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ORIGINAL PAPER

Learning research in a laboratory classroom: a reflection on complementarity and commensurability among multiple analytical accounts

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Abstract
With the myriad theories generated through research over the years, a continuing challenge for researchers is to navigate the multitude of theories in order to communicate their research, integrate empirical results, and make progress as a field by building upon empirical research. The Social Unit of Learning project was purposefully designed so that researchers from multiple disciplines with different theoretical perspectives could work together to examine the complexity of the mathematics classroom. In this paper, we reflect on the multiple analytical accounts generated from the project, drawing from the notions of complementarity and commensurability. Two parallel analyses, applying the commognitive framework and the theory of representations respectively, are used as illustrative examples for discussion regarding complementarity and commensurability. The paper addresses two focal questions, as follows: in what ways do divergence or contradiction in incommensurable analytical accounts reflect methodological discrepancies or fundamental differences in the underpinning theories? Furthermore, in what ways do the accounts generated by the parallel analyses predicated on different theories lead to differences in instructional advocacy? The answers to these questions provide empirically-grounded insights into the consideration of incommensurability in educational research, and suggest ways in which researchers and practitioners might apply the notion of complementarity to reconcile or exploit incommensurable analytical accounts that have resulted in different instructional advocacies.

Keywords Collaborative problem solving · Year 7 students · Video research · Multi-theoretic research design · Complementarity and commensurability · Instructional advocacies

1 Introduction

Theory can be thought of as "a coherent system of logically consistent and interconnected ideas used to condense and organise knowledge" (Neuman, 2014, p. 9). In this sense, a theory can be considered as "an organised system of accepted knowledge" or worldview (Mason & Waywood, 1996, p. 1055), or a discourse, that is, a special type of communication and a form of practice bound by set rules for communication and thinking, in which potentially useful stories about the world are being told (Sfard, 2008, 2021). In educational research, theory is integral to engaging with and explaining educational phenomena, and recent reviews of mathematics education research suggest that the range of theories used in mathematics continues to expand (Inglis & Foster, 2018; Lerman, 2006).

The proliferation of theories could have arisen from researchers' need to generate new theories to create identities and ascertain the novelty of their work (Lerman, 2006). This proliferation can also be explained by the move from quantitative to qualitative methods from the mid-1980s, and the need to ground qualitative methods theoretically in order to support interpretations and justify findings (Niss, 2019). Another perspective suggests that this proliferation results from research developed in different regions of the world based on different traditions, values, and practices

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(Prediger et al., 2008). Moreover, as educational phenomena are complex and multi-faceted, a single theory cannot be comprehensive. Researchers from different disciplines, such as psychology, sociology, and anthropology, all interested in the teaching and learning of mathematics, have introduced different theories from these disciplines for understanding phenomena in mathematics education.

In this paper, we aim to provide empirically-grounded insights into potential sources of incommensurability in educational research, and to suggest ways researchers and practitioners can apply the notion of complementarity to reconcile or exploit incommensurable analytical accounts that have resulted in different instructional advocacies. In this paper, we begin by outlining different ways the multiplicity of theories have been addressed in research, and then we turn to the challenges and importance of articulating theoretical choices.

2 Dealing with the multiplicity of theories in research

Educational researchers face the challenge of navigating the multitude of theories in order to communicate and integrate their findings, and to contribute to the field (e.g., Bikner-Ahsbahs & Prediger, 2014; Clarke et al., 2012). Conflating theories, however, undermines the integrity of different perspectives and overlooks the critical premises that give rise to different theorisations (Cobb, 2007). In practice, researchers have used various means to connect theoretical approaches. Prediger et al. (2008) conceptualised these different ways to connect theoretical approaches in terms of "degrees of integration", ranging from ignoring other theories, understanding others and making one's own theories understandable, comparing and contrasting, coordinating and combining, synthesising and integrating, all the way to establishing a unified global theory (pp. 170–173). The term "networking strategies" denotes the aim of reducing the number of unconnected theories while respecting the specificity of individual theories.

Avoiding theory integration is also favoured by Even and Schwarz (2003). Their comparative analyses of a mathematics lesson from Cognitivist Theory and Activity Theory perspectives demonstrates how different approaches suggest different interpretations of, and origins for, the identified learning difficulties. Their paper concludes with a cautionary note regarding any attempt to harmonise and integrate different theoretical approaches towards the development of a new radical theory which potentially undermines the complexity of different perspectives. In a multi-theoretic study of science classrooms, Clarke et al. (2012) also criticised "normative convergence", highlighting that, "In developing instructional advocacy arguments, it may be the identification of contingencies on any recommendations that offers greatest utility, by identifying combinations of context and action most likely to promote locally significant outcomes" (p. 6).

Understanding different theories and their associated instructional advocacy was an enduring interest for Clarke. His complementary accounts methodology (Clarke, 1997, 2001a) brought together the reflective voices and insights of students, teachers and researchers to facilitate multi-faceted analysis of a common dataset. Research projects that Clarke led, such as the Classroom Learning Project (Clarke, 2001b), the Learner's Perspective Study (Clarke et al., 2006), and the Causal Connections in Science Classrooms project (Clarke et al., 2012), all employed this methodology to elicit and compare different theories in researching the classroom. Central to this methodology is the notion of complementarity. Rather than aiming for consensus among the researchers, the aim of the methodology is to develop accounts that are internally coherent, consistent with the available data, and plausible. Clarke argued that such complementary accounts are important for providing a richer and more complex portrayal and understanding of classroom learning.

