Dimension of projection: Marstrand's theorem

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Tässä tutkielmassa todistetaan Marstrandin projektiolause käyttäen apuna potentiaaliteoriaa. Projektiolauseen mukaan 2-ulotteisen Borel joukon ortogonaaliprojektion Hausdorffin dimensio on luvun 1 ja kyseisen Borel joukon dimension minimi melkein kaikkiin eri suuntiin. Intuitiivisesti lause kertoo, että joukon varjon dimensio on suurin mahdollinen.

Marstrandin projektiolauseen todistamiseksi tutkielmassa rakennetaan teoria alkaen yleisen mittateorian perustuloksista. Mittateorian pohjalta määritellään Hausdorffin mitta, jonka avulla määritellään joukon Hausdorffin dimensio. Intuitiivisesti Hausdorffin dimensio kuvaa joukon geometrista kokoa ja se on yksikäsitteinen jokaiselle joukolle. Hausdorffin dimensio mahdollistaa monimutkaisten joukkojen, kuten fraktaalien, geometrian tutkimisen. Lisäksi esitellään dimensioihin liittyviä merkintöjä ja tapoja arvioida joukon Hausdorffin dimension suuruutta. Tutkielman lopussa esitellään algoritminen menetelmä, jonka avulla voidaan muodostaa esimerkkejä fraktaaleista. Lopuksi sovelletaan Marstrandin projektiolausetta erilaisiin joukkoihin.

John Marstrand todisti projektiolauseen vuonna 1954. Robert Kaufman todsti tuloksen käyttäen potentiaaliteoriaa vuonna 1968. Myöhemmin Kenneth Falconer esitteli Kaufmania mukaillen potentiaaliteoriaan perustuvan todistuksen. Tässä tutkielmassa esitellään kyseinen todistus yksityiskohtaisemmin. Marstrandin projektiolause tuli tunnetuksi, kun Mandelbrot popularisoi fraktaalin käsitteen 1970-luvulla. Lause voidaan yleistää korkeampiin dimensioihin ja se on tärkeä työkalu fraktaalien geometrian tarkastelussa. Vaikka lause on tunnettu pitkään, siihen liittyy edelleen avoimia ongelmia.

In this thesis we prove Marstrand's projection theorem using potential theoretical methods. Projection theorem claims that the Hausdorff dimension of the orthogonal projection of a Borel set in $\mathbb{R}^2$ is the minimum between 1 and the dimension of the set for almost all angles. Intuitively, the theorem gives that the shadow of the set has the highest possible dimension. This result was first proven by John Marstrand in 1954 and it became well known after Mandelbrot popularized the notion of fractal in the 1970s. Marstrand's theorem has generalizations to higher dimensions and it is an important tool to look into the geometry of fractals. Although the theorem has been known for long time, there are still open problems related to it.

Contents

Introduction

We all learn in primary school math that lines are 1-dimensional, squares and disks are 2-dimensional and cubes and balls are 3-dimensional. These shapes are often used by scientists to model the attributes of the natural world. However, as Benoit Mandelbrot famously said ”Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel a straight line.” Mandelbrot proposed that there are better shapes to model our surrounding world. Usually we consider dimension as an extension of an object in a given direction. For example a square has two directions of extension and therefore it is 2-dimensional. The usual definition of dimension is not enough when dealing with more complicated sets. The Koch Curve (see [2, Figure 0.2]) is an example that is made of straight lines. Therefore, one could consider it as 1-dimensional. However, problem comes, when we go down more levels in the construction of the curve. Going down an infinite number of levels, the length of the Koch Curve becomes infinite. To look into the geometry and dimension of these kind of complex sets and their shadows, we build a theory in this thesis.

Hausdorff dimension was first defined by Felix Hausdorff in 1918. For smooth sets the Hausdorff dimension is an integer and it coincides with the dimension in the usual way described above. Hausdorff dimension is also defined for more complicated sets. Particularly complicated sets with ’rough’ structure may have a non-integer Hausdorff dimension. In 1975 Benoit Mandelbrot named these sets ”fractals”. The word fractal comes from the Latin word frāctus, which means ’broken’ or ’shattered’ and refers to Mandelbrot’s book *The fractal geometry of nature* [8] as ”forms that Euclid leaves aside as being formless”. The concept of fractal was invented to explore complicated geometry of nature. For example a fern leaf and a lightning has this kind of complex shape.

Marstrand’s projection theorem was first proven by John Marstrand in his thesis in 1954. Marstrand proof was based on definitions and basic properties of Hausdorff dimension. Pertti Mattila generalized the result later on in 1975 [12]. In 1968 Robert Kaufman applied potential theory to prove the projection theorem [6]. Kenneth Falconer introduce a recount of Kaufman’s proof in his book *Fractal geometry: mathematical foundations and applications* [2]. In this thesis Falconer’s proof is demonstrated with more details.

The main source for this thesis is Kenneth Falconer’s book *Fractal geometry: mathematical foundations and applications* [2]. Falconer’s other books *The geometry of fractal sets* [4] and *Techniques in fractal geometry* [3] and Mattila’s book *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability* [11] are used to complete and reformulate the theory and the proofs. Mattila’s book *Fourier analysis and Hausdorff dimension* [10] and paper *Hausdorff dimension, orthogonal projections* *and intersections with planes* and Kaufman's paper *On Hausdorff dimension of projections* [6] were used to learn about the history of the projection theorem. The book *Classical Dynamics of Particles and Systems* [9] was used to learn more about the potential theoretical methods used in Chapter 4. The sources for measure theory presented in this thesis are notes from my thesis supervisor Sebastiano Nicolussi Golo [5] and from my teacher Tero Kilpeläinen [7].

The thesis begins with the basics of measure theory as foundation for the inexperienced reader to follow the thesis. In Chapter 2 we introduce Hausdorff measures. From Hausdorff measures we end up defining Hausdorff dimension in Chapter 3. Some sets that have non-integer Hausdorff dimension are introduced through self-similarity in Chapter 7. Self-similarity gives us algorithmic way of constructing examples of fractals. Another way of constructing examples is to take Cartesian product of two sets. In Chapter 5 we prove results for the dimension of these Cartesian products. In Chapter 6 we look into the projections onto the lines through the origin of $\mathbb{R}^2$ with different angles with horizontal axis. Marstrand's projection theorem claims that the dimension of the orthogonal projection of a Borel set in $\mathbb{R}^2$ is the minimum between 1 and the dimension of the set for almost all angles. This result can be generalized to higher dimensions. However there might exist exceptional angles for which the dimension drops through the projection. Chapter 8 introduces some examples of these sets and directions. There are still open problems related to the number of exceptional directions. In the University of Jyväskylä, Tuomas Orponen is working on these open problems.

The process of writing this thesis has been an occasion of learning for me. My earlier studies included only the basics of measure theory. Through this thesis I have met general measure theory. The preparation of this thesis has also improved my skills to write mathematical text and my language skills. I started studying the theory by reading the books mentioned in the bibliography and by understanding the proofs. I have been doing exercises from Falconer's book *Fractal geometry: mathematical foundations and applications* [2] to understand the theory more deeply. We have been working on the exercises and proofs together with my supervisor Sebastiano. I thank Sebastiano for this journey of learning and all the support he gave me on the way.

CHAPTER 1

Measure theory

In this chapter we introduce the basics of measure theory. The set of all the subsets of a set X is denoted by $\mathscr{P}(X) = \{A : A \subset X\}$.

DEFINITION 1.1. Let X be an arbitrary set and $\mathcal{F} \subset \mathscr{P}(X)$. The collection $\mathcal{F}$ is a *σ-algebra* on X if:

(1) $X \in \mathcal{F}$,
(2) $A \in \mathcal{F} \Rightarrow A^c = X \backslash A \in \mathcal{F}$,
(3) $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

The pair $(X, \mathcal{F})$ is called *measurable space.*

LEMMA 1.2. *If J is a set and $\{\mathcal{F}_j\}_{j\in J}$ is a collection of σ-algebras on X, then $\bigcap_{j\in J} \mathcal{F}_j$ is a σ-algebra on X.*

PROOF. (1) $X \in \mathcal{F}_j$ for every $j \in J \Rightarrow X \in \bigcap_{j\in J} \mathcal{F}_j$.

(2) $A \in \bigcap_{j\in J} \mathcal{F}_j \Rightarrow A \in \mathcal{F}_j$ for every $j \in J \Rightarrow X\backslash A \in \mathcal{F}_j$ for every $j \in J \Rightarrow X\backslash A \in \bigcap_{j\in J} \mathcal{F}_j$.

(3) $\{A_i\}_{i\in\mathbb{N}} \subset \bigcap_{j\in J} \mathcal{F}_j \Rightarrow A_i \in \mathcal{F}_j$ for every i and for every j. Therefore $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}_j$ for every $j \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \bigcap_{j\in J} \mathcal{F}_j$. □

Trivial examples of σ-algebras are $\{\emptyset, X\}$ and $\mathscr{P}(X)$. Let $X = \{a, b\}$: All the σ-algebras of X are $\{\emptyset, \{a, b\}\}$ and $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

THEOREM 1.3. *Any family $\mathcal{A} \subset \mathscr{P}(X)$ is contained in a unique smallest σ-algebra on X, which we denote by $\mathcal{F}(\mathcal{A})$ and we call* the σ-algebra generated by $\mathcal{A}$.

PROOF. By the trivial example $\mathscr{P}(X)$, there exists at least one σ-algebra on X containing $\mathcal{A}$. Let $\{\mathcal{F}_i : i \in I\}$ be the collection of all σ-algebras on X containing $\mathcal{A}$. Then $\mathcal{A} \subset \bigcap_{i\in I} \mathcal{F}_i$ and by lemma 1.2 it is a σ-algebra. Moreover, $\bigcap_{i\in I} \mathcal{F}_i$ is the smallest σ-algebra containing $\mathcal{A}$, since for any σ-algebra $\mathcal{F}$ containing $\mathcal{A}$ it holds that $\mathcal{F} \in \{\mathcal{F}_i : i \in I\}$. Hence $\mathcal{F} \supset \bigcap_{i=I} \mathcal{F}_i$. □

If X is a topological space, the *Borel σ-algebra*, or the *σ-algebra of Borel sets*, is the σ-algebra generated by the collection of all open subsets of X.

Our desire is to find a way to define "the measure" of a set. In simple cases we want that our measure matches with the geometric size of a set. For example in $\mathbb{R}^1$ we want that the measure gives us the length, in $\mathbb{R}^2$ the measure should give us the area and in $\mathbb{R}^3$ the volume. The measure should depend on the dimension of the set. If we have a curve in $\mathbb{R}^2$, its 2-dimensional measure should be 0, since curves do not have any area. However its 1-dimensional measure should give us the length of the curve. Similarly the 3-dimensional measure of a surface in $\mathbb{R}^3$ should be 0, since surfaces do not have volume. However its 2-dimensional measure should give us the area of the surface. Our measure should be the same for transferred or rotated sets. The last desired feature of a measure is called countable additivity: for countable many disjoint sets $A_1, A_2, \ldots$ the measure of the union is the sum of the measures of the sets A_j. Unfortunately this feature is something that can not be fulfilled for arbitrary families $\{A_j\}_j$. We define something weaker for all the sets.

Definition 1.4. Let X be an arbitrary set. A function $\mu : \mathscr{P}(X) \to [0, \infty]$ is called a *measure* on X if:

(1) $\mu(\emptyset) = 0$,
(2) $A_1 \subset A_2 \Longrightarrow \mu(A_1) \le \mu(A_2)$,
(3) $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(X) \Longrightarrow \mu(\bigcup_{n\in\mathbb{N}} A_n) \le \sum_{n\in\mathbb{N}} \mu(A_n)$.

Property (2) is called monotonicity and property (3) is called subadditivity. Subadditivity is something weaker than the desired property of countable additivity. For countable additivity we need to restrict the family of sets and define measurable sets.

Definition 1.5. A set $E \in \mathscr{P}(X)$ is *μ-measurable* if

$$\mu(A) = \mu(A \cap E) + \mu(A \backslash E)$$

for every $A \in \mathscr{P}(X)$.

Let $\mathscr{M}(\mu)$ be the collection of all μ-measurable subsets of X. Notice that A is an arbitrary set and that $E \backslash A = E \cap A^c$.

Remark 1.6. By the subadditivity, we have

$$\mu(A) \le \mu(A \cap E) + \mu(A \backslash E)$$

for every $A \in \mathscr{P}(X)$. Therefore

$$E \in \mathscr{M}(\mu) \Longleftrightarrow \mu(A) \ge \mu(A \cap E) + \mu(A \backslash E) \quad \forall A \in \mathscr{P}(X).$$

Lemma 1.7. *Let $E \in \mathscr{P}(X)$. $E \in \mathscr{M}(\mu)$ if and only if $\forall U \subset E$ and $\forall V \subset E^c$, $\mu(U \cup V) = \mu(U) + \mu(V)$.*

Proof. ($\Rightarrow$) Let $U \subset E$ and $V \subset E$: setting $A = U \cup V$, we have

$$\begin{aligned} \mu(U \cup V) &= \mu(E \cap (U \cup V)) + \mu(E^c \cap (U \cup V)) \\ &= \mu(U) + \mu(V). \end{aligned}$$

($\Leftarrow$) Given $A \in \mathscr{P}(X)$, set $U = A \cap E$ and $V = A \backslash E$. Hence

$$\begin{aligned} \mu(A) &= \mu(U \cup V) \\ &= \mu(U) + \mu(V) \\ &= \mu(A \cap E) + \mu(A \backslash E). \end{aligned}$$

Therefore, $E \in \mathscr{M}(\mu)$. □

We can easily construct more measurable sets from given ones.

