in ω (i.e. $\phi = const$ in ω)
  Minimize over all such ϕ and find a guaranteed error bound in the local norm

- Last two steps can be repeated.

Remark 4.6.1. *For practical purposes, it is more convenient to search for a function $\widetilde{v} = v + \phi$ rather then to find function ϕ. We call function $\widetilde{v}$ – "billet" function. Such a "billet" function should be essentially more accurate, then the actual approximation.*

From the one hand this process can be considered as a solution of the Stokes problem from the new variational formulation (4.2.2), if we use it only for finding v. From other hand, we can consider this process as solution of problem constructing v with prescribed gradient (and boundary condition). That is because the complimentary problem $\operatorname{div}\tau + f = \nabla p$ is already solved with sufficient accuracy.

This new "billet" function $(\widetilde{\psi})$ can be considered as a new solution. It is more accurate then v and its guaranteed accuracy is already estimated.

Algorithm for error estimation on several local subdomains ω_i

- Steps 1-4 are the same

- Step 5: Local estimation: add function ϕ of higher order approximation then v_h. Minimize error majorant with respect to ϕ and find much better approximation with guaranteed accuracy.

- Steps 4-5 are repeated untill the second term became sufficiently small

- Step 6: For all ω_i take ϕ such that $\nabla\phi = 0$ in ω (i.e. $\phi = const$ in ω)
  Minimize the majorant with respect to all ϕ and compute an upper bound of the error

The latter algorithm is especially favorable if it is necessary to estimate the error on several subdomains or elements (e.g., these areas can be determined by the distribution of the global estimator over the domain). Note, that minimization required in step 6 is parametric minimization of the quadratic functional.

Finally, let us comment on local error estimation for different type finite element used. In the present work we have implemented the approach suggested to different element-wise linear approximations for the Stokes problem, such as macro-elements and mini-elements. For the mini-elements, following Verfurth (see [19]), we consider only a linear part of the error, neglecting errors associated with bubble functions. The reason of this is as follows: in practical computations linear parts of approximate solution of the Stokes equation is usually a better approximation, moreover it has the same asymptotic order of convergence as the original mini-element approximation. For the conforming elements both algorithms of local error estimation can be applied without restriction. (Numerical tests of the local a posteriori error estimates are presented in the end of the this chapter). It is also possible to use functional type a posteriori error estimator for other elements. It requires proper choice of the approximation subspace of the "billet" function ϕ. Obviously, the order of approximation should be higher, then for the velocity field. However (and it is important to outline), there are no special requirements for the pair of functional spaces for the approximation for the velocity and pressure (such as Ladyzhenskaya-Babuška-Brezzi condition). For example, for Taylor-Hood approximation (quadratic for velocity, linear for pressure) we suggest to use element-wise cubical (or tetrahedral) function for the "billet" function ϕ, and a quadratic for the dual function q. Procedures of element-wise projection to the space of divergence-free function can be modified to this case.

4.7 Numerical examples

4.7.1 Example 1

Numerical experiments were made to check practical efficiency of the method proposed. We have performed various experiments using different types of finite element approximation for the Stokes problem. In all cases, we have observed robustness of the functional type error estimation with respect to type of elements, mesh structure, method and accuracy of the solution.

Let us start with an example, typical for a pôsteriori error control for the Stokes problem.

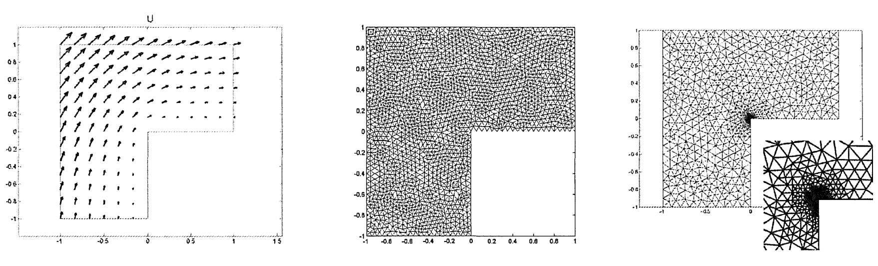

FIGURE 2 Example 1. Velocity (left); Uniform mesh (center); Adapted mesh (right).

Consider the L-shape domain

$$\mathcal{D} = (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0].$$

The boundary values are taken from the exact solution (u, p):

$$w(\phi) = (\sin((1+\alpha)\phi)\cos(\alpha\omega))/(1+\alpha) - \cos((1+\alpha)\phi) - \\ - (\sin((1-\alpha)\phi)\cos(\alpha\omega))/(1-\alpha) + \cos((1-\alpha)\phi), \quad (4.7.1)$$

$$u(r,\phi) = r^{\alpha}((1+\alpha)(\sin(\phi), -\cos(\phi))w(\phi) + (\cos(\phi), \sin(\phi))w_{\phi}(\phi)), \quad (4.7.2)$$

$$p(r,\phi) = -r^{\alpha-1}((1+\alpha)^2 w_{\phi}(\phi) + w_{\phi\phi\phi}(\phi))/(1-\alpha). \quad (4.7.3)$$

$$w(\phi) = (\sin((1+\alpha)\phi)\cos(\alpha\omega))/(1+\alpha) - \cos((1+\alpha)\phi) - \\ - (\sin((1-\alpha)\phi)\cos(\alpha\omega))/(1-\alpha) + \cos((1-\alpha)\phi). \quad (4.7.4)$$

where $f = 0$, $\alpha = 856399/1572864 \approx 0.54448$ and $\omega = 3\pi/2$,

In this example, we use mini-elements and the standard adaptation algorithm described above. Error control is obtained by using projection on the space of divergence-free functions. For guaranteed estimations of the error, we use the error majorant in form (4.2.1) and second order finite elements for approximations of the τ and q. As an initial guess for τ, we use an averaging of $\nu\nabla v$, while an initial guess for q is p^h was obtained via some numerical method. An improvement is obtained by minimization over τ and q. Table 1 contains number of elements, values of error, error majorant and the efficiency index during the adaptation process. It is easy to see, that on the each iteration step the error majorant provides a guaranteed upper bound of the error.

TABLE 1 Example 1. Error estimation.

iter	N	$\sqrt{E}$	$\sqrt{M}$	I_{eff}
5	472	0.94	1.2878	1.37
7	1151	0.053	0.06148	1.16
9	2174	0.041	0.06027	1.47
11	3714	0.031	0.04092	1.32
12	4303	0.026	0.03926	1.51
14	5734	0.013	0.01664	1.28
19	7893	0.0096	0.01373	1.43
26	12552	0.008	0.00952	1.19

If to turn to local error estimates, then we first of all should select the domain of interest around the reentrant corner. Namely, we take

$$\mathcal{D} = (-d, 1) \times (-d, d) \setminus [0, d] \times [-d, 0].$$

We use the same adaptation process, and obtain local estimates on the same mesh. Table 2 gives the effectivity index for the local adaptation between 1.5 and 2.

TABLE 2 Example 1. Estimation of local errors.

Iter	N	$\sqrt{E}$	$\sqrt{M}$	I_{eff}
5	472	0.7708	1.4876	1.93
7	1151	0.03975	0.0612	1.54
9	2174	0.03403	0.0568	1.67
11	3714	0.02387	0.0434	1.82
12	4303	0.01976	0.0362	1.83
14	5734	0.00949	0.0145	1.53
19	7893	0.00643	0.0110	1.71
26	12552	0.00536	0.0078	1.45

4.7.2 Example 2

In the second example, the data and the exact solution are smooth. Then, a priori it is not obvious, where the error should be concentrated. Consider an example from [45], which is often used in test examples. Let $\mathcal{D} = (0, 1) \times (0, 1)$, $\nu = 1$, the exact solution and effective force are defined as follows:

$$u = (-\sin(\frac{\pi}{2}\,x)\,\sin(\frac{\pi}{2}\,y), -\cos(\frac{\pi}{2}\,x)\,\cos(\frac{\pi}{2}\,y))^T,$$

$$p = \pi\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y), \quad f = (0, -\pi^2\cos(\frac{\pi}{2}\,x)\cos(\frac{\pi}{2}\,y))^T.$$

The velocity is depicted on Figure 3

This problem can be solved by different methods. We present results obtained by using the Uzawa algorithm, Hesteness-Powel algorithm, macro-elements, and Taylor-Hood elements. For the error control a similar procedure to that in Example 1 is used. But error the majorant is taken in the form (4.2.2). Estimates of C_{LBB} for the rectangular domain are known due to [38], [39].

