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# Academic Literacy Supporting Sustainability for Mathematics Education—A Case: Collaborative Working as a Meaning Making for “2/3”?

Päivi Perkkilä and Jorma Joutsenlahti

## 1. Introduction

Sustainable education is currently being developed, in many countries, in many areas of basic education, vocational education and training, polytechnics, and universities. *Education for sustainable development* (ESD) as a long-term learning process supports a better life in various areas, such as social, economic, and environmental. To support a good and sustainable learning culture in different fields of education, we should build sustainable development so that we can understand the constantly changing world and its challenges. In the field of mathematics education, this means that we must develop students' *academic literacy in mathematics* (ALM) and 21st Century competences so that students have creativity to solve and model ESD-based mathematical problems, namely in the social, economic, and environmental fields. These kinds of skills can support the lives of students in the present and future (Widiaty and Juandi 2019).

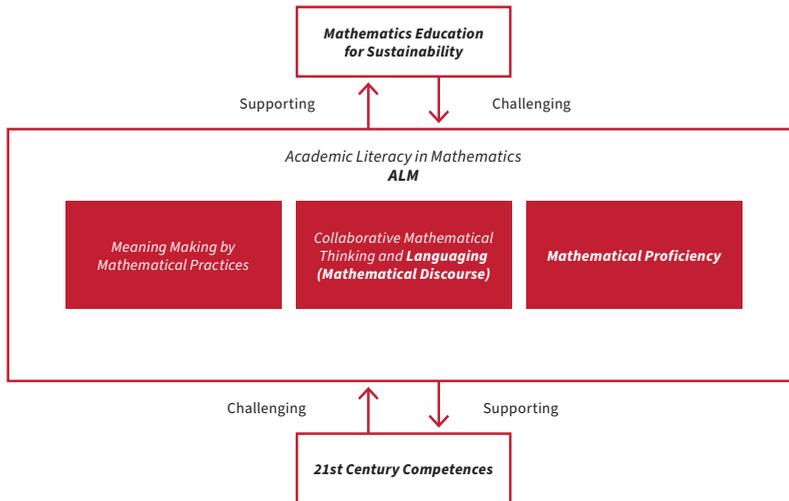
### 1.1. *Education for Sustainable Development*

Zehetmeier and Krainer (2011) have argued that sustainability mainly belongs to ecological and economical vocabulary, but is more and more employed in the educational realm too. Already in 1657, Comenius highlighted sustainability in educational in his book “*Didactica Magna*” (Flitner 1970), with a chapter about the “foundation of lasting teaching and learning”. This “foundation of lasting teaching and learning” refers to the view of ESD. Fullan (2006) observed sustainability in the light of educational change as “the capacity of a system to engage in the complexities of continuous improvement with the deep values of human purpose”. Fullan (2006) rests his definition about sustainability on Hargreaves and Hargreaves and Finks’ (2003) viewpoint, that is “sustainability does not simply mean whether something will last. It addresses how particular initiatives can be developed without compromising the development of others in the surrounding environment now and in the future”.

All these definitions of sustainability are based on durable continuation. In the educational realm, we can understand this durable continuation as lifelong learning. Concerning education, good teaching and learning that matters lasts for life, and both are inherently sustaining processes. Supporting and maintaining such deep aspects of teaching and learning, which endure and foster sophisticated understanding and lifelong learning for all, builds the main core for sustainable development in education. In this respect, we want to uncover how the collaborative aspect (working in pairs) helps prospective class teachers with meaning making for the mathematical symbol  $\frac{2}{3}$ . In 2019, we published an article about “What kind of meanings alone working prospective class teachers found for the mathematical symbol  $\frac{2}{3}$ ” (Joutsenlahti and Perkkilä 2019). In the abovementioned article, students worked alone and tried to find different meanings for the mathematical symbol  $\frac{2}{3}$  and the subject of our study was the separate answers given by each student. In the research at hand, sustainability does not simply mean whether prospective class teachers’ meaning making will last; we want to see if collaborative working enriches prospective class teachers’ skills to produce their meaning making of the symbol  $\frac{2}{3}$  compared with the situation when they were working alone (cf. Fullan 2006; Hargreaves and Finks’ 2003; Zehetmeier and Krainer 2011). In this sense, we can consider how pedagogical practices such as collaborative working methods support mathematical insights and mathematical thinking. Collaborative working methods, here meaning making collaboratively, are part of academic literacy, which is linked to ESD and 21<sup>st</sup> Century competences (see Figure 1).

### *1.2. Academic Literacy and 21st Century Competences*

In the present study, we have expanded the ALM framework to support sustainable development in mathematics education (see Figure 1). Especially we have new interpretations in the context of collaborative mathematical thinking and its impact on meaning making for the symbol  $\frac{2}{3}$ . In this way, we want to see if the collaborative aspect reinforces and enriches the students’ meaning making for the mathematical symbol  $\frac{2}{3}$  and whether collaborative mathematical thinking could be robust enough for ALM skills and for the ESD. In Figure 1, we have described how ALM relates to ESD and to 21st Century competences.



**Figure 1.** Linking Academic Literacy in Mathematics (ALM) to Education for Sustainable Development (ESD) and 21st Century Competences (adapting Moschkovich 2015a, 2015b, 2019; Widiaty and Juandi 2019).

On the one hand, in Figure 1, good skills in ALM support both ESD and 21st Century competences, but on the other hand, ESD and the mentioned competences challenge ALM skills. We see that good 21st Century competencies as future citizen skills support both the ALM and sustainable development perspectives in developing mathematics teaching and learning. These future citizen skills include the following areas: civic literacy, global awareness, and cross-cultural skills; critical and inventive thinking; communication, collaboration and information skills (Ministry of Education Singapore 2018; cf. Partnership for 21st Century Learning 2019; Valli et al. 2014). Previously mentioned competences are central abilities of thinking, working, and mastery of tools, and they are considered future competences that future citizens will need (see Valli et al. 2014). As far as ALM skills are concerned, these competences are recognizable in the subareas of ALM skills. However, both ESD and the 21st Century competences are broader than the skills of the ALM, so ESD and 21st Century competences are also challenging ALM skills.