Complementarity implies the acceptance of possible truths in different theories and perspectives. Theories may be simultaneously true within their own coherent conceptual framework although they remain disjoint and distinct. Complementarity accords parity of status to different interpretations although they are "subject to the same criteria of coherence, consistency with the [available] data, and plausibility" (Clarke, 1997, p. 111). Instead of attempting to support or reject the premises of one theory over another, complementarity obliges researchers to consider the perspectives and associated constructs each theory foregrounds and its applicability to a situation or setting. Moreover, complementarity recognises the different ways students, teachers and researchers construct and conceptualise, for example, classroom environments.

Complementarity applies to the theories of growth developed by Piaget and Vygotsky (Bruner, 1997). Piaget's structuralist approach examined qualitative changes in human cognition over time and theorised the organisation and re-organisation of mental structures within the learning process (Piaget, 1960). Vygotsky examined the connection between language and thought, the influence a more able other plays in learning, and highlighted the cultural historical context of knowledge and learning (Vygotsky, 1978, 1986). Piaget's stage theory and Vygotsky's sociocultural theory of growth are complementary as each offer a coherent, but partial, understanding of human development.

Emphasising complementarity in research, however, can encourage diversity in perspectives and an absolute relativistic stance where "anything goes" (Cobb, 2007); moreover, the choice of theory can potentially become arbitrary or create silos with little incentive to consider alternative theorisations (Prediger et al., 2008). Pragmatic realism counters this stance by considering truths as fallible human productions subject to correction by individual researchers and the research community (Cobb, 2007). This realist stance acknowledges that choices in adopting a theoretical perspective reflect researchers' values and concerns. This stance also encourages the mathematics education research community to make explicit their rationale for a particular theoretical choice, and obliges researchers to have good reasons for and discussion around their choices (Clarke & Chan, 2019; Clarke et al., 2009, 2012; Cobb, 2007).

3 Challenges in articulating theoretical choices

Challenges associated with articulating theoretical choices include the criteria used to distinguish theories and the difficulty with juxtaposing theoretical applications across research settings. These challenges are exacerbated by ill-defined notions of what constitutes a theory (Mason & Waywood, 1996; Niss, 2006; Prediger et al., 2008). In this paper, we employ Neuman's (2014) broad definition of theory as "a coherent system of logically consistent and interconnected ideas used to condense and organise knowledge" (p. 9). Unlike Prediger et al. (2008), who considered theory to have conceptual, empirical, and application components, we use the term "analytical account" to refer to interpretations arising from the application of a theory in a particular setting (Chan & Clarke, 2017a; Clarke, 1997). This distinction allows us to examine the concepts involved in a theory, and the different ways concepts are applied and used for analytical purposes in different settings.

Commensurability and compatibility also describe relationships or connections between theories. The theoretical approaches of Piaget and Vygotsky, for example, are "complementary though incommensurate" (Bruner, 1997, p. 65), as their conceptualisations of growth were arguably "incompatible" (p. 65). Cobb's (2007) examination of four theorisations of the individual in relation to learning processes illustrates the incommensurability of different perspectives. In experimental psychology, for example, an individual is statistically constructed based on attributes of a group of people, in cognitive psychology epistemic individuals reorganise their own mental activity, in sociocultural theorisations an individual is a participant in cultural practices, and in distributed cognition an individual is an element of a larger reasoning system. Cobb (2007) considered these perspectives as incommensurable because each offers a different reality in the conceptualisation of an individual.

Although Cobb's (2007) analysis usefully distinguishes among theoretical perspectives, direct comparison of analyses employing different theories, without considering the contexts or settings in which the theories are applied and the intended purpose of their application, undermines the integrity of the comparison and legitimacy of the conclusions (Chan & Clarke, 2017a). Caduri and Heyd-Metzuyanim (2015) compared the cognitive theory of Piaget (1952) and the commognitive framework of Sfard (2008) and examined the incommensurability and incompatibility of these theories in addressing learning processes. Using Sfard's (2008) definition of theories as discourses bound by specific rules of communication, incommensurability is understood as the lack of a "common language into which two contending scientific languages could be fully translated" (p. 2). Incompatibility is understood as the presence of unresolvable contradictions or conflicts, such as disagreement between positivist and interpretivist paradigms regarding truth and reality, the relationship between the knower and the known, and the inquiry approach. Caduri and Heyd-Metzuyanim suggest that while two theories may be incompatible and incommensurable at the conceptual level, they may overlap at the local level in the ways some keywords or concepts are used. Identifying common components between theories may allow researchers with different theoretical perspectives to identify common goals (e.g., what 'change' in learning constitutes) and to collaborate. Chan and Clarke (2017a) similarly argued that the commensurability of analytical accounts is discernible in the "interpretative accounts arising from the application of the theories" (p. 2719): that is, while complementarity requires theoretical articulation, commensurability requires operationalised constructs to facilitate critical considerations of instructional advocacy.