Proposition 1.8. *Let $E \in \mathscr{P}(X)$. If $E \in \mathscr{M}(\mu)$ then $E^c \in \mathscr{M}(\mu)$.*

Proof. If $A \in \mathscr{P}(X)$ and $E \in \mathscr{M}(\mu)$, then

$$\begin{aligned} \mu(A) &= \mu(A \cap E) + \mu(A \backslash E) \\ &= \mu(A \cap (E^c)^c) + \mu(A \cap E^c) \\ &= \mu(A \backslash E^c) + \mu(A \cap E^c). \end{aligned}$$

Therefore, $E^c \in \mathscr{M}(\mu)$. □

Proposition 1.9. *Let $E \in \mathscr{P}(X)$.*

$$\mu(E) = 0 \Longrightarrow E \in \mathscr{M}(\mu)$$

Proof. By the monotonicity of μ, if $A \in \mathscr{P}(X)$, then

$$\begin{aligned} \mu(A \cap E) + \mu(A \backslash E) &\leq \mu(E) + \mu(A \backslash E) \\ &= \mu(A \backslash E) \\ &\leq \mu(A). \end{aligned}$$

We conclude that $E \in \mathscr{M}(\mu)$ by Remark 1.6. □

Corollary 1.10.

$$\emptyset, X \in \mathscr{M}(\mu)$$

Proof. By the Definition 1.4, $\mu(\emptyset) = 0$. Therefore, by the Proposition 1.9, $\emptyset \in \mathscr{M}(\mu)$. Since $X = \emptyset^c$, by Proposition 1.8 we also have $X \in \mathscr{M}(\mu)$.

□

Lemma 1.11. *Let $E, F \in \mathscr{P}(X)$.*

$$E, F \in \mathscr{M}(\mu) \Longrightarrow E \cup F \in \mathscr{M}(\mu)$$

Proof. Let $A \in \mathscr{P}(X)$. It holds that

$$A \cap (E \cup F) = (A \cap E \cap F) \cup (A \cap E \cap F^c) \cup (A \cap E^c \cap F)$$

and

$$A \cap (E \cup F)^c = A \cap E^c \cap F^c.$$

If $E, F \in \mathscr{M}(\mu)$, the by the subadditivity of μ

$$\begin{aligned} \mu(A) &= \mu(A \cap E) + \mu(A \cap E^c) \\ &= \mu(A \cap E \cap F) + \mu(A \cap E \cap F^c) + \mu(A \cap E^c \cap F) + \mu(A \cap E^c \cap F^c) \\ &\geq \mu(A \cap (E \cup F)) + \mu(A \cap (E \cup F)^c). \end{aligned}$$

Therefore, $E \cup F \in \mathscr{M}(\mu)$. □

To accomplish the desired property of countable additivity for measurable sets, we need to show that the family of measurable sets forms a σ-algebra.

Theorem 1.12. *The collection $\mathscr{M}(\mu)$ is a σ-algebra on X.*

Proof. (1) Holds by Corollary 1.10.

(2) Holds by Proposition 1.8.

(3) Let $\{E_i'\}_{i=1}^{\infty} \subset \mathscr{M}(\mu)$. Let $F_n = \bigcup_{i=1}^{n} E_i'$ and $E_{n+1} = E_{n+1}' \backslash F_n$. Let $F = \bigcup_{i=1}^{\infty} E_i$. Notice that F_n is a disjoint union of E_i, $i = 1, \ldots, n$. For $A \in \mathscr{P}(X)$

$$\begin{aligned} \mu(A \cap F_n) &= \mu(A \cap F_n \cap E_n) + \mu(A \cap F_n \cap E_n^c) \\ &= \mu(A \cap E_n) + \mu(A \cap F_{n-1}). \end{aligned}$$

Repeating this gives

$$\mu(A \cap F_n) = \sum_{j=1}^{n} \mu(A \cap E_j).$$

By Lemma 1.11, $F_n \in \mathscr{M}(\mu)$. By subadditivity and monotonicity

$$\begin{aligned}\mu(A) &= \mu(A \cap F_n) + \mu(A \cap F_n^c) \\ &\geq \sum_{j=1}^{n} \mu(A \cap E_j) + \mu(A \cap F^c).\end{aligned}$$

When $n \to \infty$, by subadditivity, we get

$$\begin{aligned}\mu(A) &\geq \sum_{j=1}^{\infty} \mu(A \cap E_j) + \mu(A \cap F^c) \\ &\geq \mu\left(\bigcup_{j=1}^{\infty}(A \cap E_j)\right) + \mu(A \cap F^c) \\ &= \mu(A \cap F) + \mu(A \cap F^c).\end{aligned}$$

Therefore, by Remark 1.6, $F \in \mathscr{M}(\mu)$. □

LEMMA 1.13. *If $A \in \mathscr{M}(\mu)$ and $B \in \mathscr{M}(\mu)$, then $A \backslash B \in \mathscr{M}(\mu)$.*

PROOF. Since $A \backslash B = A \cap B^c$ and $\mathscr{M}(\mu)$ is a σ-algebra it follows that $A \backslash B \in \mathscr{M}(\mu)$. □

REMARK 1.14. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{M}(\mu)$ and $A_i \cap A_j = \emptyset, i \neq j$, then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

PROOF. We can write

$$\begin{aligned}\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \mu\left(A_1 \cap \bigcup_{n\in\mathbb{N}} A_n\right) + \mu\left(\bigcup_{n\in\mathbb{N}} A_n \backslash A_1\right) \\ &= \mu(A_1) + \mu\left(\bigcup_{n=2}^{\infty} A_n\right).\end{aligned}$$

By iterating we get

$$\begin{aligned}\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \sum_{n=1}^{k} \mu(A_n) + \mu\left(\bigcup_{n=k+1}^{\infty} A_n\right) \\ &\geq \sum_{n=1}^{k} \mu(A_n).\end{aligned}$$

Since it holds for all k, we obtain

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \geq \sum_{n=1}^{k} \mu(A_n).$$

Since the other inequality is given by subadditivity, we get the desired equality. $\square$

This property is called the *countable additivity of* μ.

PROPOSITION 1.15. *If A and B are measurable sets so that $A \supset B$, then $\mu(A\backslash B) = \mu(A) - \mu(B)$.*

PROOF. Since $A = B \cup (A\backslash B)$ is a disjoint union, Remark 1.14 implies that $\mu(A\backslash B) = \mu(A) - \mu(B)$. $\square$

PROPOSITION 1.16. *Let $E \in \mathscr{M}(\mu)$ and $E_n \in \mathscr{M}(\mu)$ for all $n \in \mathbb{N}$.*

(1) *If $E_n \nearrow E$ then $\mu(E_n) \nearrow \mu(E)$.*
(2) *If $E_n \searrow E$ and $\mu(E_1) < \infty$ then $\mu(E_n) \searrow \mu(E)$.*

Notations $\nearrow$ and $\searrow$ denotes convergence of monotonic sequences of sets or numbers.

PROOF. (1) We prove that if $A_1 \subset A_2 \subset \dots$ is an increasing sequence of μ-measurable sets then

$$\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

We write $\bigcup_{i=1}^{\infty} A_i = A_1 \cup (A_2\backslash A_1) \cup (A_3\backslash A_2) \cup \dots$ as a disjoint union. Proposition 1.15 implies

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu(A_1) + \sum_{i=1}^{\infty}(\mu(A_{i+1}) - \mu(A_i)) \\ &= \mu(A_1) + \lim_{k\to\infty}\sum_{i=1}^{k}(\mu(A_{i+1}) - \mu(A_i)) \\ &= \lim_{k\to\infty} \mu(A_k).\end{aligned}$$

Therefore, $\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$.

(2) Let $B_1 = E_1$ and $B_k = E_k\backslash E_{k-1}$ where $E = \bigcup_{n\in\mathbb{N}} B_n$ and $E_n = \bigcup_{k=1}^{n} B_k$. Note that $E_1\backslash E_n \nearrow E_1\backslash E$. Applying (1) and Proposition 1.15 we get

$$\mu(E_1) - \mu(E_n) = \mu(E_1\backslash E_n) \nearrow \mu(E_1\backslash E) = \mu(E_1) - \mu(E).$$

Therefore $\mu(E_n) \searrow \mu(E)$. $\square$

COROLLARY 1.17. *If for $\delta > 0$, A_δ are μ-measurable sets such that $A_{\delta'} \subset A_\delta$ for $0 < \delta < \delta'$, then*

$$\lim_{\delta\to 0} \mu(A_\delta) = \mu\left(\bigcup_{\delta>0} A_\delta\right).$$

PROOF. The claim follows straight from Proposition 1.16. $\square$

REMARK 1.18. The properties of Remark 1.14 and Proposition 1.16 are together called *continuity* of measures.

DEFINITION 1.19. A measure μ on a topological space X is a *Borel measure* if every open set is μ-measurable.

By defining the measure of a set we are able to discuss about the size of the sets. To discuss about the distance between the sets we definite the metric of a space. Metric spaces are an important setting for geometry and analysis. We will see how the metric geometry interacts with measure theory.

Definition 1.20. Let $X \neq \emptyset$. A function $d : X \times X \to [0, \infty[$ is a *metric* on X if:

(1) $d(x, y) = 0 \iff x = y$,
(2) $d(x, y) = d(y, x) \quad \forall x, y \in X$,
(3) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

We call (X, d) a *metric space*.

The distance between two sets is not intuitively trivial. If two sets are not disjoint then the distance between these two sets can be defined in many ways. Hausdorff distance is defined using δ-neighbourhoods.

Definition 1.21. Let $A \subset \mathbb{R}^n$ and $\delta > 0$. The *δ-neighbourhood* of A is

$$A_\delta = \{x : |x - a| \leq \delta \quad \text{for some } a \in A\}$$

Definition 1.22. Let $\mathcal{S}$ be the class of non-empty compact subsets of $\mathbb{R}^n$ and $A, B \in \mathcal{S}$. The *Hausdorff distance* between the sets A and B is

$$h(A, B) = \inf\{\delta : A \subset B_\delta \quad \text{and} \quad B \subset A_\delta\}.$$

Theorem 1.23. *Let $\mathcal{S}$ be the class of non-empty compact subsets of $\mathbb{R}^n$. The Hausdorff distance h is a metric on $\mathcal{S}$.*

Proof. We need to check the three properties listed in Definition 1.20.

(1) If $B \subset A_\delta$, for all $\delta > 0$, then every point of B is a limit point of A. Therefore, $h(A, B) = 0$ implies $B \subseteq \overline{A} = A$. Likewise $A \subseteq \overline{B} = B$.

(2) Clearly $h(A, B) = h(B, A)$ for all $A, B \in \mathcal{S}$.

(3) Let $A, B, C \in \mathcal{S}$ and $\delta > h(A, B)$ and $\varepsilon > h(B, C)$. Since $C \subset B_\varepsilon$ and $B \subset A_\delta$, then $C \subset A_{\varepsilon+\delta}$. Likewise $A \subset C_{\varepsilon+\delta}$. Thus, $h(A, C) \leq h(A, B) + h(B, C)$. ☐

Later on we need another notion of distance between sets as defined next.

Definition 1.24. Let (X, d) be a metric space. A *distance* between two sets $A, B \subset X$ is

$$\operatorname{dist}(A, B) = \inf_{a \in A, b \in B} d(a, b).$$

Notice that this distance dist is not a metric, since $\operatorname{dist}(A, B) = 0$ does not imply that $A = B$. However, if $\operatorname{dist}(A, B) = 0$ the sets A and B intersect. If $\operatorname{dist}(A, B) > 0$, then the closure of A and B are disjoint. Next we define a completeness of a metric space. First we need to define Cauchy sequences.

Definition 1.25. Let (X, d) be a metric space. A sequence $x_1, x_2, \dots \in X$ is called *Cauchy sequence* if for every positive $r > 0$ there is a positive integer N such that for all $m, n > N$,

$$d(x_n, x_m) < r.$$

Definition 1.26. Metric space (X, d) is *a complete metric space* if every Cauchy sequence of points in X has a limit in X.

Next theorem gives us a rather simple way of proving that a measure is a Borel measure.

THEOREM 1.27. *If (X, d) is a metric space and μ is a measure on X, then μ is a Borel measure if and only if*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

when dist$(A, B) > 0$.

PROOF. ($\Rightarrow$) Let $A, B \subset X$ such that $\operatorname{dist}(A, B) = \alpha > 0$. Let $C = \bigcup_{a \in A} \{x \in X : d(x, a) < \frac{\alpha}{2}\}$. Then C is open and hence μ-measurable. Therefore

$$\mu(A \cup B) = \mu((A \cup B) \cap C) + \mu((A \cup B) \backslash C) = \mu(A) + \mu(B).$$

($\Leftarrow$) Let C be a closed set: we will show that C is measurable. Let $A \subset X$. By Remark 1.6, we can assume that $\mu(A)$ is finite. For each j let $C_j = \{x \in X : \operatorname{dist}(x, C) \leq \frac{1}{j}\}$. Then

$$\operatorname{dist}(A \cap C, A \backslash C_j) > 0$$

and thus

$$\mu((A \cap C) \cup (A \backslash C_j)) = \mu(A \cap C) + \mu(A \backslash C_j).$$

Now $A \backslash C = A \cap C^c = \bigcup_j A \cap C_j^c$ and $A \cap C_j^c$ is an increasing sequence. Proposition 1.16 implies

$$\mu(A \backslash C) = \lim_{j \to \infty} \mu(A \cap C_j^c).$$

Therefore

$$\mu(A) \geq \mu(A \cap C) + \mu(A \backslash C)$$

and hence every closed set C is μ-measurable. Therefore every open set is μ-measurable by Proposition 1.8. Therefore μ is a Borel measure on X. □

De Morgan's laws express intersections and unions in terms of each other via negation.