Moschkovich (2015a, 2015b, 2019) uses the term *academic literacy* to refer to literacy in mathematics studies. Generally, the concept *literacy* has been interpreted and studied from many different perspectives and there are several related concepts, for example, technological literacy, information literacy, online literacy, image literacy, and visual literacy (Kupiainen et al. 2015). Multiliteracy has emerged

as the overarching concept that combines the different perspectives. It should be remembered that this is a diverse area of competence, not only in reading but also in writing or production (Kupiainen et al. 2015). The concept “literacy” is based on the English term multiliteracy, which means that there are different textual practices in different social contexts; for example, these social contexts may be related to different disciplines such as mathematics (Kalantzis and Cope 2012). In this article, we illustrate the connection mainly between the ALM framework (see Moschkovich 2015a) and ESD. We also partly observe the meaning of ALM and ESD in the 21st Century competences viewpoint. ALM is understood here through three integrated components: mathematical proficiency, mathematical practices, and mathematical discourse (cf. Kilpatrick et al. 2001). We concentrate, especially, on two components of ALM: mathematical practices and mathematical discourse (linguaging). Concerning sustainability in mathematics education, we see that good ALM skills in mathematics support Shulman’s (1986) three categories of teacher’s content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. We must take these three areas into account if we want to develop teachers’ and trainees’ expertise in mathematics education and sustainable development in mathematics learning (Joutsenlahti and Perkkilä 2019).

Mathematics is seen as a gatekeeper in education: Good skills in ALM (see Shulman’s (1986) three categories of teacher’s content knowledge) ensure better chances of success in studies and, thus, a commitment to lifelong learning (Díez-Palomar et al. 2018). Joutsenlahti and Perkkilä (2019) have pointed out that if the mathematical content knowledge and pedagogical content knowledge were deepened in teacher education by making them a more sustainable basis through pedagogical arrangements (e.g., collaborative thinking), this will help to build a sustainable basis for future generations’ mathematics education. Concerning pedagogical arrangements, ALM (Moschkovich and Zahner 2018) includes both sociocultural and discursive aspects of mathematical activity. These mean participation in mathematical practices and mathematical discourse. This view assumes that mathematical proficiency, mathematical practices, and mathematical discourse—the elements of ALM—work together and support collaborative mathematical thinking and linguaging to build sustainable understanding in mathematics.

## **2. The ALM Framework Supporting Sustainability in Mathematics Education**

The mission of the fourth goal of the Agenda for Sustainable Development of The United Nations (UN 2018) for 2030 is to secure inclusive and quality education for all and, in this way, promote lifelong learning. As mentioned before, mathematics

learning acts like a gatekeeper in education: those who succeed in mathematics education and have better scores in mathematics will end comprehensive school with better educational trajectories than those who underachieve in this subject (Díez-Palomar et al. 2018). To develop sustainable mathematics education, teachers' mathematical knowledge and skills for building innovative learning situations act like a gatekeeper concerning pupils' development in creating sustainable and meaningful understanding about school mathematics. In this sense, it is important to explore prospective class teachers' conceptual interpretations and, as a case example, their interpretations for the mathematical symbol " $\frac{2}{3}$ ", especially in collaborative situations, from the perspective of sociocultural situations of ALM.

### *2.1. The Components of ALM*

Widiaty and Juandi (2019) treated, in their article, education for sustainable development (ESD) from the mathematics education point of view. They interpreted mathematics as a tool for understanding, analyzing, and solving problems in the neighborhood and surrounding society. To understand the aspect of sustainable development in mathematics, we need to develop the skills and creativity of teachers to plan for the problems of the surrounding society. To guide pupils to apply mathematics in the spirit of ESD, teachers need to have a good conceptual understanding of mathematics. Through conceptual understanding, they will promote sustainability in their own mathematical thinking and professional development in mathematics. Teacher education has great responsibility because conceptual understanding should be strengthened during teacher education by paying attention to the importance of ALM and reinforcing ESD thinking with ALM and 21st Century competences. As mentioned before, Moschkovich (2015a, 2015b, 2019) has defined ALM as three integrated aspects: mathematical proficiency, mathematical practices, and mathematical discourse (cf. Kilpatrick et al. 2001). In the following figure (Figure 2), we present the ALM components from the perspective of this study.

Mathematical Proficiency	Mathematical Practices	Languageing (cf. Mathematical Discourse)
<ol style="list-style-type: none"> <li>1. <b>Conceptual understanding</b></li> <li>2. <b>Procedural fluency</b></li> <li>3. <b>Strategic competence</b></li> <li>4. <b>Adaptive reasoning</b></li> <li>5. <b>Productive disposition</b></li> </ol>	<ol style="list-style-type: none"> <li>1. <b>Meaning making</b></li> <li>2. <b>Perseverance</b></li> <li>3. <b>Abstract reasoning</b></li> <li>4. <b>Justification of the solution</b></li> <li>5. <b>Constructive criticism</b></li> <li>6. <b>Model with mathematics</b></li> <li>7. <b>Appropriate tools</b></li> <li>8. <b>Precision</b></li> <li>9. <b>Structures</b></li> <li>10. <b>Regularities</b></li> </ol>	<ol style="list-style-type: none"> <li>1. <b>Languageing</b> <ul style="list-style-type: none"> <li>– Natural language (oral or written) (NL)</li> <li>– Mathematical symbolic language (MSL)</li> <li>– Pictorial language (PL)</li> <li>– Tactile functional language (TFL)</li> </ul> </li> <li>2. <b>Collaborative Thinking</b></li> </ol>

**Figure 2.** The modified components of ALM of this study (adapting Joutsenlahti and Kulju 2016; Kilpatrick et al. 2001; Moschkovich 2015a, 2015b).