Clarke responded to the theoretical and methodological challenges of complementarity and commensurability by combining complementary accounts methodology with multi-theoretic research in the Social Unit of Learning project (Chan et al., 2018). In this paper, we examine two analytical accounts generated in the project using the notions of complementarity (Clarke, 1997) and commensurability (Chan & Clarke, 2017a), based on the data collected in Australia.[1] In light of the divergence and convergence in analytical accounts that have emerged in this project so far, we have purposefully chosen two parallel analyses to illustrate and reflect on the issue of complementarity and commensurability between theories. The two analytical accounts are from the commognitive framework (Sfard, 2007, 2008) and the theory of representations (Duval, 2006, 2017). The focal questions are as follows:

1. In what ways do divergence or contradiction in incommensurable analytical accounts reflect methodological

[1] A separate paper submitted to this special issue addresses the cross-cultural aspect of the project between Australia and China.

discrepancies or fundamental differences in the underpinning theories?

2. Furthermore, in what ways do the accounts generated by the parallel analyses predicated on different theories lead to differences in instructional advocacy?

The answers to these questions provide empirically-grounded insights into potential sources of incommensurability in educational research, and may suggest ways in which researchers and practitioners might apply the notion of complementarity to reconcile or exploit incommensurable analytical accounts that have resulted in different instructional advocacies.

4 The Social Unit of Learning project

The Social Unit of Learning project had the aim of investigating the social aspects of learning acknowledging that 'the social' represents a fundamental and useful level of explanation, modelling and instructional intervention (Chan et al., 2018). The project used the Science of Learning Research Classroom (SLRC) at the University of Melbourne in Australia and equivalent facilities in China to examine individual, dyadic, small group (four to six students) and whole class problem solving in mathematics, and the associated/consequent learning. The SLRC facility captures classroom social interactions with a rich amount of detail using advanced video technology. With 10 built-in video cameras and up to 32 audio inputs, the comprehensive and detailed recording of the activity of every participant offers the possibility for systematic examination of the processes and products of learning activities within the classroom setting.

Since 2015, eleven classes of Year 7 students (264 students in total) were filmed in the laboratory classroom engaging in mathematical problem solving activities individually, in pairs, and in groups. The data collected in the project included all written material produced by the students, video footage of student talk and interactions, and instructional material used by the teacher. Although the teacher's role in the sessions was deliberately limited to providing instructions for the task activities rather than giving direct instruction or directions regarding task completion (see Chan & Clarke, 2017b), the dataset also included video footage of the teacher, tracked throughout the session, transcripts of teacher and student speech, and pre- and post-session teacher interviews. One class of students also wore biometric wristbands during a session to measure their physiological responses such as skin conductance and heart rate. This extensive dataset allowed the examination of data from multiple perspectives by multiple researchers, as well as the reciprocal interrogation of different theoretical perspectives.

The rich data generated from the SLRC facility allowed the Social Unit of Learning project to serve the purpose of theory generation and testing through the application of a multi-theoretic research design (Chan & Clarke, 2017a; Clarke et al., 2012), which can be considered an adaption of the complementary research methodology. The multi-theoretic research design involved the construction of a complex dataset composed of video records and other supplementary data, allowing the juxtaposition of interpretive accounts arising from different theoretically-grounded analyses in order to compare and contrast the capacity of different theories or conceptual frameworks to characterise different aspects of the research setting. An international multi-disciplinary research team (combining education, cognitive and emotive psychology, learning analytics, and neuroscience perspectives) was recruited to develop analytical frames for coding the data. So far, the data have been examined in terms of meaning negotiation between students (Chan & Clarke), shared cognition (Clarke & Chan), dialogic talk (Díez-Palomar), sophistication of mathematical exchange (Tran), repertoires of participation (Moate), peer feedback (Hošpesová & Novotná), use of multiple mathematical representations (Kuntze & Friesen), interactivity (Chan & Sfard), power and agency (Nieminen), interpersonal behaviours (Haataja), and motivation desires (Tuohilampi). The data from the biometric wristbands were used to examine physiological synchrony (Cunnington & Sherwell), while multimodal learning analytic techniques were used to operationalise behavioural engagement based on video and audio data generated in the project (Ochoa). Each analysis foregrounds different aspects of student social interaction and learning processes in the mathematics classroom.

4.1 Facilitating parallel analyses

In designing the Social Unit of Learning project, Clarke had particular research directions he wanted to pursue (e.g., Chan & Clarke, 2017b; Clarke & Chan, 2020), as well as being interested in the perspectives other researchers within and outside the mathematics community could bring to the project. Through conversations with other researchers, a research team was gradually built as each researcher identified a potential area of inquiry afforded by the project data in consultation with Clarke and Chan. Analytical approaches were selected on the basis that they could address constructs, artefacts or situations distinct from those addressed in other analyses being employed, and therefore complement other analyses.

In 2016, Anna Sfard visited Melbourne and collaborated with Chan, and published the results of a study applying the commognitive framework and interactivity analysis to a pair of students from the project (Chan & Sfard, 2020). Sebastian Kuntze and Marita Friesen joined the research team in

2017 and applied representation registers in analysing the data, and separately analysed student pair interactions in collaboration with Clarke and Chan (Kuntze et al., 2022). In the following, we briefly describe the theories underpinning the two analyses before introducing the respective analytical accounts.

4.1.1 The commognitive framework

The commognitive framework was developed to understand and investigate mathematical learning as a social process. The framework assumes that "thinking is a form of communication and that learning mathematics is tantamount to modifying and extending one's discourse" (Sfard, 2007, p. 567). Communication is seen as a collective endeavour which follows specific social and linguistic rules. People engage in communication, verbally or non-verbally, through symbolic systems in order to influence the actions or feelings of another person based on their own intentions. Discourse is defined as any specific instance of communicating, whether with oneself or with other people (Sfard, 2008; Sfard & Kieran, 2001).