LEMMA 1.28. *Let A be an arbitrary set. Let I be a set and $\{E_i\}_{i \in I}$ a family of sets. Then*

$$A \backslash \bigcup_{i \in I} E_i = \bigcap_{i \in I} (A \backslash E_i)$$

and

$$A \backslash \bigcap_{i \in I} E_i = \bigcup_{i \in I} (A \backslash E_i).$$

PROOF. Let $x \in A \backslash \bigcup_{i \in I} E_i$. Then $x \in A$ and $x \notin E_i \quad \forall i \in I$. Hence $x \in A \backslash E_i \quad \forall i \in I$. Therefore $x \in \bigcap_{i \in I} (A \backslash E_i)$ and $A \backslash \bigcup_{i \in I} E_i \subseteq \bigcap_{i \in I} (A \backslash E_i)$. Let $x \in \bigcap_{i \in I} (A \backslash E_i)$. Then $x \in A \backslash E_i \quad \forall i \in I$ and therefore $x \notin E_i \quad \forall i \in I$. Hence $x \in A \backslash \bigcup_{i \in I} E_i$ and $\bigcap_{i \in I} (A \backslash E_i) \subseteq A \backslash \bigcup_{i \in I} E_i$. We conclude $A \backslash \bigcup_{i \in I} E_i = \bigcap_{i \in I} (A \backslash E_i)$.

Let $x \in A \backslash \bigcap_{i \in I} E_i$. Therefore $x \notin \bigcap_{i \in I} E_i$ so then there exists $i \in I$ so that $x \notin E_i$. Hence $x \in A \backslash E_i$. Therefore $x \in \bigcup_{i \in I} (A \backslash E_i)$ and $A \backslash \bigcap_{i \in I} E_i \subseteq \bigcup_{i \in I} (A \backslash E_i)$. Let $x \in \bigcup_{i \in I} (A \backslash E_i)$. We choose $i_0 \in I$ such that $x \in A \backslash E_{i_0}$. Since $x \notin E_{i_0}$, $x \notin \bigcap_{i \in I} E_i$, and thus $x \in A \backslash \bigcap_{i \in I} E_i$. Hence $\bigcup_{i \in I} (A \backslash E_i) \subseteq A \backslash \bigcap_{i \in I} E_i$. We conclude $A \backslash \bigcap_{i \in I} E_i = \bigcup_{i \in I} (A \backslash E_i)$. □

Next we define the convergence of a sequence of functions.

DEFINITION 1.29. Let $E \subset \mathbb{R}^n$. Sequence of functions $f_n : E \to \mathbb{R}$ *converges pointwise* on E to $f : E \to \mathbb{R}$ if for every $x \in E$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

DEFINITION 1.30. Let $E \subset \mathbb{R}^n$. The sequence $f_n : E \to \mathbb{R}$ *converges uniformly* on E if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all $n \geq N$ and for all $x \in E$.

Notice that if the sequence converges uniformly then it converges pointwise. The other direction is not necessarily true. Egorov's theorem shows that pointwise convergence is uniform on a large set.

THEOREM 1.31. *(Egorov's theorem) Let μ e a measure on a set X with $\mu(X) < \infty$. Let $\mathcal{A} \subseteq \mathscr{M}(\mu)$ be a σ-algebra and $f_k : X \to \mathbb{R}$ a sequence of $\mathcal{A}$-measurable functions pointwise converging to a $\mathcal{A}$-measurable $f : X \to \mathbb{R}$. Then, for every $\epsilon > 0$ there is $A \in \mathcal{A}$ such that $\mu(X\backslash A) < \varepsilon$ and f_k converge to f uniformly on A.*

PROOF. Let $k \in \mathbb{N}\backslash\{0\}$. When $k \to \infty$, then $\frac{1}{k} \to 0$. The pointwise convergence implies

$$X = \bigcup_{N\in\mathbb{N}} \bigcap_{n>N} E_{n,\frac{1}{k}} \quad \forall k \in \mathbb{N}\backslash\{0\},$$

where $E_{n,\frac{1}{k}} = \{x : |f_n(x) - f(x)| < \frac{1}{k}\}$. Notice that $E_{n,\frac{1}{k}} \in \mathcal{A}$ for all k and all n. Since

$$\bigcap_{n>1} E_{n,\frac{1}{k}} \subset \bigcap_{n>2} E_{n,\frac{1}{k}} \subset \dots$$

Proposition 1.16 implies

$$\mu(E) = \lim_{N\to\infty} \mu\left(\bigcap_{n>N} E_{n,\frac{1}{k}}\right).$$

As $\mu(E) < \infty$, Proposition 1.15 and Definition 1.29 implies

$$\mu\left(X\backslash \bigcap_{n>N} E_{n,\frac{1}{k}}\right) = \mu(X) - \mu\left(\bigcap_{n>N} E_{n,\frac{1}{k}}\right) \to 0,$$

when $N \to \infty$. Therefore, for all $k \in \mathbb{N}\backslash\{0\}$, we can choose $N_k \in \mathbb{N}$ such that

$$\mu\left(X\backslash \bigcap_{n>N_k} E_{n,\frac{1}{k}}\right) < \frac{\varepsilon}{2^k}.$$

Let $E_k = \bigcap_{n>N_k} E_{n,\frac{1}{k}}$. Lemma 1.28 implies

$$E\backslash \bigcap_{k\geq 1} E_k = \bigcup_{k\geq 1} E\backslash E_k.$$

It follows that

$$\mu\left(E\backslash \bigcap_{k\geq 1} E_k\right) \leq \sum_{k\geq 1} (E\backslash E_k) \leq \sum_{k\geq 1} \frac{\varepsilon}{2^k} = \varepsilon.$$

Let $A = \bigcap_{k \geq 1} E_k$. The set A is measurable and $\mu(X \backslash A) < \varepsilon$. Given any m, we have that $A = \bigcap_{k \geq 1} E_k \subset E_m$. Therefore, for all $n > N_m$ and all $x \in A$ we have

$$|f_n(x) - f(x)| < \frac{1}{m}$$

That is, $f_n \to f$ uniformly on A, as in Definition 1.30. $\square$

CHAPTER 2

Hausdorff measure

In this chapter we will define Hausdorff measure as a generalization for area and volume. Later we define Hausdorff dimensions using Hausdorff measures.

DEFINITION 2.1. Let $U \subset \mathbb{R}^n, U \neq \emptyset$. The *diameter* of U is

$$|U| = \sup\{|x-y| : x, y \in U\}.$$

DEFINITION 2.2. A δ*-cover* for F is a countable collection $\{U_i\}_{i\in\mathbb{N}}$ of sets so that $0 \leq |U_i| \leq \delta$ for each i and

$$F \subset \bigcup_{i=1}^{\infty} U_i.$$

Next we define Hausdorff measures using δ-covers.

DEFINITION 2.3. For $F \subset \mathbb{R}^n, s \geq 0$ and $\delta > 0$, let

$$\mathscr{H}^s_\delta(F) = \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i\in\mathbb{N}} \quad \text{is a } \delta\text{-cover of F}\Big\}.$$

The *s-dimensional Hausdorff measure* of F is

$$\mathscr{H}^s(F) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(F).$$

The limit $\lim_{\delta\to 0} \mathscr{H}^s_\delta(F)$ exists since, when $\delta \to 0$, the class of valid coverings is decreasing and thus the infimum is increasing. Moreover $\mathscr{H}^s$ is a Borel measure on $\mathbb{R}^n$ as we will see.

THEOREM 2.4. *$\mathscr{H}^s_\delta$ is a measure on $\mathbb{R}^n$ for every $\delta > 0$.*

PROOF. (1) Clearly $\mathscr{H}^s_\delta(\emptyset) = 0$.

(2) Let $A \subset B \subset \mathbb{R}^n$. If $\{U_i\}_{i\in\mathbb{N}}$ is a δ-cover of B, then $\{U_i\}_{i\in\mathbb{N}}$ is a δ-cover of A. Therefore

$$\mathscr{H}^s_\delta(A) \leq \mathscr{H}^s_\delta(B).$$

(3) Let $E = \bigcup_{i=1}^{\infty} E_i \subset \mathbb{R}^n$. Let $\varepsilon > 0$. For every E_i there is a δ-cover $\{U_{ij}\}_{j\in\mathbb{N}}$ such that

$$\sum_{j=1}^{\infty} |U_{ij}|^s \leq \mathscr{H}^s_\delta(E_i) + \frac{\varepsilon}{2^i}.$$

Therefore $\{U_{ij}\}_{i,\in\mathbb{N},j\in\mathbb{N}}$ is a δ-cover of E. Thus

$$\begin{aligned}\mathscr{H}^s_\delta(E) &\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|U_{ij}|^s \\ &\le \sum_{i=1}^{\infty}\left(\mathscr{H}^s_\delta(E_i)+\frac{\varepsilon}{2^i}\right) \\ &\le \sum_{i=1}^{\infty}\mathscr{H}^s_\delta(E_i)+\varepsilon.\end{aligned}$$

As $\varepsilon \to 0$, then

$$\mathscr{H}^s_\delta\left(\bigcup_{i=1}^{\infty}E_i\right) \le \sum_{i=1}^{\infty}\mathscr{H}^s_\delta(E_i).$$

By Definition 1.4, $\mathscr{H}^s_\delta$ is a measure on $\mathbb{R}^n$, for every $\delta > 0$. $\square$

THEOREM 2.5. *The s-dimensional Hausdorff measure $\mathscr{H}^s$ is a Borel measure on $\mathbb{R}^n$.*

PROOF. First, we check that $\mathscr{H}^s$ is a measure as in Definition 1.4.
(1) Clearly $\mathscr{H}^s(\emptyset) = \lim_{\delta\to 0}\mathscr{H}^s_\delta(\emptyset) = \lim_{\delta\to 0} 0 = 0$.
(2) By Theorem 2.4 (2) if $A \subset B \subset \mathbb{R}^n$ and when $\delta \to 0$, then

$$\mathscr{H}^s(A) \le \mathscr{H}^s(B).$$

(3) If $E \subset \bigcup_{i=1}^{\infty} E_i$ by Theorem 2.4

$$\mathscr{H}^s_\delta(E) \le \sum_{i=1}^{\infty}\mathscr{H}^s_\delta(E_i).$$

Since $\mathscr{H}^s_\delta$ is monotone increasing when $\delta \to 0$, then $\mathscr{H}^s(E) \le \sum_{i=1}^{\infty}\mathscr{H}^s(E_i)$. Hence by Definition 1.4 $\mathscr{H}^s$ is a measure on $\mathbb{R}^n$.

Next, we show that $\mathscr{H}^s$ is a Borel measure using Theorem 1.27. Let $A_1, A_2 \subset \mathbb{R}^n$ such that $\operatorname{dist}(A_1, A_2) > 0$. Since $\mathscr{H}^s$ is a measure it holds that

$$\mathscr{H}^s(A_1 \cup A_2) \le \mathscr{H}^s(A_1) + \mathscr{H}^s(A_2).$$

Let $\delta < \operatorname{dist}(A_1, A_2)/3$. Let $\{E_i\}_{i\in\mathbb{N}}$ be a δ-cover of $A_1 \cup A_2$ such that

$$\sum_{i=1}^{\infty}|E_i|^s \le \mathscr{H}^s_\delta(A_1 \cup A_2) + \varepsilon.$$

Since $\delta < \operatorname{dist}(A_1, A_2)/3$ each E_i intersects at most one of A_1 or A_2. Those that intersect A_j form a δ-cover of A_j. Thus, for $j \in \{1, 2\}$, we define

$$E_i^j = \begin{cases} E_i & \text{if} \quad E_i \cap A_j \neq \emptyset \\ \emptyset & \text{if} \quad E_i \cap A_j = \emptyset \end{cases},$$

so that

$$\{E_i\}_{i\in\mathbb{N}} = \bigsqcup_{j=1}^{2}\{E_i^j\}_{i\in\mathbb{N}}, \quad A_j \subset \bigcup_{i=1}^{\infty}E_i^j,$$

Hence

$$\begin{aligned}\sum_{j=1}^{2} \mathscr{H}^s_\delta(A_j) &\le \sum_{j=1}^{2}\sum_{i=1}^{\infty} |E_i^j|^s \\ &= \sum_{i=1}^{\infty} |E_i|^s \\ &\le \mathscr{H}^s_\delta(A_1 \cup A_2) + \varepsilon.\end{aligned}$$

When $\varepsilon \to 0$ and $\delta \to 0$, then

$$\mathscr{H}^s(A_1 \cup A_2) \ge \mathscr{H}^s(A_1) + \mathscr{H}^s(A_2).$$

Therefore $\mathscr{H}^s(A_1 \cup A_2) = \mathscr{H}^s(A_1) + \mathscr{H}^s(A_2)$ and $\mathscr{H}^s$ is a Borel measure on $\mathbb{R}^n$. $\square$

Hausdorff measure has a useful scaling property which we prove next.

THEOREM 2.6. *If $F \subset \mathbb{R}^n$ and $\lambda > 0$ then*

$$\mathscr{H}^s(\lambda F) = \lambda^s \mathscr{H}^s(F),$$

where $\lambda F = \{\lambda x : x \in F\}$.

PROOF. If $\{U_i\}_{i\in\mathbb{N}}$ is a δ-cover of F then $\{\lambda U_i\}_{i\in\mathbb{N}}$ is a $\lambda\delta$-cover of λF. Hence

$$\begin{aligned}\mathscr{H}^s_{\lambda\delta}(\lambda F) &\le \sum |\lambda U_i|^s \\ &= \lambda^s \sum |U_i|^s.\end{aligned}$$

By taking the infimum of the δ-covers $\{U_i\}$ of F we get $\mathscr{H}^s_{\lambda\delta}(\lambda F) \le \lambda^s \mathscr{H}^s_\delta(F)$. When $\delta \to 0$ then $\mathscr{H}^s(\lambda F) \le \lambda^s \mathscr{H}^s(F)$. We get the opposite inequality by replacing λ by $\frac{1}{\lambda}$ and F by λF:

$$\mathscr{H}^s(\lambda F) \le \lambda^s \mathscr{H}^s(F) = \lambda^s \mathscr{H}^s\left(\frac{1}{\lambda}\lambda F\right) \le \lambda^s \frac{1}{\lambda^s} \mathscr{H}^s(\lambda F) = \mathscr{H}^s(\lambda F).$$

Therefore, $\mathscr{H}^s(\lambda F) = \lambda^s \mathscr{H}^s(F)$. $\square$

The next results give similar estimates for more general transformations.