The majorant minimization requires additional computational work. Computational time spent on improvement of the estimate is determined in comparison with the time spent for finding the numerical solution (we denote this

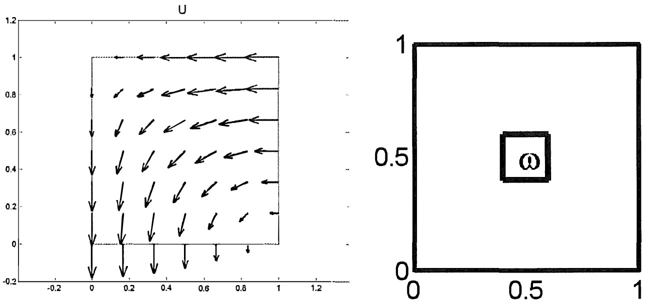

FIGURE 3 Example 2. Velocity (left); Domain of interest (right).

time by 1 TU). Table 3 demonstrates the dependence of the quality of the error estimation versus the computational time spend on improvement of the majorant. Note that $t = 0\,TU$ denotes the substitution $\tau = \nu\mathbb{G}\nabla v$, $q = p$ ($\mathbb{G}$ is operator of averaging (see, e.g., [14])). This allows us to get guaranteed bound almost without additional computational expenditures. Table 3 contains information about components of the error majorant and main characteristics, typical for a posteriori error estimation methods. They show the overestimation of the error and the quality of error indication. Theoretically, it is known that the error majorant achieves its minimum when $\tau = \nu\nabla u$ and $q = p$. By this substitution, the reliability term (the second component of (4.2.1)) turns into 0 and the error estimation turns to be equal to the error. If we take τ as the averaged gradient (i.e. $t = 0\,TU$), then the the second component in error majorant prevails and the error indication is not very accurate. But after some time spent for minimization the majorant is quite close to the error and local error indicators are also quite accurate.

In one series of tests we used linear element for the dual variables (and for the velocity). In this case, it was observed, that in certain examples it is not possible to achieve a very sharp estimation regardless of the time spend (see Table 4). If the quadratic functions are used then a much sharper estimation (see Table 5) was achieved.

Table 6 is related to the algorithm with one step retardation, presented in chapter 3. In this series of tests, the macro elements was used for approximations. This algorithm provided cheap error estimation with effeciency index around 3 for the coarse meshes. However, for fine and deeply adapted meshes, this algorithm can not provide sharp estimation.

Finally, we demonstrate robustness of the functional type error estimator in the situation, where other error indicators do not work. In Fig. 4, 5, 6 by dark (red) color we depict the zones, where the major error is concentrated.

TABLE 3 Example 2. Dependence of the quality of the error estimation on the computational time spend on majorant improvement.

	t=0	$t = 0.5TU$	$t = 1TU$	$t = 2TU$
$\sqrt{E}$	5.89 e-4	5.89 e-4	5.89 e-4	5.89 e-4
$\sqrt{M}$	0.0159	1.86 e-3	1.0 e-3	6.95 e-4
$\sqrt{M_p}$	1.3 e-4	5.3e-4	5.91 e-4	6.3 e-4
$\sqrt{M_r}$	0.0157	1.3 e-3	3.8 e-4	6.2 e-5
$\sqrt{M_{div}}$	3.1e-6	3.1e-6	3.1e-6	3.1e-6
I_{eff}	27	3.16	1.71	1.18
p_{eff}	0.63	0.87	0.96	0.97

TABLE 4 Example 2. Linear element approximations of τ and q.

	T=0	T=0,5TU	T=1TU	T=2TU	T= 3TU
$\sqrt{E}$	0.0047	0.0047	0.0047	0.0047	0.0047
$\sqrt{M}$	0.0400	0.0244	0.0113	0.0080	0.0078
$\sqrt{M_p}$	0.0002	0.0038	0.0042	0.0045	0.0045
$\sqrt{M_r}$	0.0398	0.0206	0.0071	0.0035	0.0033
β	198.7500	5.4316	1.6857	0.7756	0.7233
I_{eff}	8.50	5.20	2.40	1.70	1.65
p_{eff}	0.72	0.81	0.89	0.91	0.93

These elements are need to be refined according to the "maximum strategy". On the left side we depict such zones marked by the true error and on the right those computed according to the error majorant. It is easy to see that they display almost the same zones, what confirms good quality of the indication performed by the error majorant. In the middle we depict an error indicator based on gradient averaging. In some situations (see Fig. 4) it is also quite closed to the true error, but in some others (see Fig. 5 and 6) it is not so correct.

Let us shortly discuss some results related to local error estimation. Exact solution of the problem described in Example 2 is smooth, and the error is not localized in a small domain (as for problems with re-entrant corners). We select ω to be square domain in the center of $\mathcal{D}$ (see Figure 3, right). In this test we used a macro-element approximation. We constructed sequence of meshes and control the deviation from the exact solution in ω. The corresponding results are presented in Table 7. Effectivity indexes around 2 can be observed.

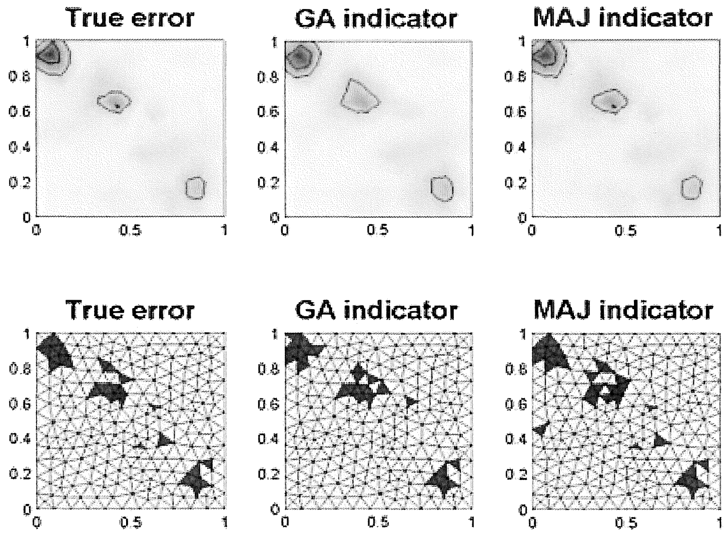

FIGURE 4 Example 2. Test1: Error indicators

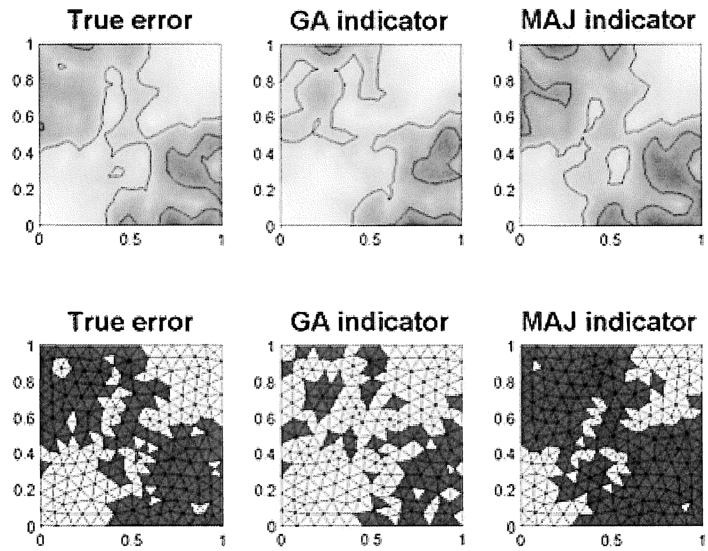

FIGURE 5 Example 2. Test2: Error indicators

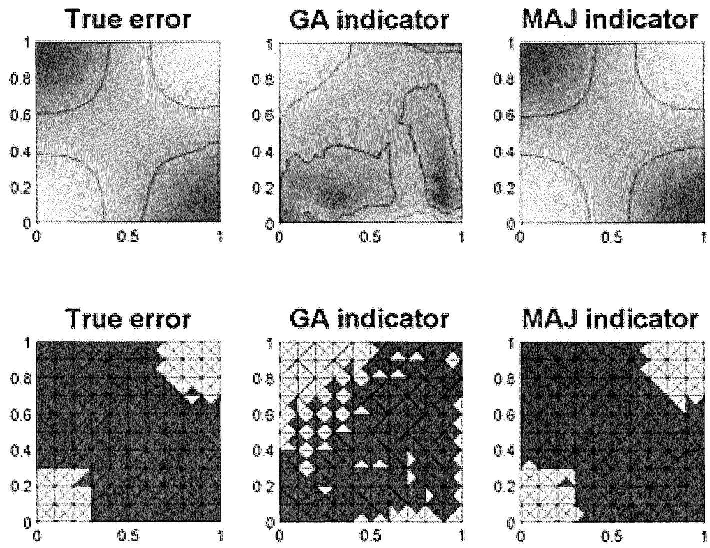

FIGURE 6 Example 2. Test3: Error indicators

4.7.3 Sensitivity of the majorant with respect to the Friedrichs' constant

Functional type error majorant contains constant from the Friedrichs' inequality. This constant is easy to obtain analytically for some simple domains (rectangular, cylinder, sphere etc.). In case of the Dirichlet boundary condition, it can be estimated by the constant of a domain that includes $\mathcal{D}$.

