JYU DISSERTATIONS 410

Keijo Mönkkönen

Integral Geometry and Unique Continuation Principles

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Foreword

I wish to thank my supervisor Joonas Ilmavirta for his support and guidance during my years as a PhD student of mathematics. He has helped me to become a little bit more mathematician after my short career in theoretical physics, but not to lose the physicist's way of thinking and calculating without thinking. I also want to thank Mikko Salo for offering me this rare but great opportunity to study inverse problems in his internationally recognized inverse problems group. I want to express my gratitude to the Department of Mathematics and Statistics of University of Jyväskylä for providing me a fruitful working environment in 2017–2021.

I thank Angkana Rüland for agreeing to be my opponent at the public examination of my dissertation. I also want to thank the pre-examiners Jürgen Frikel and Venky Krishnan for their valuable feedback. I wish to thank my colleagues Jesse Railo and Giovanni Covi for many inspiring moments in mathematics and outside mathematics. Sharing the office room with Jesse has been very beneficial for my research and career, and Giovanni's work on fractional inverse problems convinced me to become interested in that area of mathematics. I also thank Gunther Uhlmann for collaboration in a very interesting article on higher order fractional Calderón problems. I want to thank all the members in Mikko Salo's inverse problems group, former and present, for the friendly, supporting and encouraging spirit that has always existed in our group.

Finally, I want to thank my family and friends for their support in my long and undirected journey in the academic world.

Jyväskylä, July 1, 2021
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List of included articles

This dissertation consists of an introductory part and the following seven articles:

[A] J. Ilmavirta and K. Mönkkönen. Unique continuation of the normal operator of the x-ray transform and applications in geophysics. *Inverse Problems*, 36(4):045014, 2020.

[B] G. Covi, K. Mönkkönen and J. Railo. Unique continuation property and Poincaré inequality for higher order fractional Laplacians with applications in inverse problems. *Inverse Probl. Imaging*, 15(4):641–681, 2021.

[C] J. Ilmavirta and K. Mönkkönen. X-ray Tomography of One-forms with Partial Data. *SIAM J. Math. Anal.*, 53(3):3002–3015, 2021.

[D] G. Covi, K. Mönkkönen, J. Railo and G. Uhlmann. The higher order fractional Calderón problem for linear local operators: uniqueness. Preprint (2020), arXiv:2008.10227.

[E] J. Ilmavirta, K. Mönkkönen and J. Railo. On tensor decompositions and algebraic structure of the mixed and transverse ray transforms. Preprint (2020), arXiv:2009.01043.

[F] K. Mönkkönen. Boundary rigidity for Randers metrics. *Ann. Fenn. Math.*, 2020. To appear.

[G] J. Ilmavirta and K. Mönkkönen. Partial data problems and unique continuation in scalar and vector field tomography. Preprint (2021), arXiv:2103.14385.

The author has participated actively in the research of the joint articles [A, B, C, D, E, G].

Abstract

In this thesis we study inverse problems in integral geometry and non-local partial differential equations. We will study these rather different areas of mathematical inverse problems by using the theory of non-local fractional operators. This thesis mainly focuses on proving different kind of unique continuation results of fractional operators which are then used to prove uniqueness results for fractional Calderón problems and partial data problems in scalar and vector field tomography.

The introductory part of the thesis contains a general introduction and review of inverse problems arising in medical and seismic imaging. The included articles are divided into three classes which are then presented in their own sections and studied in different levels of detail.

In the articles [A, B, C, G] we consider partial data problems in the X-ray tomography of scalar and vector fields. In the first article [A] we prove unique continuation for certain Riesz potentials and apply it to partial data problems of scalar fields. In the second article [B] we prove unique continuation results for higher order fractional Laplacians which are then used in proving uniqueness for partial data problems of d-plane transforms. In the third article [C] we study partial data problems of vector fields and we prove unique continuation of the normal operator of vector fields which implies uniqueness for the partial data problems. In the seventh article [G] we generalize the unique continuation result of fractional Laplacians proved in [B] and use it to prove uniqueness for partial data problems of scalar and vector fields, extending the partial data results of the articles [A, B, C] to more general cases.

In the articles [B, D] we consider higher order fractional Calderón problems. In the second article [B] we use the unique continuation of higher order fractional Laplacians to prove uniqueness for the Calderón problem of the higher order fractional (magnetic) Schrödinger equation. In the fourth article [D] we generalize the uniqueness result proved in [B] to include general lower order local perturbations of the fractional Laplacian.

In the articles [E, F] we consider the travel time tomography problem and its different linearized versions. In the fifth article [E] we study mixing ray transforms which are generalizations of the geodesic ray transform. We prove solenoidal injectivity results for them in various different cases. In the sixth article [F] we study the boundary rigidity problem on certain non-reversible Finsler manifolds which are also called Randers manifolds. We prove that if the Randers metric consists of a boundary rigid Riemannian metric and a closed 1-form, then the boundary distances determine the Randers metric uniquely up to a natural gauge.

Tiivistelmä

Tässä väitöskirjassa tutkitaan integraaligeometrian ja epälokaalien osittaisdifferentiaaliyhtälöiden inversio-ongelmia. Näitä melko erilaisia matemaattisia inversio-ongelmia tutkitaan käyttämällä apuna epälokaalien fraktionaalisten operaattoreiden teoriaa. Väitöskirja keskittyy pääosin todistamaan fraktionaalisten operaattoreiden erilaisia yksikäsitteisen jatkon tuloksia, joita käytetään todistaessa yksikäsitteisyyttä fraktionaalisille Calderónin ongelmille sekä skalaari- ja vektorikenttien tomografian osittaisen datan ongelmille.

Väitöskirjan johdantokappale sisältää yleisen tason johdatuksen sekä kirjallisuuskatsauksen lääketieteellisessä ja seismisessä kuvantamisessa esiintyviin inversio-ongelmiin. Väitöskirjaan sisällytetyt artikkelit on jaettu kolmeen luokkaan, jotka esitellään omissa kappaleissaan ja joita tarkastellaan yksityiskohtien osalta monella eri tasolla.

Artikkelit [A, B, C, G] käsittelevät osittaisen datan ongelmia skalaari- ja vektorikenttien röntgentomografiassa. Ensimmäisessä artikkelissa [A] todistetaan yksikäsitteinen jatko tietyille Rieszin potentiaaleille ja sitä sovelletaan skalaarikenttien osittaisen datan ongelmiin. Toisessa artikkelissa [B] todistetaan yksikäsitteisen jatkon tuloksia korkeamman kertaluvun fraktionaalisille Laplace-operaattoreille ja niitä käytetään d-tasomuunnosten osittaisen datan ongelmien yksikäsitteisyyden todistamisessa. Kolmannessa artikkelissa [C] tutkitaan vektorikenttien osittaisen datan ongelmia ja todistetaan vektorikenttien normaalioperaattorin yksikäsitteinen jatko, josta seuraa yksikäsitteisyys osittaisen datan ongelmille. Seitsemännessä artikkelissa [G] yleistetään artikkelissa [B] todistettu fraktionaalisen Laplace-operaattorin yksikäsitteisen jatkon tulos ja sitä käytetään skalaari- ja vektorikenttien osittaisen datan ongelmien yksikäsitteisyyden todistamisessa, laajentaen artikkeleiden [A, B, C] osittaisen datan tuloksia yleisempiin tapauksiin.

Artikkelit [B, D] käsittelevät korkeamman kertaluvun fraktionaalisia Calderónin ongelmia. Toisessa artikkelissa [B] käytetään korkeamman kertaluvun fraktionaalisten Laplace-operaattoreiden yksikäsitteistä jatkoa todistaessa yksikäsitteisyyttä korkeamman kertaluvun fraktionaalisen (magneettisen) Schrödingerin yhtälön Calderónin ongelmalle. Neljännessä artikkelissa [D] yleistetään artikkelin [B] yksikäsitteisyystulos fraktionaalisen Laplace-operaattorin yleisille alempiasteisille lokaaleille perturbaatioille.

Artikkelit [E, F] käsittelevät matka-aikatomografiaa ja sen linearisoituja versioita. Viidennessä artikkelissa [E] tutkitaan sekoitussädemuunnoksia, jotka ovat geodeettisen sädemuunnoksen yleistyksiä. Niille todistetaan solenoidisia injektiivisyystuloksia monissa eri tilanteissa. Kuudennessa artikkelissa [F] tutkitaan reunajäykkyysongelmaa tietyillä ei-reversiibeleillä Finsler-monistoilla, joita kutsutaan myös Randers-monistoiksi. Artikkelissa todistetaan, että jos Randers-metriikka koostuu reunajäykästä Riemannin metriikasta ja suljetusta 1-muodosta, niin Randers-metriikka määräytyy reunaetäisyyksistään luonnollista mittaa vaille yksikäsitteisesti.

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1. Introduction

This thesis is about mathematical inverse problems and the main focus is in proving uniqueness results for different problems arising in tomography. As the title of the thesis suggests, some of the inverse problems appear in integral geometry. However, there are also included inverse problems which do not strictly fit under this category, but they are related to problems in integral geometry via unique continuation principles of non-local operators.

The inverse problems studied in this thesis can be roughly divided into three classes:

(I1) The travel time tomography problem and its linearized versions
(I2) Partial data problems in X-ray tomography
(I3) Fractional Calderón problems.

The classes (I1) and (I2) belong to integral geometry. In fact, problems in (I2) are linearized travel time tomography problems in Euclidean background with partial data. Hence (I2) can be seen as a subset of (I1). The class (I3) belongs to non-local partial differential equations and at first sight has nothing to do with the classes (I1) and (I2). But there is a way to get from (I3) to (I1), namely using the "intermediate step" (I2).

A unifying theme between fractional Calderón problems (I3) and partial data problems in X-ray tomography (I2) is the use of unique continuation properties of non-local operators in proving uniqueness results. The central operator of this thesis is the fractional Laplace operator $(-\Delta)^s$, $s \in (-n/2, \infty) \setminus \mathbb{Z}$, and many of the main theorems of this thesis are unique continuation results of $(-\Delta)^s$ or corollaries of them. The unique continuation of $(-\Delta)^s$ is used to prove Runge approximation and hence uniqueness for fractional Calderón problems. As a special case of fractional Laplacians we have the normal operators of different X-ray transforms whose unique continuation properties are then used to prove uniqueness for various partial data problems arising in the X-ray tomography of scalar and vector fields.

This introductory part is organized in the following way. We first discuss in section 1.1 how the different articles of this thesis are related to each other. Then we give a gentle introduction to inverse problems and forward problems in section 1.2, and in sections 1.3–1.5 we review the main three classes of inverse problems (I1)–(I3) which are studied in this thesis. In sections 2–4 we go through the main theorems of the included articles. In the beginning of each section we first introduce the inverse problem and give the main results in a general level. We then go through the needed notation in sections 2.1–4.1 before giving the main theorems with all technical details in sections 2.2–4.2. Section 4 can be read independently of sections 2 and 3. Section 3 can also be read independently of section 2 if one first goes through the notation in section 2.1.

1.1. **On the articles of this thesis.** In figure 1 we have illustrated the connection between the different articles of this thesis. In most of the articles we study inverse problems with partial data: these include fractional Calderón problems (articles [B, D]), X-ray tomography with partial data (articles [A, B, C, G]) and linearized travel time tomography with "half-local" data (article [A]). Unique continuation of fractional Laplacians has a crucial role in proving uniqueness for partial data problems studied in this thesis. Fractional Laplacians arise in fractional Calderón problems and also in X-ray tomography in the form of different normal operators. Problems in X-ray tomography in turn can be seen as linearized travel time tomography problems in Euclidean background.

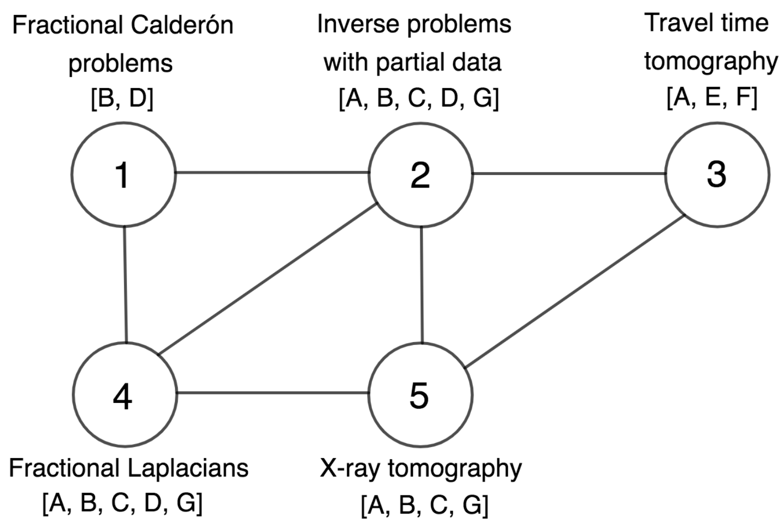

FIGURE 1. A graph illustrating the relation between the different articles of this thesis.

In fractional Calderón problems the task is to recover the potential (and more generally a perturbation) of the fractional Schrödinger equation in a bounded domain by doing measurements in the exterior of the domain. These problems are studied in the articles [B, D] and treated in section 3. In X-ray tomography we want to determine a scalar field (or a vector field) when we know its integrals over lines which intersect a given nonempty open set. This is studied in the articles [A, B, C, G] and treated in section 2. In travel time tomography one wants to recover the speed of sound (and more generally Riemannian metric or Finsler norm) by measuring travel times (geodesic distances) on the boundary of a compact manifold. This problem and its linearized versions (the geodesic ray transform and its generalizations) are studied in the articles [A, E, F] and treated in sections 2 and 4.

A remark from the point of view of graph theory: the (connected) graph presented in figure 1 has a Hamilton cycle, i.e. a closed walk such that every vertex is visited exactly once. The graph also has an Euler trail, i.e. a walk such that every edge is traversed exactly once. Formally, this “proves” that the articles of this thesis are closely related to each other. However, the graph does not admit an Euler tour (a closed walk which is an Euler trail) since not every vertex has even degree [14].

1.2. **Inverse problems and forward problems.** Inverse problems are practical or abstract problems which arise for example in medical and seismic imaging [62, 70, 78, 108, 112, 113, 114, 129, 157]. Inverse problems are often encountered when making indirect measurements. In such situations we have an object we cannot or do not want to access by invasive methods. In medical imaging the object can be a patient we want to study without doing surgical operations, and in seismic imaging the object can be the planet Earth whose deep interior we cannot reach by any practical means. The common task in both cases is that one wants to deduce the interior features of some object by making measurements on the boundary or in the exterior of the object. Usually we have some physical model which tells us how the interior properties of the object affect the measurements we make on the boundary or in the exterior. The goal is to use this physical model to deduce the interior properties of the object from the boundary or exterior measurements. The boundary and exterior measurements are often called just data.

Inverse problems are opposite to what we call direct problems or forward problems. Let us consider an example from X-ray tomography to illustrate the difference. In X-ray tomography one shoots X-rays through an object and studies the attenuation pattern of the X-rays. The attenuation of the X-rays is determined by the interior properties (the position-dependent attenuation coefficient) of the object. In the direct problem one knows the attenuation of the object and wants to determine the attenuation pattern of the X-rays. When the initial intensity of the X-rays is known, then one can easily calculate the final intensity of the X-rays by using a simple physical model [112]. Roughly saying, the direct problem corresponds to putting values for parameters in an equation and computing the result.

Inverse problems are much harder since they “operate” in the opposite direction. For example, in medical imaging one wants to determine the attenuation of the object instead of the attenuation pattern of the X-rays which can be easily measured. Since one also can control the initial intensity of the X-rays we have indirect information about the attenuation, i.e. we know the total attenuation of the X-rays and want to determine the attenuation of the object from that data. It turns out that the total attenuation corresponds to the integrals of the attenuation function along lines which intersect the object [112]. The inverse problem is to invert this integral transform which is also called the X-ray transform. The inversion of the X-ray transform is a much harder task than solving the forward problem where we already know the interior features of the object (the attenuation) and just have to calculate the end result (the final intensity of the X-rays).

Uniqueness, stability and reconstruction are important properties in the study of inverse problems. Uniqueness means that the inverse problem has a unique solution. In other words, if two objects produce the same boundary or exterior data, then they must have the same interior features. Reconstruction means that there is some way (e.g. an algorithm or formula) so that one can compute the desired physical quantity related to the interior properties of the object from the boundary or exterior data. Stability is related to how much measurement errors affect uniqueness or reconstruction. Since in practice there is always some noise in measurements, stability is important in showing that the reconstructed quantity is not too far away from the true value of that quantity. These three properties are not independent of each other since uniqueness usually follows from reconstruction and stability.

Uniqueness and stability have a connection to Hadamard's formulation of a well-posed problem [59, 60]. A mathematical problem related to a physical phenomenon is called well-posed, if the problem has unique solution which is stable with respect to the measured data (the solution depends continuously on the data) [62, 78, 108, 112, 129]. If the solution fails to exist, the solution is not unique or the solution does not depend continuously on the data, the problem is said to be ill-posed. Forward problems are often well-posed, but inverse problems tend to be ill-posed. Usually the reason for ill-posedness of inverse problems is that they lack stability which causes difficulties in numerical reconstruction [78, 80, 108, 114, 129].

In this thesis we mainly focus on uniqueness, i.e. in most of our theorems we show that the inverse problem has unique solution. Even if we do not get stability or a reconstruction formula for the problem, uniqueness is important in practical applications. Uniqueness for example increases the reliability of the results obtained in X-ray tomography when we only have a "small amount" of measurement data available.

1.3. X-ray tomography of scalar and vector fields.

1.3.1. *X-ray tomography of scalar fields.* X-ray tomography is a commonly used method in medical imaging to study interiors of objects. The main goal is to determine the attenuation of the object when one knows the initial and final intensities of X-rays, i.e. the total attenuation of the X-rays. The attenuation can be modelled as a scalar function $f\colon \mathbb{R}^n \to \mathbb{R}$. If the rays propagate parallel to x-axis, then the intensity I of the X-rays satisfies the differential equation $I'(x) = -f(x)I(x)$ and the total attenuation corresponds to the line integral [112]

$$\ln\left(\frac{I_0}{I_1}\right) = \int_\gamma f \mathrm{d}s \tag{1}$$

where I_0 is the initial intensity and I_1 is the final intensity of the X-rays, and γ is a line along which the X-ray beam propagates. The inverse problem in X-ray tomography is to solve f in equation (1) using different lines γ when the left-hand side of the equation is known.

The previous discussion motivates us to define the operator X_0 as

$$X_0 f(\gamma) = \int_\gamma f \mathrm{d}s \tag{2}$$

where γ is a line in $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a scalar field. The operator X_0 is called the X-ray transform of scalar fields and in two dimensions X_0 is also known as the Radon transform. The inverse problem is to invert the operator X_0 in equation (2) and it was first studied by Johann Radon [126]. Theoretical and practical applications to computerized tomography were studied by Cormack and Hounsfield [27, 28, 112]. There are formulas for the inversion of X_0 some of which involve the normal operator N_0 of the X-ray transform [64, 112, 129, 149]. The normal operator $N_0 = X_0^* X_0$ is defined as first applying the X-ray transform and then back-projecting $X_0 f$ from the space of all lines to a function in $\mathbb{R}^n$ using the adjoint X_0^* of the X-ray transform. Hence N_0 is a useful auxiliary operator which maps functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^n$ and one can study the X-ray transform X_0 using its normal operator N_0.

The inversion formulas for X_0 assume that we know the integrals of f over all lines in $\mathbb{R}^n$. In practical applications we only have access to a small subset of lines, and in that case we have a partial data problem. One such partial data problem is to uniquely determine f everywhere in $\mathbb{R}^n$ from its X-ray data on all lines intersecting a given open set $V \subset \mathbb{R}^n$. The integrals alone cannot determine f uniquely and one has to make additional assumptions [79, 112]. The partial data problem has unique solution, if $f|_V$ vanishes [29, 79], $f|_V$ is piecewise constant or piecewise polynomial [79, 162] or if $f|_V$ is real analytic [77]. A complementary partial data result is the Helgason support theorem where one has access to lines which do not intersect a given compact and convex set and the problem is to determine the scalar field uniquely outside that set [64]. Partial data problems are in general much harder to treat than problems with full data because the reconstruction is not stable anymore and there can be artefacts in the images even if the problem admits a unique solution. In such cases we have "invisible singularities" [83, 84, 112, 124, 125].

We can generalize the transform X_0 from lines to affine d-dimensional planes where $0 < d < n$. The d-plane transform $\mathcal{R}_d$ is defined as [64]

$$\mathcal{R}_d f(A) = \int_A f(x) \mathrm{d}m(x) \tag{3}$$

where A is an affine d-dimensional plane, m is the d-dimensional Hausdorff measure and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a scalar field. The case $d = 1$ corresponds to the X-ray transform X_0 and the case $d = n - 1$ is often called the Radon transform which coincides with the X-ray transform in two dimensions [64, 112, 129]. As before, the inverse problem is to invert the transform $\mathcal{R}_d$ in equation (3). There is an inversion formula in terms of the normal operator $\mathcal{N}_d$ of the d-plane transform which is defined in a similar way as in the case of the X-ray transform [64]. One example of partial data results for d-plane transforms is the Helgason support theorem where one knows the integrals of the scalar field over all d-planes which do not intersect a given compact and convex set [64].

The d-plane transform (also called the k-plane transform in some works) has been extensively studied after the pioneering work by Fuglede [49] and Helgason [63]. See for example [2, 55, 64, 68, 85, 127] and the works by Rubin [130, 131, 132, 133, 134].

1.3.2. *X-ray tomography of vector fields.* X-ray tomography is also used in the imaging of moving fluids which is based on Doppler backscattering or acoustic travel time measurements. If $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector field which represents the flow field of a moving fluid, then after a linearization procedure one ends up studying the transform [115, 116]

$$X_1 h(\gamma) = \int_\gamma h \cdot \mathrm{d}\overline{s}. \tag{4}$$

The operator X_1 is called the X-ray transform of vector fields and it has applications for example in medical ultrasound imaging [73, 75, 140, 148]. The inverse problem is to invert the operator X_1 in equation (4).

Unlike in the scalar case we have a natural gauge: the gradients of scalar fields which vanish at infinity are always in the kernel of X_1. For this reason one can determine the vector field h only up to potential fields from its X-ray transform, i.e. one can only determine the solenoidal part h^{s} in the Helmholtz decomposition $h = h^{\mathrm{s}} + \nabla\phi$ where $h^{\mathrm{s}}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector field such that $\operatorname{div} h^{\mathrm{s}} = 0$ and ϕ is a scalar field [140, 143, 149]. The solenoidal part can be uniquely determined from the full X-ray data and there is an inversion formula in terms of the normal operator $N_1 = X_1^* X_1$ of the X-ray transform of vector fields where X_1^* is the adjoint operator (or back-projection) [75, 115, 143, 148, 149].

Like in the scalar case, one can also study X-ray tomography of vector fields with partial data. The main goal in such problems is to determine the solenoidal part of the vector field from its partial X-ray data. Examples of such partial data results include cases where one knows the integrals of the vector field over lines which intersect a certain type of curve [42, 128, 159] or which are parallel to a finite set of planes [75, 139, 142]. There is also a vectorial version of the Helgason support theorem where one knows the integrals of the vector field over all lines not intersecting a given convex and compact set [149].

1.4. **Electrical impedance tomography and its non-local versions.**

1.4.1. *The Calderón problem.* Electrical impedance tomography (EIT) is an imaging method which has applications in geophysics and medical imaging [78, 108, 157]. EIT is based on the conductivity equation and the inverse problem is known as the Calderón problem. In the Calderón problem we have an object whose electrical properties we want to deduce by making boundary measurements. In particular, we want to determine the conductivity inside the object by applying voltages on the boundary and measuring the induced currents on the boundary which depend on the electrical properties of the interior of the object.

We can model the object as a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$. The conductivity equation is [157]

$$\begin{cases} \nabla \cdot (\eta \nabla u) &= 0 \;\text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{cases} \tag{5}$$

where f is the potential on the boundary, u is the induced potential in Ω and η is the electrical conductivity of Ω which is assumed to be sufficiently smooth positive function. The measurements are encoded in the Dirichlet-to-Neumann (DN) map Λ_η which tells how the electrical properties of the interior induce normal currents on the boundary when one applies the voltage f on the boundary. More specifically, one can write $\Lambda_\eta f = (\eta \partial_\nu u)|_{\partial\Omega}$ where ν is the outer unit normal on $\partial\Omega$. The inverse problem is to determine the conductivity η in equation (5) by applying different boundary values f (voltages) and measuring the induced currents $\Lambda_\eta f$. In particular, the uniqueness problem is the following: if $\Lambda_{\eta_1} f = \Lambda_{\eta_2} f$ for all boundary values f, does it follow that $\eta_1 = \eta_2$? This problem was first studied mathematically by Alberto Calderón and the inverse problem is therefore known as the Calderón problem [19].

Using the substitution $\tilde{u} = \sqrt{\eta} u$ one can convert the conductivity equation (5) to the following Schrödinger equation [110, 154, 157]

$$\begin{cases} (-\Delta + q)\tilde{u} &= 0 \;\text{ in } \Omega \\ \tilde{u}|_{\partial\Omega} &= \tilde{f}. \end{cases} \tag{6}$$

Here $q = (\Delta\sqrt{\eta})/\sqrt{\eta}$ now corresponds to the electric potential in Ω and $\tilde{f} = \sqrt{\eta} f$. The DN map Λ_q for equation (6) can be written as $\Lambda_q \tilde{f} = \partial_\nu \tilde{u}|_{\partial\Omega}$ assuming $\partial\Omega$ is regular enough. The interpretation of the DN map is as in the conductivity equation: the DN map tells how the applied voltage on the boundary induces normal currents on the boundary via the electrical properties of the interior of the object. The inverse problem now is to determine the potential q in equation (6) by applying different boundary values $\tilde{f}$ (voltages) and measuring the induced currents $\Lambda_q \tilde{f}$. The uniqueness problem is as for the conductivity equation: if $\Lambda_{q_1} \tilde{f} = \Lambda_{q_2} \tilde{f}$ for all boundary values $\tilde{f}$, does it follow that $q_1 = q_2$? One standard tool in proving uniqueness for the Calderón problem of the conductivity equation (5) and

Schrödinger equation (6) is the construction of complex geometrical optics solutions [8, 19, 153, 154, 157].

1.4.2. *The fractional Calderón problem.* One can study the non-local version of the Schrödinger equation (6) as follows. One replaces the Laplacian $-\Delta$ with the fractional Laplacian $(-\Delta)^s$ which is the pseudodifferential operator

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{u}), \quad s \in (-n/2, \infty) \setminus \mathbb{Z}.$$

The fractional Laplacian is a non-local operator in contrast to the ordinary Laplacian: the value $(-\Delta)^s u(x)$ depends on the values of u everywhere in $\mathbb{R}^n$ while $-\Delta u(x)$ depends only on the values of u in a small neighborhood of $x \in \mathbb{R}^n$. For example, the normal operator of the X-ray transform N_0 (and more generally the normal operator of the d-plane transform $\mathcal{N}_d$) is the fractional Laplacian $(-\Delta)^{-1/2}$ (more generally $(-\Delta)^{-d/2}$) up to a constant factor. In addition to integral geometry fractional Laplacians arise also in non-local diffusion [16, 45, 54] and in fractional quantum mechanics [90, 91].

Replacing $-\Delta$ with $(-\Delta)^s$ where $s \in (0,1)$ we obtain the fractional Schrödinger equation introduced in [54]

$$\begin{cases} ((-\Delta)^s + q)u &= 0 \ \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases} \tag{7}$$

where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the exterior of the bounded domain $\Omega \subset \mathbb{R}^n$. For such non-local equation (7) it is more natural to consider exterior values $u|_{\Omega_e} = f$ instead of boundary values. The DN map Λ_q maps the "non-local voltage" f to a non-local version of the normal current [54]: under stronger assumptions one can write $\Lambda_q f = (-\Delta)^s u|_{\Omega_e}$. In the fractional Calderón problem one wants to determine the potential q in equation (7) by applying different exterior values f and measuring the induced "exterior currents" $\Lambda_q f$. The uniqueness problem is similar as in the local case: if $\Lambda_{q_1} f = \Lambda_{q_2} f$ for all exterior values f, does it follow that $q_1 = q_2$? The fractional Calderón problem for equation (7) was first studied by Ghosh, Salo and Uhlmann [54].

In fractional Calderón problems instead of constructing complex geometrical optics solutions one can exploit the non-locality of the equation and especially the non-local behaviour of the operator $(-\Delta)^s$. One has the following unique continuation property of fractional Laplacians [54]: if $s \in (0,1)$ and $(-\Delta)^s u|_V = u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$. Clearly such property cannot hold for local operators such as $-\Delta$. The unique continuation of $(-\Delta)^s$ is in essential role in proving uniqueness for fractional Calderón problems [13, 21, 30, 54].

After the seminal work [54] there have been numerous results for different variants of the fractional Calderón problem: these include stability and instability results [135, 136], uniqueness under single measurement [53], magnetic versions of the fractional Schrödinger equation [30, 96, 97, 98], lower order local and non-local perturbations [13, 21], semilinear equations [87, 88], fractional conductivity and heat equations [31, 89, 137] and equations arising from a non-local Schrödinger-type elliptic operator [20, 52].

1.5. **Travel time tomography and its linearization.**

1.5.1. *The boundary rigidity problem.* In seismic travel time tomography the objective is to study the interior properties of the Earth by measuring travel times of seismic waves on the surface of the Earth [22, 62, 144, 152]. It is impossible to access the deep interior of the Earth by any practical means and the only way to obtain information is by doing indirect measurements on the surface. The travel times of seismic waves depend on the speed of sound in the medium where the wave propagates. Therefore the travel times contain indirect information about the physical properties of the Earth.

The Earth can be modelled as a three-dimensional compact manifold M with boundary ∂M (e.g. a closed ball). Assuming that the medium is isotropic the speed of sound depends only on position and it becomes a positive scalar function $c\colon M \to (0, \infty)$. The travel time of a seismic wave or ray can be expressed as the line integral [22]

$$T = \int_\gamma \frac{\mathrm{d}s}{c} \tag{8}$$

where γ is the ray path. The travel time tomography problem or inverse kinematic problem is to solve the speed of sound c in equation (8) when the travel times T measured on the surface are known.

The travel time tomography problem was studied first in 1900s by Herglotz, Wiechert and Zoeppritz [65, 160]. They solved the problem assuming that the speed of sound is radial $c = c(r)$ and satisfies the Herglotz condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right) > 0. \tag{9}$$

Under these assumptions the solution reduces to the inversion of an Abel-type integral transform [117, 144]. The Herglotz condition (9) is equivalent to the condition that the travel times in equation (8) are finite [36]. In geometrical terms, the Herglotz condition (9) means that one can foliate the manifold M with strictly convex hypersurfaces (i.e. spheres) [152].

The travel time tomography problem can be formulated in a more geometrical way. The speed of sound c determines the Riemannian metric $g_c = c^{-2}(x)e$ where e is the Euclidean metric. By Fermat's principle the rays propagate along geodesics of the metric g_c and the travel times correspond to lengths of these geodesics [22]. The inverse problem is to determine the scalar function c, or equivalently the metric g_c, from the lengths of all geodesics connecting points on the boundary ∂M. One sees that the problem is highly non-linear since the geodesics depend on the function c (or the metric g_c).

One can study the above geometric problem in a more general case: if g is a Riemannian metric, determine g from the distances between boundary points (boundary distances) given by g. This geometric inverse problem is known as the boundary rigidity problem [152]. In particular, one problem of interest is the uniqueness problem: if two Riemannian metrics g_1 and g_2 give the same boundary distances, does it follow that $g_1 = g_2$? The answer is no in general since there is a gauge: if $g_2 = \Psi^* g_1$ where $\Psi\colon M \to M$ is a diffeomorphism which is identity on the boundary, then g_1 and g_2 give the same boundary distances [152]. Hence without further restrictions one can determine the metric only up to a boundary preserving diffeomorphism.

The boundary rigidity problem is a difficult non-linear inverse problem and it has been solved only in certain special cases where the manifold admits strictly convex foliation [150, 152] or the manifold is known to be simple (a generalization of a Euclidean ball). Boundary rigidity holds for simple subspaces of Euclidean space [57] and simple subspaces of symmetric spaces of constant negative curvature [12]. In two dimensions examples include simple subspaces of the open hemisphere [104] and simple spaces of negative curvature [33]. If the Riemannian metrics on a compact simple Riemannian manifold are in the same conformal class, then the distances between boundary points determine the metric uniquely, i.e. the diffeomorphism Ψ becomes identity in this case [34, 109, 152]. In general, compact simple Riemannian manifolds are known to be boundary rigid in two dimensions [122], but it is conjectured that boundary rigidity holds for compact simple Riemannian manifolds of any dimension [103].

In the travel time tomography problem one usually assumes that the speed of sound c is isotropic, i.e. it only depends on position. However, anisotropies have been observed in the shallow crust, upper mantle and inner core of the Earth [32, 46, 144]. Therefore it is reasonable to consider c as a function on the tangent bundle $c\colon TM \to (0, \infty)$ so that the dependence on the direction of propagation can be taken into account. If the sound speed is anisotropic, then the seismic rays propagate along geodesics of a Finsler norm and we need Finsler geometry to treat the anisotropies [7, 161]. The travel time tomography problem can then be expressed as a boundary rigidity problem on Finsler manifolds where the fiberwise inner product depends not only on position but also on direction.

The boundary rigidity problem is much harder in the Finslerian case since there are non-isometric Finsler norms which give the same boundary distances [17, 25, 26, 72]. This means that in general Finsler norms are not rigid in the same way as Riemannian metrics. However, some rigidity results are known in certain special cases. Projectively flat Finsler norms on compact convex domains of $\mathbb{R}^2$ are uniquely determined by their boundary distances [4, 5, 86]. When we restrict ourselves to Finsler norms which are relevant in seismology, we can expect more rigidity: one can use the collection of boundary distance maps to determine the differential and topological structures of Finsler manifolds [38], and the broken scattering relation determines the isometry class of reversible Finsler manifolds which admit strictly convex foliation [37].

1.5.2. *Linearized versions of the boundary rigidity problem.* Let us study the linearization of the boundary rigidity problem. Let $\epsilon > 0$ and $s \in (-\epsilon, \epsilon)$. Assume that g^s is a family of Riemannian metrics which all give the same boundary distances where g^0 corresponds to a known "background metric". When we linearize the boundary rigidity problem, we calculate the derivative of the boundary distances at $s = 0$. Since these distances do not depend on the parameter s we obtain [143]

$$0 = \int_a^b \frac{\partial g_{ij}^s(\gamma_0(t))}{\partial s}\bigg|_{s=0} \dot{\gamma}_0^i(t)\dot{\gamma}_0^j(t)\mathrm{d}t$$

where $\gamma_0 \colon [a, b] \to M$ is a geodesic of the base manifold (M, g^0) connecting two boundary points. If the variations g^s are conformal, i.e. $g^s = f_s g_0$ where $f_s \colon M \to \mathbb{R}$ is a family of positive scalar functions such that $f_0 = 1$, then the linearization leads to

$$0 = \int_a^b \frac{\partial f_s(\gamma_0(t))}{\partial s}\bigg|_{s=0} \mathrm{d}t.$$

The previous observations motivate us to study the kernel of the geodesic ray transform of symmetric m-tensor fields where $m \geq 0$. The geodesic ray transform of a scalar field $f \colon M \to \mathbb{R}$ (or 0-tensor field) on a Riemannian manifold (M, g) is defined as

$$\mathcal{I}_0 f(\gamma) = \int_{\tau_\gamma^-}^{\tau_\gamma^+} f(\gamma(t))\mathrm{d}t \tag{10}$$

where $\gamma \colon [\tau_\gamma^-, \tau_\gamma^+] \to M$ is a geodesic defined on the maximal interval $[\tau_\gamma^-, \tau_\gamma^+]$ which can be finite or infinite. More generally, the geodesic ray transform of a symmetric (covariant) m-tensor field h is ($m \geq 1$)

$$\mathcal{I}_m h(\gamma) = \int_{\tau_\gamma^-}^{\tau_\gamma^+} h_{i_1 \ldots i_m}(\gamma(t))\dot{\gamma}^{i_1}(t) \cdots \dot{\gamma}^{i_m}(t)\mathrm{d}t \tag{11}$$

where $h_{i_1 \ldots i_m}(x)$ are the components of the m-tensor field h in local coordinates and we have used the Einstein summation convention (repeated indices which appear both as a subscript and superscript are implicitly summed over). The geodesic ray transform $\mathcal{I}_m$ can be seen as a generalization of the Euclidean X-ray transform since in Euclidean space geodesics are lines. However, Funk studied the geodesic ray transform of scalar fields on the sphere $S^2 \subset \mathbb{R}^3$ (also known as the Funk transform) before Radon introduced the Euclidean X-ray transform or Radon transform [50, 51, 64].

The inverse problem in geodesic ray tomography is to determine the m-tensor field h (or the scalar field f) from its integrals along geodesics, i.e. we want to invert the operator $\mathcal{I}_m$ in equation (11) (or in equation (10)). As in the case of vector fields in $\mathbb{R}^n$ there is a gauge for m-tensor fields of order $m \geq 1$: if h is the symmetrized covariant derivative of an $m-1$-tensor field which vanishes on the boundary (or at infinity), then h is in the kernel of $\mathcal{I}_m$. Therefore one can only determine the solenoidal part of the m-tensor field from its geodesic ray transform [69, 120, 143]; if this can be done we say that $\mathcal{I}_m$ is solenoidally injective (or s-injective) on m-tensor fields.

The solenoidal injectivity of $\mathcal{I}_m$ has been widely studied and we list only some special cases here: comprehensive treatment can be found in the reviews [69, 120]. If (M, g) is a compact simple Riemannian manifold, then the geodesic ray transform is injective on scalar fields and s-injective on 1-forms [6, 109]. Solenoidal injectivity is known for tensor fields of any order on two-dimensional compact simple manifolds [119], on simply connected compact manifolds with strictly convex boundary and non-positive curvature [118, 123, 143] and on non-compact Cartan–Hadamard manifolds under certain decay conditions on the tensor fields and on the curvature [94, 95]. If $n \geq 3$ and $m = 0, 1, 2, 4$, then solenoidal injectivity follows from foliation condition by strictly convex hypersurfaces [41, 151, 158]. There are also some partial data results for scalar and tensor fields under restrictions on the Riemannian metric [81, 151, 158]. We also mention that one of the basic general tools in studying solenoidal injectivity of $\mathcal{I}_m$ is an energy estimate also known as the Pestov identity [69, 109, 120].

An interesting generalization of the geodesic ray transform in two dimensions is the mixed ray transform [40, 143]

(12)

$$L_{k,l}h(\gamma) = \int_{\tau_\gamma^-}^{\tau_\gamma^+} h_{i_1 \dots i_k j_1 \dots j_l}(\gamma(t))(\dot{\gamma}(t)^\perp)^{i_1} \cdots (\dot{\gamma}(t)^\perp)^{i_k} \dot{\gamma}^{j_1}(t) \cdots \dot{\gamma}^{j_l}(t) \mathrm{d}t$$

where $\dot{\gamma}(t)^\perp$ denotes the rotation of $\dot{\gamma}(t)$ by 90 degrees counterclockwise and $k + l = m$. The mixed ray transform $L_{k,l}$ arises in the linearization of the elastic travel time tomography problem [39, 40, 143]. If $k = 0$, then $L_{k,l}$ reduces to the geodesic ray transform $\mathcal{I}_m$. When $l = 0$, we have the transverse ray transform [143]

$$\mathcal{I}_m^\perp h(\gamma) = \int_{\tau_\gamma^-}^{\tau_\gamma^+} h_{i_1 \dots i_m}(\gamma(t))(\dot{\gamma}(t)^\perp)^{i_1} \cdots (\dot{\gamma}(t)^\perp)^{i_m} \mathrm{d}t. \tag{13}$$

The mixed and transverse ray transforms in equations (12) and (13) can be extended to higher dimensions $n > 2$, but they become tensor-valued transforms [39, 143]. On two-dimensional orientable manifolds one can study the mixed ray transform by reducing it to the geodesic ray transform using rotations [40, 143].

The transverse ray transform was first studied by Braun and Hauck in two-dimensional Euclidean space with applications to flame analysis [15, 113, 141, 147]. Other applications of the transverse ray transform include diffraction tomography [99], polarization tomography [143] and photoelasticity [61]. The kernel of the transverse ray transform is known in $\mathbb{R}^2$ and on higher dimensional manifolds ($n \geq 3$) $\mathcal{I}_m^\perp$ is even injective under certain conditions [43, 113, 143]. There are also partial data results for the transverse ray transform [1, 82]. For the mixed ray transform some results related to solenoidal injectivity are known in $\mathbb{R}^2$, on two- and three-dimensional compact simple manifolds, and on manifolds satisfying certain curvature estimates [39, 40, 43, 143].

2. X-ray tomography with partial data: [A, B, C, G]

In the articles [A, B, C, G] we study partial data problems arising in the X-ray tomography of scalar and vector fields. The basic question in such problems is the following: can we say something about the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ if we know the integrals of f (the X-ray transform $X_0 f$) on all lines intersecting a given nonempty open set $V \subset \mathbb{R}^n$? We have focused in the uniqueness problem: if $X_0 f = 0$ on all lines intersecting V, does it follow that $f = 0$? In general, the knowledge of the integrals is not enough to determine f uniquely [112] and therefore one has to put additional assumptions on f.

We have studied the partial data problem under different assumptions. In the most general case we assume that f satisfies a constant coefficient partial differential equation in V in a weak sense. If P is a polynomial, we let $P(D)$ be the constant coefficient partial differential operator induced by P, i.e. we consider the partial derivatives D as variables in P. For example, the polynomial $P(\xi) = \xi_1^2 + \ldots + \xi_n^2$ corresponds to the Laplacian $P(D) = -\Delta$.

The main idea of the partial data problem is illustrated in figure 2: if $X_0 f = 0$ on all lines intersecting V and $P(D)f|_V = 0$ for some constant coefficient partial differential operator $P(D)$, does it follow that $f = 0$ everywhere? Using the linearity of X_0 and $P(D)$, and the commutativity of distributional derivatives we see that this is indeed a uniqueness problem in the following sense: if f_1 and f_2 are scalar fields such that $P_1(D)f_1|_V = P_2(D)f_2|_V = 0$ and $X_0 f_1 = X_0 f_2$ on all lines intersecting V, does it follow that $f_1 = f_2$ in all of $\mathbb{R}^n$? The partial data problem can be reduced to a unique continuation problem of the normal operator N_0 of the X-ray transform: if $N_0 f|_V = P(D)f|_V = 0$, does it follow that $f = 0$ everywhere?

More generally, one can replace lines with d-planes in the partial data problem of scalar fields. In this way we obtain a partial data problem for the d-plane transform $\mathcal{R}_d$: if $\mathcal{R}_d f = 0$ on all d-planes intersecting V and $P(D)f|_V = 0$, is it true that $f = 0$? The partial data problem for vector fields is formulated analogously as in the scalar case. However, for vector fields the problem is naturally formulated in terms of the curl (or the exterior derivative) of the vector field: if $X_1 h = 0$ on all lines which intersect V and $P(D)(\mathrm{d}h) = 0$ where h is a vector field and $\mathrm{d}h$ its curl, does it follow that $\mathrm{d}h = 0$? By the Poincaré lemma this is equivalent to that the solenoidal part of h vanishes [67, 100, 143]. As in the case of the X-ray transform of scalar fields, the partial data problems for $\mathcal{R}_d$ and X_1 can be reduced to the corresponding unique continuation problems of the normal operator $\mathcal{N}_d$ of the d-plane transform and the normal operator N_1 of the X-ray transform

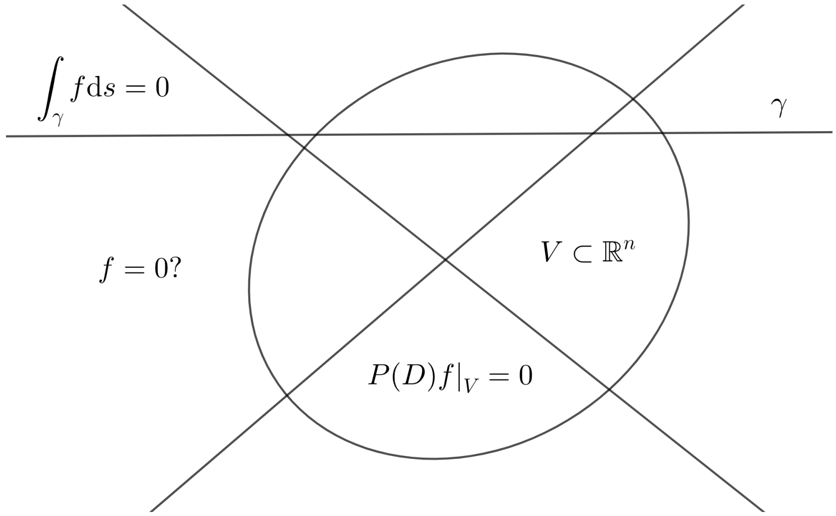

FIGURE 2. The partial data problem for the X-ray transform of scalar fields in its most general form as we have studied. Here $V \subset \mathbb{R}^n$ is a nonempty open set, $P(D)$ is a constant coefficient partial differential operator and γ is a line which intersects V.

of vector fields. We focus on studying the partial data problems from the point of view of the unique continuation of the different normal operators.

In the article [A] we study the partial data problem for X_0 under the assumption $f|_V = 0$. The main result of the article [A] is a unique continuation property of Riesz potentials which correspond to fractional Laplacians with negative exponents. The Riesz potential of a scalar function $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined as the convolution $I_\alpha f = f * |\cdot|^{-\alpha}$ where $\alpha < n$ (see section 2.1). The main theorem is the following: if $I_\alpha f$ vanishes to infinite order at some point $x_0 \in V$ where the exponent α satisfies some conditions and $f|_V = 0$, then $f = 0$. This implies a unique continuation result for the normal operator N_0: if $N_0 f|_V = f|_V = 0$, then $f = 0$. The unique continuation of N_0 can then be used to prove uniqueness for the partial data problem: if $X_0 f = 0$ on all lines intersecting V and $f|_V = 0$, then $f = 0$. We also provide an application of the partial data result to linearized travel time tomography in Euclidean background.

In the article [B] we study the partial data problem for the d-plane transform $\mathcal{R}_d$ in the case $f|_V = 0$. This is a generalization of the problem studied in the article [A] where we considered the case $d = 1$. As in the article [A], the partial data problem is studied using the normal operator $\mathcal{N}_d$ of the d-plane transform. One of the main results of the article [B] is a unique continuation property of fractional Laplacians: if $(-\Delta)^s f|_V = f|_V = 0$ where $s \in (-n/2, \infty) \setminus \mathbb{Z}$, then $f = 0$. When d is odd, this implies a unique continuation result for $\mathcal{N}_d$: if $\mathcal{N}_d f|_V = f|_V = 0$ and d is odd, then $f = 0$.

The unique continuation of $\mathcal{N}_d$ then implies uniqueness for the partial data problem: if d is odd, $\mathcal{R}_d f = 0$ on all d-planes intersecting V and $f|_V = 0$, then $f = 0$. When d is even and $\mathcal{R}_d f = 0$ on all lines which intersect V, we can locally invert the d-plane data to obtain that $f|_V = 0$.

The article [C] considers the partial data problem for X_1 under the assumption $\mathrm{d}h|_V = 0$. The approach is similar as in the scalar case, and the main result of the article [C] is a unique continuation property of N_1: if $N_1 h$ vanishes to infinite order at some point in V and $\mathrm{d}h|_V = 0$, then $\mathrm{d}h = 0$. This unique continuation result is proved by reducing it to a unique continuation problem of N_0 treated in the article [A]. The unique continuation of N_1 can then be used to prove uniqueness for the partial data problem: if $X_1 h = 0$ on all lines which intersect V and $\mathrm{d}h|_V = 0$, then $\mathrm{d}h = 0$. This is equivalent to that $h = \mathrm{d}\phi$ for some scalar field ϕ by the Poincaré lemma, or to that the solenoidal part of h vanishes. In the article [C] we also obtain partial data results for the matrix-weighted X-ray transform of vector fields which special case, the Euclidean transverse ray transform, we study in two dimensions.

The article [G] is a continuation of the articles [A, C] and the integral geometry part of the article [B]. In particular, we extend the assumptions $f|_V = 0$ and $\mathrm{d}h|_V = 0$ in the partial data problems of scalar and vector fields to the more general cases $P(D)f|_V = 0$ and $P(D)(\mathrm{d}h)|_V = 0$ where $P(D)$ is a constant coefficient partial differential operator induced by the polynomial P as above. The main result of the article [G] is a unique continuation property of fractional Laplacians which generalizes the unique continuation result proved in the article [B]: if $(-\Delta)^s f|_V = P(D)f|_V = 0$ where $s \in (-n/2, \infty) \setminus \mathbb{Z}$ and $P(D)$ is any constant coefficient partial differential operator, then $f = 0$. This unique continuation result directly implies a corresponding unique continuation property for $\mathcal{N}_d$: if d is odd and $\mathcal{N}_d f|_V = P(D)f|_V = 0$, then $f = 0$. Using reduction to the scalar case one also obtains a unique continuation result for N_1: if $N_1 h|_V = P(D)(\mathrm{d}h)|_V = 0$, then $\mathrm{d}h = 0$. These unique continuation results for $\mathcal{N}_d$ and N_1 then imply uniqueness for the most general partial data problems we have studied: if $\mathcal{R}_d f = 0$ on all d-planes intersecting V where d is odd and $P(D)f|_V = 0$ (or $X_1 h = 0$ on all lines intersecting V and $P(D)(\mathrm{d}h)|_V = 0$), then $f = 0$ (respectively $\mathrm{d}h = 0$).

2.1. **Notation.** Let us first introduce some notation before giving the main theorems. We will follow the notation conventions of the references [64, 102, 106, 112, 143, 149, 155].

We write f for a scalar function or distribution. The space of tempered distributions is denoted by $\mathscr{S}'(\mathbb{R}^n)$. We let $\mathscr{O}'_C(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ be the space of rapidly decreasing distributions. It contains as a subset all compactly supported distributions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n)$ and all continuous functions which decrease faster than any polynomial at infinity $C_\infty(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n)$. The fractional L^2-Sobolev space of order $r \in \mathbb{R}$ is defined as

$$H^r(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{f}) \in L^2(\mathbb{R}^n)\}$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$, $\hat{f} = \mathcal{F}(f)$ is the Fourier transform of tempered distributions and $\mathcal{F}^{-1}$ is the inverse Fourier transform. These spaces are nested, i.e. $H^r(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ continuously when $r \geq t$, and one can identify $H^{-r}(\mathbb{R}^n)$ with the dual $(H^r(\mathbb{R}^n))^*$ for every $r \in \mathbb{R}$. We let $H^{-\infty}(\mathbb{R}^n) = \bigcup_{r\in\mathbb{R}} H^r(\mathbb{R}^n)$ so that $\mathscr{O}'_C(\mathbb{R}^n) \subset H^{-\infty}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. The fractional Laplacian is defined via Fourier transform

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{f}), \quad s \in (-n/2, \infty) \setminus \mathbb{Z}.$$

We have that $(-\Delta)^s f$ defines a tempered distribution for $f \in \mathscr{O}'_C(\mathbb{R}^n)$ when $s \in (-n/2, \infty) \setminus \mathbb{Z}$, and for $f \in H^r(\mathbb{R}^n)$ when $s \in (-n/4, \infty) \setminus \mathbb{Z}$.

The fractional Laplacian has a connection to Riesz potentials. Let $\alpha \in \mathbb{R}$ such that $\alpha < n$. We define the Riesz potential $I_\alpha \colon \mathscr{O}'_C(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ as $I_\alpha f = f * h_\alpha$ where the kernel is $h_\alpha(x) = |x|^{-\alpha}$. If in addition $0 < \alpha < n$, then $I_\alpha = (-\Delta)^{-s}$ up to a constant factor with $s = (n-\alpha)/2$. On the other hand, if $-n/2 < s < 0$, then we can write $(-\Delta)^s f = I_{2s+n} f$ up to a constant factor. We say that $I_\alpha f$ vanishes to infinite order at a point $x_0 \in \mathbb{R}^n$, if $I_\alpha f$ is smooth in a neighborhood of x_0 and $\partial^\beta(I_\alpha f)(x_0) = 0$ for all multi-indices $\beta \in \mathbb{N}^n$.

We let $\mathcal{P}$ be the set of all polynomials on $\mathbb{R}^n$ with complex coefficients excluding the zero polynomial $P \equiv 0$. If $P \in \mathcal{P}$ is a polynomial of degree $m \in \mathbb{N}$, then it can be identified with the constant coefficient partial differential operator $P(D)$ of order $m \in \mathbb{N}$ by writing $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ where $a_\alpha \in \mathbb{C}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial_j$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index so that $|\alpha| = \alpha_1 + \ldots + \alpha_n$. If $V \subset \mathbb{R}^n$ is a nonempty open set, we define the set of admissible functions $\mathcal{A}_V$ by setting

$$\mathcal{A}_V = \{f \in H^{-\infty}(\mathbb{R}^n) : P(D)f|_V = 0 \text{ for some } P \in \mathcal{P}\}.$$

One can see that the set $\mathcal{A}_V \subset H^{-\infty}(\mathbb{R}^n)$ forms a vector space.

The X-ray transform of scalar fields is denoted by X_0 and it takes a function f and integrates it over lines. The normal operator is $N_0 = X_0^* X_0$ where X_0^* is the adjoint of X_0. If f is a distribution, then $X_0 f$ and $N_0 f$ are defined by duality. More generally, we denote by $\mathcal{R}_d$ the d-plane transform of scalar fields. The transform $\mathcal{R}_d$ takes a scalar field f and integrates it over d-dimensional planes where $0 < d < n$. The normal operator of the d-plane transform is $\mathcal{N}_d = \mathcal{R}_d^* \mathcal{R}_d$ where $\mathcal{R}_d^*$ is the adjoint of $\mathcal{R}_d$. If f is a distribution, then $\mathcal{R}_d f$ and $\mathcal{N}_d f$ are defined by using duality.

We denote by h a vector field or vector-valued distribution. We write $h \in (\mathcal{E}'(\mathbb{R}^n))^n$ if $h = (h_1, \ldots, h_n)$ where $h_i \in \mathcal{E}'(\mathbb{R}^n)$ for all $i = 1, \ldots, n$. The exterior derivative or curl of h is a matrix whose components are $(\mathrm{d}h)_{ij} = \partial_i h_j - \partial_j h_i$. The X-ray transform of vector fields is denoted by X_1 and it maps a vector field to its line integrals. The normal operator is $N_1 = X_1^* X_1$

where X_1^* is the adjoint of X_1. If h is a vector-valued distribution, then both $X_1 h$ and $N_1 h$ are defined by duality.

2.2. **Main results.** The following two theorems are the main results of the article [A]. The first one is a unique continuation result for Riesz potentials and the second one is a partial data result for the X-ray transform of scalar fields.

Theorem 2.1 ([A, Theorem 1.1])**.** *Let $\alpha = n-1$ or $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $\alpha < n$ where $n \geq 2$. Let $f \in \mathcal{E}'(\mathbb{R}^n)$, $V \subset \mathbb{R}^n$ any nonempty open set and $x_0 \in V$. If $f|_V = 0$ and $I_\alpha f$ vanishes to infinite order at x_0, then $f = 0$.*

Theorem 2.2 ([A, Theorem 1.2])**.** *Let $V \subset \mathbb{R}^n$ be any nonempty open set where $n \geq 2$. If $f \in \mathcal{E}'(\mathbb{R}^n)$ satisfies $f|_V = 0$ and $X_0 f$ vanishes on all lines that intersect V, then $f = 0$.*

Theorem 2.1 is proved by showing that one can obtain all the polynomials in a certain form by taking finite linear combinations of the derivatives of the integral kernel h_α in $I_\alpha f = f * h_\alpha$. The density of polynomials in the space of smooth functions then gives the claim since $f \in \mathcal{E}'(\mathbb{R}^n)$ belongs to the dual of that space. We give multiple proofs for theorem 2.2. Two proofs reduce the partial data problem to a unique continuation problem of normal operator: if $X_0 f = 0$ on all lines intersecting V, then $N_0 f|_V = 0$. The normal operator N_0 can be seen as the Riesz potential I_{n-1} up to a constant factor, or equivalently, as the fractional Laplacian $(-\Delta)^{-1/2}$ up to a constant factor. The partial data result then follows from theorem 2.1, or by using the unique continuation of fractional Laplacians which is proved in [54]. The third proof works directly at the level of the X-ray transform and is based on angular Fourier series and density of polynomials.

In addition, we provide an application of theorem 2.2 to linearized travel time tomography in Euclidean background. In particular, we show how one can use global shear wave splitting data to uniquely determine the difference of the S-wave speeds in weak anisotropy. We also show in the article [A] how one can use “half-local” measurements of travel times to uniquely determine the conformal factor in the linearization; this is a partial data result where we measure travel times of seismic waves in a small open subset of the surface of the Earth, but the waves can emanate from anywhere on the surface.

In the article [B] we generalize the unique continuation and partial data results proved in [A] for scalar fields to d-plane transforms. The following two theorems are the main results of the integral geometry part of the article [B].

Theorem 2.3 ([B, Corollary 1])**.** *Let $n \geq 2$ and let f belong to either $\mathcal{E}'(\mathbb{R}^n)$ or $C_\infty(\mathbb{R}^n)$. Let $d \in \mathbb{N}$ be odd such that $0 < d < n$. If $\mathcal{N}_d f|_V = 0$ and $f|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $f = 0$.*

Theorem 2.4 ([B, Corollary 2])**.** *Let $n \geq 2$, $V \subset \mathbb{R}^n$ a nonempty open set and $f \in C_\infty(\mathbb{R}^n)$ or $f \in \mathcal{E}'(\mathbb{R}^n)$. Let $d \in \mathbb{N}$ be odd such that $0 < d < n$. If $f|_V = 0$ and $\mathcal{R}_d f = 0$ for all d-planes intersecting V, then $f = 0$.*

Theorem 2.3 is proved by using a unique continuation property of fractional Laplacians which is proved in the same article [B] (see theorem 3.1). Unique continuation of fractional Laplacians can be used since the normal operator $\mathcal{N}_d$ of the d-plane transform corresponds to the fractional Laplacian $(-\Delta)^{-d/2}$ up to a constant factor. The unique continuation of $\mathcal{N}_d$ is then used to prove theorem 2.4. For this reason we have to assume that d is odd: theorem 2.3 does not hold if d is even since in that case $\mathcal{N}_d$ is the inverse of a local operator. However, if d is even, then the partial data problem for the d-plane transform is locally uniquely solvable: if $\mathcal{R}_d f = 0$ on all lines intersecting V and d is even, then $f|_V = 0$.

In the article [C] we generalize the above partial data results to vector fields. The following two main theorems of the article [C] are similar to theorems 2.1 and 2.2.

Theorem 2.5 ([C, Theorem 1.1]). *Let $h \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set where $n \geq 2$. If $\mathrm{d}h|_V = 0$ and $N_1 h$ vanishes to infinite order at $x_0 \in V$, then $h = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

Theorem 2.6 ([C, Theorem 1.2]). *Let $h \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set where $n \geq 2$. Assume that $\mathrm{d}h|_V = 0$. Then $X_1 h$ vanishes on all lines intersecting V if and only if $h = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

Instead of assuming that $N_1 h$ vanishes to infinite order in theorem 2.5 we could require that $\mathrm{d}(N_1 h)$ vanishes componentwise to infinite order at some point $x_0 \in V$. This weaker condition implies the claim since theorem 2.5 is proved by using theorem 2.1 and the fact that $\mathrm{d}(N_1 h) = N_0(\mathrm{d}h)$ holds componentwise up to a constant factor. We provide two alternative proofs for theorem 2.6. The first proof directly uses the unique continuation of the normal operator N_1 in theorem 2.5. The second proof is based on Stokes' theorem and theorem 2.2. Both proofs use the same idea: from the assumptions we deduce that $\mathrm{d}h = 0$ and the Poincaré lemma implies that $h = \mathrm{d}\phi$ for some scalar field ϕ.

In the article [C] we also study the matrix-weighted X-ray transform of vector fields $X_A = X_1 \circ A$ where A is a smooth invertible matrix field. Similar results as in theorems 2.5 and 2.6 are obtained for the transform X_A. As a special case of the transform X_A we obtain results for the Euclidean transverse ray transform in two dimensions.

In the article [G] we generalize the partial data and unique continuation results obtained in the articles [A, B, C]. The partial data results are proved by using the following unique continuation property of fractional Laplacians which is a generalization of the unique continuation result we proved in the article [B].

Theorem 2.7 ([G, Theorem 1.1]). *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $f \in \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $(-\Delta)^s f|_V = 0$, then $f = 0$. If $f \in \mathscr{O}_C'(\mathbb{R}^n) \cap \mathcal{A}_V$, then the claim holds for $s \in (-n/2, \infty) \setminus \mathbb{Z}$.*

The condition $f \in \mathcal{A}_V$ means that $f \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$ and $P(D)f|_V = 0$ for some constant coefficient partial differential operator $P(D)$. Theorem 2.7 is proved by using the unique continuation result of fractional Laplacians proved in [B] (see theorem 3.1) for the scalar field $P(D)f$. The assumptions and locality of $P(D)$ imply the conditions $P(D)f|_V = (-\Delta)^s(P(D)f)|_V = 0$ and hence f has to satisfy the global partial differential equation $P(D)f = 0$ which has only trivial solutions in the class of admissible functions $\mathcal{A}_V$.