DEFINITION 2.7. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ and $\alpha > 0$. A function $f : X \to Y$ is called *α-Hölder* if there is $c \in \mathbb{R}$ such that

$$|f(x) - f(y)| \le c|x - y|^\alpha$$

where $x, y \in X$. The function f is called a *Lipschitz function* if $\alpha = 1$.

DEFINITION 2.8. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be metric spaces. A function $f : X \to Y$ is called *bi-Lipschitz function* if there are $0 < c_1 \le c_2 < \infty$ such that

$$c_1|x - y| \le |f(x) - f(y)| \le c_2|x - y|$$

where $x, y \in X$.

PROPOSITION 2.9. *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be an α-Hölder function. Then, for each $s > 0$,*

$$\mathscr{H}^{s/\alpha}(f(F)) \le c^{s/\alpha} \mathscr{H}^s(F),$$

where c is the constant from Definition 2.7.

PROOF. Let $\{U_i\}_{i\in\mathbb{N}}$ be a δ-cover of F. Since f is a α-Hölder function $|f(F\cap U_i)| \leq c|U_i|^\alpha$. It follows that $\{f(F\cap U_i)\}_{i\in N}$ is an $c\delta^\alpha$-cover of $f(F)$. Moreover,

$$\begin{aligned}\sum_i |f(F\cap U_i)|^{s/\alpha} &\leq \sum_i (c|U_i|^\alpha)^{s/\alpha} \\ &= c^{s/\alpha}\sum_i |U_i|^s.\end{aligned}$$

By taking the infimum over δ-covers of F we get $\mathscr{H}^{s/\alpha}_{c\delta^\alpha}(f(F)) \leq c^{s/\alpha}\mathscr{H}^s_\delta(F)$. When $\delta \to 0$ then $c\delta^\alpha \to 0$ and thus

$$\mathscr{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathscr{H}^s(F).$$

□

REMARK 2.10. Notice that if f is a Lipschitz mapping ($\alpha = 1$) then

$$\mathscr{H}^s(f(F)) \leq c^s\mathscr{H}^s(F).$$

CHAPTER 3

Hausdorff dimension

In this chapter we define Hausdorff dimension using Hausdorff measures.

DEFINITION 3.1. The *Hausdorff dimension* of $F \subseteq \mathbb{R}^n$ is

$$\dim_H(F) = \inf\{s : \mathscr{H}^s(F) = 0\} = \sup\{s : \mathscr{H}^s(F) = \infty\}$$

so that

$$\mathscr{H}^s(F) = \begin{cases} \infty & \text{if} \quad s < \dim_H F \\ 0 & \text{if} \quad s > \dim_H F. \end{cases}$$

If $s = \dim_H(F)$, then $0 \leq \mathscr{H}^s(F) \leq \infty$.

DEFINITION 3.2. A Borel set F with $0 < \mathscr{H}^s(F) < \infty$ is called a *s-set*.

We obtain some estimates for Hausdorff dimension.

PROPOSITION 3.3. *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a α-Hölder function. Then*

$$\dim_H f(F) \leq \frac{1}{\alpha} \dim_H F.$$

PROOF. If $s > \dim_H F$ then $\mathscr{H}^s(F) = 0$ and by proposition 2.9

$$\mathscr{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathscr{H}^s(F) = 0.$$

Hence $\dim_H f(F) \leq s/\alpha$ for all $s > \dim_H F$, so then

$$\dim_H f(F) \leq (1/\alpha)\dim_H F.$$

□

PROPOSITION 3.4. *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a function.*

(1) *If $f : F \to \mathbb{R}^m$ is a Lipschitz function then $\dim_H f(F) \leq \dim_H F$.*
(2) *If $f : F \to \mathbb{R}^m$ is a bi-Lipschitz function then $\dim_H f(F) = \dim_H F$.*

PROOF. (1) Proposition 3.3 with $\alpha = 1$ implies that $\dim_H f(F) \leq \dim_H F$.
(2) Since f is bi-Lipschitz the function $f^{-1} : f(F) \to F$ is Lipschitz. Applying (1) to f^{-1}, we get $\dim_H f(F) \geq \dim_H F$ so then $\dim_H f(F) = \dim_H F$. □

3.1. Alternative definitions of dimension

The Hausdorff dimension defined in the previous section is the main notion of dimension we are considering in this thesis. However, in some proofs, alternative definitions of dimension can be useful. Intuitively the Box-counting dimension can be obtained by placing the set on an evenly spaced grid and then counting, how many boxes it takes to cover the set, when the grid is getting finer and finer. Later we will see that the boxes can actually be replaced with other shapes.

We will denote by $\overline{\lim}$ and $\underline{\lim}$ the upper (limsup) and the lower (liminf) limit, respectively.

DEFINITION 3.5. Let $F \subset \mathbb{R}^n$, $F \neq \emptyset$ and bounded. Let $N_\delta(F)$ be the smallest number of sets of diameter at most δ that can cover the set F. The *lower* and *upper box-counting dimensions* of F are

$$\underline{\dim}_B F = \varliminf_{\delta \to 0^+} \frac{\log N_\delta F}{-\log \delta}$$

and

$$\overline{\dim}_B F = \varlimsup_{\delta \to 0^+} \frac{\log N_\delta F}{-\log \delta}.$$

If $\underline{\dim}_B F = \overline{\dim}_B F$, then the *box-counting dimension* of F is

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta F}{-\log \delta}.$$

In Definition 3.5, we can take as $N_\delta(F)$ any iof the following numbers:

(1) the smallest number of closed balls of radius δ that cover F;
(2) the smallest number of cubes of side δ that cover F;
(3) the number of δ-mesh cubes that intersect F;
(4) the smallest number of sets of diameter at most δ that cover F;
(5) the largest number of disjoint balls of radius δ with centres in F.

The resulting box-counting dimensions will be the same (see [2, Equivalent definitions 3.1] for details).

REMARK 3.6. In the definition of lower and upper box-counting dimensions, it is enough to consider limits as $\delta \to 0$ along certain decreasing sequences δ_k. For details see [2, Equivalent definitions 3.1].

CHAPTER 4

Estimates for Hausdorff dimension using potential theoretical methods

In this chapter we consider more estimates for Hausdorff measures and dimensions. First we introduce an important covering lemma.

LEMMA 4.1. *Let $\mathcal{C}$ be a collection of balls with non-negative radius contained in a bounded region of $\mathbb{R}^n$. Then there is a finite or countable disjoint subcollection $\{B_i\}_{i\in\mathbb{N}}$ such that*

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \tilde{B}_i,$$

where $\tilde{B}$ is the closed ball with four times radius of B and the same center of B.

PROOF. We define sequences $\{\mathcal{S}_k\}_k$ and $\{\mathcal{C}_k\}_k$ of subfamilies of $\mathcal{C}$, and a sequence $\{R_k\}_k$ of non-negative numbers by induction. So, for $k = 0$ we set $\mathcal{S}_0 = \emptyset$, $\mathcal{C}_0 = \mathcal{C}$ and $R_0 = \sup\{r(B) : B \in \mathcal{C}\}$. If $\mathcal{S}_k$, $\mathcal{C}_k$ and R_k are given then we have two cases: first, there exists $\hat{B} \in \mathcal{C}_k$ such that $r(\hat{B}) > \frac{R_k}{2}$. Hence we set $\mathcal{S}_{k+1} = \mathcal{S}_k \cup \{\hat{B}\}$, $\mathcal{C}_{k+1} = \{B \in \mathcal{C}_k, B \cap \hat{B} = \emptyset\}$ and $R_{k+1} = \sup\{r(B) : B \in \mathcal{C}_{k+1}\}$; second there is no such $\hat{B}$, hence we set $\mathcal{S}_{k+1} = \mathcal{S}_k$, $\mathcal{C}_{k+1} = \mathcal{C}_k$, $R_{k+1} = R_k$. Notice that $\bigcap_k \mathcal{C}_k = \emptyset$. Next, we claim that $\mathcal{S} = \bigcup_k \mathcal{S}_k$ is the desired family of balls. First, notice that $\mathcal{S}$ is finite or countable by construction. Second, notice that balls in $\mathcal{C}_k$ have empty intersection with balls in $\mathcal{S}_k$ and that the ball $\hat{B}$ for the next step is chosen from $\mathcal{C}_k$: hence, balls in $\mathcal{S}$ are pairwise disjoint. Third, we show that $\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \tilde{B}_i$. Let $x \in \bigcup_{B\in\mathcal{C}} B$: then there is $B' \in \mathcal{C}$ containing x. Let $k' = \sup\{k, B' \in \mathcal{C}_k\}$. Since $\bigcap_k \mathcal{C}_k = \emptyset$, then $k' < \infty$. Since $B' \in \mathcal{C}_{k'}$, then $R_{k'} > 0$ and there is $\hat{B} \in \mathcal{S}_{k'+1} \backslash \mathcal{S}_k$. Since $B' \in \mathcal{C}_{k'} \backslash \mathcal{C}_{k'+1}$, then $B' \cap \hat{B} \neq \emptyset$. Moreover, $\frac{R_{k'}}{2} \leq r(\hat{B})$ and $r(B') \leq R_{k'}$. So, if $z \in B' \cap \hat{B}$,

$$\begin{aligned} |x - x(\hat{B})| &\leq |x - z| + |z - x(\hat{B})| \\ &\leq r(B') + r(\hat{B}) \\ &\leq 3r(\hat{B}). \end{aligned}$$

Therefore, $x \in \tilde{\hat{B}}$. □

From this moment on we often use mass distributions instead of measures. In physics and mechanics a mass distribution is the distribution of mass in a volume. This is the intuitive way of understanding the concept. We define mass distributions the following way.

DEFINITION 4.2. The *support* of a measure μ on $\mathbb{R}^n$, $\mathrm{supp}(\mu)$, is the smallest closed set X such that $\mu(\mathbb{R}^n \backslash X) = 0$.

Definition 4.3. A *mass distribution* on a set E is a measure μ with $\mathrm{supp}(\mu) \subset E$ and $0 < \mu(E) < \infty$.

Next we find different estimates for Hausdorff dimensions.

Proposition 4.4. *Let μ be a mass distribution on $\mathbb{R}^n$. Let $F \subset \mathbb{R}^n$ be a Borel set and $0 < c < \infty$.*

(1) *If $\overline{\lim}_{r\to 0} \frac{\mu(B_r(x))}{r^s} < c$ for all $x \in F$ then $\mathscr{H}^s(F) \geq \frac{\mu(F)}{c}$.*
(2) *If $\overline{\lim}_{r\to 0} \frac{\mu(B_r(x))}{r^s} > c$ for all $x \in F$ then $\mathscr{H}^s(F) \leq \frac{8^s \mu(\mathbb{R}^n)}{c}$.*

Proof. (1) For each $\delta > 0$ let

$$F_\delta = \{x \in F : \forall r \in (0, \delta], \quad \mu(B_r(x)) < cr^s\}.$$

Let $\{U_i\}_{i\in\mathbb{N}}$ be a δ-cover of F. Then $\{U_i\}_{i\in\mathbb{N}}$ is a δ-cover of F_δ. For each U_i such that $U_i \cap F_\delta \neq \emptyset$ it holds that $U_i \subset B_{|U_i|}(x)$ for any $x \in U_i \cap F_\delta$. Since $|U_i| < \delta$, the definition of F_δ gives

$$\mu(U_i) \leq \mu(B_{|U_i|}(x)) < c|U_i|^s.$$

Therefore

$$\mu(F_\delta) \leq \sum_{U_i \cap F_\delta \neq \emptyset} \mu(U_i) \leq \sum_{U_i \cap F_\delta \neq \emptyset} c|U_i|^s \leq c \sum_{i=1}^{\infty} |U_i|.$$

Since $\{U_i\}_{i\in\mathbb{N}}$ is any δ-cover of F, taking infimum over all δ-covers implies

$$\mu(F_\delta) \leq c\mathscr{H}^s_\delta(F) \leq c\mathscr{H}^s(F).$$

When $\delta \to 0$, F_δ increases to F and Corollary 1.17 implies $\mu(F) \leq c\mathscr{H}^s(F)$. Therefore, $\mathscr{H}^s(F) \geq \frac{\mu(F)}{c}$.

(2) Let F be bounded and $\delta > 0$ fixed. Let $\mathcal{C}$ be the collection of balls

$$\{B_r(x) : x \in F, 0 < r \leq \delta \quad \text{and} \quad \mu(B_r(x)) > cr^s\}.$$

Then $F \subset \bigcup_{B\in\mathcal{C}} B$. Lemma 4.1 implies that there is a sequence of disjoint balls $B_i \in \mathcal{C}$ such that $\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \tilde{B}_i$, where $\tilde{B}$ is the closed ball with four times radius of B and the same center of B. Hence $\{\tilde{B}_i\}$ is an 8δ-cover of F. Theorem 2.6 and the definition of $\mathcal{C}$ implies

$$\begin{aligned} \mathscr{H}^s_{8\delta}(F) &\leq \sum_i |\tilde{B}_i|^s = 4^s \sum_i |B_i|^s \\ &\leq \frac{8^s}{c} \sum_i \mu(B_i) \leq \frac{8^s}{c} \mu(\mathbb{R}^n). \end{aligned}$$

Taking the infimum of δ-covers we get

$$\mathscr{H}^s(F) \leq \frac{8^s}{c} \mu(\mathbb{R}^n).$$

□

Theorem 4.5. *Let $F \subset \mathbb{R}^n$ be a Borel set with $\mathscr{H}^s(F) = \infty$. Then there is a compact set $E \subset F$ so that $0 < \mathscr{H}^s(E) < \infty$.*

Proof. Proof is omitted, see [2, Theorem 4.10]. □

PROPOSITION 4.6. *Let F be a Borel set with $0 < \mathscr{H}^s(F) \leq \infty$. Then there exists a compact set $E \subset F$ with $0 < \mathscr{H}^s(E) < \infty$ and $b \in \mathbb{R}$ so that*

$$\mathscr{H}^s(E \cap B_r(x)) \leq br^s \tag{4.1}$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$.