Within this framework, the thinking-communication divide is resolved by equating thinking with self-communication. The framework adopts a strong *participationist* position (Sfard, 1998) focusing on learners' participation within a social context. It pays specific attention to situatedness, social interaction, social relationship, and history and culture. This perspective contrasts with an *acquisitionist* position focused on knowledge acquisition and the internal mental processes of individuals. According to the commognitive framework, mathematics is a form of discourse following particular rules for thinking and interpersonal communication. The process of mathematics learning takes place as part of an 'individualising' process in which a person not only acts according to the rules of the discourse, but exerts agency to decide how to use the discourse and proceed with it. Commognitive conflict is theorised as a source of mathematical learning when learners are exposed to a discourse that is different from their own and of which they cannot yet make sense (Cooper & Lavie, 2021), thereby requiring acceptance, customisation, and rationalisation of the divergent (incommensurable) discourses of others (Sfard, 2007). In order for the conflict to promote rather than hinder learning, a 'learning-teaching agreement' is needed. The agreement is an unspoken, often implicit, understanding that specifies the leading discourse (e.g., formal mathematics), the role of the interlocutors (e.g., with a teacher leading or a learner adapting discourses), and a realistic vision of how the discursive change will occur (Ben-Zvi & Sfard, 2007). The idea of a learning-teaching agreement accords with Vygotsky's (1978) notion of the zone of proximal development, the potential area of development that is initially realised with the assistance of a more expert other. The learning-teaching agreement essentially operates within this zone of proximal development for learners.

The commognitive framework is particularly powerful for conceptualising how mathematical learning occurs and several analytical tools have been developed for investigating learning, specifically in the classroom context, applying the framework. Interactivity analysis informed by the commognitive framework (Sfard & Kieran, 2001), for example, involves fine-grained analysis of the linguistic features of teacher-student or student–student communicative acts. The analysis enables quantitative as well as qualitative analysis through the development of participant profiles (as mathematicians and social interlocutors) and changes over time. The analysis can be used to examine the effectiveness in the communication (and therefore the thinking) process between multiple interlocutors (Sfard & Kieran, 2001).

4.1.2 Theory of representations

In contrast to the commognitive framework, Duval's (2017) theory of semiotic representations conceptualises mathematical thinking and learning in terms of representation of mathematical objects by learners. Representation is considered the core of mathematical processing which can be performed only using a semiotic system of representation, as one semiotic representation is substituted for another. As a knowledge domain, mathematics relies on a large range of semiotic representation systems, some common to natural language (e.g., 'average', 'equal', 'larger than', and 'smaller than'), and some specific to mathematics (e.g., algebraic and formal notations). Most mathematical activities require the combination and coordination of multiple semiotic representation systems. For example, geometry often involves the representation of magnitude using numerical expression, verbal description, and visualisation (Duval, 2006). Different forms, known as 'registers', of representation follow specific rules for representing mathematical objects, and provide different ways of representing mathematical objects. From this perspective, difficulty in learning mathematics can stem from difficulty with changing or translating between representational registers, for example, between algebraic and pictorial representations of a mathematical object.

Focusing on the individual, mathematics comprehension involves the coordination of at least two registers of semiotic representation. A critical threshold for progress in mathematical learning and for problem solving is a person's ability to flexibly change from one representation system to another. This threshold determines the extent to which a person can interpret representations of mathematical objects or translate between registers (Duval, 2006, 2017). When applied in the context of mathematical tasks, the possible representation registers (e.g., graphical or algebraic) that are

Table 1 A comparison of the commognitive framework (Sfard, 2007) and the theory of representation (Duval, 2017)

	Commognitive framework	Theory of representation
Theoretical focus	Communication as thinking, a collective endeavour enabled through discourse	Communication enabled through the multiplicity of representations of mathematical objects
Mathematics learning	An 'individualising' process where a person does not act blindly according to the rules of the discourse, but exerts agency in deciding how to use and proceed with the discourse	The increasing ability to flexibly use, change between and coordinate multiple registers of representation, demonstrates increasing knowledge of the related mathematical objects and strategic knowledge about different ways of representing them
Analytical focus	Linguistic features of teacher-student or student-student communicative acts and the roles played by the participants	How mathematical representation registers are used and interpreted

prescribed or governed by a particular task can be identified and compared with the actual registers used by a learner to evaluate the effectiveness of the representation registers used by the learner (Kuntze et al., 2022). Extending Duval's theory, Kuntze and colleagues (e.g., Dreher & Kuntze, 2015; Friesen & Kuntze, 2020, 2021) also developed and used representation-based analyses for a broad range of mathematics-related communication including textbook material, classroom vignettes, videos of whole-class mathematical activities and lesson plans. Table 1 compares the key concepts in the two theories.

4.2 Data sources

The commognitive and representation analyses were independently applied in the Social Unit of Learning project to examine the social interactions of a Year 7 female student–student dyad (pseudonyms Aya and Pia) as they responded to the following Household Task:

> The average age of five people living in a house is 25. One of the five people is a Year 7 student. What are the ages of the other four people and how are the five people in the house related? Write a paragraph explaining your answer.