In the numerical test discussed below we investigate the influence of possible overestimation of Friedrichs' constant. For the square domain, we know its analytical value (which refers to a "sharp estimation" case). The corresponding results are presented in Table 8. In Table 9, we use the value $\widetilde{c_{\mathcal{D}}} = 3c_{\mathcal{D}}$.

One can observe, that overestimation of $c_{\mathcal{D}}$ increases the time required for the minimization the majorant. We observe that the results in the column $t = 0\,TU$ and $t = 0.5\,TU$ are quite different (i.e. in these cases an overestimation of $c_{\mathcal{D}}$ plays a significant role). However, for $t = 2\,TU$ the influence of an overestimation is not so large because the value of the reliability term is small.

4.8 Chapter 4: concluding remarks

Numerical results exposed in Chapter 4 confirm the efficiency of the functional type error estimates for approximate solutions of the Stokes problem. It was demonstrated that the error majorant can be used for a quick (preliminary) estimation of the error as well as for a sharp error estimation that also provides an accurate and robust error indication valid not only to Galerkin solutions but to any other conforming approximation. The influence of the overestimation of the global constant ($c_{\mathcal{D}}$) to the majorant was also studied. It was shown, that its overestimation does not critically affect the quality of the estimate if a minimization of the majorant is used. It was demonstrated that the best error estimate can be achieved by the direct minimization of the error majorant.

Local a posteriori error estimates were also tested. They allow to estimate the error related to a certain subdomain of interest. For this task, we suggest a new computational technology based on the so-called "billet" function. In the respective numerical examples, the efficiency index around 2 was achieved.

TABLE 5 Example 2. Quadratic elements approximations of τ and q.

	T=0	T=0.5TU	T=1TU	T=2TU	T= 3TU
$\sqrt{E}$	0.0047	0.0047	0.0047	0.0047	0.0047
$\sqrt{M}$	0.0397	0.0103	0.0063	0.0062	0.0060
$\sqrt{M_p}$	0.0002	0.0046	0.0047	0.0046	0.0048
$\sqrt{M_r}$	0.0395	0.0057	0.0016	0.0016	0.0012
β	197.5750	1.2478	0.3500	0.3487	0.2533
I_{eff}	8.45	2.20	1.35	1.32	1.28
p_{eff}^{max}	0.72	0.90	0.91	0.94	0.94

TABLE 6 Example 2. Algorithm with one step retardation.

N	$\sqrt{E}$ (%)	$\sqrt{M}$ (%)	I_{eff}
253	3.74	10.0	2.9
468	2.0	6.4	3.2
590	1.82	5.1	2.8
854	1.34	4.1	3.1
1278	0.98	3.0	3.05
2074	0.75	0.94	-

TABLE 7 Example 2.Estimation of local errors.

Iter	N	$\sqrt{E_w}$	$\sqrt{M_{local}}$	I_{eff}
1	266	0.021	0.0483	2.3
2	470	0.017	0.0323	1.9
3	926	0.013	0.0208	1.6
4	2052	0.0095	0.0171	1.8
5	3564	0.0072	0.01728	2.4
6	4820	0.0063	0.01323	2.1
7	7008	0.0041	0.00697	1.7
8	10430	0.0054	0.01134	2.1

TABLE 8 Example 2. Sharp constant $c_{\mathcal{D}}$.

	t=0	t=0.5 TU	t=1 TU	t=2 TU
$\sqrt{E}$	0.0014	0.0014	0.0014	0.0014
$\sqrt{M}$	0.036	0.003	0.0024	0.0016
$\sqrt{M_p}$	0.0001	0.0015	0.0014	0.0015
$\sqrt{M_r}$	0.0356	0.0015	0.0011	0.00018
$\sqrt{M_{div}}$	1.1e-5	1.1e-5	1.1e-5	1.1e-5
I_{eff}	24	2.16	1.71	1.07
p_{eff}	0.63	0.90	0.98	0.98

TABLE 9 Example 2. Overestimated constant $c_{\mathcal{D}}$.

	t=0	t=0.5 TU	t=1 TU	t=2 TU
$\sqrt{E}$	0.0014	0.0014	0.0014	0.0014
$\sqrt{M}$	0.107	0.059	0.0041	0.00197
$\sqrt{M_p}$	0.0001	0.0015	0.0015	0.0015
$\sqrt{M_r}$	0.1068	0.0044	0.003	0.0005
$\sqrt{M_{div}}$	1.1e-5	1.1e-5	1.1e-5	1.1e-5
I_{eff}	76	4.2	3.14	1.41
p_{eff}	0.63	0.85	0.96	0.97

5 A POSTERIORI ERROR ESTIMATES FOR THE SEMI-DISCRETE APPROXIMATIONS FOR THE EVOLUTIONARY STOKES PROBLEM

5.1 Formulation of the problem

Let $\mathcal{D}$ be an open bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\mathcal{D}$ and $Q_T := \mathcal{D} \times (0, T)$ be a space-time cylinder, where $T > 0$ is a given number determining the upper boundary of the observation interval. The surface of the space-time cylinder is denoted $S_T := \partial\mathcal{D} \times [0, T)$.

The classical statement of the non-stationary Stokes problem is to find a continuous vector-valued function $u(x,t)$ (velocity) and a scalar-valued function $p(x,t)$ (pressure) in $\overline{Q_T}$ such that

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - \nu \triangle u(x,t) = f - \nabla p(x,t) && \text{in} Q_T, && (5.1.1a)\\
&\operatorname{div} u(x,t) = 0 && \text{in } Q_T, && (5.1.1b)\\
&u(x,t) = 0 && \text{on } S_T, && (5.1.1c)\\
&u(x,0) = \varphi(x) && \text{in } \mathcal{D}. && (5.1.1d)
\end{aligned}
$$

Here,the ν is a positive constant (viscosity), f presents volume forces, and φ is the initial velocity.

For any t, $0 < t \leq T$ and any separable Banach space X provided with the norm $\|\cdot\|_X$, we denote by $L_2(0, t; X)$ the space of measurable functions v

from (0, t) in X such that

$$\|v\|_{L_2(0,t;X)} = \left(\int_0^t \|\nabla v\|_X^2 \right)^{\frac{1}{2}} < \infty. \tag{5.1.2}$$

Introduce the space $H^m(0,t;X)$ of functions in $L_2(0,t;X)$ such that all their time derivatives of order m (positive integer) belong to $L_2(0,t;X)$. We also use the space of continuous functions v from $[0,t]$ in X ($C^0(0,t;X)$) and the space $\widetilde{L}_2(0;t;X)$ of functions in with zero mean value on $\mathcal{D}$.

Finally, let $\overset{o}{J}{}^{\infty}(Q_T,\mathbb{R}^n)$ be the space of continuous divergence-free functions on the Q_T vanishing on $\partial\mathcal{D}$ in the sense of traces. Let V be a closure of $\overset{o}{J}{}^{\infty}(Q_T,\mathbb{R}^n)$ with respect to the norm of the space $L_2(0,T,\overset{o}{H}{}^1(\mathcal{D},\mathbb{R}^n))$ and $\overset{o}{J}{}^{1}_{2}(Q_T,\mathbb{R}^n)$ be a closure of $\overset{o}{J}{}^{\infty}(Q_T,\mathbb{R}^n)$ with respect to the norm of the space $\overset{o}{H}{}^1(Q_T,\mathbb{R}^n)$.

Generalized solution of the problem (5.1.1) is a function $u(x,t) \in V \cap C^0(0,t;L_2(\mathcal{D},\mathbb{R}^n))$ such that

$$u(x,0) = \varphi \quad \text{a.e. in } \mathcal{D}, \tag{5.1.3}$$

and

$$\begin{aligned} &\int_{Q_T} \nu\nabla u : \nabla w + \int_{\mathcal{D}} (u(x,T)\cdot w(x,T) - u(x,0)\cdot w(x,0)) - \\ &\qquad - \int_{Q_T} u\cdot\frac{\partial w}{\partial t} = \int_{Q_T} f\cdot w \quad \forall w \in \overset{o}{J}{}^{1}_{2}(Q_T,\mathbb{R}^n). \end{aligned} \tag{5.1.4}$$

On $L_2(0,t;\overset{o}{H}{}^1(\mathcal{D},\mathbb{R}^n)) \cap C^0(0,t;L_2(\mathcal{D},\mathbb{R}^n))$, we introduce the following norm

$$[v](t) = \left(\|v(\cdot,t)\|_{\mathcal{D}}^2 + \nu\int_0^t \|\nabla v\|_{\mathcal{D}} \right)^{\frac{1}{2}}. \tag{5.1.5}$$

It is known (see, e.g., [34], [46], [47]), that under proper assumptions on the integrability of the right-hand side and initial conditions, the generalized solution u exists and unique.