In this article, we will focus especially on the sociocultural aspects (see the mathematical practices and languageing in Figure 2) of the ALM framework in the context of prospective class teachers' mathematics education. These sociocultural aspects include participation in mathematical practices and participation in mathematical discourse. Mathematical proficiency includes the traditional cognitive aspects of mathematical activity such as mathematical reasoning, thinking, conceptual development, and metacognition (Kilpatrick et al. 2001; Moschkovich 2019; Moschkovich and Zahner 2018). We agree with Moschkovich (2019) and Moschkovich and Zahner (2018) that the sociocultural aspects of the ALM framework will not be separated from mathematical proficiency. The sociocultural aspects more likely assume that all the fields of ALM work together. When these three aspects are socioculturally included in mathematics learning, it will make the learning situations dynamic and will improve the meaning making of conceptual understanding in mathematics. These situations involve multiple modes of languageing like oral and written texts, gestures, drawings, objects, tables, graphs, symbols, etc. We need to account for all three ALM categories that develop mathematics teaching toward greater sustainability in mathematical meaning making and in achieving better learning results. Our focus is on education for sustainable development in prospective class teachers' mathematics education because as in-service teachers, they will act as key roles in building the quality of mathematics education for all and promoting sustainable meaningful learning in mathematics education. Next, we will clarify the components of ALM in this study.

### 2.1.1. Mathematical Proficiency

Moschkovich (2015b) stated that the mathematical competence model published in 2001 (Kilpatrick et al. 2001) is still valid for describing the cognitive domain of

academic literacy. This model by Kilpatrick et al. (2001) presents mathematical proficiency, consisting of five features: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (see also Joutsenlahti 2005).

The feature of conceptual understanding includes an understanding about mathematical concepts and relationships between them, as well as mathematical operations and their relationship to mathematical concepts (Joutsenlahti 2005; Kilpatrick et al. 2001). Conceptual understanding is reflected in the meanings that the mathematical problem solver gives to the solution (What does the result mean for the assignment?), the solution process (Why do the selected procedures and methods work in this solution?), and the final question (Why is the answer correct for this problem?) (Moschkovich 2015b). Procedural fluency is demonstrated by the ability to use mathematical procedures flexibly, carefully, efficiently, and expediently (Joutsenlahti 2005; Kilpatrick et al. 2001). In schools, the role of mechanical computing is particularly emphasized, but computational competence is often an essential part of conceptual understanding, and vice versa (Moschkovich 2015b). Strategic competence is the ability to formulate, present, and solve non-routine mathematical problems. The feature described above is central to problem solving. Adaptive reasoning is logical thinking, reflection, finding explanations, and witnessing. The last mentioned aspect of mathematical competence, productive disposition (the view of mathematics), reflects the learner's perception of the importance and usefulness of mathematics, as well as his or her own diligence and effectiveness in mathematics study (Joutsenlahti 2005; Kilpatrick et al. 2001). The five characteristics of mathematical proficiency (Figure 2) are in fact the cognitive component of academic literacy (Joutsenlahti and Kulju 2016). We can see that the cognitive component of academic literacy works like a tool which supports ESD and gives tools to formulate, present, model, and solve, for example, non-routine problems in the neighborhood and surrounding society. It is also connected to the meanings of 21st Century competences. This means that students should not only have the skills to think mathematically but they should have sensitivity to the problems found in the surrounding society, especially in the social, economic, and environmental fields (Widiaty and Juandi 2019); this is a real challenge for sustainable mathematics education.

### 2.1.2. Meaning Making by Mathematical Practices

In the United States, the Common Core State Standards (2019) define eight mathematical practices (see Figure 2) that can be interpreted in teaching mathematics from preschool to high school. The practices described guide students starting and

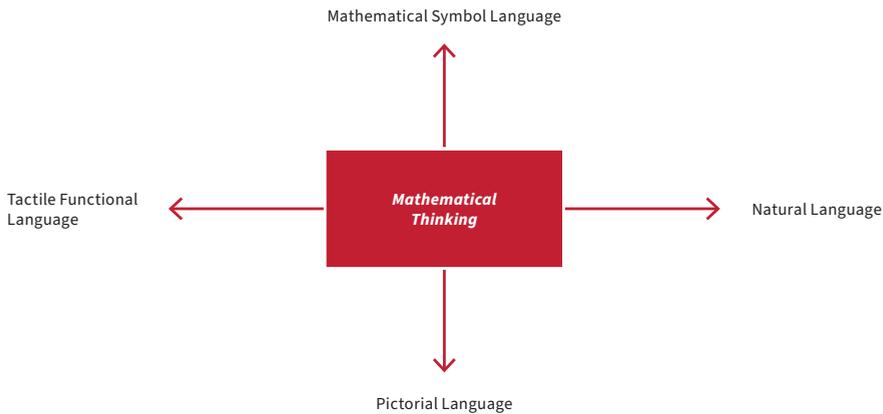
mastering mathematical problem-solving processes. The mathematical practices are (the Common Core State Standards 2019): make sense of the problems and persevere in solving them, reason abstractly and quantitatively, construct viable arguments and critique the reasoning of others, model with mathematics, use appropriate tools strategically, attend to precision, look for and make of use structure, and look for and express regularity in repeated reasoning.

The first goal is to understand a given problem, whereby a mathematically proficient student begins problem-solving by first explaining to themselves the relevant features of the problem. The student uses typical problem-solving methods (for example, analogue problems, and trying special cases and simpler forms of the original problem in order to gain insight into the solution) in a consistent and persevering manner while monitoring and evaluating his/her own solution process. Secondly, the student must draw abstract and quantitative conclusions from the problem assignment. A mathematically proficient student understands the significance of the numbers or variables given in the problem assignment and their relationship to the problem assignment. He/she can contextualize the problem in such a way that he/ she is able to describe mathematical symbols in the problem and the relationships between them, as well as to simplify expressions and solve equations (the Common Core State Standards 2019; Moschkovich 2015b). Meaning making by mathematical practices includes several important competences, which researchers have related to sustainable development: 1. problem solving, critical thinking (e.g., construct viable arguments and critique the reasoning of others), action competence, and system thinking (e.g., model with mathematics, use appropriate tools strategically, attend to precision); 2. imagination, critical thinking and reflection, system of thinking, partnership, learning to work together, and participation in decision-making (e.g., meaning making, perseverance, constructive criticism); 3. systems thinking—the ability to see the interconnections between different dimensions and the complexity of systems and situations (e.g., abstract reasoning, precision, structures). The previously mentioned competences related to Sustainable Development are those skills which strengthen facing and solving mathematical problems, namely in the social, economic, and environmental fields, to support better life in the present and future (cf. Widiaty and Juandi 2019).