The students were part of an intact class of 26 students asked to complete mathematical tasks individually, in pairs, and in small groups in the SLRC. The full session ran for 60 min. The students were grouped based on the teacher's knowledge of their academic achievement in mathematics and students were paired with peers of moderately different ability levels to encourage collaboration.

The data available included the full video capturing the student pair conversations and behaviour as they completed the Household Task over a 15-min period. Student speech was fully transcribed based on the video. The student written work, including their working sheets and final solution, was collected and scanned and made available to the research team members for their analysis.

Based on Clarke and Chan's knowledge of the dataset, Aya and Pia were chosen for detailed analysis based on the richness of the pair interactions and the written materials that they produced. Pia was one of the highest performing students in class and Aya was above average in class. To briefly describe the pair interaction, Aya initially led the response to the task by drawing on the working out sheet. Her drawings and her conversation with Pia, however, indicated that she interpreted average as the median. Pia, who appeared to fully grasp the mathematical concepts and the requirements of the task, questioned Aya's understanding of the task and tried to complete the task with Aya's input, but Aya found ways to deflect and control the conversation. Occasionally Pia dropped out of the conversation and doodled while Aya worked on her own before Pia re-engaged with the task. The pair produced the final solution of a family with two siblings of ages 5 and 17, Year 7 student of 13 years, and the parents 46 and 44 years respectively (see Fig. 1). The full transcript of the student conversation is available as an appendix in Chan and Sfard's (2020) paper.

4.2.1 Interactivity analysis

The interactivity analysis conducted as part of the Social Unit of Learning project followed the procedure developed by Sfard and Kieren (2001). The analysis involved partitioning the transcribed student speech into utterances (the smallest communicative interactive unit) and indicating for each of the utterances whether it was a private or interpersonal communication, whether the utterance was reactive (in response to a previous utterance) and/or proactive (inviting a further response), and the focus of the utterance as either the object-level or meta-level mathematising, or subjectifying. These different types of utterances are considered to offer the participants opportunities for learning, in terms of invitation to change the participants' own command of the discourse or changing the discourse between the participants. The analysis allows the identification of opportunities for learning and responses to such opportunities in terms of patterns

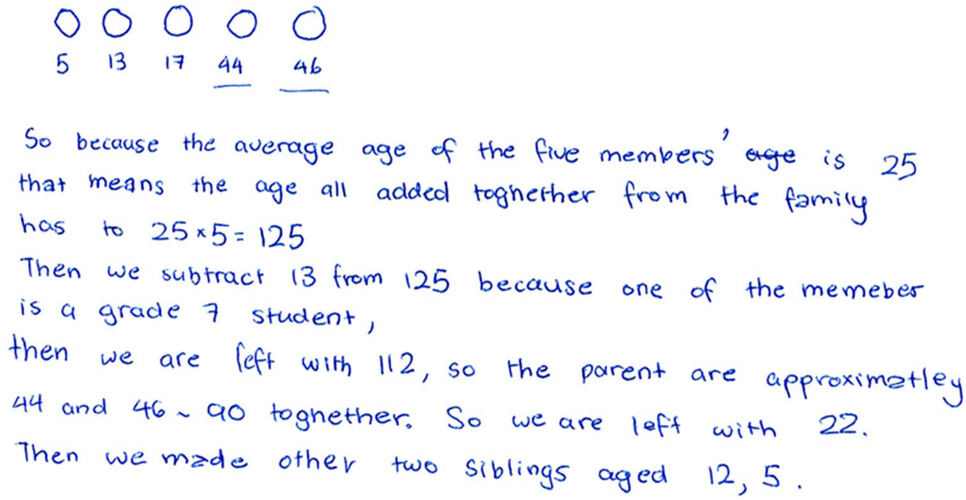

Fig. 1 Final solution produced by Aya and Pia

of interaction between the students, such as instances where one person consistently ignores or responds (reacts) to the other person's utterances, or frequently initiates new topics (Sfard & Kieran, 2001).

4.2.2 Case interpretation—interactivity analysis

From the perspective of the interactivity analysis, the interactions between Aya and Pia were often non-productive. Figure 2 shows an excerpt of the interaction diagram of the students. In the leftmost column the student utterances are numbered. The Action column contains the behaviours of the students as seen on the video, and the two Speech columns contain the students' transcribed speech based on the video. The two columns between the speech columns contain circles that represent the student utterances, where a black circle indicates object-level mathematising, a grey circle indicates meta-level mathematising, and a white circle indicates subjectifying. The diagonal arrows indicate that the student's utterance was either reactive (upward diagonal arrow) and/or proactive (downward diagonal arrow). // indicates overlapping speech.

Based on the interactivity analysis, Pia and Aya did not appear to have a working learning-teaching agreement (Chan & Sfard, 2020). Despite working with a more capable other (Pia), Aya appeared often to miss the learning opportunities available in her interaction with Pia by giving responses that suggested understanding (e.g., [51] "Oh okay, okay. That makes sense then."; [59] "I know, I know.") but then she showed no adoption or accommodation of her partner's discourse subsequently. At the same time Pia also appeared to show reluctance in taking on a leading role in her interactions with Aya. Based on the analysis, Aya did not appear to have learned from her partner in the sense of endorsing the discourse of the more capable other (Pia).