As before, the unique continuation of fractional Laplacians in theorem 2.7 can be used to prove partial data results for scalar and vector fields. The following two theorems of the article [G] are generalizations of theorems 2.2 and 2.6.

Theorem 2.8 ([G, Theorem 1.4])**.** *Let $n \geq 2$ and $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $X_0 f = 0$ on all lines intersecting V, then $f = 0$.*

Theorem 2.9 ([G, Theorem 1.7])**.** *Let $n \geq 2$ and $h \in (\mathcal{E}'(\mathbb{R}^n))^n$ such that $(\mathrm{d}h)_{ij} \in \mathcal{A}_V$ for all $i, j = 1, \ldots, n$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $X_1 h = 0$ on all lines intersecting V, then $\mathrm{d}h = 0$. Especially, $h = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

Theorems 2.8 and 2.9 are proved in the following way. Theorem 2.7 implies a corresponding unique continuation result for the normal operator N_0. The unique continuation of N_0 is then used to prove the partial data result in theorem 2.8. Further, using again the fact that $\mathrm{d}(N_1 h) = N_0(\mathrm{d}h)$ holds componentwise up to a constant factor we can prove a unique continuation property for N_1, which in turn implies the partial data result in theorem 2.9. When d is odd, theorem 2.7 implies a corresponding unique continuation result for the normal operator $\mathcal{N}_d$, which in turn implies a similar partial data result as in theorem 2.8 for the d-plane transform $\mathcal{R}_d$.

3. Higher order fractional Calderón problems: [B, D]

In the articles [B, D] we study uniqueness for higher order fractional Calderón problems. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ its exterior and $s \in (0, \infty) \setminus \mathbb{Z}$. We consider the Calderón problem for the fractional Schrödinger equation

$$\begin{cases} ((-\Delta)^s + q)u &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases} \tag{14}$$

and for the more general equation involving lower order local perturbations of the fractional Laplacian

$$\begin{cases} ((-\Delta)^s + P(x,D))u &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases} \tag{15}$$

where $P(x, D)$ is a variable coefficient partial differential operator of order $m \in \mathbb{N}$. We can write $P(x, D)$ as

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

where the coefficients $a_\alpha = a_\alpha(x)$ are functions in Ω (or more generally Sobolev multipliers in $\mathbb{R}^n$). We assume that $m < 2s$ so that $P(x, D)$ can be considered as a lower order perturbation to $(-\Delta)^s$. We see that equation (14) is a special case of equation (15) and the potential q can be treated as a zeroth order perturbation to $(-\Delta)^s$.

The inverse problem for equations (14) and (15) is illustrated in figure 3. Formally, we put some "non-local voltage" in the open set $W_1 \subset \Omega_e$ and measure "non-local currents" in the open set $W_2 \subset \Omega_e$. More precisely, the fractional Calderón problem for equation (15) is formulated as follows: if the DN maps Λ_{P_1} and Λ_{P_2} agree in W_2 for all exterior values $f \in C_c^\infty(W_1)$, does it follow that the partial differential operators P_1 and P_2 are equal in Ω? This problem was first introduced by Ghosh, Salo and Uhlmann in their seminal work [54] where the authors studied equation (14) in the case $s \in (0, 1)$. We can think the inverse problem as a partial data problem: instead of having data in the full exterior Ω_e we only have information or make measurements in the (possibly small) open subsets $W_1, W_2 \subset \Omega_e$.

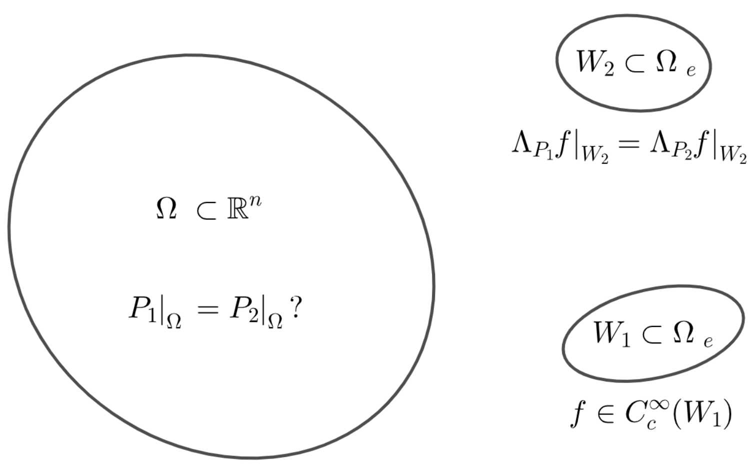

Figure 3. The fractional Calderón problem in its most general form as we have studied. Here $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ its exterior. The "measurements" are done in the (possibly disjoint) subsets $W_1, W_2 \subset \Omega_e$ of the exterior.

The basic tools in proving uniqueness for fractional Calderón problems are the unique continuation property and fractional Poincaré inequality for the fractional Laplacian $(-\Delta)^s$. We already saw the importance of unique continuation of fractional Laplacians in partial data problems of scalar and vector fields. In fractional Calderón problems the unique continuation of $(-\Delta)^s$ implies Runge approximation: one can approximate functions in certain Sobolev spaces arbitrarily well by solutions of the fractional equation under study (see section 3.2). The fractional Poincaré inequality is a norm estimate involving the L^2-norms of a function and its fractional Laplacian, and it is an important inequality in proving well-posedness for the forward problem (the coercivity of the bilinear form). These basic tools (unique continuation and Poincaré inequality) were proved in [54] in the case $s \in (0, 1)$.

In the article [B] we study the higher order fractional Calderón problem for equation (14) when $s \in (0, \infty) \setminus \mathbb{Z}$. We prove higher order unique continuation result for fractional Laplacians: if $(-\Delta)^s u|_V = u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$ and $s \in (-n/2, \infty) \setminus \mathbb{Z}$, then $u = 0$. This generalizes the result proved in [54]. We also prove higher order fractional Poincaré inequality for $s \in (0, \infty) \setminus \mathbb{Z}$ which says that the L^2-norm of u can be bounded from above by the L^2-norm of $(-\Delta)^s u$. We provide five possible proofs for the Poincaré inequality and some of the proofs also give information about the constant in the inequality. Using the Poincaré inequality we prove well-posedness of the forward problem, and unique continuation of $(-\Delta)^s$ implies Runge approximation for equation (14). Using Runge approximation and the so-called Alessandrini identity (see section 3.2) for suitable test functions we prove uniqueness for the inverse problem: if $W_1, W_2 \subset \Omega_e$ are some open sets such that the DN maps satisfy $\Lambda_{q_1} f = \Lambda_{q_2} f$ in W_2 for all exterior values $f \in C_c^\infty(W_1)$, then $q_1 = q_2$ in Ω. This is done for certain singular potentials q which can be viewed as Sobolev multipliers. We also study the magnetic counterpart of equation (14) (the higher order fractional magnetic Schrödinger equation) in the article [B] and prove uniqueness (up to a gauge) under certain assumptions on the electric and magnetic potentials, generalizing the results in [30] to higher order cases.

The article [D] is a continuation of the article [B] in the sense that we replace the potential q in equation (14) with a general lower order local perturbation $P(x, D)$. In the article [D] we study the fractional Calderón problem for equation (15) when $s \in (0, \infty) \setminus \mathbb{Z}$ and $m < 2s$. We consider two different classes of coefficients a_α of the partial differential operator $P(x, D)$: coefficients which belong to certain L^∞-Bessel potential spaces, and coefficients which are certain Sobolev multipliers. The same tools that we develop in the article [B], i.e. the higher order unique continuation property and fractional Poincaré inequality for $(-\Delta)^s$, are applicable in proving uniqueness in the article [D]. In addition to the Poincaré inequality we also need the Kato–Ponce inequality in proving well-posedness of the forward problem. The Kato–Ponce inequality is a fractional Leibniz rule in terms of L^p-norms [56, 58, 76]. As in the article [B], the unique continuation of $(-\Delta)^s$ implies Runge approximation for equation (15). Using the Runge approximation and the corresponding Alessandrini identity for suitable test functions we prove uniqueness for the inverse problem: if $W_1, W_2 \subset \Omega_e$ are some open sets such that the DN maps satisfy $\Lambda_{P_1} f = \Lambda_{P_2} f$ in W_2 for all exterior values $f \in C_c^\infty(W_1)$, then $P_1 = P_2$ in Ω. This uniqueness result is shown for both classes of coefficients a_α, i.e. coefficients with bounded fractional derivatives and coefficients which are Sobolev multipliers.

3.1. **Notation.** We use the same notation that we introduced in section 2.1, but we also introduce some additional notation. We follow the notation conventions of the references [11, 23, 101, 102, 106, 156].

If $1 \leq p \leq \infty$, we define the fractional L^p-Bessel potential space of order $r \in \mathbb{R}$ as

$$H^{r,p}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^p(\mathbb{R}^n)\}$$

and we equip it with the norm

$$\|u\|_{H^{r,p}(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u})\right\|_{L^p(\mathbb{R}^n)}.$$

These spaces are nested, i.e. $H^{r,p}(\mathbb{R}^n) \hookrightarrow H^{t,p}(\mathbb{R}^n)$ continuously when $r \geq t$. We see that $H^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Omega \subset \mathbb{R}^n$ is an open set, then we define the spaces $H^{r,p}(\Omega)$ as restrictions

$$H^{r,p}(\Omega) = \{u|_\Omega : u \in H^{r,p}(\mathbb{R}^n)\}$$

and we use the quotient norm

$$\|w\|_{H^{r,p}(\Omega)} = \inf\{\|u\|_{H^{r,p}(\mathbb{R}^n)} : u \in H^{r,p}(\mathbb{R}^n) \text{ such that } u|_\Omega = w\}.$$

It follows that the inclusions $H^{r,p}(\Omega) \hookrightarrow H^{t,p}(\Omega)$ are continuous when $r \geq t$. The spaces $H^{r,p}(\Omega)$ are not to be confused with the Sobolev-Slobodeckij spaces $W^{r,p}(\Omega)$ which are defined by using weak derivatives of L^p-functions and which in general are different from the Bessel potential spaces we have introduced [44]. If $r \geq 0$ and $p = 2$, then $H^{r,2}(\mathbb{R}^n) = W^{r,2}(\mathbb{R}^n)$ and $H^{r,2}(\Omega) = W^{r,2}(\Omega)$ when Ω is a Lipschitz domain.

The following spaces are special cases of the above Bessel potential spaces

$$\begin{aligned}
H_F^{r,p}(\mathbb{R}^n) &= \{u \in H^{r,p}(\mathbb{R}^n) : \mathrm{spt}(u) \subset F\} \\
\widetilde{H}^{r,p}(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\cdot\|_{H^{r,p}(\mathbb{R}^n)} \\
H_0^{r,p}(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\cdot\|_{H^{r,p}(\Omega)}
\end{aligned}$$

where $F \subset \mathbb{R}^n$ is some closed set. Observe that $\widetilde{H}^{r,p}(\Omega) \subset H^{r,p}(\mathbb{R}^n)$ and $H_0^{r,p}(\Omega) \subset H^{r,p}(\Omega)$. One also sees that $\widetilde{H}^{r,p}(\Omega) \subset H_0^{r,p}(\Omega)$ and $\widetilde{H}^{r,p}(\Omega) \subset H_{\overline{\Omega}}^{r,p}(\mathbb{R}^n)$. When $p = 2$, we simply write $H^{r,2}(\mathbb{R}^n) = H^r(\mathbb{R}^n)$, $H^{r,2}(\Omega) = H^r(\Omega)$ and so on. It follows that $(\widetilde{H}^r(\Omega))^* = H^{-r}(\Omega)$ and $(H^r(\Omega))^* = \widetilde{H}^{-r}(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$ and $r \in \mathbb{R}$. If in addition Ω is a Lipschitz domain and $r \geq 0$ such that $r \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \dots\}$, then $\widetilde{H}^r(\Omega) = H_0^r(\Omega)$.

We define the space of Sobolev multipliers $M(H^r \to H^t) \subset \mathcal{D}'(\mathbb{R}^n)$ by saying that the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $M(H^r \to H^t)$ if the multiplier norm

$$\|f\|_{r,t} = \sup\{|\langle f, uv\rangle| : u, v \in C_c^\infty(\mathbb{R}^n),\ \|u\|_{H^r(\mathbb{R}^n)} = \|v\|_{H^{-t}(\mathbb{R}^n)} = 1\}$$

is finite. We let $M_0(H^r \to H^t)$ be the closure of $C_c^\infty(\mathbb{R}^n)$ in $M(H^r \to H^t)$ with respect to the norm $\|\cdot\|_{r,t}$. The elements of the space $M(H^r \to H^t)$ are called Sobolev multipliers since each $f \in M(H^r \to H^t)$ induces a map $m_f\colon H^r(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ defined as

$$\langle m_f(u), v\rangle = \langle f, uv\rangle$$

for all $u \in H^r(\mathbb{R}^n)$ and $v \in H^{-t}(\mathbb{R}^n)$. As a special case of multipliers we write $Z^{-s}(\mathbb{R}^n) = M(H^s \to H^{-s})$ and $Z_0^{-s}(\mathbb{R}^n) = M_0(H^s \to H^{-s})$ whose elements we also call singular potentials.

We say that 0 is not a Dirichlet eigenvalue of the operator $(-\Delta)^s + q$, if the following condition holds:

(16) If $u \in H^s(\mathbb{R}^n)$ solves $((-\Delta)^s + q)u = 0$ in Ω and $u|_{\Omega_e} = 0$, then $u = 0$.

Analogously, we say that 0 is not a Dirichlet eigenvalue of the operator $(-\Delta)^s + P(x, D)$ if condition (16) holds when q is replaced with the partial differential operator $P(x, D)$. When the forward problem for equation (14) is well-posed, we can define the DN map $\Lambda_q\colon H^s(\Omega_e) \to (H^s(\Omega_e))^*$ as $\langle \Lambda_q f_1, f_2\rangle = B_q(u_{f_1}, f_2)$ where $B_q(\cdot,\cdot)$ is the bilinear form associated to equation (14) and u_{f_1} is the unique solution to equation (14) with exterior value $u|_{\Omega_e} = f_1$. The DN map Λ_P for equation (15) is defined similarly.

3.2. **Main results.** One of the main theorems of the article [B] is the following unique continuation property of fractional Laplacians.

Theorem 3.1 ([B, Theorem 1.1]). *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $u \in H^r(\mathbb{R}^n)$ where $r \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$. The claim holds also for $s \in (-n/2, -n/4] \setminus \mathbb{Z}$ if $u \in H^{r,1}(\mathbb{R}^n)$ or $u \in \mathscr{O}_C'(\mathbb{R}^n)$.*

Theorem 3.1 is proved by reducing the claim to the case $s \in (0, 1)$ and using the unique continuation result proved in [54]. The reduction can be done by using the simple relation $(-\Delta)^k(-\Delta)^s = (-\Delta)^s(-\Delta)^k = (-\Delta)^{s+k}$ when $k \in \mathbb{N}$ and $s \in (-n/2, \infty)\setminus\mathbb{Z}$. The assumptions on s in theorem 3.1 are put so that $(-\Delta)^s$ is a non-local operator and $(-\Delta)^s u$ is well-defined as a tempered distribution. We also prove in the article [B] many other versions of the unique continuation of $(-\Delta)^s$ in different Sobolev spaces, including homogeneous Sobolev spaces and certain Bessel potential spaces.

The next theorem of the article [B] is called the (fractional) Poincaré inequality. It has an essential role in proving well-posedness for the forward problems of equations (14) and (15).

Theorem 3.2 ([B, Theorem 1.2]). *Let $n \geq 1$, $s \geq t \geq 0$, $K \subset \mathbb{R}^n$ a compact set and $u \in H_K^s(\mathbb{R}^n)$. There exists a constant $c = c(n, K, s) > 0$ such that*

$$\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} \leq c \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}. \tag{17}$$

In well-posedness we only need the cases $t = 0$ and $s \in (0,\infty) \setminus \mathbb{Z}$ of theorem 3.2. Note that the inequality (17) holds for all exponents $s \geq t \geq 0$, not just fractional ones. The interpretation of theorem 3.2 is that the norms of lower order derivatives of u are bounded from above by the norms of higher order derivatives of u. When $t = 0$ and $s = 1$, then the inequality (17) reduces to the classical Poincaré inequality.

We provide five different proofs for theorem 3.2. Two of the simplest proofs are based on Fourier analysis: the first uses splitting of frequencies on the Fourier side and the second uses uncertainty inequalities proved in [48]. Two other proofs are based on a reduction argument similar to what we did in proving the unique continuation of higher order fractional Laplacians. The fifth proof considers the case $s \geq 1$ and it uses interpolation in homogeneous Sobolev spaces and the classical Poincaré inequality. This proof also gives an explicit constant for the inequality: if $s \geq 1$ and $u \in \widetilde{H}^s(\Omega)$, then in theorem 3.2 we can take $c = C^{s-t}$ where C is the classical Poincaré constant. This is expected since on the left-hand side of equation (17) we take t derivatives and on the right-hand side we take s derivatives.

The next theorem of the article [B] gives uniqueness for the higher order fractional Schrödinger equation with a singular potential.

Theorem 3.3 ([B, Theorem 1.3])**.** *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ a bounded open set, $s \in (0,\infty) \setminus \mathbb{Z}$, and $q_1, q_2 \in Z_0^{-s}(\mathbb{R}^n)$ such that 0 is not a Dirichlet eigenvalue of the operators $(-\Delta)^s + q_i$. Let $W_1, W_2 \subset \mathbb{R}^n \setminus \overline{\Omega}$ be open sets. If the DN maps for the equations $(-\Delta)^s u + m_{q_i}(u) = 0$ in Ω satisfy $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$, then $q_1|_\Omega = q_2|_\Omega$.*

The proof of theorem 3.3 follows from Runge approximation for equation (14) and choosing suitable test functions in the Alessandrini identity. The Runge approximation says that we can approximate functions in $\widetilde{H}^s(\Omega)$ arbitrarily well by solutions of the fractional Schrödinger equation (14), and it can be proved by using the unique continuation of $(-\Delta)^s$ in theorem 3.1 and the well-posedness of the forward problem. The Alessandrini identity is an integral identity showing how the DN maps Λ_{q_i} and the corresponding potentials q_i are related in terms of exterior values f and solutions u_f of equation (14).

The article [D] generalizes the higher order fractional Schrödinger equation studied in the article [B] to include more general lower order local perturbations. The following two theorems are the main results of the article [D].

Theorem 3.4 ([D, Theorem 1.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set where $n \geq 1$. Let $s \in (0,\infty) \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j = \sum_{|\alpha| \leq m} a_{j,\alpha} D^\alpha, \quad j = 1, 2,$$

be partial differential operators of order m where $a_{j,\alpha} \in M_0(H^{s-|\alpha|} \to H^{-s})$ such that 0 is not a Dirichlet eigenvalue of the operators $(-\Delta)^s + P_j$. Given any two open sets $W_1, W_2 \subset \mathbb{R}^n \setminus \overline{\Omega}$, suppose that the DN maps Λ_{P_j} for the equations $((-\Delta)^s + P_j)u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1|_\Omega = P_2|_\Omega$.

Theorem 3.5 ([D, Theorem 1.2])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain where $n \geq 1$. Let $s \in (0,\infty) \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j(x, D) = \sum_{|\alpha| \leq m} a_{j,\alpha}(x) D^\alpha, \quad j = 1, 2,$$

be partial differential operators of order m with coefficients $a_{j,\alpha} \in H^{r_\alpha,\infty}(\Omega)$ where

$$r_\alpha = \begin{cases} 0 & \text{if} \quad |\alpha| - s < 0, \\ |\alpha| - s + \delta & \text{if} \quad |\alpha| - s \in \{1/2, 3/2, ...\}, \\ |\alpha| - s & \text{if} \quad \text{otherwise} \end{cases}$$

for any fixed $\delta > 0$ and assume that 0 is not a Dirichlet eigenvalue of the operators $(-\Delta)^s + P_j(x, D)$. Given any two open sets $W_1, W_2 \subset \mathbb{R}^n \setminus \overline{\Omega}$, suppose that the DN maps Λ_{P_j} for the equations $((-\Delta)^s + P_j(x, D))u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1(x, D) = P_2(x, D)$.

It is not known whether the spaces $M_0(H^{s-|\alpha|} \to H^{-s})$ and $H^{r_\alpha,\infty}(\Omega)$ are contained in each other. If this is not the case, then theorems 3.4 and 3.5 are distinct and neither claim implies the other. In theorem 3.4 we consider multipliers which can be approximated in the multiplier norm by smooth compactly supported functions and for this reason we do not need to assume anything about the boundary of Ω. In theorem 3.5 we have put some conditions on $\partial\Omega$ and for the exponent r_α which are needed in proving well-posedness in the case of coefficients with bounded fractional derivatives. The assumptions that 0 is not a Dirichlet eigenvalue and $2s > m$ (i.e. we consider perturbations to $(-\Delta)^s$) are also crucial in both theorems when proving well-posedness of the forward problem. It follows that $M(H^{s-|\alpha|} \to H^{-s}) = \{0\}$ if $s - |\alpha| < -s$. Partly because of this reason theorem 3.4 is formulated only for $2s > m$ since the multiplier coefficients for higher order derivatives are zero, i.e. $a_\alpha = 0$ for all $|\alpha| > 2s$.

Theorems 3.4 and 3.5 are proved in the same way, even though the exact details are a little bit different. The proofs follow the same ideas as in the article [B] where we proved uniqueness for zeroth order perturbations, and we see that theorem 3.3 is in fact a special case of theorem 3.4. The well-posedness of the forward problem is proved by using the higher order fractional Poincaré inequality in theorem 3.2 and interpolation inequality in non-homogeneous Sobolev spaces. In the case of coefficients with bounded fractional derivatives we also need the Kato–Ponce inequality in proving well-posedness. The higher order unique continuation of $(-\Delta)^s$ in theorem 3.1 together with well-posedness implies Runge approximation for equation (15) (and for the adjoint equation of (15)): one can approximate functions in $\widetilde{H}^s(\Omega)$ arbitrarily well by solutions of equation (15). We can prove uniqueness for the inverse problem by using the Runge approximation and suitable test functions in the Alessandrini identity which gives the relation between the DN maps Λ_{P_i} and the partial differential operators P_i in terms of exterior values f and solutions u_f of equation (15) (and the adjoint equation of (15)). It is important to notice that in theorems 3.4 and 3.5 we recover the coefficients a_α uniquely and there is no gauge in contrast to the perturbed local Schrödinger equation [71, 111, 138].

Note that even though we consider partial differential operators in theorems 3.4 and 3.5, the results apply for more general local linear operators. In fact, Peetre's theorem implies that any local linear operator $L\colon C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ which satisfies $\mathrm{spt}(Lf) \subset \mathrm{spt}(f)$ for all $f \in C_c^\infty(\Omega)$ can be identified with a partial differential operator [105, 121]. Hence our results hold for any such local operator satisfying the assumptions in theorems 3.4 and 3.5.

4. Travel time tomography on Riemannian and Finsler manifolds: [E, F]

In the articles [E, F] we study the travel time tomography or boundary rigidity problem and its linearized versions on Riemannian manifolds and more general Finsler manifolds where the fiberwise inner product depends on direction. The basic idea of the boundary rigidity problem is illustrated in figure 4. Suppose we have two Finsler norms F_1 and F_2 (which can be for example two Riemannian metrics) on a manifold M with boundary ∂M. We assume that between any two boundary points $x, x' \in \partial M$ there is unique geodesic γ_i of the Finsler norm F_i going from x to x'. The length of the geodesic γ_i with respect to F_i is denoted by $L_{F_i}(\gamma_i)$ and it gives the (not necessarily symmetric) distance from $x \in \partial M$ to $x' \in \partial M$. The boundary rigidity problem is the following: if the Finsler norms F_1 and F_2 give the same distances between all boundary points $x, x' \in \partial M$, does it follow that $F_1 = F_2$ up to a natural gauge?

If the Finsler norms F_i are induced by Riemannian metrics g_i (the fiberwise inner product does not depend on direction), then the natural gauge is a boundary preserving diffeomorphism: if $g_2 = \Psi^* g_1$ where $\Psi\colon M \to M$ is a diffeomorphism such that $\Psi|_{\partial M} = \mathrm{Id}$, then g_1 and g_2 give the same boundary distances. For a special class of non-reversible Finsler norms called Randers metrics the gauge is similar: if $F_1 = F_g + \beta$ where F_g is a Finsler norm

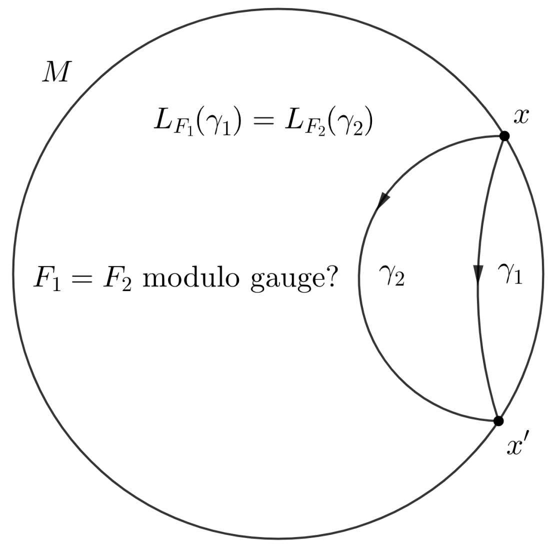

FIGURE 4. An illustration of the boundary rigidity problem on Finsler manifolds (M, F). Here $x, x' \in \partial M$ are two boundary points, γ_1 and γ_2 are the unique geodesics of the Finsler norms F_1 and F_2 connecting x to x', and $L_{F_i}(\gamma_i)$ denotes the length of the geodesic γ_i with respect to F_i (adapted from [F, Figure 1]).

induced by the Riemannian metric g and β is a 1-form whose norm with respect to g is small, then F_1 and $F_2 = \Psi^* F_1 + \mathrm{d}\phi$ give the same boundary distances where $\Psi\colon M \to M$ is a diffeomorphism which is identity on the boundary and ϕ is a scalar field vanishing on the boundary (and $\mathrm{d}\phi$ is considered as a small perturbation to $\Psi^* F_1$).

On Riemannian manifolds (M, g) the linearization of the boundary rigidity problem leads to the geodesic ray transform $\mathcal{I}_m$ of symmetric (covariant) m-tensor fields [143]: if h and h' are two symmetric m-tensor fields such that $\mathcal{I}_m h = \mathcal{I}_m h'$, does it follow that $h = h'$ up to a natural gauge? When $m \geq 1$, the gauge is given by the derivative of a lower order tensor field: if $h' = h + \sigma\nabla v$ where h is symmetric m-tensor field, v is an $m-1$-tensor field vanishing on the boundary (or at infinity) and $\sigma\nabla$ is the symmetrized covariant derivative, then $\mathcal{I}_m h = \mathcal{I}_m h'$. Since the problem is linear ($\mathcal{I}_m$ is a linear operator) it is enough to study the kernel of $\mathcal{I}_m$: if h is a symmetric m-tensor field such that $\mathcal{I}_m h = 0$, does it follow that $h = \sigma\nabla v$ where v is an $m-1$-tensor field vanishing on the boundary (or at infinity)? If this is true for all sufficiently regular symmetric m-tensor fields, we say that $\mathcal{I}_m$ is solenoidally injective (or s-injective).

In the article [E] we study the mixed ray transform and more general mixing ray transforms on Riemannian manifolds. These integral transforms are generalizations of the geodesic ray transform and they arise in the linearization of the elastic travel time tomography problem [39, 40, 143]. The main focus in the article [E] is on the algebraic properties of mixing ray transforms and decompositions of tensor fields with respect to these transforms. We have various corollaries of a main idea how to study the kernel characterization and solenoidal injectivity of the mixing ray transforms using correct notion of symmetry and reduction.

The mixing ray transform of m-tensor fields ($m \geq 1$) is defined as the composition $\mathcal{I}_A h = (\mathcal{I}_m \circ A)h$ where $\mathcal{I}_m$ is the geodesic ray transform of m-tensor fields and A is a smooth linear invertible map on m-tensor fields (see section 4.1). If A is the identity map, then $\mathcal{I}_A$ reduces to the geodesic ray transform $\mathcal{I}_m$. One can think that the transform $\mathcal{I}_A$ first rotates the tensor field h and then takes the geodesic ray transform of the rotated m-tensor field Ah.

The mixing ray transforms are matrix-weighted geodesic ray transforms and they have a different kind of kernel than the geodesic ray transform. We prove in the article [E] that every m-tensor field h can be written as the direct sum $h = \widehat{\sigma}_A h + (h - \widehat{\sigma}_A h)$ where $\widehat{\sigma}_A$ is the symmetrization map with respect to the transform $\mathcal{I}_A$ (see section 4.1) and $h - \widehat{\sigma}_A h \in \mathrm{Ker}(\mathcal{I}_A)$. Here $\widehat{\sigma}_A h$ is the “symmetric part” of h and $h - \widehat{\sigma}_A h$ is the “trivial part” of h from the point of view of the transform $\mathcal{I}_A$. We show that if $\mathcal{I}_m$ is s-injective on symmetric m-tensor fields and $\mathcal{I}_A h = 0$, then $\widehat{\sigma}_A h = \widehat{\sigma}_A \nabla^A v$ for some $m-1$-tensor field v vanishing on the boundary (or at infinity) where $\nabla^A = A^{-1} \circ \nabla$ is the weighted covariant derivative associated to $\mathcal{I}_A$. This property is referred as the solenoidal injectivity of $\mathcal{I}_A$ and it allows us to write the kernel of $\mathcal{I}_A$ as the direct sum $\mathrm{Ker}(\mathcal{I}_A) = \mathrm{Im}(\mathcal{H}) \oplus \mathrm{Im}(\widehat{\sigma}_A \nabla^A)$ where $\mathcal{H} = \mathrm{Id} - \widehat{\sigma}_A$ is the projection onto the “trivial part” of $\mathrm{Ker}(\mathcal{I}_A)$ (see sections 4.1 and 4.2).

In addition to solenoidal injectivity results we prove in the article [E] numerous corollaries of the algebraic approach to mixing ray transforms and related transforms such as the mixed ray transform and the light ray transform. For example, we show that previous results for the light ray transform on Lorentzian manifolds and the mixed ray transform on simple Riemannian manifolds in [40, 47] can be seen as solenoidal injectivity results when we have a correct notion of symmetry. We also prove some stability results for the mixed ray transform, and show that the geodesic ray transform and the transverse ray transform together determine 1-forms uniquely on certain two-dimensional compact and non-compact manifolds.

In the article [F] we study the boundary rigidity problem for certain non-reversible Finsler norms called Randers metrics. Finsler norms are non-negative functions on the tangent bundle $F\colon TM \to [0, \infty)$ so that for every $x \in M$ the map $y \mapsto F(x, y)$ is a positively homogeneous norm in $T_x M$. The Finsler norm F is reversible, if $F(x, -y) = F(x, y)$ for all $x \in M$ and $y \in T_x M$. In this case the map $y \mapsto F(x, y)$ defines a norm in $T_x M$. In general, the distance function given by F is not necessarily symmetric in contrast to the Riemannian distance function. Finsler norms induce a fiberwise inner product which depends not only on position but also on direction. Riemannian metrics are a special case of reversible Finsler norms where the inner product does not depend on direction.

Randers metrics are Finsler norms of the form $F = F_g + \beta$ where F_g is a Finsler norm induced by the Riemannian metric g and β is a 1-form whose norm with respect to g is small enough. Randers metrics are non-reversible since $F(x, -y) = F(x, y)$ for all $x \in M$ and $y \in T_xM$ if and only if $\beta \equiv 0$. Randers metrics arise naturally in Zermelo's navigation problem [10, 146]. Roughly saying, Zermelo's problem asks what is the shortest path in time for a moving object to travel from point A to point B when an external force field is acting on the object. Basic example is a ship which is sailing on a sea under the influence of wind or current.

In the article [F] we prove two boundary rigidity results. If F is a Finsler norm and $x, x' \in \partial M$, denote by $d_F(x, x')$ the (non-symmetric) geodesic distance from x to x' (see section 4.1). The first theorem is the following: if F_1 and F_2 are Finsler norms of the form $F_i = F_{r,i} + \beta_i$ where $F_{r,i}$ is a reversible Finsler norm and β_i is a closed 1-form ($\mathrm{d}\beta_i = 0$) such that $d_{F_1}(x, x') = d_{F_2}(x, x')$ for all $x, x' \in \partial M$, then $\beta_2 = \beta_1 + \mathrm{d}\phi$ where ϕ is a scalar field vanishing on the boundary and $d_{F_{r,1}}(x, x') = d_{F_{r,2}}(x, x')$ for all $x, x' \in \partial M$. This is done by using projective equivalence of the Finsler norms F_i and $F_{r,i}$: since the 1-form β_i is closed F_i and $F_{r,i}$ have the same geodesics as point sets, and the geodesics of F_i remain geodesics (as point sets) if their orientation is reversed. The second theorem is a boundary rigidity result for Randers metrics and it is a corollary of the first theorem: if $F_1 = F_{g_1} + \beta_1$ and $F_2 = F_{g_2} + \beta_2$ are Randers metrics where g_1 and g_2 are boundary rigid Riemannian metrics, β_i is a closed 1-form and $d_{F_1}(x, x') = d_{F_2}(x, x')$ for all $x, x' \in \partial M$, then $F_2 = \Psi^* F_1 + \mathrm{d}\phi$ where ϕ is a scalar field vanishing on the boundary and $\Psi\colon M \to M$ is a diffemorphism which is identity on the boundary. In other words, the equality of the boundary distances implies that the Randers metrics F_1 and F_2 are equal up to the natural gauge. Using Zermelo's navigation problem we provide an application of the second theorem to seismology where the seismic wave propagates in a moving medium.

4.1. **Notation.** Let us first go through the notation used in the article [E]. We follow the notation conventions of the references [92, 93, 95, 120, 143]. We will use the Einstein summation convention so that every repeated index appearing both as a subscript and superscript is implicitly summed over.

Let M be an n-dimensional smooth manifold where $n \geq 2$. We usually assume that M is compact and has a boundary ∂M or that M is non-compact without boundary. If (M, g) is a Riemannian manifold, we denote by $K(x)$ the Gaussian curvature at $x \in M$. We say that a compact Riemannian manifold (M, g) with boundary is simple (or that the Riemannian metric g is simple) if it is non-trapping (maximal geodesics have finite length), geodesics have no conjugate points and the boundary ∂M is strictly convex with respect to g (the second fundamental form on ∂M is positive definite). A compact simple manifold is always diffeomorphic to a ball. We say that a non-compact manifold (M, g) without boundary is a Cartan–Hadamard manifold if it is simply connected, complete and its sectional curvature is nonpositive. Cartan–Hadamard manifolds are diffeomorphic to $\mathbb{R}^n$ and basic examples are the Euclidean space and hyperbolic spaces.

Let $m \geq 1$. We denote by $\mathfrak{X}(T_m M)$ the space of all covariant m-tensor fields and $S_m M \subset \mathfrak{X}(T_m M)$ is the space of symmetric covariant m-tensor fields. The notations $C^\infty(T_m M) := C^\infty(\mathfrak{X}(T_m M))$ and $C^\infty(S_m M)$ mean that the corresponding tensor fields are smooth. The pointwise norm of a covariant m-tensor field h is $|h|_{g_x} = \sqrt{g_x(h,h)}$ where $g_x(\cdot,\cdot)$ is the fiberwise inner product of m-tensor fields. We define the following sets of polynomially and exponentially decaying tensor fields which are mainly used on Cartan–Hadamard manifolds

$$
\begin{aligned}
E_\eta(T_m M) = \{h \in C^1(T_m M) :& \\
&|h|_{g_x} \leq C e^{-\eta d(x,o)} \text{ for some } C > 0\},\\
E^1_\eta(T_m M) = \{h \in C^1(T_m M) :& \\
&|h|_{g_x} + |\nabla h|_{g_x} \leq C e^{-\eta d(x,o)} \text{ for some } C > 0\},\\
P_\eta(T_m M) = \{h \in C^1(T_m M) :& \\
&|h|_{g_x} \leq C(1 + d(x,o))^{-\eta} \text{ for some } C > 0\},\\
P^1_\eta(T_m M) = \{h \in C^1(T_m M) :& \\
&|h|_{g_x} \leq C(1 + d(x,o))^{-\eta} \text{ and}\\
&|\nabla h|_{g_x} \leq C(1 + d(x,o))^{-\eta-1} \text{ for some } C > 0\}
\end{aligned}
$$

where $o \in M$ is a fixed point and $\eta > 0$.

Let (M, g) be a non-trapping compact Riemannian manifold with boundary ∂M. Let $x \in \partial M$ and $\xi \in T_x M$ be an inward-pointing unit vector. Denote by $\gamma_{x,\xi}$ the geodesic starting at x in the direction ξ and let $\tau(x, \xi)$ be the first time when the geodesic hits the boundary again. The geodesic ray transform of a sufficiently regular m-tensor field h is defined as

$$
\mathcal{I}_m h(x,\xi) = \int_0^{\tau(x,\xi)} h_{i_1 \dots i_m}(\gamma_{x,\xi}(t)) \dot\gamma^{i_1}_{x,\xi}(t) \cdots \dot\gamma^{i_m}_{x,\xi}(t) \mathrm{d}t.
$$

Similarly, if (M, g) is a Cartan–Hadamard manifold and $x \in M$ and $\xi \in T_x M$ has unit length, then we define

$$
\mathcal{I}_m h(x,\xi) = \int_{-\infty}^{\infty} h_{i_1 \dots i_m}(\gamma_{x,\xi}(t)) \dot\gamma^{i_1}_{x,\xi}(t) \cdots \dot\gamma^{i_m}_{x,\xi}(t) \mathrm{d}t
$$

whenever the m-tensor field h decays rapidly enough at infinity. By completeness geodesics are defined on all times on Cartan–Hadamard manifolds.

We define $A\colon C^\infty(T_mM) \to C^\infty(T_mM)$ as a smooth linear invertible map on m-tensor fields which operates as

$$(Ah)_x(\xi_1, \dots, \xi_m) = h_x(A_1(x)\xi_1, \dots, A_m(x)\xi_m)$$

where $\xi_i \in T_xM$ and each $A_i(x)$ is a linear bijection in T_xM. Such map A is called a mixing of degree $m \geq 1$. If A is a mixing of degree m, we define the mixing ray transform $\mathcal{I}_A$ by setting $\mathcal{I}_A = \mathcal{I}_m \circ A$ where $\mathcal{I}_m$ is the geodesic ray transform of m-tensor fields. On orientable two-dimensional manifolds an important special case of the mixing ray transforms is the mixed ray transform $L_{k,l} = \mathcal{I}_m \circ A_{k,l}$ where the components A_i of the mixing $A_{k,l}$ satisfy $A_i = \star$ when $i = 1, \dots, k$ and $A_i = \mathrm{Id}$ when $i = k+1, \dots, k+l = m$. Here $\star$ is the Hodge star operating on 1-forms (and hence on vector fields via the musical isomorphisms) and on orientable two-dimensional manifolds it corresponds to rotation by 90 degrees counterclockwise.

If A is a mixing of degree m, we define the generalized symmetrization operator $\widehat{\sigma}_A = A^{-1} \circ \sigma \circ A$ where σ is the usual symmetrization of tensor fields. Then $\widehat{\sigma}_A$ is a projection onto $A^{-1}(S_mM)$ and we have the direct decomposition $h = \widehat{\sigma}_A h + (h - \widehat{\sigma}_A h)$ where $\widehat{\sigma}_A h \in A^{-1}(S_mM)$ and $h - \widehat{\sigma}_A h \in \mathrm{Ker}(\mathcal{I}_A)$. We denote by ∇^A the weighted covariant derivative $\nabla^A = A^{-1} \circ \nabla$. We say that the mixing ray transform $\mathcal{I}_A$ is s-injective on a compact Riemannian manifold (M, g) with boundary, if for every $h \in C^\infty(T_mM)$ we have that $\mathcal{I}_A h = 0$ if and only if $\widehat{\sigma}_A h = \widehat{\sigma}_A \nabla^A v$ for some $v \in C^\infty(S_{m-1}M)$ vanishing on the boundary.

Then we shortly introduce the additional notation used in the article [F]; these basic notions of Finsler geometry can be found in [3, 9, 24, 145].

Let $F\colon TM \to [0, \infty)$ be a Finsler norm and denote by F_r a reversible Finsler norm, i.e. $F_r(x, -y) = F_r(x, y)$ for all $x \in M$ and $y \in T_xM$. If g is a Riemannian metric, then it defines a reversible Finsler norm F_g as $F_g(x, y) = \sqrt{g_{ij}(x)y^iy^j}$. We denote by β a smooth 1-form and say that β is closed, if $\mathrm{d}\beta = 0$ where d is the exterior derivative of differential forms. We define the dual norm of β as $\|\beta\|_{F^*} = \sup_{x \in M} F^*(x, \beta_x)$ where F^* is the co-Finsler norm in T^*M. More specifically, $F^*(x, \beta_x) = \sup_{y \in T_xM, F(x,y)=1} \beta_x(y)$. If F is a Finsler norm and β is a 1-form such that $\|\beta\|_{F^*} < 1$, then $F + \beta$ also defines a Finsler norm.

We define admissible Finsler norms as follows: F is admissible, if for any two points $x, x' \in \partial M$ there exists unique geodesic γ of F going from x to x' having finite length. When F is admissible, we define the map $d_F(\cdot, \cdot)\colon \partial M \times \partial M \to [0, \infty)$ as $d_F(x, x') = L_F(\gamma)$ where $L_F(\gamma)$ is the length of the geodesic γ with respect to F. In general the map $d_F(\cdot, \cdot)$ is not symmetric. We say that the Riemannian metrics g_1 and g_2 on M are boundary rigid, if $d_{g_1}(x, x') = d_{g_2}(x, x')$ for all $x, x' \in \partial M$ if and only if $g_2 = \Psi^* g_1$ where $\Psi\colon M \to M$ is a diffeomorphism which is identity on the boundary.

4.2. **Main results.** In the article [E] we study linearized travel time tomography. We have numerous corollaries of the algebraic approach to mixing ray transforms and here we only present the most important results considering solenoidal injectivity. The first result says that s-injectivity of one mixing ray transform implies s-injectivity for all mixing ray transforms.

Theorem 4.1 ([E, Corollary 3.4])**.** *Let $m \geq 1$ and (M, g) be a compact Riemannian manifold with boundary so that the transform I_A is s-injective for some A of degree m. Then $I_{\widetilde{A}}$ is s-injective for all $\widetilde{A}$ of degree m.*

Theorem 4.1 holds in all dimensions $n \geq 2$ and it is proved by using the definition of s-injectivity and the properties of mixings A and the corresponding projections $\widehat{\sigma}_A$. The following theorem is a special case of theorem 4.1 in two dimensions.

Theorem 4.2 ([E, Corollary 4.1])**.** *Let $m \geq 1$. Let (M, g) be a compact two-dimensional orientable Riemannian manifold with boundary such that the geodesic ray transform is s-injective on $C^\infty(S_m M)$ and let $h \in C^\infty(T_m M)$. Then $L_{k,l}h = 0$ if and only if $\widehat{\sigma}_{A_{k,l}}h = \widehat{\sigma}_{A_{k,l}}\nabla^{A_{k,l}}v$ for some $v \in C^\infty(S_{m-1}M)$ vanishing on the boundary ∂M.*

S-injectivity of the geodesic ray transform is known for example on compact simple surfaces [119] and on simply connected compact surfaces with strictly convex boundary and non-positive sectional curvature [118, 143]. Hence we obtain many new s-injectivity results for the mixed ray transform in two dimensions using theorem 4.2. The assumption that (M, g) is a two-dimensional orientable manifold is needed so that the mixed ray transform is well-defined, i.e. we can use the Hodge star $\star$ to rotate vector fields.

The next theorem of the article [E] shows that s-injectivity holds for the mixed ray transform also on certain non-compact Cartan–Hadamard manifolds.

Theorem 4.3 ([E, Corollary 4.2])**.** *Let (M, g) be a two-dimensional Cartan–Hadamard manifold and let $m \geq 1$. The following claims are true:*

(a) Let $-K_0 \leq K \leq 0$ for some $K_0 > 0$ and $h \in E^1_\eta(T_m M)$ for some $\eta > \frac{3}{2}\sqrt{K_0}$. Then $L_{k,l}h = 0$ if and only if $\widehat{\sigma}_{A_{k,l}}h = \widehat{\sigma}_{A_{k,l}}\nabla^{A_{k,l}}v$ for some $v \in S_{m-1}M$ such that $v \in E_{\eta-\epsilon}(T_{m-1}M)$ for all $\epsilon > 0$.

(b) Let $K \in P_\kappa(M)$ for some $\kappa > 2$ and $h \in P^1_\eta(T_m M)$ for some $\eta > 2$. Then $L_{k,l}h = 0$ if and only if $\widehat{\sigma}_{A_{k,l}}h = \widehat{\sigma}_{A_{k,l}}\nabla^{A_{k,l}}v$ for some $v \in S_{m-1}M \cap P_{\eta-1}(T_{m-1}M)$.

Theorem 4.3 follows from the corresponding s-injectivity result for the geodesic ray transform proved in [95]. Before we can use the results in [95] we show that the mixing $A_{k,l}$ in the mixing ray transform $L_{k,l} = \mathcal{I}_m \circ A_{k,l}$ maps tensor fields in $E^1_\eta(T_m M)$ to tensor fields in $E^1_\eta(T_m M)$, and similarly tensor fields in $P^1_\eta(T_m M)$ to tensor fields in $P^1_\eta(T_m M)$. We do not need to assume orientability in theorem 4.3 since Cartan–Hadamard manifolds are always orientable.

Theorem 4.2 implies that we can write the kernel of the mixed ray transform on compact orientable surfaces with boundary admitting s-injectivity of the geodesic ray transform as the direct sum

$$\text{Ker}(L_{k,l}|_{C^\infty(T_m M)}) = \text{Im}(\mathcal{H}|_{C^\infty(T_m M)}) \oplus \text{Im}(\widehat{\sigma}_{A_{k,l}} \nabla^{A_{k,l}}|_Y) \tag{18}$$

where $\mathcal{H} = \text{Id} - \widehat{\sigma}_{A_{k,l}}$ is the projection onto the trivial part of $\text{Ker}(L_{k,l})$ and $Y = \{v \in C^\infty(S_{m-1}M) : v|_{\partial M} = 0\}$. Similar decomposition as in (18) holds for non-compact Cartan–Hadamard manifolds by theorem 4.3 using the sets of polynomially and exponentially decaying tensor fields.

In the article [F] we study the non-linear travel time tomography or boundary rigidity problem. The following theorem is the first main result of the article [F].

Theorem 4.4 ([F, Theorem 1.3]). *Let M be a compact and simply connected smooth manifold with boundary. For $i \in \{1,2\}$ let $F_i = F_{r,i} + \beta_i$ be admissible Finsler norms where $F_{r,i}$ is an admissible and reversible Finsler norm and β_i is a smooth closed 1-form such that $\|\beta_i\|_{F^*_{r,i}} < 1$. Then the following are equivalent:*

- *(i) $d_{F_1}(x,x') = d_{F_2}(x,x')$ for all $x, x' \in \partial M$.*
- *(ii) There is unique scalar field ϕ vanishing on the boundary such that $\beta_2 = \beta_1 + \mathrm{d}\phi$, and $d_{F_{r,1}}(x,x') = d_{F_{r,2}}(x,x')$ for all $x, x' \in \partial M$.*

One can take F_r to be for example a simple Riemannian metric in theorem 4.4 since they are admissible and reversible. Since F_r is reversible for any curve γ we can obtain $L_{F_r}(\gamma)$ from the symmetric part and $\int_\gamma \beta$ from the antisymmetric part of the length functional $L_F(\gamma)$. In other words, the data for β and F_r "decouple". Closedness of the 1-form β is in essential role in proving theorem 4.4: $\mathrm{d}\beta = 0$ implies that F_r and $F = F_r + \beta$ have the same geodesics up to orientation preserving reparametrizations, and geodesics of F remain geodesics as point sets when their parametrization is reversed. Simply connectedness of M implies that $\beta_i = \mathrm{d}\phi_i$ for some scalar field ϕ_i and this fact also plays a role in the proof.

The next theorem is the second main result of the article [F] and it gives a boundary rigidity result for certain Randers metrics.

Theorem 4.5 ([F, Theorem 1.5]). *Let M be a compact and simply connected smooth manifold with boundary. For $i \in \{1,2\}$ let $F_i = F_{g_i} + \beta_i$ be admissible Finsler norms where g_i is an admissible Riemannian metric and β_i is a smooth closed 1-form such that $\|\beta_i\|_{g_i} < 1$. Assume that (M, g_i) is boundary rigid. Then the following are equivalent:*

- *(a) $d_{F_1}(x,x') = d_{F_2}(x,x')$ for all $x, x' \in \partial M$.*
- *(b) There is unique scalar field ϕ vanishing on the boundary and a diffeomorphism Ψ which is identity on the boundary such that $\beta_2 = \beta_1 + \mathrm{d}\phi$ and $g_2 = \Psi^* g_1$.*

(c) *There is unique scalar field ϕ vanishing on the boundary and a diffeomorphism Ψ which is identity on the boundary such that $\beta_2 = \Psi^*\beta_1 + \mathrm{d}\phi$ and $g_2 = \Psi^* g_1$.*

Theorem 4.5 is proved by using theorem 4.4 and the rigidity assumption on the Riemannian metrics g_i. Theorem 4.5 part (c) implies that $F_2 = \Psi^* F_1 + \mathrm{d}\phi$. Finsler norms satisfying such relation are sometimes called almost isometric Finsler norms and the diffeomorphism $\Psi\colon (M, F_2) \to (M, F_1)$ is called an almost isometry [18, 35, 66, 74]. We note that Ψ cannot be an isometry since this would require that $\Psi^*\beta_1 = \beta_2$. Theorem 4.5 can be seen as a generalization of the Riemannian boundary rigidity results to non-reversible Randers manifolds. If $n = 2$, then one can take g_i to be a simple Riemannian metric in theorem 4.5 since in two dimensions simple Riemannian metrics are boundary rigid [122].

Theorem 4.5 has the following application in seismology. Assume that $M = \overline{B}(0, R) \subset \mathbb{R}^n$ is a closed ball of radius $R > 0$ equipped with the Riemannian metric $g = c^{-2}(r)e$ where $c = c(r)$ is a radial sound speed satisfying the Herglotz condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right) > 0, \quad r \in [0, R],$$

and e is the Euclidean metric. In addition, let us assume that g has no conjugate points so that g becomes a simple Riemannian metric [107, 152]. Suppose that the seismic wave propagates in a moving medium which velocity field is given by the vector field W. Using Zermelo's navigation problem and a first-order approximation we obtain that if the scaled flow field W/c^2 is irrotational ($\mathrm{d}(W/c^2) = 0$), then one can uniquely determine the speed of sound c and the velocity field W up to potential fields from travel time measurements of seismic waves which are done on the boundary $\partial M = S^{n-1}(0, R) \subset \mathbb{R}^n$.

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Included articles

[A]

Unique continuation of the normal operator of the x-ray transform and applications in geophysics

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UNIQUE CONTINUATION OF THE NORMAL OPERATOR OF THE X-RAY TRANSFORM AND APPLICATIONS IN GEOPHYSICS

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ABSTRACT. We show that the normal operator of the X-ray transform in $\mathbb{R}^d$, $d \geq 2$, has a unique continuation property in the class of compactly supported distributions. This immediately implies uniqueness for the X-ray tomography problem with partial data and generalizes some earlier results to higher dimensions. Our proof also gives a unique continuation property for certain Riesz potentials in the space of rapidly decreasing distributions. We present applications to local and global seismology. These include linearized travel time tomography with half-local data and global tomography based on shear wave splitting in a weakly anisotropic elastic medium.

1. Introduction

Linearized travel time tomography of shear waves reduces mathematically to a version of the X-ray tomography problem under a suitable model. We are interested in shear wave splitting of waves travelling through the mantle, leading us to a partial data problem. The partial data problem of the X-ray transform can then be reduced to a unique continuation problem of the normal operator of the X-ray transform. We study the unique continuation property of the normal operator mathematically and apply it to show that our partial data problems arising from geophysics have unique solutions.

Consider the following X-ray tomography problem with partial data. Assume we have a compactly supported function or distribution f on $\mathbb{R}^d$, $d \geq 2$, and an open set $V \subset \mathbb{R}^d$. Suppose we only know the integrals of f over the lines through V and the values of f in V. Does this information determine f uniquely? In terms of the X-ray transform X, if $Xf(\gamma) = 0$ for all lines γ intersecting V and $f|_V = 0$, is it true that $f = 0$? The answer is positive and even more is true.

The partial data problem can be recast into a unique continuation problem of the normal operator $N = X^*X$ of the X-ray transform. In other words, if $Nf|_V = 0$ and $f|_V = 0$, does it imply that $f = 0$? The answer is 'yes', and we prove a stronger unique continuation property for N where we only require that Nf vanishes to infinite order at some point in V. The proof also applies to some Riesz potentials of rapidly decreasing distributions. As a corollary we get the uniqueness result for the X-ray tomography problem with partial data.

It is well known that the partial data problem or region of interest (ROI) problem has important applications in medical imaging (see e.g. [21, 22, 34, 57, 58]). We

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introduce two possibly new applications in theoretical seismology. Namely, we show that one can uniquely solve a linearized travel time problem with receivers only in a small open subset of the Earth's surface. In addition, we describe how to use shear wave (S-wave) splitting measurements to determine the difference of the S-wave speeds. See section 1.2 for details on these applications.

Similar partial data results are known in $\mathbb{R}^2$ for compactly supported smooth functions, compactly supported L^1-functions and compactly supported distributions [4, 21, 22]. Our method of proof applies to all dimensions $d \geq 2$. An important novelty is in looking at the partial data result from the point of view of unique continuation of the normal operator. The theorem can be seen as a complementary result to the Helgason support theorem (see lemma 2.3) where one requires that the lines do not intersect the set in question. Our result can also be seen as a unique continuation property for the inverse operator of the fractional Laplacian $(-\Delta)^s$.

We present two alternative proofs for the partial data problem. The first proof uses the unique continuation property of the normal operator of the X-ray transform. The second proof is more direct and uses spherical symmetry. However, both proofs rely on a similar idea, differentiation of an integral kernel and density of polynomials. We also present an alternative proof for the unique continuation of the Riesz potential which is based on unique continuation of the fractional Laplacian.

1.1. **The main results.** Denote by $\mathcal{D}(\mathbb{R}^d)$ the set of compactly supported smooth functions and by $\mathcal{D}'(\mathbb{R}^d)$ the space of all distributions in $\mathbb{R}^d$, $d \geq 2$. Also denote by $\mathcal{E}'(\mathbb{R}^d)$ the set of compactly supported distributions in $\mathbb{R}^d$. Let $\alpha = d-1$ or $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $\alpha < d$. We define the Riesz potential $I_\alpha f = f * h_\alpha$ for $f \in \mathcal{E}'(\mathbb{R}^d)$ where $h_\alpha(x) = |x|^{-\alpha}$ and the convolution is understood in the sense of distributions. If $\alpha = d-1$, then I_α reduces to the normal operator of the X-ray transform up to a constant factor 2. We say that $I_\alpha f$ vanishes to infinite order at a point x_0 if $\partial^\beta (I_\alpha f)(x_0) = 0$ for all $\beta \in \mathbb{N}^d$. Our main result is the following (see also theorem 5.1 and theorem 5.2).

Theorem 1.1. *Let $f \in \mathcal{E}'(\mathbb{R}^d)$, $V \subset \mathbb{R}^d$ any nonempty open set and $x_0 \in V$. If $f|_V = 0$ and $I_\alpha f$ vanishes to infinite order at x_0, then $f = 0$. In particular, this holds for the normal operator of the X-ray transform.*

The condition $f|_V = 0$ guarantees that $I_\alpha f$ is smooth in a neighborhood of x_0. The pointwise derivatives $\partial^\beta (I_\alpha f)(x_0)$ therefore exist, see the proof of theorem 1.1 for details. The condition of vanishing derivatives at a point only makes sense under the assumption that f vanishes (or is smooth) in V.

Theorem 1.1 can be seen as a unique continuation property of the Riesz potential I_α. The result resembles a strong unique continuation property but the roles in the decay conditions are interchanged. As an immediate corollary we obtain the following partial data results for the X-ray tomography problem. The first one is similar compared to the uniqueness results in [21, 22]. For the definition of the X-ray transform on distributions, see section 3.

Theorem 1.2. *Let $V \subset \mathbb{R}^d$ be any nonempty open set. If $f \in \mathcal{E}'(\mathbb{R}^d)$ satisfies $f|_V = 0$ and Xf vanishes on all lines that intersect V, then $f = 0$.*

Corollary 1.3. *Let $R > r > 0$ and $f \in \mathcal{E}'(\mathbb{R}^d)$ such that $\operatorname{spt}(f) \subset \overline{B}(0,R) \setminus B(0,r)$. If Xf vanishes on all lines that intersect $B(0,r)$, then $f = 0$.*

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, smooth and strictly convex set and $\Sigma \subset \partial\Omega$ any nonempty open subset of its boundary. If $f \in \mathcal{E}'(\mathbb{R}^d)$ is supported in $\overline{\Omega}$ and its X-ray transform vanishes on all lines that meet Σ, then $f = 0$.*

Proofs of the theorems and corollaries can be found in section 2.3 (see also the alternative proofs in section 5). Some of our assumptions are crucial for the theorems to be true. Theorem 1.2 is clearly false if $d = 1$. The function f cannot be determined from its integrals over the lines through the ROI only [22, 34, 48]. Thus one needs some information of f in the open set V which the lines all meet. Especially we need the assumption $f|_V = 0$ when we use the Kelvin transform and density of polynomials. Our proof also exploits the assumption of compact support which is motivated by the physical setting and is needed to define the Riesz potential on distributions. However, one can relax that assumption to rapid decay at infinity (see theorem 5.1 and theorem 5.2). Theorem 1.2 and corollaries 1.3 and 1.4 have important applications in theoretical seismology and medical imaging. This is discussed in more depth in the next section.

1.2. **Applications.** Our results have theoretical applications in seismology. Applications include linearization of anisotropies in S-wave splitting and linearized travel time tomography. Even though there exist many different types of seismic data, we only use linearized travel time data without reflections in our models. For the following treatment of splitting of S-waves we refer to [5, 28, 29, 45, 47].

In linear elasticity in $\mathbb{R}^3$ there are three polarizations of seismic waves which correspond to the eigenvectors of the symmetric Christoffel matrix. The eigenvalues correspond to wave speeds. In the isotropic case the largest eigenvalue is simple with the eigenvector parallel to the direction of propagation, corresponding to a P-wave. The other eigenvalue is degenerate with eigenvectors orthogonal to the P-wave polarization. These eigenvectors correspond to S-waves. In anisotropic medium this degeneracy is typically lost and the degenerate S polarization splits to two quasi-S (qS) polarizations. The data in the imaging method based on S-wave splitting is the arrival time difference between the two qS-waves.

One common type of anisotropy is hexagonally symmetric anisotropy. This means that there is a preferred direction or a symmetry axis and the velocities vary only with the angle from the axis, i.e. there is rotational symmetry. For example sedimentary layering and aligned crystals or cracks can cause hexagonal anisotropy. If the seismic wavelength is substantially larger than the layer or crack spacing, then the material appears to be anisotropic [1]. The widely used one-dimensional Preliminary Reference Earth Model (PREM) indicates this kind of anisotropy between the depths 80–220 km in the upper mantle [10, 47]. In the PREM-model the symmetry axis is radial and all the physical parameters of the Earth depend only on the depth. Anisotropies have also been observed in the shallow crust and in the inner core where the fastest direction is parallel to the rotation axis of the Earth [6, 47].

Our results pertain to so-called weak anisotropy, where we consider the anisotropy as a small perturbation to an isotropic reference model. In the isotropic background model S-waves have a speed $c_0(x)$ for all directions and polarizations. When we add a small anisotropic perturbation, the speeds become $c_i(x, v) = c_0(x) + \delta c_i(x, v)$, $i = 1, 2$. Here $v \in S^2$ is the direction of propagation of the wave. In the linearized regime $|\delta c_i| \ll |c_0|$ we have

$$\frac{1}{c_i(x,v)} = \frac{1}{c_0(x) + \delta c_i(x,v)} \approx \frac{1}{c_0(x)} - \frac{\delta c_i(x,v)}{c_0^2(x)}.$$

If we only measure small differences in the arrival times, our data is roughly

$$\delta t \approx \int_\gamma \frac{\mathrm{d}s}{c_1(x,v)} - \int_\gamma \frac{\mathrm{d}s}{c_2(x,v)} \approx \int_\gamma \frac{\delta c_2(x,v) - \delta c_1(x,v)}{c_0^2(x)} \mathrm{d}s.$$

Thus upon linearization, the data is the X-ray transform of $c_0^{-2}(\delta c_2 - \delta c_1)$. To simplify this problem, we assume the function to depend on x but not on v. If the splitting occurs in a layer near the surface (see figure 1), we are in the setting of corollary 1.3. The corollary implies that the linearized shear wave splitting data determines $\delta c_2 - \delta c_1$ and thus $c_2 - c_1$ uniquely in the outermost layer.