Our goal is to find computable upper bounds of the error that we evaluate in terms of the quantity

$$[w]^2_{(\gamma^*,\delta^*)} := \gamma^*\,\|w(\cdot,T))\|_{\mathcal{D}}^2 + \delta^*\nu\,\|\nabla w\|_{Q_T}^2, \tag{5.1.6}$$

where γ^* and δ^* are some positive numbers and function w satisfies the zero boundary conditions on $\partial\mathcal{D}$ in the sense of traces.

Let v be some approximate solution, satisfying the prescribed boundary condition. In the following sections we will obtain the upper bound of the difference $u - v$ in the norm (5.1.6).

5.2 Functional type a posteriori error estimates

Functional type a posteriori error estimate for the semi-discrete solenoidal and non solenoidal approximations of the evolutionary Stokes problem were achieved in the paper **PVI**. In the paper, we have implemented to the Evolutionary Stokes problem the ideas from the functional estimate of the parabolic equation (see [48] and [49]). Below, we state the main result of the paper.

The main results can be formulated in the following theorems:

Theorem 5.2.1 (Estimate for divergence-free approximations). *For any positive β, $v \in \overset{o}{H}{}^1(Q_T, \mathbb{R}^n)$, $q \in L_2(0,T;H^1(\mathcal{D}))$, $\boldsymbol{\tau} \in L_2(0,T;\Sigma_{\mathrm{div}}(\mathcal{D}))$ the following estimate holds:*

$$[u - v]^2_{(\gamma^*,\delta^*)} \leq \mathcal{M}^{\oplus}_{sol}(v, \beta(t), \boldsymbol{\tau}, q) = M_i + M_p + M_r \tag{5.2.1}$$

where $\delta_* = (2 - \delta)$, $\gamma_* = 1$ and M_i, M_r, M_p are defined by the following relations:

$$M_i := \|\varphi - v(0)\|^2_{\mathcal{D}} \tag{5.2.2a}$$

$$M_p := \int\limits_0^T \frac{1}{\nu\delta}(1 + \beta(t)) \, \|\boldsymbol{\tau} - \nu\nabla v\|^2_{\mathcal{D}} \tag{5.2.2b}$$

$$M_r := \int\limits_0^T \frac{1}{\nu\delta}\left(1 + \frac{1}{\beta(t)}\right) c^2_{\mathcal{D}} \left\|\mathrm{Div}\,\boldsymbol{\tau} + f - \nabla q - \frac{\partial v}{\partial t}\right\|^2_{\mathcal{D}}. \tag{5.2.2c}$$

Here $c_{\mathcal{D}}$ is a constant from the inequality

$$\|w\|_{\mathcal{D}} \leq c_{\mathcal{D}} \, \|\nabla w\|_{\mathcal{D}} \quad \forall w \in \overset{o}{H}{}^1(\mathcal{D}). \tag{5.2.3}$$

Theorem 5.2.2 (Estimate for non divergence-free approximations). *For any positive β, $\tilde{v} \in \overset{o}{J}{}^1_2(Q_T, \mathbb{R}^n)$, $q \in L_2(0,T;H^1(\mathcal{D}))$, $\boldsymbol{\tau} \in L_2(0,T;\Sigma_{\mathrm{div}}(\mathcal{D}))$ then* *following estimate holds:*

$$
\begin{aligned}
[u-v]^2_{(\gamma^*,\delta^*)} \leq \mathcal{M}^{\oplus}(\tilde{v},\beta(t),\boldsymbol{\tau},q) = (1+\gamma)\left((1+\gamma_1)M_i + \right.\\
\left. + (1+\gamma_2)M_p + (1+\gamma_3)M_r\right) + \\
+(1+\gamma)\left((1+\frac{1}{\gamma_1})M^0_{div} + (1+\frac{1}{\gamma_2})M^{Q_T2}_{div} + (1+\frac{1}{\gamma_3})M^e_{div}\right) \\
+(1+\frac{1}{\gamma})\left(M^{QT1}_{div} + M^T_{div}\right),
\end{aligned}
\tag{5.2.4}
$$

where $\delta_* = (2-\delta)$, $\gamma_* = 1$, $\gamma, \gamma_1, \gamma_2, \gamma_3 \geq 0$ *and* M_i, M_r, M_p, M^0_{div}, M^T_{div}, $M^{Q_T1}_{div}$, $M^{Q_T2}_{div}$, M^e_{div} *are defined by the following relations:*

$$M_i := \|\varphi - v(0)\|^2_{\mathcal{D}}, \tag{5.2.5a}$$

$$M_p := \int_0^T \frac{1}{\nu\delta}(1+\beta(t))\,\|\boldsymbol{\tau} - \nu\nabla v\|^2_{\mathcal{D}}, \tag{5.2.5b}$$

$$M_r := \int_0^T \frac{1}{\nu\delta}\left(1+\frac{1}{\beta(t)}\right)c^2_{\mathcal{D}}\left\|\mathrm{Div}\,\boldsymbol{\tau} + f - \nabla q - \frac{\partial v}{\partial t}\right\|^2_{\mathcal{D}}, \tag{5.2.5c}$$

$$M^0_{div} = c^2_{\mathcal{D}}\,\|\mathrm{div}\,\tilde{v}(x,0)\|^2_{\mathcal{D}} \tag{5.2.5d}$$

$$M^T_{div} = \frac{c^2_{\mathcal{D}}}{C^2_{LBB}}\,\|\mathrm{div}\,\tilde{v}(x,T)\|^2_{\mathcal{D}}, \tag{5.2.5e}$$

$$M^{Q_T1}_{div} = (2-\delta)\frac{\nu^2}{C^2_{LBB}}\,\|\mathrm{div}\,\tilde{v}\|^2_{Q_T}, \tag{5.2.5f}$$

$$M^{Q_T2}_{div} = \int_0^T \frac{1}{\nu\delta}(1+\beta(t))\frac{\nu^2}{C^2_{LBB}}\,\|\mathrm{div}\,\tilde{v}\|^2_{\mathcal{D}}, \tag{5.2.5g}$$

$$M^e_{div} = \frac{1}{\nu\delta}\sum_{k=1}^{N}\left(1+\frac{1}{\beta(t_{k+\frac{1}{2}})}\right)\frac{1}{\varrho_k}c^2_{\mathcal{D}}\,\|\mathrm{div}\,(\tilde{v}_{k+1}-\tilde{v}_k)\|^2_{\mathcal{D}}. \tag{5.2.5h}$$

Here C_{LBB} *is a constant from Ladyzhenskaya-Babuška-Brezzi inequality*

5.3 Chapter 5: concluding remarks

In the Chapter 5 we have presented new a posteriori error estimates for the non-stationary Stokes problem. The error is measured in a special norm that is natural for the non stationary problems. Varying the parameter δ it is possible to consider different norms, taking into account the error related to the whole period of time $(0, T)$, or emphasizing the error related to the very last time step. In the case of divergence-free approximation the error estimate obtained is similar to the estimate for the linear parabolic problem (see [49]). For of stationary flows, it can be transformed to a known estimate for the Stokes problem (see [1]). In addition to the "primary term" (M_p) and the "reliability term" (M_r) (that is similar to the stationary Stokes problem) the error majorant for the non-stationary problem contains also the "initial term" (M_i), which penalizes for possible violations of the initial conditions.

The error majorant for non divergence-free approximations $\mathcal{M}^{\oplus}$ has the same principal structure, as the error majorant for the divergence-free approximations $\mathcal{M}^{\oplus}_{sol}$. In addition to the M_i, M_r, M_p, it contains the terms, that penalize for possible violation of the solenoidality condition. In the case of $\tilde{v}$ being a divergence-free approximation, it can be easily transformed to the $\mathcal{M}^{\oplus}_{sol}$, by putting all γ to zero.

Both of these estimates can be considered as new variational formulations of the non-stationary Stokes problem. Indeed, the functionals $\mathcal{M}^{\oplus}_{sol}$and $\mathcal{M}^{\oplus}$attain their minimal value (zero) only on the solution of the problem. It is also possible to construct numerical methods based on the minimization of the functionals $\mathcal{M}^{\oplus}_{sol}$and $\mathcal{M}^{\oplus}$.