4.2.3 Analysis of the use of representations

The Social Unit of Learning project provided the testing ground for Kuntze and Friesen to develop an analytical framework for understanding social interactions through the lens of representations. Their analysis of representation use involved viewing the video, the transcript, and the written notes of the students to create an inventory of representation registers of mathematical objects between the two students, which included verbal, written, and gestural expressions. Using the video, the researchers re-constructed the time sequence in which the students each contributed to a particular representation register. The inventory allowed the researchers to explore the mathematical potential of the registers, that is, the possible ways of solving the task based on the particular register. The researchers also reconstructed the 'rules' of individual representation registers of the students in order to find out how they conceived these possibilities of representing the mathematical objects. The ways in which the students dealt with the representations were then examined (Kuntze et al., 2022).

4.2.4 Case interpretation—analysis of the use of representations

The analysis of representation use showed a different picture of Aya and Pia's interaction. Figure 3 shows the working out sheet of the students as they attempted the task. The writing of interest is enclosed with ovals in dashed and solid lines. Initially the students appeared to translate the given text register each into a different register (Kuntze et al., 2022): Aya used a graphical representation register by drawing five circles (circled in a dashed line on the top left of the page) representing the five household members, with the middle person 25 years old. Pia, on the other hand, reasoned both in written and spoken form that

	Action	Aya Speech			Pia Speech
47b		Twenty-five.			
48a	*P's pen is hovering over the page, brings other hand close to ask question*				Why are you saying that dude's 25?
48b					They don't have to be 25.
49	*A pointing with pen*	It - it - this one is 25 because that's the average.			
50	*P appears to adjust what is written on paper*				Average doesn't have to - doesn't mean that one guy has to be 25.
51		Oh okay, okay. That makes sense then.			
52	*P writing on paper then moves hand away*				Altogether it's 125 because like ...
53		Yeah, yeah, yeah.			
54					And ...
55a		Now, I get it.			
55b		I thought that was //just 25.			
56a	*P writing on paper then moves hand away. A watches and then glances at P before asking question*				//Yeah, yeah.
56b					So one dude's 13. That means the other four is 112.
57a	*A's hand hovering over the paper*	What do you mean?			
57b		No. It can't - they can't all be like so equal.			
58a	*P raises both hands momentarily as though in despair then takes paper to explain*				They're not. Oh my God.
58b					Look, so 25's one guy,
58c					right.
58d					No. It's like for, you know, average means like ...
59	*A raises hand to cheek*	I know, I know.			
60a	*Both girls look at paper, P points with pen*				Yeah.
60b					So 25 times five is the total,
60c					right?
61	*A brings pen round to point at the same point*	Yeah. I know.			
62					So everyone's 125. And one guy is 13.
63	*A writes on paper*	I know, one guy. So ...			

Fig. 2 Excerpt from the interactive diagram of Aya and Pia

the average age of 25 means that "Twenty-five times five is 125" (circled in a dashed line in the top middle of the page in writing: $25 \times 5 = 125$). From Aya's drawing and their conversation, Pia appeared to realise that Aya might not understand the concept of average (see Fig. 2 [50] and [58a]), and she worked towards connecting with Aya's register by adopting her graphical representation in their discussion. The approach seemed to have worked, with Aya using Pia's representation register in adding/subtracting of ages from the sum (circled in a dashed line in the bottom right corner with the five circles where 13 was subtracted from 112, and further subtracting by 90, the total of the other two people's ages, namely, 44 and 46). In the analysis, it was found that the students reached a solution by combining their registers (circled in solid lines), where Pia's register for the calculation was used in combination with Aya's initial register of representing people using circles. From the analysis, there is clear evidence of learning