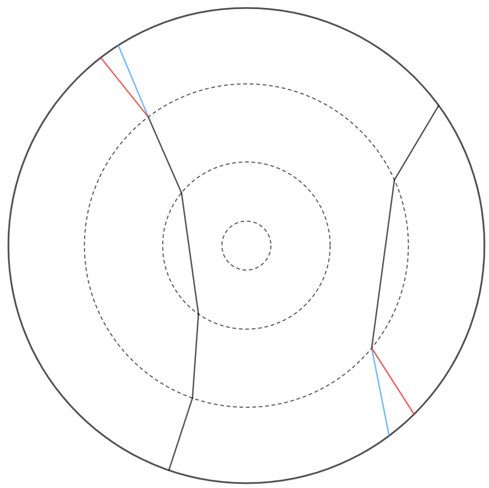

FIGURE 1. A highly simplified picture of the setting in the linearized model. The splitting can occur at every interface but we only care about the splitting near the surface with smallest difference in the arrival times. There may exist different polarization states during the propagation of the initial wave, we only assume that the second to last part is an S-type wave. Our data consists purely of the branched parts of the waves.

Travel time tomography has a close relationship to the boundary rigidity problem where the aim is to reconstruct the metric of a manifold from boundary distance measurements [50, 53]. In seismology these distances correspond to travel times of seismic waves which are assumed to propagate along geodesics or straightest possible paths in the manifold. This problem is highly nonlinear and difficult to solve in full generality. Thus it is relevant to consider the first-order approximation and linearize the problem. When we linearize the general travel time tomography problem assuming our manifold to be $\mathbb{R}^d$ and that the variations in the metric are conformally Euclidean, the geodesics become lines and the problem reduces to the X-ray tomography problem of a scalar function.

Linearized travel time tomography motivates the following application of observing earthquakes by seismic arrays on the surface of the Earth. In the context of corollary 1.4 one can ideally think that some open set of the surface is covered densely by seismometers (see figure 2). One detects earthquakes only in this set and measures travel times of seismic waves originating anywhere on the surface. In geometrical terms, our geodesics have one endpoint in this open set and the other endpoint can freely vary. In contrast to "local data" where both endpoints are in the small set, we call this setting "half-local data". The interesting question then is whether this limited set of travel time data can determine the inner structure of the Earth uniquely. When we do the usual conformal linearization in the Euclidean background, we end up with partial X-ray tomography problem of a scalar function. Corollary 1.4 then tells that in principle one can use these kind of seismic arrays to uniquely determine the conformal factor in the linearization.

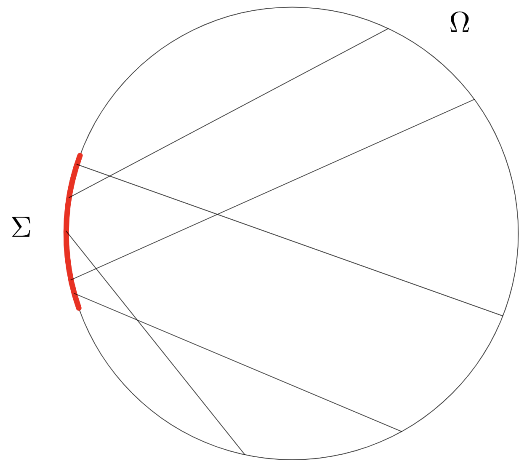

FIGURE 2. The setting as in corollary 1.4. Here Σ (thick) represents the seismic array where one measures the travel times of seismic waves and Ω represents the Earth.

In addition to theoretical seismology one important application is medical imaging, see [21, 22, 34, 57, 58] and the references therein. Suppose we want to reconstruct a specific part of the human body, a region of interest (ROI). Is it possible to reconstruct the image by shooting X-rays only through the ROI? If this was possible it would be unnecessary to give a higher dose of X-rays to the patient and radiate regions outside the ROI which do not contribute significantly to the image. We can interpret the function f in theorem 1.2 as the attenuation of X-rays which to a good approximation travel along straight lines inside a body. Somehow surprisingly theorem 1.2 tells us that if we know the values of f in a small open set inside the ROI and the integrals of f over the lines going through the ROI, then f is uniquely determined everywhere (see figure 3). It is important to note that arbitrary attenuation cannot be determined from the line integrals only even in the ROI but one can always recover the singularities in the ROI [22, 34, 48].

1.3. **Related results.** The partial data problem for the X-ray transform has been solved earlier in $\mathbb{R}^2$ under a variety of assumptions [4, 21, 22, 55]. The uniqueness result is known for C_c^∞-functions and compactly supported L^1-functions if one assumes the knowledge of f inside an open set in the ROI [4, 22]. One also obtains uniqueness without knowing the exact values of f in the ROI; if f is piecewise constant or piecewise polynomial in the ROI, then the X-ray data determines f uniquely [22, 55]. If f is polynomial in the ROI, then one obtains stability as well [21]. Closest to our theorem is the uniqueness result in [21] (see also [22] where the authors mention in the proof of lemma 2.4 that their method applies also to compactly supported distributions which are piecewise constant in the ROI). According to that result, if $f \in \mathcal{E}'(\mathbb{R}^2)$ integrates to zero over all lines intersecting V and $f|_V$ is real analytic, then $f = 0$.

Our result for the partial data problem uses stronger assumption $f|_V = 0$. This assumption is needed so that the Kelvin transformed function will be compactly supported and we can use density of polynomials. However, our theorem applies to any dimension $d \geq 2$. Another difference is in the point of view; we consider the normal operator and observe that the same result holds for a larger class of

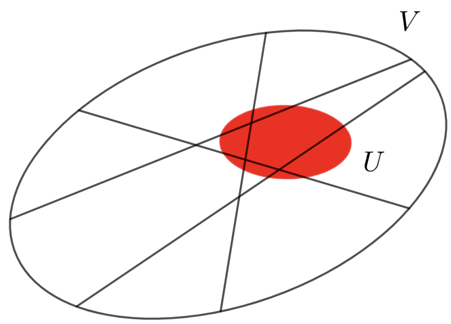

FIGURE 3. Basic idea of ROI-tomography in the context of theorem 1.2. Here V is the region of interest and $U \subset V$ some open subset. If one knows the attenuation f in U and the integrals of f over the lines through V, then one can construct f uniquely from the data.

Riesz potentials. Also our alternative proofs (theorem 5.1 and theorem 5.2) imply uniqueness for the partial data problem without assumption of compact support, rapid decay at infinity is enough. We remark that the X-ray data alone does not uniquely determine the attenuation in general. One cannot even construct C_c^∞-functions only from the integrals but one can always recover the singularities, which is equivalent with recovering the function up to a smooth error [22, 34, 48].

Unlike in [4, 21] our method is very unstable and concrete reliable reconstructions are basically hopeless. Our instability comes from the differentiation of the data and approximation of test functions by polynomials up to arbitrary order. However, our method of proof is not the only reason for instability. Instability is an intrinsic property of partial data problems. When we have limited X-ray data it is not guaranteed that we can see all the singularities of f from the data. Singularities which are invisible in the microlocal sense are related to the instability of inverting f from its limited X-ray data [25, 33, 35, 36]. See also [24, 37] for discussion of which part of the wave front set is visible in limited data tomography. Even though our theorem loses stability it gives uniqueness which is relevant for applications.

Our theorem is related to travel time tomography and the inverse kinematic problem. For a review of these, see [50, 53] and also [17, 54] for the original works by Herglotz, Wiechert and Zoeppritz. Specifically our result is a contribution to local and global theoretical seismology (see section 1.2). For example one can uniquely determine the difference of the anisotropic perturbations of the S-wave speeds by measuring the arrival time differences of the split S-waves. From the point of view of ROI tomography these seismic applications are new to the best of our knowledge.

It is also worth mentioning that our result is in a sense complementary to the famous support theorem by Helgason (see lemma 2.3). Helgason's theorem states that if $C \subset \mathbb{R}^d$ is a convex compact set and $f \in \mathcal{E}'(\mathbb{R}^d)$ such that $f|_C = 0$ and the X-ray transform Xf vanishes on all lines not meeting C, then $f = 0$. Compared to theorem 1.2, Helgason's result uses complementary data but gives the same conclusion. Helgason's theorem holds also for rapidly decreasing continuous functions; our partial data result is true for this function class as well (see section 5 and the discussion after theorem 5.1).

Our theorem has a connection to the fractional Laplacian $(-\Delta)^s$. The operator $(-\Delta)^s$ can be defined in many equivalent ways and one way is to consider it as the inverse of a Riesz potential [26]. In our notation $I_\alpha f = (-\Delta)^{-s} f$ where $s = (d-\alpha)/2$ assuming $0 < \alpha < d$. For example from equation (2) we see that in Euclidean space the normal operator of the X-ray transform N is the inverse of the fractional Laplacian $(-\Delta)^{1/2}$. Thus our result can be seen as a unique continuation property for the operator $(-\Delta)^{-\delta/2}$ where δ is any positive non-integer or $\delta = 1$. There are several unique continuation results for the operator $(-\Delta)^s$ when $0 < s < 1$ and they have been recently used in fractional Calderón problems [14, 15, 39, 42]. One version of our theorem can be proved using unique continuation of $(-\Delta)^s$ (see theorem 5.2). The fractional Laplacian even admits a strong unique continuation property if one assumes more regularity from the function [11, 41]. Here "strong" means that the function does not need to be zero in an open set, it only has to vanish to infinite order at some point. Theorem 1.1 has similar vanishing assumption for $I_\alpha f$ instead of f. There are also (strong) unique continuation results for the higher order Laplacian $(-\Delta)^t$ where t is a positive non-integer exponent [12, 13, 56].

In Euclidean space one can reconstruct a compactly supported distribution uniquely from its X-ray transform [48]. There even exist explicit inversion formulas using the formal adjoint X^* and the normal operator N. It is also known that the X-ray transform is injective on compact simple Riemannian manifolds with boundary [19]. Interesting injectivity results considering seismic applications have been obtained for conformally Euclidean metrics which satisfy the Herglotz condition [7]. See also how the length spectrum can be obtained from the Neumann spectrum of the Laplace-Beltrami operator or from the toroidal modes on these kind of manifolds in three dimensions [8]. This has a connection to the free oscillations of the Earth.

There are some partial data results for certain manifolds. If (M, g) is a two-dimensional compact simple Riemannian manifold with boundary and a real-analytic metric g, then one can reconstruct L^2-functions locally from their geodesic X-ray transform [23]. In dimensions $d \geq 3$ one can relax the analyticity condition to smoothness using a convexity assumption on the boundary [52]. Furthermore one can even invert the X-ray transform locally in a stable way and obtain a reconstruction formula based on Neumann series. Both of the results in [23, 52] rely on microlocal analysis. One can also locally invert, up to potential fields, tensors of order 1 and 2 near a strictly convex boundary point [49]. We remark that there is a similar distinction between analyticity and smoothness for the injectivity of the weighted X-ray transform in Euclidean space. When $d = 2$ the analyticity of the weight is required for injectivity while in higher dimensions smoothness is enough [2, 3, 48].

1.4. **Organization of the paper.** We begin our treatment by proving the main results in section 2. We also discuss the assumptions used in the results and applications. In section 3 we recall some basic theory of distributions and integral geometry in $\mathbb{R}^d$. Section 4 is devoted to the proof of lemma 2.2 which says that one can express all the polynomials in a certain form as a finite linear combination of the derivatives of the kernel of the Riesz potential I_α. Section 5 contains alternative proofs for theorem 1.1 and theorem 1.2.

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the fractional Laplacian. We are grateful to the anonymous referees for insightful remarks and suggestions.

2. Proofs of the main results

2.1. **An overview of the proof.** The rough idea of the proof of theorem 1.1 is the following. We may assume that $x_0 = 0$. The function $I_\alpha f$ is smooth in V, and by assumption all of its derivatives vanish at the origin. By a convolution argument these derivatives can be computed explicitly. The vanishing of these derivatives amounts to f integrating to zero against a set of functions. After a change of variables and suitable rescaling, one can use density of polynomials to show that this set is dense. Therefore f has to vanish.

The proofs of the corollaries are more straightforward. Detailed proofs of these main results are given in section 2.3 below. The reader who is not familiar with the theory of distributions and integral geometry can first read section 3. See section 5 for alternative proofs of theorems 1.1 and 1.2.

2.2. **Auxiliary results.** In this section we give a few auxiliary results which are needed in our proofs. The first one is a known theorem in distribution theory.

Lemma 2.1 ([51, p.160 Corollary 4])**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then the polynomials form a dense subspace of $\mathcal{E}(\Omega)$.*

Recall the kernel of the Riesz potential $h_\alpha(x) = |x|^{-\alpha}$. The next lemma is proved in section 4.

Lemma 2.2. *If $d \geq 2$ and $\alpha > d-2$ or $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then for any polynomial p one can express the product $p(K(x))h_\alpha(x)$ as a finite linear combination of derivatives of h_α. Here $K(x) = x\,|x|^{-2}$ is the Kelvin transform.*

We also need the following support theorem to prove corollary 1.4. The proof can be found for example in [16, 48].

Lemma 2.3 (Helgason's support theorem)**.** *Let $C \subset \mathbb{R}^d$ be a compact convex set and $f \in \mathcal{E}'(\mathbb{R}^d)$. If Xf vanishes on all lines not meeting C, then* $\operatorname{spt}(f) \subset C$.

2.3. **Proofs of the results.** Now we are ready to prove our main theorem and its corollaries. Let $d \geq 2$. Recall the definition of the Riesz potential $I_\alpha f = f * h_\alpha$ for $f \in \mathcal{E}'(\mathbb{R}^d)$ where $\alpha = d-1$ or $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $\alpha < d$. The kernel h_α has an expression $h_\alpha(x) = |x|^{-\alpha}$. We denote by K the Kelvin transform $K(x) = x\,|x|^{-2}$. See section 3 for basic results on distribution theory used in the proof.

Proof of theorem 1.1. We have to show that if $f \in \mathcal{E}'(\mathbb{R}^d)$ and $V \subset \mathbb{R}^d$ is any nonempty open set such that $f|_V = 0$ and $\partial^\beta(I_\alpha f)(x_0) = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}^d$, then $f = 0$. Because the problem is translation invariant we can assume that $x_0 = 0$. Since f has compact support and it vanishes in a neighborhood of the origin, we have that $\operatorname{spt}(f) \subset A$ for some open annulus A centered at the origin. Let $g \in \mathcal{D}(\mathbb{R}^d)$ be a symmetric smooth version of h_α such that $g|_A = h_\alpha|_A$. Choosing small enough $\epsilon > 0$ we have $I_\alpha f|_{B(0,\epsilon)} = (f * g)|_{B(0,\epsilon)}$ where $f * g \in \mathcal{D}(\mathbb{R}^d)$ by lemma 3.4. Since $I_\alpha f$ vanishes to infinite order at 0 lemmas 3.4 and 3.5 give us $\partial^\beta(f * g)(0) = (f * (\partial^\beta g))(0) = \left\langle f, \tau_0\widetilde{\partial^\beta g}\right\rangle = \left\langle f, \widetilde{\partial^\beta g}\right\rangle = 0$ for all multi-indices $\beta \in \mathbb{N}^d$. Since g is symmetric we get the condition $\left\langle f, \partial^\beta g\right\rangle = 0$.

Let $\eta \in C_c^\infty(A)$ be such that $\eta = 1$ in $\operatorname{spt}(f)$. By lemma 3.1 and the definition of restriction $f|_A$ we have $0 = \left\langle f, \partial^\beta g\right\rangle = \left\langle f, \eta\partial^\beta g\right\rangle = \left\langle f|_A, \eta\partial^\beta g\right\rangle$. Since $g|_A = h_\alpha|_A$ by lemma 2.2 we obtain all the polynomials p in the form $p(K(x))h_\alpha(x)$ restricted to A by taking finite linear combinations of the derivatives of g. Using linearity we obtain $\langle f|_A, \eta h_\alpha(p \circ K)\rangle = 0$ for all polynomials p. Taking the pullback we get $\langle f|_A \circ K, \eta_1 p \rangle = 0$ where $\eta_1 = ((\eta\,|J_{K^{-1}}|^{-1}) \circ K)h_\alpha^{-1}$. Let $\psi \in \mathcal{E}(K^{-1}(A))$. By lemma 2.1 there exists a sequence of polynomials p_k such that $p_k \to \psi$ in $\mathcal{E}(K^{-1}(A))$. This implies $\eta_1 p_k \to \eta_1 \psi$ in $\mathcal{E}(K^{-1}(A))$ because $\text{spt}(\eta_1) \subset\subset K^{-1}(A)$. Since $f|_A \circ K \in \mathcal{E}'(K^{-1}(A))$ by continuity $\langle \eta_1(f|_A \circ K), \psi \rangle = \langle f|_A \circ K, \eta_1 \psi \rangle = 0$, i.e. $\eta_1(f|_A \circ K) = 0$. But now $\eta_1 \neq 0$ in $K^{-1}(\text{spt}(f)) = \text{spt}(f|_A \circ K)$ and hence $f|_A \circ K = 0$ by lemma 3.3. Again using lemma 3.2 we obtain $f|_A = 0$ which implies $f = 0$. □

As an immediate consequence we obtain the proofs for the X-ray tomography problem with partial data.

Proof of theorem 1.2. We have to show that if $f \in \mathcal{E}'(\mathbb{R}^d)$ and $V \subset \mathbb{R}^d$ is any nonempty open set such that $f|_V = 0$ and $Xf|_{\Gamma_V} = 0$ where Γ_V is the set of all lines that intersect V, then $f = 0$. We can assume that V is a ball centered at the origin. Let $\varphi \in \mathcal{D}(V)$. From the definition of the normal operator of the X-ray transform we obtain $\langle Nf, \varphi \rangle = \langle Xf, X\varphi \rangle = 0$ since $X\varphi \in \mathcal{D}(\Gamma_V)$. Hence $Nf|_V = 0$ and the claim follows from theorem 1.1 by taking $\alpha = d - 1$. □

Proof of corollary 1.3. We have to show that if $R > r > 0$ and $f \in \mathcal{E}'(\mathbb{R}^d)$ such that $\text{spt}(f) \subset \overline{B}(0,R) \setminus B(0,r)$ and Xf vanishes on all lines that meet $B(0,r)$, then $f = 0$. Take a nonempty open set $V \subset\subset B(0,r)$. Then we have $f|_V = 0$ and Xf vanishes on all lines that intersect V. Theorem 1.2 implies that $f = 0$. □

Proof of corollary 1.4. Let $\Omega \subset \mathbb{R}^d$ be a bounded, smooth and strictly convex set and $\Sigma \subset \partial\Omega$ nonempty open subset of the boundary. We have to show that if $f \in \mathcal{E}'(\mathbb{R}^d)$ is supported in $\overline{\Omega}$ and Xf vanishes on all lines that meet Σ, then $f = 0$. We can assume that Σ is connected by passing to a connected component. Denote by $\text{ch}(\Sigma)$ the convex hull of Σ (see figure 4). By the Helgason support theorem (lemma 2.3) the function f vanishes in $\text{ch}(\Sigma)$. Take open set $V \subset \text{ch}(\Sigma)$, $V \neq \varnothing$. Then $f|_V = 0$ and Xf vanishes on all lines that intersect V. We can apply theorem 1.2 to conclude that $f = 0$. □

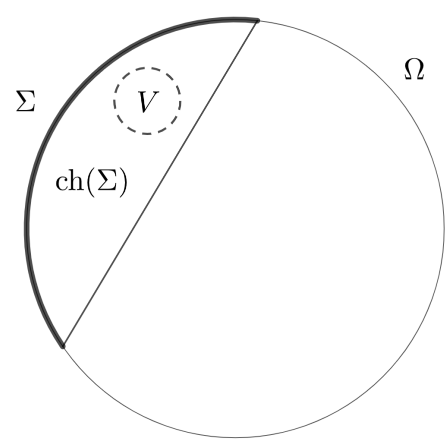

FIGURE 4. Idea of the proof of corollary 1.4. Here Σ (thick arc) is a connected open subset of $\partial\Omega$ and $\text{ch}(\Sigma)$ (segment) its convex hull. Helgason's support theorem (lemma 2.3) implies that f vanishes in $\text{ch}(\Sigma)$ and then theorem 1.2 is used for the dashed set V to conclude that $f = 0$.

2.4. **Discussion of assumptions and methods.** We assume that $f|_V = 0$ so as to ensure that $I_\alpha f|_V$ is smooth and the differentiation makes sense. For this purpose alone it would have been enough to assume that $f|_V$ is smooth. However, if $f|_V$ is non-zero, our method of proof appears to become untractable. Especially the Kelvin transformed function $f \circ K$ is not compactly supported anymore and we can not use density of polynomials in the proof. If $f|_V$ is polynomial (or real analytic) and $d = 2$, the method of [21] can be applied to prove the partial data result for the X-ray transform directly. Our method has the additional freedom that the Riesz potential need not be exactly the normal operator and that the dimension is not restricted to two. Moreover, in the physical application of shear wave splitting in the mantle, only the anisotropy in the mantle will matter and the perturbation can thus be taken to be supported outside the core.

The assumptions in theorem 1.1 are not optimal. The assumption of compact support is needed to define the Riesz potential I_α on distributions and is crucial in the proof when we use density of polynomials. However, compact support can be replaced with rapid decay at infinity (see theorem 5.1 and theorem 5.2). Theorem 1.2 is clearly false if $d = 1$. Also one cannot construct arbitrary C_c^∞-functions from the integrals over the lines through the ROI only [22, 34, 48]. Therefore one needs some information of the function f in the open set V; our method of proof especially requires the assumption $f|_V = 0$. In corollary 1.4 it is enough to assume that only the subset $\Sigma \subset \partial\Omega$ is strictly convex and the convex hull of the rest of the boundary does not cover all of Σ. The constraint $\alpha < d$ comes from the requirement that the kernel h_α determines a distribution. The other constraints for α come from the proof of lemma 2.2.

It would be interesting to know whether we could weaken the decay assumption in theorem 1.2 in the smooth case. Does there exist $f \in C^\infty(\mathbb{R}^d)$ such that $f|_V = 0$ and $Xf = 0$ for all lines through V but f is not identically zero? By theorem 5.1 the result in theorem 1.2 holds when f decreases faster that any polynomial at infinity. There also exists a counterexample for the Helgason support theorem where the function does not decay rapidly enough [16, 34]. Since our theorem is similar in spirit, we would expect a counterexample also in our case.

The normal operator of the X-ray transform $N = X^*X$ is an elliptic pseudo-differential operator. Therefore it would be natural to try methods of microlocal analysis to prove our main theorem. But the usual microlocal approach does not work here in the following sense. First, if we do the identification $f \sim g$ if and only if $f - g \in C^\infty(\mathbb{R}^d)$, then the claim of theorem 1.1 is not true. Namely, the assumptions $f|_V \in C^\infty(V)$ and $Nf|_V \in C^\infty(V)$ do not imply that necessarily $f \in C^\infty(\mathbb{R}^d)$. Thus our result is not true modulo C^∞. Second, from the assumptions of theorem 1.2 it is clear that some of the singularities of f are not visible in the data. These invisible singularities are usually difficult to reconstruct from the limited set of data [35, 36, 37]. The surprising thing here is that even though our data is local and smooth, we can still recover a distribution.

Our theorem considers the unique continuation of the normal operator of the X-ray transform. It is then natural to ask the following question: when does the normal operator of the geodesic X-ray transform on a manifold satisfy the unique continuation property? At the moment no results are known expect in the Euclidean case. Also there does not exist any simple relationship between the normal operator and the fractional Laplacian on general manifolds. In the context of seismic applications, it would be very beneficial to generalize the result to manifolds which are equipped with a conformally Euclidean metric satisfying the Herglotz condition [17, 54]. For example the widely used model of spherically symmetric

Earth (PREM model) satisfies the Herglotz condition to a good accuracy excluding discontinuity zones [10, 47]. But our method of proof seems to fit only to the Euclidean case, i.e. to zero curvature. Our proof was heavily based on a density argument using polynomials and polynomials were obtained by differentiating the kernel of the Riesz potential. Our preliminary calculations suggest that we cannot obtain all the polynomials even in the constant negative curvature case. In fact the procedure fails in the very first steps: we cannot even construct polynomials of order 2. Therefore we would need a different approach if we wanted to generalize our result to non-Euclidean manifolds.

There is another proof for theorem 1.2 which is based on spherical symmetry and angular Fourier series (see section 5.3). This method could perhaps generalize to some sort of spherically symmetric manifolds but it is not studied in a great detail yet. The big problem of general manifolds is that one cannot do explicit calculations. Especially we would need to express the Chebyshev polynomials in a nice form and show properties of them. The integral kernel is known in the conformally Euclidean case [7]. However, the issue becomes to calculate the derivatives of the kernel up to any order since the idea in the alternative proof is also to obtain all the polynomials and use density.

In section 1.2 we studied the applications of our results to seismology. We discussed about a model where we measure arrival time differences of split S-waves in a thin annulus. We did a linearization of the anisotropies of the S-wave speeds in isotropic background and made an (artificial) assumption that the difference of the perturbations is independent of direction of propagation. One could also consider a more general linearization in the elastic theory. This means that we have a known isotropic elastic model and a small anisotropic perturbation in the stiffness tensor c_{ijkl} to be determined from travel time measurements. It is shown in [46] that this kind of linearization leads to the X-ray tomography problem of a tensor field of degree 4 for P-waves. For S-waves one needs to study the so-called mixed ray transform of tensor fields of degree 4. There exists a kernel characterization for the full mixed ray transform of tensors of arbitrary order on 2-dimensional compact simple Riemannian manifolds with boundary [9]. But there are no known partial data results for the mixed ray transform. These would be highly beneficial and interesting considering applications in seismology.

If one treats the annulus as a thin layer with respect to the radius of the Earth ("flat Earth"), the situation resembles the X-ray tomography problem in a periodic slab $[0, \epsilon] \times \mathbb{T}^2$, $\epsilon > 0$. There is a kernel characterization for the X-ray transform of L^2-regular tensor fields of any order on periodic slabs of type $[0, 1] \times \mathbb{T}^d$ where d is any non-negative integer [20]. In particular the X-ray transform has a nontrivial kernel even for scalar fields in contrast to our result.

3. Integral geometry and distributions

3.1. **Distribution theory.** Let us review some basic distribution theory. A more detailed treatment can be found in a number of introductory books on distribution theory and functional analysis, e.g. [18, 31, 40, 43, 51]. This introduction is included for the benefit of readers less familiar with the theory and for the sake of easy reference later on. All the lemmas of this subsection are either well known or trivial and are therefore not proven.

Consider an open domain $\Omega \subset \mathbb{R}^d$. We denote by $\mathcal{E}(\Omega)$ the space of all smooth functions $\Omega \to \mathbb{C}$ and by $\mathcal{D}(\Omega)$ the subspace consisting of compactly supported functions. These spaces are equipped with the topology of uniform convergence of derivatives of any order on compact sets. The topological duals of these function spaces are denoted by $\mathcal{E}'(\Omega)$ and $\mathcal{D}'(\Omega)$, respectively, and their elements are called distributions. The space $\mathcal{E}'(\Omega)$ can be identified with the subspace of $\mathcal{D}'(\Omega)$ consisting of compactly supported distributions.

A multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ is a d-tuplet of natural numbers. We use the convention that $0 \in \mathbb{N}$. We write $|\beta| \coloneqq \beta_1 + \dots + \beta_d$ and

$$\partial^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\beta_d}.$$

The distributional derivative of order β of $u \in \mathcal{D}'(\Omega)$ is defined so that

$$\left\langle \partial^\beta u, \varphi \right\rangle = (-1)^{|\beta|} \left\langle u, \partial^\beta \varphi \right\rangle$$

for all $\varphi \in \mathcal{D}(\Omega)$ and similarly for $\mathcal{D}'$ and $\mathcal{D}$ replaced with $\mathcal{E}'$ and $\mathcal{E}$.

The value of a distribution evaluated at a test function only depends on the values of the test functions in the support of the distribution as stated in the next lemma.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in \mathcal{E}'(\Omega)$. If $\psi_1, \psi_2 \in \mathcal{E}(\Omega)$ are such that $\psi_1|_{\mathrm{spt}(u)} = \psi_2|_{\mathrm{spt}(u)}$, then $\langle u, \psi_1 \rangle = \langle u, \psi_2 \rangle$. The corresponding result also holds with $\mathcal{E}'$ and $\mathcal{E}$ replaced with $\mathcal{D}'$ and $\mathcal{D}$.*

It will be convenient to make a change of variables for distributions. Let $F\colon \Omega_1 \to \Omega_2$ be a C^∞-diffeomorphism between two domains $\Omega_1, \Omega_2 \subset \mathbb{R}^d$. The pullback $F^*u = u \circ F \in \mathcal{D}'(\Omega_1)$ of $u \in \mathcal{D}'(\Omega_2)$ is defined so that

$$\langle u \circ F, \varphi \rangle = \left\langle u, (\varphi \circ F^{-1}) \left| J_{F^{-1}} \right| \right\rangle$$

for all $\varphi \in \mathcal{D}(\Omega_1)$. Here $|J_{F^{-1}}|$ denotes the absolute value of the Jacobian determinant of F^{-1}. The same definition can be applied to $u \in \mathcal{E}'(\Omega_2)$ with $\varphi \in \mathcal{E}(\Omega_1)$. The supports behave naturally under pullbacks as stated in the next lemma.

Lemma 3.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open and $F\colon \Omega_1 \to \Omega_2$ be a C^∞-diffeomorphism. If $u \in \mathcal{D}'(\Omega_2)$, then $\mathrm{spt}(u \circ F) = F^{-1}(\mathrm{spt}(u))$. In particular, $u = 0$ if and only if $u \circ F = 0$.*

We will make use of the Kelvin transform or the inversion $K\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \setminus \{0\}$ given by $K(x) = |x|^{-2} x$. The Kelvin transform is its own inverse.

Any element of the spaces $\mathcal{E}(\Omega)$, $\mathcal{E}'(\Omega)$, $\mathcal{D}(\Omega)$, and $\mathcal{D}'(\Omega)$ can be multiplied by an element of $\mathcal{E}(\Omega)$. Such multiplication has an injectivity property we will need:

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^d$ be open, $u \in \mathcal{E}'(\Omega)$ and $g \in C^\infty(\Omega)$ such that $g \neq 0$ in $\mathrm{spt}(u)$. Then $u = 0$ if and only if $gu = 0$.*

For test functions $\varphi \in \mathcal{E}(\mathbb{R}^d)$ we define translation τ_{x_0} by $x_0 \in \mathbb{R}^d$ so that $(\tau_{x_0}\varphi)(x) = \varphi(x - x_0)$. The reflection $\widetilde{\varphi}$ is defined by $\widetilde{\varphi}(x) = \varphi(-x)$. Naturally $\tau_{x_0}\varphi, \widetilde{\varphi} \in \mathcal{E}(\mathbb{R}^d)$. Translations and reflections can be defined on distributions by duality.

Convolutions can also be defined for distributions (see e.g. [43]):

Lemma 3.4. *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then $u * \varphi$ has a representative $g_1 \in \mathcal{E}(\mathbb{R}^d)$ which is given by the formula $g_1(x) = \langle u, \tau_x \widetilde{\varphi} \rangle$. Additionally, if $v \in \mathcal{E}'(\mathbb{R}^d)$, then $v * \varphi$ has a representative $g_2 \in \mathcal{D}(\mathbb{R}^d)$ which is given by the formula $g_2(x) = \langle v, \tau_x \widetilde{\varphi} \rangle$.*

Lemma 3.5. *Let $u \in \mathcal{E}'(\mathbb{R}^d)$ and $v \in \mathcal{D}'(\mathbb{R}^d)$. Then $u * v \in \mathcal{D}'(\mathbb{R}^d)$ is defined via the formula*

$$\langle u * v, \varphi \rangle = \langle u, \widetilde{v} * \varphi \rangle$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, and for every $\beta \in \mathbb{N}^d$ the derivatives satisfy

$$\partial^\beta (u * v) = (\partial^\beta u) * v = u * (\partial^\beta v)$$

in the sense of distributions.

3.2. **Integral geometry and the normal operator.** In this section we introduce basic theory of integral geometry in $\mathbb{R}^d$. For this we mainly follow the books [16, 34, 48], see also [38]. We define the Riesz potential I_α and discuss about its connection to the normal operator of the X-ray transform N.

Denote by Γ the set of all oriented lines in $\mathbb{R}^d$. The X-ray transform of a function f is the map $Xf\colon \Gamma \to \mathbb{R}$,

$$Xf(\gamma) = \int_\gamma f \mathrm{d}s$$

for all lines $\gamma \in \Gamma$ assuming that the integrals exists. The integrals are finite whenever f decays fast enough at infinity. If the lines are parametrized by the set

$$\{(z,\theta) : \theta \in S^{d-1},\ z \in \theta^\perp\},$$

the X-ray transform may be written as

$$Xf(z,\theta) = \int_{\mathbb{R}} f(z + s\theta)\mathrm{d}s.$$

It is a continuous linear map $X\colon \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\Gamma)$. The set Γ can be freely identified with TS^{d-1}. As Γ is a smooth manifold, the test function and distribution spaces on it can be defined similarly to the Euclidean setting.

The formal adjoint $X^*\colon \mathcal{E}(\Gamma) \to \mathcal{E}(\mathbb{R}^d)$ is given by

$$X^*\psi(x) = \int_{S^{d-1}} \psi(x - (x\cdot\theta)\theta, \theta)\mathrm{d}\theta.$$

The function ψ can be interpreted as a function in the set of all lines. The value $X^*\psi(x)$ is obtained by integrating ψ over all lines going through the point x. The formal adjoint does not preserve compact supports, but the integrals in its definition are taken over compact sets.

The operators X and X^* can be defined on distributions by duality. That is, $X\colon \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(\Gamma)$ and $X^*\colon \mathcal{D}'(\Gamma) \to \mathcal{D}'(\mathbb{R}^d)$ are defined so that they satisfy

$$\langle Xf, \eta\rangle = \langle f, X^*\eta\rangle$$

for all $f \in \mathcal{E}'(\mathbb{R}^d)$ and $\eta \in \mathcal{E}(\Gamma)$, and

$$\langle X^*g, \varphi\rangle = \langle g, X\varphi\rangle$$

for all $g \in \mathcal{D}'(\Gamma)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We say that Xf vanishes on all lines which intersect an open set V, if $Xf|_{\Gamma_V} = 0$ as a distribution where Γ_V is the set of all parametrized lines intersecting V.

It is often convenient to study the X-ray transform X by way of its normal operator $N = X^*X$. This is not suited for all partial data scenarios and our proof in section 5.3 works directly at the level of X, but we make use of the normal operator elsewhere. Due to the mapping properties established above, the normal operator maps $N\colon \mathcal{E}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$. It is a pseudodifferential operator of order -1, but our problem is not well suited for a microlocal approach as discussed in section 2.4. For a test function $f \in \mathcal{D}(\mathbb{R}^d)$ the normal operator can be expressed conveniently as [48]

$$Nf(x) = 2\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-1}}\mathrm{d}y = 2(f * |\cdot|^{1-d})(x). \tag{1}$$

The convolution formula holds for a distribution $f \in \mathcal{E}'(\mathbb{R}^d)$ by a duality argument, and it holds also for continuous functions which decrease rapidly enough at infinity [16].

The normal operator of the X-ray transform can be inverted by the formula [48]

$$f = c_d(-\Delta)^{1/2}Nf, \quad c_d = (2\pi\left|S^{d-2}\right|)^{-1} \tag{2}$$

for any $f \in \mathcal{E}'(\mathbb{R}^d)$. Here the fractional Laplacian $(-\Delta)^s$ is defined via the inverse Fourier transform $(-\Delta)^s f = \mathcal{F}^{-1}(|\cdot|^{2s}\hat{f})$ and it is a non-local operator. As can be seen in equation (2), the normal operator of the X-ray transform is essentially $(-\Delta)^{-1/2}$ and is inverted by $(-\Delta)^{1/2}$.

Let $h_\alpha(x) = |x|^{-\alpha}$ where $\alpha = d-1$ or $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $\alpha < d$. We define the Riesz potential $I_\alpha \colon \mathcal{E}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ as

$$\langle I_\alpha f, \varphi \rangle = \langle f * h_\alpha, \varphi \rangle \tag{3}$$

for all $f \in \mathcal{E}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. When $\alpha < d$ then h_α is locally integrable and thus defines a tempered distribution. The convolution between two distributions is well-defined when at least one of them has compact support. This implies that $I_\alpha f$ is always defined as a distribution when $f \in \mathcal{E}'(\mathbb{R}^d)$. Especially if $f \in \mathcal{D}(\mathbb{R}^d)$, then

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\alpha} \mathrm{d}y.$$

We call h_α the kernel of the Riesz potential I_α. If $\alpha = d-1$, then equation (3) defines the normal operator of the X-ray transform N up to a constant factor 2, see equation (1). Extensive treatment of Riesz potentials can be found in many books, see e.g. [16, 27, 32, 44].

4. Proof of lemma 2.2

In this section we give a rather technical proof of lemma 2.2. The proof is based on induction and algebraic relations between certain functions and their derivatives.

Proof of lemma 2.2. We need to show that if $d \geq 2$ and $\alpha > d-2$ or $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, then for any polynomial p one can express $p(x\,|x|^{-2})\,|x|^{-\alpha}$ as a finite linear combination of derivatives of $h_\alpha(x) = |x|^{-\alpha}$. Let us denote

$$A_i = x_i B, \quad B = |x|^{-2} \quad \text{and} \quad C = |x|^{-\alpha}.$$

Then one can calculate the relations

$$\partial_j A_i = \delta_{ij} B - 2A_i A_j, \quad |A|^2 = B \quad \text{and} \quad \partial_i C = -\alpha A_i C.$$

Let us also define

$$D_{i_1 \dots i_n} = A_{i_1} \cdot \ldots \cdot A_{i_n} \cdot C = \left(\prod_{l=1}^{n} A_{i_l} \right) C, \quad i_k \in \{1, \dots, d\}.$$

We would like to express $D_{i_1 \dots i_n}$ for all $n \in \mathbb{N}$ as a finite linear combination of derivatives of h_α. The constant polynomials are given by h_α itself. The first derivative is

$$\partial_i h_\alpha(x) = -\alpha x_i \,|x|^{-\alpha-2} = -\alpha D_i.$$

Whence D_i can be obtained from first-order derivatives of h_α. Differentiating D_i gives

$$\partial_j D_i = \delta_{ij} BC - (2+\alpha) D_{ij}$$

and the divergence is

$$\sum_{i=1}^{d} \partial_i D_i = (d-2-\alpha) BC.$$

Combining these we obtain

$$D_{ij} = \frac{1}{2+\alpha}\left(\left(\frac{\delta_{ij}}{d-2-\alpha} \sum_{i=1}^{d} \partial_i D_i \right) - \partial_j D_i \right).$$

We have thus expressed the terms D_{ij} as a finite linear combination of the derivatives of the terms D_i which were multiples of the first-order derivatives of h_α. Hence D_{ij} can be expressed as a finite linear combination of second-order derivatives of h_α.

We claim that $D_{i_1\dots i_n}$ is a finite linear combination of nth order derivatives of h_α for all $n \in \mathbb{N}$ and we have shown this for $n = 0, 1, 2$. The lemma follows from this claim. Let us assume that the claim holds for some $m-1 \in \mathbb{N}$. Then $D_{i_1\dots i_{m-1}}$ is a finite linear combination of $(m-1)$th order derivatives of h_α. Thus $\partial_{i_m} D_{i_1\dots i_{m-1}}$ is a finite linear combination of mth order derivatives of h_α and a calculation shows that

$$\partial_{i_m} D_{i_1\dots i_{m-1}} = (2-2m-\alpha) D_{i_1\dots i_m} + \sum_{j=1}^{m-1} \left(\delta_{i_m i_j} BC \prod_{\substack{l=1 \\ l\neq j}}^{m-1} A_{i_l} \right). \tag{4}$$

Let us then calculate the divergence from equation (4). We get

$$\sum_{i_k=1}^{d} \partial_{i_k} D_{i_1\dots i_k \dots i_{m-1}} = (d-m-\alpha) BC \prod_{\substack{l=1 \\ l\neq k}}^{m-1} A_{i_l}. \tag{5}$$

From equations (4) and (5) we obtain the following expression for $D_{i_1\dots i_m}$

$$\frac{1}{2-2m-\alpha} \left(\partial_{i_m} D_{i_1\dots i_{m-1}} - \frac{1}{d-m-\alpha} \sum_{j=1}^{m-1} \left(\delta_{i_m i_j} \sum_{i_j=1}^{d} \partial_{i_j} D_{i_1\dots i_j \dots i_{m-1}} \right) \right)$$

which is by the induction assumption a finite linear combination of mth order derivatives of h_α. Thus the claim follows for all $n \in \mathbb{N}$. □

5. Alternative proofs of the main theorems

In this section we give alternative proofs to our main theorems, theorem 1.1 and theorem 1.2. We believe that presenting several proofs opens more possibilities to generalize the results and gives more tools for solving similar unique continuation problems and partial data problems.

We prove theorem 1.1 under the stronger assumption $I_\alpha f|_V = 0$ for a slightly larger class of distributions, i.e. rapidly decreasing distributions. We do it in two alternative ways. First proof is based on convolution approximation and density of polynomials. The second approach uses the unique continuation property of the fractional Laplacian. The second proof is short since it relies on a strong result. The unique continuation of $(-\Delta)^s$, $s \in (0,1)$, is based on technical results about Carleman estimates and Caffarelli-Silvestre extensions [15].

We then prove theorem 1.2 first for compactly supported smooth functions using angular Fourier series and density argument based on differentiation of an integral kernel. By a standard mollification argument we obtain the same result for compactly supported distributions. The proof works directly at the level of the X-ray transform and does not use the normal operator at all. Therefore we do not need to use any unique continuation results in the proof of the partial data problem.

We briefly go through our notations. We denote by $\mathscr{O}_M(\mathbb{R}^d)$ the space of polynomially increasing smooth functions, by $\mathscr{S}'(\mathbb{R}^d)$ the space of tempered distributions, by $\mathscr{O}'_C(\mathbb{R}^d)$ the space of rapidly decreasing distributions and by $H^r(\mathbb{R}^d)$ the fractional L^2-Sobolev space of order $r \in \mathbb{R}$. For precise definitions see [18, 30, 43, 51]. For us it is enough to know that $\mathcal{E}'(\mathbb{R}^d) \subset \mathscr{O}'_C(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ and

$$\mathscr{O}'_C(\mathbb{R}^d) \subset \bigcup_{r\in\mathbb{R}} H^r(\mathbb{R}^d).$$

Rapidly decreasing continuous functions, i.e. continuous functions which decrease faster than any polynomial at infinity, are contained in $\mathscr{O}'_C(\mathbb{R}^d)$. The convolution operator $*$ is a separately continuous map $*\colon \mathscr{O}'_C(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$. This implies that the Riesz potential $I_\alpha f = f * |\cdot|^{-\alpha}$ is defined as a distribution when $f \in \mathscr{O}'_C(\mathbb{R}^d)$ and $\alpha < d$. The Fourier transform is a bijective map from $\mathscr{O}'_C(\mathbb{R}^d)$ onto $\mathscr{O}_M(\mathbb{R}^d)$ and the usual convolution formula $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ holds in the sense of distributions when $f \in \mathscr{O}'_C(\mathbb{R}^d)$ and $g \in \mathscr{S}'(\mathbb{R}^d)$.

5.1. **Using convolution approximation.** In this section we prove theorem 1.1 under the assumption $I_\alpha f|_V = 0$ first for Schwartz functions. The result follows also for rapidly decreasing distributions by considering the mollifications $f * j_\epsilon$.

Theorem 5.1. *Let $\alpha = d-1$ or $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ and $\alpha < d$. Let $f \in \mathscr{O}'_C(\mathbb{R}^d)$ and $V \subset \mathbb{R}^d$ any nonempty open set. If $f|_V = I_\alpha f|_V = 0$, then $f = 0$.*

Proof. We can assume that $0 \in V$. Let first $f \in \mathscr{S}(\mathbb{R}^d)$. Like in the proof of theorem 1.1 we smoothen the kernel h_α near the origin, let this smoothened version be $g \in C^\infty(\mathbb{R}^d)$. There is $\epsilon > 0$ such that $(f * g)|_{B(0,\epsilon)} = (f * h_\alpha)|_{B(0,\epsilon)}$. It holds that $\partial^\beta(f * g) = f * \partial^\beta g$ where by lemma 2.2 one obtains all the polynomials p in the form $p(K(x))h_\alpha(x)$ by taking finite linear combinations of $\partial^\beta g$. Since f is not supported in a ball B centered at the origin, we can use the Kelvin transform to obtain

$$0 = \int_{B^c} f(y)p(y\,|y|^{-2})\,|y|^{-\alpha}\,\mathrm{d}y = \int_{\widetilde{B}\setminus\{0\}} f(x\,|x|^{-2})p(x)\,|x|^{\alpha}\,|J_K(x)|\,\mathrm{d}x$$

where $\widetilde{B}$ is some closed ball centered at the origin. One can calculate that $|J_K(x)| = |x|^{-2d}$ (see [15, Remark 4.2]). Since f goes rapidly to zero at infinity, we can extend the function $x \mapsto f(x\,|x|^{-2})\,|x|^{\alpha}\,|J_K(x)|$ continuously to zero and we call this extension $\widetilde{f}$. We obtain

$$\int_{\widetilde{B}} \widetilde{f}(x)p(x)\mathrm{d}x = 0$$

for all polynomials p. Since $\widetilde{f}$ is continuous and $\widetilde{B}$ is compact, by the Stone-Weierstrass theorem $\widetilde{f} = 0$. This implies $f = 0$.

Then let $f \in \mathscr{O}'_C(\mathbb{R}^d)$. Denote by $j_\epsilon \in \mathcal{D}(\mathbb{R}^d)$ the standard mollifier and consider the mollifications $f_\epsilon = f * j_\epsilon \in \mathscr{S}(\mathbb{R}^d)$. Since $I_\alpha(f * j_\epsilon) = I_\alpha f * j_\epsilon$ it follows that $f_\epsilon|_W = I_\alpha f_\epsilon|_W = 0$ for small enough $\epsilon > 0$ and $W \subset V$ open. By the first part of the proof $f_\epsilon = 0$ for small $\epsilon > 0$. This implies $f = 0$ since $f_\epsilon \to f$ as distributions in $\mathscr{S}'(\mathbb{R}^d)$ when $\epsilon \to 0$. ☐

We remark that theorem 5.1 implies uniqueness for the partial data problem (theorem 1.2) when f is a continous function which decreases faster than any polynomial. We can thus relax the assumption of compact support to rapid decay at infinity in theorem 1.2.

5.2. **Using unique continuation of the fractional Laplacian.** Here we give an alternative proof for a modified version of theorem 1.1 using Fourier analysis and unique continuation of $(-\Delta)^s$ in $H^r(\mathbb{R}^d)$, $r \in \mathbb{R}$, when $0 < s < 1$. The unique continuation of $(-\Delta)^s$ is proved in [15].

Theorem 5.2. *Let $f \in \mathscr{O}'_C(\mathbb{R}^d)$, $V \subset \mathbb{R}^d$ any nonempty open set and $0 < \alpha < d$ such that $(\alpha - d)/2 \notin \mathbb{Z}$. If $f|_V = 0$ and $I_\alpha f|_V = 0$, then $f = 0$.*

Proof. There is $k \in \mathbb{N}$ such that $-k < (\alpha - d)/2 < -k + 1$. Using the convolution property of the Fourier transform we can write

$$I_\alpha f = f * |\cdot|^{-\alpha} = c_d \mathcal{F}^{-1}(\mathcal{F}(f * |\cdot|^{-\alpha})) = c_d \mathcal{F}^{-1}(\hat{f}\,|\cdot|^{\alpha-d}) = c_d(-\Delta)^{\frac{\alpha-d}{2}} f,$$

where $c_d > 0$ is a constant depending on dimension. Since $(-\Delta)^{\frac{\alpha-d}{2}} f$ is a tempered distribution, again by the properties of the Fourier transform it follows that $(-\Delta)^k(-\Delta)^{\frac{\alpha-d}{2}} f = (-\Delta)^{k+\frac{\alpha-d}{2}} f = (-\Delta)^s f$ where $s = k + (\alpha - d)/2 \in (0, 1)$. Since $(-\Delta)^k$ is a local operator and $(-\Delta)^{\frac{\alpha-d}{2}} f$ vanishes in the open set V, we obtain the conditions $f|_V = 0$ and $(-\Delta)^s f|_V = 0$. Now $f \in \mathscr{O}'_C(\mathbb{R}^d)$ which implies $f \in H^r(\mathbb{R}^d)$ for some $r \in \mathbb{R}$. By [15, Theorem 1.2] we obtain $f = 0$. □

We remark that theorem 5.2 implies the unique continuation of the normal operator of the X-ray transform in dimensions $d \geq 2$ since in that case $0 < d-1 = \alpha < d$ and $(\alpha - d)/2 = -1/2 \notin \mathbb{Z}$.

5.3. **Angular Fourier series approach.** In this section we give another proof of theorem 1.2. We assume without loss of generality that f is supported in $\overline{B}(0, R') \setminus B(0, R)$ for some $R' > R > 0$ and that $0 \in V$. The proof is based on a similar idea as before, differentiation of an integral kernel and density of polynomials. However, now we study the X-ray transform directly and exploit the underlying spherical symmetry by using angular Fourier series expansion.

In the next theorem, when $f \in C_c(\mathbb{R}^d)$ it would be enough to assume that the X-ray transform Xf vanishes to infinite order on all lines through the origin, i.e. $\partial_r^n (Xf)(r, \theta)|_{r=0} = 0$ for all $n \in \mathbb{N}$. This is a similar assumption that we used in theorem 1.1.

Theorem 5.3. *Fix any $0 < \epsilon < R < R'$. Let $f \in \mathcal{E}'(\mathbb{R}^d)$ such that $\operatorname{spt}(f) \subset \overline{B}(0, R') \setminus B(0, R)$. If f integrates to zero over all lines in $B(0, R')$ that meet $B(0, \epsilon)$, then $f = 0$.*

Proof. Without loss of generality we can assume that $R' = 1$. Let first $f \in C_c(\mathbb{R}^d)$. By intersecting the origin with 2-planes it is enough to prove the result in two dimensions. The function f can be expressed as an angular Fourier series

$$f(r, \theta) = \sum_{k \in \mathbb{Z}} e^{ik\theta} a_k(r).$$

Our goal is to show that $a_k = 0$ for all $k \in \mathbb{Z}$. When we parameterize the lines in $\mathbb{R}^2$ by their closest point to the origin and use polar coordinates for these points, we find

$$Xf(r, \theta) = \sum_{k \in \mathbb{Z}} e^{ik\theta} \mathcal{A}_{|k|} a_k(r),$$

where $\mathcal{A}_k$ is the generalized Abel transform defined by

$$\mathcal{A}_k g(z) = 2 \int_z^1 K_k(z, y) g(y) \mathrm{d}y. \tag{6}$$

Here the kernel is $K_k(z, y) = T_k(z/y)[1 - (z/y)^2]^{-1/2}$ and T_k are the Chebyshev polynomials.

We know that $f(r, \theta) = 0$ when $r < R$ and $Xf(r, \theta) = 0$ when $r < \varepsilon$. For the Fourier components $a_k(r)$ this means that for every $k \in \mathbb{Z}$ we have $a_k(r) = 0$ for $r < R$ and $\mathcal{A}_k a_k(r) = 0$ for $r < \varepsilon$. Hence we get

$$\int_R^1 K_k(z, y) a_k(y) \mathrm{d}y = 0 \tag{7}$$

for every $z \in [0, \varepsilon)$. Like in the proof of theorem 1.1, we differentiate the integral kernel n times in (7) with respect to z and evaluate at $z = 0$ to obtain

$$\int_R^1 D_k^n(y) a_k(y) \mathrm{d}y = 0 \tag{8}$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, where $D_k^n(y) = \partial_z^n K_k(z, y)|_{z=0}$.

By scaling arguments $D_k^n(y) = A_k^n y^{-n}$ for some numbers A_k^n. The term $k = 0$ is

$$A_0^n = \begin{cases} (n-1)!!^2, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{9}$$

We denote the coefficient of x^l in $T_k(x)$ by t_k^l. The lth derivative of $T_k(x)$ at $x = 0$ is $l! t_k^l$. The coefficients also satisfy

$$\sum_{l=0}^{k} t_k^l = T_k(1) = 1. \tag{10}$$

By basic properties of Chebyshev polynomials $t_k^l = 0$ if $l - k$ is odd or $l > k$. Using $K_k(z, y) = T_k(z/y) K_0(z, y)$ and the product rule of higher order derivatives we find

$$A_k^n = \sum_{l=0}^{n} \binom{n}{l} l! t_k^l A_0^{n-l}.$$

By parity properties it is clear that A_k^n vanishes unless both n and l are even or both are odd.

We will show that for any $k \in \mathbb{N}$ there is a number $N(k)$ so that $A_k^n > 0$ when $n \geq N(k)$ and parity is right. For $k = 0$ this follows from equation (9) with $N(k) = 0$. Consider first the case when n and k are both even and assume $n > k$. A calculation shows that

$$A_k^n = n! \frac{(n-1)!!}{n!!} \sum_{m=0}^{k/2} \left[t_k^{2m} + t_k^{2m} \left(\frac{(n-2m-1)!! n!!}{(n-2m)!!(n-1)!!} - 1 \right) \right].$$

There are only finitely many terms in the sum, and for every m we have

$$\lim_{n\to\infty} \frac{(n-2m-1)!! n!!}{(n-2m)!!(n-1)!!} = 1.$$

Equation (10) implies $\sum_{m=0}^{k/2} t_k^{2m} = 1$ so that

$$\lim_{n\to\infty} \sum_{m=0}^{k/2} \left[t_k^{2m} + t_k^{2m} \left(\frac{(n-2m-1)!! n!!}{(n-2m)!!(n-1)!!} - 1 \right) \right] = 1.$$

Therefore $A_k^n > 0$ for sufficiently large n as claimed. Similarly one can show for odd indices that

$$A_k^n = n! \frac{(n-2)!!}{(n-1)!!} \sum_{m=0}^{(k-1)/2} \left[t_k^{2m+1} + t_k^{2m+1} \left(\frac{(n-2m-2)!!(n-1)!!}{(n-2m-1)!!(n-2)!!} - 1 \right) \right].$$

With the same limit argument we get $A_k^n > 0$ for large n.

We fix any $k \in \mathbb{Z}$ and use (8) to show that $a_k = 0$. By symmetry it suffices to consider $k \geq 0$. We found $N(k)$ so that $A_k^n \neq 0$ for $n \geq N(k)$ when $n - N(k)$ is even. We find

$$\int_R^1 y^{-N(k)-2m} a_k(y) \mathrm{d}y = 0$$

for every $m \in \mathbb{N}$. By linearity

$$\int_R^1 y^{-N(k)} p(y^{-2}) a_k(y) \mathrm{d}y = 0$$

for any polynomial p. Changing variable to $s = y^{-2}$ and defining new coefficients $\tilde{a}_k(s) = s^{N(k)/2-3/2} a_k(s^{-1/2})$, we obtain

$$\int_1^{R^{-1/2}} p(s) \tilde{a}_k(s) \mathrm{d}s = 0.$$

By density of polynomials $\tilde{a}_k(s) = 0$ for all $s \in [1, R^{-1/2}]$. This implies $a_k = 0$ for all $k \in \mathbb{Z}$ and hence $f = 0$.

Then let $f \in \mathcal{E}'(\mathbb{R}^d)$ and consider the mollifications $f * j_\epsilon \in \mathcal{D}(\mathbb{R}^d)$. Following Helgason [16] we define the "convolution"

$$(g \times \varphi)(z, \theta) = \int_{\mathbb{R}^d} g(y) \varphi(z - y, \theta) \mathrm{d}y$$

where $g \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\Gamma)$. By a simple calculation one can show that $X^*(g \times \varphi) = g * X^* \varphi$. Using the properties of the convolutions $*$ and $\times$ we obtain

$$\langle X(f * j_\epsilon), \varphi \rangle = \langle f * j_\epsilon, X^* \varphi \rangle = \langle f, j_\epsilon * X^* \varphi \rangle = \langle f, X^*(j_\epsilon \times \varphi) \rangle = \langle X f, j_\epsilon \times \varphi \rangle .$$

Thus for small enough $\epsilon > 0$ and $\tilde{R} > 0$ we get that $(f * j_\epsilon)|_{B(0,\tilde{R})} = 0$ and $X(f * j_\epsilon)$ vanishes on all lines which intersect $B(0, \epsilon)$. The first part of the proof implies $f * j_\epsilon = 0$ for small $\epsilon > 0$. The claim follows since $f * j_\epsilon \to f$ in $\mathcal{E}'(\mathbb{R}^d)$ when $\epsilon \to 0$. □

We remark that the assumption that f is supported away from the origin is crucial since it turns a Volterra integral equation into a Fredholm integral equation. This simplifies the derivatives of expression (6).

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Unique continuation property and Poincaré inequality for higher order fractional Laplacians with applications in inverse problems

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UNIQUE CONTINUATION PROPERTY AND POINCARÉ INEQUALITY FOR HIGHER ORDER FRACTIONAL LAPLACIANS WITH APPLICATIONS IN INVERSE PROBLEMS

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ABSTRACT. We prove a unique continuation property for the fractional Laplacian $(-\Delta)^s$ when $s \in (-n/2, \infty) \setminus \mathbb{Z}$ where $n \geq 1$. In addition, we study Poincaré-type inequalities for the operator $(-\Delta)^s$ when $s \geq 0$. We apply the results to show that one can uniquely recover, up to a gauge, electric and magnetic potentials from the Dirichlet-to-Neumann map associated to the higher order fractional magnetic Schrödinger equation. We also study the higher order fractional Schrödinger equation with singular electric potential. In both cases, we obtain a Runge approximation property for the equation. Furthermore, we prove a uniqueness result for a partial data problem of the d-plane Radon transform in low regularity. Our work extends some recent results in inverse problems for more general operators.

1. INTRODUCTION

The fractional Laplacian $(-\Delta)^s$, $s \in (-n/2, \infty) \setminus \mathbb{Z}$, is a non-local operator by definition and thus differs substantially from the ordinary Laplacian $(-\Delta)$. The non-local behaviour can be exploited when solving fractional inverse problems. In section 3.1, we prove that $(-\Delta)^s$ admits a unique continuation property (UCP) for open sets, that is, if u and $(-\Delta)^s u$ both vanish in a nonempty open set, then u vanishes everywhere. Clearly this property cannot hold for local operators. We give many other versions of UCPs as well.

We have also included a quite comprehensive discussion of the Poincaré inequality for the higher order fractional Laplacian $(-\Delta)^s$, $s \geq 0$, in section 3.2. We give many proofs for the higher order fractional Poincaré inequality based on various different methods in the literature. The higher order fractional Poincaré inequality appears earlier at least in [84] for functions in $C_c^\infty(\Omega)$ where Ω is a bounded Lipschitz domain. Also similar inequalities are proved in the book [4] for homogeneous Sobolev norms but without referring to the fractional Laplacian. However, we have extended some known results, given alternative proofs, and studied a connection between the fractional and the classical Poincaré constants. We believe that section 3.2 will serve as a helpful reference on fractional Poincaré inequalities in the future.

Our main applications are fractional Schrödinger equations with and without a magnetic potential, and the d-plane Radon transforms with partial data. We apply the UCP result and the Poincaré inequality for higher order fractional Laplacians to show uniqueness for the associated fractional Schrödinger equation and the Runge approximation properties. UCPs have also applications in integral geometry since certain partial data inverse problems for the Radon transforms can be reduced to unique continuation problems of the normal operators. We remark that the normal operators of the Radon transforms are negative order fractional Laplacians (Riesz potentials) up to constant coefficients.

In this section, we introduce our models, discuss some related results and present our main theorems and corollaries. We start with the classical Calderón problem as a motivation.

1.1. The Calderón problem.

We will study a non-local version of the famous Calderón problem called the fractional Calderón problem. A survey of the fractional Calderón problem is given in [79]. The Calderón problem is a classical inverse problem where one wants to determine the

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electrical conductivity on some sufficiently smooth domain by boundary measurements [77, 83]. Suppose that $\Omega \subset \mathbb{R}^n$ is a domain with regular enough boundary $\partial\Omega$. The electrical conductivity is usually represented as a bounded positive function γ, and the conductivity equation is

$$\begin{cases} \nabla \cdot (\gamma \nabla u) &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{cases} \tag{1}$$

where f is the potential on the boundary $\partial\Omega$ and u is the induced potential in Ω. The data in this problem is the Dirichlet-to-Neumann (DN) map $\Lambda_\gamma(f) = (\gamma \partial_\nu u)|_{\partial\Omega}$, where ν is the outer unit normal on the boundary. The DN map basically tells how the applied voltage on the boundary induces normal currents on the boundary by the electrical properties of the interior. The inverse problem is to determine γ from the DN map Λ_γ. One of the associated basic questions is the uniqueness problem, that is, whether $\gamma_1 = \gamma_2$ follows from $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$.

Equation (1) can be reduced to a Schrödinger equation

$$\begin{cases} (-\Delta + q)u &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{cases} \tag{2}$$

where $q = (\Delta\sqrt{\gamma})/\sqrt{\gamma}$ now represents the electric potential in Ω. One typically assumes that 0 is not a Dirichlet eigenvalue of the operator $(-\Delta + q)$ to obtain unique solutions to equation (2). The inverse problem then is to know whether one can determine the electric potential q uniquely from the DN map Λ_q, which can be expressed in terms of the normal derivative $\Lambda_q f = \partial_\nu u|_{\partial\Omega}$ for regular enough boundaries. For more details on the classical Calderón problem and its applications to medical, seismic and industrial imaging, see [77, 83].