### 2.1.3. Linguaging and Collaborative Mathematical Thinking

Moschkovich (2015a, 2015b) highlights mathematical discourse as the third component of ALM. She emphasizes that mathematical discourse is a broader concept than the language used to study mathematics. Moschkovich (2015b) defines

mathematical discourse more precisely as communication competence that enables participation in mathematical practices. According to her, mathematical discourse includes not only oral and written texts, but also many modes (or symbol systems) such as gestures, activity materials, drawings, tables, graphs, and mathematical symbols. Interaction involves various registers such as school mathematics and home language. Moschkovich (2015b) emphasizes that in defining mathematical discourse, confrontation between the use of formal mathematics, such as the textbook definitions of concepts, and the student's own everyday register should be avoided. However, when looking at academic literacy and its components, we have chosen the third component as the "expression of mathematical languaging", instead of the student studying mathematics and discourse. In other words, the features of mathematical competence describe the cognitive potential of said learner, and the adoption of the mathematical practices given to the learner describes mathematical activity in the study and decision processes. The expression of mathematical languaging controls the abovementioned activity. It is based on the pupil's mathematical thinking, which is built on existing knowledge and skills (see Joutsenlahti 2005). This construction of thinking is also accompanied by the expression of thinking, where the student expresses his or her thinking in typical manners for mathematics by utilizing languages in a versatile way (e.g., Joutsenlahti and Rättyä 2015). Mathematical activity is also guided by collaborative working with other students and this interaction can generate mathematical knowledge and understanding. From the literacy point of view, the student expresses his or her mathematical thinking through natural language, mathematical symbolic language, pictorial language, and/or tactile functional language (Joutsenlahti and Kulju 2016; Joutsenlahti and Rättyä 2015). Likewise, mathematical thinking can be expressed using different symbolic systems (or languages) in different texts, for example, in verbal assignments, narratives, or in the oral presentation of a lesson (Joutsenlahti and Kulju 2017). By collaborative mathematical thinking, we mean here common work in pairs, where students share their ideas with each other (the other pair member) and produce joint solutions. In Figure 3, we clarify languaging as mathematical thinking. As mentioned before, languaging is divided into four different parts.



**Figure 3.** Linguaging as multimodality in expressing mathematical thinking (adapting Joutsenlahti and Perkkilä 2019; Joutsenlahti and Rättyä 2015).

We call the process in which, for example, a pupil expresses his/her thinking as “linguaging”. We describe linguaging in mathematics as referring to expressing one’s mathematical thinking by different modes, either orally (by natural language) or in writing (by natural language, mathematical symbolic language, or pictorial language) (Joutsenlahti and Kulju 2017). In learning situations, especially with elementary school students, a tactile, functional language (working with manipulatives) is also involved as a fourth language, referring to expressing one’s mathematical thinking. In mathematics textbooks, we can recognize three languages, which mathematics textbooks use as meaning making tools for mathematical concepts and procedures. Linguaging of mathematical thinking helps pupils to structure their thinking and, in that way, understand mathematical concepts and procedures (Joutsenlahti and Perkkilä 2019; Kilpatrick et al. 2001; Moschkovich 2015b).

Collaborative learning aims at a relatively lasting change in a student’s knowledge, collaboration, thinking, and attitudes. Collaborative thinking in groups/pairs facilitates verbal inference, explanation, evaluation, and reflection on what the individual knows. Through a shared reflection process, knowledge becomes meaningful and integrates into meaning structures (Hellström et al. 2015). In our study, collaborative mathematical thinking was manifested in pair work, where students must express their thinking to each other by linguaging the meanings of the mathematical symbol “ $\frac{2}{3}$ ”.

### 3. Case: What Kind of Meanings Do the Prospective Class Teachers Find for the Mathematical Symbol “ $\frac{2}{3}$ ”?

We have chosen the concept of fractions and the symbol “ $\frac{a}{b}$ ” as a case from school mathematics. Fractions are one of the most challenging areas in school mathematics (e.g., Martin et al. 2008; Morgan 2001; Perkkilä 2001; Vosniadou 1999). Pupils have trouble with fractions, especially understanding fractions as numbers that extend the whole number system to rational numbers (e.g., Joutsenlahti and Kulju 2017; Siegler et al. 2013). One must have a clear picture of the different meanings of fractions to understand this extension. The different meanings of fractions are introduced separately without clear context in the Finnish mathematics textbooks, in which the fractions are often taught emphasizing a procedural perspective. The conceptual understanding about fractions and their different meanings is left with less attention (cf. Lemke 2002).

#### 3.1. Literature Review of Meanings of the Symbol “ $\frac{a}{b}$ ”

Researchers have found different subconstructs which are related to the symbol “ $\frac{a}{b}$ ”. Each subconstruct creates a context for the fraction that gives it a contextual meaning and these subconstructs also refer to the extension of whole numbers to rational numbers. Pantziara and Philippou (2012), for example, highlighted that the diverse constructs of fractions make it difficult for pupils understanding the concept of fractions. They described the following five subconstructs: part-whole, ratio, quotient, measure, and operator. We collected, from the literature, some typical approaches to the mathematical symbol “ $\frac{a}{b}$ ” in Table 1.

**Table 1.** The approaches and the meanings of the symbol “ $\frac{a}{b}$ ”.

Approach to “ $\frac{a}{b}$ ”	Fraction	Rational number	Division	Ratio	Others
<i>Mathematical and historical approach</i> (Klein 1968; Park et al. 2013)	part-whole, set-theoretical	measurement	division	ratio	
<i>Pedagogical approach A</i> (Kieren 1976; Stewart 2005; Pantziara and Philippou 2012)	part-whole, set-operator	measurement	quotient	ratio	
<i>Pedagogical approach B</i> (Joutsenlahti et al. 2017)	part out of a sum of parts, a fraction of given whole	rational number	division	ratio	probability

In Table 1, we separate two different approaches: (1) mathematical and historical and (2) pedagogical. The first one is based on how the symbol “ $\frac{a}{b}$ ” has been described and used from the point of view of mathematics and takes into account the

historical development of the meanings of fractions. The pedagogical approach is based on analysis of the concept “fraction” in school mathematics and, especially, how it appears in mathematics textbooks. The more detailed description of the meanings of the symbol “ $a/b$ ” can be found in the writers’ earlier article (Joutsenlahti and Perkkilä 2019).