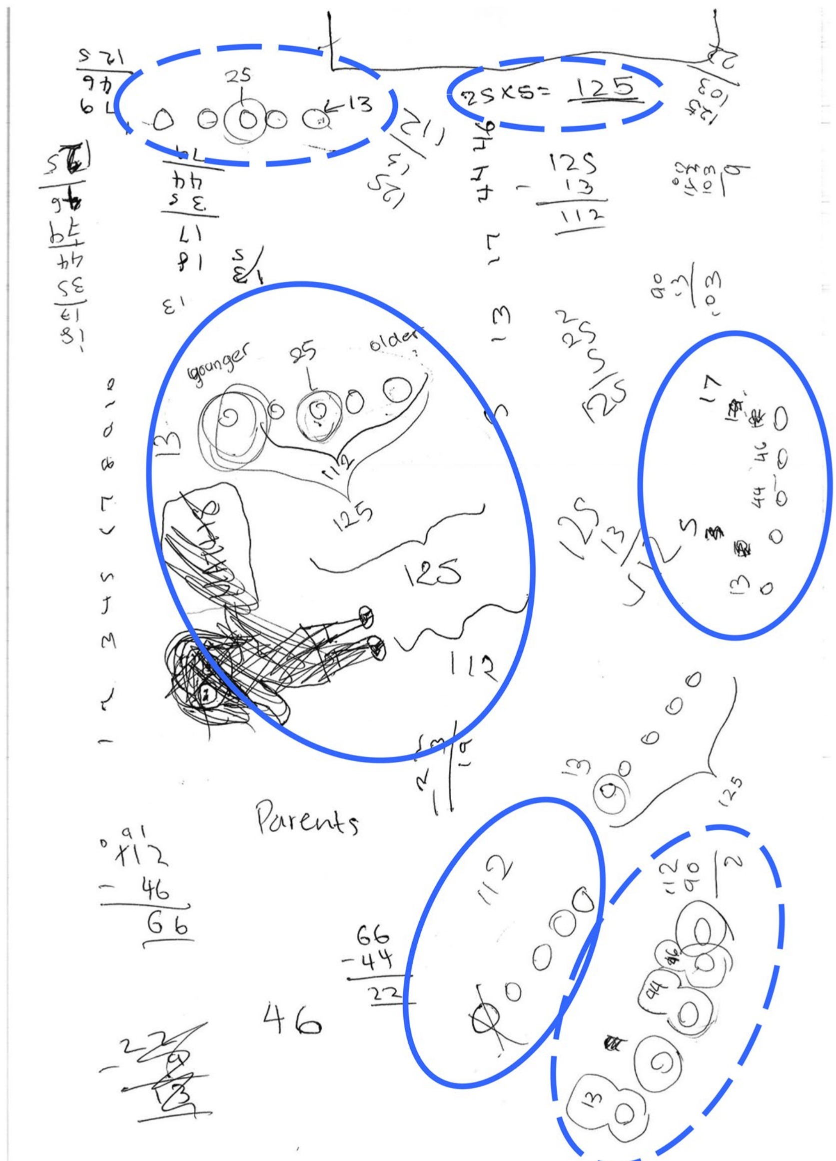

Fig. 3 The working out sheet of Aya and Pia

for the two students in relation to the use of and change between representations.

5 Discussion

Despite drawing from the same dataset, the two analyses provided incommensurable accounts of the interactions between Aya and Pia and their mathematical learning. Based on the interactivity analysis, Aya and Pia were demonstrating non-productive learning interaction as Aya failed to endorse Pia's discourse and thus advance her understanding of what average means. Aya appeared often to dominate the conversation and Pia had to resort to self-talk at times, despite being the more capable other. From the analysis of the pair's use of representations, however, more learning appears to have taken place through their interaction. Although the students initially used different registers (Aya's five dots and Pia's numerical representation) and appeared to struggle to understand each other because their initial registers and related thinking were so different, they eventually integrated each other's suggestions into their own representations. The analysis indicated that the solution could finally be worked out by the two students when they managed to adopt each other's representation registers and were able to create a joint register. The two analyses offered an interesting case for our reflection of complementarity and commensurability between theories.

To answer the first focal question, the complementarity and incommensurability of the two perspectives is evident in terms of the different constructs, artefacts or situations that each analysis highlighted. The interactivity analysis privileged verbal communication between the students and the underlying meaning conveyed by the students' utterances. The statement [51] "Oh okay, okay. That makes sense then." was interpreted as a way in which Aya dismissed Pia's suggestion and continued with her own interpretation of the task. On the other hand, in the analysis of representation use based on Duval's (2017) theory, the analysis brought together students' use and coordination of multiple representations displayed in their verbal utterances and written notes, to further the understanding of their mathematical thinking. Utterances that do not clearly refer to mathematics do not form the focus of the analysis. In contrast, while the drawing and writing of the students during the task were taken into consideration in the interactivity analysis, they were supplementary to the transcript analysis providing the context for the interpretation of the students' verbal exchanges. Each analysis therefore foregrounded different aspects of the student interaction, drawing from different evidence, and produced internally coherent, yet incommensurable analytical accounts.

In terms of the second question, the commognitive perspective identified unhelpful ways in which a learner may avoid learning from a more capable partner in a mathematics classroom. There are subjectifying, face-saving moves that a learner may use to mitigate the role conflict and to avoid feelings of inferiority. At the same time, the more capable peer may also be reluctant to change the power relation and to take on a leadership role. With regard to instructional advocacy based on this account, adopting a more dialogic communication style in the classroom to facilitate and consider multiple voices and perspectives through communicative acts (e.g., speech and writing) to create shared meaning and explore new meanings, has been promoted as a beneficial way for students to learn from peers (e.g., Mercer et al., 2020; Resnick et al., 2015).

The representation perspective, on the other hand, focused on how the students engaged with mathematics objects through different representation registers. The analysis showed how the use of multimodal communication through speech, writing, and graphical means allowed the students to demonstrate and try to clarify their knowledge and incorporate the representation registers of their peers in communicating their understanding. Instructional advocacy based on this account highlights the usefulness of encouraging students to be aware and make use of different representation registers and become fluent in their integration and conversion in the mathematics classroom (Duval, 2006; Kuntze et al., 2022).

These two incommensurable accounts offer different ways of understanding student social interaction and mathematics learning with associated instructional advocacies. Both accounts are true within their own coherent frameworks, but they both offer only a partial vision of the students' social interaction and mathematics learning (Clarke, 2011). Based on Cobb's (2007) suggestion, we could identify sources of incommensurability between the theories, which therefore allowed us to understand each of the perspectives more fully. In the case of the two analyses, one source of incommensurability appears to be how the concept of mathematical learning is conceptualised and evaluated. From the commognitive framework perspective, mathematics learning involves how students communicate mathematical understanding and facilitate and influence each other's discourse and the leading or following role that they play when interacting with each other (Chan & Sfard, 2020). From the perspective of representation theory, mathematics learning involves the different representation registers of mathematical objects that students separately or jointly employ to solve mathematical tasks. Individual researchers or educators may choose to follow the instructional implications arising from either perspective. From an absolute relativist stance, these different instructional advocacies are not problematic, especially if each perspective is applied to different cases in different settings. However, as both perspectives were applied in the same setting based on the interaction of the same student pair, any discrepancy obliges us to think more deeply about how to deal with the divergence in perspectives based on the 'same' but differently constructed and portrayed situations.