The estimates proposed are valid for the general class of approximation and do not depend on the method of solution used. We considered piecewise linear with respect to time non-solenoidal approximations. The same approach can be applied for any other semi-discrete approximations, e.g., quadratic or cubic with respect to time.

6 A POSTERIORI ERROR ESTIMATION FOR THE PROBLEM WITH ROTATION

6.1 Formulation of the problem

6.1.1 Rotating Reference Frame

Use of intermediate reference frame, with a coordinate system fixed in that frame, is often advantageous in the analysis of particle and rigid body dynamics. In the analysis of rotating fluid based on Eulerian formulation of mechanics, the reference frame is more then just advantageous – it is often essential for a successful analysis. Let us consider the rotating coordinate system with angular velocity Ω. Let $\mathcal{D} = \mathcal{D}_h \times \mathcal{D}_z \in \mathbb{R}^n$ be a bounded domain with Lipschitz continious boundary $\partial\mathcal{D}$. Navier-Stokes equation written for the use of reference frame becomes

$$\partial_t \widetilde{u} + \widetilde{u} \cdot \nabla \widetilde{u} - \nu \triangle \widetilde{u} + 2\widetilde{\Omega} \times \widetilde{u} + \widetilde{\Omega} \times \widetilde{\Omega} \times r = \widetilde{f} - \nabla \widetilde{p} \quad \text{in } \mathcal{D}, \qquad (6.1.1)$$

$$\operatorname{div} \widetilde{u} = 0 \quad \text{in } \mathcal{D}. \qquad (6.1.2)$$

The system (6.3.1-6.3.2) includes two extra forces: the Coriolis force $2\widetilde{\Omega} \times \widetilde{u}$ and centrifugal force $\widetilde{\Omega} \times \widetilde{\Omega} \times r$. The centripetal and Coriolis term quantify the major phenomena that distinguish fluid with large-scale rotations. In the rotating Eulerian coordinate system, the centripetal term is computed using the angular velocity of the rotating coordinate system (Ω) and fixed locations (r) in the rotating Eulerian grid. It can be represented by a radial pressure distribution imposed on that region, independent of fluid motion relative to the rotating grid, and computed as the gradient of a scalar function. Because of this, the term can be included in the pressure term and usually disappears from rotating fluid computations (see, e.g., [50]). The value of the Coriolis term depends on the value of the velocity of the fluid relative to the rotating grid. This term never disappears unless $\widetilde{\Omega}$ or $\widetilde{u}$ are zero or parallel.

The dimensionless equations have the following form:

$$\partial_t u + u \cdot \nabla u - \nu \triangle u + \frac{1}{\varepsilon} B \times u = f - \frac{1}{\varepsilon} \nabla p \quad \text{in } \mathcal{D}, \tag{6.1.3}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{D}, \tag{6.1.4}$$

where $B = b e_z$ is the adimensionalized rotation vector. Denote by U and L the characteristic velocity and length scales of the motion and by ϕ the vertical component of the Earth rotation. The parameter

$$\varepsilon = \frac{U}{2\phi L}$$

is known as the Rossby number. In (6.1.3), we have also re-scaled the pressure field by the factor $\frac{1}{\varepsilon} p$ in order to cancel the Coriolis acceleration caused by the horizontal part of the pressure gradient (see, e.g., [51])

Problems with rotations arises in various applications. In particular, they are motivated by the geophysical flow problem related to large-scale water basin (e.g. oceans). Equations (6.1.3-6.1.4) present an evolutionary model describing the atmosphere or oceans at mid-latitude that have been studied by a number of authors (see [52], [53], [54], [55], [56]).

6.1.2 Model Problem

In this work, we derive an posteriori error estimate for the following simplified system:

$$-\nu \triangle u + \Omega \times u = f - \nabla p \quad \text{in } \mathcal{D}, \tag{6.1.5}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{D}, \tag{6.1.6}$$

$$u = u_g \quad \text{on } \partial\mathcal{D}. \tag{6.1.7}$$

Here $\nu > 0$ is the viscosity parameter, $f \in L_2(\mathcal{D}, \mathbb{R}^n)$ is a given vector-valued function, p is the pressure function and $u_g \in H^1(\mathcal{D}, \mathbb{R}^n)$ defines the Dirichlet boundary conditions on $\partial\mathcal{D}$. It is assumed that $\operatorname{div} u_g = 0$.

We recall that $\overset{0}{J}{}_2^1(\mathcal{D}, \mathbb{R}^n)$ denotes the closure of smooth solenoidal functions with compact supports in $\mathcal{D}$ with respect to $H^1(\mathcal{D}, \mathbb{R}^n)$ norm.

Generalized solution of the problem (6.1.5)-(6.1.7) is defined by the integral identity

$$\int_{\mathcal{D}} \nu \nabla u : \nabla w + (\Omega \times u) \cdot w = \int_{\mathcal{D}} f \cdot w \quad \forall w \in \overset{o}{J}{}^1_2(\mathcal{D}, \mathbb{R}^n). \tag{6.1.8}$$

Besides, we can define generalized solution as a function from a wider space, $\overset{o}{H}{}^1(\mathcal{D}, \mathbb{R}^n) + u_g$ that satisfies the integral relations:

$$\nu \int_{\mathcal{D}} \nabla u : \nabla w + (\Omega \times u) \cdot w = \int_{\mathcal{D}} (f - \nabla p) \cdot w \quad \forall w \in \overset{o}{H}{}^1(\mathcal{D}, \mathbb{R}^n), \tag{6.1.9a}$$

$$-\int_{\mathcal{D}} q \operatorname{div} u = 0 \quad \forall q \in \widetilde{L}_2(\mathcal{D}, \mathbb{R}^n) = \{q \in L_2(\mathcal{D}) | \int_{\mathcal{D}} q = 0\}. \tag{6.1.9b}$$

In the articles **PIII**, **PIV**, **PV** we deduced the functional type a posteriori error estimates for approximate solution of the system (6.1.5)-(6.1.7). Bellow we discuss them and present a generalized form of the estimate that can be used for the case of different vertical and horizontical viscosity.

6.2 Functional type a posteriori error estimate

Functional type a posteriori error estimates for the problem with rotation were first obtained theoretically and investigated numerically in [57] (**PIII** included in the thesis).

The main results are formulated in the following theorems:

Theorem 6.2.1 (Estimate for divergence-free approximations). *For any* $v \in \overset{o}{J}{}^1_2(\mathcal{D}, \mathbb{R}^n) + u_g$, $\tau \in \Sigma_{\operatorname{div}}(\mathcal{D})$, $q \in H^1(\mathcal{D})$ *the following estimate holds:*

$$\|\nu \nabla (u - v)\| \leq \|\tau - \nu \nabla v\| + c_{\mathcal{D}} \|f + \operatorname{div} \tau - \Omega \times v - \nabla q\|. \tag{6.2.1}$$

As for the Stokes problem, it is possible to transform (6.2.2) into a form with lower requirements on q, namely $q \in \widetilde{L}_2(\mathcal{D})$:

$$\nu \|\nabla (u - v)\| \leq \|\eta - \nu \nabla v + q\mathbb{I}\| + c_{\mathcal{D}} \|f + \operatorname{div} \eta - \Omega \times v\|, \tag{6.2.2}$$

where $\eta \in \Sigma_{\operatorname{div}}(\mathcal{D})$.

Theorem 6.2.2 (Estimate for non divergence-free approximations). *For any* $v \in \overset{o}{H}{}^1(\mathcal{D}, \mathbb{R}^n) + u_g$, $\eta \in \Sigma_{\text{div}}(\mathcal{D})$, $q \in \widetilde{L}_2(\mathcal{D})$ *the following estimate holds:*

$$\nu\|\nabla(u - \tilde{v})\| \leq c_{\mathcal{D}}\|f + \operatorname{div}\eta - \Omega \times \tilde{v}\| + \\ + \|\eta - \nu\nabla\tilde{v} + q\mathbb{I}\| + C(\mathcal{D}, \nu, \Omega)\|\operatorname{div}\tilde{v}\|, \quad (6.2.3)$$

where $\mathbb{I}$ *is the identity tensor, and*

$$C(\mathcal{D}, \nu, \Omega) = \frac{(2\nu + |\Omega| c_{\mathcal{D}})}{C_{LBB}}. \quad (6.2.4)$$

Theorem 6.2.3 (Estimate for the pressure). *For any* $\tilde{v} \in \overset{o}{H}{}^1(\mathcal{D}, \mathbb{R}^n) + u_g$, $q \in \widetilde{L}_2(\mathcal{D})$ *and* $\eta \in \Sigma_{\text{div}}(\mathcal{D})$ *the following estimate holds:*

$$\|p - q\| \leq \frac{2}{C_{LBB}}\|\nu\nabla\tilde{v} - \eta + q\mathbb{I}\| + \frac{2}{C_{LBB}} c_{\mathcal{D}}\|\operatorname{div}\eta + f - \Omega \times \tilde{v})\| + \\ + \frac{(2 + \frac{1}{\nu}|\Omega| c_{\mathcal{D}})}{C_{LBB}^2}\|\operatorname{div}\tilde{v}\|. \quad (6.2.5)$$

6.3 Functional type a posteriori error estimate for the rotation system with different vertical and horizontal viscosities

Models with different vertical and horizontal viscosity

$$-\nu_H \triangle_H u - \nu_V \partial_z^2 u + \Omega \times u = f - \nabla p \quad \text{in } \mathcal{D}, \quad (6.3.1)$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{D} \quad (6.3.2)$$

are considered, e.g., in [55]. Here ν_H and ν_V are horizontal and vertical kinematic viscosities and $\triangle_H$ is the horizontally components of the Laplacian.