PROOF. If $\mathscr{H}^s(F) = \infty$ then Theorem 4.5 implies that there is a compact set $G \subset F$ so that $0 < \mathscr{H}^s(G) < \infty$. Applying the following result on G, we can assume $0 < \mathscr{H}^s(F) < \infty$. Let

$$F' = \{x \in \mathbb{R}^n : \overline{\lim_{r\to 0}} \frac{\mathscr{H}^s(F \cap B_r(x))}{r^s} > 8^{s+1}\}.$$

If $\mathscr{H}^s(F) < \infty$ then by Proposition 4.4 (2)

$$\mathscr{H}^s(F') \leq \frac{8^s}{8^{s+1}} \mathscr{H}^s(F) \leq \frac{1}{8} \mathscr{H}^s(F).$$

Thus

$$\mathscr{H}^s(F \backslash F') \geq \frac{7}{8} \mathscr{H}^s(F) > 0.$$

Let $E' = F \backslash F'$. Then $\mathscr{H}^s(E') > 0$ and for all $x \in E'$

$$f_\infty(x) = \overline{\lim_{r\to 0}} \frac{\mathscr{H}^s(F \cap B_r(x))}{r^s} \leq 8^{s+1}.$$

Let

$$f_n(x) = \sup\left\{ \frac{\mathscr{H}^s(F \cap B_r(x))}{r^s}, r \in \left(0, \frac{1}{n}\right] \right\}.$$

For all $x \in E'$, we have $f_n(x) \searrow f_\infty(x) \leq 8^{s+1}$. Theorem 1.31 implies that there is $E \subset E'$ such that $f_n(x) \searrow f_\infty(x)$ uniformly on E and $\mathscr{H}^s(E) > 0$. Thus there exists $N \in \mathbb{N}$ such that for all $n > N$ and $x \in E$

$$|f_n(x) - f_\infty(x)| < 8^{s+1}.$$

Thus

$$f_n(x) \leq f_\infty(x) + |f_n(x) - f_\infty(x)| < 2 \cdot 8^{s+1}.$$

Therefore for all $r \leq \frac{1}{n}$, we have $\mathscr{H}^s(E \cap B_r(x)) \leq 2 \cdot 8^{s+1} r^s$. By Theorem 1.31 it follows that there is a compact set $E \subset E'$ with $\mathscr{H}^s(E) > 0$ and a number $r_0 > 0$ such that

$$\frac{\mathscr{H}^s(F \cap B_r(x))}{r^s} \leq 2 \cdot 8^{s+1}$$

for all $x \in E$ and all $0 < r \leq r_0$. However, if $r \geq r_0$, we have

$$\frac{\mathscr{H}^s(F \cap B_r(x))}{r^s} \leq \frac{\mathscr{H}^s(F)}{r_0^s}.$$

Therefore, (4.1) holds for all $r > 0$. □

Using potential theoretical methods, we are able to obtain more estimates for Hausdorff measures. Newton's gravitational law states that each mass particle attracts every other particle with force F such that

$$F = -G \frac{mM}{r^2} \mathbf{u},$$

where G is a constant, r is a distance between masses m and M and $\mathbf{u}$ is a unit vector pointing from the mass M to mass m. This law can be applied only to point particles. However, if we replace the particle of mass M with a body $\subseteq \mathbb{R}^3$ with mass density ρ and assume that the gravitational force field is a linear field, the force F applied to a mass m at x can be written

$$F = Gm \int_V \frac{\mathbf{u}(x-y)}{|x-y|^2} \rho(y) dy,$$

where $\mathbf{u} = \frac{v}{|v|}$. Dividing F by the mass m we get the gravitational field vector g as

$$g(x) = G \int_V \frac{\mathbf{u}(x-y)}{|x-y|^2} \rho(y) dy. \tag{4.2}$$

The magnitude of g is better known as the gravitational acceleration constant (at the surface of earth $g \approx 9,81 m/s^2$). The vector g can be represented as

$$g = -\nabla \Phi$$

and Φ is the gravitational potential

$$\Phi = G \int_V \frac{1}{|x-y|} \rho(y) dy.$$

In our case we consider s-dimensional potentials and next we will define s-potentials.

Definition 4.7. Let $s \geq 0$ and μ be a mass distribution on $\mathbb{R}^n$. The s-*potential* of μ at a point $x \in \mathbb{R}^n$ is

$$\phi_s^\mu(x) = \int \frac{d\mu(y)}{|x-y|^s}.$$

Let dW' be the work per unit mass that has to be done by an outside force on a body in a gravitational field to move the body a distance dr. The work done on the body per unit mass is

$$dW' = -g \cdot dr = (\nabla \Phi) \cdot dr = d\Phi.$$

According to this we define s-energy as it follows.

Definition 4.8. Let $s \geq 0$ and μ be a mass distribution on $\mathbb{R}^n$. The s-*energy* of μ is

$$I_s(\mu) = \int \phi_s^\mu(x) d\mu(x) = \int\int \frac{d\mu(x) d\mu(y)}{|x-y|^s}.$$

Definition 4.9. Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be measurable spaces. A function $f : X \to Y$ is *measurable function* if

$$f^{-1}(N) \in \mathscr{M}$$

for every $N \in \mathscr{N}$.

Lemma 4.10. *Let μ be a measure on X. Let $f : X \to [0, \infty]$ μ-measurable and $r > 0$. Then*

$$\int_X f(x) d\mu(x) = \int_0^\infty \mu(\{x : f(x) \geq r\}) \, dr.$$

Proof. The proof is omitted, see [7, Theorem 7.6]. $\square$

THEOREM 4.11. *Let $F \subset \mathbb{R}^n$.*

(1) *If there exists a mass distribution μ on F with $I_s(\mu) < \infty$ then $\mathscr{H}^s(F) = \infty$ and $dim_H(F) \geq s$.*
(2) *If F is a Borel set with $\mathscr{H}^s(F) > 0$ then there exists a mass distribution μ on F with $I_t(\mu) < \infty$ for all $t < s$.*

PROOF. (1) Let

$$F' = \{x \in F : \overline{\lim_{r\to 0}} \frac{\mu(B_r(x))}{r^s} > 0\}$$

and fix $x \in F'$. Then, there is $\varepsilon > 0$ and a decreasing sequence $r_i \to 0$ such that

$$\mu(B_{r_i}(x)) \geq \varepsilon r_i^s$$

for all i. Notice that, if $\mu(\{x\}) > 0$, then $I_s(\mu) = \infty$; therefore $\lim_{r\to\infty} \mu(B_r(x)) = \mu(\{x\}) = 0$. In particular, for every $i \in \mathbb{N}$ there is $0 < q_i < r_i$ such that $\mu(B_{q_i}(x)) < \frac{\varepsilon r_i^s}{2}$. Set $A_i = B_{r_i}(x) \backslash B_{q_i}(x)$. Proposition 1.15 implies

$$\mu(A_i) = \mu(B_{r_i}(x)) - \mu(B_{q_i}(x)) \geq \frac{1}{2}\varepsilon r_i^s,$$

where $i = 1, 2, \ldots$. Taking subsequence we have $r_{i+1} < q_i$ for all i so that the A_i are disjoint and centered on x. Hence for $x \in F'$

$$\begin{aligned}\phi_s^\mu(x) &= \int \frac{d\mu(y)}{|x-y|^s} \\ &\geq \sum_{i=1}^\infty \int_{A_i} \frac{d\mu(y)}{|x-y|^s} \\ &\geq \sum_{i=1}^\infty \frac{1}{2}\varepsilon r_i^s r_i^{-s} = \infty,\end{aligned}$$

since $|x-y|^{-s} \geq r_i^{-s}$ on A_i. Since, $I_s(\mu) = \int \phi_s^\mu(x) d\mu(x) < \infty$, so $\phi_s^\mu(x) < \infty$ for μ-almost every x. We conclude that $\mu(F') = 0$. If $x \in F\backslash F'$, then $\overline{\lim}_{r\to 0} \frac{\mu(B_r(x))}{r^s} = 0$. Hence Proposition 4.4 (1) implies that, for all $c > 0$, we have

$$\mathscr{H}^s(F) \geq \mathscr{H}^s(F\backslash F') \geq \frac{\mu(F\backslash F')}{c} = \frac{\mu(F) - \mu(F')}{c} = \frac{\mu(F)}{c}.$$

Hence $\mathscr{H}^s(F) = \infty$.

(2) Proposition 4.6 implies that there exist a compact set $E \subset F$ with $0 < \mathscr{H}^s(E) < \infty$ and $r > 0$ such that, for all $r > 0$ and all $x \in \mathbb{R}^n$,

$$\mathscr{H}^s(E \cap B_r(x)) \leq br^s.$$

Let μ be the restriction of $\mathscr{H}^s$ to E: therefore, μ is a mass distribution on F. For a fixed $x \in \mathbb{R}^n$, let

$$m(r) = \mu(B_r(x)) = \mathscr{H}^s(E \cap B_r(x)) \leq br^s.$$

Notice that $|x-y|^{-t} \geq r$ implies $|x-y| \leq r^{-\frac{1}{t}}$. Let $f : \mathbb{R} \to \{0,1\}, f(y) = |x-y|^{-t}\mathbf{1}_{B(x,1)}(y)$. Let $\alpha > 1$. If $0 \leq t < s$, then Lemma 4.10 implies

$$\phi_t^\mu(x) = \int \frac{1}{|x-y|^t} d\mu(y)$$

$$
\begin{aligned}
&= \int_0^\infty \mu\left(\left\{y : \frac{1}{|x-y|^t} \geq r\right\}\right) dr \\
&= \int_0^\infty \mu\left(\left\{y : |x-y|^t \leq r^{-1/t}\right\}\right) dr \\
&= \int_0^\infty \mu\left(B(x, r^{-1/t}\right) dr \\
&= \int_0^1 \mu\left(B(x, r^{-1/t}\right) dr + \int_1^\infty \mu\left(B(x, r^{-1/t}\right) dr \\
&\leq \mu(\mathbb{R}^n) + \int_1^\infty b r^{-s/t} dr \\
&= \mu(\mathbb{R}^n) + \frac{bt}{s-t},
\end{aligned}
$$

where the last step is possible, because $t < s$. Thus $\phi_t^\mu(x) < \mu(\mathbb{R}^n) + \frac{bt}{s-t} < \infty$ and

$$I_t(\mu) = \int \phi_t^\mu(x) d\mu(x) \leq \left(\mu(\mathbb{R}^n) + \frac{bt}{s-t}\right) \mu(\mathbb{R}^n) < \infty.$$

$\square$

CHAPTER 5

Dimension of Cartesian products

One way of obtaining new sets from given ones is to take a Cartesian product. In this chapter we introduce results on these Cartesian products and their dimensions.

DEFINITION 5.1. Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$. The *Cartesian product* $E \times F$ is defined as

$$E \times F = \{(x, y) \in \mathbb{R}^{n+m} : x \in E, y \in F\}.$$

To get more information about the mass spread along the set we need to define densities.

DEFINITION 5.2. The *lower* and *upper s-densities* of a set F at a point $x \in \mathbb{R}^n$ are defined as

$$\underline{D}^s(F, x) = \varliminf_{r \to 0} \frac{\mathscr{H}^s(F \cap B(x, r))}{(2r)^s}$$

and

$$\overline{D}^s(F, x) = \varlimsup_{r \to 0} \frac{\mathscr{H}^s(F \cap B(x, r))}{(2r)^s}.$$

If $\underline{D}^s(F, x) = \overline{D}^s(F, x)$ then the *s-density* of F at x exists and we denote it by $D^s(F, x)$.

Before we can have estimates for the dimension of Cartesian product we prove some estimates for densities. For this purpose, we introduce a well known Vitali's covering theorem.

DEFINITION 5.3. Let μ be a measure on X. The measure μ is *Borel regular* if it is Borel measure and if for every $A \subset X$ there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

PROPOSITION 5.4. *The Hausdorff measure $\mathscr{H}^s$ is Borel regular.*

PROOF. By Theorem 2.5 $\mathscr{H}^s$ is a Borel measure. If $\mathscr{H}^s(A) = +\infty$ then we choose $B = \mathbb{R}^n$. We assume that $\mathscr{H}^s(A) < \infty$. Therefore, $\mathscr{H}^s_\delta(A) < \infty$ for all $\delta > 0$. For every $i \in \mathbb{N}$, let $\{E^i_j\}_{j \in \mathbb{N}}$ be a covering of A by closed sets such that

$$\mathscr{H}^s_{1/i}(A) \le \sum_{j=1}^{\infty} |E_j|^s \le \mathscr{H}^s_{1/i}(A) + \frac{1}{i}.$$

Let $B_i = \bigcup_{j=1}^{\infty} E_j$ and $B = \bigcap_{i=1}^{\infty} B_i$. Hence $A \subseteq B$ and B is Borel. Moreover,

$$\begin{aligned}
\mathscr{H}^s(A) \le \mathscr{H}^s(B) &= \lim_{i \to \infty} \mathscr{H}^s_{i/1}(B) \\
&\le \lim_{i \to \infty} \mathscr{H}^s_{i/1}(B_i) \\
&\le \lim_{i \to \infty} \mathscr{H}^s_{1/i}(A) + \frac{1}{i} \\
&= \mathscr{H}^s(A).
\end{aligned}$$

Thus $\mathscr{H}^s(B) = \mathscr{H}^s(A)$. $\square$

DEFINITION 5.5. Let μ be a measure on X and $A \subset X$. The measure μ_A is the *restriction of* μ *to* A if $\mu_A = \mu(A \cap E)$ for every $E \subset X$. We denote the restriction of μ to A as $\mu \llcorner A$.