The meanings of the concept could be presented in many different ways: mathematical symbolic language (e.g., “ $2/3$ ”), natural language (e.g., “two of three parts”), or pictorial language (e.g., Figure 3) (Joutsenlahti and Perkkilä 2019; Shaughnessy 2005). Student’s own meaning making processes need the use of natural language (most often, the student’s mother tongue) and visual representations in learning activities (e.g., in studying new mathematical concepts and doing exercises) (Joutsenlahti and Perkkilä 2019; Lemke 2002; Morgan 2001).

If we want to learn mathematical concepts by understanding, we must have an awareness of how the new concepts are related to other concepts, and the ability to use them meaningfully in new contexts. We can distinguish the following expressions when we see mathematical symbolic language, mathematics by natural language, or by pictorial language: (1) from the point of view of a concept, we can speak of representations of the concept (Díez-Palomar et al. 2018; Joutsenlahti and Perkkilä 2019; Vosniadou 1999); (2) when a student expresses mathematical thinking, the student can use different multimodal approaches; and (3) when a reader makes meanings for a mathematical text, the languages can be seen as a multisemiotic approach (Joutsenlahti and Perkkilä 2019). In this article, we concentrate on connecting mathematical symbolic language and pictorial language by interpreting students’ expressions in natural language.

### *3.2. Research Questions*

In this article, we study the same phenomena related to the concepts (see Table 1) fraction, rational number, ratio, division, and probability as we did in our earlier article (Joutsenlahti and Perkkilä 2019). Now we have new data, which have been collected by a different study design: the prospective teachers (class teachers) answered the research questions in small groups (mainly two students in a group). We think that it is interesting to research how collaborative work helps students find meanings for the mathematical symbol “ $2/3$ ” in different given contexts. If we consider prospective class teachers, we can understand the difficulties in the conceptual learning of fractions and find new development targets for teacher education. The modified components of ALM (Figure 2) give us the useful theoretical frame, because they take into account the features of mathematical proficiency and practices. The third

component of ALM is central from the point view of the interpretations and what kind of meanings the students found for the symbol “ $2/3$ ”.

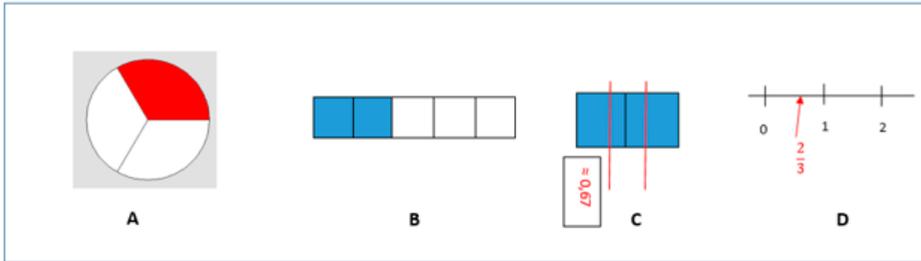
In this article, the focus was to compare the answers given, by a single person (one student in a group) and in pairs (two students in a group), to the questions (Joutsenlahti and Perkkilä 2019):

1. What meanings do students give spontaneously for the symbol “ $2/3$ ”?
2. What relationships do students find for the given pictures and the symbol “ $2/3$ ”?
3. What kind of influence does the multisemiotic approach have on students’ interpretations in collaborative thinking?

### 3.3. *Materials and Methods*

Our data were collected in two phases: in 2017, first-year students ( $N_{2017} = 102$ ) completed the research in single-person groups and in 2018, first-year students ( $N_{2018} = 136$ ) completed the research in two-person groups. The students were from the University of Tampere and the University of Jyväskylä. The data were collected during the mathematics didactics course for first-year students by questionnaires in spring 2017 and in spring 2018. In 2017, the questionnaire had two pages: on the first page students gave their opinions spontaneously about what different meanings the mathematical symbol “ $2/3$ ” can have and on the second page the students were asked to describe in natural language (Finnish) how the pictures A–D (Figure 4) were connected with the mathematical symbol “ $2/3$ ”. In 2018, the questionnaire was answered in the computer environment, but the questions were mostly the same (there were some more questions). The idea of the questionnaire was based on our theoretical background (see Table 1). In both controlled tests, the students had the same time (1 h) to give answers. From the answers to the questionnaire, we wanted to obtain an understanding about students’ perceptions of the meanings of the mathematical symbol “ $2/3$ ”, and, on the other hand, how a multisemiotic approach supports the interpretations of collaborative thinking.

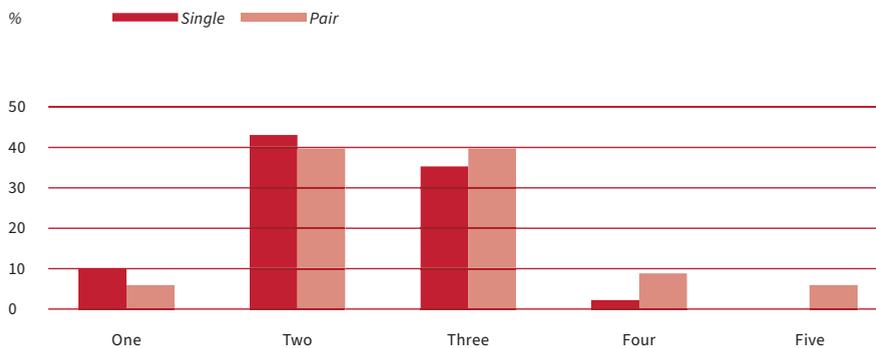
The data were analyzed by mixed methods. We used the IBM SPSS statistics 24 program for typical statistical analysis (e.g., mean and standard deviation). The qualitative analysis was done by theory-guided content analysis (e.g., categorizations). We used the classification into the six categories presented earlier in the theoretical part of the article and, on the other hand, the categories generated by the answers.



**Figure 4.** The second question of the research questionnaire: How the figures A–D describe mathematical symbol “ $2/3$ ” (Joutsenlahti and Perkkilä 2019).