To unify estimates presented to the system (6.3.1)-(6.3.2) we introduce the energy norm

$$|||u - v||| = ||\Lambda(u - v)||, \quad (6.3.3)$$

where Λu can be either ∇u or $\varepsilon(u)$ (deviatoric part of the stress). We note that these norms are equivalent due to incompressibility condition.

Let us introduce tensor function of 4th order $\mathbb{A}_{kmst}$, unit operator in the above space we will denote by $\mathcal{I}$ ($\mathcal{I}\tau = \tau$). Then the rheological low of the generalized system with different horizontal and vertical viscosity is as follows:

$$\sigma = \mathbb{A}\Lambda u + p\mathbb{I}. \tag{6.3.4}$$

In the standard case, $\mathbb{A} = \nu\mathcal{I}$. Note, the the viscosity can also depend on the coordinate. In case of different viscosities $\mathbb{A}$ can written be in the way

$$\mathbb{A}_{1111} = \mathbb{A}_{2222} = \nu_H, \quad \mathbb{A}_{3333} = \nu_V, \tag{6.3.5}$$

$$\mathbb{A}_{1212}\mathbb{A}_{2121} = \nu_H, \quad \mathbb{A}_{1313} = \mathbb{A}_{2323} = \frac{1}{2}(\nu_H + \nu_V) \tag{6.3.6}$$

while all the other coefficient is equal to zero.

Thus, we consider generalized system

$$-\operatorname{div} \mathbb{A}\Lambda u + \Omega \times u = f - \nabla p \quad \text{in } \mathcal{D}, \tag{6.3.7}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{D}, \tag{6.3.8}$$

$$u = u_0 \quad \text{on } \partial\mathcal{D}. \tag{6.3.9}$$

We introduce the energy norm for the system (6.3.7)-(6.3.9) as follows:

$$|||u|||_*^2 := \int_{\mathcal{D}} \mathbb{A}\Lambda u : \Lambda u. \tag{6.3.10}$$

The error estimate of the energy norm of the system (6.3.7)-(6.3.9) can be formulated in the following theorem.

Theorem 6.3.1. *For any* $v \in \overset{o}{J}{}^1_2(\mathcal{D}, \mathbb{R}^n) + u_g$, $\beta > 0$, $\tau \in \Sigma_{\text{div}}(\mathcal{D})$, $q \in H^1(\mathcal{D})$ *the following estimate holds:*

$$\begin{aligned} |||u - v|||_* \leq (1+\beta) \int_{\mathcal{D}} (\mathbb{A}\Lambda(u-v) - \boldsymbol{\tau}) : (\Lambda v - \mathbb{A}^{-1}\boldsymbol{\tau}) + \\ +(1+\frac{1}{\beta})\frac{c_{\mathcal{D}}^2}{\min\{\nu_H, \nu_V\}} \int_{\mathcal{D}} |\operatorname{Div}\boldsymbol{\tau} + f + \Omega \times v - \nabla q|^2 . \end{aligned} \tag{6.3.11}$$

PROOF. *Similar to the Proof of the estimate for the system (6.1.5)-(6.1.7) introduced in Section 3 publication* ***PIII****.*

6.4 Numerical examples

Consider an axisymmetric domain, rotating around the z-axis. Evidently, it is reasonable to use the Cylindrical coordinate system.

If the symmetry of the problem ($\frac{\partial \cdot}{\partial \phi} = 0$) is taken into account, then the basic operators are given by the relations:

Gradient of scalar function:

$$\nabla a = \begin{pmatrix} \frac{\partial a_r}{\partial r} \\ 0 \\ \frac{\partial a}{\partial z} \end{pmatrix}.$$

Divergence of vector function:

$$\nabla \cdot u = \frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{\partial u_z}{\partial z}.$$

Gradient of vector function:

$$\nabla u = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\phi}{\partial r} & \frac{\partial u_z}{\partial r} \\ -\frac{u_\phi}{r} & \frac{u_r}{r} & 0 \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\phi}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix}.$$

Divergence of tensor function:

$$\nabla \cdot A = \begin{pmatrix} \frac{\partial a_{rr}}{\partial r} + \frac{a_{rr}}{r} - \frac{a_{\phi\phi}}{r} + \frac{\partial a_{zr}}{\partial z} \\ \frac{\partial a_{r\phi}}{\partial r} + \frac{a_{r\phi}}{r} + \frac{a_{\phi r}}{r} + \frac{\partial a_{z\phi}}{\partial z} \\ \frac{\partial a_{rz}}{\partial r} + \frac{a_{rz}}{r} + \frac{\partial u_{zz}}{\partial z} \end{pmatrix}.$$

Laplacian:

$$\triangle u = \begin{pmatrix} \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} \\ \frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} + \frac{\partial^2 u_\phi}{\partial z^2} - \frac{u_\phi}{r^2} \\ \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_r}{\partial z^2} \end{pmatrix}.$$

The Rotating system in cylindrical coordinates has the form:

$$-\nu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_r}{\partial r}) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}) + \Omega\, u_\phi = f_r - \frac{\partial p}{\partial r}, \tag{6.4.1}$$

$$-\nu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_\phi}{\partial r}) + \frac{\partial^2 u_phi}{\partial z^2} - \frac{u_\phi}{r^2}) - \Omega\, u_r = f_\phi, \tag{6.4.2}$$

$$-\nu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) + \frac{\partial^2 u_z}{\partial z^2}) = f_z - \frac{\partial p}{\partial z}, \tag{6.4.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0. \tag{6.4.4}$$

The system (6.4.1)-(6.4.4) is solved by finite element method and the error is estimated by the majorant, which is also presented in the Cylindrical coordinate system.

6.4.1 Example 3

Let $\mathcal{D}$ be a cylinder with radius 1 and hight 2. The system is rotates around the axis z. The boundary conditions are assumed to be of the Dirichlet type.

We take problems whose exact solution are given by the relations:

$$
\begin{aligned}
u_r &= U_r + \gamma u_r^*, \\
u_\phi &= U_\phi + \gamma u_\phi^*, \\
u_z &= U_z + \gamma u_z^*.
\end{aligned}
$$

Here

$$
\begin{aligned}
U_r = &\ \ 0, \\
U_\phi = &\ \ \Omega\, r, \\
U_z = &\ \ R^2 - r^2
\end{aligned}
$$

are the component of the Poiseuille flow under the pressure fall, and

$$
\begin{aligned}
u_r^* = &\ \ \frac{\partial w}{\partial z}, \\
u_z^* = &\ \ -\frac{\partial w}{\partial r} - \frac{w}{r}, \\
w = &\ \ r^3\,(r-R)^3\,z^2\,(z-H)^2, \\
u_\phi^* = &\ \ r^2\,(r-R)^2\,z\,(z-H).
\end{aligned}
$$

The exact solution of the problem considered is $u = U + \gamma u^*$. Approximate solutions are computed by the Finite Element Method using quadratic approximation for the velocity and linear for the pressure (in the cylindrical coordinate system). After that we project the approximate solution to the space of divergence-free functions. Then, we use the error majorant. We find a tensor-valued functions τ and scalar-valued function q (using quadratic approximations for both of them) by a minimization of the right-hand side of (6.2.2). We compute guaranteed upper bound and make an error indication over the domain.

In the Tables 10-12, we collect the results of several iteration steps. M_p and M_r denote the "primary" and "reliability" terms of the error majorant, respectively.

From Tables 10 and 11 one can see that error majorant gives a guaranteed and sharp upper bound and provides a reliable error indication.