THEOREM 5.6. *Let μ be a measure on X and $A \subset X$. Let μ_A be a restriction of μ to A.*

(1) *Every μ-measurable set is μ_A-measurable.*
(2) *If μ is Borel regular and A is μ-measurable with $\mu(A) < \infty$, then μ_A is Borel regular.*

PROOF. (1) Let $M \in \mathscr{M}(X)$ and $E \subseteq X$. We have

$$\begin{aligned}\mu_A(E) &= \mu(A \cap E) = \mu(A \cap E \cap M) + \mu((A \cap E)\backslash M)\\ &= \mu_A(E \cap M) + \mu(A \cap (E\backslash M)) = \mu_A(E \cap M) + \mu_A(E\backslash M).\end{aligned}$$

Therefore, M is μ_A-measurable.

(2) By part (1), Borel sets are μ_A-measurable and so μ_A is Borel measurable. Let $C \subseteq X$. Then there is a Borel set B with $A \cap C \subseteq B$ and $\mu(A \cap C) = \mu(B)$. Notice that $\mu(A \cap C) = \mu(B) \geq \mu(A \cap B) \geq \mu(A \cap C)$, because $A \cap C \subseteq A \cap B$. Therefore, $\mu_A(C) = \mu_A(B)$. We conclude that μ_A is Borel regular. $\square$

THEOREM 5.7. *Let μ be a Borel regular measure on X, A a μ-measurable set and $\varepsilon > 0$.*

(1) *If $\mu(A) < \infty$, there is a closed set $C \subset A$ such that $\mu(A\backslash C) < \varepsilon$.*
(2) *If there are open sets $V_1, V_2, \dots$ such that $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu(V_i) < \infty$ for all i, then there exists open set V such that $A \subset V$ and $\mu(V\backslash A) < \varepsilon$.*

PROOF. (1) The proof is omitted here, but it can be found in [5, Theorem 1.2.6].

(2) Applying (1) to sets $V_i\backslash A$ we find closed sets $C_i \subset V_i\backslash A$ so that $\mu(V_i\backslash C_i)\backslash A) < \frac{\varepsilon}{2^i}$ for $i = 1, 2, \dots$. Then $A \subset V = \bigcup_i (V_i\backslash C_i)$. Now V is an open set with $\mu(V\backslash A) < \varepsilon$. $\square$

THEOREM 5.8 (Vitali's covering theorem). *Let μ be a Radon measure on $\mathbb{R}^n$ and $F \subset \mathbb{R}^n$. Let $\mathcal{B}$ be a family of closed balls such that each point of F is the centre of arbitrarily small balls of $\mathcal{B}$. Then there are disjoint balls $B_i \in \mathcal{B}$ such that*

$$\mu\left(F\backslash \bigcup_i B_i\right) = 0$$

PROOF. The proof is omitted, but it can be found in Mattila's book [11, Theorem 2.8]. $\square$

Next we will obtain the estimates for the densities mentioned earlier.

PROPOSITION 5.9. *Let F be a set so that $0 < \mathscr{H}^s(F) < \infty$. Then*

(1) $\underline{D}^s(F, x) = \overline{D}^s(F, x) = 0$ *for $\mathscr{H}^s$-almost all $x \notin F$.*
(2) $2^{-s} \leq \overline{D}^s(F, x) \leq 1$ *for $\mathscr{H}^s$-almost all $x \in F$.*

PROOF. (1) We show that for any $t > 0$ for the set

$$B = \{x \in \mathbb{R}^n \backslash F : \overline{D}^s(F, x) > t\}$$

it holds that $\mathscr{H}^s(B) = 0$. Let $\varepsilon > 0$. Since $(\mathscr{H}^s \llcorner F)(B) = 0$, Theorem 5.7 (2) implies that there is open set U such that $B \subset U$ and $\mathscr{H}^s(A \cap U) < \varepsilon$. For every $x \in B$ there is $r_x > 0$ so that $B(x, r_x) \subset U$ and

$$\mathscr{H}^s(F \cap B(x, r_x)) > t(2r_x)^s.$$

By Lemma 4.1 there are $x_1, x_2, \dots \in B$ such that the balls $B_i = B(x_i, r_{x_i})$ are disjoint and the balls $\tilde{B}_i$ with four times a radius of B_i cover B. Then

$$\begin{aligned} t\mathscr{H}^s_\infty(B) &\leq t\sum_i |\tilde{B}_i|^s = 4^s t \sum_i |B_i|^s \\ &< 4^s \sum_i \mathscr{H}^s(F \cap B_i) \leq 4^s \mathscr{H}^s(F \cap U) < 4^s \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we get $\mathscr{H}^s_\infty(B) = 0$. Therefore, it follows that $\mathscr{H}^s(B) = 0$.

(2) We first show the left-hand inequality. Let μ be a restriction of $\mathscr{H}^s$ to F. If

$$F_c = \left\{ x \in F : \overline{D}(F, x) = \overline{\lim_{r \to 0}} \frac{\mathscr{H}^s(F \cap B(x, r))}{(2r)^s} < \frac{c}{2^s} \right\}$$

for some constant c, then Proposition 4.4 (1) implies

$$\mathscr{H}^s(F_c) \geq \frac{\mathscr{H}^s(F_c)}{c}. \tag{5.1}$$

If $0 < c < 1$ then (5.1) holds only if $\mathscr{H}^s(F_c) = 0$. Thus for $\mathscr{H}^s$-almost all $x \in F$ we have $\overline{D}(F, x) \geq 2^{-s}$.

Now we show the right-hand inequality. We assume that the set F is a Borel set: we can do this assumption because the measure $\mathscr{H}^s$ is Borel regular by Proposition 5.4. Let $c > 1$ and

$$F_c = \{x \in F : \overline{D}(F, x) > c\}.$$

We show $\mathscr{H}^s(F_c) = 0$. Let $\varepsilon > 0$ and $\delta > 0$. Applying Theorem 5.7 to the measure μ we find an open set U such that $F_c \subset U$ and $\mathscr{H}^s(F \cap U) < \mathscr{H}^s(F_c) + \varepsilon$. Let $\mathscr{O}$ be the family of balls $\mathscr{O} = \{B(x, r) \subseteq U : x \in F_c, r < \delta/2, \mathscr{H}^s(F \cap B(x, r)) > c(2r)^s\}$. If $x \in F_c$, then there are arbitrarily small radius r such that $B(x, r) \in \mathscr{O}$. Therefore, we can apply Theorem 5.8 and obtain a countable family of balls $B_1, B_2, \dots$ in $\mathscr{O}$ that are pairwise disjoint such that $\mathscr{H}^s\left(F_c \backslash \bigcup_j B_j\right) = 0$. Thus

$$\begin{aligned} \mathscr{H}^s(F_c) + \varepsilon > \mathscr{H}^s(F \cap U) &\geq \sum \mathscr{H}^s(F \cap B_i) \\ &> c\sum_i |B_i|^s \geq c\mathscr{H}^s_\delta(F_c \cap \bigcup_i B_i) \\ &= c\mathscr{H}^s_\delta(F_c). \end{aligned}$$

If $\mathscr{H}^s(A) = 0$, then $\mathscr{H}^s_\delta(A) = 0$ for all $\delta > 0$. Therefore,

$$\begin{aligned}\mathscr{H}^s_\delta(F_c) &\leq \mathscr{H}^s_\delta\left(F_c \cap \bigcup_i B_i\right) + \mathscr{H}^s_\delta\left(F_c \backslash \bigcup_i B_i\right)\\ &= \mathscr{H}^s_\delta\left(F_c \cap \bigcup_i B_i\right) \leq \mathscr{H}^s_\delta(F_c).\end{aligned}$$

Hence $\mathscr{H}^s_\delta(F_c) = \mathscr{H}^s_\delta\left(F_c \cap \bigcup_i B_i\right)$. So when $\varepsilon \to 0$ and $\delta \to 0$ we get $\mathscr{H}^s(F_c) \geq c\mathscr{H}^s(F_c)$, which implies $\mathscr{H}^s(F_c) = 0$ for $c > 1$. Hence $\overline{D}(F, x) \leq 1$. □

Now we are able to obtain the desired results for the dimension of Cartesian product.

PROPOSITION 5.10. *If $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ are Borel sets with $\mathscr{H}^s(E), \mathscr{H}^t(F) < \infty$, then*

$$\mathscr{H}^{s+t}(E \times F) \geq c\mathscr{H}^s(E)\mathscr{H}^t(F), \tag{5.2}$$

where $c > 0$ depends only on s and t.

PROOF. We prove the claim for $n = m = 1$. Let $E, F \subset \mathbb{R}$. Then $E \times F \subset \mathbb{R}^2$. If $\mathscr{H}^s(E) = 0$ or $\mathscr{H}^t(F) = 0$ then (5.2) is trivial. We suppose that $0 < \mathscr{H}^s(E) < \infty$ and $0 < \mathscr{H}^t(F) < \infty$. We define a measure μ on $\mathbb{R}^2$ as follows: if $I, J \subset \mathbb{R}$, we define μ on $I \times J$ by

$$\mu(I \times J) = \mathscr{H}^s(E \cap I)\mathscr{H}^t(F \cap J).$$

It can be shown that this defines a mass distribution μ on $E \times F$ with $\mu(\mathbb{R}^2) = \mathscr{H}^s(E)\mathscr{H}^t(F)$. By the Proposition 5.9 we have that

$$\overline{\lim_{r\to 0}} \frac{\mathscr{H}^s(E \cap B(x, r))}{(2r)^s} \leq 1 \tag{5.3}$$

for $\mathscr{H}^s$-almost all $x \in E$ and

$$\overline{\lim_{r\to 0}} \frac{\mathscr{H}^t(F \cap B(y, r))}{(2r)^t} \leq 1 \tag{5.4}$$

for $\mathscr{H}^t$-almost all $y \in F$. By the definition of μ, (5.3) and (5.4) hold for μ-almost all $(x, y) \in E \times F$. We have

$$\mu(B((x, y), r)) \leq \mu(B(x, r) \times B(y, r)) = \mathscr{H}^s(E \cap B(x, r))\mathscr{H}^t(F \cap B(y, r))$$

so

$$\frac{\mu(B((x, y), r)}{(2r)^{s+t}} \leq \frac{\mathscr{H}^s(E \cap B(x, r))}{(2r)^s}\frac{\mathscr{H}^t(F \cap B(y, r))}{(2s)^t}.$$

Using (5.3) and (5.4) it follows that

$$\overline{\lim_{r\to 0}} \frac{\mu(B((x, y), r)}{(2r)^{s+t}} \leq 1$$

for μ-almost all $(x, y) \in E \times F$. By Proposition 4.4 (1)

$$\mathscr{H}^s(E \times F) \geq \frac{\mu(E \times F)}{2^{s+t}} = \frac{\mathscr{H}^s(E)\mathscr{H}^t(F)}{2^{s+t}}.$$

Thus, we have proven (5.2) with $c = \frac{1}{2^{s+t}}$. □

LEMMA 5.11. *If $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ are Borel sets then*

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F).$$

PROOF. Let $s < \dim_H(E)$ and $t < \dim_H(F)$. Then $\mathscr{H}^s(E) = \mathscr{H}^s(F) = \infty$. By Theorem 4.5 there exist Borel sets $E_0 \subset E$ and $F_0 \subset F$ with $0 < \mathscr{H}^s(E_0) < \infty$ and $0 < \mathscr{H}^s(F_0) < \infty$. Proposition 5.10 implies that there is $c > 0$ so that

$$\mathscr{H}^{s+t}(E \times F) \geq \mathscr{H}^{s+t}(E_0 \times F_0) \geq c\mathscr{H}^s(E_0)\mathscr{H}^t(F_0).$$

Hence $\dim_H(E \times F) \geq s + t$. Since we can choose s and t to be arbitrarily close to $\dim_H(E)$ and $\dim_H(F)$, we obtain $\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)$. □

LEMMA 5.12. *For any sets $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$*

$$\dim_H(E \times F) \leq \dim_H(E) + \overline{\dim}_B(F). \tag{5.5}$$

PROOF. We prove the claim for $n = m = 1$. Let $E \subset \mathbb{R}$ and $F \subset \mathbb{R}$. We choose $s > \dim_H(E)$ and $t > \overline{\dim}_B(F)$. Then there is $\delta_0 > 0$ such that F can be covered by $N_\delta(F) \leq \delta^{-t}$ intervals of length δ for all $\delta \leq \delta_0$. Fix $0 < \delta \leq \delta_0$. Since $\mathscr{H}^s(E) = 0$, there is a δ-cover $\{U_i\}$ of E with $\sum_i |U_i|^s < 1$. For each i, since $|U_i| \leq \delta \leq \delta_0$, there is a $|U_i|$-cover $U_{i,j}$ of F by $N_{|U_i|}(F)$ intervals. Then $U_i \times F$ is covered by $N_{|U_i|}(F)$ squares $\{U_i \times U_{i,j}\}$ of side $|U_i|$. Therefore $E \times F \subset \bigcup_i \bigcup_j (U_i \times U_{i,j})$ and

$$\begin{aligned}
\mathscr{H}^{s+t}_{\delta\sqrt{2}}(E \times F) &\leq \sum_i \sum_j |U_i \times U_{i,j}|^{s+t} \leq \sum_i N_{|U_i|}(F) 2^{(s+t)/2} |U_i|^{s+t} \\
&\leq 2^{(s+t)/2} \sum_i |U_i|^{-t} |U_i|^{s+t} < 2^{(s+t)/2}.
\end{aligned}$$

When $\delta \to 0$, we obtain $\mathscr{H}^{s+t}(E \times F) < \infty$ whenever $s > \dim_H(E)$ and $t > \overline{\dim}_B(F)$. Hence $\dim_H(E \times F) \leq s + t$ and (5.5) follows. □

COROLLARY 5.13. *Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$. If $\dim_H(F) = \overline{\dim}_B(F)$ then*

$$\dim_H(E \times F) = \dim_H(E) + \dim_H(F).$$

PROOF. Using Lemma 5.11 and Lemma 5.12 we get

$$\dim_H(E) + \dim_H(F) \leq \dim_H(E \times F) \leq \dim_H(E) + \overline{\dim}_B(F).$$

□

CHAPTER 6

Marstrand's projection theorem

In this chapter we prove the main theorem of the thesis: Marstrand's projection theorem. We apply potentials discussed in Chapter 4. First we define the pushforward of a measure and introduce projections.