The approach that Caduri and Heyd-Metzuyanim (2015) undertook was to acknowledge the potential incommensurability and incompatibility of theories at the global level, while identifying local common constructs between the perspectives in order to resolve the tensions between the perspectives. They also illustrated the usefulness of further interrogating the data to see if there are aspects being omitted from another perspective. In the case of the application of the commognitive and representation perspectives in the Social Unit of Learning project, we can identify the ways in which each perspective foregrounded different evidence in conceptualising student learning: one heavily relied on verbal discourse, and the other on multimodal representations of mathematical objects. This difference in focus appears to have created divergence in the analytical accounts when the verbal interactions between the students provided one interpretation of the situation in terms of non-collaboration, while the focus on the students' representation registers suggested potential transfer and reciprocal influence of ideas.

In addition, the analysis drawing from the commognitive framework focused on the Vygotskian perspective (Vygotsky, 1978) in terms of the expected knowledge transfer direction (from the more to the less capable peer).[2] Role conflict was suggested to arise when the more capable peer

[2] The recent work by Abtahi, Graven, and Lerman (2017), reconceptualises such knowledge transfer as multidirectional.

did not maintain the leader role in mathematising (Cooper & Lavie, 2021). Productive collaboration and learning takes place when the less knowledgeable peer endorses and accepts the discourse of the more knowledgeable peer. While the mathematical appropriateness of a representation can be determined based on representation theory, the theory does not prescribe the expected direction of knowledge transfer. Collaboration and learning were judged based on whether the pair jointly constructed representations that are productive for solving the task. Role conflict in the sense of learning-teaching agreement (Ben-Zvi & Sfard, 2007) is not part of the formulation of representation theory (Duval, 2017), which appears to create divergence in the evaluation of whether 'productive' collaboration and learning occurred, based on the student–student interaction compared to the commognitive perspective.

With the divergence in analytical accounts based on the two different theories, how should we reconcile their incommensurability? For example, should we place more emphasis on representations in interactivity analysis or examine role conflict in the analysis of representations? We would argue that refining theories and analytical approaches drawing from other perspectives is certainly an option and potentially useful, provided that the adjustments cohere with the theoretical framework and do not compromise the integrity of the framework (Even & Schwarz, 2003; Prediger et al., 2008). Vygotsky (1978, 1986), for example, wrote about the importance of written language as a tool for thinking. The incorporation of the students' writing in the interactivity analysis would be consistent with Vygotsky's perspective, though it may require modification of the analytical approach to incorporate multimodal communication. The concept of role conflict from a social theory perspective appears to fall outside the cognitive theory of representation (Duval, 2017). Although we cannot deny the value of considering role conflict in students' representation use, we would argue that a strength of the analysis of representation use is the clear focus on mathematical objects (Kuntze et al., 2022). Introducing new elements and foci may overcomplicate analyses that are already highly complex, and thus distract from the priority of the theorisation.

The divergence in interpretive accounts demonstrates the challenges involved in investigating student learning. By nature, student learning is a dynamic and implicit process. As pointed out by Berliner (2002), educators often need the knowledge of the particular, the local, in making decisions about their practices (also see Nuthall, 2005). Over the years, researchers created different conceptual tools (e.g., learning theories) and physical tools (e.g., assessment or observation tools) to indirectly infer the process and outcome of learning. Each theory and method provides a partial perspective of the learning process. This is accentuated in the Social Unit of Learning project as designed by Clarke and supported by the research team. The laboratory classroom setting offers researchers the opportunity to compare and contrast their perspectives within the same setting as well as to have conversations with each other. It is the acknowledgement of our own limited perspective that makes collaboration and conversations with other researchers all the more important.

In terms of offering research-based instructional advocacies, the current paper highlighted that this endeavour is fraught with challenges. Rather than assuming that research can offer definitive answers to inform teaching, we take the stance of Kuhn (1970) and Wiliam (2016), who emphasise research as an evolving undertaking that can only provide the best understanding to date and which is highly situated. Teachers' knowledge of their students and their classroom context is essential to determining the appropriateness of any instructional advocacy in their classrooms. As Bruner (1997) suggested, the divergence and incommensurability in our perspectives should be celebrated rather than avoided. At the same time, instructional advocacies based on research should be seen as tools and strategies that researchers offer educators in response to the diversity and complexity of student learning. These advocacies offer opportunities for educators to reflect on and complement their existing practice.

6 Conclusion

With the abundance of theories and perspectives generated through research over the years, a continuing challenge that researchers face relates to the difficulty of navigating the multitude of theories available (Bikner-Ahsbahs & Prediger, 2014; Cobb, 2007). The Social Unit of Learning project was purposefully designed so that researchers from multiple disciplines holding different theoretical perspectives could work together to examine the complexity of the mathematics classroom. The notion of complementarity is central to its design. However, rather than assuming a relativistic stance, the possibility in the project to juxtapose multiple analytical accounts predicated on different theories in relation to the same research setting creates the opportunity for researchers to examine the connections, tensions, and even potential contradictions and incompatibilities between these different accounts. Through the discussion of complementarity and commensurability, we intend to build on the work of the Project Leader, Professor David Clarke, to contribute to the continuing research into the complexity of the mathematics classroom in terms of theory, methodology, and practice.

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