In order to test the behavior of the error majorant in case of inaccurate solutions we add an "oscillation" proportional to u^* with a parameter δ. Thus,

TABLE 10 Example 3. $\Omega = 1, \gamma = 2$

Iteration	1	2	3	4
N	312	405	608	973
$\sqrt{E}(\%)$	0.47	0.35	0.243	0.15
$\sqrt{E}$	0.25	0.019	0.0132	0.008
$\sqrt{M_p}$	0.029	0.019	0.013	0.008
$\sqrt{M_r}$	0.0009	0.0012	0.0009	0.00076
β	0.17	0.25	0.2	0.29
$\sqrt{M}$	0.0303	0.021	0.014	0.0095
I_{eff}	1.18	1.11	1.09	1.14
p_{eff}^{bulk}	0.96	0.96	0.94	0.97
p_{eff}^{max}	0.94	0.9	0.93	0.94

TABLE 11 Example 3. $\Omega = 100, \gamma = 2$

Iter	1	2	3	4
N	312	436	612	1100
$\sqrt{E}(\%)$	0.0089	0.0068	0.0048	0.0032
$\sqrt{E}$	0.0316	0.024	0.0171	0.0115
$\sqrt{M_p}$	0.034	0.0277	0.01873	0.0129
$\sqrt{M_r}$	0.0025	0.00086	0.00098	0.00097
β	0.275	0.17	0.229	0.27
$\sqrt{M}$	0.036	0.028	0.01972	0.0138
I_{eff}	1.16	1.18	1.15	1.2
p_{eff}^{bulk}	0.92	0.93	0.96	0.94
p_{eff}^{max}	0.90	0.90	0.95	0.92

v is a quadratic interpolant of $U + \gamma u^* + \delta u^*$ to be an approximate solution. Thus, in this case we do not solve the problem, but only estimate the error of the approximate solution. As for the rest, the algorithm is the same.

One can observe from the Tables 12 and 13, that for big values of the rotation parameter the quality of the error estimation is not so high. Possible explanation of this phenomena consists of that the majorant contains the term $\Omega \times v$. So that if $\Omega \gg 1$, then the error encompassed in v is multiplied by a large factor.

Error indication provided by the error majorant is presented on Figure 7, where the elements which need to be refined are depicted. The left-hand picture is related to the "true error" and the right-hand one to the error computed by the majorant. One can see, that the pictures is quite similar to each

TABLE 12 Example 3. $\Omega = 1.\ \gamma = 2.\ \delta = 2$

$Iter$	1	2	3	4
N	312	452	706	1052
$\sqrt{E}(\%)$	0.47	0.36	0.25	0.17
$\sqrt{E}$	0.025	0.019	0.013	0.009
$\sqrt{M}$	0.03	0.022	0.016	0.011
I_{eff}	1.2	1.16	1.21	1.18
p_{eff}^{bulk}	0.94	0.91	0.92	0.92
p_{eff}^{max}	0.89	0.92	0.88	0.90

TABLE 13 Example 3. $\Omega = 100.\ \gamma = 2.\ \delta = 2$

$Iter$	1	2	3	4
N	312	468	673	1140
$\sqrt{E}(\%)$	0.0089	0.0069	0.0005	0.0036
$\sqrt{E}$	0.031	0.024	0.017	0.129
$\sqrt{M}$	0.09	0.075	0.039	0.037
I_{eff}	2.85	3.1	2.84	2.92
p_{eff}^{bulk}	0.90	0.88	0.92	0.89
p_{eff}^{max}	0.81	0.81	0.84	0.84

adaptation step.

6.4.2 Example 4

Consider the flow of viscous incompressible fluid in the container show on Figure 8 which is rotated around i_z axis.

In this example the "true solution" (which is unknown) is replaced by the so-called "reference solution" obtained on the very fine mesh.

In Tables 14 and 16, we presented result for different parameters Ω, R_{top} and R_{bottom}. In all the cases, error the error majorant provides a realistic upper bound. Decreasing of the quality of the error majorant in the case of big rotation parameter, observed in Tables 14 and 16, is naturally to await.

Figure 10 depict a sequence of adapted meshes. In such type of examples, refinement of the mesh near reentrant corners is always observed. This phenomenon is clearly seen on Figure 10.

On Figure 9 the optimal error indication and the error indication computed by the error majorant are depicted (for the very last mesh). It can be

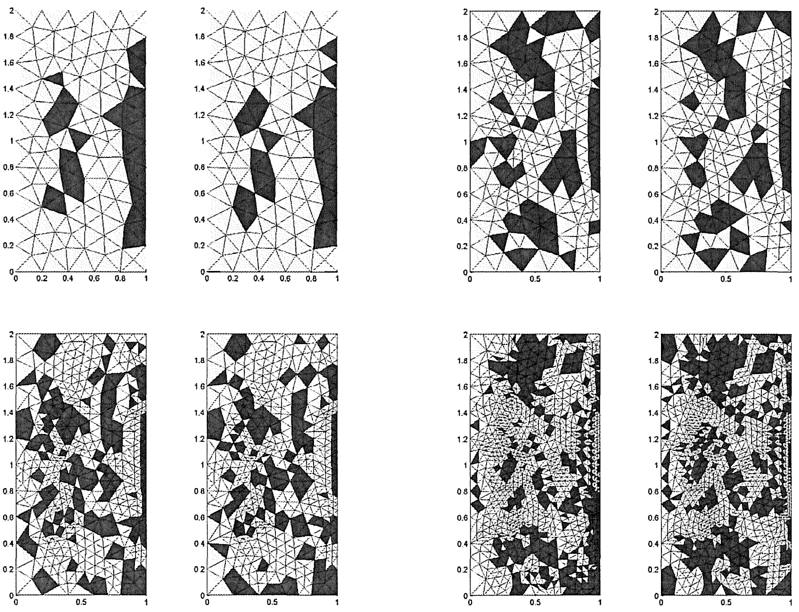

FIGURE 7 Example 3. Some adaptation steps. Etalon error indication (left) and error majorant indication (right).

TABLE 14 Example 4. $\Omega = 1$, $R_{top} = 0.8$, $R_{bottom} = 0.8$

$Iter$	1	2	3	4
N	425	554	828	1326
$\sqrt{E}(\%)$	16.3	12.21	8.42	5.32
$\sqrt{E}$	1.66	1.23	0.85	0.54
$\sqrt{M_p}$	1.806	1.29	0.89	0.567
$\sqrt{M_r}$	0.0068	0.005	0.0057	0.008
β	0.019	0.02	0.029	0.019
$\sqrt{M}$	1.82	1.30	0.0905	0.057
I_{eff}	1.09	1.05	1.09	1.14
p_{eff}^{bulk}	0.98	0.98	0.97	0.96
p_{eff}^{max}	0.96	0.95	0.94	0.92

observed, that the indicator based on the error majorant provides a reliable indication on unstructured meshes.

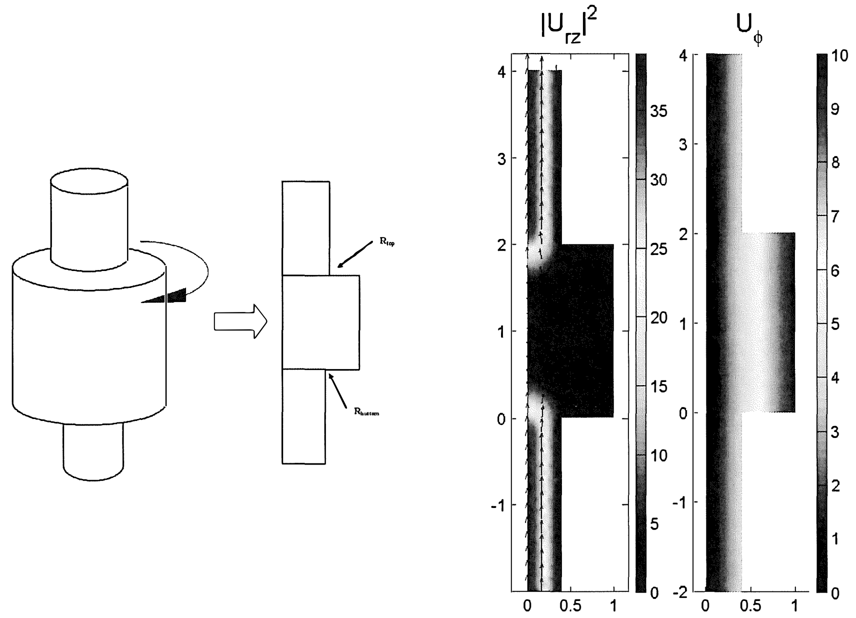

FIGURE 8 Example 4. Computational domain (left); Velocity of the fluid(right).