LEMMA 6.1. *Let $F \subset \mathbb{R}^n$. Let Π be any subspace and let $proj_\Pi$ be an othogonal projection of F onto Π. Then*

$$\dim_H (proj_\Pi(F)) \leq \dim_H(F).$$

PROOF. Since

$$|proj_\Pi(x) - proj_\Pi(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^n$, $proj_\Pi$ is a Lipschitz function. Proposition 3.4 implies $\dim_H(proj_\Pi(F)) \leq \dim_H(F)$. $\square$

DEFINITION 6.2. Let $(X, \mathcal{F}_1, \mu)$ be a measure space. Let $(Y, \mathcal{F}_2)$ be a measurable space and $f : X \to Y$ a measurable map. Then the mapping

$$f_\# \mu(B) = \mu\left(f^{-1}(B)\right),$$

where $B \subseteq Y$, is called *pushforward measure* on Y.

LEMMA 6.3. *Let $(X, \mathcal{F}_1, \mu)$ be a measure space, $(Y, \mathcal{F}_2)$ a measurable space and $f : X \to Y$ a measurable map. The pushforward measure $f_\#\mu$ is a measure on Y.*

PROOF. We need to check the three properties listed in Defintion 1.4.

(1) $f_\#\mu(\emptyset) = \mu\left(f^{-1}(\emptyset)\right) = \mu(\emptyset) = 0$.

(2) Let $B_1 \subset B_2 \subset Y$. Then $f^{-1}(B_1) \subset f^{-1}(B_2)$. Thus $f_\#\mu(B_1) = \mu\left(f^{-1}(B_1)\right) \leq \mu\left(f^{-1}(B_2)\right) = f_\#\mu(B_2)$.

(3) Let $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(Y)$.

$$\begin{aligned} f_\#\mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) &= \mu\left(f^{-1}(\bigcup_{n\in\mathbb{N}} B_n)\right) \\ &= \mu\left(\bigcup_{n\in\mathbb{N}} f^{-1}(B_n)\right) \\ &\leq \sum_{n=1}^{\infty} \mu\left(f^{-1}(B_n)\right) \\ &= \sum_{n=1}^{\infty} f_\#\mu(B_n). \end{aligned}$$

$\square$

Let l_θ be the line through the origin of $\mathbb{R}^2$ that makes an angle θ with the horizontal axis. We denote orthogonal projection onto l_θ by π_θ. Therefore $\theta \in [0, \pi)$ and $\pi_\theta(x) \in l_\theta$.

THEOREM 6.4. *(Marstrand's projection theorem) Let $F \subset \mathbb{R}^2$ be a Borel set. Then $\dim_H(\pi_\theta(F)) = \min\{\dim_H(F), 1\}$ for almost all $\theta \in [0, \pi)$.*

PROOF. Proposition 3.4 and the fact that $\dim_H(l_\theta) = 1$, imply $\dim_H(\pi_\theta(F)) \leq \min\{\dim_H(F), 1\}$ for all $\theta \in [0, \pi)$. We need to show the opposite inequality. Let $0 < t < \dim_H(F)$. Theorem 4.11 implies that there is a mass distribution μ on F with $0 < \mu(F) < \infty$

$$I_t(\mu) = \int_F \int_F \frac{d\mu(x)d\mu(y)}{|x-y|^t} < \infty.$$

For each θ we project μ onto the line l_θ and we get a mass distribution $\mu_\theta = \pi_{\theta\#}\mu$ on $\pi_\theta(F)$. Then

$$\begin{aligned} I_t(\mu_\theta) &= \int_{l_\theta} \int_{l_\theta} \frac{d\mu_\theta(u)d\mu_\theta(v)}{|u-v|^t} \\ &= \int_F \int_F \frac{d\mu(x)d\mu(y)}{|x \cdot \theta - y \cdot \theta|^t} = \int_F \int_F \frac{d\mu(x)d\mu(y)}{|(x-y) \cdot \theta|^t}. \end{aligned}$$

Applying Fubini's Theorem to $[0, \pi) \times \mathbb{R}^2 \times \mathbb{R}^2$ we get

$$\begin{aligned} \int_0^\pi I_t(\mu_\theta) d\theta &= \int_0^\pi \int_F \int_F \frac{d\mu(x)d\mu(y)}{|(x-y) \cdot \theta|^t} d\theta \\ &= \int_F \int_F \left[\int_0^\pi \frac{1}{|(x-y) \cdot \theta|^t} d\theta \right] d\mu(x)d\mu(y) \\ &= \int_F \int_F \left[\int_0^\pi \frac{1}{|\frac{(x-y)}{|x-y|} \cdot \theta|^t} d\theta \right] \frac{d\mu(x)d\mu(y)}{|x-y|^t} \\ &= \int_F \int_F \left[\int_0^\pi \frac{1}{|\tau \cdot \theta|^t} d\theta \right] \cdot \frac{d\mu(x)d\mu(y)}{|x-y|^t} \\ &= \int_F \int_F c_t \cdot \frac{d\mu(x)d\mu(y)}{|x-y|^t} \\ &= c_t I_t(\mu), \end{aligned}$$

where $c_t = \int_0^\pi \frac{1}{|\tau \cdot \theta|^t} d\theta$ is independent of the unit vector τ.

Now we will check when c_t is finite. With $\tau = (0, 1)$ and $(x, y) = (\cos\theta, \sin\theta)$. Hence

$$\begin{aligned} c_t &= \int_0^\pi \frac{1}{|\sin\theta|^t} d\theta \\ &= 2 \int_0^{\pi/2} \frac{1}{(\sin\theta)^t} d\theta \\ &= 2 \int_0^{\varepsilon_0} \frac{1}{|\sin\theta|^t} d\theta + 2 \int_{\varepsilon_0}^{\pi/2} \frac{1}{|\sin\theta|^t} d\theta, \end{aligned}$$

where $\int_{\varepsilon_0}^{\pi/2} \frac{1}{|\sin\theta|^t} d\theta < \infty$ for all t. There exists ε_0 such that $\frac{1}{2}\theta < \sin\theta < 2\theta$ for $0 < \theta < \varepsilon_0$. Hence

$$\int_0^{\varepsilon_0} \frac{1}{2^t \theta^t} d\theta \leq \int_0^{\varepsilon_0} \frac{1}{|\sin\theta|^t} d\theta \leq \int_0^{\varepsilon_0} \frac{2^t}{\theta^t} d\theta$$

Therefore, $\int_0^{\varepsilon_0} \frac{1}{|\sin\theta|^t} d\theta < \infty$ if and only if $\int_0^{\varepsilon_0} \frac{1}{\theta^t} d\theta < \infty$ and this holds if and only if $t < 1$. In the other words $c_t < \infty$ if and only if $t < 1$.

Thus, when $t < 1$, $I_t(\mu_\theta) < \infty$ for almost all $\theta \in [0, \pi)$. Theroem 4.11 implies $\dim_H(\pi_\theta(F)) \geq t$. Finally, take $t_n = \min\{1, \dim_H(F)\} - \frac{1}{n}$. Then, for every n there is $E_n \subseteq [0, \pi)$ such that $\mathscr{H}^1(E_n) = 0$ and $\dim_H(\pi_\theta(F)) \geq t_n$ for all $\theta \in [0, \pi) \backslash E_n$. Therefore, $E = \bigcup_n E_n$ is so that $\mathscr{H}^1(E) = 0$ and, for all $\theta \in [0, \pi) E$, $\dim_H(\pi_\theta(F)) \geq t_n$ for all n. We conclude $\dim_H(\pi_\theta(F)) \geq \min\{1, \dim_H(F)\}$. $\square$

In this Theorem we can find a set $F \subset \mathbb{R}^2$ for which there are zero, one or two directions θ so the dimension of the projection is less than $\dim_H(F)$. These examples are discussed more in Chapter 8.

CHAPTER 7

Self-similar sets

In this chapter we find an algorithmic way of constructing examples of fractals.

LEMMA 7.1. *Let F be a set covered by n_k sets of diameter at most δ_k with $\delta_k \to 0$ as $k \to \infty$. Then*

$$\dim_H(F) \leq \underline{\dim}_B(F) \leq \underline{\lim}_{k\to\infty} \frac{\log(n_k)}{-\log(\delta_k)}.$$

If $n_k \delta_k^s$ remains bounded when $k \to \infty$, then $\mathscr{H}^s(F) < \infty$. If $\delta_k \to 0$ but $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then

$$\overline{\dim}_B(F) \leq \overline{\lim}_{k\to\infty} \frac{\log(n_k)}{-\log(\delta_k)}.$$

PROOF. Definition 3.5 implies $\underline{\dim}_B(F) \leq \underline{\lim}_{k\to\infty} \frac{\log(n_k)}{-\log(\delta_k)}$. If $n_k \delta_k^s$ remains bounded when $k \to \infty$ it holds that $\mathscr{H}^s_{\delta_k}(F) \leq n_k \delta_k^s$. Thus $\mathscr{H}^s_{\delta_k}(F) \to \mathscr{H}^s(F)$ when $k \to \infty$ and $\mathscr{H}^s(F)$ is finite. If $\delta_k \to 0$ but $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then Remark 3.6 implies $\overline{\dim}_B(F) \leq \overline{\lim}_{k\to\infty} \frac{\log(n_k)}{-\log(\delta_k)}$. Since the set F can be covered by $N_\delta(F)$ sets of diameter δ then Definition 2.3 implies

$$\mathscr{H}^s_\delta(F) \leq N_\delta(F)\delta^s.$$

If $1 < \mathscr{H}^s(F) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(F)$ then $\log(N_\delta(F)) + s\log(\delta) > 0$, when δ is small enough. Hence $s < \lim_{\delta\to 0} \frac{\log(N_\delta(F))}{-\log(\delta)}$ and therefore $s \leq \underline{\lim}_{\delta\to 0} \frac{\log(N_\delta(F))}{-\log(\delta)}$. Therefore

$$s = \dim_H(F) \leq \underline{\dim}_B(F)$$

for every $F \subset \mathbb{R}^n$. □

THEOREM 7.2. *Let μ be a mass distribution on F. Suppose that for some s there are numbers $c > 0$ and $\varepsilon > 0$ such that*

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| < \varepsilon$. Then $\mathscr{H}^s(F) \geq \frac{\mu(F)}{c}$ and

$$s \leq \dim_H(F).$$

PROOF. Let $0 < \delta < \varepsilon$ and $\{U_i\}_i$ a δ-cover of F. Then we have

$$\mu(F) \leq \mu\left(\bigcup_i U_i\right) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s.$$

Taking the infimum over all the δ-covers, we get $\mathscr{H}^s_\delta(F) \geq \frac{\mu(F)}{c}$. Hence $\mathscr{H}^s(F) \geq \frac{\mu(F)}{c}$. Since $\mu(F) > 0$, it follows that $\dim_H(F) \geq s$. □

Definition 7.3. A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is called *contraction* on $\mathbb{R}^n$ if there is a number c with $0 < c < 1$ such that

$$|S(x) - S(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}^n$. If S satisfies $|S(x) - S(y)| = c|x - y|$ for all $x, y \in \mathbb{R}^n$, then S is called *similarity.*

Definition 7.4. A finite family of contractions $\{S_1, S_2, \ldots, S_m\}$, where $m \geq 2$, is called an *iterated function system* or *IFS.*

Definition 7.5. Let $F \subset \mathbb{R}^n$ be a non-empty and compact. The set F is called an *attractor* for the IFS $\{S_1, S_2, \ldots, S_m\}$ if

$$F = \bigcup_{i=1}^{m} S_i(F).$$

In Theorem 7.9 we prove that a IFS determines a unique attractor. This unique attractor is usually a fractal.

Definition 7.6. Let $S_1, \ldots, S_m : \mathbb{R}^n \to \mathbb{R}^n$ be similarities such that

$$|S_i(x) - S_i(y)| = c_i|x - y|$$

for all $x, y \in \mathbb{R}^n$, where $0 < c_i < 1$ is called the *ratio* of S_i. The attractor of such a collection of similarities is called a *self-similar set.*

Definition 7.7. A collection of similarities $\{S_1, S_2, \ldots, S_m\}$ satisfies the *open set condition* if there exists a non-empty bounded open set V such that

(1) $\bigcup_{i=1}^{m} S_i(V) \subset V$
(2) $S_i(V) \cap S_j(V) \neq \emptyset$ for $i \neq j$.

Lemma 7.8. *Let $\{V_i\}_{i\in\mathbb{N}}$ be a collection of disjoint open subsets of $\mathbb{R}^n$ such that each V_i contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$ for some $0 < a_1 < a_2$ and a fixed $r > 0$. Then any ball B of radius r intersects at most $(\frac{1+2a_2}{a_1})^n$ of the closures $\overline{V}_i$.*

Proof. If $\overline{V}_i$ and the ball B intersect, then $\overline{V}_i$ is contained in the ball concentric with B of radius $(1+2a_2)r$. We assume that q of the sets $\overline{V}_i$ intersect B. By summing the volumes of the interior balls it follows that $q(a_1 r)^n \leq (1 + 2a_2)^n r^n$. Therefore $q \leq (\frac{1+2a_2}{a_1})^n$. □

Theorem 7.9. *Consider the iterated function system given by the contractions $\{S_1, \ldots, S_m\}$ on $\mathbb{R}^n$, so that*

$$|S_i(x) - S_i(y)| \leq c_i|x - y|,$$

where $x, y \in \mathbb{R}^n$ with $c_i < 1$ for each i. Then there is a unique attractor F for the IFS $\{S_1, S_2, \ldots, S_m\}$.