#### 4. Results

In the first question of the questionnaire, the first-year students ( $N_{2017} = 102$  and  $N_{2018} = 136$ ) gave as many meanings as they discovered for the mathematical symbol “ $2/3$ ”. If we study the results from the point of view of groups, we can see that, in 2017, there were 102 students who worked alone (single/one student per group) and, in 2018, there were 68 groups (two students per group). Student groups spontaneously found different numbers of meanings for the symbol “ $2/3$ ”. The proportional quantities of the frequencies for each number (one, two, ..., five) are shown in Figure 5. We can see that working in pairs produced relatively more different meanings.

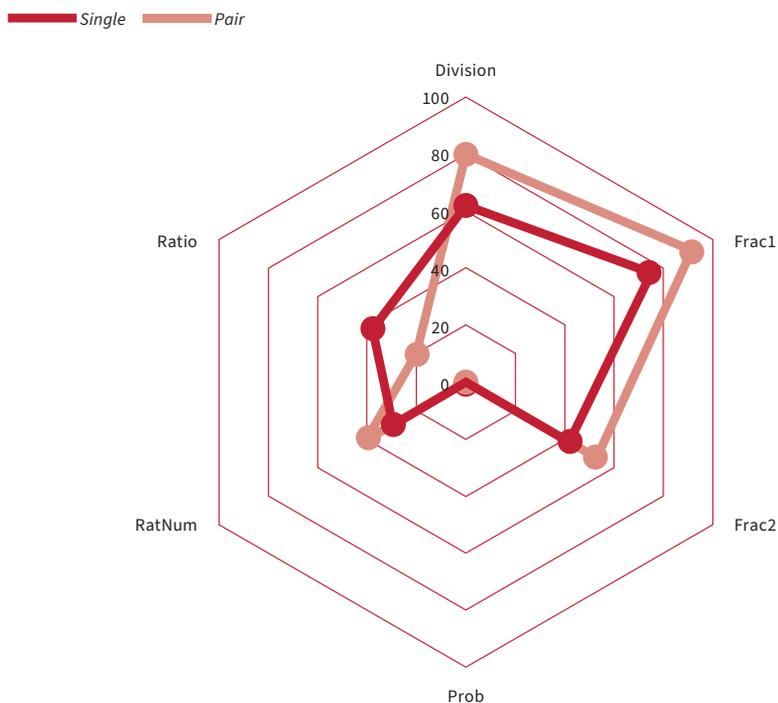


**Figure 5.** The proportional quantity of different meanings per a group that found single-student groups (single) and two-student groups (pair).

In the first question of the questionnaire, groups’ ( $N_{2017} = 102$  and  $N_{2018} = 68$ ) spontaneous understanding about the mathematical symbol “ $2/3$ ” was, in most cases, as a fraction ( $N_{2017} = 77$  and  $N_{2018} = 62$  groups gave “two thirds of a given whole”

(Frac1) and  $N_{2017} = 43$  and  $N_{2018} = 36$  groups gave “two of three parts” (Frac2)), which are typical also in the Finnish mathematics textbooks. Ratio, rational numbers, and probability were mentioned the least.

In Figure 6, we can see that two-student groups found relatively more meanings “Division”, “two thirds of a given whole” (Frac1), “two of three parts” (Frac2), and rational number (RatNum), but less meanings for probability (Prob) and ratio.



**Figure 6.** Proportional quantity (percent) of different meanings per a group (Single—single-student groups ( $N_{2017} = 102$ ); Pair—two-student groups ( $N_{2018} = 68$ ); Frac1—“two thirds of a given whole”; Frac2—“two of three parts”; Prob—probability; RatNum—rational number).

In the second question of the questionnaire, the groups’ problem was writing how four figures (Figure 4) described the symbol “ $2/3$ ”. In Table 2, proportional frequencies (percent) are calculated as the observed frequencies for each group (single or pair). From the table, we can see that the groups’ interpretations are mostly the same (highlighted with yellow), but the two-student groups have interpretations focused more on typical meanings (fraction in Figure A, division in Figure C, and rational

number in Figure D). Figure B (Table 2) is an exception because single-student groups found relatively more ratio meanings. “Other” meanings (Table 2) are meanings other than what have been given in the table. Most of these “other” meanings were vague descriptions that could not be meaningfully interpreted in this study. On the other hand, it is interesting to notice that two-student groups found, concerning Figure B, several different meaningful meanings. For example, the interpretation of the connection between Figure B and Frac1: *“The figure has one whole, or three white squares, next to each other. Next to it is two-thirds of it, or two blue squares”* (Two-student group 13). Collaborative thinking seems to be creative.

**Table 2.** Proportional frequencies (quantity per each group) of the interpretations how mathematical symbol “ $2/3$ ” is connected to figures A–D (Figure 3). (S—single-student groups ( $N_{2017} = 102$ ); P—two-student groups ( $N_{2018} = 68$ ); Frac1—“two thirds of a given whole”; Frac2—“two of three parts”; Prob—probability; RatNum—rational number).

Meaning	Figure A		Figure B		Figure C		Figure D	
	S(%)	P(%)	S(%)	P(%)	S(%)	P(%)	S(%)	P(%)
Frac1	57.8	69.1	0	4.4	17.6	4.4	44.1	29.4
Frac2	37.3	30.9	0	1.5	0	0	0	0
Ratio	2	0	42.2	35.3	0	0	0	0
Division	1	0	1	0	54.9	66.2	1	1.5
Rational	2.9	0	0	2.9	1	0	40.2	64.7
Other	15.7	0	48	55.9	14.7	29.4	7.8	4.4

Two-student groups ( $N_{2018} = 68$ ) had a new problem where they had to consider if the given Figure A or B (see Figure 3) could somehow describe the symbol “ $2/3$ ” and if they answered in the affirmative, they gave an example about the meaning. In Table 3, we can see that the groups invented, by collaborative thinking, mostly good argumentations for Figures A and B, but, particularly, the meanings “two of three parts” (Frac2) and division were difficult to connect meaningfully to Figure B. Some of the argumentations for Figure B (e.g., Frac1 and division) show creative thinking, probably supported by collaborative thinking in the groups. The students invented new contexts guided by the Figures and they unequivocally adapted the content of the chosen concept to themselves.