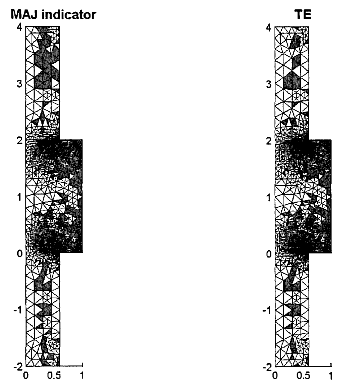

FIGURE 9 Example 4. Final mesh. $R_{top} = R_{bottom} = 0,6$.

6.5 Chapter 6: concluding remarks

New a posteriori error estimates for the rotating system were derived and numerically tested. The results demonstrate that the error majorant is efficient in

TABLE 15 Example 4. $\Omega = 1$, $R_{top} = 0.6$, $R_{bottom} = 0.4$

Iter	1	2	3
N	358	465	698
$\sqrt{E}(\%)$	9.34	6.96	4.8
$\sqrt{E}$	2.43	1.81	1.25
$\sqrt{M}$	3.7	2.76	2.05
I_{eff}	1.51	1.52	1.6
p_{eff}^{bulk}	0.96	0.94	0.93
p_{eff}^{max}	0.82	0.8	0.78

TABLE 16 Example 4. $\Omega = 100$, $R_{top} = 0.8$, $R_{bottom} = 0.8$

Iter	1	2	3	4
N	425	637	956	1147
$\sqrt{E}(\%)$	0.38	0.28	0.19	0.126
$\sqrt{E}$	1.93	1.44	0.997	0.63
$\sqrt{M}$	2.63	1.87	1.18	0.86
I_{eff}	1.35	1.29	1.19	1.36
p_{eff}^{bulk}	0.97	0.95	0.96	0.92
p_{eff}^{max}	0.87	0.86	0.85	0.84

the estimation of the overall accuracy of an approximate solution as well as in the error indication. A particular form of the majorant associated to cylindrical coordinate system was observed and studied. Numerical examples demonstrate the robustness of the estimates in all the cases considered. In the case of a big rotation parameter, the quality of the error estimation is not so high (I_{eff} is around 3), but quite sufficient for engineering purposes.

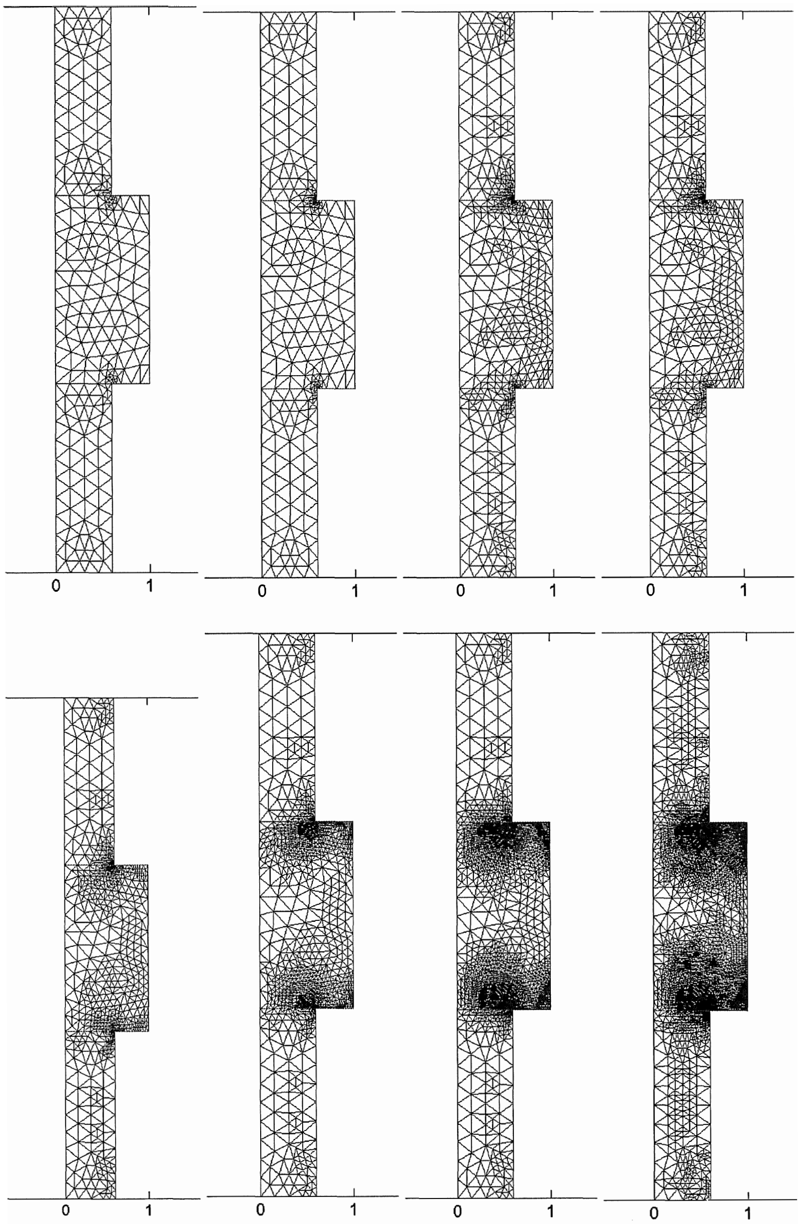

FIGURE 10 Example 4. Adaptive meshes. $R_{top} = R_{bottom} = 0,6$

7 SUMMARY AND CONCLUSIONS

The aim of this thesis was to present and numerically check functional type a posteriori error estimates for the different linearizations of the Navier-Stokes equation. The major attention was paid to the Stokes problem, the evolutionary Stokes problem and a systems with rotation.

The analysis is based on the so-called functional a posteriori estimates that are valid for any conforming approximation of a boundary value problem. In addition, error majorant gives a new variational functional related to a problem whose minimum is equal to zero and achieved on the exact solution.

For the system with rotation and semi-discrete approximations of the evolutionary Stokes problem such type of estimates were derived in the thesis.

For the Stokes problem and a system with rotation, the estimates proposed were tested in application to different numerical strategies and problems in Cartesian and Cylindrical coordinate systems. Minimization of the error majorants was performed with the help of the relaxation method and a direct solver implemented in MATLAB.

The influence of the overestimation of the constant in the error majorant was also studied. It was found, that an overestimation of this constant does not crucially worsen the estimate, but may nevertheless increase the computational time, necessary to achieve a good accuracy. It was demonstrated, that the best error estimate can be archived by a direct minimization of the error majorant.

A special attention was paid on the quality of the error indication provided by the majorants for approximations of various types. It was shown that element-wise quantities contained in the majorant can be used to create efficient and robust indicators.

Local a posteriori error estimates were also tested. They allow to estimate the error related to a certain subdomain of interest. For this task, we suggest a new computational technology based on the so-called "billet function. In the respective numerical examples, the efficiency index around 2 was achieved.

The main results of this thesis are as follows:

- A multi-side study of practically efficiency of error estimation technologies based on functional a posteriori estimates was performed.
- A new method of local error estimation was developed and tested.
- New functional error majorants were derived for problems with rotation and for semi-discrete approximations of the parabolic Stokes problem.

YHTEENVETO (FINNISH SUMMARY)

Tämä väitöskirja keskittyy sellaisen modernin laskennan metodologian kehitykseen ja numeeriseen todistukseen, jonka avulla voidaan taata yläraja virheen energianormille. Ehdotettu metodologia perustuu niinsanottuun funktionaalisen tyyppiseen a posteriooriin virheen estimointiin. Työssä on käytetty useampaa Navier-Stokes-yhtälön linearisaatiota. Ensimmäinen käytetty linearisaatio on Stokes-ongelma, toinen ajasta riippuva Stokes ongelma ja kolmas rotaatio Stokes-ongelma. Virheen estimointi ajasta riippuvalle Stokes- ongelmalle ja rotaatio Stokes-ongelmalle on esitetty työssä ensimmäistä kertaa.

Stokes-ongelmalle ja rotaatio Stokes-ongelmalle on työssä implementoitu ja testattu useita erilaisia numeerisia algoritmeja. Numeeriset testit on suoritettu karteesisessa ja sylinterimäisessä koordinaatiosysteemissä. Työssä esitetään, että funktionaalisen tyyppisen a posterioorisen virheen estimointimetodin avulla kyetään aina etsimään luotettava yläraja virheen suuruudelle ja tarkka virheen esiintymispaikka.

Ehdotettu virheen estimoinnin lähestymistapa mahdollistaa tehokkaiden adaptiivisten mesh-tyyppisten algoritmien muodostamisen ja takaa luotettavan tarkkuuden työssä esitetyille tehtäville.

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