1.2. **Fractional Schrödinger equation.** In this article, we focus on the fractional Schrödinger equation and its generalization, the fractional magnetic Schrödinger equation. The main difference between the classical and fractional Schrödinger operators is that the first one is local and the second one is non-local. This can be seen since the Laplacian $(-\Delta)$ is local as a differential operator while the fractional counterpart $(-\Delta)^s$, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, is a non-local Fourier integral operator. In other words, the value $(-\Delta)^s u(x)$, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, depends on the values of u everywhere, not just in a small neighbourhood of $x \in \mathbb{R}^n$. Fractional Laplacians have a close connection to Levý processes and have been used in many areas of mathematics and physics, for example to model anomalous and nonlocal diffusion, and also in the formulation of fractional quantum mechanics where the fractional Schrödinger equation arises naturally as a generalization of the ordinary Schrödinger equation [3, 7, 18, 28, 50, 51, 58, 71].

Since the fractional Laplacian is a non-local operator, it is more natural to fix exterior values for the solutions of the equation instead of just boundary values. This motivates the study of the following exterior value problem, first introduced in [28],

$$\begin{cases} ((-\Delta)^s + q)u &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases} \tag{3}$$

where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the exterior of Ω. The associated DN map for equation (3) is a bounded linear operator $\Lambda_q \colon H^s(\Omega_e) \to (H^s(\Omega_e))^*$ which, under stronger assumptions, has an expression $\Lambda_q f = (-\Delta)^s u|_{\Omega_e}$ [28]. We assume that the potential q is such that the following holds:

$$\text{If } u \in H^s(\mathbb{R}^n) \text{ solves } ((-\Delta)^s + q)u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \text{ then } u = 0. \tag{4}$$

In other words, condition (4) requires that 0 is not a Dirichlet eigenvalue of the operator $((-\Delta)^s + q)$.

In section 5, we will prove that, under certain assumptions, one can uniquely determine the potential q in equation (3) from exterior measurements when $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and we also prove a Runge approximation property for equation (3) (see also section 1.5). These generalize the results in [28, 75] to higher fractional powers of s. The proofs basically reduce to the fact that the operator $(-\Delta)^s$ has the following UCP: if $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$ everywhere. This reflects the fact that $(-\Delta)^s$ is a non-local operator since such UCP can never hold for local operators.

Unique continuation of the fractional Laplacian has been extensively studied and used to show uniqueness results for fractional Schrödinger equations [14, 27, 28, 75]. One version was already proved by Riesz [28, 70] and similar methods were used in [41] to show a UCP of Riesz potentials I_α which can be seen as fractional Laplacians with negative exponents. See also [45] for a unique continuation result of Riesz potentials. UCP of $(-\Delta)^s$ for functions in $H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, was proved in [28] when $s \in (0,1)$. The proof is based on Carleman estimates from [72] and on Caffarelli-Silvestre extension [8, 9]. Using the known result for $s \in (0,1)$, we provide an elementary proof which generalizes the UCP for all $s \in (-n/2, \infty) \setminus \mathbb{Z}$. With the same trick we obtain several other unique continuation results. There are also strong unique continuation results for $s \in (0,1)$ if one assumes more regularity from the function [22, 72]. In the strong UCP, one replaces the condition $u|_V = 0$ by the requirement that u vanishes to infinite order at some point $x_0 \in V$. The higher order case $s \in \mathbb{R}^+ \setminus (\mathbb{Z} \cup (0,1))$ has been studied recently by several authors [23, 26, 86]. These results however assume some special conditions on the function u, i.e. they require that u is in a Sobolev space which depends on the power s of the fractional Laplacian $(-\Delta)^s$. We only require that u is in some Sobolev space $H^r(\mathbb{R}^n)$ where $r \in \mathbb{R}$ can be an arbitrarily small (negative) number.

See also [45] where the author proves a higher order Runge approximation property by s-harmonic functions in the unit ball when $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ (compare to theorem 1.7). Here s-harmonicity simply means that $(-\Delta)^s u = 0$ in some domain Ω. The s-harmonic approximation in the case $s \in (0,1)$ was already studied in [17]; similar higher regularity approximation results are proved in [11, 28] for the fractional Schrödinger equation.

1.3. **Fractional magnetic Schrödinger equation.** Section 6 of this paper extends the study of the fractional magnetic Schrödinger equation (FMSE) begun in [14], expanding the uniqueness result for the related inverse problem to the cases when $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. The direct problem for the classical magnetic Schrödinger equation (MSE) consists in finding a function u satisfying

$$\begin{cases} (-\Delta)_A u + qu &= -\Delta u - i\nabla \cdot (Au) - iA \cdot \nabla u + (|A|^2 + q)u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is some bounded open set with Lipschitz boundary representing a medium, f is the boundary value for the solution u, and A, q are the vector and scalar potentials of the equation. In the associated inverse problem, we are given measurements on the boundary in the form of a DN map $\Lambda_{A,q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, and we are asked to recover A, q in Ω using this information. It was shown in [60] that this is only possible up to a natural gauge: one can uniquely determine the potential q and the magnetic *field* curl$A$, but the magnetic *potential* A can not be determined in greater detail. The inverse problem for MSE is of great interest, because it generalizes the non-magnetic case by adding some first order terms, and shows a quite different behavior. It also possesses multiple applications in the sciences: the papers [60, 62, 56, 20, 61] and [35] give some examples of this, treating the inverse scattering problem with a fixed energy, isotropic elasticity, the Maxwell, Schrödinger and Dirac equations and the Stokes system. We refer to the survey [76] for many more references on inverse boundary value problems related to MSE.

We are interested in the study of a high order fractional version of the MSE. There have been many studies in this direction (see for instance [54, 52, 53]). In our work, we will build upon the results from [14] and generalize them to higher order. Thus, for us the direct problem for FMSE asks to find a function u which satisfies

$$\begin{cases} (-\Delta)^s_A u + qu &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases}$$

where Ω, f, A and q play a similar role as in the local case, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $(-\Delta)^s_A$ is the magnetic fractional Laplacian. This is a fractional version of $(-i\nabla + A) \cdot (-i\nabla + A)$, the magnetic Laplacian from which MSE arises. In section 6, we will construct the fractional magnetic Laplacian based on the fractional gradient operator ∇^s. The fractional gradient is based on the framework laid down in [18, 19], and has been studied in the papers [15, 14]. One should keep in mind that for $s > 1$ the fractional gradient is a tensor of order $\lfloor s \rfloor$ rather than a vector. In the corresponding inverse problem, we assume to know the DN map $\Lambda^s_{A,q} : H^s(\Omega_e) \to (H^s(\Omega_e))^*$, and we wish to recover A, q in Ω. In the cases when $s \in (0, 1)$, it has been shown that the pair A, q can only be recovered up to a natural gauge [14]. We generalize this result to the case $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. This is achieved by first proving a weak UCP and the Runge approximation property for FMSE, and then testing the Alessandrini identity for the equation with suitably chosen functions.

Remark 1.1. *The case of the high order magnetic Schrödinger equation, that is the one in which $s \in \mathbb{N}$, $s \neq 1$, is still open at the time of writing to the best of the authors' knowledge. Our methods are purely nonlocal, and thus cannot be applied to the integer case. It was however showed in* [60]*, as cited above, that a uniqueness result up to a natural gauge holds when $s = 1$.*

1.4. **Radon transforms and region of interest tomography.** Unique continuation results have also applications in integral geometry. It was proved in [41] that the normal operator of the X-ray transform admits a UCP in the class of compactly supported distributions. This was done by considering the normal operator as a Riesz potential. We generalize the result for the normal operator of the d-plane transform R_d where $d \in \mathbb{N}$ is odd such that $0 < d < n$. In the case $d = 1$ the transform R_d corresponds to the X-ray transform and in the case $d = n - 1$ to the Radon transform. The UCP of the normal operator $N_d = R_d^* R_d$ implies uniqueness for the following partial data problem: if f integrates to zero over all d-planes which intersect some nonempty open set V and $f|_V = 0$, then $f = 0$. This can be seen as a complementary result to the Helgason support theorem for the d-plane transform [36]. Helgason's theorem says that if f integrates to zero over all d-planes not intersecting a convex and compact set K and $f|_K = 0$, then $f = 0$. The d-plane transform R_d is injective on continuous functions which decay rapidly enough at infinity and also on compactly supported distributions [36]. The d-plane transform has been recently studied in the periodic case on the flat torus [2, 40, 67] but also in other settings [16, 37, 69]. Weighted and limited data Radon transforms ($d = n - 1$) have been studied recently for example in [25, 29, 30, 31].

When $d = 1$, partial data problems as discussed above arise for example in seismology and medical imaging. In [41], it is explained how one can use shear wave splitting data to uniquely determine the difference of the anisotropic perturbations in the S-wave speeds, and also how one can use local measurements of travel times of seismic waves to uniquely determine the conformal factor in the linearization. Both of these problems reduce to the following partial data result: if f integrates to zero over all lines which intersect some nonempty open set V and $f|_V = 0$, then $f = 0$. In medical imaging, one typically wants to reconstruct a specific part of the human body. Can this be done by using only X-rays which go though our region of interest (ROI)? Generally this is not possible even for C_c^∞-functions [43, 63, 81], but if we know some information of f in the ROI, then the reconstruction can be done. For example, if the function f is piecewice constant, piecewice polynomial or analytic in the ROI, then f can be uniquely determined from the X-ray data [42, 43, 85]. Also, if we know the X-ray data through the ROI and the values of f in an arbitrarily small open set inside the ROI, then f is uniquely determined everywhere [13, 41]. For practical applications of ROI tomography in medical imaging, see for example [87, 88]. See also [44, 65, 66] for a discussion of the difficulties of obtaining stable reconstruction in partial data problems for the X-ray transform (visible and invisible singularities).

1.5. **Main results.** We briefly introduce the basic notation; more details can be found in sections 2, 4, 5 and 6. Let $H^r(\mathbb{R}^n)$ be the L^2 Sobolev space of order $r \in \mathbb{R}$ and $\widetilde{H}^r(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^r(\mathbb{R}^n)$ when Ω is an open set. The L^1 Bessel potential space is denoted by $H^{r,1}(\mathbb{R}^n)$. We define $H^r_K(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ to be those Sobolev functions which have support in the compact set K. The fractional Laplacian is defined via the Fourier transform $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{u})$. Then $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is a continuous operator when

$s \in \mathbb{R}^+ \setminus \mathbb{Z}$. The d-plane transform R_d takes a function which decreases rapidly enough at infinity and integrates it over d-dimensional planes where $0 < d < n$. The normal operator of the d-plane transform is defined as $N_d = R_d^* R_d$ where R_d^* is the adjoint operator. Further, we denote by $\mathcal{D}'(\mathbb{R}^n)$ the space of all distributions, $\mathcal{E}'(\mathbb{R}^n)$ the space of compactly supported distributions, $\mathscr{O}_C'(\mathbb{R}^n)$ the space of rapidly decreasing distributions and $C_\infty(\mathbb{R}^n)$ the set of rapidly decreasing continuous functions. The space of singular potentials $Z_0^{-s}(\mathbb{R}^n)$ is a certain subset of distributions $\mathcal{D}'(\mathbb{R}^n)$ and can be interpreted as a set of bounded multipliers from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$.

The following theorem extends a result in [28] and has a central role in this article. We call it the UCP of the operator $(-\Delta)^s$.

Theorem 1.2. *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $u \in H^r(\mathbb{R}^n)$ where $r \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$. The claim holds also for $s \in (-n/2, -n/4] \setminus \mathbb{Z}$ if $u \in H^{r,1}(\mathbb{R}^n)$ or $u \in \mathscr{O}_C'(\mathbb{R}^n)$.*

Theorem 1.2 is proved in section 3.1. The UCP of $(-\Delta)^s$ implies corresponding UCP for Riesz potentials (see corollary 3.2 and [41, Theorem 5.2]). This in turn implies the following UCP for the normal operator of the d-plane transform N_d when d is odd; the case $d = 1$ was already studied in [41].

Corollary 1.3. *Let $n \geq 2$ and let f belong to either $\mathcal{E}'(\mathbb{R}^n)$ or $C_\infty(\mathbb{R}^n)$. Let $d \in \mathbb{N}$ be odd such that $0 < d < n$. If $N_d f|_V = 0$ and $f|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $f = 0$.*

From the UCP of N_d we obtain the next result which is in a sense complementary to the Helgason support theorem for the d-plane transform [36, Theorem 6.1]. It extends a result in [41] where the authors prove a similar uniqueness property for the X-ray transform.

Corollary 1.4. *Let $n \geq 2$, $V \subset \mathbb{R}^n$ a nonempty open set and $f \in C_\infty(\mathbb{R}^n)$. Let $d \in \mathbb{N}$ be odd such that $0 < d < n$. If $f|_V = 0$ and $R_d f = 0$ for all d-planes intersecting V, then $f = 0$. The claim holds also for $f \in \mathcal{E}'(\mathbb{R}^n)$ when the assumption $R_d f = 0$ for all d-planes intersecting V is understood in the sense of distributions.*

If d is even, then f is uniquely determined in V by its integrals over d-planes which intersect V, i.e. $R_d f = 0$ for all d-planes intersecting V implies $f|_V = 0$ (see remark 4.2). The authors do not know if the result of corollary 1.4 holds when d is even. However, if d is even, then the result of corollary 1.3 cannot be true as the normal operator N_d is the inverse of a local operator. See section 4 for the proofs and the definition of the d-plane transform of distributions.

The following result is a general version of the Poincaré inequality which we need for the well-posedness of the inverse problem for the fractional Schrödinger equation.

Theorem 1.5. *Let $n \geq 1$, $s \geq t \geq 0$, $K \subset \mathbb{R}^n$ a compact set and $u \in H_K^s(\mathbb{R}^n)$. There exists a constant $\tilde{c} = \tilde{c}(n, K, s) > 0$ such that*

$$\left\| (-\Delta)^{t/2} u \right\|_{L^2(\mathbb{R}^n)} \leq \tilde{c} \left\| (-\Delta)^{s/2} u \right\|_{L^2(\mathbb{R}^n)}.$$

The constant $\tilde{c}$ can be expressed in terms of the classical Poincaré constant when $s \geq 1$ (see theorem 3.17. See section 3.2 for several proofs of the Poincaré inequality. From the unique continuation of $(-\Delta)^s$ we obtain results for the higher order fractional Schrödinger equation with singular electric potential. The following theorems generalize the results in [28, 75] for higher exponents $s \in \mathbb{R}^+ \setminus (\mathbb{Z} \cup (0, 1))$.

Theorem 1.6. *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ a bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and $q_1, q_2 \in Z_0^{-s}(\mathbb{R}^n)$ which satisfy condition (4). Let $W_1, W_2 \subset \Omega_e$ be open sets. If the DN maps for the equations $(-\Delta)^s u + m_{q_i}(u) = 0$ in Ω satisfy $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$, then $q_1|_\Omega = q_2|_\Omega$.*

Theorem 1.7. *Let $n \geq 1$ and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\Omega_1 \supset \Omega$ any open set such that int$(\Omega_1 \setminus \Omega) \neq \varnothing$. If $q \in Z_0^{-s}(\mathbb{R}^n)$ satisfies condition (4), then any $g \in \widetilde{H}^s(\Omega)$ can be approximated arbitrarily well in $\widetilde{H}^s(\Omega)$ by solutions $u \in H^s(\mathbb{R}^n)$ to the equation $(-\Delta)^s u + m_q(u) = 0$ in Ω such that* $\mathrm{spt}(u) \subset \overline{\Omega}_1$.

We remark that the approximation property in theorem 1.7 also holds in $L^2(\Omega)$ when one takes restrictions of the solutions (see [28, Theorem 1.3]). In [17, 45] the authors prove similar approximation results: C^k-functions can be approximated (in the C^k-norm) in the unit ball by s-harmonic functions, i.e. functions u which satisfy $(-\Delta)^s u = 0$ in $B_1(0)$ (see also [28, Remark 7.3]). Theorems 1.6 and 1.7 are proved in section 5. The proofs are almost identical to those in [28, 75] and only slight changes need to be done. We will present the main ideas of the proofs for clarity and in order to make a comparison to the more complicated case of FMSE.

We have achieved the following result on the Calderón problem for FMSE:

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and let A_i, q_i verify assumptions (a1)-(a5) in section 6 for $i = 1, 2$. Let $W_1, W_2 \subset \Omega_e$ be open sets. If the DN maps for the FMSEs in Ω relative to (A_1, q_1) and (A_2, q_2) satisfy*

$$\Lambda^s_{A_1,q_1}[f]|_{W_2} = \Lambda^s_{A_2,q_2}[f]|_{W_2} \quad \text{for all } f \in C_c^\infty(W_1),$$

then $(A_1, q_1) \sim (A_2, q_2)$, that is, the potentials coincide up to gauge.

An in-depth clarification of the assumptions and the definition of the gauge involved in the proof are presented in section 6.

1.6. **Organization of the article.** This article is organized as follows. Section 2 is devoted to preliminaries. We introduce our notation and definitions of relevant quantities. In sections 3.1 and 3.2 we prove the unique continuation property of $(-\Delta)^s$ for $s \in (-n/2, \infty) \setminus \mathbb{Z}$ and give several proofs for the fractional Poincaré inequality. We introduce some applications in integral geometry and partial data problems of the d-plane transform in section 4. In section 5, we show the uniqueness and the Runge approximation results for the higher order fractional Schrödinger equation with singular electric potential. We prove the uniqueness result up to a gauge for the higher order fractional magnetic Schrödinger equation in section 6. Finally, in section 7, we discuss other problems that would now naturally continue our work. There are many potential recent results in inverse problems which perhaps can be generalized to higher order fractional Laplacians using our unique continuation result and fractional Poincaré inequality.

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2. Preliminaries

In this section, we will go through our basic notations and definitions. The following theory of distributions, Fourier analysis and Sobolev spaces can be found in many books (see for example [1, 4, 6, 38, 39, 57, 59, 78, 82]). We write $|\cdot|$ for both the Euclidean norm of vectors and the absolute value of complex numbers. We denote by $\mathbb{N}_0$ the set of natural numbers including zero.

2.1. **Distributions and Fourier transform.** We denote by $\mathcal{E}(\mathbb{R}^n)$ the set of smooth functions equipped with the topology of uniform convergence of derivatives of all order on compact sets. We also denote by $\mathcal{D}(\mathbb{R}^n)$ the set of compactly supported smooth functions with the topology of uniform convergence of derivatives of all order in a fixed compact set. The topological duals of these spaces are denoted by $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$. Elements in the space $\mathcal{E}'(\mathbb{R}^n)$ can be identified as distributions in $\mathcal{D}'(\mathbb{R}^n)$ with compact support.

We also use the space of rapidly decreasing smooth functions, i.e. Schwartz functions. Define the Schwartz space as

$$\mathscr{S}(\mathbb{R}^n) = \left\{ \varphi \in C^\infty(\mathbb{R}^n) : \left\| \langle \cdot \rangle^N \partial^\beta \varphi \right\|_{L^\infty(\mathbb{R}^n)} < \infty \text{ for all } N \in \mathbb{N} \text{ and } \beta \in \mathbb{N}_0^n \right\},$$

where $\langle x \rangle = (1+|x|^2)^{1/2}$, equipped with the topology induced by the seminorms $\left\| \langle \cdot \rangle^N \partial^\beta \varphi \right\|_{L^\infty(\mathbb{R}^n)}$. The continuous dual of $\mathscr{S}(\mathbb{R}^n)$ is denoted by $\mathscr{S}'(\mathbb{R}^n)$ and its elements are called tempered distributions. We have the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. The Fourier transform of $u \in L^1(\mathbb{R}^n)$ is defined as

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \mathrm{d}x$$

and it is an isomorphism $\mathcal{F} \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. By duality the Fourier transform is also an isomorphism $\mathcal{F} \colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. By density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ the Fourier transform can be extended to an isomorphism $\mathcal{F} \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The following subset of Schwartz space

$$\mathscr{S}_0(\mathbb{R}^n) = \{\varphi \in \mathscr{S}(\mathbb{R}^n) : \hat{\varphi}|_{B(0,\epsilon)} = 0 \text{ for some } \epsilon > 0\}$$

is used to define fractional Laplacians on homogeneous Sobolev spaces.

Finally, we denote by $\mathscr{O}_C'(\mathbb{R}^n)$ the space of rapidly decreasing distributions. One has that $T \in \mathscr{O}_C'(\mathbb{R}^n)$ if and only if for any $N \in \mathbb{N}$ there exist $M(N) \in \mathbb{N}$ and continuous functions g_β such that

$$T = \sum_{|\beta| \leq M(N)} \partial^\beta g_\beta \ ,$$

where $\langle \cdot \rangle^N g_\beta$ is a bounded function for every $|\beta| \leq M(N)$. Alternatively one can characterize $\mathscr{O}_C'(\mathbb{R}^n)$ via the Fourier transform: it holds that $\mathcal{F} \colon \mathscr{O}_C'(\mathbb{R}^n) \to \mathscr{O}_M(\mathbb{R}^n)$ is a bijective map where $\mathscr{O}_M(\mathbb{R}^n)$ is the space of smooth functions with polynomially bounded derivatives of all orders. We have the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. For example $C_\infty(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$, where $f \in C_\infty(\mathbb{R}^n)$ if and only if f is continuous and $\langle \cdot \rangle^N f$ is bounded for every $N \in \mathbb{N}$. The convolution formula for the Fourier transform $\widehat{f * g} = \hat{f}\hat{g}$ holds in the sense of distributions when $f \in \mathscr{O}_C'(\mathbb{R}^n)$ and $g \in \mathscr{S}'(\mathbb{R}^n)$. For more details on distributions, see the classic books [38, 39, 82].

2.2. **Fractional Laplacian on Sobolev spaces.** Let $r \in \mathbb{R}$. We define the inhomogeneous fractional L^2 Sobolev space of order r to be the set

$$H^r(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \in L^2(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{H^r(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \right\|_{L^2(\mathbb{R}^n)}.$$

The spaces $H^r(\mathbb{R}^n)$ are Hilbert spaces for all $r \in \mathbb{R}$. It follows that both $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}_0(\mathbb{R}^n)$ are dense in $H^r(\mathbb{R}^n)$ for all $r \in \mathbb{R}$. Note that

$$\mathscr{O}_C'(\mathbb{R}^n) \subset \bigcup_{r \in \mathbb{R}} H^r(\mathbb{R}^n).$$

If $s \in (0,1)$, the fractional Laplacian can be defined in several equivalent ways [46]. We will take the Fourier transform approach which allows us to define it as a continuous map on Sobolev spaces for all $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. Define the fractional Laplacian of order $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ as $(-\Delta)^s \varphi = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{\varphi})$ for $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then $(-\Delta)^s \colon \mathscr{S}(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is linear and continuous with respect to the norm $\|\cdot\|_{H^r(\mathbb{R}^n)}$ by a simple calculation. Thus we can uniquely extend it to a continuous linear operator $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ as $(-\Delta)^s u = \lim_{k\to\infty} (-\Delta)^s \varphi_k$, where $\varphi_k \in \mathscr{S}(\mathbb{R}^n)$ is such that $\varphi_k \to u$ in $H^r(\mathbb{R}^n)$.

On the other hand, if $s > -n/4$, one can always define $(-\Delta)^s u$ for $u \in H^r(\mathbb{R}^n)$ as the tempered distribution $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{u})$, note that we also allow integer values of s here. This can be seen in the following way: let $\varphi_k \in \mathscr{S}(\mathbb{R}^n)$ such that $\varphi_k \to 0$ in $\mathscr{S}(\mathbb{R}^n)$. It holds that

$|\cdot|^{-\beta} \in L^1_{loc}(\mathbb{R}^n)$ if and only if $\beta < n$. Taking $N \in \mathbb{N}$ large enough and using Cauchy-Schwartz we obtain

$$\int_{\mathbb{R}^n} |x|^{2s} |\hat{u}(x)| |\varphi_k(x)| \,\mathrm{d}x \leq \left(\int_{\mathbb{R}^n} \langle x \rangle^{2r} |\hat{u}(x)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^n} |x|^{4s} \langle x \rangle^{-2r} |\varphi_k(x)|^2 \,\mathrm{d}x \right)^{1/2}$$
$$\leq C \left(\int_{\mathbb{R}^n} \frac{|x|^{4s}}{\langle x \rangle^{2N}} \mathrm{d}x \right)^{1/2} \left\| \langle \cdot \rangle^{N-r} \varphi_k \right\|_{L^\infty(\mathbb{R}^n)} \to 0.$$

Hence $|\cdot|^{2s} \hat{u} \in \mathscr{S}'(\mathbb{R}^n)$ and also $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{u}) \in \mathscr{S}'(\mathbb{R}^n)$. The definition can be relaxed to $s > -n/2$ if we assume that $\langle \cdot \rangle^t \hat{u} \in L^\infty(\mathbb{R}^n)$ for some $t \in \mathbb{R}$. This holds for example if $u \in \mathscr{O}'_C(\mathbb{R}^n)$ or $u \in H^{r,1}(\mathbb{R}^n)$ (see the definition of Bessel potential spaces below). When $s \geq 0$, we again obtain that $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is continuous. It follows from the properties of the Fourier transform that $(-\Delta)^k(-\Delta)^s = (-\Delta)^{k+s}$ when $s > -n/2$ and $k \in \mathbb{N}$. This relation will be used many times.

Fractional Laplacians with negative powers s have a connection to Riesz potentials. Let $\alpha \in \mathbb{R}$ such that $0 < \alpha < n$. We define the Riesz potential $I_\alpha \colon \mathscr{O}'_C(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ as $I_\alpha f = f * h_\alpha$, where the kernel is $h_\alpha(x) = |x|^{-\alpha}$. It follows that I_α is continuous in the distributional sense and $I_\alpha = (-\Delta)^{-s}$, up to a constant factor, where $s = (n - \alpha)/2$. On the other hand, if $-n/2 < s < 0$, then one can write $(-\Delta)^s f = f * |\cdot|^{-2s-n} = I_{2s+n} f$, also up to a constant factor. Hence fractional Laplacians with negative powers correspond to Riesz potentials and vice versa.

Following [4], one can define fractional Laplacians and Riesz potentials on homogeneous Sobolev spaces. Let us define

$$\dot{H}^r(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \hat{u} \in L^1_{loc}(\mathbb{R}^n) \text{ and } |\cdot|^r \hat{u} \in L^2(\mathbb{R}^n)\}$$

and equip it with the norm

$$\|u\|_{\dot{H}^r(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \right)^{1/2}.$$

The norm $\|u\|_{\dot{H}^r(\mathbb{R}^n)}$ is homogeneous with respect to scaling $\xi \to \lambda\xi$ in contrast to the norm $\|u\|_{H^r(\mathbb{R}^n)}$. We have the inclusions $\dot{H}^r(\mathbb{R}^n) \subsetneq H^r(\mathbb{R}^n)$ for $r < 0$ and $H^r(\mathbb{R}^n) \subsetneq \dot{H}^r(\mathbb{R}^n)$ for $r > 0$. If $r < n/2$, then $\dot{H}^r(\mathbb{R}^n)$ is a Hilbert space and $\mathscr{S}_0(\mathbb{R}^n)$ is dense in $\dot{H}^r(\mathbb{R}^n)$. Let $s \geq 0$ and define $(-\Delta)^s \varphi = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{\varphi})$ for $\varphi \in \mathscr{S}_0(\mathbb{R}^n)$. Then $(-\Delta)^s \colon \mathscr{S}_0(\mathbb{R}^n) \to \dot{H}^{r-2s}(\mathbb{R}^n)$ is an isometry with respect to the norm $\|\cdot\|_{\dot{H}^r(\mathbb{R}^n)}$ and by density can be extended to a continuous map $(-\Delta)^s \colon \dot{H}^r(\mathbb{R}^n) \to \dot{H}^{r-2s}(\mathbb{R}^n)$ when $r < n/2$. Similarly one obtains that $I_\alpha \colon \dot{H}^r(\mathbb{R}^n) \to \dot{H}^{r+n-\alpha}(\mathbb{R}^n)$ is a continuous map for $r < \alpha - n/2$ and corresponds to fractional Laplacians with negative powers, up to a constant factor.

The fractional Laplacian can also be defined on Bessel potential spaces. Let $1 \leq p < \infty$. We define

$$H^{r,p}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \in L^p(\mathbb{R}^n)\}$$

and equip it with the norm

$$\|u\|_{H^{r,p}(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \right\|_{L^p(\mathbb{R}^n)}.$$

It follows that $H^{r,p}(\mathbb{R}^n)$ is a Banach space and $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^{r,p}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$. By the Mikhlin multiplier theorem, one obtains that the operator $(-\Delta)^s \colon H^{r,p}(\mathbb{R}^n) \to H^{r-2s,p}(\mathbb{R}^n)$ is continuous for $s \geq 0$ and $1 < p < \infty$. The fractional Laplacian is also defined in the space $H^{r,1}(\mathbb{R}^n)$ since $H^{r,1}(\mathbb{R}^n) \hookrightarrow H^{\frac{2r-n-\epsilon}{2}}(\mathbb{R}^n)$ for any $\epsilon > 0$ by the continuity of the Fourier transform $\mathcal{F} \colon L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.

One can define fractional Laplacians on more general spaces. It follows that if $s \in (-n/2, 1]$, then $(-\Delta)^s \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}_s(\mathbb{R}^n)$ is continuous where $\mathscr{S}_s(\mathbb{R}^n)$ is the set

$$\mathscr{S}_s(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \langle \cdot \rangle^{n+2s} \partial^\beta \varphi \in L^\infty(\mathbb{R}^n) \text{ for all } \beta \in \mathbb{N}_0^n\}$$

equipped with the topology induced by the seminorms $\left\|\langle\cdot\rangle^{n+2s}\partial^\beta\varphi\right\|_{L^\infty(\mathbb{R}^n)}$. One can then extend $(-\Delta)^s$ by duality to a continuous map $(-\Delta)^s\colon (\mathscr{S}_s(\mathbb{R}^n))^* \to \mathscr{S}'(\mathbb{R}^n)$. See [28, 80] for more details and a characterization of the dual $(\mathscr{S}_s(\mathbb{R}^n))^*$.

2.3. **Trace spaces and singular potentials.** Let U, $F \subset \mathbb{R}^n$ be an open and a closed set. We define the following Sobolev spaces

$$\begin{aligned} H^r(U) &= \{u|_U : u \in H^r(\mathbb{R}^n)\} \\ \widetilde{H}^r(U) &= \text{ closure of } C_c^\infty(U) \text{ in } H^r(\mathbb{R}^n) \\ H_0^r(U) &= \text{ closure of } C_c^\infty(U) \text{ in } H^r(U) \\ H_F^r(\mathbb{R}^n) &= \{u \in H^r(\mathbb{R}^n) : \mathrm{spt}(u) \subset F\}. \end{aligned}$$

It is obvious that $\widetilde{H}^r(U) \subset H^r_{\overline{U}}(\mathbb{R}^n)$ and $\widetilde{H}^r(U) \subset H^r_0(U)$. In nonlocal problems, we impose exterior values for the equation instead of boundary values. Therefore exterior values are considered to be the same if their difference is in the space $\widetilde{H}^r(U)$. For example, in equation (3) the condition $u|_{\Omega_e} = f$ means that $u - f \in \widetilde{H}^s(\Omega)$, i.e. u and f are equal outside $\overline{\Omega}$, where Ω is bounded open set. This motivates the definition of the abstract trace space $X = H^r(\mathbb{R}^n)/\widetilde{H}^r(\Omega)$ which identifies functions in Ω_e. If Ω is a Lipschitz domain, then we have $H^r_0(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$ when $r > -1/2$, $r \notin \{1/2, 3/2, \dots\}$, $\widetilde{H}^r(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$, $X = H^r(\Omega_e)$ and $X^* = H^{-r}_{\overline{\Omega}_e}(\mathbb{R}^n)$. Thus for more regular domains it could be more convenient to work with the spaces $H^r_{\overline{\Omega}}(\mathbb{R}^n)$, but in this article we do not assume any regularity of the set Ω. For more theory of Sobolev spaces on (non-Lipschitz) domains and their properties, see [12, 57].

We also use some properties of singular potentials which were introduced in [75]. Let $t \geq 0$ and define $Z^{-t}(U)$ as a subspace of distributions $\mathcal{D}'(U)$ equipped with the norm

$$\|f\|_{Z^{-t}(U)} = \sup\{|\langle f, u_1u_2\rangle_U| : u_i \in C_c^\infty(U),\ \|u_i\|_{H^t(\mathbb{R}^n)} = 1\}\ ,$$

where $\langle\cdot,\cdot\rangle_U$ is the dual pairing. We denote by $Z_0^{-t}(U)$ the closure of $C_c^\infty(U)$ in $Z^{-t}(U)$. Elements in $Z^{-t}(\mathbb{R}^n)$ can be seen as multipliers: every $f \in Z^{-t}(\mathbb{R}^n)$ induces a map $m_f\colon H^t(\mathbb{R}^n) \to H^{-t}(\mathbb{R}^n)$ defined as $\langle m_f(u), v\rangle_{\mathbb{R}^n} = \langle f, uv\rangle_{\mathbb{R}^n}$. Also $|\langle f, uv\rangle_{\mathbb{R}^n}| \leq \|f\|_{Z^{-t}(\mathbb{R}^n)}\|u\|_{H^t(\mathbb{R}^n)}\|v\|_{H^t(\mathbb{R}^n)}$, and this inequality can be seen as a motivation for the definition of the space $Z^{-t}(\mathbb{R}^n)$. Clearly we have $Z_0^{-t}(\mathbb{R}^n) \subset Z^{-t}(\mathbb{R}^n)$. If U is bounded, then $L^{\frac{n}{2t}}(U) \subset Z_0^{-t}(\mathbb{R}^n)$ for $0 < t < n/2$ and $L^\infty(U) \subset Z_0^{-t}(\mathbb{R}^n)$ in the sense of zero extensions. Further, it holds that $L^p(U) \subset Z_0^{-t}(\mathbb{R}^n)$ when $p > \max\{1, n/2t\}$ (see section 6). We will only need these basic inclusions. For a more detailed treatment of the space of singular potentials $Z^{-t}(U)$, see [55, 75].

3. Unique continuation property and Poincaré inequality

3.1. **Unique continuation results.** In this section, we prove theorem 1.2 and give several other unique continuation results for fractional Laplacians and Riesz potentials in inhomogeneous and homogeneous Sobolev spaces. Even though we do not need all the results to solve the inverse problems considered in this article, we still state those variants since they are not given in earlier literature to the best of our knowledge. The strategy to prove results in this chapter is straightforward: if something is true for $(-\Delta)^s$ when $s \in (0,1)$, then by the splitting $(-\Delta)^s = (-\Delta)^k(-\Delta)^{s-k}$ it should also be true for all powers s whenever the operations and claims are meaningful.

First we need a basic lemma for polyharmonic distributions, i.e. distributions which satisfy $(-\Delta)^k g = 0$ for some integer $k \in \mathbb{N}$. We sketch the proof since it reflects the method of reduction we repeatedly use in this section.

Lemma 3.1. *Let $V \subset \mathbb{R}^n$ be any nonempty open set. If $g \in \mathcal{D}'(\mathbb{R}^n)$ satisfies $(-\Delta)^k g = 0$ and $g|_V = 0$ for some $k \in \mathbb{N}$, then $g = 0$.*

Proof. The proof is by induction. The case $k = 1$ is true since harmonic distributions are harmonic functions and therefore analytic [59]. Assume that the lemma holds for some $k =$

$m \in \mathbb{N}$. If $(-\Delta)^{m+1}g = 0$ and $g|_V = 0$, then $(-\Delta)^m((-\Delta)g) = 0$ and $(-\Delta)g|_V = 0$ since $(-\Delta)$ is a local operator. The induction assumption implies $(-\Delta)g = 0$, and since also $g|_V = 0$, we obtain $g = 0$ by harmonicity. This implies the claim. Alternatively one could use the fact that polyharmonic distributions are analytic [59, Theorem 7.30]. □

Now we can prove theorem 1.2. The idea is to reduce the general case back to the one where $s \in (0, 1)$ and use the UCP proved in [28]. Note that the corresponding UCP cannot hold for local operators such as $(-\Delta)^k$ when $k \in \mathbb{N}$. Therefore we have to assume that $s \in \mathbb{R} \setminus \mathbb{Z}$. For the proof of the case $s \in (0, 1)$, see [28, Theorem 1.2].

Proof of theorem 1.2. Because of our assumptions for u, the fractional Laplacian $(-\Delta)^s u$ for $s \in (-n/2, \infty) \setminus \mathbb{Z}$ is well-defined, see section 2.2. Assume that $k - 1 < s < k$ for some $k \in \mathbb{N}$. Now we can split $(-\Delta)^s u = (-\Delta)^{s-(k-1)}((-\Delta)^{k-1}u)$ where $s - (k - 1) \in (0, 1)$. Since the operator $(-\Delta)^{k-1}$ is local, we obtain $(-\Delta)^{s-(k-1)}((-\Delta)^{k-1}u)|_V = 0$ and $(-\Delta)^{k-1}u|_V = 0$ where $(-\Delta)^{k-1}u \in H^{r-2(k-1)}(\mathbb{R}^n)$. By the UCP of $(-\Delta)^{s-(k-1)}$, we have $(-\Delta)^{k-1}u = 0$. Since u is polyharmonic and $u|_V = 0$, lemma 3.1 implies $u = 0$.

If $-n/2 < s < 0$, $s \notin \mathbb{Z}$, choose $k \in \mathbb{N}$ such that $k + s > 0$. Then by the locality of $(-\Delta)^k$ we obtain $(-\Delta)^{k+s}u|_V = 0$ and $u|_V = 0$. The first part of the proof implies the claim. □

Note that theorem 1.2 implies UCP for equations of the type $(-\Delta)^s u + Lu = 0$ where L is any local operator. Especially, this holds if $L = P(x, D)$ where

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

is a differential operator of order m.

The following unique continuation result of Riesz potentials was presented in [41]. We use it to show uniqueness for partial data problems of the d-plane transform in section 4. We recall the short proof since it relies on the UCP of the fractional Laplacian.

Corollary 3.2. *Let $\alpha \in \mathbb{R}$ such that $0 < \alpha < n$ and $(\alpha - n)/2 \in \mathbb{R} \setminus \mathbb{Z}$. Let $f \in \mathscr{O}'_C(\mathbb{R}^n)$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $I_\alpha f|_V = 0$ and $f|_V = 0$, then $f = 0$.*

Proof. Recall that $f \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. We can write $I_\alpha f = (-\Delta)^{-s} f$ where $s = (n - \alpha)/2$. Choose $k \in \mathbb{N}$ such that $k - s > 0$. By locality of $(-\Delta)^k$ we obtain the conditions $(-\Delta)^{k-s} f|_V = 0$ and $f|_V = 0$. Theorem 1.2 implies $f = 0$. □

It is also independently proved in [41], without using the UCP of $(-\Delta)^s$, that if $f \in \mathcal{E}'(\mathbb{R}^n)$, then one can replace the condition $I_\alpha f|_V = 0$ by the requirement $\partial^\beta (I_\alpha f)(x_0) = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}_0^n$. In fact, this can be used to prove a slightly stronger result for $(-\Delta)^s$ in the case of compact support.

Corollary 3.3. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$, $V \subset \mathbb{R}^n$ some nonempty open set and $s \in (-n/2, \infty) \setminus \mathbb{Z}$. If $\partial^\beta((-\Delta)^s u)(x_0) = 0$ and $u|_V = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}_0^n$, then $u = 0$.*

Proof. Let $k-1 < s < k$ where $k \in \mathbb{N}$. Now $(-\Delta)^s = (-\Delta)^k(-\Delta)^{s-k} = (-\Delta)^k I_\alpha$ where $\alpha = n+2s-2k \in (n-2, n)$. Furthermore, $\partial^\beta(-\Delta)^s u = \partial^\beta I_\alpha(-\Delta)^k u$ since the Riesz potential commutes with derivatives. By the locality of $(-\Delta)^k$ we obtain the conditions $\partial^\beta(I_\alpha(-\Delta)^k u)(x_0) = 0$ and $(-\Delta)^k u|_V = 0$ where $(-\Delta)^k u \in \mathcal{E}'(\mathbb{R}^n)$. By [41, Theorem 1.1], we must have $(-\Delta)^k u = 0$. Since also $u|_V = 0$, we obtain $u = 0$ by lemma 3.1.

Let then $s \in (-n/2, 0)$, $s \notin \mathbb{Z}$, and pick $k \in \mathbb{N}$ such that $s + k > 0$. All the derivatives $\partial^\beta((-\Delta)^s u)(x_0)$ vanish, and hence $((-\Delta)^k \partial^\beta)((-\Delta)^s u)(x_0) = 0$. Now $((-\Delta)^k \partial^\beta)((-\Delta)^s u) = \partial^\beta((-\Delta)^{s+k} u)$ and we get the conditions $\partial^\beta((-\Delta)^{s+k}u)(x_0) = 0$ and $u|_V = 0$. The first part of the proof gives the claim. □

The UCP of $(-\Delta)^s$ also extends to homogeneous Sobolev spaces. The following result is a simple consequence of theorem 1.2. See [22, 23] for related results (strong UCP and measurable UCP in some special cases).

Corollary 3.4. *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $u \in \dot{H}^r(\mathbb{R}^n)$, $r < n/2$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$.*

Proof. If $r < 0$, then $u \in H^r(\mathbb{R}^n)$ and the claim follows from theorem 1.2. Let $r > 0$ and choose $k \in \mathbb{N}$ such that $r - 2k < 0$. Now $(-\Delta)^k(-\Delta)^s = (-\Delta)^s(-\Delta)^k$ holds in $\mathscr{S}_0(\mathbb{R}^n)$ so by the density of $\mathscr{S}_0(\mathbb{R}^n)$ and the locality of $(-\Delta)^k$ we obtain $(-\Delta)^s((-\Delta)^k u)|_V = 0$ and $(-\Delta)^k u|_V = 0$, where $(-\Delta)^k u \in \dot{H}^{r-2k}(\mathbb{R}^n) \subset H^{r-2k}(\mathbb{R}^n)$. Hence $(-\Delta)^k u = 0$ by theorem 1.2 and since $u|_V = 0$ we obtain $u = 0$ by lemma 3.1. □

Since $(-\Delta)^k(-\Delta)^{-s} = (-\Delta)^{k-s}$ also holds by the density of $\mathscr{S}_0(\mathbb{R}^n)$, one can reduce the case of negative exponents to the case of positive exponents. Thus one obtains the corresponding UCP for the Riesz potential I_α in $\dot{H}^r(\mathbb{R}^n)$ where $r < \alpha - n/2$. By the Sobolev embedding theorem we obtain the following unique continuation result for Bessel potential spaces when $1 \leq p \leq 2$.

Corollary 3.5. *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $1 \leq p \leq 2$ and $u \in H^{r,p}(\mathbb{R}^n)$, $r \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$.*

Proof. If $p = 1$, then $\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^1(\mathbb{R}^n)$ which implies $\langle\cdot\rangle^r \hat{u} \in L^\infty(\mathbb{R}^n)$ since $\mathcal{F}\colon L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is continuous. Hence $u \in H^t(\mathbb{R}^n)$ for some $t \in \mathbb{R}$ and the claim follows from theorem 1.2. Let then $1 < p \leq 2$. By the Sobolev embedding theorem (see e.g. [6, Theorem 6.5.1]) $H^{r,p}(\mathbb{R}^n) \hookrightarrow H^{r_1,p_1}(\mathbb{R}^n)$ when $r_1 \leq r$, $1 < p \leq p_1 < \infty$ and

$$r - \frac{n}{p} = r_1 - \frac{n}{p_1}.$$

Choose $p_1 = 2$. Then for any $1 < p \leq 2$ the previous equality holds when

$$r_1 = \frac{2rp + n(p-2)}{2p} \leq r.$$

Hence $u \in H^{r_1,2}(\mathbb{R}^n) = H^{r_1}(\mathbb{R}^n)$ and by theorem 1.2 we obtain $u = 0$. □

For higher exponents p, we can prove the following version of unique continuation considering the Fourier transform.

Corollary 3.6. *Let $r \geq 0$, $2 \leq p < \infty$ and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. Let $u \in H^{r,p}(\mathbb{R}^n)$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $(-\Delta)^s \hat{u}|_V = 0$ and $\hat{u}|_V = 0$, then $u = 0$.*

Proof. By the inclusion $H^{r,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $r \geq 0$, we can assume $u \in L^p(\mathbb{R}^n)$. If $p = 2$, then $\hat{u} \in L^2(\mathbb{R}^n)$. By theorem 1.2, we obtain $\hat{u} = 0$ and hence $u = 0$. If $2 < p < \infty$, then we have that $\hat{u} \in H^{-t}(\mathbb{R}^n)$ where $t > n(1/2 - 1/p)$ by [38, Theorem 7.9.3]. Again we obtain $\hat{u} = 0$ by theorem 1.2 and eventually $u = 0$. □

Note that if u has compact support, then by the Paley-Wiener theorem the condition $\hat{u}|_V = 0$ already implies that $u = 0$.

3.2. **The fractional Poincaré inequality.** This subsection is dedicated to the proofs of a fractional Poincaré inequality. It serves the goal of estimating the L^2-norm of $u \in \widetilde{H}^s(\Omega)$ with that of its fractional Laplacian $(-\Delta)^{s/2}u$. We give five possible proofs for the fractional Poincaré inequality. We believe that giving several proofs will be helpful in subsequent works. This also illustrates some connections between methods which might have been unnoticed before.

The first proof is the most direct one and is based on splitting of frequencies on the Fourier side. The second proof utilizes several estimates (most importantly Hardy-Littlewood-Sobolev inequalities). This proof is motivated by the approach taken in [28]. Third proof uses a reduction argument to extend the inequality proved in [11] for all powers $s \geq 0$. Fourth proof is based on interpolation of homogeneous Sobolev spaces and it also gives an explicit constant in terms of the classical Poincaré constant. Fifth proof uses uncertainty inequalities which are treated in [24].

We begin our first proof by dividing the Fourier side into high and low frequencies. We only use simple estimates in the proof. In this approach we also get a control on the Poincaré constant. The result is basically the same as [4, Proposition 1.55].

Theorem 3.7 (Poincaré inequality)**.** *Let $s \geq 0$, $K \subset \mathbb{R}^n$ compact set and $u \in H^s_K(\mathbb{R}^n)$. There exists a constant $c = c(n, K, s) > 0$ such that*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}.$$

Proof. We divide the integration into high and low frequencies

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \int_{|\xi|\leq\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi + \int_{|\xi|>\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi$$

where $\epsilon > 0$ is determined later on. Let us analyze the first part. Since $u \in L^2(\mathbb{R}^n)$ and has support in K, Hölder's inequality implies

$$|\hat{u}(\xi)| \leq \|u\|_{L^1(\mathbb{R}^n)} \leq |K|^{1/2} \|u\|_{L^2(\mathbb{R}^n)}.$$

Thus we have

$$\int_{|\xi|\leq\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \leq \int_{|\xi|\leq\epsilon} |K| \|u\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}\xi = \epsilon^n |K| |B(0,1)| \|u\|^2_{L^2(\mathbb{R}^n)}$$

where $|K|$ and $|B(0,1)|$ are the measures of K and the unit ball $B(0,1)$. For high frequencies we can do the following trick

$$\int_{|\xi|>\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi = \int_{|\xi|>\epsilon} \frac{|\xi|^{2s} |\hat{u}(\xi)|^2}{|\xi|^{2s}} \mathrm{d}\xi \leq \epsilon^{-2s} \left\|(-\Delta)^{s/2} u\right\|^2_{L^2(\mathbb{R}^n)}.$$

Now choose $0 < \epsilon < (|K| |B(0,1)|)^{-1/n}$. Then one obtains the inequality

$$\|u\|_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon^{-s}}{\sqrt{1 - \epsilon^n |K| |B(0,1)|}} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}. \quad \square$$

Remark 3.8. *Choosing $\epsilon = (2 |K| |B(0,1)|)^{-1/n}$ one obtains the following inequality in theorem 3.7*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq \sqrt{2}(2 |K| |B(0,1)|)^{s/n} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}.$$

If K is a ball, the constant in this inequality has the same scaling with respect to the diameter of the set as in theorem 3.17, i.e. $c \approx (diam(K))^s$. Further, one can use similar method of proof as in theorem 3.7 to show Poincaré inequalities for more general pseudodifferential operators on certain manifolds. See [84] *for details.*

Provided we have the Poincaré inequality, we can prove the generalized version of it. See also [4, Corollary 1.56] for a similar inequality when K is a ball. In that case one can take $\tilde{c} \approx (\mathrm{diam}(K))^{s-t}$. The cases $s \geq t \geq 1$ and $s \geq 1 \geq t \geq 0$ are also proved for $u \in \widetilde{H}^s(\Omega)$ in theorem 3.17.

Proof of theorem 1.5. Since $s \geq t \geq 0$ we have the continuous embeddings $H^t(\mathbb{R}^n) \hookrightarrow \dot{H}^t(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$. Using the Poincaré inequality in theorem 3.7 we obtain

$$\begin{aligned}
\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} = \|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq \|u\|_{H^t(\mathbb{R}^n)} \leq \|u\|_{H^s(\mathbb{R}^n)} &\leq 2^{\frac{s+1}{2}} \Big(\|u\|_{L^2(\mathbb{R}^n)} + \|u\|_{\dot{H}^s(\mathbb{R}^n)} \Big) \\
&\leq 2^{\frac{s+1}{2}} \left(c \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)} + \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)} \right) \\
&= \tilde{c} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

where the constants c and $\tilde{c}$ do not depend on u. In the fourth step we used the elementary inequality $(a+b)^r \leq 2^r(a^r + b^r)$ for $a, b \geq 0$. This concludes the proof. $\square$

We then start preparation for our second proof by stating some known lemmas:

- lemma 3.9 is the continuity of Riesz potentials,
- lemma 3.10 is the L^2 boundedness of inverse of elliptic second order operators,
- lemma 3.11 is a convolution L^p estimate from below by an inhomogeneous Hölder norm,
- lemma 3.12 is a specific form of the Poincaré inequality for fractional Laplacians, and
- lemma 3.13 is a simple commutation property for the gradient and a Fourier multiplier.

Lemma 3.9 (Theorem 4.5.3 in [38])**.** *Let $t \geq 0$, $1 < p < \infty$ be such that $n > tp$, and define $q = \frac{np}{n-tp}$. Then the Riesz potential $(-\Delta)^{-t/2} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous.*

Lemma 3.10 (Section 6 in [21])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in L^2(\Omega)$. If $w \in H^1_0(\Omega)$ is the unique solution of the problem*

$$\begin{cases} (-\Delta)w &= f \quad \text{in } \Omega \\ \quad w|_{\partial\Omega} &= 0 \end{cases},$$

then there exists a constant $C = C(\Omega)$ such that

$$\|w\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \ . \tag{5}$$

Lemma 3.11 (Theorem 4.5.10 in [38])**.** *Let $\psi \in C^1(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree $-n/a$, $p \in [1, \infty]$ and $\gamma = n(1 - 1/a - 1/p)$ be such that $\gamma \in (0, 1)$. Then if $v \in L^p(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$ we have*

$$\sup_{x \neq y} \left\{ \frac{|(\psi * v)(x) - (\psi * v)(y)|}{|x-y|^\gamma} \right\} \leq C\|v\|_{L^p(\mathbb{R}^n)},$$

where C does not depend on w.

Lemma 3.12 (Formula (1.3) in [64])**.** *Let $1 < p \leq q < \infty$ and $f \in W^{n/p,p}(\mathbb{R}^n)$. There is a constant $C = C(n, p)$ such that*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq Cq^{1-1/p}\|(-\Delta)^{n/2p} f\|^{1-p/q}_{L^p(\mathbb{R}^n)}\|f\|^{p/q}_{L^p(\mathbb{R}^n)} \ . \tag{6}$$

This estimate is proved using sharp Hardy-Littlewood-Sobolev inequalities.

Lemma 3.13. *Let $t \geq 0$ and $f \in H^{1+2t}(\mathbb{R}^n)$. Then $[\nabla, (-\Delta)^t]f = 0$, that is, the gradient and the fractional Laplacian of exponent t commute.*

Proof. The proof is just a trivial computation with Fourier symbols:

$$\mathcal{F}(\nabla(-\Delta)^t f) = i\xi|\xi|^{2t}\hat{f}(\xi) = |\xi|^{2t}i\xi\hat{f}(\xi) = \mathcal{F}((-\Delta)^t(\nabla f)) \ . \qquad \square$$

We are now ready to state and prove the fractional Poincaré inequality.

Theorem 3.14 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $s \in [0, \infty)$ and $u \in \widetilde{H}^s(\Omega)$. There exists a constant $c = c(n, \Omega, s)$ such that*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)} \ .$$

Proof. In the inequalities the constants (usually denoted by c, C, etc.) do not depend on the function which is being estimated and can change from line to line. We let the symbol $s' = s - \lfloor s \rfloor$ indicate the fractional part of the exponent s, with the convention that $s' \in [0, 1)$. First observe that by using lemma 3.9 with $p = 2$ and Hölder's inequality we get the following useful estimate

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_\Omega\|u\|_{L^q(\mathbb{R}^n)} \leq c\|(-\Delta)^{t/2}u\|_{L^2(\mathbb{R}^n)} \tag{7}$$

when $u \in \widetilde{H}^t(\Omega)$ where q and t are as in lemma 3.9. Our proof is divided in several cases.

Case 1: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' = 0$.
Recall that $\widetilde{H}^{2h}(\Omega) \subset H^{2h}_0(\Omega)$. We show that if $u \in H^{2h}_0(\Omega)$ and $h \in \mathbb{N}$ then there exists a constant $c = c(n, \Omega, h)$ such that

$$\|(-\Delta)^h u\|_{L^2(\mathbb{R}^n)} \geq c\|u\|_{L^2(\mathbb{R}^n)} \ . \tag{8}$$

The estimate (8) holds trivially if $h = 0$, while if $h = 1$ then (8) follows from the boundedness of the inverse lemma 3.10. Assume now that $h \geq 2$, and by induction that (8) holds for $h-1$. Then $(-\Delta)u \in H_0^{2h-2}(\Omega)$, so we can apply (8) and (5) to get

$$\|(-\Delta)^h u\|_{L^2(\mathbb{R}^n)} = \|(-\Delta)^{h-1}(-\Delta u)\|_{L^2(\mathbb{R}^n)} \geq c\|(-\Delta)u\|_{L^2(\mathbb{R}^n)} \geq c'\|u\|_{L^2(\mathbb{R}^n)} .$$

In the next steps we consider $s \notin \mathbb{N}$.

Case 2: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' \in (0, 1/2)$ or $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' \in [1/2, 1)$, $n \geq 2$.
Now it holds that $n > 2s'$, and there exists $k \in \mathbb{N}$ such that $s \in (2k, 2k+1)$ and we can write $(-\Delta)^{s/2}u = (-\Delta)^{s'/2}(-\Delta)^k u$. Since $(-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega) = \widetilde{H}^{s'}(\Omega)$, we can apply formula (7)

$$\|(-\Delta)^k u\|_{L^2(\mathbb{R}^n)} \leq c\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)} .$$

Since $u \in H_0^s(\Omega) \subset H_0^{2k}(\Omega)$, we can get the result using formula (8).

Case 3: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' \in (1/2, 1)$, $n = 1$.
As in the second case, there exists $k \in \mathbb{N}$ such that $s \in (2k, 2k+1)$ and we can write $(-\Delta)^{s/2}u = (-\Delta)^{s'/2}(-\Delta)^k u$. However, since now $n < 2s'$, we cannot directly use formula (7).
Assume first that $w \in C_c^\infty(\Omega)$. Then we can take $y_0 \in \Omega$ such that $w(y_0) = 0$ and $x_0 \in \Omega$ such that $w(x_0) = \|w\|_{L^\infty(\Omega)}$. With these choices and for any $\gamma > 0$ we can write

$$\|w\|_{L^2(\mathbb{R}^n)} \leq C\|w\|_{L^\infty(\Omega)} \leq C\frac{w(x_0) - w(y_0)}{|x_0 - y_0|^\gamma} . \tag{9}$$

We now let $\gamma = s' - n/2 = s' - 1/2 \in (0, 1/2)$, and define $\psi = |x|^{s'-1}$, $v = (-\Delta)^{s'/2}w$. By the mapping properties of the fractional Laplacian and the Mikhlin theorem, we can observe that $v \in L^p(\mathbb{R})$ for all $1 < p < \infty$ (see [1, Theorem 7.2]). Using the continuity of the Riesz potential in lemma 3.9, we see that for a constant $c = c(n, s)$ the following holds almost everywhere:

$$w = (-\Delta)^{-s'/2}((-\Delta)^{s'/2}w) = (-\Delta)^{-s'/2}v = cI_{1-s'}v = c|x|^{s'-1} * v = c(\psi * v) .$$

Let χ_R be the characteristic function of the ball B_R of radius $R > 0$, and define $w_R = c(\psi * (\chi_R v))$, with c as above. We see that

$$w_R(x) = c(\psi * (\chi_R v))(x) = c\int_{\mathbb{R}} \psi(x-y)\chi_R(y)v(y)dy ,$$

and the integrand is dominated by $|\psi(x-y)v(y)|$. This is an integrable function, since

$$\int_{\mathbb{R}} |\psi(x-y)v(y)|dy = \int_{\mathbb{R}} \psi(x-y)|v(y)|dy = I_{1-s'}(|v|)(x) ,$$

and the Riesz potential is well defined almost everywhere on $L^p(\mathbb{R})$ for any $1 < p < 1/s'$. Now the dominated convergence theorem gives that $w_R(x) \to w(x)$ as $R \to \infty$ for almost every fixed $x \in \mathbb{R}$.

Let $\epsilon > 0$ and $x_0', y_0' \in \mathbb{R}$ be such that $|x_0 - x_0'| < \epsilon$, $|y_0 - y_0'| < \epsilon$ and $w_R(x_0')$, $w_R(y_0')$ converge to $w(x_0')$, $w(y_0')$ as $R \to \infty$. Applying lemma 3.11 with $p = 2$, $n = 1$ and $a = 1 - s'$, we see that

$$\begin{aligned} \frac{w_R(x_0') - w_R(y_0')}{|x_0 - y_0|^\gamma} &\leq \sup_{x \neq y}\left\{\frac{w_R(x) - w_R(y)}{|x-y|^\gamma}\right\} \\ &= c\sup_{x \neq y}\left\{\frac{(\psi * (\chi_R v))(x) - (\psi * (\chi_R v))(y)}{|x-y|^\gamma}\right\} \\ &\leq C\|\chi_R v\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})} = C\|(-\Delta)^{s'/2}w\|_{L^2(\mathbb{R})} . \end{aligned}$$

We now first take the limit for $R \to \infty$ and then the one for $\epsilon \to 0$. By the smoothness of w, this gives

$$\frac{w(x_0) - w(y_0)}{|x_0 - y_0|^\gamma} \leq C\|(-\Delta)^{s'/2}w\|_{L^2(\mathbb{R})} . \tag{10}$$

Combining formulas (9) and (10) we get $\|w\|_{L^2(\mathbb{R}^n)} \leq C\|(-\Delta)^{s'/2}w\|_{L^2(\mathbb{R}^n)}$, and the same inequality holds for $w \in \widetilde{H}^{s'}(\Omega)$ by density. Let now $w := (-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega) = \widetilde{H}^{s'}(\Omega)$. The result is then obtained applying formula (8).

Case 4: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' = 1/2$, $n = 1$.
Let $w := (-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega) = \widetilde{H}^{s'}(\Omega)$. Here we make use of formula (6) with $p = 2$, $q = 3$ in order to estimate

$$\|w\|_{L^2(\mathbb{R}^n)} = \|w\|^3_{L^2(\mathbb{R}^n)}\|w\|^{-2}_{L^2(\mathbb{R}^n)} \leq \|w\|^3_{L^3(\mathbb{R}^n)}\|w\|^{-2}_{L^2(\mathbb{R}^n)} \leq C\|(-\Delta)^{n/4}w\|_{L^2(\mathbb{R}^n)} \,. \tag{11}$$

Since $n/4$ equals $s'/2$ for $n = 1$, the results follows from (11) and (8).

Case 5: $\lfloor s \rfloor \notin 2\mathbb{Z}$.
Let $u \in C_c^\infty(\Omega)$. In this case $s = s' + 2k + 1$ for some $k \in \mathbb{N}$, therefore we can calculate

$$\begin{aligned} \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)} &= \|(-\Delta)^{1/2}(-\Delta)^{(s'+2k)/2}u\|_{L^2(\mathbb{R}^n)} \\ &= \|\nabla(-\Delta)^{(s'+2k)/2}u\|_{L^2(\mathbb{R}^n)} \\ &= \|(-\Delta)^{(s'+2k)/2}\nabla u\|_{L^2(\mathbb{R}^n)} \\ &\geq C\|\nabla u\|_{L^2(\mathbb{R}^n)} \geq C\|u\|_{L^2(\mathbb{R}^n)} \,. \end{aligned} \tag{12}$$

The second equality in (12) is just an L^2 property of the gradient and the $(-\Delta)^{1/2}$ operator. The third equality in (12) follows from lemma 3.13. The first inequality in (12) follows from the even cases, given that $\lfloor s' + 2k \rfloor \in 2\mathbb{Z}$ and $\nabla u \in \widetilde{H}^{s'+2k}(\Omega)$ componentwise. The last inequality follows from the classical Poincaré inequality. The rest follows by approximation. □

Remark 3.15. *Third way to prove the Poincaré inequality is using the known result in the case $n \geq 1$ and $s \in (0,1)$ [11, Lemma 2.2]. This result is proved using Caffarelli-Silvestre extension. Then one can use similar reduction argument to prove it for all $s \geq 0$ and $u \in C_c^\infty(\Omega)$. Namely, one shows using the classical Poincaré inequality that the claim holds for all $s \in [0,2)$. The higher order fractional cases are obtained by splitting the fractional Laplacian as $(-\Delta)^s = (-\Delta)^k(-\Delta)^{t/2}$ where $t \in (0,2)$. Boundedness of the inverse and the fractional Poincaré inequality for $t \in (0,2)$ imply the claim for fractional exponents. Integer order exponents are obtained from the boundedness of the inverse as before. The inequality for $u \in \widetilde{H}^s(\Omega)$ follows by approximation.*

For the fourth proof we use the following interpolation lemma of homogeneous Sobolev spaces which is a simple consequence of Hölder's inequality, see [4, Proposition 1.32].