**Table 3.** Two-student groups' ( $N_{2018} = 68$ ) problem: Is it possible connect the given figure (A or B in Figure 3) to the symbol " $2/3$ "? If the answer is "yes", then explain how. (N("yes"): frequency of "yes" answers, N("good"): frequency of "good" answers).

" $2/3$ "	Figure A		Figure B	
				
	N("yes")	Example of a good argumentation, (N("good"))	N("yes")	Example of a good argumentation, (N("good"))
FRAC1	67	The pattern is one complete pattern divided into three parts, two of which are white. (Group 29), (65)	14	The illustration shows a sticker sheet divided into five sections. If the sheet folds between the blue and white sections, the blue section will cover two thirds of the white label sheet. (Group 52), (4)
FRAC2	66	The figure has three parts, two of which are colored in white, i.e., two of the three are colored in white. (Group 30), (66)	12	(0)
RATIO	39	The ratio of two white parts to the whole circle is the ratio of two to three. (Group 49), (27)	63	Oh yes, because the image can be interpreted as making juice sealant, adding two parts blue and three parts white water. (Group 16), (54)
DIVISION	31	If the circle is set to two, the sectors can be used to represent a division by two divided by three. (Group 47), (7)	28	Two blue boxes can be shredded into three equal parts and divided into three white boxes evenly. This can illustrate the division of two into three and at the same time it is noticed that one blue box does not go full blue, so $2:3$ must be less than one whole. (Group 58), (2)

## 5. Discussion

### 5.1. Collaborative Aspect

The students interpreted the symbol “ $2/3$ ” most frequently as a fraction or division (Figure 6). The two-student groups (pair) found, spontaneously, a proportionally bigger quantity of different meanings (Figure 5) and mainly gave, proportionally, more different meanings (Figure 6) than the single-student groups. Collaborative thinking and discussions with the peer (linguaging) in two-student groups obviously strengthened the student’s conceptual understanding and strategic competence, which they needed in the problem-solving situation.

Figures A–D (Figure 4) were interpreted typically, with Figure A meaning fraction (Frac1 and Frac2), Figure B meaning ratio, Figure C meaning division, and Figure D meaning rational number (RatNum) or fraction (Frac1) (Table 2). The two-student groups found proportionally more of those typical meanings for the figures (especially A, C, and D), but Figure B was an exception (Table 2). In the interpretations of the meanings of Figures A–D, we noticed that the students connected three different languages (Figure 3): pictorial and mathematical symbolic languages were present and in the students’ task to combine the meaning they found the two languages and expressed them by natural language. Collaborative thinking and peer discussion helped solve the problem of how the three languages are connected by the given information. Students need meaning making, justification skills, modeling with mathematics, structure understanding, and skills to find regularities during the solution process (mathematical practices in ALM).

The two-student groups thought systemically that the given figures A or B could somehow be interpreted as symbol “ $2/3$ ”. Table 3 shows that the groups found a lot of possible contexts for almost every option. Collaborative thinking in each group brought good examples for the typical connections (compare Table 2). It is interesting that the students invented contexts of new kinds for the familiar meanings of symbol “ $2/3$ ” by oral languaging, in which they reached a shared view. At the same time, when students create their own contexts for the different meanings of the symbol “ $2/3$ ”, they deepen their own and their mutual understanding about the concept and concept network.

Contradictions the students faced in the problems (e.g., Table 2) illustrated the narrowness of the learning materials, which were mainly constructed as definition-based, without an inquiry approach to concepts. These exercises help students understand that, e.g., the meanings of mathematical symbols are constructed only in their contexts. The curriculum (2014) emphasizes wide-ranging entities as

part of sustainable development, which requires multiliteracy skills (e.g., languaging skills). Therefore, the learning materials should also include a variety of contexts for concepts (including symbols) and guide students to recognize different meanings. Working as described above enables students to deepen the concepts they have already learned as well as to acquire new concepts and integrate them into existing conceptual networks.

In general, the results show that collaborative mathematical thinking in pairs helped students find more meaning for the mathematical symbol " $\frac{2}{3}$ " (cf. Moschkovich 2015a, 2015b, 2019). Students took part in a math discussion that gave them good tips on the meaning of the discussion and how to share their thinking in mathematics learning situations (cf. Joutsenlahti and Kulju 2016). By considering the meaning of the mathematical symbol " $\frac{2}{3}$ " together, they also deepened their understanding about the meanings of the fraction (cf. Joutsenlahti and Perkkilä 2019).

## 5.2. *ALM Supporting Sustainability*

Re-examining our research from a languaging point of view in the data collection situation, we noted that when the students were answering our questionnaire they were codeswitching between pictorial, natural language, and mathematical symbolic language. At the same time, they explained their mathematical thinking to each other. There was a relationship between language and mathematical thinking in this situation. We can imagine that by solving the problems of our questionnaire, students had to have literacy in understanding pictorial language and understanding about mathematical language. Solving these problems together, students had to have proficiency in the content of mathematics but also competences in collaborative mathematical thinking and languaging, and mathematical practices. (cf. Moschkovich 2019; see also Figures 1 and 2). We can cautiously think that the ALM components were present in the research situation, helping students to deepen their mathematical thinking (see Figures 1 and 2). Students took part in mathematical discussions that gave them experiences on the meaning of discussion about sharing their thinking in mathematics learning situations (see Figures 1 and 2; collaborative mathematical thinking and languaging; mathematical practices). By considering the meaning of the mathematical symbol " $\frac{2}{3}$ " together, they also deepened their understanding of the meanings of the fraction (see Figures 1 and 2; mathematical proficiency). Moschkovich (2019) and Moschkovich and Zahner (2018) highlighted that sociocultural aspects of the ALM framework work together with mathematical proficiency. We can assume that these aspects were socioculturally