Moreover, the attractor F has the following description. Let $\mathcal{S}$ be the class of non-empty compact subsets of $\mathbb{R}^n$. We define a transformation S on $\mathcal{S}$ by

$$S(E) = \bigcup_{i=1}^{m} S_i(E)$$

for $E \in \mathcal{S}$. We write S^k for the kth iterate of S. Then

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

for every set $E \in \mathcal{S}$ such that $S_i(E) \subset E$ for all i.

Proof. Notice that the sets in $\mathcal{S}$ are transformed by S into other sets of $\mathcal{S}$. Recall the Hausdorff distance h (Definition 1.22). Theorem 1.23 implies that h is a metric on $\mathcal{S}$. Moreover, $(\mathcal{S}, h)$ is a complete metric space (see [13, Theorem 4.9]). For each i,

$$h(S_i(A), S_i(B)) \leq c_i h(A, B).$$

Therefore,

$$h(S(A), S(B)) \leq \left(\max_i c_i\right) h(A, B),$$

where $c = \max_i c_i < 1$. Banach's fixed point theorem (see [1, Theorem 1.171]) implies that S has a unique fixed point, that is there is a unique set $F \in \mathcal{S}$ so that $S(F) = F$, where

$$F = \bigcup_{i=1}^{m} S_i(F)$$

is an attractor of the IFS. Since $h(F, S(A)) < ch(F, A)$, then $\lim_{k\to\infty} S^k(E) = F$ for every $E \in \mathcal{S}$. In particular, if $S(E) \subseteq E$, then $\lim_{k\to\infty} S^k(E) = \bigcap_{k=1}^{\infty} S^k(E)$. ☐

The next theorem gives us one way of finding the dimension of self-similar fractals.

Theorem 7.10. *Let $\{S_i : 1 \leq i \leq m\}$ be similarities satisfying the open set condition on $\mathbb{R}^n$. Let $0 < c_i < 1$ be the ratio of S_i. If F is the attractor of the IFS $\{S_1, \dots S_m\}$, then $\dim_H(F) = \dim_B(F) = s$, where s is given by*

$$\sum_{i=1}^{m} c_i^s = 1.$$

Moreover, for this value of s, $0 < \mathscr{H}^s(F) < \infty$.

Proof. Let s satisfy $\sum_{i=1}^m c_i^s = 1$. Let $\mathcal{I}_k$ be the set of all k-tuples $(i_1, \dots, i_k)$ with $1 \leq i_j \leq m$. For an arbitrary set A and $(i_1, \dots, i_k) \in \mathcal{I}_k$ we write $A_{i_1,\dots,i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(A)$. Using iteratively the fact $F = \bigcup_{i=1}^m S_i(F)$, it follows that

$$F = \bigcup_{(i_1,\dots,i_k)\in\mathcal{I}_k} F_{i_1,\dots,i_k}.$$

We will prove an upper and a lower estimates for the Hausdorff measure $\mathscr{H}^s(F)$. Upper estimate for the Hausdorff measure: We claim that

$$\mathscr{H}^s(F) \leq |F|^s. \tag{7.1}$$

Notice that the mapping $S_{i_1} \circ \cdots \circ S_{i_k}$ is a similarity of ratio $c_{i_1} \cdots c_{i_k}$. Since s satisfies $\sum_{i=1}^m c_i^s = 1$ it follows that

$$\sum_{(i_1,\dots,i_k)\in\mathcal{I}_k} |F_{i_1,\dots,i_k}|^s = \sum_{(i_1,\dots,i_k)\in\mathcal{I}_k} (c_{i_1} \cdots c_{i_k})^s |F|^s = \left(\sum_{i_1} c_{i_1}^s\right) \cdots \left(\sum_{i_k} c_{i_k}^s\right) |F|^s = |F|^s.$$

For every $\delta > 0$ we can choose k such that

$$|F_{i_1,\dots i_k}| \le \left(\max_i c_i\right)^k |F| \le \delta,$$

for every $(i_1, \dots, i_k) \in \mathcal{I}_k$. Therefore, $\mathscr{H}^s_\delta(F) \le |F|^s$ and hence we have (7.1).

Lower estimate for the Hausdorff measure: We claim that there exists $q > 0$ such that

$$\mathscr{H}^s(F) > \frac{1}{q}. \tag{7.2}$$

Let $\mathcal{I}$ be the set of all infinite sequences, $\mathcal{I} = \{(i_1, i_2, \dots) : 1 \le i_j \le m\}$. Let $I_{i_1,\dots,i_k} = \{(i_1, \dots, i_k, q_{k+1}, \dots) : 1 \le q_j \le m\}$ be the 'cylinder' consisting sequences belonging to $\mathcal{I}$ that have initial terms $i_1, \dots i_k$. One can show that there exists a measure μ on $\mathcal{I}$ such that $\mu(I_{i_1,\dots,i_k}) = (c_{i_1} \cdots c_{i_k})^s$ for $(i_1, \dots, i_k) \in \mathcal{I}_k$. Notice that $\mu(\mathcal{I}) = 1$ and, since

$$(c_{i_1} \cdots c_{i_k})^s = \sum_{i=1}^m (c_{i_1} \cdots c_{i_k} c_i)^s,$$

then

$$\mu(I_{i_1,\dots,i_k}) = \sum_{i=1}^m \mu(I_{i_1,\dots,i_k,i}).$$

We define $\phi : \mathcal{I} \to \mathbb{R}^n$, $\phi(i_1, \dots, i_j, \dots) = \lim_{j\to\infty} S_{i_1} \circ \cdots \circ S_{i_j}(0)$, which can be shown to be a measurable function. We define a mass distribution $\tilde{\mu}$ on F as $\tilde{\mu} = \phi_\# \mu$, that is, $\tilde{\mu}(A) = \mu\{(i_1, i_2, \dots) : x_{i_1,i_2,\dots} \in A\}$ for $A \subset \mathbb{R}^n$. In particular $\tilde{\mu}(F) = 1$.

Next we show that $\tilde{\mu}$ satisfies the conditions of the Theorem 7.2. Let U be an open set with diameter $|U| < 1$. Then $U \subseteq B$, where B is a ball of radius $r = |U| < 1$. Let V be the open set of Definition 7.7. It holds that $\overline{V} \supset S(\overline{V}) = \bigcup_{i=1}^m S_i(\overline{V})$. The decreasing sequence of iterates $S^k(\overline{V})$ converges to F by Theorem 7.9. Moreover $\overline{V} \supset F$ and $\overline{V}_{i_1,\dots i_k} \supset F_{i_1,\dots i_k}$ for each finite sequence $(i_1, \dots, i_k)$. We estimate $\tilde{\mu}(B)$ by considering the sets $V_{i_1,\dots,i_k}$ with diameters comparable with that of B and with closures intersecting $F \cap B$.

We cut each infinite sequence $(i_1, i_2, \dots) \in \mathcal{I}$ after the first term i_k for which

$$\left(\min_{1\le i\le m} c_i\right) r \le c_{i_1} c_{i_2} \cdots c_{i_k} \le r. \tag{7.3}$$

Let $\mathcal{Q}$ denote the finite set of all finite sequences obtained in this way. For every infinite sequence $(i_1, i_2, \dots) \in \mathcal{I}$ there is one value of k with $(i_1, \dots, i_k) \in \mathcal{Q}$.

Notice that, if $k \le c$, then $V_{i_1,\dots,i_k} \cap V_{j_1,\dots j_c} \ne \emptyset$ if and only if $i_a = j_a$ for $a = 1, \dots, k$. It follows that the collection of open sets $\{V_{i_1,\dots,i_k} : (i_1, \dots, i_k) \in \mathcal{Q}\}$ is disjoint. Moreover,

$$F \subset \bigcup_{(i_1,\dots,i_k)\in\mathcal{Q}} F_{i_1,\dots,i_k} \subset \bigcup_{(i_1,\dots,i_k)\in\mathcal{Q}} \overline{V}_{i_1,\dots,i_k}.$$

We choose a_1 and a_2 so that V contains a ball of radius a_1 and is contained in a ball of radius a_2. For all $(i_1, \dots, i_k) \in \mathcal{Q}$ the set $V_{i_1,\dots,i_k}$ contains a ball of radius $c_{i_1} \cdots c_{i_k} a_1$. Therefore, it contains a ball of radius $(\min_i c_i) a_1 r$ by (7.3). Moreover, $V_{i_1,\dots,i_k}$ is contained in a ball of radius $c_{i_1} \cdots c_{i_k} a_2$ and hence it is contained in a ball of radius $a_2 r$, again by (7.3). We denote by $\mathcal{Q}_1$ the collection of sequences

$(i_1, \dots, i_k)$ such that B intersects $\overline{V}_{i_1,\dots,i_k}$. Lemma 7.8 implies that there are at most $q = (\frac{1+2a_2}{a_1 \min_i c_i})^n$ sequences in $\mathcal{Q}_1$. Notice that

$$\phi^{-1}(F \cap B) = \{(j_1, j_2, \dots) : \phi(j_1, j_2, \dots) \in F \cap B\} \subseteq \bigcup_{(i_1,\dots,i_k)\in\mathcal{Q}_1} I_{i_1,\dots,i_k}.$$

Thus

$$\begin{aligned}\tilde{\mu}(B) = \mu(\phi^{-1}(F \cap B)) &\leq \sum_{(i_1,\dots,i_k)\in\mathcal{Q}_1} \mu(I_{i_1,\dots,i_k}) \\ &= \sum_{(i_1,\dots,i_k)\in\mathcal{Q}_1} (c_{i_1} \cdots c_{i_k})^s \leq \sum_{(i_1,\dots,i_k)\in\mathcal{Q}_1} r^s \leq r^s q.\end{aligned}$$

Going back to U, since $U \subseteq B$ and $|B| = 2r = 2|U|$, we have

$$\tilde{\mu}(U) \leq \tilde{\mu}(B) \leq c|B|^s \leq \frac{c}{2^s}|U|^s,$$

that is, $\tilde{\mu}(U) \leq |U|^s q$. Theorem 7.2 gives

$$\mathcal{H}^s(F) \geq \frac{1}{q} > 0.$$

The estimates (7.1) and (7.2) yeld $\dim_H(F) = s$.

We now compute the box dimension of F. If $\mathcal{Q}$ is any set of finite sequences such that for every $(i_1, i_2, \dots) \in \mathcal{I}$ there is one integer k with $(i_1, \dots, i_k) \in \mathcal{Q}$. It follows inductively from $\sum_{i=1}^m c_i^s = 1$ that $\sum_{\mathcal{Q}_1} (c_{i_1} c_{i_2} \cdots c_{i_k})^s = 1$. If $\mathcal{Q}$ is chosen as above, $\mathcal{Q}$ contains at most $\frac{1}{(\min_i c_i r)^s}$ sequences. For each sequence $(i_1, \dots, i_k) \in \mathcal{Q}$ we have $|\overline{V}_{i_1,\dots,i_k}| = c_{i_1} c_{i_2} \cdots c_{i_k} |\overline{V}| \leq r|\overline{V}|$. Therefore F may be covered by $\frac{1}{(\min_i c_i r)^s}$ sets of diameter $r|\overline{V}|$ for each $r < 1$. Definition 3.5 implies that $\overline{\dim}_B(F) \leq s$. Noting that $s = \dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F) \leq s$, the proof is complete. $\square$

CHAPTER 8

Examples

Let l_θ be the line through the origin of $\mathbb{R}^2$ with angle θ with horizontal axis. We denote orthogonal projection onto l_θ by π_θ. Therefore $\pi_\theta(F)$ denotes the projection of $F \subset \mathbb{R}^2$ onto the line l_θ.

EXAMPLE 8.1. Let F be a unit square in $\mathbb{R}^2$. For all the lines l_θ the set $\pi_\theta(F)$ is a segment and $\dim_H(\pi_\theta(F)) = 1$.

EXAMPLE 8.2. Let F be a segment in $\mathbb{R}^2$. Let the line l_{θ_1} be orthogonal to the segment F. Then $\dim_H(\pi_{\theta_1}(F)) = 0$. Notice that there is only one possible θ_1. Otherwise $\dim_H(\pi_\theta(F)) = 1$.

EXAMPLE 8.3. Let F be a middle third Cantor set. Let $S_1, S_2 : [0,1] \to \mathbb{R}$ be given by $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$. Now $F = S_1(F) \cup S_2(F)$, where $S_1(F)$ gives the left half and $S_2(F)$ gives the right half of F. Hence F is an attractor of IFS $\{S_1, S_2\}$. Let $E = [0,1]$. First step of the middle third Cantor set is obtained by $S_1(E) \cup S_2(E)$. Second step is obtained by

$$S_1(S_1(E)) = \frac{1}{9}x, \quad S_1(S_2(E)) = \frac{1}{9}x + \frac{2}{9}$$
$$S_2(S_1(E)) = \frac{1}{9}x + \frac{2}{3}, \quad S_2(S_2(E)) = \frac{1}{9}x + \frac{8}{9}$$

and so on. The sets $S^k(E) = \bigcup_{i=1,2} S_{i_1} \circ \cdots \circ S_{i_k}(E)$ gives increasingly good approximations of the set F. Theorem 7.10 implies that $\dim_H(F) = s$, where s is given by $(\frac{1}{3})^s + (\frac{1}{3})^s = 1$. Hence $s = \frac{\log 2}{\log 3}$. Let $F \times F$ be a product of two middle third Cantor sets. By the Corollary 5.13 $\dim_H(F \times F) = \frac{\log 2}{\log 3} + \frac{\log 2}{\log 3} = \frac{\log 4}{\log 3}$. For almost every θ $\dim_\theta(F \times F) = 1$. There is two angles $\theta_1 = 0$ and $\theta_2 = \pi$ for which $\dim(\pi_{\theta_i}(F \times F)) = \dim(F)$, where $i = 1,2$. Hence there is two exceptional angles for which the dimension drops.

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