Lemma 3.16. *Let $s_0 \leq r \leq s_1$ and $f \in \dot{H}^{s_0}(\mathbb{R}^n) \cap \dot{H}^{s_1}(\mathbb{R}^n)$. Then $f \in \dot{H}^r(\mathbb{R}^n)$ and*

$$\|f\|_{\dot{H}^r(\mathbb{R}^n)} \leq \|f\|^{1-\theta}_{\dot{H}^{s_0}(\mathbb{R}^n)} \|f\|^{\theta}_{\dot{H}^{s_1}(\mathbb{R}^n)} \,, \quad r = (1-\theta)s_0 + \theta s_1.$$

Using the interpolation lemma and the usual Poincaré inequality we can easily prove the following theorem. Note that we also obtain explicit constant from the proof.

Theorem 3.17 (Poincaré inequality)**.** *Let $s \geq t \geq 1$ or $s \geq 1 \geq t \geq 0$, $\Omega \subset \mathbb{R}^n$ bounded open set and $u \in \widetilde{H}^s(\Omega)$. The following inequality holds*

$$\left\|(-\Delta)^{t/2}u\right\|_{L^2(\mathbb{R}^n)} \leq C^{s-t}\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}$$

where $C = C(n,\Omega)$ is the classical Poincaré constant.

Proof. Let $s \geq t \geq 1$ and $u \in C_c^\infty(\Omega)$. The usual Poincaré inequality can be written in terms of the homogeneous Sobolev norm as

$$\|u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\Omega)} \leq C\,\|\nabla u\|_{L^2(\Omega)} = C\,\|\nabla u\|_{L^2(\mathbb{R}^n)} = C\,\|u\|_{\dot{H}^1(\mathbb{R}^n)}$$

where $C = C(n, \Omega)$. We use the interpolation lemma 3.16 twice. First choose $s_0 = 0$, $r = 1$ and $s_1 = t \geq 1$. Now $\theta = 1/t$ and by the classical Poincaré inequality we obtain

$$\|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{\theta} \leq C^{1-\theta} \|u\|_{\dot{H}^1(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{\theta}.$$

From this we get the following inequality

$$\|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq C^{\frac{1-\theta}{\theta}} \|u\|_{\dot{H}^t(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\Omega)$. Hence

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C \|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq C^t \|u\|_{\dot{H}^t(\mathbb{R}^n)}.$$

Then choose $s_0 = 0$, $r = t$ and $s_1 = s \geq t$ in lemma 3.16. Now $\theta = t/s$ and by the previous inequality

$$\|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta} \leq C^{t(1-\theta)} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta}.$$

From this we obtain

$$\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} \leq C^{s-t} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}$$

for $u \in C_c^\infty(\Omega)$.

Let then $s \geq 1 \geq t \geq 0$ and $u \in C_c^\infty(\Omega)$. First interpolate for $s \geq 1 \geq t$ to obtain

$$\|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta}, \quad \theta = \frac{1-t}{s-t}.$$

Second, interpolate for $1 \geq t \geq 0$ and use the previous inequality and the classical Poincaré inequality to get

$$\|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}^{1-\tilde{\theta}} \|u\|_{\dot{H}^1(\mathbb{R}^n)}^{\tilde{\theta}} \leq C^{1-\tilde{\theta}} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta}, \quad \tilde{\theta} = t,$$

which eventually implies the inequality

$$\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} = \|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq C^{s-t} \|u\|_{\dot{H}^s(\mathbb{R}^n)} = C^{s-t} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\Omega)$.

Then let $u \in \widetilde{H}^s(\Omega)$. By definition there is a sequence $\varphi_k \in C_c^\infty(\Omega)$ such that

$$\varphi_k \to u \quad \text{in } H^s(\mathbb{R}^n).$$

The continuity of $(-\Delta)^{t/2}$ implies that

$$(-\Delta)^{t/2} \varphi_k \to (-\Delta)^{t/2} u \quad \text{in } H^{s-t}(\mathbb{R}^n).$$

The embedding $H^{s-t}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is continuous and thus

$$(-\Delta)^{t/2} \varphi_k \to (-\Delta)^{t/2} u \quad \text{in } L^2(\mathbb{R}^n).$$

By the continuity of the norm and $(-\Delta)^{s/2}$ we finally obtain

$$\begin{aligned} \left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} = \lim_k \left\|(-\Delta)^{t/2} \varphi_k\right\|_{L^2(\mathbb{R}^n)} &\leq C^{s-t} \lim_k \left\|(-\Delta)^{s/2} \varphi_k\right\|_{L^2(\mathbb{R}^n)} \\ &= C^{s-t} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

□

We remark that the case $t = 0$ and $s = 1$ corresponds to the classical Poincaré inequality since $\|\nabla u\|_{L^2(\mathbb{R}^n)} = \left\|(-\Delta)^{1/2} u\right\|_{L^2(\mathbb{R}^n)}$. Also the constant C^{s-t} is the expected one. In the usual Poincaré inequality we take one derivative and the constant is C. In the higher order version we take t and s derivatives and the constant naturally becomes C^{s-t}. The constant C can be taken to be proportional to the diameter of the set, $C \approx \operatorname{diam}(\Omega)$.

Remark 3.18. *Fifth way to prove the Poincaré inequality is using uncertainty inequalities. If $u \in L^2(\mathbb{R}^n)$, then there is a constant $c = c(n,s)$ such that*

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leq c \, \||\cdot|^s \, u\|_{L^2(\mathbb{R}^n)} \, \||\cdot|^s \, \hat{u}\|_{L^2(\mathbb{R}^n)}\,, \tag{13}$$

see the discussion after theorem 4.1 in [24]*. We can interpret this inequality as*

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leq c \left\|(-\Delta)^{s/2}(\mathcal{F}^{-1}(u))\right\|_{L^2(\mathbb{R}^n)} \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}$$

whenever the terms on the right hand side of equation (13) *are finite. If u is supported in some fixed compact set K, then one obtains similar inequality as in theorem 3.7, i.e.*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c' \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}$$

holds for all $u \in H^s_K(\mathbb{R}^n)$ and for some constant $c' = c'(n, K, s)$.

Remark 3.19. *The Poincaré inequality for the operator $(-\Delta)^{s/2}$ implies also Poincaré inequality for the fractional gradient $\nabla^s \colon H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s \rfloor+1})$ which is defined as*

$$\nabla^s u(x,y) := \frac{\mathcal{C}^{1/2}_{n,s'}}{\sqrt{2}} \frac{\nabla^{\lfloor s \rfloor}u(x) - \nabla^{\lfloor s \rfloor}u(y)}{|y-x|^{n/2+s'+1}} \otimes (y-x),$$

see section 6 for more details. If $s \geq t \geq 0$ and $u \in C^\infty_c(\Omega)$, then

$$\left\|\nabla^t u\right\|_{L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s \rfloor+1})} = \left\|(-\Delta)^{t/2}u\right\|_{L^2(\mathbb{R}^n)} \leq \tilde{c} \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)} = \tilde{c} \left\|\nabla^s u\right\|_{L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s \rfloor+1})}\,,$$

where the constant $\tilde{c}$ does not depend on u. By approximation and the continuity of ∇^s the previous inequality is also true for $u \in \widetilde{H}^s(\Omega)$.

4. Applications to integral geometry

In this section we discuss how the UCP of Riesz potentials can be used in partial data problems in integral geometry. We follow [36] for the treatment of the d-plane transform, theory of X-ray transform and Radon transform can also be found in [63, 68, 81]. Let $d \in \{1, \dots, n-1\}$ and denote by $\mathbf{P}^d$ the space of all d-dimensional affine planes in $\mathbb{R}^n$. We define the d-plane transform of a function f to be

$$R_d f(A) = \int_{x \in A} f(x) \mathrm{d}m(x)$$

where $A \in \mathbf{P}^d$ and m is the Hausdorff measure on A. The adjoint of R_d is defined as

$$R^*_d g(x) = \int_{A \ni x} g(A) \mathrm{d}\mu(A)$$

where g is a function on $\mathbf{P}^d$ and μ is the associated measure. These transforms are defined for functions such that the integrals exist. The case $d = 1$ corresponds to the usual X-ray transform and $d = n-1$ to the Radon transform. The normal operator of the d-plane transform $N_d = R^*_d R_d$ has an expression $N_d f = c_{n,d}(f * |\cdot|^{-(n-d)})$ where $c_{n,d}$ is a constant depending on n and d. The normal operator is well defined if f is a function that decreases rapidly enough at infinity [36]. This holds for example if $f \in C_\infty(\mathbb{R}^n)$ where $C_\infty(\mathbb{R}^n)$ is the space of continuous functions which decrease faster than any polynomial at infinity (see section 2.1 for a precise definition). Thus, up to a constant factor, N_d can be represented as a Riesz potential $N_d = I_\alpha = (-\Delta)^{-d/2}$ where $\alpha = n - d \in \{1, \dots, n-1\}$.

The transforms R_d and R^*_d can be extended to distributions by duality. Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and $g \in \mathcal{D}'(\mathbb{R}^n)$. Since $R_d \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbf{P}^d)$ and $R^*_d \colon \mathcal{E}(\mathbf{P}^d) \to \mathcal{E}(\mathbb{R}^n)$ are continuous [32], we can define the following operations

$$\langle R_d f, \psi \rangle = \langle f, R^*_d \psi \rangle\,, \quad \psi \in \mathcal{E}(\mathbf{P}^d)$$
$$\langle R^*_d g, \varphi \rangle = \langle g, R_d \varphi \rangle\,, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Therefore $R_d f \in \mathcal{E}'(\mathbf{P}^d)$ and $R_d^* g \in \mathcal{D}'(\mathbb{R}^n)$. This shows that the normal operator $N_d = R_d^* R_d \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is always defined and $N_d f = c_{n,d}(f * |\cdot|^{-(n-d)})$ holds in the sense of distributions. Let $V \subset \mathbb{R}^n$ be a nonempty open set and $f \in \mathcal{E}'(\mathbb{R}^n)$. We say that $R_d f$ vanishes on all d-planes intersecting V, if $\langle R_d f, \varphi \rangle = 0$ for all $\varphi \in C_c^\infty(\mathbf{P}_V^d)$ where $\mathbf{P}_V^d$ is the set of all d-planes intersecting V. If $V = B(0,R)$ is a ball, $\varphi \in C_c^\infty(\mathbf{P}_V^d)$ means that φ is smooth and $\varphi(A) = 0$ for all d-planes A for which $d(0,A) > r$ for some $r < R$. For more details on the range of the d-plane transform and duality in integral geometry, see [32] and [36, Chapter II].

Remark 4.1. *The UCP of Riesz potentials (corollary 3.2) immediately implies the UCP of the normal operator of the d-plane transform when d is odd (corollary 1.3) since $N_d \approx I_{n-d}$ and $d/2 \notin \mathbb{Z}$. However, such UCP cannot hold if d is even, which can be shown by contradiction. Assume that corollary 1.3 holds when d is even. Take any nonzero $f \in C_c^\infty(\mathbb{R}^n)$. By the properties of the Fourier transform and Riesz potentials we have $(-\Delta)^{d/2} f = (-\Delta)^{-d/2}((-\Delta)^d f) = N_d(-\Delta)^d f$ up to a constant factor. Since d is even $(-\Delta)^{d/2}$ is a local operator and we obtain $N_d(-\Delta)^d f|_V = (-\Delta)^d f|_V = 0$ where $V \subset \mathbb{R}^n$ is an open set outside the support of f and $(-\Delta)^d f \in C_c^\infty(\mathbb{R}^n)$. By the assumption we would get that $(-\Delta)^d f = 0$, i.e. f is polyharmonic. But this implies $f = 0$ by lemma 3.1, which is a contradiction. Hence the UCP cannot hold for N_d when d even.*

Using the UCP of N_d we can now prove corollary 1.4.

Proof of corollary 1.3. Consider first $f \in C_\infty(\mathbb{R}^n)$. Taking the adjoint, we get the conditions $N_d f|_V = 0$ and $f|_V = 0$. By corollary 1.3 we obtain $f = 0$ whenever d is odd. Then let $f \in \mathcal{E}'(\mathbb{R}^n)$. We can assume that $V = B(0,R)$ is a ball of radius R centered at the origin. As in [36] we define the "convolution"

$$(g \times \varphi)(A) = \int_{\mathbb{R}^n} g(y)\varphi(A-y)\mathrm{d}y$$

where $g \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbf{P}^d)$, $A \in \mathbf{P}^d$ and $A - y$ is a d-plane shifted by $y \in \mathbb{R}^n$. Then one can calculate that $R_d^*(g \times \varphi) = g * R_d^* \varphi$ (see [36, Proof of theorem 5.4]). Let $j_\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the standard mollifier and consider the mollifications $f * j_\epsilon \in C_c^\infty(\mathbb{R}^n)$. By the properties of the convolutions

$$\langle R_d(f * j_\epsilon), \varphi \rangle = \langle f * j_\epsilon, R_d^* \varphi \rangle = \langle f, j_\epsilon * R_d^* \varphi \rangle = \langle f, R_d^*(j_\epsilon \times \varphi) \rangle = \langle R_d f, j_\epsilon \times \varphi \rangle . \tag{14}$$

Take $r > 0$ and $\epsilon > 0$ small enough so that $r + \epsilon < R$. Let $\varphi \in C_c^\infty(\mathbf{P}^d)$ such that $\varphi(A) = 0$ for all d-planes which satisfy $d(0,A) > r$. Then $(j_\epsilon \times \varphi)(A) = 0$ for all d-planes for which $d(0,A) > r + \epsilon$. Thus $j_\epsilon \times \varphi \in C_c^\infty(\mathbf{P}_V^d)$ and by (14) it follows that $R_d(f * j_\epsilon) = 0$ for all d-planes intersecting $B(0,r)$. We also have $(f * j_\epsilon)|_{B(0,r)} = 0$ and the first part of the proof implies the claim for $f * j_\epsilon$ for small $\epsilon > 0$. Since $f * j_\epsilon \to f$ in $\mathcal{E}'(\mathbb{R}^n)$ when $\epsilon \to 0$, we obtain the claim for f. □

Remark 4.2. *When d is even, corollary 1.4 does not say that the result is false. It only indicates that we cannot use the UCP of the normal operator in the proof. This boils down to the fact that $(-\Delta)^s$ does not admit the UCP for $s \in \mathbb{Z}$. However, if d is even, then the function f is determined uniquely in V by its integrals over d-planes which intersect V. Namely, if $R_d f = 0$ on all d-planes intersecting V, then $N_d f|_V = 0$. Since $N_d \approx (-\Delta)^{-d/2}$, one can invert $N_d f$ by the local operator $(-\Delta)^{d/2}$ to obtain $f|_V = 0$. Hence the ROI problem is uniquely solvable in this case without the additional knowledge of f in an open set inside the ROI.*

Remark 4.3. *We also note that unlike in the global data case lower dimensional data does not determine higher dimensional data. In other words, $R_k f = 0$ for all k-planes intersecting V does not necessarily imply that $R_d f = 0$ for all d-planes which intersect V where $0 < k < d < n$. Thus one cannot reduce the partial data problem for k-planes to the partial data problem for d-planes.*

5. Higher order fractional Schrödinger equation with singular potential

In this section, we study the fractional Schrödinger equation with higher order fractional Laplacian and singular potential. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and consider the equation

$$\begin{cases} ((-\Delta)^s + q)u &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases} \tag{15}$$

where $u \in H^s(\mathbb{R}^n)$, $f \in H^s(\mathbb{R}^n)$ is the exterior value of u and $q \in L^\infty(\Omega)$ represents the electric potential. When the potential q is more singular one has to interpret the product qu in a suitable way. If $q \in Z_0^{-s}(\mathbb{R}^n)$, then q acts as a multiplier and induces a map $m_q \colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ defined by $\langle m_q(u), v\rangle_{\mathbb{R}^n} = \langle q, uv\rangle_{\mathbb{R}^n}$. Then equation (15) becomes

$$\begin{cases} (-\Delta)^s u + m_q(u) &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f. \end{cases} \tag{16}$$

We will prove that the generalized DN map Λ_q for equation (16) determines the restriction of the potential $q \in Z_0^{-s}(\mathbb{R}^n)$ to Ω uniquely from exterior measurements. We also obtain the Runge approximation property for equation (16): any function $g \in \widetilde{H}^s(\Omega)$ can be approximated arbitrarily well by solutions of (16).

Similar results were proved in [75] when $0 < s < 1$. Our theorems generalize those results for higher order fractional Laplacians. The proofs rely essentially on the same thing: the UCP of the operator $(-\Delta)^s$ which was proved for $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ in section 3.1. Also the higher order Poincaré inequality is needed for the well-posedness of the inverse problem. In this section, we provide the basic ideas for the proofs of the lemmas, which are reminescent of the ones in [75] and [28]. We will mainly follow the same notation as in those articles.

The strategy to prove theorems 1.6 and 1.7 is the following. First one constructs a bilinear form and proves that unique weak solutions are obtained in the complement of a countable set of eigenvalues. One also proves that 0 is not an eigenvalue when (4) holds. Then one defines the abstract DN map and proves the Alessandrini identity using it. Using the UCP of $(-\Delta)^s$ one obtains the Runge approximation property for equation (16). From the Runge approximation and the Alessandrini identity, one can prove the uniqueness result for the inverse problem.

If $U \subset \mathbb{R}^n$ is open and $u, v \in L^2(U)$, we denote the inner product by

$$\langle u, v\rangle_U = \int_U uv \mathrm{d}x.$$

We also use the same notation $\langle \cdot, \cdot\rangle_U$ for dual pairing.

The following lemma guarantees the existence of unique weak solutions (see [75, Lemma 2.6]).

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $q \in Z_0^{-s}(\mathbb{R}^n)$. For $v, w \in H^s(\mathbb{R}^n)$ define the bilinear form B_q as*

$$B_q(v, w) = \left\langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w \right\rangle_{\mathbb{R}^n} + \langle m_q(v), w\rangle_{\mathbb{R}^n}.$$

The following claims hold:

(a) There is a countable set $\Sigma = \{\lambda_i\}_{i=1}^\infty \subset \mathbb{R}$, $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$, with the following property: if $\lambda \notin \Sigma$, then for any $F \in (\widetilde{H}^s(\Omega))^*$ and $f \in H^s(\mathbb{R}^n)$ there is unique $u \in H^s(\mathbb{R}^n)$ satisfying

$$B_q(u, w) - \lambda \langle u, w\rangle_{\mathbb{R}^n} = F(w) \quad \text{for } w \in \widetilde{H}^s(\Omega), \quad u - f \in \widetilde{H}^s(\Omega)$$

with the norm estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\Big(\|F\|_{(\widetilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)} \Big)$$

where C is independent of F and f.

(b) *The function* u *in* (a) *is the unique* $u \in H^s(\mathbb{R}^n)$ *satisfying*

$$((-\Delta)^s u + m_q(u) - \lambda u)|_\Omega = F$$

in the sense of distributions with $u - f \in \widetilde{H}^s(\Omega)$.

(c) *One has* $0 \notin \Sigma$ *if* (4) *holds. If* $q \in L^\infty(\Omega)$ *and* $q \geq 0$, *then* $\Sigma \subset (0, \infty)$ *and* (4) *always holds.*

Proof. The constants in the inequalities do not depend on the function v in the proof. It is enough to solve the problem in (a) for $u - f = v \in \widetilde{H}^s(\Omega)$. Using the fractional Poincaré inequality (theorem 1.5) we obtain

$$\|v\|^2_{H^s(\mathbb{R}^n)} \leq C\left(\left\|(-\Delta)^{s/2}v\right\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)}\right) \leq C'\left\|(-\Delta)^{s/2}v\right\|^2_{L^2(\mathbb{R}^n)}.$$

Let $0 < \epsilon < 1/C'$ where the constant $C' > 0$ comes from the previous inequality. Since $q \in Z_0^{-s}(\mathbb{R}^n)$, we can find $q_s \in C_c^\infty(\mathbb{R}^n)$ and $q_r \in Z^{-s}(\mathbb{R}^n)$ such that $q = q_s + q_r$ and $\|q_r\|_{Z^{-s}(\mathbb{R}^n)} < \epsilon$. When we take $\mu = \|q_s^-\|_{L^\infty(\mathbb{R}^n)}$ where $q_s^- = -\min\{0, q_s(x)\}$, we obtain the coercivity condition

$$B_q(v,v) + \mu\langle v, v\rangle_{\mathbb{R}^n} \geq \left\|(-\Delta)^{s/2}v\right\|^2_{L^2(\mathbb{R}^n)} + \langle q_r, vv\rangle_{\mathbb{R}^n} \geq \frac{1}{C'}\|v\|^2_{H^s(\mathbb{R}^n)} - \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}.$$

Hence $v, w \mapsto B_q(v,w) + \mu\langle v, w\rangle_{\mathbb{R}^n}$ defines an equivalent inner product in $\widetilde{H}^s(\Omega)$. The proof is then completed as in [28]: the Riesz representation theorem implies that for every $\widetilde{F} \in (\widetilde{H}^s(\Omega))^*$ there is unique $v = G_\mu \widetilde{F} \in \widetilde{H}^s(\Omega)$ such that $B_q(v,w) + \mu\langle v, w\rangle_{\mathbb{R}^n} = \widetilde{F}(w)$ for all $w \in \widetilde{H}^s(\Omega)$. The map $G_\mu\colon (\widetilde{H}^s(\Omega))^* \to \widetilde{H}^s(\Omega)$ induces a compact, self-adjoint and positive definite operator $\widetilde{G}_\mu\colon L^2(\Omega) \to L^2(\Omega)$ by the compact Sobolev embedding theorem. The spectral theorem for the self-adjoint compact operator $\widetilde{G}_\mu$ implies the claim in (a). Part (b) holds since $C_c^\infty(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$. The first claim in (c) follows from the Fredholm alternative. The second claim in (c) is essentially the same as in [28, Lemma 2.3] and is proved by replacing $\widetilde{H}^s(\Omega)$ with $H_{\overline{\Omega}}(\mathbb{R}^n)$ in the proof of (a). □

Recall the definition of the abstract trace space $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ which we equip with the quotient norm

$$\|[f]\|_X = \inf_{\varphi \in \widetilde{H}^s(\Omega)} \|f - \varphi\|_{H^s(\mathbb{R}^n)}, \quad f \in H^s(\mathbb{R}^n).$$

The following lemma implies that the DN map is well-defined and continuous. The result follows immediately from the definition of the bilinear form $B_q(\cdot,\cdot)$ and from the continuity of $(-\Delta)^{s/2}\colon H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (see [28, Lemma 2.4]).

Lemma 5.2. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded open set,* $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ *and* $q \in Z_0^{-s}(\mathbb{R}^n)$ *which satisfies* (4). *Then the map* $\Lambda_q\colon X \to X^*$, $\langle \Lambda_q[f], [g]\rangle = B_q(u_f, g)$, *is linear and continuous, where* $u_f \in H^s(\mathbb{R}^n)$ *solves* $(-\Delta)^s u + m_q(u) = 0$ *in* Ω *with* $u - f \in \widetilde{H}^s(\Omega)$. *One also has the self-adjointness property* $\langle \Lambda_q[f], [g])\rangle = \langle [f], \Lambda_q[g]\rangle$ *for* $f, g \in H^s(\mathbb{R}^n)$.

Proof. Since u_f is a solution to $(-\Delta)^s u + m_q(u) = 0$ in Ω with $u_f - f \in \widetilde{H}^s(\Omega)$ and solutions are unique, we obtain $B_q(u_{f+\varphi}, g + \psi) = B_q(u_f, g)$ for $\varphi, \psi \in \widetilde{H}^s(\Omega)$. This implies that Λ_q is well-defined. Further, using continuity of $(-\Delta)^{s/2}$ and the norm estimate for solution u_f from lemma 5.1, we obtain

$$\begin{aligned}|\langle \Lambda_q[f], [g]\rangle| &\leq \left\|(-\Delta)^{s/2}u_f\right\|_{L^2(\mathbb{R}^n)}\left\|(-\Delta)^{s/2}g\right\|_{L^2(\mathbb{R}^n)} + \|q\|_{Z^{-s}(\mathbb{R}^n)}\|u_f\|_{H^s(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)} \\ &\leq C\|f\|_{H^s(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)},\end{aligned}$$

where C does not depend on f and g. By the definition of the quotient norm $|\langle \Lambda_q[f], [g]\rangle| \leq C\|[f]\|_X\|[g]\|_X$, so Λ_q is continuous. Choosing $g = u_g$ we obtain by symmetry of $B_q(\cdot,\cdot)$

$$\langle \Lambda_q[f], [g]\rangle = B_q(u_f, u_g) = \langle \Lambda_q[g], [u_f]\rangle = \langle [f], \Lambda_q[g]\rangle$$

where we used the fact that $u_f - f \in \widetilde{H}^s(\Omega)$. $\square$

We immediately obtain the Alessandrini identity from lemma 5.2 (see [75, Lemma 2.7]). We use some abuse of notation and write f instead of $[f]$.

Lemma 5.3 (Alessandrini identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $q_1, q_2 \in Z_0^{-s}(\mathbb{R}^n)$ which satisfy (4). For any $f_1, f_2 \in X$ one has*

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle = \langle m_{q_1 - q_2}(u_1), u_2 \rangle_{\mathbb{R}^n}$$

where $u_i \in H^s(\mathbb{R}^n)$ solves $(-\Delta)^s u_i + m_{q_i}(u_i) = 0$ in Ω with $u_i - f_i \in \widetilde{H}^s(\Omega)$.

Proof. Using the self-adjointness of Λ_q and the property $B_q(u_i, g+\psi) = B_q(u_i, g)$ for $\psi \in \widetilde{H}^s(\Omega)$, we obtain

$$\begin{aligned}
\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle &= \langle \Lambda_{q_1} f_1, f_2 \rangle - \langle f_1, \Lambda_{q_2} f_2 \rangle = B_{q_1}(u_1, f_2) - B_{q_2}(u_2, f_1) \\
&= B_{q_1}(u_1, f_2 + (u_2 - f_2)) - B_{q_2}(u_2, f_1 + (u_1 - f_1)) \\
&= B_{q_1}(u_1, u_2) - B_{q_2}(u_1, u_2) = \langle m_{q_1 - q_2}(u_1), u_2 \rangle_{\mathbb{R}^n}
\end{aligned}$$

which gives the claim. $\square$

From the UCP of $(-\Delta)^s$ (theorem 1.2), we obtain the following approximation result (see [75, Lemma 8.1]).

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $q \in Z_0^{-s}(\mathbb{R}^n)$ which satisfies (4). Denote by $P_q \colon X \to H^s(\mathbb{R}^n)$, $P_q f = u_f$, where $u_f \in H^s(\mathbb{R}^n)$ is the unique solution to $(-\Delta)^s u + m_q(u) = 0$ in Ω with $u - f \in \widetilde{H}^s(\Omega)$ given by lemma 5.1. Let $W \subset \Omega_e$ be any open set and define the set*

$$\mathcal{R} = \{P_q f - f : f \in C_c^\infty(W)\}.$$

Then $\mathcal{R}$ is dense in $\widetilde{H}^s(\Omega)$.

Proof. By the Hahn-Banach theorem it is enough to show that if $F \in (\widetilde{H}^s(\Omega))^*$ and $\langle F, v \rangle = 0$ for all $v \in \mathcal{R}$, then $F = 0$. Let $F \in (\widetilde{H}^s(\Omega))^*$ and assume that

$$\langle F, P_q f - f \rangle = 0, \quad f \in C_c^\infty(W).$$

Let $\varphi \in \widetilde{H}^s(\Omega)$ be the solution to

$$(-\Delta)^s \varphi + m_q(\varphi) = F \text{ in } \Omega, \quad \varphi|_{\Omega_e} = 0$$

which exists by lemma 5.1. This means that $B_q(\varphi, w) = \langle F, w \rangle$ for all $w \in \widetilde{H}^s(\Omega)$. Let $u_f = P_q f \in H^s(\mathbb{R}^n)$ where $u_f - f \in \widetilde{H}^s(\Omega)$. Now

$$\langle F, P_q f - f \rangle = B_q(\varphi, u_f - f) = -B_q(\varphi, f)$$

since u_f is a solution to $(-\Delta)^s u + m_q(u) = 0$ in Ω and $\varphi \in \widetilde{H}^s(\Omega)$. Thus $B_q(\varphi, f) = 0$ for all $f \in C_c^\infty(W)$. Using the fact that $\mathrm{spt}(\varphi)$ and $\mathrm{spt}(f)$ are disjoint, we obtain

$$0 = \left\langle (-\Delta)^{s/2} \varphi, (-\Delta)^{s/2} f \right\rangle_{\mathbb{R}^n} = \langle (-\Delta)^s \varphi, f \rangle_{\mathbb{R}^n}.$$

Here we used that $\langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v \rangle_{\mathbb{R}^n} = \langle (-\Delta)^s u, v \rangle_{\mathbb{R}^n}$ for $u, v \in \mathscr{S}(\mathbb{R}^n)$ and the equality holds also in $H^s(\mathbb{R}^n)$ by density. Hence $\varphi|_W = (-\Delta)^s \varphi|_W = 0$ and theorem 1.2 implies $\varphi = 0$ and eventually $F = 0$. $\square$

We remark that exactly the same proof gives the density of $r_\Omega \mathcal{R}$ in $L^2(\Omega)$ where r_Ω is the restriction to Ω (see [28, Lemma 5.1]). Now it is easy to prove theorems 1.6 and 1.7.

Proof of theorem 1.6. Since we can always shrink the sets W_i, we can assume without loss of generality that $\overline{W}_1 \cap \overline{W}_2 = \varnothing$ and $(\overline{W}_1 \cup \overline{W}_2) \cap \overline{\Omega} = \varnothing$. Using the Alessandrini identity (lemma 5.3), we obtain

$$\langle m_{q_1-q_2}(u_1), u_2 \rangle_{\mathbb{R}^n} = 0$$

for any $u_i \in H^s(\mathbb{R}^n)$ which solves $(-\Delta)^s u_i + m_{q_i}(u_i) = 0$ in Ω with exterior values in $C_c^\infty(W_i)$. Let $v_1, v_2 \in \widetilde{H}^s(\Omega)$. By lemma 5.4 there are sequences of exterior values $f_1^k \in C_c^\infty(W_1)$ and $f_2^k \in C_c^\infty(W_2)$ with sequences of solutions $u_1^k, u_2^k \in H^s(\mathbb{R}^n)$ such that

- $(-\Delta)^s u_i^k + m_{q_i}(u_i^k) = 0$ in Ω
- $u_i^k - f_i^k \in \widetilde{H}^s(\Omega)$
- $u_i^k = f_i^k + v_i + r_i^k$ where $r_i^k \xrightarrow{k\to\infty} 0$ in $\widetilde{H}^s(\Omega)$.

When we insert the solutions u_i^k into the Alessandrini identity, use the support conditions and take the limit $k \to \infty$, we obtain

$$\langle m_{q_1-q_2}(v_1), v_2 \rangle_{\mathbb{R}^n} = 0.$$

Let $\varphi \in C_c^\infty(\Omega)$. Choose $v_1 = \varphi$ and $v_2 \in C_c^\infty(\Omega)$ such that $v_2 = 1$ in a neighborhood of $\mathrm{spt}(\varphi)$. This implies

$$0 = \langle m_{q_1-q_2}(v_1), v_2 \rangle_{\mathbb{R}^n} = \langle q_1 - q_2, v_1 v_2 \rangle_{\mathbb{R}^n} = \langle q_1 - q_2, \varphi \rangle_{\mathbb{R}^n}$$

which is equivalent to that $q_1|_\Omega = q_2|_\Omega$ as distributions. □

Proof of theorem 1.7. Since $\mathrm{int}(\Omega_1 \setminus \Omega) \neq \varnothing$, there is open set $W \subset \Omega_e$ such that $\overline{W} \subset \Omega_1 \setminus \overline{\Omega}$. By lemma 5.4, the set $\mathcal{R}$ is dense in $\widetilde{H}^s(\Omega)$. Hence, we can approximate any $g \in \widetilde{H}^s(\Omega)$ arbitrarily well by solutions $u \in H^s(\mathbb{R}^n)$ to the equation $(-\Delta)^s u + m_q(u) = 0$ in Ω with $u - f \in \widetilde{H}^s(\Omega)$. Since $f \in C_c^\infty(W)$ we especially have $\mathrm{spt}(u) \subset \overline{\Omega}_1$. □

6. Higher order fractional magnetic Schrödinger equation

In this section we will be dealing with the definition of the FMSE, as well as with the proof of the injectivity result for the corresponding inverse problem. For the sake of simplicity, let us fix the convention throughout this section that the symbol $\langle \cdot, \cdot \rangle$ indicates both the scalar product (duality pairing) on $L^2(\mathbb{R}^n)$ and the one on $L^2(\mathbb{R}^{2n})$, the distinction between the two being always possible by checking the number of free variables inside the brackets. We also let the norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^s}$ etc. to denote the norms over the whole $\mathbb{R}^n$ or $\mathbb{R}^{2n}$ when the base set is not specified.

6.1. **High order bivariate functions.** Let $l, n \in \mathbb{N}$, and consider a family A of scalar two-point functions indexed over the set $\{1, ..., n\}^l$. A generic member of the family is determined by a vector $(i_1, ..., i_l)$ such that $i_j \in \{1, ..., n\}$ for all $j \in \{1, ..., l\}$, and it is a function $A_{i_1,...,i_l} : \mathbb{R}^{2n} \to \mathbb{R}$. We call such family A a *bivariate function of order l*. One can see the family A as a function $A : \mathbb{R}^{2n} \to \mathbb{M}^l$, where $\mathbb{M}^l$ is the set of all $n \times ... \times n = n^l$ arrays of real numbers, i.e. tensors of order l.

Let $a, b \in \mathbb{N}$, and let A, B be bivariate functions of orders a and b respectively, in the variables x_1, x_2. The *tensor product* of A and B is the bivariate function of order $a + b$ given by

$$(A \otimes B)_{i_1,...,i_a,j_1,...,j_b}(x_1, x_2) := A_{i_1,...,i_a}(x_1, x_2) B_{j_1,...,j_b}(x_1, x_2)\ .$$

In particular, for a vector $\xi \in \mathbb{R}^n$ we let $\xi^{\otimes 0} = 0$, $\xi^{\otimes 1} = \xi$ and recursively $\xi^{\otimes m} = \xi^{\otimes(m-1)} \otimes \xi$. Let A, B as before, but assume now that $a \geq b$. The *contraction* of A and B is the bivariate function of order $a - b$ given by

$$(A \cdot B)_{i_1,...,i_{a-b}}(x_1, x_2) := \sum_{j_1,...,j_b=1}^{n} A_{i_1,...,i_{a-b},j_1,...,j_b}(x_1, x_2) B_{j_1,...,j_b}(x_1, x_2)\ .$$

If $A = B$, then of course $a = b$, so that $A \cdot A$ is a scalar function of the variables (x_1, x_2). One sees that $|A| := (A \cdot A)^{1/2}$ defines a norm for fixed x_1 and x_2, and that this coincides with the usual one when A is a vector function.

Lemma 6.1. *Let $a, b \in \mathbb{N}$, and let A, B, v be bivariate functions of orders a, b and 1 respectively, in the variables x_1, x_2. Assume that $a \geq b+1$; then*

$$A \cdot (B \otimes v) = (A \cdot v) \cdot B \ .$$

Proof. The proof is just a simple computation:

$$\begin{aligned}
[A \cdot (B \otimes v)]_{i_1,\dots,i_{a-b-1}} &= \sum_{j_1,\dots,j_{b+1}=1}^{n} A_{i_1,\dots,i_{a-b-1},j_1,\dots,j_{b+1}} (B \otimes v)_{j_1,\dots,j_{b+1}} \\
&= \sum_{j_1,\dots,j_{b+1}=1}^{n} A_{i_1,\dots,i_{a-b-1},j_1,\dots,j_{b+1}} B_{j_1,\dots,j_b} v_{j_{b+1}} \\
&= \sum_{j_1,\dots,j_b=1}^{n} B_{j_1,\dots,j_b} \sum_{j_{b+1}=1}^{n} A_{i_1,\dots,i_{a-b-1},j_1,\dots,j_{b+1}} v_{j_{b+1}} \\
&= \sum_{j_1,\dots,j_b=1}^{n} B_{j_1,\dots,j_b} (A \cdot v)_{i_1,\dots,i_{a-b-1},j_1,\dots,j_b} \\
&= [(A \cdot v) \cdot B]_{i_1,\dots,i_{a-b-1}} \ .
\end{aligned}$$

□

Let A be a bivariate function of any order. Following [14], we recall the definitions of the *symmetric* and *antisymmetric parts of A with respect to the variables x and y* and the *L^2 norms of A with respect to the first and second variable at point x*:

$$A_s(x, y) := \frac{A(x, y) + A(y, x)}{2} \ , \qquad A_a(x, y) := A(x, y) - A_s(x, y) \ ,$$

$$\mathcal{J}_1 A(x) := \left(\int_{\mathbb{R}^n} |A(y, x)|^2 \, dy \right)^{1/2} \ , \quad \mathcal{J}_2 A(x) := \left(\int_{\mathbb{R}^n} |A(x, y)|^2 \, dy \right)^{1/2} \ .$$

It is easily seen that $A \in L^2$ implies $A_s, A_a \in L^2$, since

$$\|A_s\|_{L^2} = \left\| \frac{A(x, y) + A(y, x)}{2} \right\|_{L^2} \leq \|A\|_{L^2} \ , \quad \|A_a\|_{L^2} = \left\| \frac{A(x, y) - A(y, x)}{2} \right\|_{L^2} \leq \|A\|_{L^2} \ . \tag{17}$$

A bivariate function A of any order will be called *symmetric* if $A = A_s$ almost everywhere, and *antisymmetric* if $A = A_a$ almost everywhere.

Lemma 6.2. *Let $A \in L^1(\mathbb{R}^{2n}, \mathbb{M}^l)$ be an antisymmetric bivariate function of order l for some $l \in \mathbb{N}$. Then $\int_{\mathbb{R}^{2n}} A(x, y) \, dy dx = 0$.*

Proof. Let D^+, D^- be the sets respectively above and under the diagonal $D := \{(x, y) \in \mathbb{R}^{2n} : x = y\}$ of $\mathbb{R}^{2n}$. Since $\int_{D^\pm} A(x, y) \, dy dx \leq \int_{D^\pm} |A(x, y)| \, dy dx \leq \|A\|_{L^1} < \infty$, we can decompose the integral as $\int_{\mathbb{R}^{2n}} A(x, y) \, dy dx = \int_{D^+} A(x, y) \, dy dx + \int_{D^-} A(x, y) \, dy dx$. Given the symmetry of the sets D^+ and D^-, this can be rewritten as $\int_{\mathbb{R}^{2n}} A(x, y) \, dy dx = \int_{D^+} (A(x, y) + A(y, x)) \, dy dx$, which vanishes by virtue of the antisymmetry of A. □

6.2. **Fractional operators.** Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $u \in C_c^\infty(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$. Let $\lfloor s \rfloor := \sup\{n \in \mathbb{N} : n < s\}$ and $s' := s - \lfloor s \rfloor$, so that by definition $s' \in (0, 1)$. The *fractional gradient of order s of u at points x and y* is the following symmetric bivariate function of order $\lfloor s \rfloor + 1$:

$$\nabla^s u(x, y) := \frac{\mathcal{C}_{n,s'}^{1/2}}{\sqrt{2}} \frac{\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)}{|y - x|^{n/2 + s' + 1}} \otimes (y - x) \ .$$

Observe that this definition coincides with the usual one for $s \in (0,1)$, since in this case $\lfloor s \rfloor = 0$ and $s' = s$. One can compute

$$\begin{aligned}
\|\nabla^s u\|^2_{L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1})} &= \frac{\mathcal{C}_{n,s'}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\nabla^{\lfloor s\rfloor}u(x)-\nabla^{\lfloor s\rfloor}u(y)|^2}{|x-y|^{n+2s'}}dx\,dy \\
&= \frac{\mathcal{C}_{n,s'}}{2}[\nabla^{\lfloor s\rfloor}u]^2_{\dot{H}^{s'}(\mathbb{R}^n)} = \left\|(-\Delta)^{s'/2}\nabla^{\lfloor s\rfloor}u\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \||\xi|^{s'}\xi^{\otimes\lfloor s\rfloor}\hat{u}(\xi)\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^s\hat{u}(\xi)\|^2_{L^2(\mathbb{R}^n)} \\
&= \left\|(-\Delta)^{s/2}u\right\|^2_{L^2(\mathbb{R}^n)} \leq \|u\|^2_{H^s(\mathbb{R}^n)}\ .
\end{aligned}$$

Thus, by the density of C^∞_c in H^s, ∇^s can be extended to a continuous operator $\nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1})$. One sees by density that the formula given for $\nabla^s u$ in the case $u \in C^\infty_c(\mathbb{R}^n)$ still holds almost everywhere for $u \in H^s(\mathbb{R}^n)$. Thus if $u, v \in H^s$, by the above computation,

$$\langle\nabla^s u, \nabla^s u\rangle = \|(-\Delta)^{s/2}u\|^2_{L^2} = \langle(-\Delta)^{s/2}u, (-\Delta)^{s/2}u\rangle = \langle(-\Delta)^s u, u\rangle\ ,$$

so that by the polarization identity and the self-adjointness of $(-\Delta)^s$,

$$\begin{aligned}
\langle\nabla^s u, \nabla^s v\rangle &= \frac{\langle\nabla^s(u+v),\nabla^s(u+v)\rangle - \langle\nabla^s u,\nabla^s u\rangle - \langle\nabla^s v,\nabla^s v\rangle}{2} \\
&= \frac{\langle(-\Delta)^s(u+v),u+v\rangle - \langle(-\Delta)^s u,u\rangle - \langle(-\Delta)^s v,v\rangle}{2} \\
&= \frac{\langle(-\Delta)^s u,v\rangle + \langle(-\Delta)^s v,u\rangle}{2} = \langle(-\Delta)^s u,v\rangle\ .
\end{aligned}$$

This proves that if the *fractional divergence* $(\nabla\cdot)^s : L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1}) \to H^{-s}(\mathbb{R}^n)$ is defined as the adjoint of ∇^s, then weakly $(\nabla\cdot)^s\nabla^s = (-\Delta)^s$ for $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. This result was already proved in [15], but only for the case $s \in (0,1)$. If we define the antisymmetric bivariate vector function

$$\alpha(x,y) := \frac{\mathcal{C}^{1/2}_{n,s'}}{\sqrt{2}}\frac{y-x}{|y-x|^{n/2+s'+1}}$$

then for $u \in H^s$ the identity

$$\nabla^s u(x,y) = (\nabla^{\lfloor s\rfloor}u(x) - \nabla^{\lfloor s\rfloor}u(y)) \otimes \alpha(x,y)$$

holds almost everywhere.

We now define the magnetic versions of the above operators. Fix $p > \max\{1, n/2s\}$, and let A be a bivariate function of order $\lfloor s\rfloor + 1$ such that

(a1) $\mathcal{J}_2 A \in L^{2p}(\mathbb{R}^n)$
(a2) $\mathrm{spt}(A) \subset \Omega \times \Omega$.

With such choice of p, the embedding $H^s \times L^{2p} \hookrightarrow L^2$ always holds by [5, Theorem 6.1], recall that $W^r(\mathbb{R}^n) = H^r(\mathbb{R}^n)$ with equivalent norms when $r \in \mathbb{R}$ and $W^r(\mathbb{R}^n)$ is the L^2 Sobolev-Slobodecki space [5, 57]. Therefore, if $u \in H^s$,

$$\begin{aligned}
\|A(x,y)u(x)\|_{L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1})} &= \left(\int_{\mathbb{R}^n}|u(x)|^2\int_{\mathbb{R}^n}|A(x,y)|^2dy\,dx\right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n}|u(x)|^2\ |\mathcal{J}_2A(x)|^2\ dx\right)^{1/2} = \|u\,\mathcal{J}_2A\|_{L^2(\mathbb{R}^n)} \\
&\leq c\|u\|_{H^s}\|\mathcal{J}_2A\|_{L^{2p}} < \infty,
\end{aligned}$$

where c does not depend on u and A. This allows the definition of $\nabla^s_A u(x,y) := \nabla^s u(x,y) + A(x,y)u(x)$ and its adjoint $(\nabla\cdot)^s_A$ just as in [14], in such a way that $\nabla^s_A : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1})$ and $(\nabla\cdot)^s_A : L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1}) \to H^{-s}(\mathbb{R}^n)$. By definition, the *magnetic fractional Laplacian* $(-\Delta)^s_A : H^s \to H^{-s}$ will be the composition $(\nabla\cdot)^s_A\nabla^s_A$. Let now q be a scalar field such that (a3) $q \in L^p(\Omega)$.

By [5, Theorem 8.3] we have the embedding $H^s \times L^p \hookrightarrow H^{-s}$ and hence $qu \in H^{-s}$ holds for $u \in H^s$. We can thus define the magnetic Schrödinger operator $(-\Delta)^s_A + q : H^s \to H^{-s}$ and the fractional magnetic Schrödinger equation (FMSE)

$$(-\Delta)^s_A u + qu = 0 \ .$$

In the next Lemma we write $(-\Delta)^s_A$ in a more convenient form. To this scope, we introduce the bivariate function of order $\lfloor s \rfloor$ given by $S(x,y) := A(x,y) \cdot \alpha(x,y)$, for which we assume that

(a4) $|S(x,y)| \leq \tilde{S}(y)$ for a.e. $x, y \in \mathbb{R}^n$, with $\tilde{S} \in L^2$,
(a5) $S(x,y) \in H^{\lfloor s \rfloor}(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s \rfloor})$.

Remark 6.3. *Assumption (a4) is really relevant only when $\lfloor s \rfloor \neq 0$, as it will be clear from the proof; in the case $s \in (0,1)$, this assumption can be reduced. We refer to* [14] *for a set of sufficient conditions in that regime. Moreover, with a more careful analysis, one could reduce the exponent of the space to which $\tilde{S}$ belongs. However, we decided to keep L^2 for the sake of simplicity.*

Lemma 6.4. *Let A be a bivariate function of order $\lfloor s \rfloor + 1$ satisfying conditions (a1), (a2) (a4), (a5), and let $u \in H^s$. There exist linear operators $\mathfrak{N}, \mathfrak{M}_\beta$ acting on bivariate functions of order $\lfloor s \rfloor$, with β a multi-index of length $|\beta| \leq \lfloor s \rfloor$, such that the equation*

$$\begin{aligned}(-\Delta)^s_A u(x) =& (-\Delta)^s u(x) + \sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta u(x) (\mathfrak{M}_\beta(S))(x) + \\ &+ \int_{\mathbb{R}^n} u(y) (\mathfrak{N}(S))(x,y)\, dy + u(x) \int_{\mathbb{R}^n} |A(x,y)|^2 dy\end{aligned}$$

holds in weak sense.

Proof. If $v \in H^s$, then in weak sense

$$\langle (-\Delta)^s_A u, v \rangle = \langle \nabla^s u, \nabla^s v \rangle + \langle \nabla^s u, Av \rangle + \langle \nabla^s v, Au \rangle + \langle Au, Av \rangle \ , \tag{18}$$

where all the terms on the right hand side are finite, as observed above.

Step 1. Let us start by computing the third term on the right hand side of (18). The bivariate function $\nabla^s v(x,y)[A(x,y)u(x)]_a$ is antisymmetric, and by Cauchy-Schwartz and formula (17) we have $\|\nabla^s v\, (Au)_a\|_{L^1} \leq \|\nabla^s v\|_{L^2} \|(Au)_a\|_{L^2} \leq \|v\|_{H^s} \|Au\|_{L^2} < \infty$. Therefore Lemma 6.2 gives $\langle \nabla^s v, (Au)_a \rangle = 0$, and we can use Lemma 6.1 to write

$$\begin{aligned}\langle \nabla^s v, Au \rangle &= \langle \nabla^s v, Au \rangle - \langle \nabla^s v, (Au)_a \rangle = \langle \nabla^s v, (Au)_s \rangle \\ &= \langle (\nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y)) \otimes \alpha, (Au)_s \rangle \\ &= \langle \nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y), (Au)_s \cdot \alpha \rangle \\ &= \langle \nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y), (A \cdot \alpha u)_a \rangle \\ &= \langle \nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y), (Su)_a \rangle \ .\end{aligned} \tag{19}$$

The bivariate function $[\nabla^{\lfloor s \rfloor} v(x) + \nabla^{\lfloor s \rfloor} v(y)][S(x,y)u(x)]_a$ is antisymmetric, and we can estimate its L^1 norm by means of the triangle inequality as

$$\|(\nabla^{\lfloor s \rfloor} v(x) + \nabla^{\lfloor s \rfloor} v(y))(Su)_a\|_{L^1} \leq \|(\nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y))(Su)_a\|_{L^1} + \|2\nabla^{\lfloor s \rfloor} v(x)(Su)_a\|_{L^1} \ .$$

The first term on the right hand side equals $\|\nabla^s v\, (Au)_s\|_{L^1}$ by computation (19), so that it is finite by $\|\nabla^s v\, (Au)_s\|_{L^1} \leq \|\nabla^s v\|_{L^2} \|(Au)_s\|_{L^2} \leq \|v\|_{H^s} \|Au\|_{L^2} < \infty$. We estimate the other term again by triangular inequality as

$$\|2\nabla^{\lfloor s \rfloor} v(x)(Su)_a\|_{L^1} \leq \|\nabla^{\lfloor s \rfloor} v(x) S(x,y) u(x)\|_{L^1} + \|\nabla^{\lfloor s \rfloor} v(x) S(y,x) u(y)\|_{L^1} \ . \tag{20}$$

The estimation of the second term on the right hand side of (20) can be done as follows, and similarly for the other one:

$$
\begin{aligned}
\|\nabla^{\lfloor s\rfloor}v(x)S(y,x)u(y)\|_{L^1} &= \int_{\mathbb{R}^n}|\nabla^{\lfloor s\rfloor}v(x)|\int_{\mathbb{R}^n}|S(y,x)|\,|u(y)|\,dydx \\
&\leq \int_{\mathbb{R}^n}|\nabla^{\lfloor s\rfloor}v(x)|\tilde{S}(x)\int_{\Omega}|u(y)|\,dydx \\
&\leq c\|u\|_{L^2}\int_{\mathbb{R}^n}|\nabla^{\lfloor s\rfloor}v(x)|\tilde{S}(x)dx \\
&\leq c\|u\|_{L^2}\|\nabla^{\lfloor s\rfloor}v(x)\|_{L^2}\|\tilde{S}\|_{L^2} \\
&\leq c\|u\|_{H^s}\|v\|_{H^s}\|\tilde{S}\|_{L^2}<\infty,
\end{aligned} \tag{21}
$$

where the constant c can change from line to line and does not depend on v,u and S.
Thus we have proved that $\|2\nabla^{\lfloor s\rfloor}v(x)(Su)_a\|_{L^1}<\infty$, which in turn implies that $\|(\nabla^{\lfloor s\rfloor}v(x)+\nabla^{\lfloor s\rfloor}v(y))(Su)_a\|_{L^1}<\infty$. Now we can use again Lemma 6.2 to conclude that $\langle\nabla^{\lfloor s\rfloor}v(x)+\nabla^{\lfloor s\rfloor}v(y),(Su)_a\rangle=0$. From this fact and formula (19), integrating by parts,

$$
\begin{aligned}
\langle\nabla^s v,Au\rangle &= \langle\nabla^{\lfloor s\rfloor}v(x)-\nabla^{\lfloor s\rfloor}v(y),(Su)_a\rangle+\langle\nabla^{\lfloor s\rfloor}v(x)+\nabla^{\lfloor s\rfloor}v(y),(Su)_a\rangle \\
&= 2\langle\nabla^{\lfloor s\rfloor}v(x),(Su)_a\rangle=\langle\nabla^{\lfloor s\rfloor}v(x),S(x,y)u(x)-S(y,x)u(y)\rangle \\
&= \langle\nabla^{\lfloor s\rfloor}v(x),S(x,y)u(x)\rangle-\langle\nabla^{\lfloor s\rfloor}v(x),S(y,x)u(y)\rangle \\
&= (-1)^{\lfloor s\rfloor}\left\langle v,(\nabla\cdot)^{\lfloor s\rfloor}_x\left(u(x)\int_{\mathbb{R}^n}S(x,y)dy\right)\right\rangle \\
&\quad -(-1)^{\lfloor s\rfloor}\left\langle v,(\nabla\cdot)^{\lfloor s\rfloor}_x\int_{\mathbb{R}^n}S(y,x)u(y)dy\right\rangle .
\end{aligned}
$$

In the last term the derivatives can pass under the integral sign by means of the dominated convergence theorem, since $|S(x,y)u(y)|\leq\tilde{S}(y)|u(y)|$, and $\int_{\mathbb{R}^n}\tilde{S}(y)|u(y)|dy\leq\|\tilde{S}\|_{L^2}\|u\|_{L^2}<\infty$. Eventually,

$$
\begin{aligned}
\langle\nabla^s v,Au\rangle &= (-1)^{\lfloor s\rfloor}\left\langle v,(\nabla\cdot)^{\lfloor s\rfloor}_x\left(u(x)\int_{\mathbb{R}^n}S(x,y)dy\right)\right\rangle \\
&\quad +(-1)^{\lfloor s\rfloor+1}\left\langle v,\int_{\mathbb{R}^n}u(y)(\nabla\cdot)^{\lfloor s\rfloor}_x S(y,x)dy\right\rangle .
\end{aligned} \tag{22}
$$

Step 2. Next we compute the second term on the right hand side of (18). With a computation similar to (19), we obtain $\langle\nabla^s u,Av\rangle=\langle\nabla^{\lfloor s\rfloor}u(x)-\nabla^{\lfloor s\rfloor}u(y),S(x,y)v(x)\rangle$; moreover, we have estimates similar to the ones in (21), and so we can split the integral. Eventually, we integrate by parts and get

$$
\begin{aligned}
\langle\nabla^s u,Av\rangle &= \langle\nabla^{\lfloor s\rfloor}u(x),S(x,y)v(x)\rangle-\langle\nabla^{\lfloor s\rfloor}u(y),S(x,y)v(x)\rangle \\
&= \left\langle v(x),\nabla^{\lfloor s\rfloor}u(x)\cdot\int_{\mathbb{R}^n}S(x,y)dy\right\rangle-\left\langle v(x),\int_{\mathbb{R}^n}\nabla^{\lfloor s\rfloor}u(y)\cdot S(x,y)dy\right\rangle \\
&= \left\langle v(x),\nabla^{\lfloor s\rfloor}u(x)\cdot\int_{\mathbb{R}^n}S(x,y)dy\right\rangle+(-1)^{\lfloor s\rfloor+1}\left\langle v(x),\int_{\mathbb{R}^n}u(y)(\nabla\cdot)^{\lfloor s\rfloor}_y S(x,y)dy\right\rangle .
\end{aligned} \tag{23}
$$

Step 3. The properties $\langle(-\Delta)^s u,v\rangle=\langle\nabla^s u,\nabla^s v\rangle$ and $\langle Au,Av\rangle=\left\langle v,u\int_{\mathbb{R}^n}|A(x,y)|^2dy\right\rangle$ hold, as proved in [14]. Using this information and formulas (22), (23) we can write the fractional magnetic Schrödinger operator as

$$
\begin{aligned}
&\langle(-\Delta)^s u,v\rangle+\left\langle\nabla^{\lfloor s\rfloor}u(x)\cdot\int_{\mathbb{R}^n}S(x,y)dy+(-1)^{\lfloor s\rfloor}(\nabla\cdot)^{\lfloor s\rfloor}_x\left(u(x)\int_{\mathbb{R}^n}S(x,y)dy\right),v\right\rangle+ \\
&+(-1)^{\lfloor s\rfloor+1}\left\langle\int_{\mathbb{R}^n}u(y)\left((\nabla\cdot)^{\lfloor s\rfloor}_x S(y,x)+(\nabla\cdot)^{\lfloor s\rfloor}_y S(x,y)\right)dy,v\right\rangle+\left\langle u\int_{\mathbb{R}^n}|A|^2dy,v\right\rangle .
\end{aligned}
$$

Let us compute the left hand side of the second bracket and collect the resulting terms according to the order of their derivatives of u. For every multi-index β such that $|\beta| \leq \lfloor s \rfloor$ we can find a linear operator $\mathfrak{M}_\beta$ such that

$$\nabla^{\lfloor s \rfloor} u(x) \cdot \int_{\mathbb{R}^n} S(x,y)dy + (-1)^{\lfloor s \rfloor} (\nabla\cdot)^{\lfloor s \rfloor}_x \left(u(x) \int_{\mathbb{R}^n} S(x,y)dy \right) = \sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta u(x) \mathfrak{M}_\beta(S) \ .$$

We can also define the following linear operator:

$$\mathfrak{N}(S) = (-1)^{\lfloor s \rfloor + 1} \left((\nabla\cdot)^{\lfloor s \rfloor}_x S(y,x) + (\nabla\cdot)^{\lfloor s \rfloor}_y S(x,y) \right) \ .$$

With these new definitions, we can rewrite the fractional magnetic Schrödinger operator as in the statement of the Lemma. □

6.3. **The bilinear form and the DN map.** For every $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $u, v \in H^s$ we define the bilinear form $B^s_{A,q} : H^s \times H^s \to \mathbb{R}$ as in [14]:

$$B^s_{A,q}(u,v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla^s_A u \cdot \nabla^s_A v \, dydx + \int_{\mathbb{R}^n} quv \, dx \ .$$

Lemma 6.5. *There are constants $\mu', k' > 0$ such that, for all $u \in H^s$,*

$$B^s_{A,q}(u,u) + \mu' \langle u, u \rangle \geq k' \|u\|^2_{H^s} \ .$$

Proof. The formula we want to prove is called *coercivity estimate*. Using (18), we can write

$$\begin{aligned} B^s_{A,q}(u,u) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla^s_A u \cdot \nabla^s_A u \, dydx + \int_{\mathbb{R}^n} qu^2 \, dx \\ &= \int_{\mathbb{R}^n} u(-\Delta)^s_A u \, dx + \int_{\mathbb{R}^n} qu^2 \, dx = \langle (-\Delta)^s_A u, u \rangle + \langle qu, u \rangle \\ &= \langle (-\Delta)^s u, u \rangle + 2 \langle \nabla^s u, Au \rangle + \left\langle \left(q + \int_{\mathbb{R}^n} |A(x,y)|^2 dy \right) u, u \right\rangle \\ &= \langle (-\Delta)^s u, u \rangle + 2 \left\langle \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy, u \right\rangle + \langle Qu, u \rangle \ , \end{aligned} \tag{24}$$

where $Q(x) := q(x) + \int_{\mathbb{R}^n} |A(x,y)|^2 dy$ belongs to L^p since Cauchy-Schwartz and assumptions (a1) and (a3) imply the embedding $L^{2p} \times L^{2p} \hookrightarrow L^p$. Since we always have $L^p \times H^s \hookrightarrow H^{-s}$, we get $\langle Qu, u \rangle \leq \|u\|_{H^s} \|Qu\|_{H^{-s}} \leq \|Q\|_{L^p} \|u\|^2_{H^s}$. For the second term on the right hand side of (24) we first perform an estimate by means of the Young inequality

$$2 \left\langle \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy, u \right\rangle \leq \epsilon^{-1} \|u\|^2_{L^2} + \epsilon \left\| \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy \right\|^2_{L^2} \ ,$$

then estimate the second term with the Cauchy-Schwartz inequality, in light of (a4):

$$\begin{aligned} \epsilon \left\| \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy \right\|^2_{L^2} &= \epsilon \left\| \int_{\mathbb{R}^n} \left((\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)) \otimes \alpha \right) \cdot A \, dy \right\|^2_{L^2} \\ &= \epsilon \left\| \int_{\mathbb{R}^n} (\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)) \cdot (A \cdot \alpha) \, dy \right\|^2_{L^2} \\ &= \epsilon \left\| \int_\Omega (\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)) \cdot S(x,y) \, dy \right\|^2_{L^2(\Omega)} \\ &\leq \epsilon \left\| \left(\int_\Omega |\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)|^2 dy \right)^{1/2} \left(\int_\Omega |S(x,y)|^2 \, dy \right)^{1/2} \right\|^2_{L^2(\Omega)} \\ &= \epsilon \int_\Omega \left(\int_\Omega |\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)|^2 dy \int_\Omega |S(x,y)|^2 \, dy \right) dx \end{aligned}$$

$$\leq \epsilon \int_\Omega \left(\int_\Omega (|\nabla^{\lfloor s \rfloor} u(x)| + |\nabla^{\lfloor s \rfloor} u(y)|)^2 dy \int_\Omega \tilde{S}^2(y)\, dy \right) dx$$
$$= \epsilon \|\tilde{S}\|^2_{L^2(\Omega)} \int_\Omega \int_\Omega (|\nabla^{\lfloor s \rfloor} u(x)| + |\nabla^{\lfloor s \rfloor} u(y)|)^2 dy dx$$
$$\leq 2\epsilon \|\tilde{S}\|^2_{L^2(\Omega)} \int_\Omega \int_\Omega (|\nabla^{\lfloor s \rfloor} u(x)|^2 + |\nabla^{\lfloor s \rfloor} u(y)|^2) dy dx$$
$$\leq 4|\Omega|\epsilon \|\tilde{S}\|^2_{L^2(\Omega)} \|\nabla^{\lfloor s \rfloor} u\|^2_{L^2} \leq c\,\epsilon \|\nabla^{\lfloor s \rfloor} u\|^2_{H^{s'}} \leq c\,\epsilon \|u\|^2_{H^s},$$

where the constant c can change from line to line and does not depend on u.
Eventually

$$2\left\langle \int_{\mathbb{R}^n} \nabla^s u \cdot A\, dy, u \right\rangle \leq \epsilon^{-1} \|u\|^2_{L^2} + c\,\epsilon \|u\|^2_{H^s},$$

which leads to

$$B^s_{A,q}(u,u) \geq B^s_{0,Q}(u,u) - \epsilon^{-1}\|u\|^2_{L^2} - c\,\epsilon\|u\|^2_{H^s}\ . \tag{25}$$

Since $C^\infty_c(\Omega)$ is dense in $L^p(\Omega)$, for every $\delta > 0$ we can find functions Q_s, Q_r such that $Q_s \in C^\infty_c(\Omega)$, $\|Q_r\|_{L^p(\Omega)} \leq \delta$ and $Q = Q_s + Q_r$. Also, if $\phi_j \in C^\infty_c(\Omega)$ and $\|\phi_j\|_{H^s} = 1$ for $j = 1, 2$, then $|\langle Q_r\phi_1, \phi_2\rangle| \leq c\|\phi_1\|_{H^s}\|\phi_2\|_{H^s}\|Q_r\|_{L^p} \leq c\delta$ by the embedding $L^p \times H^s \hookrightarrow H^{-s}$. Therefore,

$$\|Q_r\|_{Z^{-s}} = \sup_{\|\phi_j\|_{H^s}=1} \{|\langle Q_r\phi_1, \phi_2\rangle|\} \leq c\delta\ ,$$

which means that Q belongs to the closure of $C^\infty_c(\Omega)$ in $Z^{-s}(\mathbb{R}^n)$, that is $Q \in Z^{-s}_0(\mathbb{R}^n)$. Now by Lemma 5.1 we know the coercivity estimate for the non-magnetic high exponent case; this lets us write (25) as

$$B^s_{A,q}(u,u) + (\mu + \epsilon^{-1})\langle u, u\rangle \geq (k - c\,\epsilon)\|u\|^2_{H^s}\ ,$$

which is the coercivity estimate for $B^s_{A,q}$ as soon as ϵ is fixed small enough and $\mu' := \mu + \epsilon^{-1}$, $k' := k - c\,\epsilon$ are defined. □

By means of the lemma above, if we assume 0 is not an eigenvalue for the equation, we can proceed as in the proof of Lemma 2.6 from [75] and get the well-posedness of the direct problem for FMSE. This can be stated as follows: if $F \in (\widetilde{H}^s(\Omega))^*$, there exists unique solution $u \in H^s(\mathbb{R}^n)$ to $B^s_{A,q}(u,v) = F(v)$ for all $v \in \widetilde{H}^s(\Omega)$, i.e. unique $u \in H^s(\mathbb{R}^n)$ such that $(-\Delta)^s_A u + qu = F$ in Ω, $u|_{\Omega_e} = 0$. This is also true for non-vanishing exterior value $f \in H^s(\mathbb{R}^n)$ (see [15] and [28]), and the following estimate holds:

$$\|u\|_{H^s(\mathbb{R}^n)} \leq c(\|F\|_{(\widetilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)}), \tag{26}$$

where c does not depend on F and f.
One can prove (see Lemma 3.11 from [14]) that $B^s_{A,q}$ also enjoys these properties:

(1) $B^s_{A,q}(v,w) = B^s_{A,q}(w,v)$, for all $v, w \in H^s$,
(2) $|B^s_{A,q}(v,w)| \leq c\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}$ for all $v, w \in H^s$, where c does not depend on v and w.
(3) $B^s_{A,q}(u_1, e_2) = B^s_{A,q}(u_2, e_1)$, for $u_j \in H^s$ solution to the direct problem for FMSE with exterior value $f_j \in H^s(\Omega_e)$ and e_j any extension of f_j to H^s, $j = 1, 2$.