included in our data gathering situations and made the learning situations dynamic and possibly improved the students' meaning making of conceptual understanding in mathematics, especially about the meaning making of fractions (see Figures 1 and 2). In this way, the sociocultural aspects supported the deepening of mathematical thinking and, at the same time, broadened the students' perspective of fractions. Through collaborative mathematical thinking, students promoted sustainability in their own mathematical thinking and professional development in mathematics (cf. (Widiaty and Juandi 2019)). Collaborative mathematical thinking has connected ALM with 21st Century competences. Pair-working in the research situations included fields of the 21st Century competences subareas: civic literacy, global awareness, and cross-cultural skills; critical and inventive thinking; and communication, collaboration and information skills. In the meaning making situations, students had to use civic literary (languaging), global awareness and cross-cultural skills (different learners worked together), critical and inventive thinking (meaning making for "2/3"), collaboration (collaborative mathematical thinking), and information skills (multimodality in expressing mathematical thinking) (cf. Figure 1). Based on our study, we noted that during teacher education, prospective class teachers should have experiences (e.g., collaborative mathematical thinking, languaging) that promote their mathematical knowledge and understanding and contribute to the building of their and future generations' basis of sustainable mathematics understanding. In addition, this will support sustainable education and 21st Century competences. We summarize, in Table 4, the factors that support the understanding of key concepts and concept networks in mathematics education from the perspective of 21st Century skills. The objectives presented in Table 4 are objectives for both teacher education, teachers, and students, and they support ALM and ESD in mathematics learning (See Figures 1 and 2).

**Table 4.** Sustainable development goals for teacher education, teachers and students in mathematics teaching and studies.

Understanding Learning of Mathematical Concepts And Concept Networks		
<i>21st Century Competences</i>	<i>Teacher</i>	<i>Student</i>
<i>Contextualization</i>		
<ul style="list-style-type: none"> <li>- Global awareness and cross-cultural skills</li> </ul>	<ul style="list-style-type: none"> <li>- <i>to understand</i> students' cultural knowledge and its use in constructing the meanings of mathematical concepts (in meaning making)</li> <li>- <i>to contextualize</i> mathematical concepts in phenomena familiar to students</li> </ul>	<ul style="list-style-type: none"> <li>- <i>to highlight contextualization</i> in own examples and self-invented assignments in relation to mathematical concepts at hand</li> </ul>
<hr/>		
<i>Mathematical thinking</i>		
<ul style="list-style-type: none"> <li>- Critical and Innovative thinking</li> </ul>	<ul style="list-style-type: none"> <li>- <i>critical use</i> of the textbook in teaching</li> <li>- <i>to create and tailor</i> appropriate teaching approaches on mathematical concepts to be studied for students and small groups</li> </ul>	<ul style="list-style-type: none"> <li>- <i>the courage to question and to ask</i> questions about mathematical concepts and solutions</li> <li>- <i>to create</i> analogies and applications of the mathematical concepts to be studied</li> </ul>
<hr/>		
<i>Languaging</i>		
<ul style="list-style-type: none"> <li>- Communication</li> <li>- Collaborative and information skills</li> </ul>	<ul style="list-style-type: none"> <li>- <i>to understand</i> languaging as a teaching model</li> <li>- <i>to work together</i> on concept building, problem solving and game-based learning</li> <li>- <i>functional learning</i> as part of the concept understanding process</li> </ul>	<ul style="list-style-type: none"> <li>- <i>to describe</i> concepts and solution processes in their own words (including by pictures)</li> <li>- <i>to build</i> knowledge together with a peer group</li> <li>- <i>to give constructive feedback</i> to others (to other students or to peer group etc.)</li> </ul>

From a teacher's standpoint, contextualization (see Table 4) means that a teacher's knowledge of his/her students' mathematical skills must be good in order to contextualize mathematical concepts in phenomena that are familiar to students. When the learning atmosphere is good and a teacher knows his/her students, the teacher can focus on the potential for progress in what students say and do. Therefore, the students are free to bring multiple perspectives to learning situations,

e.g., by contextualizing their own examples and self-invented assignments in relation to the mathematical concepts at hand, and also interpreting mathematics as a tool for understanding, analyzing, and solving the problems in the neighborhood and surrounding society (cf. Widiaty and Juandi 2019). Often (e.g., in Finland) mathematics teaching is 'textbook driven'. Due to this, the nature of mathematics and the contents of school mathematics might appear for pupils only through textbooks. Then, there is no room left for the pupils to question and to ask questions about mathematical concepts and solutions and express their own thinking (cf. Joutsenlahti and Perkkilä 2019). The critical use of the textbook in teaching (see Table 4) gives students more time and space to question and to ask questions about mathematical concepts and solutions. Students also have time to construct their own mathematical concepts solution processes and *create* analogies and applications for the mathematical concepts to be studied. Asking questions, reasoning, logical thought, creating analogies and applications for mathematical concepts, describing and explaining, and justification are closely related to conceptual understanding, which is one of the five intertwined strands of the description of mathematical proficiency (see Kilpatrick et al. 2001; Moschkovich 2019). As mentioned in Section 2.1.1, mathematical proficiency includes following intertwined strands: procedural fluency, strategic competence, adaptive reasoning, and productive disposition (see Figures 1 and 2). These strands are included in Table 4. Languaging as a teaching method provides teachers and students space to express and explain mathematical concepts in versatile ways. Therefore, codeswitching can serve as a resource during teaching and learning (tactile functional language, pictorial language, natural language, and mathematical symbolic language; see Figure 3). Languaging of mathematical thinking helps students to analyze and structure their thinking and, in that way, understand mathematical concepts, procedures, and solution processes (Joutsenlahti and Perkkilä 2019; Kilpatrick et al. 2001; Moschkovich 2015b). From the sociocultural perspective, we can think that languaging as a teaching model involves students in discipline-based practices in mathematics learning situations that involve reasoning and justification, sharing different views, understanding, and communication. Collaborative mathematical thinking is involved in languaging situations, and by working in small groups or in pairs, students share their ideas with each other, produce their joint solutions and learn to give feedback to each other. In Table 4, the sustainable development goals for teacher education, teachers, and students in mathematics teaching and studies involve the elements of ALM. These goals are created to give tools for teaching and learning mathematical concepts and mathematical networks in sustainable ways. ALM supports ESD and 21st Century competences but, on the other hand, ESD and

21st Century competences also challenge ALM skills (see Figure 1). The goals in Table 4 give versatile ways to build a solid mathematical knowledge base, so that teacher education, teachers, and students can have better tools to solve and model ESD-based mathematical problems, e.g., in the social, economic, and environmental fields. The results (e.g., in Table 4) could be a base for mathematics education for a new curriculum of teacher education in universities.

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