Lemma 6.6. *Let $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ be the abstract quotient space, and let $u_1 \in H^s$ be the solution to the direct problem for FMSE with exterior value $f_1 \in H^s(\Omega_e)$. Then*

$$\langle \Lambda^s_{A,q}[f_1], [f_2]\rangle = B^s_{A,q}(u_1, f_2), \qquad f_j \in H^s,\ j = 1, 2$$

defines a bounded, linear, self-adjoint map $\Lambda^s_{A,q} : X \to X^*$. We call $\Lambda^s_{A,q}$ the DN map.

Proof. The proof follows trivially from properties (1)-(3) of $B^s_{A,q}$ and (26). □

6.4. **The gauge.** Consider two couples of potentials (A_1, q_1) and (A_2, q_2). We say that $(A_1, q_1) \sim (A_2, q_2)$ if and only if the following conditions are met:

- $\mathfrak{N}(S_1 - S_2) = 0$ for almost every $x, y \in \mathbb{R}^n$
- $\mathfrak{M}_{(0,\dots,0)}(S_1 - S_2) + \int_{\mathbb{R}^n}(|A_1|^2 - |A_2|^2)dy + (q_1 - q_2) = 0$ for almost every $x \in \mathbb{R}^n$
- $\mathfrak{M}_\beta(S_1 - S_2) = 0$ for all $1 \leq |\beta| \leq \lfloor s \rfloor$ and almost every $x \in \mathbb{R}^n$.

It is clear from the linearity of $\mathfrak{N}$ and $\mathfrak{M}_\alpha$ that $\sim$ is an equivalence relation, and so the set of all couples of potentials is divided into equivalence classes by $\sim$. We call these *gauge classes*, and if $(A_1, q_1) \sim (A_2, q_2)$ we say that (A_1, q_1) and (A_2, q_2) *are in gauge*.

Observe that this gauge $\sim$ coincides with the one defined in [14] if $s \in (0, 1)$, although it looks quite different. Since in this case $\lfloor s \rfloor = 0$, there is no third condition. In the language of that paper, the first condition reads

$$\begin{aligned} 0 &= -\mathfrak{N}(S_1 - S_2) = S_1(y, x) + S_1(x, y) - S_2(y, x) - S_2(x, y) \\ &= (A_1(x, y) - A_2(x, y)) \cdot \alpha(x, y) + (A_1(y, x) - A_2(y, x)) \cdot \alpha(y, x) \\ &= (A_1(x, y) - A_1(y, x) - A_2(x, y) + A_2(y, x)) \cdot \alpha(x, y) \\ &= 2(A_1 - A_2)_a \cdot \alpha = 2(A_1 - A_2)_{a\|} \cdot \alpha \ , \end{aligned}$$

which is equivalent to $(A_1)_{a\|} = (A_2)_{a\|}$, since the two vectors in the last scalar product have the same direction. Given this fact, for any $v \in H^s$ the first term in the second condition weakly is

$$\begin{aligned} \langle \mathfrak{M}_{(0,\dots,0)}(S_1 - S_2), v \rangle &= 2\langle S_1 - S_2, v \rangle = 2\langle \alpha \cdot (A_1 - A_2), v \rangle = 2\langle \alpha \cdot (A_1 - A_2)_\|, v \rangle \\ &= 2\langle \alpha \cdot (A_1 - A_2)_{s\|}, v \rangle = 2\langle \alpha v, (A_1 - A_2)_{s\|} \rangle = 2\langle (\alpha v)_s, (A_1 - A_2)_{s\|} \rangle \\ &= \langle \alpha(x, y)v(x) + \alpha(y, x)v(y), (A_1 - A_2)_{s\|} \rangle \\ &= \langle \alpha(x, y)(v(x) - v(y)), (A_1 - A_2)_{s\|} \rangle \\ &= \langle \nabla^s v, (A_1 - A_2)_{s\|} \rangle = \langle v, (\nabla\cdot)^s((A_1 - A_2)_{s\|}) \rangle \ , \end{aligned}$$

which lets us rewrite the second condition as

$$(\nabla\cdot)^s(A_1)_{s\|} + \int_{\mathbb{R}^n} |A_1|^2 dy + q_1 = (\nabla\cdot)^s(A_2)_{s\|} + \int_{\mathbb{R}^n} |A_2|^2 dy + q_2 \ .$$

Remark 6.7. *Observe that the gauge enjoyed by the FMSE is quite different from the one holding for the MSE. For the sake of simplicity, we shall compare the classical case with the fractional one in the regime $s \in (0, 1)$, following section 3 in* [14].

Given lemma 6.4, one sees that the following is an equivalent definition for the gauge $\sim$ above:

$$(A_1, q_1) \sim (A_2, q_2) \quad \Leftrightarrow \quad (-\Delta)^s_{A_1} u + q_1 u = (-\Delta)^s_{A_2} u + q_2 u \ ,$$

for all $u \in H^s(\mathbb{R}^n)$. One may also define the accessory gauge $\approx$ as

$$(A_1, q_1) \approx (A_2, q_2) \quad \Leftrightarrow \quad \exists \phi \in G : (-\Delta)^s_{A_1}(u\phi) + q_1 u\phi = \phi((-\Delta)^s_{A_2} u + q_2 u) \ ,$$

for all $u \in H^s(\mathbb{R}^n)$, where $G := \{\phi \in C^\infty(\mathbb{R}^n) : \phi > 0, \phi|_{\Omega_e} = 1\}$. These definitions can be extended to the MSE in the natural way. It was proved in lemmas 3.9 and 3.10 of [14] *that the FMSE enjoys the gauge $\sim$, but not $\approx$. In the same discussion, it was argued that the opposite holds for MSE. The reason for this surprising discrepancy should be looked for in the nonlocal structure of the FMSE. As apparent in formula (10) in* [14]*, the coefficient of the gradient term in FMSE is not related to the whole vector potential A itself, but only to its antisymmetric part A_a. It is such antisymmetry requirement what eventually does not allow the FMSE to enjoy $\approx$ as the MSE. As a result, the scalar potential q can not be in general uniquely determined as in the classical case.*

6.5. **Main result.**

Remark 6.8. *Assume $W \subseteq \Omega_e$ is an open set and $u \in H^s$ satisfies $u = 0$ and $(-\Delta)^s_A u + qu = 0$ in W. We say that the fractional magnetic Schrödinger operator enjoys the weak unique continuation property (WUCP) if we can deduce that $u = 0$ in Ω. This was proved in* [14] *by using the UCP of the fractional Laplacian for $s \in (0, 1)$; since we know by Theorem 1.2 that*

UCP still holds for $(-\Delta)^s$ in the regime $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, we can deduce WUCP for $(-\Delta)^s_A + q$ by the same proof.

Proof of theorem 1.8. **Step 1.** Without loss of generality, let $W_1 \cap W_2 = \emptyset$. Let $f_i \in C_c^\infty(W_i)$, and let $u_i \in H^s(\mathbb{R}^n)$ solve $(-\Delta)^s_{A_i} u_i + q_i u_i = 0$ with $u_i - f_i \in \widetilde{H}^s(\Omega)$ for $i = 1, 2$. Knowing that the DN maps computed on $f \in C_c^\infty(W_1)$ coincide when restricted to W_2, using Lemmas 6.4 and 6.6 we write this integral identity

$$\begin{aligned}
0 &= \langle (\Lambda^s_{A_1,q_1} - \Lambda^s_{A_2,q_2}) f_1, f_2 \rangle = B^s_{A_1,q_1}(u_1, u_2) - B^s_{A_2,q_2}(u_1, u_2) \\
&= \left\langle u_2, \sum_{|\beta| \le \lfloor s \rfloor} \partial^\beta u_1 \mathfrak{M}_\beta (S_1 - S_2) \right\rangle + \left\langle u_2, \int_{\mathbb{R}^n} u_1(y) \mathfrak{N}(S_1 - S_2)\, dy \right\rangle + \\
&\quad + \left\langle u_2, u_1 \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right) \right\rangle .
\end{aligned}$$

Since if $x \notin \Omega$ or $y \notin \Omega$ we have $A_1(x, y) = A_2(x, y)$ and $q_1(x) = q_2(x)$, we can restrict u_1, u_2 and $\partial^\beta u_1$ over Ω in the previous formula; it is also true that $(\partial^\beta u_1)|_\Omega = \partial^\beta (u_1|_\Omega)$, and therefore

$$\begin{aligned}
0 &= \left\langle u_2|_\Omega, \sum_{|\beta| \le \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) \right\rangle + \left\langle u_2|_\Omega, \int_{\mathbb{R}^n} u_1|_\Omega(y) \mathfrak{N}(S_1 - S_2)\, dy \right\rangle + \\
&\quad + \left\langle u_2|_\Omega, u_1|_\Omega \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right) \right\rangle .
\end{aligned}$$

This is the Alessandrini identity, which now we will test with certain solutions in order to obtain information about the potentials. The appropriate test solutions will be produced by means of the Runge approximation property (RAP), which holds for the FMSE because of Remark 6.8 and Lemma 3.15 in [14]. This property says that the set $\mathcal{R} = \{u_f|_\Omega : f \in C_c^\infty(W)\} \subset L^2(\Omega)$ of the restrictions to Ω of those functions u_f solving FMSE for some smooth exterior value f supported in W is dense in $L^2(\Omega)$.

Step 2. Given any $f \in L^2(\Omega)$, by the RAP we can find a sequence of solutions $(u_2)_k \to f$ in L^2 sense as $k \to \infty$. Substituting these in the Alessandrini identity and taking limits, by the arbitrarity of f we can deduce that

$$\begin{aligned}
0 &= \sum_{|\beta| \le \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) + \int_{\mathbb{R}^n} u_1|_\Omega(y) \mathfrak{N}(S_1 - S_2)\, dy + \\
&\quad + u_1|_\Omega \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right)
\end{aligned}$$

holds for every solution $u_1 \in H^s$ and almost every point $x \in \Omega$. Fix $x \in \Omega$. Consider now any $\psi \in C_c^\infty(\Omega)$ and let $g(y) := e^{-1/|x-y|}\psi(y)$, $g(x) = 0$. Since $e^{-1/|x-y|}$ is smooth, it is easy to see that $g \in C_c^\infty(\Omega) \subset L^2(\Omega)$; also, by the properties of $e^{-1/|x-y|}$ one has that $\partial^\beta g(x) = 0$ for all multi-indices β. By the RAP we can find a sequence of solutions $(u_1)_k \to g$ in L^2 sense as $k \to \infty$. Substituting these in the above identity and taking limits, we get

$$\int_{\mathbb{R}^n} e^{-1/|x-y|} \psi(y) \mathfrak{N}(S_1 - S_2)\, dy = 0 \ ,$$

which by the arbitrarity of ψ and the positivity of the exponential now implies $\mathfrak{N}(S_1 - S_2) = 0$ for almost all $x, y \in \Omega$. We can now return to the above equation with this new information: for every solution $u_1 \in H^s$ and almost every $x \in \Omega$,

$$0 = \sum_{|\beta| \le \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) + u_1|_\Omega \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right) .$$

For every multi-index β we can consider the function $h_\beta(x) = x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$, which belongs to $L^2(\Omega)$. Let $(h_\beta)_k$ be a sequence of solutions approximating h_β in L^2, which exists by the RAP. We will first substitute $(h_{(0,\dots,0)})_k$ into the last formula, take limits and deduce

$$\mathfrak{M}_{(0,\dots,0)}(S_1 - S_2) + \int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2)dy + (q_1 - q_2) = 0 \ ,$$

which has the effect of reducing the equation to

$$\sum_{1 \leq |\beta| \leq \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) = 0.$$

If $\lfloor s \rfloor \geq 1$, we will repeat the last steps with each h_β such that $|\beta| = 1$, deducing $\mathfrak{M}_\beta(S_1 - S_2) = 0$ for every such β, and subsequently

$$\sum_{2 \leq |\beta| \leq \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) = 0.$$

Repeating this process for a total of $\lfloor s \rfloor$ times eventually leads to

$$\mathfrak{M}_\beta (S_1 - S_2) = 0 \quad \forall \ 1 \leq |\beta| \leq \lfloor s \rfloor \ ,$$

which proves the theorem by the definition of the gauge $\sim$. □

7. Possible generalizations and applications beyond this article

We discuss some possible directions for the future research on higher order fractional inverse problems, fractional Poincaré inequalities and unique continuation properties. It seems that now it would be the most natural to reconsider many of the recent developments in fractional inverse problems for higher order operators. We outline here some problems which we would like to see solved in the future.

We have split this section in three in order to emphasize some open problems which we find especially interesting. We do not claim that answers to all questions are positive and it would be interesting to see why and where the greatest difficulties, or even counterexamples, would show up. We first list the most natural directions to continue our work on higher order fractional Calderón problems. One could study for example the following cases:

(i) Is reconstruction from a single measurement [15, 27] possible also in the higher order cases?
(ii) Is there stability [75] in the higher order cases?
(iii) Is there exponential instability [73] in the higher order cases?
(iv) Is there uniqueness for the Calderón problem for fractional semilinear Schrödinger equations [47, 48] in the higher order cases?
(v) Do the monotonicity methods [33, 34] extend to the higher order cases?
(vi) Is there uniqueness for the conductivity type fractional Calderón problems [10, 15] in the higher order cases?
(vii) Could recent results on fractional heat equations [49, 74] be generalized to the higher order cases?
(viii) Does the higher regularity Runge approximation in [11, 28] generalize to higher order cases?

7.1. Unique continuation problems.

We state here some unique continuation problems, which do not follow directly from the earlier results and the techniques that we have developed for this article.

Question 7.1 (UCP for Bessel potentials). *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $p \in [1, \infty)$ and $r \in \mathbb{R}$. Let $V \subset \mathbb{R}^n$ be an open set. Suppose that $f \in H^{r,p}(\mathbb{R}^n)$, $f|_V = 0$ and $(-\Delta)^s f|_V = 0$. Show that $f \equiv 0$ or give a counterexample.*

The positive answer to question 7.1 is known when $p \in [1, 2]$ (see corollary 3.5). If f has compact support, then the answer is positive for all $p \in [1, \infty)$ (see corollary 3.3). Question 7.1 is also open for the exponents $s \in (0, 1)$ when $p > 2$. See section 3.1 for details.

Question 7.2 (Measurable UCP)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $r \in \mathbb{R}$. Let $V \subset \mathbb{R}^n$ be an open set and $E \subset V$ a measurable set with positive measure. Suppose that $f \in H^r(\mathbb{R}^n)$, $f|_E = 0$ and $(-\Delta)^s f|_V = 0$. Show that $f \equiv 0$ or give a counterexample.*

The positive answer to question 7.2 is known when $s \in (0, 1)$ [27]. Question 7.2 with a potential q from a suitable class of functions is also an interesting and more challenging problem. See [27, Proposition 5.1] for more details.

Question 7.3 (Alternative strong UCP)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $r \in \mathbb{R}$. Let $V \subset \mathbb{R}^n$ be an open set. Suppose that $f \in H^r(\mathbb{R}^n)$, $f|_V = 0$ and $\partial^\beta((-\Delta)^s f)(x_0) = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}_0^n$. Show that $f \equiv 0$ or give a counterexample.*

Question 7.3 can be seen as a version of the strong unique continuation property (see e.g. [22, 26, 72]) with interchanged decay conditions. When f has compact support, the answer to question 7.3 is positive for $s \in (-n/2, \infty) \setminus \mathbb{Z}$ (see corollary 3.3).

The problems posed in questions 7.1–7.3 for the fractional Laplacian are interesting mathematical problems on their own right, but they also have important applications in inverse problems. The UCPs can be used to show Runge approximation properties for nonlocal equations such as the fractional Schrödinger equation (see theorem 1.7), which in turn can be used to show uniqueness for the corresponding nonlocal inverse problem (see theorem 1.6). The UCPs have also applications in integral geometry, where the uniqueness of the ROI problem for the d-plane transform can be reduced to a unique continuation problem for the fractional Laplacian (see remark 4.1 and corollaries 1.3 and 1.4).

7.2. **Fractional Poincaré inequality for L^p-norms.** In section 3.2 we prove the fractional Poincaré inequality for L^2-norms in multiple ways. The inequality is needed for the well-posedness of the inverse problem for the fractional Schrödinger equation. One could try to extend the Poincaré inequality for general L^p-norms. This suggests the following natural question which is also interesting from the pure mathematical point of view.

Question 7.4. *Let $s \geq 0$, $1 \leq p < \infty$, $K \subset \mathbb{R}^n$ compact set and $u \in H^{s,p}(\mathbb{R}^n)$ such that* $\mathrm{spt}(u) \subset K$*. Show that there exists a constant $c = c(n, K, s, p)$ such that*

$$\|u\|_{L^p(\mathbb{R}^n)} \leq c \left\| (-\Delta)^{s/2} u \right\|_{L^p(\mathbb{R}^n)} \tag{27}$$

or give a counterexample.

Since we have presented several proofs for the Poincaré inequality in the case $p = 2$, one could try some of our methods to solve question 7.4. However, some of our proofs are heavily based on Fourier analysis and those approaches might be difficult to generalize to the L^p-case when $p \neq 2$. Like in theorem 1.5 and in theorem 3.17, another interesting question is whether one can replace u in the left-hand side of equation (27) with $(-\Delta)^{t/2} u$ when $0 \leq t \leq s$, and whether the constant c in equation (27) can be expressed in terms of the classical Poincaré constant when $s \geq 1$.

7.3. **The Calderón problem for determining a higher order PDO.** In this discussion, we try to make as simple assumptions as possible. The whole point is to introduce a new inverse problem that we think is a very natural and interesting one, at least from a pure mathematical point of view. Therefore the optimal regularity in the statement of the problem is not as important. Let Ω be a domain with smooth boundary. Suppose that $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ is a partial differential operator (PDO) of order m with smooth coefficients on Ω. We argue in section 3.1 that the operator $(-\Delta)^s + P(x, D)$ admits the UCP (in open sets).

It is shown in the seminal work of Ghosh, Uhlmann and Salo [28] that if $P(x, D)$ is of order $m = 0$, then one can determine the zeroth order coefficient (i.e. the potential q) from the associated DN map. It was then later shown in [11] that if $P(x,D)$ is of order $m=1$, then one can also determine the coefficients (i.e. the potential q and the magnetic drift b) from the associated DN map whenever the order of $(-\Delta)^s$ is large enough, namely when $2s>1$. This and our work on higher order Calderón problems motivate the following inverse problem.

Question 7.5. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary. Let $P_j(x,D)$, $j=1,2$, be smooth PDOs of order $m \in \mathbb{N}$ in Ω. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ be such that $2s>m$. Given any two open sets $W_1, W_2 \subset \Omega_e$, suppose that the DN maps Λ_{P_i} for the equations*

$$((-\Delta)^s + P_j(x,D))u_j = 0 \ \text{ in } \Omega$$

satisfy $\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$. Show that $P_1(x,D) = P_2(x,D)$ or give a counterexample.

Another interesting question is whether the strong UCP [26] can be extended to higher order PDOs.

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X-ray Tomography of One-forms with Partial Data

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X-RAY TOMOGRAPHY OF ONE-FORMS WITH PARTIAL DATA

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ABSTRACT. If the integrals of a one-form over all lines meeting a small open set vanish and the form is closed in this set, then the one-form is exact in the whole Euclidean space. We obtain a unique continuation result for the normal operator of the X-ray transform of one-forms, and this leads to one of our two proofs of the partial data result. Our proofs apply to compactly supported covector-valued distributions.

1. INTRODUCTION

Let f be a one-form on $\mathbb{R}^n$ where $n \geq 2$. We define the X-ray transform (also known as the Doppler transform in this case) of f by the formula

$$X_1 f(\gamma) = \int_\gamma f \tag{1}$$

where γ is a line in $\mathbb{R}^n$. We freely identify one-forms with vector fields, so the differential of a scalar field corresponds to its gradient. We are interested in the problem of reconstructing f from $X_1 f$. One-forms of the form $f = \mathrm{d}\phi$ where ϕ goes to zero at infinity are always in the kernel of X_1. Thus one can only try to recover the solenoidal part f^{s} of the decomposition $f = f^{\mathrm{s}} + \mathrm{d}\phi$ from the data $X_1 f$. The transform X_1 is known to be solenoidally injective [31, 40], i.e. $X_1 f = 0$ implies $f = \mathrm{d}\phi$ for some scalar function ϕ. We study whether this implication holds in the whole space also in the case where we know $X_1 f$ only for a subset of lines.

We consider the following partial data problem for X_1. Let $V \subset \mathbb{R}^n$ be a nonempty open set. Assume that we know $\mathrm{d}f|_V$ and $X_1 f$ on all lines intersecting V, where $\mathrm{d}f$ is the exterior derivative or the curl of the one-form f. Can we determine the solenoidal part f^{s} – find f modulo potential fields – from this data? We will study the uniqueness of the partial data problem: If $\mathrm{d}f|_V = 0$ and $X_1 f = 0$ on all lines intersecting V, does it follow that $f^{\mathrm{s}} = 0$?

The partial data problem for X_1 can be reduced to the following unique continuation problem for the normal operator $N_1 = X_1^* X_1$: if $\mathrm{d}f|_V = 0$ and

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$N_1 f|_V = 0$, does it follow that $f^{\mathrm{s}} = 0$? We prove that such unique continuation property holds for compactly supported covector-valued distributions under the weaker assumption that $N_1 f$ vanishes to infinite order at some point in V. The unique continuation of the normal operator implies uniqueness for the partial data problem: The solenoidal part of a one-form f is uniquely determined whenever one knows the curl of the one-form in V and the integrals of f over all lines intersecting V.

For scalar fields the uniqueness of a corresponding partial data problem and the unique continuation of the normal operator were proved in [18]. We generalize the results to one-forms using the results for scalar fields in our proofs. We also obtain partial data results and unique continuation results for the generalized X-ray transform of one-forms $X_A = X_1 \circ A$ where A is a smooth invertible matrix-valued function. As a special case of this transform we study the transverse ray transform in $\mathbb{R}^2$.

We give two alternative proofs for the partial data results. The first one uses the unique continuation of the normal operator while the second one works directly at the level of the X-ray transform and is based on Stokes' theorem.

The X-ray transform of one-forms or vector fields has applications in the determination of velocity fields of moving fluids using acoustic travel time measurements [29] or Doppler backscattering measurements [30]. Medical applications include ultrasound imaging of blood flows [20, 21, 42]. The transverse ray transform of one-forms has applications in the temperature measurements of flames [4, 38]. For two-tensors the applications include also diffraction tomography [24], photoelasticity [14] and polarization tomography [40]. For a more comprehensive treatment see the reviews [36, 37, 41] and the references therein.

We will give our main results in section 1.1 and discuss related results in section 1.2. The preliminaries are covered in section 2 and finally the theorems are proven in section 3.

1.1. **Main results.** Here we give the main results of this paper. The proofs can be found in section 3. First we briefly go through our notation; for more detailed definitions see section 2.

Let $\mathcal{E}'(\mathbb{R}^n)$ be the space of compactly supported distributions. By $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ we mean that $f = (f_1, \dots, f_n)$ where $f_i \in \mathcal{E}'(\mathbb{R}^n)$ for all $i = 1, \dots, n$. We call $(\mathcal{E}'(\mathbb{R}^n))^n$ the space of compactly supported covector-valued distributions. We denote by X_1 the X-ray transform of one-forms and by $N_1 = X_1^* X_1$ its normal operator; see equation (20) for an explicit formula.

We say that $N_1 f$ vanishes to infinite order at $x_0 \in \mathbb{R}^n$ if it is smooth in a neighborhood of x_0 and $\partial^\beta (N_1 f)_i(x_0) = 0$ for all $\beta \in \mathbb{N}^n$ and $i = 1, \dots, n$. We denote the exterior derivative of differential forms by d. When acting on scalars, it corresponds to the gradient.

Our first result is a unique continuation property for the normal operator N_1. The corresponding result for scalar fields and the normal operator $N_0 = X_0^* X_0$ of the scalar X-ray transform X_0 (see equation (13)) was proven in [18, Theorem 1.1].

Theorem 1.1. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $\mathrm{d}f|_V = 0$ and $N_1 f$ vanishes to infinite order at $x_0 \in V$, then $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

We point out that as $\mathrm{d}f$ vanishes in V, the distribution $N_1 f$ is smooth in V by lemma 3.3 and the vanishing condition at a point is well-defined.

Theorem 1.1 is also true under the weaker assumption that $\mathrm{d}f|_V = 0$ and $\mathrm{d}(N_1 f)$ vanishes to infinite order at x_0 (see the proof in section 3.1). The condition that f is closed in V (i.e. $\mathrm{d}f|_V = 0$) is satisfied if, for example, $f|_V = 0$. When f is solenoidal (i.e. $\mathrm{div}(f) = 0$), theorem 1.1 gives the following unique continuation property: if $f|_V = N_1 f|_V = 0$, then $f = 0$.

The next result is stated directly at the level of the X-ray transform. The corresponding problem with full data was solved in [40, Theorem 2.5.1].

Theorem 1.2. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. Assume that $\mathrm{d}f|_V = 0$. Then $X_1 f$ vanishes on all lines intersecting V if and only if $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

Remark 1.3. *In theorems 1.1 and 1.2 the support of the potential ϕ is contained in the convex hull of* $\mathrm{spt}(f)$.

Remark 1.4. *We can combine the partial data result for vector fields (theorem 1.2) with the partial data result for scalar fields (lemma 3.4) to obtain the following partial data result. Let $F\colon S\mathbb{R}^n \to \mathbb{R}$ be a function on the sphere bundle $S\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$ defined as $F(x,\xi) = g(x) + f(x)\cdot\xi$ where $g\colon \mathbb{R}^n \to \mathbb{R}$ is a function on $\mathbb{R}^n$ and f is a one-form on $\mathbb{R}^n$. We define the X-ray transform $X_{S\mathbb{R}^n}$ of F as*

$$X_{S\mathbb{R}^n} F(\gamma) = \int_{\mathbb{R}} F(\gamma(t), \dot{\gamma}(t))\mathrm{d}t = X_0 g(\gamma) + X_1 f(\gamma) \tag{2}$$

where γ is an oriented line in $\mathbb{R}^n$ and X_0 is the X-ray transform of scalar fields (see section 2.2).

Assume that $V \subset \mathbb{R}^n$ is a nonempty open set such that $g|_V = \mathrm{d}f|_V = 0$ and $X_{S\mathbb{R}^n} F(\gamma) = 0$ on all lines γ intersecting V. Denote by $\overleftarrow{\gamma}$ the reversed line. Since $X_0 g(\overleftarrow{\gamma}) = X_0 g(\gamma)$ and $X_1 f(\overleftarrow{\gamma}) = -X_1 f(\gamma)$ we obtain $X_0 g(\gamma) = \frac{1}{2}(X_{S\mathbb{R}^n} F(\gamma) + X_{S\mathbb{R}^n} F(\overleftarrow{\gamma}))$ and $X_1 f(\gamma) = \frac{1}{2}(X_{S\mathbb{R}^n} F(\gamma) - X_{S\mathbb{R}^n} F(\overleftarrow{\gamma}))$. Hence the partial data problem for $X_{S\mathbb{R}^n} F$ decouples to separate partial data problems for $X_0 g$ and $X_1 f$. Using theorem 1.2 and lemma 3.4 one obtains that $g = 0$ and $f = \mathrm{d}\phi$ for some scalar field ϕ. This means that $F = \mathrm{d}\phi$, i.e. $F(x,\xi) = \mathrm{d}\phi(x)\cdot\xi$. See [3, 34] *for similar results in the case of full data.*

One can view theorems 1.1 and 1.2 in terms of the global solenoidal decomposition $f = f^{\mathrm{s}} + \mathrm{d}\phi$ (see section 2.1 and equation (5)). The conclusion $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$ is equivalent to $f^{\mathrm{s}} = 0$.

From theorem 1.2 we obtain the following local partial data result in a bounded domain $\Omega \subset \mathbb{R}^n$. The X-ray transform of $f \in (L^2(\Omega))^n$ is defined to be $X_1 f := X_1 \widetilde{f}$ where $\widetilde{f}$ is the zero extension of f to $\mathbb{R}^n$.

Theorem 1.5. *Let $f \in (L^2(\Omega))^n$ where $\Omega \subset \mathbb{R}^n$ is a bounded and smooth convex domain and let $V \subset \Omega$ be some nonempty open set. Assume that $\mathrm{d}f|_V = 0$. Then $X_1 f = 0$ on all lines intersecting V if and only if $f = \mathrm{d}\phi$ for some $\phi \in H^1_0(\Omega)$.*

In terms of the local solenoidal decomposition $f = f^{\mathrm{s}}_\Omega + \mathrm{d}\phi_\Omega$ (see section 2.1 and equation (6)) the conclusion $f = \mathrm{d}\phi$ for some $\phi \in H^1_0(\Omega)$ is equivalent to that $f^{\mathrm{s}}_\Omega = 0$.

From theorem 1.1 we also obtain the following unique continuation and partial data results for the transform $X_A = X_1 \circ A$ where $A = A(x)$ is smooth invertible matrix field. We denote by $N_A = A^T \circ N_1 \circ A$ the normal operator of X_A. When B is the constant matrix field $B(v_1, v_2) = (v_2, -v_1)$ where $(v_1, v_2) \in \mathbb{R}^2$ we write $X_B = X_\perp$ and call $X_\perp$ the transverse ray transform.

Corollary 1.6. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $\mathrm{d}(Af)|_V = 0$ and $N_A f|_V = 0$, then $f = A^{-1}(\mathrm{d}\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$.*

Corollary 1.7. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. Assume that $\mathrm{d}(Af)|_V = 0$. Then $X_A f$ vanishes on all lines intersecting V if and only if $f = A^{-1}(\mathrm{d}\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$.*

In corollaries 1.6 and 1.7 the distribution $\psi \in \mathcal{E}'(\mathbb{R}^n)$ is the potential part of the solenoidal decomposition of $Af \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $\mathrm{spt}(\psi)$ is contained in the convex hull of $\mathrm{spt}(f)$. As a special case of the transform X_A we obtain the next partial data result for the transverse ray transform $X_\perp$ which is similar to the full data result in [4, 11] (see also [2]).

Corollary 1.8. *Let $f \in (\mathcal{E}'(\mathbb{R}^2))^2$ and $V \subset \mathbb{R}^2$ some nonempty open set. Assume that $\mathrm{div}(f)|_V = 0$. Then $X_\perp f$ vanishes on all lines intersecting V if and only if $\mathrm{div}(f) = 0$.*

In particular, if $\mathrm{d}f|_V = \mathrm{div}(f)|_V = 0$ and both $X_1 f$ and $X_\perp f$ vanish on all lines intersecting V, then $f = 0$.

Alternatively, one can conclude in the first claim of corollary 1.8 that $f = \mathrm{curl}(\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$ where $\mathrm{curl}(\psi) = (\partial_2\psi, -\partial_1\psi)$. In terms of the global solenoidal decomposition this is equivalent to that $f = f^{\mathrm{s}}$. Also in the latter claim it is enough to know the partial data of $X_1 f$ for $V \subset \mathbb{R}^2$ and the partial data of $X_\perp f$ for $W \subset \mathbb{R}^2$ where V and W can be disjoint.

Remark 1.9. *Some of the results above can be slightly generalized. Using the same proof as in theorem 1.5 one can show that corollaries 1.7 and 1.8 hold also in the local case when $f \in (L^2(B))^n$. Also in corollary 1.6 one can replace the condition $N_A f|_V = 0$ with the requirement that $N_A f$ vanishes to infinite order at $x_0 \in V$ when A is a constant matrix field. Especially, this holds for the normal operator of the transverse ray transform. One can also see from theorem 1.2 and corollary 1.8 that the X-ray transform and the transverse ray transform provide complementary information about the one-form in $\mathbb{R}^2$.*

We note that if $A = A(x)$ is not invertible for all $x \in \mathbb{R}^n$, we can still conclude in corollary 1.7 that $Af = \mathrm{d}\psi$ for some potential $\psi \in \mathcal{E}'(\mathbb{R}^n)$. Thus we obtain the "pointwise projection" Af modulo potentials from the local data for $X_A f$. We also remark that in all of our results which consider the X-ray transform in $\mathbb{R}^n$ we could replace the assumption of compact support with rapid decay at infinity. If all the derivatives of the matrix field $A = A_{ij}(x)$ grow at most polynomially, then the results are true for one-forms whose components are Schwartz functions. This follows since the corresponding partial data result for scalar fields holds for Schwartz functions [18] and our method of proof is based on reducing the problem of one-forms to the problem of scalar fields.

1.2. **Related results.** Similar partial data results as in theorems 1.2 and 1.5 are previously known for scalar fields. If one knows the values of the scalar function f in an open set V, then one can uniquely reconstruct f from its local X-ray data [5, 18, 23]. In $\mathbb{R}^2$ uniqueness is also obtained under weaker assumptions: if f is piecewise constant, piecewise polynomial or analytic in V, then one can recover f uniquely from its integrals over the lines going through V [22, 23, 48]. A complementary partial data result is the Helgason support theorem [15]. According to Helgason's theorem, if $f|_C = 0$ and the integrals of f vanish on all lines not intersecting a compact and convex set C, then $f = 0$.

The normal operator of the X-ray transform of scalar fields admits a similar unique continuation property as in theorem 1.1. If f is a function which satisfies $f|_V = 0$ and $N_0 f$ vanishes to infinite order at some point in V, then $f = 0$ [18]. This is a special case of a more general unique continuation result for Riesz potentials [18] (see equation (13)). Unique continuation of Riesz potentials is related to unique continuation of fractional Laplacians [6, 12, 18] (see also equation (14)).

Unique reconstruction of the solenoidal part of a one-form or vector field with full data is known in $\mathbb{R}^n$ [21, 29, 42, 43] and on compact simple Riemannian manifolds with boundary [17, 31]. In $\mathbb{R}^n$ uniqueness holds for compactly supported covector-valued distributions as well [40]. Some partial data results are known for one-forms. The solenoidal part can be reconstructed by knowing $X_1 f$ on all lines parallel to a finite set of planes [21, 35, 39]. When $n \geq 3$, one can locally recover one-forms up to potential fields near a strictly convex boundary point [44], and the solenoidal part can be determined from the knowledge of $X_1 f$ on all lines intersecting a certain type of curve [47] (see also [10]). One can also obtain information about the singularities of the curl of a compactly supported covector-valued distribution from its X-ray data on lines intersecting a fixed curve [33]. There is a Helgason-type support theorem for the X-ray transform of one-forms which is in a sense complementary to our result. If f integrates to zero over all lines not intersecting a compact and convex set C, then $\mathrm{d}f = 0$ in the complement of C [43, Theorem 7.5]. If we further assume that $\mathrm{d}f|_C = 0$, then the one-form f is closed in the whole space which implies that f is exact and the solenoidal part of f vanishes. See also the discussion after the alternative proof in section 3.2.

The transverse ray transform has been studied earlier with full data in $\mathbb{R}^2$ [4, 11, 28] and also on Riemannian manifolds [19, 40] (see also [1] for a support theorem). The transverse ray transform is a special case of a more general mixed ray transform [7, 8, 11, 40]. In higher dimensions the transverse ray transform is related to the normal Radon transform [41, 45]. In $\mathbb{R}^2$ and on certain Riemannian manifolds the knowledge of $X_1 f$ and $X_\perp f$ fully determines the one-form [4, 11, 19]. By theorem 1.2 and corollary 1.8 this is true in $\mathbb{R}^2$ also in the case of partial data. In higher dimensions f is determined by $X_1 f$ and the normal Radon transform of f [45]. A similar transform to X_A was studied in [19, 32]. Recently in [2] the authors studied the so-called V-line transform of vector fields which is a generalization of the X-ray transform to V-shaped "lines" which consist of one vertex and two rays (half-lines).

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2. Preliminaries

In this section we give a brief introduction to the theory of X-ray tomography of scalar fields and one-forms in $\mathbb{R}^n$. We also define the generalized X-ray transform of one-forms. First we recall the definition and solenoidal decomposition of covector-valued distributions. We mainly follow the conventions of [9, 16, 27, 40, 43, 46] and refer the reader to them for further details.

2.1. **Covector-valued distributions and solenoidal decomposition.**

We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of compactly supported smooth functions, by $\mathscr{S}(\mathbb{R}^n)$ the space of rapidly decreasing smooth functions (Schwartz space) and by $\mathcal{E}(\mathbb{R}^n)$ the space of smooth functions. All spaces are equipped with their standard topologies. The spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are the corresponding topological duals. Elements of $\mathcal{D}'(\mathbb{R}^n)$ are called distributions and $\mathcal{E}'(\mathbb{R}^n)$ can be seen as the space of compactly supported distributions. We have the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. We write the dual pairing as $\langle f, \varphi \rangle$ when f is a distribution and φ is a test function.

We define the vector-valued test function space $(\mathcal{D}(\mathbb{R}^n))^n$ such that $\varphi \in (\mathcal{D}(\mathbb{R}^n))^n$ if and only if $\varphi = (\varphi_1, \dots, \varphi_n)$ and $\varphi_i \in \mathcal{D}(\mathbb{R}^n)$ for all $i = 1, \dots, n$. The topology of the space $(\mathcal{D}(\mathbb{R}^n))^n$ is defined as follows: a sequence φ_k converges to zero in $(\mathcal{D}(\mathbb{R}^n))^n$ if and only if $(\varphi_k)_i$ converges to zero in $\mathcal{D}(\mathbb{R}^n)$ for all $i = 1, \dots, n$. We then define the space of covector-valued distributions $(\mathcal{D}'(\mathbb{R}^n))^n$ so that $f \in (\mathcal{D}'(\mathbb{R}^n))^n$ if and only if $f = (f_1, \dots, f_n)$ and $f_i \in \mathcal{D}'(\mathbb{R}^n)$ for all $i = 1, \dots, n$. The duality pairing of $f \in (\mathcal{D}'(\mathbb{R}^n))^n$ and $\varphi \in (\mathcal{D}(\mathbb{R}^n))^n$ becomes

$$\langle f, \varphi \rangle = \sum_{i=1}^{n} \langle f_i, \varphi_i \rangle . \tag{3}$$

The spaces $(\mathcal{E}(\mathbb{R}^n))^n$, $(\mathscr{S}(\mathbb{R}^n))^n$, $(\mathcal{E}'(\mathbb{R}^n))^n$ and $(\mathscr{S}'(\mathbb{R}^n))^n$ are defined in a similar way and we call $(\mathcal{E}'(\mathbb{R}^n))^n$ the space of compactly supported covector-valued distributions. Covector-valued distributions are a special case of currents which are continuous linear functionals in the space of differential forms [9, Section III]. The components of the exterior derivative or the curl of a one-form or covector-valued distribution are

$$(\mathrm{d}f)_{ij} = \partial_i f_j - \partial_j f_i. \tag{4}$$

One can split certain covector-valued distributions into a divergence-free part and a potential part. If $f \in (\mathcal{E}'(\mathbb{R}^n))^n$, then we have the unique decomposition [40]

$$f = f^{\mathrm{s}} + \mathrm{d}\phi, \quad \mathrm{div}(f^{\mathrm{s}}) = 0 \tag{5}$$

where $\phi \in \mathscr{S}'(\mathbb{R}^n)$ and $f^{\mathrm{s}} \in (\mathscr{S}'(\mathbb{R}^n))^n$ are smooth outside $\mathrm{spt}(f)$ and go to zero at infinity. Here ϕ is defined so that it solves the equation $\Delta\phi = \mathrm{div}(f)$ in the sense of distributions and $f^{\mathrm{s}} = f - \mathrm{d}\phi$. The decomposition (5) is known as solenoidal decomposition or Helmholtz decomposition and it holds also for $f \in (\mathscr{S}(\mathbb{R}^n))^n$ [40]. We call f solenoidal if $\mathrm{div}(f) = 0$. For the decomposition (5) this means that $f = f^{\mathrm{s}}$.

If f is supported in a fixed set, we can do the decomposition locally in that set. If $\Omega \subset \mathbb{R}^n$ is a regular enough bounded domain and $f \in (L^2(\Omega))^n$, we let ϕ_Ω to be the unique weak solution to the Poisson equation

$$\begin{cases} \Delta\phi = \mathrm{div}(f) \text{ in } \Omega \\ \phi \in H^1_0(\Omega). \end{cases} \tag{6}$$

Then we have $f = f^{\mathrm{s}}_\Omega + \mathrm{d}\phi_\Omega$ where $f^{\mathrm{s}}_\Omega = f - \mathrm{d}\phi_\Omega \in (L^2(\Omega))^n$ and $\mathrm{div}(f^{\mathrm{s}}_\Omega) = 0$. If $f \in (C^{1,\alpha}(\overline{\Omega}))^n$ for some $0 < \alpha < 1$, then there is unique classical solution $\phi_\Omega \in C^{2,\alpha}(\overline{\Omega})$ to the boundary value problem (6) and the solenoidal decomposition holds pointwise [13].

2.2. **The X-ray transform of scalar fields.** Let Γ be the set of all oriented lines in $\mathbb{R}^n$. The X-ray transform of a function f is defined as

$$X_0 f(\gamma) = \int_\gamma f \mathrm{d}s, \quad \gamma \in \Gamma \tag{7}$$

whenever the integrals exist. The set Γ can be parameterized as

$$\Gamma = \{(z,\theta) : \theta \in S^{n-1},\ z \in \theta^\perp\}. \tag{8}$$

Then the X-ray transform becomes

$$X_0 f(z,\theta) = \int_{\mathbb{R}} f(z + t\theta)\mathrm{d}t \tag{9}$$

and it is a continuous map $X_0 \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\Gamma)$. One can define the adjoint using the formula

$$X_0^*\psi(x) = \int_{S^{n-1}} \psi(x - (x\cdot\theta)\theta, \theta)\mathrm{d}\theta \tag{10}$$

and it follows that $X_0^* \colon \mathcal{E}(\Gamma) \to \mathcal{E}(\mathbb{R}^n)$ is continuous. By duality we can define $X_0 \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\Gamma)$ and $X_0^* \colon \mathcal{D}'(\Gamma) \to \mathcal{D}'(\mathbb{R}^n)$ as

$$\langle X_0 f, \varphi\rangle = \langle f, X_0^*\varphi\rangle \tag{11}$$

$$\langle X_0^* g, \eta\rangle = \langle g, X_0\eta\rangle. \tag{12}$$

The normal operator $N_0 = X_0^* X_0$ is useful in studying the properties of the X-ray transform since it takes functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^n$. It has an expression

$$N_0 f(x) = 2\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\mathrm{d}y = 2(f * |\cdot|^{1-n})(x) \tag{13}$$

for continuous functions f decreasing rapidly enough at infinity. By duality the formula $N_0 f = 2(f * |\cdot|^{1-n})$ holds also for compactly supported distributions and the normal operator becomes a map $N_0\colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$. One can invert f from its X-ray transform using the normal operator by

$$f = c_{0,n}(-\Delta)^{1/2} N_0 f, \tag{14}$$

where $c_{0,n} = (2\pi \left|S^{n-2}\right|)^{-1}$ is a constant depending on the dimension and $(-\Delta)^{1/2}$ is the fractional Laplacian of order $1/2$. The inversion formula (14) holds for $f \in \mathcal{E}'(\mathbb{R}^n)$ and for continuous functions f decreasing rapidly enough at infinity.

2.3. **The X-ray transform of one-forms.** Let f be a one-form on $\mathbb{R}^n$. We define its X-ray transform as

$$X_1 f(\gamma) = \int_\gamma f, \quad \gamma \in \Gamma \tag{15}$$

whenever the integrals exist. The formula (15) is understood as the integral of the one-form f over the (oriented) one-dimensional submanifold γ. Using the parametrization (8) for Γ we can write

$$X_1 f(z,\theta) = \int_{\mathbb{R}} f(z + t\theta) \cdot \theta \mathrm{d}t. \tag{16}$$

It follows that $X_1\colon (\mathcal{D}(\mathbb{R}^n))^n \to \mathcal{D}(\Gamma)$ is continuous. The adjoint is defined as

$$(X_1^*\psi)_i(x) = \int_{S^{n-1}} \theta_i \psi(x - (x\cdot\theta)\theta, \theta) \mathrm{d}\theta \tag{17}$$

and $X_1^*\colon \mathcal{E}(\Gamma) \to (\mathcal{E}(\mathbb{R}^n))^n$ is also continuous. Thus we can define $X_1\colon (\mathcal{E}'(\mathbb{R}^n))^n \to \mathcal{E}'(\Gamma)$ and $X_1^*\colon \mathcal{D}'(\Gamma) \to (\mathcal{D}'(\mathbb{R}^n))^n$ as

$$\langle X_1 f, \varphi\rangle = \langle f, X_1^*\varphi\rangle \tag{18}$$

$$\langle X_1^* g, \eta\rangle = \langle g, X_1 \eta\rangle . \tag{19}$$

If $f \in (L^p(\Omega))^n$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $p \geq 1$, we define its X-ray transform as $X_1 f := X_1 \widetilde{f}$ where $\widetilde{f} \in (\mathcal{E}'(\mathbb{R}^n))^n$ is the zero extension of f.

Like in the scalar case we define the normal operator $N_1 = X_1^* X_1$ and it satisfies the formula

$$(N_1 f)_i = \sum_{j=1}^n \frac{2x_i x_j}{|x|^{n+1}} * f_j. \tag{20}$$

The normal operator can be extended to a map $N_1\colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{D}'(\mathbb{R}^n))^n$ and the formula (20) holds for $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and also for continuous one-forms decreasing rapidly enough at infinity. One can invert the solenoidal part of f using the normal operator by

$$f^{\mathrm{s}} = c_{1,n}(-\Delta)^{1/2} N_1 f, \tag{21}$$

where $c_{1,n} = |S^n|$ is a constant depending on the dimension and $(-\Delta)^{1/2}$ operates componentwise. The formula (21) holds for $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and also for continuous one-forms decreasing rapidly enough at infinity.

2.4. **The generalized X-ray transform of one-forms.** Let $A = A(x)$ be a smooth matrix-valued function on $\mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ the matrix $A(x)$ is invertible. We define the transform X_A of a one-form f as

$$X_A f(\gamma) = \int_{-\infty}^{\infty} A(\gamma(t)) f(\gamma(t)) \cdot \dot{\gamma}(t) \mathrm{d}t = X_1(Af)(\gamma), \quad \gamma \in \Gamma. \tag{22}$$

Thus X_A can be seen as the X-ray transform of the "rotated" one-form Af. The transform X_A can also be defined on compactly supported covector-valued distributions. We first let $\langle Af, \varphi \rangle = \langle f, A^T \varphi \rangle$ for $f \in (\mathcal{D}'(\mathbb{R}^n))^n$ and a test function φ where A^T is the pointwise transpose of A and $(A^T\varphi)(x) = A^T(x)\varphi(x)$. Then clearly A is a map $A\colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{E}'(\mathbb{R}^n))^n$. Therefore we can define $X_A\colon (\mathcal{E}'(\mathbb{R}^n))^n \to \mathcal{D}'(\Gamma)$ as $X_A f = X_1(Af)$. One easily sees that the adjoint is $X_A^* = A^T \circ X_1^*$ and the normal operator becomes $N_A = A^T \circ N_1 \circ A$. By the discussion above the normal operator can be extended to a map $N_A\colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{D}'(\mathbb{R}^n))^n$.

Let B be the constant matrix field on $\mathbb{R}^2$ defined as $B(v_1e_1 + v_2e_2) = v_2e_1 - v_1e_2$ where $\{e_1, e_2\}$ is any orthonormal basis of $\mathbb{R}^2$. The matrix B corresponds to a clockwise rotation by 90 degrees. We then define the transverse ray transform $X_\perp$ by letting $X_\perp = X_B$. It follows that the transverse ray transform provides complementary information about the solenoidal decomposition compared to the X-ray transform, i.e. X_1 determines the solenoidal part and $X_\perp$ determines the potential part of a one-form [4, 11] (see also theorem 1.2 and corollary 1.8).

3. Proofs of the main results

We give two alternative proofs for the partial data results. The first proof uses the unique continuation of the normal operator and the second proof works directly at the level of the X-ray transform. Both proofs are based on the corresponding results for scalar fields.

3.1. **Proofs using the unique continuation of the normal operator.** In this section we prove our main results using the unique continuation property of the normal operator. We need the following lemmas in our proofs.

Lemma 3.1 ([18, Theorem 1.1])**.** *Let $V \subset \mathbb{R}^n$ be some nonempty open set and $g \in \mathcal{E}'(\mathbb{R}^n)$. If $g|_V = 0$ and $\partial^\beta(N_0 g)(x_0) = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}^n$, then $g = 0$.*

Lemma 3.2 (Poincaré lemma)**.** *Let $g \in (\mathcal{D}'(\mathbb{R}^n))^n$ such that $\mathrm{d}g = 0$. Then there is $\eta \in \mathcal{D}'(\mathbb{R}^n)$ such that $\mathrm{d}\eta = g$. If $g \in (\mathcal{E}'(\mathbb{R}^n))^n$, then $\eta \in \mathcal{E}'(\mathbb{R}^n)$.*

The proof of lemma 3.2 can be found in [16, 25]. We first prove the unique continuation result for the normal operator. The proof is based on the fact that we can reduce the unique continuation problem of N_1 to a unique continuation problem of N_0 acting on the components of $\mathrm{d}f$.

The assumptions of theorem 1.1 come in two stages. We first assume that $\mathrm{d}f|_V = 0$. To make sense of the next assumption that $N_1 f$ vanishes at x_0 to infinite order, we need to ensure that it is smooth near this point. This is given by the next lemma.

Lemma 3.3. *Let $V \subset \mathbb{R}^n$ be an open set and $f \in (\mathcal{E}'(\mathbb{R}^n))^n$. If $\mathrm{d}f|_V = 0$, then $N_1 f|_V$ is smooth.*

Proof. Take any $x_0 \in V$ and a small open ball B centered at it and contained in V. As $\mathrm{d}f|_B = 0$, the Poincaré lemma applied in the ball B (lemma 3.2 is applicable because B is diffeomorphic to $\mathbb{R}^n$) gives $f|_B = \mathrm{d}h$ for some $h \in \mathcal{D}'(B)$. Let $B' \subset B$ be a smaller ball with the same center, and let $\chi \in \mathcal{D}(B)$ be a bump function so that $\chi|_{B'} \equiv 1$. If we let $h' = \chi h \in \mathcal{E}'(\mathbb{R}^n)$, then $f = \mathrm{d}h' + g$, where $g \in (\mathcal{E}'(\mathbb{R}^n))^n$ with $g|_{B'} = 0$.

As $X_1(\mathrm{d}h') = 0$ (cf. (27)), we have $N_1 f = N_1 g$. Because $g|_{B'} = 0$, it follows from properties of convolutions that $N_1 f$ is smooth in B'. Now that $N_1 f$ is smooth in a neighborhood of any point in V, the claim follows. □

Proof of theorem 1.1. The normal operator has an expression

$$(N_1 f)_i = \sum_{j=1}^{n} \frac{2x_i x_j}{|x|^{n+1}} * f_j. \tag{23}$$

We can write the kernel as

$$\frac{2x_i x_j}{|x|^{n+1}} = \frac{2}{n-1}\left(\delta_{ij}\,|x|^{1-n} - \partial_i(x_j\,|x|^{1-n})\right) \tag{24}$$

and we obtain

$$(N_1 f)_i = \frac{2}{n-1}\left(\frac{1}{2} N_0 f_i - \sum_{j=1}^{n} x_j\,|x|^{1-n} * \partial_i f_j\right). \tag{25}$$

We can calculate that

$$\partial_k (N_1 f)_i - \partial_i (N_1 f)_k = \frac{1}{n-1} N_0(\partial_k f_i - \partial_i f_k). \tag{26}$$

This can be interpreted as $\mathrm{d}(N_1 f) = (n-1)^{-1} N_0(\mathrm{d}f)$, where the scalar normal operator N_0 acts on the 2-form $\mathrm{d}f$ componentwise to produce another 2-form. The normal operator commutes with the exterior derivative in this sense.

Since $N_1 f$ vanishes to infinite order at $x_0 \in V$ also $N_0(\partial_k f_i - \partial_i f_k)$ vanishes to infinite order at x_0. Using lemma 3.1 we obtain $\mathrm{d}f = 0$. By lemma 3.2 there is $\phi \in \mathcal{E}'(\mathbb{R}^n)$ such that $\mathrm{d}\phi = f$. This concludes the proof. □

Lemma 3.1 is false if no restrictions are imposed on $g|_V$ [23, 27], and the assumption $g|_V = 0$ is the most convenient. Consequently, the assumption $\mathrm{d}f|_V = 0$ in theorem 1.1 is important. This condition is invariant under gauge transformations of the field f.

If $f^{\mathrm{s}}|_V = N_1 f|_V = 0$, then one can alternatively use the unique continuation of the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$, to prove the unique continuation of the normal operator [12]. This follows since $(-\Delta)^{1/2} f^{\mathrm{s}} = c_{1,n}(-\Delta) N_1 f$ where $f^{\mathrm{s}} \in (H^r(\mathbb{R}^n))^n$ for some $r \in \mathbb{R}$ when $f \in (\mathcal{E}'(\mathbb{R}^n))^n$. One can also make use of the fact that $(-\Delta)^{-1/2}$ is a Riesz potential and use its unique continuation properties [18] (see equation (21)).

The rest of the results follow easily from theorem 1.1.

Proof of theorem 1.2. Let $f = \mathrm{d}\phi$ where $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Then $\mathrm{d}\phi \in (\mathcal{E}'(\mathbb{R}^n))^n$ and using the definition of the X-ray transform on distributions we obtain

$$\langle X_1(\mathrm{d}\phi), \varphi\rangle = \langle \mathrm{d}\phi, X_1^*\varphi\rangle = \langle \phi, \mathrm{div}(X_1^*\varphi)\rangle = 0. \tag{27}$$

Here we used the fact that $\mathrm{div}(X_1^*\varphi) = 0$ which follows from a straightforward computation. This shows that $X_1 f = X_1(\mathrm{d}\phi) = 0$, and especially $X_1 f$ vanishes on all lines intersecting V. Assume then that $\mathrm{d}f|_V = 0$. Since $X_1 f = 0$ on all lines intersecting V we obtain $N_1 f|_V = 0$. Theorem 1.1 implies that $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$. This concludes the proof. □

Proof of theorem 1.5. If $f = \mathrm{d}\phi$ where $\phi \in H_0^1(\Omega)$, then using the same argument as in the proof of theorem 1.2 and the fact that $H_0^1(\Omega) \subset \mathcal{E}'(\mathbb{R}^n)$ in the sense of zero extension we obtain that $X_1 f = 0$, and especially $X_1 f$ vanishes on all lines intersecting V. Then assume that $\mathrm{d}f|_V = 0$ and $X_1 f = 0$ on all lines intersecting V. Let $\widetilde{f} \in (\mathcal{E}'(\mathbb{R}^n))^n$ be the zero extension of f. The assumptions imply that $\mathrm{d}\widetilde{f}|_V = 0$ and $X_1\widetilde{f} = 0$ on all lines intersecting V. Theorem 1.2 implies that $\widetilde{f} = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Since $\Delta\phi = \mathrm{div}(\widetilde{f}) \in H^{-1}(\mathbb{R}^n)$ we have $\phi \in H^1(\mathbb{R}^n)$ by elliptic regularity. On the other hand, $\mathrm{spt}(\phi) \subset \overline{\Omega}$ and hence $\phi \in H_0^1(\Omega)$ [26, Theorem 3.33]. The claim follows from the fact that $\mathrm{d}\phi = \widetilde{f} = f$ in Ω. □

Proof of corollary 1.6. We know that the normal operator is $N_A = A^T \circ N_1 \circ A$. The assumptions imply that $\mathrm{d}(Af)|_V = N_1(Af)|_V = 0$. By theorem 1.1 we obtain that $Af = \mathrm{d}\psi$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$. This gives the claim. □

Proof of corollary 1.7. The claim follows directly from corollary 1.6 and from the fact that $X_A = X_1 \circ A$. □

In theorems 1.1 and 1.2 one has $\mathrm{spt}(\phi) \subset \mathrm{Conv}(\mathrm{spt}(f))$ where $\mathrm{Conv}(\mathrm{spt}(f))$ is the convex hull of $\mathrm{spt}(f)$. This follows from the fact that ϕ has compact support and $\mathrm{d}\phi$ vanishes in the connected set $\mathrm{Conv}(\mathrm{spt}(f))^c$. This was pointed out in remark 1.3.

In corollaries 1.6 and 1.7 one also has $\mathrm{spt}(\psi) \subset \mathrm{Conv}(\mathrm{spt}(f))$. This holds since $\mathrm{d}\psi$ vanishes in the connected set $\mathrm{Conv}(\mathrm{spt}(Af))^c$ and $\mathrm{spt}(Af) = \mathrm{spt}(f)$.

Proof of corollary 1.8. Assume first that $\mathrm{div}(f) = 0$. Since f is a covector-valued distribution in $\mathbb{R}^2$ we can identify $\mathrm{d}f = \partial_1 f_2 - \partial_2 f_1$. It follows that $\mathrm{d}(Bf) = -\mathrm{div}(f) = 0$ and thus $Bf = \mathrm{d}\eta$ for some $\eta \in \mathcal{E}'(\mathbb{R}^n)$ by lemma 3.2. Therefore $X_\perp f = X_1(Bf) = X_1(\mathrm{d}\eta) = 0$. Assume then that $\mathrm{div}(f)|_V = 0$ and $X_\perp f = 0$ on all lines intersecting V. As above we obtain that $\mathrm{d}(Bf)|_V = 0$ and $X_\perp f = 0$ on all lines intersecting V. Corollary 1.7 implies that $f = B^{-1}(\mathrm{d}\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$. From this we obtain that $\mathrm{div}(f) = 0$.

Assume then that $\mathrm{d}f|_V = \mathrm{div}(f)|_V = 0$ and both $X_1 f$ and $X_\perp f$ vanish on all lines intersecting V. By the discussion above we obtain that $\mathrm{div}(f) = 0$. On the other hand, theorem 1.2 implies that $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^2)$. Therefore $\Delta\phi = 0$ and since ϕ has compact support we must have $\phi = 0$, i.e. $f = 0$. □

3.2. **Proofs based on Stokes' theorem.** In this section we give alternative proofs for the partial data results using Stokes' theorem in $\mathbb{R}^n$. A similar approach was used in [21, 42] in the case of full data, and also recently in [2] for the generalized V-line transform. We prove the results first for compactly supported smooth one-forms and then use standard mollification argument to prove them for compactly supported covector-valued distributions. We only need to prove theorem 1.2 since the rest of the partial data results follow from it. We will use the following lemma.

Lemma 3.4 ([18, Theorem 1.2])**.** *Let $V \subset \mathbb{R}^n$ be some nonempty open set and $g \in \mathcal{E}'(\mathbb{R}^n)$. If $g|_V = 0$ and $X_0 g = 0$ on all lines intersecting V, then $g = 0$.*

Alternative proof of theorem 1.2. By lemma 3.2 it suffices to show that $\mathrm{d}f = 0$. Assume first that $n = 2$ and $f \in (\mathcal{D}(\mathbb{R}^2))^2$. Let γ be any (oriented) line going through V and ν the counterclockwise rotated normal to γ. We denote by $\gamma_h = h\nu + \overleftarrow{\gamma}$ the reversed parallel line shifted by $h > 0$ in the direction of ν so that γ_h also intersects V. By assumption $\int_\gamma f = \int_{\gamma_h} f = 0$.

We form a closed loop $\widetilde{\gamma}_h$ enclosing counterclockwise a rectangular region R_h such that the ends are outside $\mathrm{spt}(f)$ (see figure 1). When considered as chains, we have $\widetilde{\gamma}_h = \partial R_h$. As the chains $\gamma - \gamma_h$ and $\widetilde{\gamma}_h$ differ only outside the support of f, the integrals coincide. By Stokes' theorem

$$0 = \int_\gamma f - \int_{\gamma_h} f = \int_{\widetilde{\gamma}_h} f = \int_{\partial R_h} f = \int_{R_h} \mathrm{d}f = \int_{R_h} \star \mathrm{d}f \, \mathrm{d}\mu, \tag{28}$$

where $\star$ is the Hodge star and μ is the 2-Hausdorff measure.

We aim to show that the scalar function $\star\mathrm{d}f$ vanishes. Scaling with h, we find

$$0 = \lim_{h\to 0} \frac{1}{h} \int_{R_h} \star\mathrm{d}f \, \mathrm{d}\mu = \int_\gamma \star\mathrm{d}f \, \mathrm{d}s. \tag{29}$$

Now that $\star\mathrm{d}f|_V = 0$ and $X_0(\star\mathrm{d}f)(\gamma) = 0$ for all lines γ meeting V, lemma 3.4 implies that $\star\mathrm{d}f = 0$ and thus also $\mathrm{d}f = 0$ in the whole plane.

Consider then the case $n \geq 3$ for a compactly supported smooth one-form f. Let $P \subset \mathbb{R}^n$ be any two-plane meeting V and $\iota_P \colon P \to \mathbb{R}^n$ the corresponding inclusion. By the argument above for the two-form $\iota_P^* f$ in the plane P we have that $\iota_P^* \mathrm{d}f = 0$ for all such planes.

Take any point $z \in \mathbb{R}^n$. For any plane P through z that intersects V we have $\iota_P^* \mathrm{d}f = 0$. This is an open subset of the Grassmannian of 2-planes through z, so $\mathrm{d}f(z) = 0$. As the point z was arbitrary, we have $\mathrm{d}f = 0$.

Finally, let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and define $f_\epsilon = f * j_\epsilon = (f_1 * j_\epsilon, \ldots, f_n * j_\epsilon)$ where $j_\epsilon \in \mathcal{D}(\mathbb{R}^n)$ is the standard mollifier. Then $f_\epsilon \in (\mathcal{D}(\mathbb{R}^n))^n$ and $\langle X_1(f * j_\epsilon), \varphi\rangle = \langle X_1 f, X_0 j_\epsilon \circledast \varphi\rangle$ where

$$(h \circledast g)(z, \theta) = \int_{\theta^\perp} h(z - y, \theta) g(y, \theta) \mathrm{d}y. \tag{30}$$

Hence there is a nonempty open set $W \subset V$ such that for small $\epsilon > 0$ we have $f_\epsilon|_W = 0$ and $X_1 f_\epsilon = 0$ on all lines intersecting W. Using the above reasoning for smooth one-forms we obtain $0 = \mathrm{d}f_\epsilon = \mathrm{d}f * j_\epsilon$ for small $\epsilon > 0$. Taking $\epsilon \to 0$ we get $\mathrm{d}f = 0$. $\square$

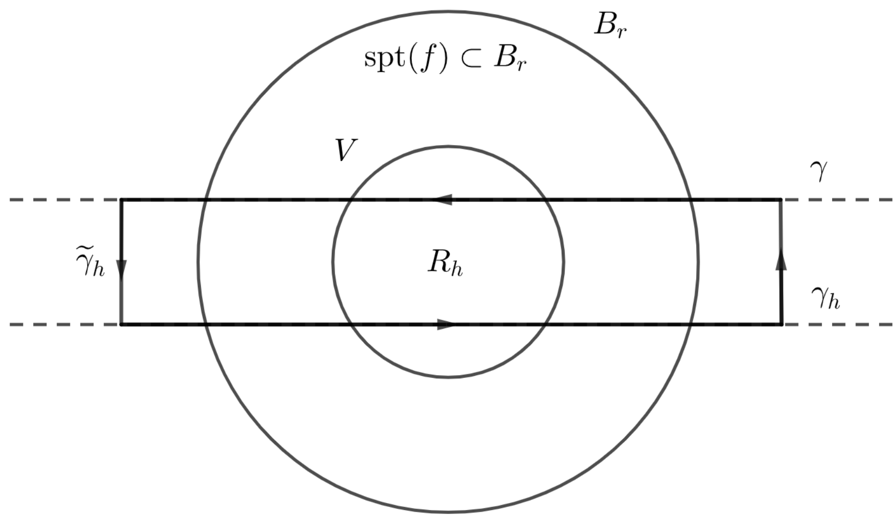

FIGURE 1. Basic idea of the alternative proof of theorem 1.5 when $n = 2$. We may assume that f is supported in a ball B_r. We form a closed loop $\widetilde{\gamma}_h$ with the lines γ and γ_h (dashed) enclosing the rectangular region R_h. Then we apply Stokes' theorem and a limit argument $h \to 0$ together with a known partial data result for scalar fields to obtain that $\mathrm{d}f = 0$.

Now the proof of theorem 1.5 follows in the same way from theorem 1.2 as before using the zero extension $\widetilde{f}$. Corollaries 1.7 and 1.8 are also direct consequences of theorem 1.2 since $X_A = X_1 \circ A$.

Moreover, the above alternative proof can be used to prove a complementary support theorem for the transform X_A: if $\mathrm{d}(Af)|_C = 0$ and $X_A f = X_1(Af) = 0$ on all lines not intersecting a convex and compact set C, then $f = A^{-1}(\mathrm{d}\psi)$ for some potential ψ (see [43, Theorem 7.5] for a similar support theorem for the X-ray transform X_1). Indeed, if γ is any line not intersecting C, then we can form a closed loop $\widetilde{\gamma}_h$ as in figure 1 so that the loop is completely contained in C^c and the ends are outside the support of f. Using Stokes' theorem and a limit argument $h \to 0$ as in the alternative proof above we obtain that $X_0(\star \mathrm{d}(Af)) = 0$ on all lines not intersecting C. Now we can use the Helgason support theorem for scalar fields (see e.g. [15, Corollary 6.1] and [43, Section 5.2]) to conclude that $\mathrm{d}(Af) = 0$ in C^c. Since also $\mathrm{d}(Af)|_C = 0$ we get that Af is a closed one-form and thus exact, i.e. there is a scalar field ψ such that $Af = \mathrm{d}\psi$.

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The higher order fractional Calderón problem for linear local operators: uniqueness

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THE HIGHER ORDER FRACTIONAL CALDERÓN PROBLEM FOR LINEAR LOCAL OPERATORS: UNIQUENESS

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Abstract. We study an inverse problem for the fractional Schrödinger equation (FSE) with a local perturbation by a linear partial differential operator (PDO) of the order smaller than the order of the fractional Laplacian. We show that one can uniquely recover the coefficients of the PDO from the Dirichlet-to-Neumann (DN) map associated to the perturbed FSE. This is proved for two classes of coefficients: coefficients which belong to certain spaces of Sobolev multipliers and coefficients which belong to fractional Sobolev spaces with bounded derivatives. Our study generalizes recent results for the zeroth and first order perturbations to higher order perturbations.

1. Introduction

Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $\Omega \subset \mathbb{R}^n$ a bounded open set where $n \geq 1$, $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ its exterior and $P(x, D)$ a linear partial differential operator (PDO) of order $m \in \mathbb{N}$

$$P(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

where the coefficients $a_\alpha = a_\alpha(x)$ are functions defined in Ω. We study a nonlocal inverse problem for the perturbed fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + P(x,D)u = 0 \text{ in } \Omega \\ u = f \text{ in } \Omega_e \end{cases} \tag{1}$$

where $(-\Delta)^s$ is a nonlocal pseudo-differential operator $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{u})$ in contrast to the local operator $P(x, D)$. In the inverse problem, one aims to recover the local operator P from the associated Dirichlet-to-Neumann map.

We always assume that 0 is not a Dirichlet eigenvalue of the operator $((-\Delta)^s + P(x, D))$, i.e.

$$\text{If } u \in H^s(\mathbb{R}^n) \text{ solves } ((-\Delta)^s + P(x,D))u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \text{ then } u = 0.$$

Our data for the inverse problem is the Dirichlet-to-Neumann (DN) map $\Lambda_P \colon H^s(\Omega_e) \to (H^s(\Omega_e))^*$ which maps Dirichlet exterior values to a nonlocal version of the Neumann boundary value (see section 2 and 3.1). The main question that we study in this article is whether the DN map Λ_P determines uniquely the coefficients a_α in Ω. In other words, does $\Lambda_{P_1} = \Lambda_{P_2}$ imply that $a_{1,\alpha} = a_{2,\alpha}$ in Ω for all $|\alpha| \leq m$? We prove that the answer is positive under certain restrictions on the coefficients a_α and the order of the PDOs.

This gives positive answer to the uniqueness problem [10, Question 2.5] posed by the first three authors in a previous work. The precise statement in [10] asks to prove uniqueness for the higher order fractional Calderón problem in the case of a bounded domain with smooth boundary and PDOs with smooth coefficients (up to the boundary). The positive answer to this question follows from theorem 1.2. The study of the fractional Calderón problem was initiated by Ghosh, Salo and Uhlmann in the work [15] where the uniqueness for the associated inverse problem is proved when $m = 0$, $s \in (0, 1)$ and $a_0 \in L^\infty(\Omega)$.

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We briefly note that by Peetre's theorem any linear operator $L\colon C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ which does not increase supports, i.e. $\mathrm{spt}(Lf) \subset \mathrm{spt}(f)$ for all $f \in C_c^\infty(\Omega)$, is in fact a differential operator [30] (see also the original work [32]). Therefore our results apply to any local operator satisfying such properties and it is enough to study PDOs only. For a more general formulation of Peetre's theorem on the level of vector bundles, see [31].

1.1. **Main results.** We denote by $M(H^{s-|\alpha|} \to H^{-s})$ the space of all bounded Sobolev multipliers between the Sobolev spaces $H^{s-|\alpha|}(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$. We denote by $M_0(H^{s-|\alpha|} \to H^{-s}) \subset M(H^{s-|\alpha|} \to H^{-s})$ the space of bounded Sobolev multipliers that can be approximated with smooth compactly supported functions in the multiplier norm of $M(H^{s-|\alpha|} \to H^{-s})$. We also write $H^{r,\infty}(\Omega)$ for the local Bessel potential space with bounded derivatives. See section 2 for more detailed definitions.

Our first theorem is a generalization of [36, Theorem 1.1] which considered the case $m = 0$ with $s \in (0, 1)$. It also generalizes [10, Theorem 1.5] which considered the higher order cases $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ when $m = 0$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set where $n \geq 1$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j = \sum_{|\alpha| \leq m} a_{j,\alpha} D^\alpha, \quad j = 1, 2,$$

be linear PDOs of order m with coefficients $a_{j,\alpha} \in M_0(H^{s-|\alpha|} \to H^{-s})$. Given any two open sets $W_1, W_2 \subset \Omega_e$, suppose that the DN maps Λ_{P_i} for the equations $((-\Delta)^s + P_j)u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1|_\Omega = P_2|_\Omega$.

In theorem 1.1 one can pick the lower order coefficients ($|\alpha| < s$) from $L^p(\Omega)$ for high enough p (especially from $L^\infty(\Omega)$) and higher order coefficients ($s < |\alpha| < 2s$) from the closure of $C_c^\infty(\Omega)$ in $H^{r,\infty}(\Omega)$ for certain values of $r \in \mathbb{R}$. Lemmas 2.8 and 2.9 give more examples of Sobolev spaces which belong to the space of multipliers $M_0(H^{s-|\alpha|} \to H^{-s})$. We also note that when $|\alpha| = 0$, then the space of multipliers $M_0(H^s \to H^{-s})$ coincides with the one studied in [36].

It follows that the space of multipliers is trivial for higher order operators, i.e. $M(H^{s-|\alpha|} \to H^{-s}) = \{0\}$ when $s - |\alpha| < -s$. It would be possible to state theorem 1.1 for higher order PDOs, but that forces $a_\alpha = 0$ for all $|\alpha| > 2s$. For this reason we only consider PDOs whose order is $m < 2s$. See lemma 2.5 and the related remarks for more details.

Our second theorem generalizes [7, Theorem 1.1] and [15, Theorem 1.1] where similar results are proved when $m = 0, 1$ and $s \in (0, 1)$. It also generalizes [10, Theorem 1.5] where the case $m = 0$ and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ was studied.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain where $n \geq 1$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j(x, D) = \sum_{|\alpha| \leq m} a_{j,\alpha}(x) D^\alpha, \quad j = 1, 2,$$

be linear PDOs of order m with coefficients $a_{j,\alpha} \in H^{r_\alpha,\infty}(\Omega)$ where

$$r_\alpha := \begin{cases} 0 & \text{if} \quad |\alpha| - s < 0, \\ |\alpha| - s + \delta & \text{if} \quad |\alpha| - s \in \{1/2, 3/2, ...\}, \\ |\alpha| - s & \text{if} \quad \text{otherwise} \end{cases} \tag{2}$$

for any fixed $\delta > 0$. Given any two open sets $W_1, W_2 \subset \Omega_e$, suppose that the DN maps Λ_{P_i} for the equations $((-\Delta)^s + P_j(x, D))u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1(x, D) = P_2(x, D)$.

Our first theorem is formulated for general bounded open sets and the second theorem for Lipschitz domains. The difference arises in the proof of the well-posedness of the inverse problem. We note that theorem 1.2 holds for coefficients a_α which are smooth up to the boundary ($a_\alpha = g|_\Omega$ where $g \in C^\infty(\mathbb{R}^n)$). The conditions (2) imply that one can choose $a_\alpha \in L^\infty(\Omega)$ for every α such that $|\alpha| < s$. The case $|\alpha| = s$ never happens, as s is assumed not to be an integer. If $|\alpha| > s$, we have $a_\alpha \in H^{|\alpha|-s,\infty}(\Omega)$ when $|\alpha| - s \notin \{1/2, 3/2, ...\}$. Thus the conditions (2) coincide with [7, 15] when $m = 0, 1$ and $s \in (0, 1)$.

Our article is roughly divided into two parts. The first part of the article (theorem 1.1 and section 3) generalizes the study of the uniqueness problem for singular potentials in [36] and the second part (theorem 1.2 and section 4) generalizes the uniqueness problem for bounded first order perturbations in [7].

The approach to prove theorems 1.1 and 1.2 is the following. First one shows that the inverse problem is well-posed and the corresponding bilinear forms are bounded. This leads to the boundedness of the DN maps and an Alessandrini identity. By a unique continuation property of the higher order fractional Laplacian one obtains a Runge approximation property for equation (1). Using the Runge approximation and the Alessandrini identity for suitable test functions one proves the uniqueness of the inverse problem.

1.2. **On the earlier literature.** Equation (1) and theorems 1.1 and 1.2 are related to the Calderón problem for the fractional Schrödinger equation first introduced in [15]. There one tries to uniquely recover the potential q in Ω by doing measurements in the exterior Ω_e. This is a nonlocal (fractional) counterpart of the classical Calderón problem arising in electrical impedance tomography where one obtains information about the electrical properties of some bounded domain by doing voltage and current measurements on the boundary [39, 40]. In [36] the study of the fractional Calderón problem is extended for "rough" potentials q, i.e. potentials which are in general bounded Sobolev multipliers. First order perturbations were studied in [7] assuming that the fractional part dominates the equation, i.e. $s \in (1/2, 1)$, and that the perturbations have bounded fractional derivatives. A higher order version ($s \in \mathbb{R}^+ \setminus \mathbb{Z}$) of the fractional Calderón problem was introduced and studied in [10]. These three articles [7, 10, 36] motivate the study of higher order (rough) perturbations to the fractional Laplacian $(-\Delta)^s$ in equation (1). The natural restriction for the order of $P(x, D)$ in theorems 1.1 and 1.2 is then $2s > m$ so that the fractional part governs the equation (1).

The fractional Calderón problem for $s \in (0, 1)$ has been studied in many settings. We refer to the survey [37] for a more detailed treatment. In the work [36] stability was proved for singular potentials, and in [34] the related exponential instability was shown. The fractional Calderón problem has also been solved under single measurement [14]. The perturbed equation is related to the fractional magnetic Schrödinger equation which is studied in [9, 24, 25, 26]. See also [4] for a fractional Schrödinger equation with a lower order nonlocal perturbation. Other variants of the fractional Calderón problem include semilinear fractional (magnetic) Schrödinger equation [19, 20, 24, 25], fractional heat equation [21, 35] and fractional conductivity equation [8] (see also [6, 13] for equations arising from a nonlocal Schrödinger-type elliptic operator). In the recent work [10], the first three authors of this article studied higher order versions ($s \in \mathbb{R}^+ \setminus \mathbb{Z}$) of the fractional Calderón problem and proved uniqueness for the Calderón problem for the fractional magnetic Schrödinger equation (up to a gauge). This article continues these studies by showing uniqueness for the fractional Schrödinger equation with higher order perturbations and gives positive answer to the question 2.5 posed in [10].

1.3. **Examples of fractional models in the sciences.** Equations involving fractional Laplacians like (1) have applications in mathematics and natural sciences. Fractional Laplacians appear in the study of anomalous and nonlocal diffusion, and these diffusion phenomena can be used in many areas such as continuum mechanics, graph theory and ecology just to mention a few [2, 5, 12, 27, 33]. Another place where the fractional counterpart of the classical Laplacian naturally shows up is the formulation of fractional quantum mechanics [22, 23]. For more applications of fractional mathematical models, see [5] and the references therein.

1.4. **The organization of the article.** In section 2 we introduce the notation and give preliminaries on Sobolev spaces and fractional Laplacians. We also define the spaces of rough coefficients (Sobolev multipliers) and discuss some of the basic properties. In section 3 we prove theorem 1.1 in detail. Finally, in section 4 we prove theorem 1.2 but as the proofs of both theorems are very similar we do not repeat all identical steps and we keep our focus in the differences of the proofs.

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2. Preliminaries

In this section we recall some basic theory of Sobolev spaces, Fourier analysis and fractional Laplacians on $\mathbb{R}^n$. We also introduce the spaces of Sobolev multipliers and prove a few properties for them. Some auxiliary lemmas which are needed in the proofs of our main theorems are given as well. We follow the references [1, 15, 29, 28, 38, 41] (see also section 3 in [10]).

2.1. **Sobolev spaces.** The (inhomogeneous) fractional L^2-based Sobolev space of order $r \in \mathbb{R}$ is defined to be

$$H^r(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^2(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{H^r(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u})\right\|_{L^2(\mathbb{R}^n)}.$$

Here $\hat{u} = \mathcal{F}(u)$ is the Fourier transform of a tempered distribution $u \in \mathscr{S}'(\mathbb{R}^n)$, $\mathcal{F}^{-1}$ is the inverse Fourier transform and $\langle x\rangle = (1+|x|^2)^{1/2}$. We define the fractional Laplacian of order $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ as $(-\Delta)^s\varphi = \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{\varphi})$ where $\varphi \in \mathscr{S}(\mathbb{R}^n)$ is a Schwartz function. Then $(-\Delta)^s$ extends to a bounded operator $(-\Delta)^s\colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$ by density of $\mathscr{S}(\mathbb{R}^n)$ in $H^r(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $F \subset \mathbb{R}^n$ a closed set. We define the following Sobolev spaces

$$\begin{aligned} H^r_F(\mathbb{R}^n) &= \{u \in H^r(\mathbb{R}^n) : \mathrm{spt}(u) \subset F\} \\ \widetilde{H}^r(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^r(\mathbb{R}^n) \\ H^r(\Omega) &= \{u|_\Omega : u \in H^r(\mathbb{R}^n)\} \\ H^r_0(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^r(\Omega). \end{aligned}$$

It follows that $\widetilde{H}^r(\Omega) \subset H^r_0(\Omega)$, $\widetilde{H}^r(\Omega) \subset H^r_{\overline{\Omega}}(\mathbb{R}^n)$, $(\widetilde{H}^r(\Omega))^* = H^{-r}(\Omega)$ and $(H^r(\Omega))^* = \widetilde{H}^{-r}(\Omega)$ for any open set Ω and $r \in \mathbb{R}$. If Ω is in addition a Lipschitz domain, then we have $\widetilde{H}^r(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$ and $H^r_0(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$ when $r > -1/2$ such that $r \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\ldots\}$.

More generally, let $1 \leq p \leq \infty$ and $r \in \mathbb{R}$. We define the Bessel potential space

$$H^{r,p}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^p(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{H^{r,p}(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u})\right\|_{L^p(\mathbb{R}^n)}.$$

We also write $\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) =: J^r u$ where the Fourier multiplier $J = (\mathrm{Id}-\Delta)^{1/2}$ is called the Bessel potential. We have the continuous inclusions $H^{r,p}(\mathbb{R}^n) \hookrightarrow H^{t,p}(\mathbb{R}^n)$ whenever $r \geq t$ [41]. By the

Mikhlin multiplier theorem one can show that $(-\Delta)^s\colon H^{r,p}(\mathbb{R}^n) \to H^{r-2s,p}(\mathbb{R}^n)$ is continuous whenever $s \geq 0$ and $1 < p < \infty$. The local version of the space $H^{r,p}(\mathbb{R}^n)$ is defined as earlier by the restrictions

$$H^{r,p}(\Omega) = \{u|_\Omega : u \in H^{r,p}(\mathbb{R}^n)\}$$

where $\Omega \subset \mathbb{R}^n$ is any open set. This space is equipped with the quotient norm

$$\|v\|_{H^{r,p}(\Omega)} = \inf\{\|w\|_{H^{r,p}(\mathbb{R}^n)} : w \in H^{r,p}(\mathbb{R}^n),\ w|_\Omega = v\}.$$

We have the continuous inclusions $H^{r,p}(\Omega) \hookrightarrow H^{t,p}(\Omega)$ whenever $r \geq t$ by the definition of the quotient norm.

We also define the spaces

$$\begin{aligned} H^{r,p}_F(\mathbb{R}^n) &= \{u \in H^{r,p}(\mathbb{R}^n) : \mathrm{spt}(u) \subset F\} \\ \widetilde{H}^{r,p}(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^{r,p}(\mathbb{R}^n) \\ H^{r,p}_0(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^{r,p}(\Omega) \end{aligned}$$

where $F \subset \mathbb{R}^n$ is a closed set. Note that $\widetilde{H}^{r,p}(\Omega) \subset H^{r,p}_0(\Omega)$ since the restriction map $|_\Omega\colon H^{r,p}(\mathbb{R}^n) \to H^{r,p}(\Omega)$ is by definition continuous. One can also see that $\widetilde{H}^{r,p}(\Omega) \subset H^{r,p}_{\overline{\Omega}}(\mathbb{R}^n)$. If Ω is a bounded C^∞-domain and $1 < p < \infty$, then we have [38, Theorem 1 in section 4.3.2]

$$\begin{aligned} \widetilde{H}^{r,p}(\Omega) &= H^{r,p}_{\overline{\Omega}}(\mathbb{R}^n), \quad s \in \mathbb{R} \\ H^{r,p}_0(\Omega) &= H^{r,p}(\Omega), \quad s \leq \frac{1}{p}. \end{aligned}$$

Some authors (especially in [7, 36]) use the notation $W^{r,p}(\Omega)$ for Bessel potential spaces. We have decided to use the notation $H^{r,p}(\Omega)$ so that these spaces are not confused with the Sobolev-Slobodeckij spaces which are in general different from the Bessel potential spaces [11].

The equation (1) we study is nonlocal. Instead of putting boundary conditions we impose exterior values for the equation. This can be done by saying that $u = f$ in Ω_e if $u - f \in \widetilde{H}^s(\Omega)$. Motivated by this we define the (abstract) trace space $X = H^r(\mathbb{R}^n)/\widetilde{H}^r(\Omega)$, i.e. functions in X are the same (have the same trace) if they agree in Ω_e. If Ω is a Lipschitz domain, then we have $X = H^r(\Omega_e)$ and $X^* = H^{-r}_{\overline{\Omega}_e}(\mathbb{R}^n)$.

2.2. **Properties of the fractional Laplacian.** The fractional Laplacian admits two important properties which we need in our proofs. The first one is unique continuation property (UCP) which is used in proving the Runge approximation property.

Lemma 2.1 (UCP)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $r \in \mathbb{R}$ and $u \in H^r(\mathbb{R}^n)$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$.*

Lemma 2.1 is proved in [10] for $s > 1$ by reducing the problem to the UCP result for $s \in (0,1)$ in [15]. Note that such property is not true for local operators like the classical Laplacian $(-\Delta)$. The second property we need is the Poincaré inequality, which is used in showing that the fractional Calderón problem is well-posed.

Lemma 2.2 (Poincaré inequality)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $K \subset \mathbb{R}^n$ compact set and $u \in H^s_K(\mathbb{R}^n)$. There exists a constant $c = c(n, K, s) > 0$ such that*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}.$$

Many different proofs for lemma 2.2 are given in [10]. We note that in the literature, the fractional Poincaré inequality is typically considered only when $s \in (0,1)$.

Finally, we recall the fractional Leibniz rule, also known as the Kato-Ponce inequality. It is used to show the boundedness of the bilinear forms associated to the perturbed fractional Schrödinger equation in the case when the coefficients of the PDO have bounded fractional derivatives.

Lemma 2.3 (Kato-Ponce inequality)**.** *Let* $s \geq 0$, $1 < r < \infty$, $1 < q_1 \leq \infty$ *and* $1 < p_2 \leq \infty$ *such that* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. *If* $f \in L^{p_2}(\mathbb{R}^n)$, $J^s f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{q_1}(\mathbb{R}^n)$ *and* $J^s g \in L^{q_2}(\mathbb{R}^n)$, *then* $J^s(fg) \in L^r(\mathbb{R}^n)$ *and*

$$\|J^s(fg)\|_{L^r(\mathbb{R}^n)} \leq C(\|J^s f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{q_1}(\mathbb{R}^n)} + \|f\|_{L^{p_2}(\mathbb{R}^n)} \|J^s g\|_{L^{q_2}(\mathbb{R}^n)})$$

where J^s *is the Bessel potential of order* s *and* $C = C(s, n, r, p_1, p_2, q_1, q_2)$.

The proof of lemma 2.3 can be found in [17] (see also [16, 18]).

2.3. **Spaces of rough coefficients.** Following [28, Ch. 3], we introduce the space of multipliers $M(H^r \to H^t)$ between pairs of Sobolev spaces. Here we are assuming that $r, t \in \mathbb{R}$. The coefficients of $P(x, D)$ in theorem 1.1 will be picked from such spaces of multipliers.

If $f \in \mathcal{D}'(\mathbb{R}^n)$ is a distribution, we say that $f \in M(H^r \to H^t)$ whenever the norm

$$\|f\|_{r,t} := \sup\{|\langle f, uv\rangle| \; ; \, u, v \in C_c^\infty(\mathbb{R}^n), \|u\|_{H^r(\mathbb{R}^n)} = \|v\|_{H^{-t}(\mathbb{R}^n)} = 1\}$$

is finite. Here $\langle \cdot, \cdot \rangle$ is the duality pairing. By $M_0(H^r \to H^t)$ we indicate the closure of $C_c^\infty(\mathbb{R}^n)$ in $M(H^r \to H^t) \subset \mathcal{D}'(\mathbb{R}^n)$. If $f \in M(H^r \to H^t)$ and $u, v \in C_c^\infty(\mathbb{R}^n)$ are both non-vanishing, we have the multiplier inequality

(3)
$$|\langle f, uv\rangle| = \left| \left\langle f, \frac{u}{\|u\|_{H^r(\mathbb{R}^n)}} \frac{v}{\|v\|_{H^{-t}(\mathbb{R}^n)}} \right\rangle \right| \|u\|_{H^r(\mathbb{R}^n)} \|v\|_{H^{-t}(\mathbb{R}^n)} \leq \|f\|_{r,t} \|u\|_{H^r(\mathbb{R}^n)} \|v\|_{H^{-t}(\mathbb{R}^n)}.$$

By density (3) can be extended to act over $u \in H^r(\mathbb{R}^n), v \in H^{-t}(\mathbb{R}^n)$. Moreover, each $f \in M(H^r \to H^t)$ gives rise to a multiplication map $m_f : H^r(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ defined as

$$\langle m_f(u), v\rangle := \langle f, uv\rangle \quad \text{for all} \quad u \in H^r(\mathbb{R}^n), v \in H^{-t}(\mathbb{R}^n).$$

We have as well the unique adjoint multiplication map $m_f^* : H^{-t}(\mathbb{R}^n) \to H^{-r}(\mathbb{R}^n)$ such that

$$\langle m_f^*(v), u\rangle := \langle f, uv\rangle \quad \text{for all} \quad u \in H^r(\mathbb{R}^n), v \in H^{-t}(\mathbb{R}^n).$$

Since one sees that the adjoint of m_f is m_f^*, the chosen notation is justified. For convenience, in the rest of the paper we will just write fu for both $m_f(u)$ and $m_f^*(u)$.

Remark 2.4. *The spaces of rough coefficients we use are generalizations of the ones considered in* [36]. *In fact, the space* $Z^{-s}(\mathbb{R}^n)$ *used there coincides with our space* $M(H^s \to H^{-s})$.

In the next lemma we state some elementary properties of the spaces of multipliers. Other interesting properties may be found in [28].

Lemma 2.5. *Let* $\lambda, \mu \geq 0$ *and* $r, t \in \mathbb{R}$. *Then*

(i) $M(H^r \to H^t) = M(H^{-t} \to H^{-r})$, *and the norms associated to the two spaces also coincide.*

(ii) $M(H^{r-\lambda} \to H^{t+\mu}) \hookrightarrow M(H^r \to H^t)$ *continuously.*

(iii) $M(H^r \to H^t) = \{0\}$ *whenever* $r < t$.

Proof. (i) Let $f \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution. Then by just using the definition we see that

$$\begin{aligned}\|f\|_{r,t} &= \sup\{|\langle f, uv\rangle| \; ; \, u, v \in C_c^\infty(\mathbb{R}^n), \|u\|_{H^r(\mathbb{R}^n)} = \|v\|_{H^{-t}(\mathbb{R}^n)} = 1\}\\ &= \sup\{|\langle f, vu\rangle| \; ; \, v, u \in C_c^\infty(\mathbb{R}^n), \|v\|_{H^{-t}(\mathbb{R}^n)} = \|u\|_{H^{-(-r)}(\mathbb{R}^n)} = 1\} = \|f\|_{-t,-r}.\end{aligned}$$

(ii) Observe that the given definition of $\|f\|_{r,t}$ is equivalent to the following:

$$\|f\|_{r,t} = \sup\{|\langle f, uv\rangle| \; ; \, u, v \in C_c^\infty(\mathbb{R}^n), \|u\|_{H^r(\mathbb{R}^n)} \leq 1, \|v\|_{H^{-t}(\mathbb{R}^n)} \leq 1\}.$$

Since $\lambda, \mu \geq 0$, we also have

$$\|u\|_{H^{r-\lambda}(\mathbb{R}^n)} \leq \|u\|_{H^r(\mathbb{R}^n)}, \quad \|v\|_{H^{-(t+\mu)}(\mathbb{R}^n)} \leq \|v\|_{H^{-t}(\mathbb{R}^n)}.$$

This implies $\|f\|_{r,t} \leq \|f\|_{r-\lambda,t+\mu}$, which in turn gives the wanted inclusion.

(iii) If $0 \leq r < t$, then this was considered in [28, Ch. 3]. The proof given there recalls the easier one for Sobolev spaces ([28, Sec. 2.1]), which is based on the explicit computation of derivatives of aptly chosen exponential functions.

If $r < t \leq 0$, then by point (i) we have $M(H^r \to H^t) = M(H^{-t} \to H^{-r})$. We need to show that $M(H^{-t} \to H^{-r}) = \{0\}$ whenever $0 \leq -t < -r$. This reduces the problem back to the case of non-negative Sobolev scales.

If $r \leq 0 < t$, then $-r \geq 0$. Now by point (ii), we have $M(H^r \to H^t) \subseteq M(H^{r+(-r)} \to H^t) = M(L^2 \to H^t)$. It is therefore enough to show that this last space is trivial, which again immediately follows from the case of non-negative Sobolev scales.

If $r < 0 \leq t$, then the problem can be reduced again to the earlier cases. $\square$

Remark 2.6. *We also have $M_0(H^{r-\lambda} \to H^{t+\mu}) \subseteq M_0(H^r \to H^t)$ whenever $\lambda, \mu \geq 0$, since the inclusion in (ii) is continuous.*

Remark 2.7. *In light of lemma 2.5 (ii) we are only interested in $M(H^r \to H^t)$ in the case $r \geq t$, the case $r < t$ being trivial. For our theorem 1.1, this translates into the condition $m \leq 2s$. We decided not to consider the limit case $m = 2s$ in this work, as our machinery (in particular, the coercivity estimate* (25)*) breaks down in this case. However, it should be noted that since by assumption we have $m \in \mathbb{Z}$ and $s \notin \mathbb{Z}$, the equality $m = 2s$ can only arise if m is odd, which forces $s = 1/2 + k$ with $k \in \mathbb{Z}$. This case was excluded in* [7, 15] *as well.*

The next lemmas relate our spaces of multipliers with some special Bessel potential spaces. This is interesting since in the coming section 3 we will consider the inverse problem for coefficients coming from such spaces. We start with a general result.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $t \in \mathbb{R}$ and $r \in \mathbb{R}$ be such that $t > \max\{0, r\}$. The following inclusions hold:*

(i) $\widetilde{H}^{r',\infty}(\Omega) \subset M_0(H^{-r} \to H^{-t})$ whenever $r' \geq \max\{0, r\}$.

(ii) $H_0^{r',\infty}(\Omega) \subset M_0(H^{-r} \to H^{-t})$ whenever $r' \geq \max\{0, r\}$ such that $r' \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$ and Ω is a Lipschitz domain.

(iii) $\widetilde{H}^{r'}(\Omega) \subset M_0(H^{-r} \to H^{-t})$ whenever $r' \geq t$ and $r' > n/2$. The same holds true for $H^{r'}_{\overline{\Omega}}(\mathbb{R}^n)$ if Ω is a Lipschitz domain, and for $H_0^{r'}(\Omega)$ when Ω is a Lipschitz domain and $r' \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$.

Proof. Throughout the proof we assume that $u, v \in C_c^\infty(\mathbb{R}^n)$ such that $\|u\|_{H^{-r}(\mathbb{R}^n)} = \|v\|_{H^t(\mathbb{R}^n)} = 1$. In parts (i) and (ii) we can assume that $r' < t$ since if $r' \geq t$, then we have the continuous inclusion $H^{r',\infty}(\Omega) \hookrightarrow H^{r'',\infty}(\Omega)$ where $\max\{0, r\} \leq r'' < t$ (such r'' always exists since $t > \max\{0, r\}$).

(i) Let $f \in \widetilde{H}^{r',\infty}(\Omega)$. Now $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\|f_2\|_{H^{r',\infty}(\mathbb{R}^n)} \leq \epsilon$. Then

$$\begin{aligned}|\langle f_2, uv\rangle| &\leq \|f_2 v\|_{H^{r'}(\mathbb{R}^n)} \|u\|_{H^{-r'}(\mathbb{R}^n)} \leq C \|f_2\|_{H^{r',\infty}(\mathbb{R}^n)} \|v\|_{H^{r'}(\mathbb{R}^n)} \|u\|_{H^{-r}(\mathbb{R}^n)} \\ &\leq C\epsilon \|v\|_{H^t(\mathbb{R}^n)} = C\epsilon.\end{aligned}$$

Here we used the Kato-Ponce inequality (lemma 2.3)

$$\begin{aligned}\left\|J^{r'}(f_2 v)\right\|_{L^2(\mathbb{R}^n)} &\leq C(\|f_2\|_{L^\infty(\mathbb{R}^n)} \left\|J^{r'} v\right\|_{L^2(\mathbb{R}^n)} + \left\|J^{r'} f_2\right\|_{L^\infty(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}) \\ &\leq C \|f_2\|_{H^{r',\infty}(\mathbb{R}^n)} \|v\|_{H^{r'}(\mathbb{R}^n)}\end{aligned}$$

and the assumption $\max\{0, r\} \leq r' < t$. Therefore $\|f - f_1\|_{-r,-t} = \|f_2\|_{-r,-t} \leq C\epsilon$ which shows that $f \in M_0(H^{-r} \to H^{-t})$.

(ii) Let $f \in H_0^{r',\infty}(\Omega)$. Now $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\|f_2\|_{H^{r',\infty}(\Omega)} \leq \epsilon$. By the definition of the quotient norm $\|\cdot\|_{H^{r',\infty}(\Omega)}$ we can take $F \in H^{r',\infty}(\mathbb{R}^n)$ such that $F|_\Omega = f_2$ and $\|F\|_{H^{r',\infty}(\mathbb{R}^n)} \leq 2 \|f_2\|_{H^{r',\infty}(\Omega)}$. The assumptions imply the duality $(H^{-r'}(\Omega))^* = H_0^{r'}(\Omega) \subset$

$H^{r'}(\Omega)$. Using the Kato-Ponce inequality for the extension F we obtain as in the proof of part (i) that

$$\left\| J^{r'}(Fv) \right\|_{L^2(\mathbb{R}^n)} \leq C \|F\|_{H^{r',\infty}(\mathbb{R}^n)} \|v\|_{H^{r'}(\mathbb{R}^n)} \leq 2C \|f_2\|_{H^{r',\infty}(\Omega)} \|v\|_{H^t(\mathbb{R}^n)} \leq 2C\epsilon$$

and hence

$$\begin{aligned} |\langle f_2, uv\rangle| &\leq \|f_2 v\|_{(H^{-r'}(\Omega))^*} \|u\|_{H^{-r'}(\Omega)} \leq \|f_2 v\|_{H^{r'}(\Omega)} \|u\|_{H^{-r}(\mathbb{R}^n)} \\ &\leq \left\| J^{r'}(Fv) \right\|_{L^2(\mathbb{R}^n)} \leq 2C\epsilon. \end{aligned}$$

This shows that $f \in M_0(H^{-r} \to H^{-t})$.

(iii) Let $f \in \widetilde{H}^{r'}(\Omega)$. Now $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\|f_2\|_{H^{r'}(\mathbb{R}^n)} \leq \epsilon$. Now [3, Theorem 7.3] implies the continuity of the multiplication $H^{r'}(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ when $r' \geq t$ and $r' > n/2$. We obtain

$$|\langle f_2, uv\rangle| \leq \|f_2 v\|_{H^t(\mathbb{R}^n)} \|u\|_{H^{-t}(\mathbb{R}^n)} \leq C \|f_2\|_{H^{r'}(\mathbb{R}^n)} \|v\|_{H^t(\mathbb{R}^n)} \|u\|_{H^{-r}(\mathbb{R}^n)} \leq C\epsilon.$$

Hence $f \in M_0(H^{-r} \to H^{-t})$. If Ω is a Lipschitz domain, then $H^{r'}_{\overline{\Omega}}(\mathbb{R}^n) = \widetilde{H}^{r'}(\Omega)$. If in addition $r' \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$, we also have $H_0^{r'}(\Omega) = \widetilde{H}^{r'}(\Omega)$. □

Note that the assumptions in theorem 1.1 satisfy the conditions of the previous lemma since then $r = |\alpha| - s$ and $t = s$. The following lemma gives examples of spaces of lower order coefficients ($|\alpha| \leq s$).

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $t > 0$. The following inclusions hold:*

(i) $L^p(\Omega) \subset M_0(H^0 \to H^{-t})$ whenever $2 \leq p < \infty$ and $p > n/t$. Especially, if Ω is bounded, then $L^\infty(\Omega) \subset M_0(H^0 \to H^{-t})$.

(ii) $\widetilde{H}^r(\Omega) \subset M_0(H^0 \to H^{-t})$ whenever $r \geq 0$ and $r > n/2 - t$. The same holds true for $H^r_{\overline{\Omega}}(\mathbb{R}^n)$ if Ω is a Lipschitz domain, and for $H_0^r(\Omega)$ when Ω is Lipschitz domain and $r \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$.

Proof. Throughout the proof we assume that $u, v \in C_c^\infty(\mathbb{R}^n)$ such that $\|u\|_{L^2(\mathbb{R}^n)} = \|v\|_{H^t(\mathbb{R}^n)} = 1$.

(i) Let $f \in L^p(\Omega)$. By density of $C_c^\infty(\Omega)$ in $L^p(\Omega)$ we have $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\left\| \widetilde{f}_2 \right\|_{L^p(\mathbb{R}^n)} \leq \epsilon$ where $\widetilde{f}_2$ is the zero extension of $f_2 \in L^p(\Omega)$. The assumptions on p imply the continuity of the multiplication $L^p(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ ([3, Theorem 7.3]) and we have

$$\left| \left\langle \widetilde{f}_2, uv \right\rangle \right| \leq \left\| \widetilde{f}_2 v \right\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)} \leq C \left\| \widetilde{f}_2 \right\|_{L^p(\mathbb{R}^n)} \|v\|_{H^t(\mathbb{R}^n)} \leq C\epsilon.$$

This gives that $f \in M_0(H^0 \to H^{-t})$. If Ω is bounded, we have $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$, giving the second claim.

(ii) Let $f \in \widetilde{H}^r(\Omega)$. Now we have $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\|f_2\|_{H^r(\mathbb{R}^n)} \leq \epsilon$. The assumptions on r imply that the multiplication $H^r(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is continuous ([3, Theorem 7.3]). We obtain

$$|\langle f_2, uv\rangle| \leq \|f_2 v\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)} \leq C \|f_2\|_{H^r(\mathbb{R}^n)} \|v\|_{H^t(\mathbb{R}^n)} \leq C\epsilon$$

and therefore $f \in M_0(H^0 \to H^{-t})$. The claims for $H^r_{\overline{\Omega}}(\mathbb{R}^n)$ and $H_0^r(\Omega)$ follow as in the proof of part (iii) of lemma 2.8 from the usual identifications for Lipschtiz domains. □

As mentioned above we put $t = s > 0$ in theorem 1.1 and the condition in lemma 2.9 is satisfied. Note that under the assumption $|\alpha| \leq s$ we have $M_0(H^0 \to H^{-s}) \subset M_0(H^{s-|\alpha|} \to H^{-s})$. Hence we can choose the lower order coefficients from a less regular space in theorem 1.1 (compare to lemma 2.8).

3. Main theorem for singular coefficients

In this section, to shorten the notation, we will write $\|\cdot\|_{H^s}$, $\|\cdot\|_{L^2}$ and so on for the global norms in $\mathbb{R}^n$ when the base set is not written explicitly.

3.1. **Well-posedness of the inverse problem.** Consider the problem

$$(-\Delta)^s u + \sum_{|\alpha|\le m} a_\alpha (D^\alpha u) = F \quad \text{in } \Omega, \qquad u = f \quad \text{in } \Omega_e \tag{4}$$

and the corresponding *adjoint-problem*

$$(-\Delta)^s u^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_\alpha u^*) = F^* \quad \text{in } \Omega, \qquad u^* = f^* \quad \text{in } \Omega_e. \tag{5}$$

Note that if $u, u^* \in H^s(\mathbb{R}^n)$ and $a_\alpha \in M(H^{s-|\alpha|} \to H^{-s}) = M(H^s \to H^{|\alpha|-s})$, then $a_\alpha(D^\alpha u) \in H^{-s}(\mathbb{R}^n)$ and $D^\alpha(a_\alpha u^*) \in H^{-s}(\mathbb{R}^n)$ matching with $(-\Delta)^s u, (-\Delta)^s u^* \in H^{-s}(\mathbb{R}^n)$.

The problems (4) and (5) are associated to the bilinear forms

$$B_P(v, w) := \langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w\rangle_{\mathbb{R}^n} + \sum_{|\alpha|\le m} \langle a_\alpha, (D^\alpha v) w\rangle_{\mathbb{R}^n} \tag{6}$$

and

$$B_P^*(v, w) := \langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w\rangle_{\mathbb{R}^n} + \sum_{|\alpha|\le m} \langle a_\alpha, v(D^\alpha w)\rangle_{\mathbb{R}^n}, \tag{7}$$

defined on $v, w \in C_c^\infty(\mathbb{R}^n)$. In the latter terms of the bilinear forms we have written the dual pairing as $\langle \cdot, \cdot\rangle_{\mathbb{R}^n}$ since a_α is now a distribution in the whole space $\mathbb{R}^n$ in contrast to section 4 where a_α is an object defined only in Ω.

Remark 3.1. *Observe that B_P is not symmetric, which motivates the introduction of the bilinear form B_P^*. Moreover, one sees by simple inspection that $B_P(v, w) = B_P^*(w, v)$ for all $v, w \in C_c^\infty(\mathbb{R}^n)$. This identity holds for $v, w \in H^s(\mathbb{R}^n)$ as well by density, thanks to the following boundedness lemma.*

Lemma 3.2 (Boundedness of the bilinear forms)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ such that $2s \ge m$, and let $a_\alpha \in M(H^{s-|\alpha|} \to H^{-s})$. Then B_P and B_P^* extend as bounded bilinear forms on $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$.*

Proof of lemma 3.2. We only prove the boundedness of B_P, as for B_P^* one can proceed in the same way. The proof is a simple calculation following from inequality (3). Let $u, v \in C_c^\infty(\mathbb{R}^n)$. We can then estimate that

$$\begin{aligned}
|B_P(v, w)| &\le |\langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w\rangle_{\mathbb{R}^n}| + \sum_{|\alpha|\le m} |\langle a_\alpha, D^\alpha v w\rangle_{\mathbb{R}^n}| \\
&\le \|w\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)} + \sum_{|\alpha|\le m} \|a_\alpha\|_{s-|\alpha|,-s} \|D^\alpha v\|_{H^{s-|\alpha|}(\mathbb{R}^n)} \|w\|_{H^s(\mathbb{R}^n)} \\
&\le \left(1 + \sum_{|\alpha|\le m} \|a_\alpha\|_{s-|\alpha|,-s}\right) \|w\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Now the claim follows from the density of $C_c^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$. □

Next we shall prove existence and uniqueness of solutions for the problems (4) and (5). To this end, we will use the following form of Young's inequality, which holds for all $a, b, \eta \in \mathbb{R}^+$

and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$:

$$ab \leq \frac{(q\eta)^{-p/q}}{p} a^p + \eta b^q. \tag{8}$$

The validity of (8) is easily proved by choosing $a_1 = a(q\eta)^{-1/q}$ and $b_1 = b(q\eta)^{1/q}$ in Young's inequality $a_1 b_1 \leq a_1^p/p + b_1^q/q$.

Lemma 3.3 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$, and let $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$. There exist a real number $\mu > 0$ and a countable set $\Sigma \subset (-\mu, \infty)$ of eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \to \infty$ such that if $\lambda \in \mathbb{R} \setminus \Sigma$, for any $f \in H^s(\mathbb{R}^n)$ and $F \in (\widetilde{H}^s(\Omega))^*$ there exists unique $u \in H^s(\mathbb{R}^n)$ such that $u - f \in \widetilde{H}^s(\Omega)$ and*

$$B_P(u, v) - \lambda \langle u, v \rangle_\Omega = F(v) \quad \text{for all} \quad v \in \widetilde{H}^s(\Omega).$$

One has the estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C \left(\|f\|_{H^s(\mathbb{R}^n)} + \|F\|_{(\widetilde{H}^s(\Omega))^*} \right).$$

The function u is also the unique $u \in H^s(\mathbb{R}^n)$ satisfying

$$r_\Omega \left((-\Delta)^s + \sum_{|\alpha| \leq m} a_\alpha D^\alpha - \lambda \right) u = F$$

in the sense of distributions in Ω and $u - f \in \widetilde{H}^s(\Omega)$. Moreover, if (14) *holds then $0 \notin \Sigma$.*

Proof. Let $\tilde{u} := u - f$. The above problem is reduced to finding a unique $\tilde{u} \in \widetilde{H}^s(\Omega)$ such that $B_P(\tilde{u}, v) - \lambda \langle \tilde{u}, v \rangle_\Omega = \tilde{F}(v)$, where $\tilde{F} := F - B_P(f, \cdot) + \lambda \langle f, \cdot \rangle_\Omega$. Observe that the modified functional $\tilde{F}$ belongs to $(\widetilde{H}^s(\Omega))^*$ as well, since by lemma 3.2 we have for all $v \in \widetilde{H}^s(\Omega)$

$$|\tilde{F}(v)| \leq |F(v)| + |B_P(f, v)| + |\lambda| \, |\langle f, v \rangle_\Omega| \leq (\|F\|_{(\widetilde{H}^s(\Omega))^*} + (C + |\lambda|)\|f\|_{H^s(\mathbb{R}^n)})\|v\|_{H^s(\mathbb{R}^n)}.$$

Since $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$, for any $\epsilon > 0$ we can write $a_\alpha = a_{\alpha,1} + a_{\alpha,2}$, where $a_{\alpha,1} \in C_c^\infty(\mathbb{R}^n) \cap M(H^{s-|\alpha|} \to H^{-s})$ and $\|a_{\alpha,2}\|_{s-|\alpha|,-s} < \epsilon$. Thus by formula (3), the continuity of the multiplication $H^r(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n)$ for large enough $r \in \mathbb{R}$ (see [3, Theorem 7.3]) and the fact that $a_{\alpha,1} \in C_c^\infty(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ for all $r \in \mathbb{R}$ we obtain

(9)

$$\begin{aligned}
|\langle a_\alpha, D^\alpha v w \rangle| &\leq |\langle a_{\alpha,1}, D^\alpha v w \rangle| + |\langle a_{\alpha,2}, D^\alpha v w \rangle| \\
&\leq \|a_{\alpha,1}\|_{H^r(\mathbb{R}^n)} \|D^\alpha v\|_{H^{-s}(\mathbb{R}^n)} \|w\|_{H^s(\mathbb{R}^n)} + \|a_{\alpha,2}\|_{s-|\alpha|,-s} \|D^\alpha v\|_{H^{s-|\alpha|}(\mathbb{R}^n)} \|w\|_{H^s(\mathbb{R}^n)} \\
&\leq c\|w\|_{H^s(\mathbb{R}^n)} \left(\|a_{\alpha,1}\|_{H^r(\mathbb{R}^n)} \|v\|_{H^{|\alpha|-s}(\mathbb{R}^n)} + \epsilon \|v\|_{H^s(\mathbb{R}^n)} \right)
\end{aligned}$$

where $r \in \mathbb{R}$ is large enough ($r > \max\{s, n/2\}$ is sufficient). If $|\alpha| < s$, from formulas (9) and (8) with $p = q = 2$ we get directly

$$\begin{aligned}
|\langle a_\alpha, D^\alpha v v \rangle| &\leq C \left(\|v\|_{H^s(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} + \epsilon \|v\|^2_{H^s(\mathbb{R}^n)} \right) \\
&\leq C(\epsilon^{-1} \|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon \|v\|^2_{H^s(\mathbb{R}^n)})
\end{aligned} \tag{10}$$

for a constant C independent of v, w, ϵ. If instead $|\alpha| > s$ (observe that we can not have $|\alpha| = s$, because s can not be an integer), we use the interpolation inequality

$$\|v\|_{H^{|\alpha|-s}(\mathbb{R}^n)} \leq C \|v\|^{1-(|\alpha|-s)/s}_{L^2(\mathbb{R}^n)} \|v\|^{(|\alpha|-s)/s}_{H^s(\mathbb{R}^n)} = C \|v\|^{2-|\alpha|/s}_{L^2(\mathbb{R}^n)} \|v\|^{|\alpha|/s-1}_{H^s(\mathbb{R}^n)}$$

in order to get

$$|\langle a_\alpha, D^\alpha v w \rangle| \leq C \|w\|_{H^s(\mathbb{R}^n)} \left(\|v\|^{2-|\alpha|/s}_{L^2(\mathbb{R}^n)} \|v\|^{|\alpha|/s-1}_{H^s(\mathbb{R}^n)} + \epsilon \|v\|_{H^s(\mathbb{R}^n)} \right).$$

Then by formula (8) with

$$a=\|v\|_{L^2(\mathbb{R}^n)}^{2-|\alpha|/s},\quad b=\|v\|_{H^s(\mathbb{R}^n)}^{|\alpha|/s-1},\quad p=\frac{s}{2s-|\alpha|},\quad q=\frac{s}{|\alpha|-s},\quad \eta=\epsilon$$

we obtain

$$|\langle a_\alpha, D^\alpha vw\rangle|\le C\|w\|_{H^s(\mathbb{R}^n)}\left(\epsilon^{\frac{s-|\alpha|}{2s-|\alpha|}}\|v\|_{L^2(\mathbb{R}^n)}+\epsilon\|v\|_{H^s(\mathbb{R}^n)}\right)$$

for a constant C independent of v, w, ϵ. Now we use formula (8) again, but this time we choose

$$a=\|v\|_{L^2(\mathbb{R}^n)},\quad b=\|v\|_{H^s(\mathbb{R}^n)},\quad q=p=2,\quad \eta=\epsilon^{s/(2s-|\alpha|)}.$$

This leads to

$$\begin{aligned}
|\langle a_\alpha, D^\alpha vv\rangle| &\le C\left(\epsilon^{\frac{s-|\alpha|}{2s-|\alpha|}}\|v\|_{L^2(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}+\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\le C\left(\epsilon^{\frac{-|\alpha|}{2s-|\alpha|}}\|v\|^2_{L^2(\mathbb{R}^n)}+2\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\le C\left(\epsilon^{\frac{-|\alpha|}{2s-|\alpha|}}\|v\|^2_{L^2(\mathbb{R}^n)}+\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\le C'\left(\epsilon^{\frac{-m}{2s-m}}\|v\|^2_{L^2(\mathbb{R}^n)}+\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)
\end{aligned}\tag{11}$$

where C, C' are constants changing from line to line. Observe that C' can be taken independent of α. Eventually, using (10) and (11) we get

$$\begin{aligned}
B_P(v,v) &\ge \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)}-\sum_{|\alpha|\le m}|\langle a_\alpha, D^\alpha vv\rangle|\\
&\ge \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)}-C'\left((\epsilon^{\frac{-m}{2s-m}}+\epsilon^{-1})\|v\|^2_{L^2(\mathbb{R}^n)}+\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right).
\end{aligned}\tag{12}$$

By the higher order Poincaré inequality (lemma 2.2) (24) turns into

$$\begin{aligned}
B_P(v,v) &\ge c\left(\|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)}+\|v\|^2_{L^2(\mathbb{R}^n)}\right)-C'\left((\epsilon^{\frac{-m}{2s-m}}+\epsilon^{-1})\|v\|^2_{L^2(\mathbb{R}^n)}+\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\ge c\|v\|^2_{H^s(\mathbb{R}^n)}-C'\left((\epsilon^{\frac{-m}{2s-m}}+\epsilon^{-1})\|v\|^2_{L^2(\mathbb{R}^n)}+\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)
\end{aligned}$$

for some constant $c=c(\Omega,n,s)$ changing from line to line. For ϵ small enough, this eventually gives the coercivity estimate

$$B_P(v,v)\ge c_0\|v\|^2_{H^s(\mathbb{R}^n)}-\mu\|v\|^2_{L^2(\mathbb{R}^n)}\tag{13}$$

for some constants $c_0, \mu>0$ independent of v.

As a consequence of the coercivity estimate, $B_P(\cdot,\cdot)+\mu\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}$ is an equivalent inner product on $\widetilde{H}^s(\Omega)$. Therefore, by the Riesz representation theorem there exists a bounded linear operator $G_\mu : (\widetilde{H}^s(\Omega))^*\to\widetilde{H}^s(\Omega)$ associating each functional in $(\widetilde{H}^s(\Omega))^*$ to its unique representative in the inner product $B_P(\cdot,\cdot)+\mu\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}$ on $\widetilde{H}^s(\Omega)$. Thus $\tilde{u}:=G_\mu\tilde{F}$ verifies

$$B_P(\tilde{u},v)+\mu\langle\tilde{u},v\rangle_{L^2(\mathbb{R}^n)}=\tilde{F}(v)\quad\text{for all}\quad v\in\widetilde{H}^s(\Omega)$$

and it is the required unique solution $\tilde{u}\in\widetilde{H}^s(\Omega)$. Moreover, G_μ induces a compact, self-adjoint and positive operator $\tilde{G}_\mu : L^2(\Omega)\to L^2(\Omega)$ by the compact Sobolev embedding theorem. The remaining claims follow from the spectral theorem for $\tilde{G}_\mu$ and from the Fredholm alternative as in [15]. □

By the above lemma 3.3, both problems (4) and (5) have a countable set of Dirichlet eigenvalues. Throughout the paper we will assume that the coefficients a_α are such that 0 is not a

Dirichlet eigenvalue for either of the problems. That is, we assume that

$$\begin{cases} \text{if } u \in H^s(\mathbb{R}^n) \text{ solves } (-\Delta)^s u + \sum_{|\alpha|\leq m} a_\alpha D^\alpha u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \\ \text{then } u \equiv 0 \end{cases} \tag{14}$$

and

$$\begin{cases} \text{if } u^* \in H^s(\mathbb{R}^n) \text{ solves } (-\Delta)^s u^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha u^*) = 0 \text{ in } \Omega \text{ and } u^*|_{\Omega_e} = 0, \\ \text{then } u^* \equiv 0. \end{cases} \tag{15}$$

With this in mind, we shall define the DN maps. Consider the abstract trace space $X := H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ equipped with the quotient norm

$$\|[f]\|_X := \inf_{\phi \in \widetilde{H}^s(\Omega)} \|f - \phi\|_{H^s(\mathbb{R}^n)}, \quad f \in H^s(\mathbb{R}^n)$$

and its dual space X^*. We use these in order to define the DN maps associated to the problems (4) and (5), which we study in the following lemma.

Lemma 3.4 (DN maps)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ such that $2s > m$, and let $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$. There exist two continuous linear maps*

$$\Lambda_P : X \to X^* \quad \textit{defined by} \quad \langle \Lambda_P[f], [g] \rangle := B_P(u_f, g)$$

and

$$\Lambda_P^* : X \to X^* \quad \textit{defined by} \quad \langle \Lambda_P^*[f], [g] \rangle := B_P^*(u_f^*, g)$$

where u_f, u_f^* are the unique solutions to the equations

$$(-\Delta)^s u + \sum_{|\alpha|\leq m} a_\alpha D^\alpha u = 0 \quad \textit{in } \Omega, \quad u - f \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha u^*) = 0 \quad \textit{in } \Omega, \quad u^* - f \in \widetilde{H}^s(\Omega)$$

with $f, g \in H^s(\mathbb{R}^n)$. Moreover, the identity $\langle \Lambda_P[f], [g] \rangle = \langle [f], \Lambda_P^*[g] \rangle$ holds.

Proof. We show well-definedness and continuity only for Λ_P, the proof being similar for Λ_P^*. We note that such unique solutions exist by lemma 3.3.

If $\phi \in \widetilde{H}^s(\Omega)$, then $u_f|_{\Omega_e} = f = u_{f+\phi}|_{\Omega_e}$, and also u_f, $u_{f+\phi}$ both solve $(-\Delta)^s u + Pu = 0$ in Ω. By unicity of solutions, we must then have that u_f and $u_{f+\phi}$ coincide. On the other hand, if $\psi \in \widetilde{H}^s(\Omega)$, then $\psi|_{\Omega_e} = 0$. These two facts imply the well-definedness of Λ_P, since

$$B_P(u_{f+\phi}, g + \psi) = B_P(u_f, g) + B_P(u_f, \psi) = B_P(u_f, g).$$

The continuity of Λ_P is an easy consequence of lemma 3.2 and the estimate in lemma 3.3. If $f, g \in H^s(\mathbb{R}^n)$ and $\phi, \psi \in \widetilde{H}^s(\Omega)$, then

$$|\langle \Lambda_P[f], [g] \rangle| = |B_P(u_{f-\phi}, g - \psi)| \leq C \|u_{f-\phi}\|_{H^s} \|g - \psi\|_{H^s} \leq C \|f - \phi\|_{H^s} \|g - \psi\|_{H^s}.$$

By taking the infimum on both sides with respect to ϕ and ψ, we end up with

$$|\langle \Lambda_P[f], [g] \rangle| \leq C \inf_{\phi \in \widetilde{H}^s(\Omega)} \|f - \phi\|_{H^s} \inf_{\psi \in \widetilde{H}^s(\Omega)} \|g - \psi\|_{H^s} = C \|[f]\|_X \|[g]\|_X.$$

The well-posedness result proved above implies that for all $f, g \in H^s(\mathbb{R}^n)$ we have $\langle \Lambda_P[f], [g] \rangle = B_P(u_f, e_g)$, where e_g is a generic extension of $g|_{\Omega_e}$ from Ω_e to $\mathbb{R}^n$. In particular, $\langle \Lambda_P[f], [g] \rangle = B_P(u_f, u_g^*)$. By lemma 3.2 this leads to

$$\langle \Lambda_P[f], [g] \rangle = B_P(u_f, u_g^*) = B_P^*(u_g^*, u_f) = \langle \Lambda_P^*[g], [f] \rangle$$

which conlcudes the proof. $\square$

Remark 3.5. *We should observe at this point that a priori Λ_P^* has no reason to be the adjoint of Λ_P, as the symbols would suggest. However, the identity we proved in lemma 3.4 shows that this is in fact true, and thus there is no abuse of notation.*

3.2. **Proof of injectivity.** The proof of injectivity is based on an Alessandrini identity and the Runge approximation property for our operator, following the scheme developed in [15].

Lemma 3.6 (Alessandrini identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ such that $2s > m$. For $j = 1, 2$, let $a_{j,\alpha} \in M_0(H^{s-|\alpha|} \to H^{-s})$. For any $f_1, f_2 \in H^s(\mathbb{R}^n)$, let $u_1, u_2^* \in H^s(\mathbb{R}^n)$ respectively solve*

$$(-\Delta)^s u_1 + \sum_{|\alpha|\leq m} a_{1,\alpha} D^\alpha u_1 = 0 \quad in\ \ \Omega, \quad u_1 - f_1 \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u_2^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_{2,\alpha} u_2^*) = 0 \quad in\ \ \Omega, \quad u_2^* - f_2 \in \widetilde{H}^s(\Omega).$$

Then we have the integral identity

$$\langle (\Lambda_{P_1} - \Lambda_{P_2})[f_1], [f_2] \rangle = \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha u_1 u_2^* \rangle.$$

Proof. The proof is a simple computation following from lemma 3.4

$$\begin{aligned} \langle (\Lambda_{P_1} - \Lambda_{P_2})[f_1], [f_2] \rangle &= \langle \Lambda_{P_1}[f_1], [f_2] \rangle - \langle \Lambda_{P_2}[f_1], [f_2] \rangle = \langle \Lambda_{P_1}[f_1], [f_2] \rangle - \langle [f_1], \Lambda_{P_2}^*[f_2] \rangle \\ &= B_{P_1}(u_1, u_2^*) - B_{P_2}^*(u_2^*, u_1) = \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha u_1 u_2^* \rangle. \end{aligned}$$ □

Lemma 3.7 (Runge approximation property)**.** *Let $\Omega, W \subset \mathbb{R}^n$ respectively be a bounded open set and a non-empty open set such that $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$, and let $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$. Moreover, let $\mathcal{R} := \{ u_f - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$ where u_f solves*

$$(-\Delta)^s u_f + \sum_{|\alpha|\leq m} a_\alpha D^\alpha u_f = 0 \quad in\ \ \Omega, \quad u_f - f \in \widetilde{H}^s(\Omega)$$

and $\mathcal{R}^* := \{ u_f^* - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$ where u_f^* solves

$$(-\Delta)^s u_f^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha u_f^*) = 0 \quad in\ \ \Omega, \quad u_f^* - f \in \widetilde{H}^s(\Omega).$$

Then $\mathcal{R}$ and $\mathcal{R}^*$ are dense in $\widetilde{H}^s(\Omega)$.

Proof. The proofs of the two statements are similar, so we show only the density of $\mathcal{R}$ in $\widetilde{H}^s(\Omega)$. By the Hahn-Banach theorem, it is enough to prove that any functional F acting on $\widetilde{H}^s(\Omega)$ that vanishes on $\mathcal{R}$ must be identically 0. Thus, let $F \in (\widetilde{H}^s(\Omega))^*$ and assume $F(u_f - f) = 0$ for all $f \in C_c^\infty(W)$. Let ϕ be the unique solution of

$$(-\Delta)^s \phi + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha \phi) = -F \quad \text{in } \Omega, \quad \phi \in \widetilde{H}^s(\Omega). \tag{16}$$

In other words, ϕ is the unique function in $\widetilde{H}^s(\Omega)$ such that $B_P^*(\phi, w) = -F(w)$ for all $w \in \widetilde{H}^s(\Omega)$. Then we can compute

$$\begin{aligned} 0 = F(u_f - f) &= -B_P^*(\phi, u_f - f) = B_P^*(\phi, f) \\ &= \langle (-\Delta)^{s/2} f, (-\Delta)^{s/2} \phi \rangle + \sum_{|\alpha|\leq m} \langle a_\alpha, D^\alpha f \phi \rangle \\ &= \langle f, (-\Delta)^s \phi \rangle. \end{aligned} \tag{17}$$

On the first line of (17) we used that $\phi \in \widetilde{H}^s(\Omega)$ and u_f solves the equation in Ω, and on the last line we used the support condition for f. By the arbitrariety of $f \in C_c^\infty(W)$ we have obtained that $(-\Delta)^s \phi = 0$ in W, and on the same set we also have $\phi = 0$. Using the unique continuation result for the higher order fractional Laplacian given in lemma 2.1 we deduce $\phi \equiv 0$ on all of $\mathbb{R}^n$. The vanishing of the functional F now follows easily from the definition of ϕ. $\square$

Remark 3.8. *We remark that using the same proof one can show that $r_\Omega \mathcal{R} \subset L^2(\Omega)$ and $r_\Omega \mathcal{R}^* \subset L^2(\Omega)$ are dense in $L^2(\Omega)$, where r_Ω is the restriction to Ω. If $F \in L^2(\Omega)$, then F induces an element in $(\widetilde{H}^s(\Omega))^*$ via the inner product $F(w) := \langle F, r_\Omega w\rangle_\Omega$, where $w \in \widetilde{H}^s(\Omega)$. Hence one can choose the solution ϕ in equation* (16) *with F as a source term and complete the proof as in equation* (17) *showing that $(r_\Omega \mathcal{R})^\perp = \{0\}$ in $L^2(\Omega)$ (similarly $(r_\Omega \mathcal{R}^*)^\perp = \{0\}$).*

We are ready to prove the main result of the paper.

Proof of theorem 1.1. **Step 1.** Since one can always shrink the sets W_1 and W_2 if necessary, we can assume without loss of generality that $\overline{W_1} \cap \overline{W_2} = \emptyset$. Let $v_1, v_2 \in C_c^\infty(\Omega)$. By the Runge approximation property proved in lemma 3.7 we can find two sequences of functions $\{f_{j,k}\}_{k\in\mathbb{N}} \subset C_c^\infty(W_j)$, $j = 1, 2$, such that

$$u_{1,k} = f_{1,k} + v_1 + r_{1,k}, \quad u_{2,k}^* = f_{2,k} + v_2 + r_{2,k}$$

where $u_{1,k}$, $u_{2,k}^* \in \widetilde{H}^s(\Omega)$ respectively solve

$$(-\Delta)^s u_{1,k} + \sum_{|\alpha|\le m} a_{1,\alpha} D^\alpha u_{1,k} = 0 \quad \text{in } \Omega, \quad u_{1,k} - f_{1,k} \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u_{2,k}^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_{2,\alpha} u_{2,k}^*) = 0 \quad \text{in } \Omega, \quad u_{2,k}^* - f_{2,k} \in \widetilde{H}^s(\Omega)$$

and $r_{1,k}$, $r_{2,k} \to 0$ in $\widetilde{H}^s(\Omega)$ as $k \to \infty$. By the assumption on the DN maps and the Alessandrini identity from lemma 3.6 we have

$$\begin{aligned}
(18) \qquad 0 = \langle (\Lambda_{P_1} - \Lambda_{P_2})[f_{1,k}], [f_{2,k}]\rangle &= \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha u_{1,k} u_{2,k}^*\rangle \\
&= \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha r_{1,k} u_{2,k}^*\rangle + \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 r_{2,k}\rangle \\
&\quad + \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 v_2\rangle.
\end{aligned}$$

However, for the first two terms on the right hand side of (18) we can deduce

$$\begin{aligned}
\left| \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha r_{1,k} u_{2,k}^*\rangle \right| &\le \sum_{|\alpha|\le m} |\langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha r_{1,k} u_{2,k}^*\rangle| \\
&\le C \|u_{2,k}^*\|_{H^s} \|r_{1,k}\|_{H^s} \sum_{|\alpha|\le m} \|a_{1,\alpha} - a_{2,\alpha}\|_{s-|\alpha|,-s} \to 0
\end{aligned}$$

and

$$\begin{aligned}
| \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 r_{2,k}\rangle | &\le \sum_{|\alpha|\le m} |\langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 r_{2,k}\rangle| \\
&\le C \|r_{2,k}\|_{H^s} \|v_1\|_{H^s} \sum_{|\alpha|\le m} \|a_{1,\alpha} - a_{2,\alpha}\|_{s-|\alpha|,-s} \to 0
\end{aligned}$$

as $k \to \infty$. Thus by taking the limit in formula (18) we obtain

$$(19) \qquad \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 v_2\rangle = 0 \quad \text{for all} \quad v_1, v_2 \in C_c^\infty(\Omega).$$

Step 2. Assume that we have $a_{1,\alpha}|_\Omega = a_{2,\alpha}|_\Omega$ for all α such that $|\alpha| < N$ for some $N \in \mathbb{N}$. We show that the equality of the coefficients also holds for α for which $|\alpha| = N$ and this will prove the theorem by the principle of complete induction.

To this end, consider $v_2 \in C_c^\infty(\Omega)$, and then take $v_1 \in C_c^\infty(\Omega)$ such that $v_1(x) = x^\alpha$ on $\text{supp}(v_2) \Subset \Omega$. Recall that since $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ is a multi-index and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, the symbol x^α is intended to mean $x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$. With this choice of v_1, v_2, equation (19) becomes

$$
\begin{aligned}
(20) \qquad 0 &= \sum_{|\beta|\le m} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta v_1 v_2 \rangle = \sum_{N\le|\beta|\le m} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta (x^\alpha) v_2 \rangle \\
&= \sum_{N<|\beta|\le m} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta (x^\alpha) v_2 \rangle + \sum_{|\beta|=N,\, \beta\neq\alpha} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta (x^\alpha) v_2 \rangle \\
&\quad + \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha (x^\alpha) v_2 \rangle.
\end{aligned}
$$

If $|\beta| > N = |\alpha|$, then there must exist $k \in \{1, 2, ..., n\}$ such that $\beta_k > \alpha_k$. This is true also if $|\beta| = N$ with $\beta \neq \alpha$. In both cases we can compute

$$D^\beta (x^\alpha) = (\partial_{x_1}^{\beta_1} x_1^{\alpha_1})\ (\partial_{x_2}^{\beta_2} x_2^{\alpha_2})\ ...\ (\partial_{x_n}^{\beta_n} x_n^{\alpha_n}) = 0$$

because $\partial_{x_k}^{\beta_k} x_k^{\alpha_k} = 0$. Therefore formula (20) becomes

$$0 = \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha (x^\alpha) v_2 \rangle_{\mathbb{R}^n} = \alpha! \langle a_{1,\alpha} - a_{2,\alpha}, v_2 \rangle_{\mathbb{R}^n}$$

which by the arbitrariety of $v_2 \in C_c^\infty(\Omega)$ implies $a_{1,\alpha}|_\Omega = a_{2,\alpha}|_\Omega$ also for α for which $|\alpha| = N$.

Step 3. We have proved that $a_{1,\alpha}|_\Omega = a_{2,\alpha}|_\Omega$ for all α of the order $|\alpha| \le m$. Since this entails $P_1|_\Omega = P_2|_\Omega$, the proof is complete. □

4. Main theorem for bounded coefficients

We shall now study the case when the coefficients of PDOs are from the bounded spaces $H^{r_\alpha,\infty}(\Omega)$. It should be noted, however, that most of the considerations of the previous section still apply identically.

4.1. Well-posedness of the inverse problem.

We shall define the bilinear forms for the problems (4) and (5) respectively by (6) and (7), just as in the case of singular coefficients. These will turn out to be bounded in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ as well, but the proof we give of this fact is *a fortiori* different. Since now we assume that $a_\alpha \in H^{r_\alpha,\infty}(\Omega) \subset L^\infty(\Omega)$ for $r_\alpha \ge 0$, the duality pairing $\langle a_\alpha, D^\alpha v w \rangle$ becomes an inner product over Ω and we write $\langle a_\alpha(x) D^\alpha v, w \rangle_\Omega$ to emphasize that the coefficients $a_\alpha = a_\alpha(x)$ are now functions defined in Ω.

Lemma 4.1 (Boundedness of the bilinear forms). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in* (2)*. Then B_P and B_P^* extend as bounded bilinear forms on $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$.*

Remark 4.2. *Since $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $|\alpha| \le m < 2s$, we also have that* $\max(0, |\alpha| - s) \le r_\alpha < s$ *for $\delta > 0$ small (see equation* (2)*).*

Proof of lemma 4.1. We only prove the boundedness of B_P, as for B_P^* one can proceed in the same way. If $v, w \in C_c^\infty(\mathbb{R}^n)$, then

$$|\langle a_\alpha(x) D^\alpha v, w \rangle_\Omega| = \left| \int_\Omega a_\alpha w D^\alpha v \, dx \right| \le \|a_\alpha w\|_{(H^{-r_\alpha}(\Omega))^*} \|D^\alpha v\|_{H^{-r_\alpha}(\Omega)}.$$

Since Ω is a Lipschitz domain and $r_\alpha > -1/2$, $r_\alpha \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2} ...\}$, we have $(H^{-r_\alpha}(\Omega))^* = H_0^{r_\alpha}(\Omega) \subset H^{r_\alpha}(\Omega)$. Therefore

$$
\begin{aligned}
(21) \quad |\langle a_\alpha(x) D^\alpha v, w \rangle_\Omega| &\le C \|a_\alpha w\|_{H^{r_\alpha}(\Omega)} \|D^\alpha v\|_{H^{-r_\alpha}(\Omega)} \le C \|A_\alpha w\|_{H^{r_\alpha}(\mathbb{R}^n)} \|D^\alpha v\|_{H^{-r_\alpha}(\Omega)} \\
&\le C \|J^{r_\alpha}(A_\alpha w)\|_{L^2(\mathbb{R}^n)} \|v\|_{H^{|\alpha|-r_\alpha}(\Omega)}
\end{aligned}
$$

where $J = (\mathrm{Id} - \Delta)^{1/2}$ is the Bessel potential and A_α is an extension of a_α from Ω to $\mathbb{R}^n$ such that $A_\alpha|_\Omega = a_\alpha$ and $\|A_\alpha\|_{H^{r_\alpha,\infty}(\mathbb{R}^n)} \leq 2\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}$. Since $r_\alpha \geq 0$, we may estimate the last term of (21) by the Kato-Ponce inequality given in lemma 2.3

$$\begin{aligned}\|J^{r_\alpha}(A_\alpha w)\|_{L^2(\mathbb{R}^n)} &\leq C\left(\|A_\alpha\|_{L^\infty(\mathbb{R}^n)}\|J^{r_\alpha}w\|_{L^2(\mathbb{R}^n)} + \|J^{r_\alpha}A_\alpha\|_{L^\infty(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}\right)\\ &\leq C\|A_\alpha\|_{H^{r_\alpha,\infty}(\mathbb{R}^n)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)} \leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)}.\end{aligned}$$

Substituting this into (21) gives

$$\begin{aligned}|\langle a_\alpha(x)D^\alpha v, w\rangle_\Omega| &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)}\|v\|_{H^{|\alpha|-r_\alpha}(\Omega)}\\ &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}\end{aligned} \tag{22}$$

given that both $r_\alpha < s$ and $|\alpha| - r_\alpha \leq s$ hold by remark 4.2. Eventually we obtain

$$\begin{aligned}|B_P(v,w)| &\leq |\langle(-\Delta)^{s/2}v, (-\Delta)^{s/2}w\rangle_{\mathbb{R}^n}| + \sum_{|\alpha|\leq m}|\langle a_\alpha D^\alpha v, w\rangle_{\mathbb{R}^n}|\\ &\leq \|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \sum_{|\alpha|\leq m} C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}\\ &\leq C\|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$ □

Next we shall prove existence and uniqueness of solutions for the problems (4) and (5). The reasoning is similar to the one for the proof of lemma 3.3, but the details of the computations are quite different.

Lemma 4.3 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in (2). There exist a real number $\mu > 0$ and a countable set $\Sigma \subset (-\mu, \infty)$ of eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \to \infty$ such that if $\lambda \in \mathbb{R} \setminus \Sigma$, for any $f \in H^s(\mathbb{R}^n)$ and $F \in (\widetilde{H}^s(\Omega))^*$ there exists a unique $u \in H^s(\mathbb{R}^n)$ such that $u - f \in \widetilde{H}^s(\Omega)$ and*

$$B_P(u,v) - \lambda\langle u,v\rangle_\Omega = F(v) \quad \textit{for all} \quad v \in \widetilde{H}^s(\Omega).$$

One has the estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\left(\|f\|_{H^s(\mathbb{R}^n)} + \|F\|_{(\widetilde{H}^s(\Omega))^*}\right).$$

The function u is also the unique $u \in H^s(\mathbb{R}^n)$ satisfying

$$r_\Omega\left((-\Delta)^s + \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha - \lambda\right)u = F$$

in the sense of distributions in Ω and $u - f \in \widetilde{H}^s(\Omega)$. Moreover, if (14) holds then $0 \notin \Sigma$.

Proof. Again it is enough to find unique $\tilde{u} \in \widetilde{H}^s(\Omega)$ such that $B_P(\tilde{u},v) - \lambda\langle\tilde{u},v\rangle_\Omega = \tilde{F}(v)$, where $\tilde{F} := F - B_P(f,\cdot) + \lambda\langle f,\cdot\rangle_\Omega$. Consider $v,w \in C_c^\infty(\Omega)$ and $r_\alpha \neq 0$. Since $0 < r_\alpha < s$, the interpolation inequality

$$\|w\|_{H^{r_\alpha}(\mathbb{R}^n)} \leq C\|w\|_{L^2(\mathbb{R}^n)}^{1-r_\alpha/s}\|w\|_{H^s(\mathbb{R}^n)}^{r_\alpha/s}$$

holds. Using this and formula (22) we get, for a constant $C = C(\Omega, n, s, r_\alpha)$ which may change from line to line,

$$\begin{aligned}|\langle a_\alpha(x)D^\alpha v, w\rangle_\Omega| &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)}\\ &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}^{1-r_\alpha/s}\|w\|_{H^s(\mathbb{R}^n)}^{r_\alpha/s}\\ &\leq \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\left(C\epsilon^{r_\alpha/(r_\alpha-s)}\|w\|_{L^2(\mathbb{R}^n)} + \epsilon\|w\|_{H^s(\mathbb{R}^n)}\right).\end{aligned} \tag{23}$$

In the last step of (23) we used formula (8) with

$$q = \frac{s}{r_\alpha}, \quad p = \frac{s}{s - r_\alpha}, \quad b = \|w\|_{H^s(\mathbb{R}^n)}^{r_\alpha/s}, \quad a = C\|w\|_{L^2(\mathbb{R}^n)}^{1-r_\alpha/s}, \quad \eta = \epsilon.$$

If instead $r_\alpha = 0$, just by formula (22) we already have

$$|\langle a_\alpha(x) D^\alpha v, w\rangle_\Omega| \leq C\|a_\alpha\|_{L^\infty(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}.$$

Moreover, the two estimates above also hold for $v, w \in \widetilde{H}^s(\Omega)$ by the density of $C_c^\infty(\Omega)$ in $\widetilde{H}^s(\Omega)$. Now we use formula (8) again, but this time we choose

$$q = p = 2, \quad b = \|v\|_{H^s(\mathbb{R}^n)}, \quad a = \|v\|_{L^2(\mathbb{R}^n)}, \quad \eta = \epsilon^{s/(s-r_\alpha)}.$$

This leads to

$$\begin{aligned}
|\langle a_\alpha(x) D^\alpha v, v\rangle_\Omega| &\leq \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)} \left(C\epsilon^{r_\alpha/(r_\alpha - s)}\|v\|_{L^2(\mathbb{R}^n)} + \epsilon\|v\|_{H^s(\mathbb{R}^n)} \right) \\
&= \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)} \left(C\epsilon^{r_\alpha/(r_\alpha - s)}\|v\|_{L^2(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)} \right) \\
&\leq \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)} \left(C\epsilon^{\frac{r_\alpha+s}{r_\alpha-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon(C+1)\|v\|^2_{H^s(\mathbb{R}^n)} \right) \\
&\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)} \left(\epsilon^{\frac{r_\alpha+s}{r_\alpha-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)} \right) \\
&\leq C'\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)} \left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)} \right)
\end{aligned}$$

where $C = C(\Omega, n, s, r_\alpha)$ and $C' = C'(\Omega, n, s)$ are constants changing from line to line and $M \in [0, s)$ is defined by $M := \max_{|\alpha|\leq m} r_\alpha$. Eventually

$$\begin{aligned}
(24)\quad B_P(v,v) &\geq \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - \sum_{|\alpha|\leq m} |\langle a_\alpha(x) D^\alpha v, v\rangle_\Omega| \\
&\geq \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - C' \left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)} \right) \sum_{|\alpha|\leq m} \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)} \\
&= \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - C'C'' \left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)} \right)
\end{aligned}$$

where $C'' := \sum_{|\alpha|\leq m} \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}$ is a constant independent of ϵ and v. By the higher order Poincaré inequality (lemma 2.2) (24) turns into

$$\begin{aligned}
B_P(v,v) &\geq c\left(\|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)} \right) - C'C'' \left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)} \right) \\
&\geq c\|v\|^2_{H^s(\mathbb{R}^n)} - C'C'' \left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)} \right)
\end{aligned}$$

for some constant $c = c(\Omega, n, s)$ changing from line to line. For ϵ small enough (notice that $M - s < 0$), this eventually gives the coercivity estimate

$$(25)\qquad B_P(v,v) \geq c_0\|v\|^2_{H^s(\mathbb{R}^n)} - \mu\|v\|^2_{L^2(\mathbb{R}^n)}$$

for some constants $c_0, \mu > 0$ independent of v. The proof is now concluded as in lemma 3.3. □

Assuming as in Section 3 that both (14) and (15) hold, by means of the above lemma 4.3 we can define the DN-maps Λ_P, Λ_P^* just as in lemma 3.4. We also arrive at the same Alessandrini identity and Runge approximation property which we get in lemmas 3.6 and 3.7.

Lemma 4.4 (DN maps)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in (2). There exist two continuous linear maps*

$$\Lambda_P : X \to X^* \quad \textit{defined by} \quad \langle \Lambda_P[f], [g]\rangle := B_P(u_f, g)$$

and

$$\Lambda_P^* : X \to X^* \quad \textit{defined by} \quad \langle \Lambda_P^*[f], [g]\rangle := B_P^*(u_f^*, g)$$

where u_f, u_f^* are the unique solutions to the equations

$$(-\Delta)^s u + \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u = 0 \quad \textit{in } \Omega, \quad u - f \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) u^*) = 0 \quad \text{in } \Omega, \quad u^* - f \in \widetilde{H}^s(\Omega)$$

with $f, g \in H^s(\mathbb{R}^n)$. Moreover, the identity $\langle \Lambda_P[f], [g]\rangle = \langle [f], \Lambda_P^*[g]\rangle$ holds.

Lemma 4.5 (Alessandrini identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in (2). For any $f_1, f_2 \in H^s(\mathbb{R}^n)$, let $u_1, u_2^* \in H^s(\mathbb{R}^n)$ respectively solve*

$$(-\Delta)^s u_1 + \sum_{|\alpha|\leq m} a_{1,\alpha}(x) D^\alpha u_1 = 0 \quad \text{in } \Omega, \quad u_1 - f_1 \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u_2^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_{2,\alpha}(x) u_2^*) = 0 \quad \text{in } \Omega, \quad u_2^* - f_2 \in \widetilde{H}^s(\Omega).$$

Then we have the integral identity

$$\langle (\Lambda_{P_1} - \Lambda_{P_2})[f_1], [f_2]\rangle = \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha u_1, u_2^* \rangle_\Omega.$$

Lemma 4.6 (Runge approximation property)**.** *Let $\Omega, W \subset \mathbb{R}^n$ respectively be a bounded Lipschitz domain and a non-empty open set such that $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in (2). Moreover, let $\mathcal{R} := \{ u_f - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$, where u_f solves*

$$(-\Delta)^s u_f + \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u_f = 0 \quad \text{in } \Omega, \quad u_f - f \in \widetilde{H}^s(\Omega)$$

and $\mathcal{R}^* := \{ u_f^* - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$, where u_f^* solves

$$(-\Delta)^s u_f^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) u_f^*) = 0 \quad \text{in } \Omega, \quad u_f^* - f \in \widetilde{H}^s(\Omega).$$

Then $\mathcal{R}$ and $\mathcal{R}^*$ are dense in $\widetilde{H}^s(\Omega)$.

4.2. **Proof of injectivity.**

Proof of theorem 1.2. The proof is virtually identical to the one of theorem 1.1, the unique difference being in the way the error terms of the Runge approximation are estimated. We make use of (22), which relied on the Kato-Ponce inequality instead of multiplier space estimates. In this way we get

$$\begin{aligned}\left| \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha r_{1,k}, u_{2,k}^* \rangle_{\mathbb{R}^n} \right| &\leq \sum_{|\alpha|\leq m} |\langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha r_{1,k}, u_{2,k}^* \rangle_{\mathbb{R}^n}| \\ &\leq C \|u_{2,k}^*\|_{H^s(\mathbb{R}^n)} \|r_{1,k}\|_{H^s(\mathbb{R}^n)} \sum_{|\alpha|\leq m} \|a_{1,\alpha} - a_{2,\alpha}\|_{H^{r_\alpha,\infty}(\Omega)} \to 0\end{aligned}$$

and

$$\begin{aligned}| \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha v_1, r_{2,k} \rangle_{\mathbb{R}^n} | &\leq \sum_{|\alpha|\leq m} |\langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha v_1, r_{2,k} \rangle_{\mathbb{R}^n}| \\ &\leq C \|r_{2,k}\|_{H^s(\mathbb{R}^n)} \|v_1\|_{H^s(\mathbb{R}^n)} \sum_{|\alpha|\leq m} \|a_{1,\alpha} - a_{2,\alpha}\|_{H^{r_\alpha,\infty}(\Omega)} \to 0. \square\end{aligned}$$

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On tensor decompositions and algebraic structure of the mixed and transverse ray transforms

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ON TENSOR DECOMPOSITIONS AND ALGEBRAIC STRUCTURE OF THE MIXED AND TRANSVERSE RAY TRANSFORMS

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Abstract. The geodesic ray transform, the mixed ray transform and the transverse ray transform of a tensor field on a manifold can all be seen as what we call mixing ray transforms, compositions of the geodesic ray transform and an invertible linear map on tensor fields. We show that the characterization of the kernel and the stability of a mixing ray transform can be reduced to the same properties of any other mixing ray transform. Our approach applies to various geometries and ray transforms, including the light ray transform. In particular, we extend studies in de Hoop–Saksala–Zhai (2019) from compact simple surfaces to orientable surfaces with solenoidally injective geodesic ray transform. Our proofs are based on algebraic arguments.

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1. Introduction

This article gives an algebraic point of view to various geodesic ray transforms of tensor fields, unifying the Riemannian X-ray transform, the transverse ray transform, and the mixed ray transform, and the light ray transforms on Lorentzian manifolds. This comes with a natural notion of symmetry, which is not the same as the symmetry of the covariant tensor field whose integral transforms are under study, but arises from the structure of the relevant transform. For example, in the case of light ray tomography of 2-tensor fields on Lorentzian manifolds, the concept of symmetry automatically includes the "conformal gauge freedom".

When two transforms differ from each other by a so-called mixing, they have the same injectivity properties by theorem 3.3. Mixings turn mixed ray transforms into regular tensor transforms in two dimensions. In corollary 3.7 we recast the injectivity result [7] of the mixed ray transform on simple surfaces in our language and we provide a reproof in corollary 4.1. These results are also extended to Cartan–Hadamard manifolds in corollary 4.2.

The tensor tomography results [8] on globally hyperbolic Lorentzian manifolds have a different kind of kernel than their Riemannian counterpart. The kernel, when operating on symmetric tensor fields of order $m \geq 2$, contains both potential fields and conformal multiples of the metric. In the present approach the conformal gauge is absorbed into the concept of symmetry, making the statements of solenoidal injectivity (s-injectivity) fully analogous on Riemannian and Lorentzian manifolds; see corollary 3.9.

A number of corollaries of the method are given in this article, and we refrain from listing them all here. Consequently we have a great amount of notation, and we have collected the key items in appendix A to help the reader.

1.1. **Mixing ray transforms.** Let M be a Riemannian manifold of dimension $n \geq 2$. Let $f \in \mathfrak{X}(T_m M)$ be a covariant m-tensor field (not necessarily symmetric) where $m \geq 1$. We completely exclude the scalar case $m = 0$ from our discussion. Let $A\colon \mathfrak{X}(T_m M) \to \mathfrak{X}(T_m M)$ be an invertible linear map such that

$$(Af)_x(v_1, \ldots, v_m) = f_x(A_1(x)v_1, \ldots, A_m(x)v_m), \tag{1.1}$$

where $A_i(x)\colon T_x M \to T_x M$ are linear isomorphisms. The linear maps $\mathfrak{X}(T_m M) \to \mathfrak{X}(T_m M)$ of this form are called *mixings* in this article.

We study the class of geodesic ray transforms, called *mixing ray transforms*, defined by the formula

(1.2)

$$\begin{aligned} I_A f(x, v) &:= \int_{\tau_-(x,v)}^{\tau_+(x,v)} (Af)_{\gamma_{x,v}(t)}(\dot{\gamma}_{x,v}(t)^{\otimes m}) \mathrm{d}t \\ &= \int_{\tau_-(x,v)}^{\tau_+(x,v)} f_{\gamma_{x,v}(t)}(A_1(\gamma_{x,v}(t))\dot{\gamma}_{x,v}(t), \ldots, A_m(\gamma_{x,v}(t))\dot{\gamma}_{x,v}(t)) \mathrm{d}t, \end{aligned}$$

where $\gamma_{x,v}\colon [\tau_-(x,v), \tau_+(x,v)] \to M$ is the maximal unit speed geodesic through $(x, v) \in SM$. Formula (1.2) is invariant under the geodesic flow $\varphi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$, that is, $I_A f(x, v) = I_A f(\varphi_t(x, v))$ for any $t \in$

$\mathbb{R}$ in the maximal domain of $\gamma_{x,v}$. This definition allows to define I_A on Riemannian manifolds without boundary, provided that the tensor field f is sufficiently integrable. We remark that if $A_i = \mathrm{Id}$ for every $i = 1, \ldots, m$, then I_A is the usual geodesic ray transform of tensor fields. Other special cases of the mixing ray transforms in two dimensions have been studied earlier in [5, 7, 26], and somewhat related geodesic ray transforms in higher dimensions have been studied recently in [1, 6, 17]. We remark that the mixing ray transforms are defined for all $n \geq 2$ but they do not include the higher dimensional transforms ($n \geq 3$) studied in [1, 6, 17].

The main problems that we study are uniqueness and stability for recovering $f \in \mathfrak{X}(T_m M)$ from the knowledge of $I_A f$. The main point of this work is an algebraic view of the mixing ray transforms. We present many applications of the method and instead of having a main theorem we have a main idea how to study the mixing ray transforms. We show in theorem 3.3 and corollary 3.4 that the related inverse problems for I_A and $I_{\widetilde{A}}$ with two different mixings A and $\widetilde{A}$ can be reduced to each other. Especially, this allows us to derive new uniqueness and stability results for the mixed and transverse ray transforms in two dimensions using the known results for the geodesic ray transform. These results are given in corollaries 4.1, 4.2, 4.5 and 4.7. Moreover, we show in corollaries 4.9 and 4.13 that on compact simple surfaces and on certain Cartan–Hadamard manifolds the geodesic ray transform and the transverse ray transform together determine one-forms uniquely. This extends results in [5] to more general Riemannian manifolds.

Furthermore, we study tensor decompositions and their symmetries with respect to these integral transforms. These considerations lead us to corollaries 3.7 and 3.9 which show how the earlier kernel characterizations of the mixed ray transform on compact simple surfaces and the light ray transform on static globally hyperbolic Lorentzian manifolds can be seen as s-injectivity results under the correct notions of symmetry.

1.2. **Related problems.** The geodesic ray transform has been studied extensively on Riemannian manifolds and s-injectivity is known in many cases. For example, the geodesic ray transform is s-injective on tensor fields of any order on two-dimensional compact simple manifolds [21] and on simply connected compact manifolds with strictly convex boundary and non-positive curvature [20, 23, 26]. S-injectivity is also known on non-compact Cartan–Hadamard manifolds for all tensor fields which satisfy certain decay conditions [15, 16]. We refer to the surveys [12, 22] for a more comprehensive treatment of the geodesic ray transform and s-injectivity. The mixed ray transform has been studied mainly on two- and three-dimensional compact simple manifolds, and the kernel is known in these cases for a certain class of tensor fields [6, 7] (see also [26]). There are a few results for the transverse ray transform: in $\mathbb{R}^2$ the kernel of the transverse ray transform consists of curls of scalar fields [19] and in higher dimensions the transform is even injective on certain manifolds [26] (see also [1] for a support theorem).

The usual applications of the geodesic ray transform are medical imaging [18, 19], Doppler tomography [24, 27] and seismic imaging [26, 30]. The transverse ray transform has applications in polarization tomography [26], photoelasticity [11], diffraction tomography [17] and also in the determination of the refractive index of gases [5, 25]. The mixed ray transform arises in seismology as a linearization of elastic travel time tomography problem [6, 26].

Organization of the article. In section 2 we recall the preliminaries on the geodesic ray transform and the mixed ray transform. In section 3 we define the mixing ray transforms and study their basic properties using an algebraic approach. In section 4 we apply our methods for the mixed ray transform and the transverse ray transform on orientable two-dimensional Riemannian manifolds which admit s-injectivity of the geodesic ray transform. We have included some of our notation in appendix A.

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2. Preliminaries

We mainly follow the reference [26] for the integral geometry part of this section. Basic theory of differential geometry can be found in [13, 14] and basic theory of Sobolev spaces of tensor fields on manifolds can be found for example in [3, 33]. We always assume that (M, g) is a connected Riemannian manifold, and we can sometimes allow it to be pseudo-Riemannian.

2.1. **Notation.** If E is a vector bundle, we denote by $\mathfrak{X}(E)$ the space of all smooth sections of E. We use this notation whenever the regularity is unimportant.

We let $T^{m_2}_{m_1}M = T^*M^{\otimes m_1} \otimes TM^{\otimes m_2}$ be the bundle of tensors of type (m_2, m_1) over M. Then $\mathfrak{X}(T^{m_2}_{m_1}M)$ is the space of all (m_2, m_1)-tensor fields on M. We also write $\mathfrak{X}(T_m M) \coloneqq \mathfrak{X}(T^0_m M)$.

We denote by $S_m M \subset \mathfrak{X}(T_m M)$ the space of all symmetric covariant tensor fields. When we want to emphasize the regularity of the tensor field, we replace $\mathfrak{X}$ with the regularity in question; for example $C^q(T_m M) \subset \mathfrak{X}(T_m M)$, $q \in \mathbb{N}$, is the space of all C^q-smooth $(0, m)$-tensor fields on M. For symmetric tensor fields we write $C^q(S_m M)$ and so on. We use the Einstein summation convention, where every repeated index (both as a subscript and superscript) is implicitly summed over.

2.2. **Sobolev norms of tensor fields.** Let $f, h \in \mathfrak{X}(T_m M)$ be tensor fields and $m \geq 1$. We define the fiberwise inner product as

$$g_x(f, h) = g^{i_1 j_1}(x) \dots g^{i_m j_m}(x) f_{i_1 \dots i_m}(x) h_{j_1 \dots j_m}(x) \tag{2.1}$$

and the fiberwise norm is denoted by $|f|_{g_x} = \sqrt{g_x(f, f)}$. If $m = 0$, we simply let $|f|_{g_x} \coloneqq |f(x)|$.

Let $\mathrm{d}V_g(x)$ be the Riemannian volume measure on M. If M is orientable, then $\mathrm{d}V_g(x)$ is given by the Riemannian volume form and $\mathrm{d}V_g(x) = \sqrt{\det g(x)}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n$ where $(x^1, \dots, x^n)$ are any positively oriented smooth coordinates. We define the L^p-norm, $1 \leq p < \infty$, of a tensor field $f \in \mathfrak{X}(T_m M)$ by

$$\|f\|_p = \left(\int_M |f|_{g_x}^p \, \mathrm{d}V_g(x) \right)^{1/p} \tag{2.2}$$

whenever the integral exists.

Denote by $\nabla^k f \in C^{q-k}(T_{m+k}M)$ the kth iterated covariant derivative of the tensor field $f \in C^q(T_m M)$ whenever $q \geq k \geq 0$ and $k, q \in \mathbb{N}$. We define the Sobolev norm $\|\cdot\|_{k,p}$ as

$$\|f\|_{k,p} = \left(\sum_{i=0}^{k} \left\|\nabla^i f\right\|_p^p \right)^{1/p} \tag{2.3}$$

where $\nabla^0 f \coloneqq f$. Let $C^\infty_{k,p}(T_m M)$ be the set of smooth tensor fields f for which $\|f\|_{k,p} < \infty$. The Sobolev space $W^{k,p}(T_m M)$ is defined to be the completion of $C^\infty_{k,p}(T_m M)$ with respect to the norm $\|\cdot\|_{k,p}$. We are mainly interested in the space $W^{k,2}(T_m M) =: H^k(T_m M)$. Then $H^k(T_m M)$ is a Hilbert space with the inner product

$$\langle f, h\rangle_{H^k(T_m M)} = \sum_{i=0}^{k} \left\langle \nabla^i f, \nabla^i h\right\rangle_{L^2(T_{m+i}M)} = \sum_{i=0}^{k} \int_M g_x(\nabla^i f, \nabla^i h) \mathrm{d}V_g(x). \tag{2.4}$$

Similarly one defines the Sobolev space $H^k(S_m M) \subset H^k(T_m M)$ as the completion of $C^\infty_{k,2}(S_m M)$ with respect to the norm induced by the inner product $\langle\cdot,\cdot\rangle_{H^k(T_m M)}$.

2.3. **Hodge star on orientable Riemannian surfaces.** Assume that (M, g) is two-dimensional orientable Riemannian manifold. For example, M is orientable if there is a smooth mapping $F\colon M \to N$ such that F is a local diffeomorphism and N is orientable, or if M is simply connected [13]. The Hodge star $\star$ is an operator on one-forms $\star\colon \mathfrak{X}(T_1 M) \to \mathfrak{X}(T_1 M)$ and it corresponds to a 90 degree rotation counterclockwise. Orientability of M guarantees that $\star$ is a well-defined global operator. Since we can identify one-forms with vector fields by the musical isomorphisms $\flat$ and $\sharp$, we can also rotate vector fields. To shorten the notation, we simply let $\sharp \star \flat =: \star$ and locally we have

$$\star\,(v^1 e_1 + v^2 e_2) \coloneqq -v^2 e_1 + v^1 e_2 \tag{2.5}$$

in any positively oriented local orthonormal frame $\{e_1, e_2\}$.

2.4. **The geodesic ray transform.** For any set X we denote by $\mathcal{F}(X)$ the space of all complex-valued functions $X \to \mathbb{C}$. We define the map $\lambda\colon \mathfrak{X}(T_m M) \to \mathcal{F}(TM)$ as

$$(\lambda f)(x, v) \coloneqq f_x(v, \ldots, v) = f_{i_1 \ldots i_m}(x) v^{i_1} \ldots v^{i_m} \tag{2.6}$$

where $f_{i_1 \ldots i_m}(x)$ are the components of the tensor field $f \in \mathfrak{X}(T_m M)$ in any local coordinates. We let $SM = \bigcup_{x \in M} S_x M$ be the sphere bundle where the fibers are the unit spheres $S_x M = \{v \in T_x M : |v|_{g_x} = 1\}$ of the tangent spaces $T_x M$. The unit sphere bundle SM is not to be confused with the space $S_m M$ of symmetric covariant tensor fields of order m. The geodesic flow is defined as $\varphi_t(x,v) = (\gamma_{x,v}(t), \dot\gamma_{x,v}(t))$ where $\gamma_{x,v}(t)$ is the unique geodesic such that $(\gamma_{x,v}(0), \dot\gamma_{x,v}(0)) = (x,v) \in SM$. If M has boundary ∂M, we denote by $\tau(x,v)$ the first time when the geodesic $\gamma_{x,v}$ reaches ∂M.

Assume that (M,g) is compact and non-trapping Riemannian manifold with boundary. Non-trapping means that $\tau(x,v) < \infty$ for all $(x,v) \in SM$. We denote by $\partial_{\text{in}} SM \subset \partial SM$ the inward-pointing unit vectors. We define the geodesic ray transform to be the operator $I : \mathcal{X} \to \mathcal{F}(\partial_{\text{in}} SM)$ given by the formula

$$If(x,v) = \int_0^{\tau(x,v)} (\lambda f)(\varphi_t(x,v))\mathrm{d}t, \quad (x,v) \in \partial_{\text{in}} SM \tag{2.7}$$

where $\mathcal{X} \subset \mathfrak{X}(T_m M)$ is any set such that the integral in (2.7) is well-defined. Typically we choose $\mathcal{X} = C_c^\infty(T_m M)$ or $\mathcal{X} = H^k(T_m M)$. We note that two definitions (1.2) and (2.7) agree when $A = \text{Id}$ and $(x,v) \in \partial_{\text{in}} SM$ (in that case $\tau_+(x,v) = \tau(x,v)$ and $\tau_-(x,v) = 0$). One can also write $If = I_{SM}(\lambda f|_{SM})$ where the geodesic ray transform of a function $h \colon SM \to \mathbb{R}$ is

$$I_{SM} h(x,v) = \int_0^{\tau(x,v)} h(\varphi_t(x,v))\mathrm{d}t, \quad (x,v) \in \partial_{\text{in}} SM. \tag{2.8}$$

One can then define an adjoint I^* by duality using an L^2-inner product. However, there are different measures on $\partial_{\text{in}} SM$ which lead to different adjoints. We use the weighted measure defined in [22] which is invariant under the scattering relation, and the normal operator $N = I^* I$ is defined with respect to this measure.

If $f \in H^k(S_m M)$ and M is a compact Riemannian manifold with boundary, then there is the solenoidal decomposition [26, Theorem 3.3.2]

$$f = f^{\mathrm{s}} + \sigma\nabla p, \quad \operatorname{div}(f^{\mathrm{s}}) = 0, \quad p|_{\partial M} = 0 \tag{2.9}$$

where $f^{\mathrm{s}} \in H^k(S_m M)$, $p \in H^{k+1}(S_{m-1} M)$ and $m \geq 1$. Moreover, if $f \in C^\infty(S_m M)$, then $f^{\mathrm{s}} \in C^\infty(S_m M)$ and $p \in C^\infty(S_{m-1} M)$. Here σ is the symmetrization of tensor fields (see section 3.1 for details) and $\operatorname{div}(\cdot)$ is the covariant divergence. The tensor field f^{s} is the solenoidal part and $\sigma\nabla p$ is the potential part of f. By the fundamental theorem of calculus one sees that $I(\sigma\nabla p) = 0$ since p vanishes on the boundary. Therefore potentials are always in the kernel of I and we can only try to recover the solenoidal part of f from I. When $m \geq 1$ we say that I is solenoidally injective (s-injective) if for sufficiently regular $f \in S_m M$ it holds that $If = 0$ if and only if $f = \sigma\nabla p$ for some (sufficiently regular) $p \in S_{m-1} M$ vanishing on the boundary.

One particular class of manifolds where one usually studies the geodesic ray transform is the class of compact simple manifolds. The manifold (M,g) is simple if it is non-trapping, has no conjugate points and the boundary ∂M is strictly convex (the second fundamental form on ∂M is positive definite). Each compact simple manifold is diffeomorphic to the Euclidean unit ball. It also follows that compact simple manifolds are simply connected and hence orientable [21, 32].

One can also study the geodesic ray transform on certain non-compact manifolds. The manifold (M,g) without boundary is a Cartan–Hadamard manifold if it is complete, simply connected and its sectional curvature is nonpositive. Cartan–Hadamard manifolds are always non-compact, orientable and diffeomorphic to $\mathbb{R}^n$. Basic examples of Cartan–Hadamard manifolds are Euclidean and hyperbolic spaces. On such manifolds the geodesic ray transform is defined as

$$If(x,v) = \int_{-\infty}^{\infty} (\lambda f)(\varphi_t(x,v))\mathrm{d}t, \quad (x,v) \in SM. \tag{2.10}$$

Note that completeness implies that geodesics are defined on all times by the Hopf–Rinow theorem [14]. We will use the following classes of tensor fields on Cartan–Hadamard manifolds:

$$\begin{aligned} E_\eta(T_m M) &= \{f \in C^1(T_m M) : \\ &\qquad |f|_{g_x} \le Ce^{-\eta d(x,o)} \text{ for some } C > 0\}, \\ E^1_\eta(T_m M) &= \{f \in C^1(T_m M) : \\ &\qquad |f|_{g_x} + |\nabla f|_{g_x} \le Ce^{-\eta d(x,o)} \text{ for some } C > 0\}, \\ P_\eta(T_m M) &= \{f \in C^1(T_m M) : \\ &\qquad |f|_{g_x} \le C(1 + d(x,o))^{-\eta} \text{ for some } C > 0\}, \\ P^1_\eta(T_m M) &= \{f \in C^1(T_m M) : \\ &\qquad |f|_{g_x} \le C(1 + d(x,o))^{-\eta} \text{ and} \\ &\qquad |\nabla f|_{g_x} \le C(1 + d(x,o))^{-\eta-1} \text{ for some } C > 0\}. \end{aligned} \tag{2.11}$$

Here $o \in M$ is fixed reference point and $\eta > 0$. The spaces defined above are independent of the choice of this point.

2.5. **The mixed and transverse ray transforms.** Define $S_k M \otimes S_l M \subset \mathfrak{X}(T_{k+l}M)$ to be the set of $(k+l)$-tensor fields which are symmetric in the first k and last l variables. Let $S_k(T_x M)$ denote the space of symmetric $(0,k)$-tensors on $T_x M$ for any fixed $x \in M$. If $f \in S_k M \otimes S_l M$, then $f_x \in S_k(T_x M) \otimes S_l(T_x M)$. Let $\pi\colon \partial_{\mathrm{in}} SM \to M$ be the restriction of the projection of the tangent bundle. Let $\pi^*(S_k M)$ be the pullback bundle of symmetric k-tensor fields so that for every $\varphi \in \mathfrak{X}(\pi^*(S_k M))$ and $(x,v) \in \partial_{\mathrm{in}} SM$ we have $\varphi_{x,v} \in S_k(T_x M)$.

Let $v \in S_x M$. We define the projection operator $p_v : T_x M \to v^\perp \subset T_x M$ as

$$p_v(w) \coloneqq w - g_x(w,v)v = \left(\delta^i_j - v_j v^i\right) w^j e_i \tag{2.12}$$

where the latter formula holds in any local coordinates. We then define the projection operator $P^k_v\colon S_k(T_x M) \to S_k(T_x M)$ by the formula

$$(P^k_v h)(v_1, \dots, v_k) \coloneqq h(p_v(v_1), \dots, p_v(v_k)) \tag{2.13}$$

for any $v_1, \dots, v_k \in T_x M$, and one can write in any local coordinates that

$$(P^k_v h)_{i_1 \dots i_k} = (\delta^{j_1}_{i_1} - v^{j_1} v_{i_1}) \dots (\delta^{j_k}_{i_k} - v^{j_k} v_{i_k}) h_{j_1 \dots j_k}. \tag{2.14}$$

We can identify p_v as a $(1,1)$-tensor by setting $\tilde{p}_v(\alpha, w) \coloneqq \alpha(p_v(w))$ where $w \in T_x M$ and $\alpha \in T^*_x M$. We note that also $P^k_v h = \tilde{p}_v^{\otimes k} h$ where the product on the right hand side is a contraction of $\tilde{p}_v^{\otimes k}$ by h.

We define the contraction of $f \in S_k(T_xM) \otimes S_l(T_xM)$ by $v \in T_xM$ with respect to the last l arguments as a mapping $\Lambda_v^l\colon S_k(T_xM) \otimes S_l(T_xM) \to S_k(T_xM)$ by

$$(\Lambda_v^l f)_{i_1\dots i_k} = f_{i_1\dots i_k j_1 \dots j_l} v^{j_1} \dots v^{j_l}. \tag{2.15}$$

Let us denote by $\mathcal{T}_\gamma^{t\to s}$ the parallel transport along γ from $\gamma(t)$ to $\gamma(s)$ whenever $s,t \in \mathbb{R}$ belong to the maximal domain of γ. The mixed ray transform is the map $L_{k,l}\colon S_kM \otimes S_lM \to \pi^*(S_kM)$ defined as

$$L_{k,l}f(x,v) := \int_0^{\tau(x,v)} \mathcal{T}_{\gamma_{x,v}}^{t\to 0}(P^k_{\dot\gamma_{x,v}(t)} \Lambda^l_{\dot\gamma_{x,v}(t)} f_{\gamma_{x,v}(t)}) \mathrm{d}t \tag{2.16}$$

for any $(x,v) \in \partial_{\mathrm{in}}SM$, whenever the integral is well-defined. We note that

$$\begin{aligned} &P^k_{\dot\gamma_{x,v}(t)} \Lambda^l_{\dot\gamma_{x,v}(t)} f_{\gamma_{x,v}(t)}(w_1,\dots,w_k) \\ &= f_{\gamma_{x,v}(t)}(p_{\dot\gamma_{x,v}(t)}w_1,\dots,p_{\dot\gamma_{x,v}(t)}w_k, \dot\gamma_{x,v}(t),\dots,\dot\gamma_{x,v}(t)) \end{aligned} \tag{2.17}$$

for any $w_1,\dots,w_k \in T_{\gamma_{x,v}(t)}M$.

Using (2.17) and the definition of the parallel transport $\mathcal{T}_\gamma^{t\to s}$, one can show that the mixed ray transform acts on $(x,v) \in \partial_{\mathrm{in}}SM$ as

$$\begin{aligned} &\left\langle L_{k,l}f(x,v), (\eta + av)^{\otimes k} \right\rangle \\ &= \int_0^{\tau(x,v)} f_{i_1\dots i_k j_1\dots j_l}(\gamma_{x,v}(t)) \eta^{i_1}_{x,v}(t)\dots \eta^{i_k}_{x,v}(t) \dot\gamma^{j_1}_{x,v}(t)\dots\dot\gamma^{j_l}_{x,v}(t)\mathrm{d}t \end{aligned} \tag{2.18}$$

where $(\eta+av)^{\otimes k}$ is the tensor product of $\eta+av$ with itself k times, $a \in \mathbb{R}$ and $\eta_{x,v}(t)$ is the parallel transport of a vector $\eta = \eta_{x,v}(0) \in T_xM$ orthogonal to $v = \dot\gamma_{x,v}(0)$, see [26, Chapter 5.2] for details.

The mixed ray transform is considerably simpler when M is orientable and $n = 2$. Then $v^\perp$ is one-dimensional for all $(x,v) \in \partial_{\mathrm{in}}SM$ and there is only one possible choice (modulo sign) for the vector η which is parallel transported along γ. We choose the orthogonal vector field as $\eta(t) = (\star\dot\gamma)(t)$. It is clear that $\star\dot\gamma \perp \dot\gamma$ at every point on the geodesic γ and that $D_t^\gamma(\star\dot\gamma) = 0$ where D_t^γ is the covariant derivative along the geodesic γ. Therefore $\star\dot\gamma$ is parallel along γ. Now using formula (2.18) the mixed ray transform can be seen as a composition $L_{k,l} = I \circ A_{k,l}$ where

$$(A_{k,l}f)_x(v_1,\dots v_m) = f_x(A_1v_1,\dots, A_mv_m) \tag{2.19}$$

and $A_i = \star$ when $i = 1,\dots,k$ and $A_i = \mathrm{Id}$ when $i = k+1,\dots,k+l$. Thus in two dimensions the mixed ray transform operates as

$$L_{k,l}f(x,v) = \int_0^{\tau(x,v)} (\lambda(A_{k,l}f))(\varphi_t(x,v))\mathrm{d}t, \quad (x,v) \in \partial_{\mathrm{in}}SM, \tag{2.20}$$

and with these choices of $A_{k,l}$ we have $L_{k,l} = I_{A_{k,l}}$ where the transform $I_{A_{k,l}}$ is given by formula (1.2). If $k = 0$, then $L_{0,l}$ reduces to the geodesic ray transform I. If $l = 0$, we call $L_{k,0}$ the transverse ray transform and use the notation $I_\perp := L_{k,0}$. In higher dimensions $n > 2$ the operator $\star$ cannot be used to define the mixed ray transform since it maps k-forms into $(n-k)$-forms.

3. The algebraic structure of mixing ray transforms

3.1. **Decompositions of tensor fields.** Let $\sigma\colon \mathfrak{X}(T_mM) \to S_mM$ be the usual symmetrization map of tensor fields where $m \geq 2$. We remind that if $m = 1$, then any $f \in \mathfrak{X}(T_mM)$ is symmetric. The components of σf at a point $x \in M$ are

$$(\sigma f)_{i_1\dots i_m}(x) = \frac{1}{m!}\sum_{\tau\in\Pi_m} f_{i_{\tau(1)}\dots i_{\tau(m)}}(x) \tag{3.1}$$

where Π_m is the group of permutations. The symmetrization σ is a projection $\mathfrak{X}(T_mM) \to S_mM$, and it turns out to be orthogonal at every point with respect to any Riemannian metric by proposition 3.1. In particular, σ is idempotent and we can decompose the space $\mathfrak{X}(T_mM)$ as

$$\mathrm{Ker}(\sigma) \oplus \mathrm{Im}(\sigma) = \mathfrak{X}(T_mM) \tag{3.2}$$

by letting $f = (f - \sigma f) + \sigma f$. The decomposition (3.2) can be done on any differentiable manifold M. The set $\mathrm{Ker}(\sigma)$ can be identified with antisymmetric tensor fields when $m = 2$ and for $m > 2$ the antisymmetric tensor fields are a strict subset of $\mathrm{Ker}(\sigma)$.

Recall that the map $\lambda\colon \mathfrak{X}(T_mM) \to \mathcal{F}(TM)$ was defined as

$$(\lambda f)(x, v) = f_x(v, \dots , v) \tag{3.3}$$

where $\mathcal{F}(TM)$ is the space of all complex-valued functions on TM. We note that the restriction $\lambda f|_{SM}$ determines λf completely since f_x is homogeneous of degree m. It follows directly from the definitions that $\lambda \circ \sigma = \lambda$. It is true that $\mathrm{Ker}(\sigma) = \mathrm{Ker}(\lambda)$ (see proposition 3.1) and $\mathrm{Ker}(\lambda) \subset \mathrm{Ker}(I)$. Hence we call $\mathrm{Ker}(\lambda) \subset \mathfrak{X}(T_mM)$ the set of λ-*antisymmetric tensor fields* or *trivial part of the kernel of the geodesic ray transform* depending on the context.

We denote by $\lambda_x\colon (T^*_xM)^{\otimes m} \to \mathcal{F}(T_xM)$ the map $(\lambda_x\omega)(v) = \omega(v, \dots , v)$, i.e. $(\lambda f)(x, v) = (\lambda_x f_x)(v)$. We let σ_x be the symmetrization of m-tensors in $(T^*_xM)^{\otimes m}$ and $S_m(T_xM)$ is the space of symmetric m-tensors in $(T^*_xM)^{\otimes m}$. We have the following proposition which summarizes some important connections between the different concepts introduced above.

Proposition 3.1. *Suppose that $m \geq 2$ and let M be a Riemannian (or pseudo-Riemannian) manifold. Let $x \in M$ and define the sets*
(a) $V_1 = S_m(T_xM)$, $V_2 = \mathrm{Im}(\sigma_x)$ and $V_3 = \mathrm{Ker}(\lambda_x)^\perp$.
(b) $W_1 = (S_m(T_xM))^\perp$, $W_2 = \mathrm{Ker}(\sigma_x)$ and $W_3 = \mathrm{Ker}(\lambda_x)$.
Then $V_1 = V_2 = V_3$, $W_1 = W_2 = W_3$, and $V_i \oplus W_j = (T^*_xM)^{\otimes m}$ for any $i, j = 1, 2, 3$.

Proof. It follows directly from the definitions that $V_1 = V_2$. Suppose that $W_2 = W_3$ and $V_3 \subset V_1$. This implies that

$$(T^*_xM)^{\otimes m} = V_2 \oplus W_2 = V_3 \oplus W_3 = V_3 \oplus W_2. \tag{3.4}$$

Since $V_3 \subset V_1 = V_2$, we get that $V_2 = V_3$. It then follows that $V_1 = V_2 = V_3$, $W_1 = W_2 = W_3$, and $V_i \oplus W_j = (T^*_xM)^{\otimes m}$ for any $i, j = 1, 2, 3$. Hence it is sufficient to show that $W_2 = W_3$ and $V_3 \subset V_1$.

Let us first prove that $W_2 = W_3$. It is clear that $\mathrm{Ker}(\sigma_x) \subset \mathrm{Ker}(\lambda_x)$ since $\lambda_x \circ \sigma_x = \lambda_x$. Let $f \in \mathrm{Ker}(\lambda_x)$. It now follows that $\sigma_x f \in \mathrm{Ker}(\lambda_x)$. The polarization identity for symmetric multilinear maps [31, Theorem 1] states that a symmetric multilinear map is uniquely determined by its restriction to the diagonal. Since $\lambda_x \sigma_x f$ is the restriction of $\sigma_x f\colon (T_xM)^m \to \mathbb{C}$ to the diagonal of $(T_xM)^m$, $\sigma_x f \in S_m(T_xM)$ and $\lambda_x\sigma_x f = 0$, we obtain that $\sigma_x f = 0$. This shows that $\mathrm{Ker}(\lambda_x) \subset \mathrm{Ker}(\sigma_x)$, and we conclude that $W_2 = W_3$.

Let us then prove that $V_3 \subset V_1$. Let $f \in V_3$. Fix some indices $j_1', \dots, j_m'$ and define the components of the tensor $h \in (T_x^*M)^{\otimes m}$ as

$$h_{j_1\dots j_m} = \big(\delta^{j_1'}_{j_1}\delta^{j_m'}_{j_m} - \delta^{j_1'}_{j_m}\delta^{j_m'}_{j_1}\big)\delta^{j_2'}_{j_2}\cdots\delta^{j_{m-1}'}_{j_{m-1}}. \tag{3.5}$$

Then $h \in \mathrm{Ker}(\lambda_x)$ and $g_x(f,h) = 0$ implies that $f_{i_1\dots i_m} = f_{i_m\dots i_1}$. By switching the order of the indices j_k in the definition of h one sees that $f_{i_1\dots i_m}$ has to be symmetric with respect to all indices. Hence $V_3 \subset V_1$. This completes the proof. □

Remark 3.2. We obtain a somewhat surprising implication that the orthogonal complement of $S_m(T_xM)$ does not depend on the Riemannian metric g_x at the point $x \in M$. This follows from proposition 3.1 since the mapping σ_x does not depend on g_x. Proposition 3.1 also shows that the symmetrization $\sigma\colon \mathfrak{X}(T_mM) \to S_mM$ is an orthogonal projection when M is equipped with any Riemannian or pseudo-Riemannian metric.

By proposition 3.1 we have many choices for the decomposition of the space $(T_x^*M)^{\otimes m}$. We will use the orthogonal complement so that $\mathrm{Ker}(\lambda_x) \oplus \mathrm{Ker}(\lambda_x)^\perp = (T_x^*M)^{\otimes m}$. This allows us to decompose the space $\mathfrak{X}(T_mM)$ in the following way. We define the space $\mathrm{Ker}(\lambda)^\perp$ by saying that $f \in \mathrm{Ker}(\lambda)^\perp$ if and only if $f_x \in \mathrm{Ker}(\lambda_x)^\perp$ for all $x \in M$. Define the projection $\widehat{\sigma}\colon \mathfrak{X}(T_mM) \to \mathrm{Ker}(\lambda)^\perp$ such that $(\widehat{\sigma} f)_x = P_{\mathrm{Ker}(\lambda_x)^\perp} f_x$ where $P_{\mathrm{Ker}(\lambda_x)^\perp}$ is the orthogonal projection $P_{\mathrm{Ker}(\lambda_x)^\perp}\colon (T_x^*M)^{\otimes m} \to \mathrm{Ker}(\lambda_x)^\perp$. Then $f = (f - \widehat{\sigma} f) + \widehat{\sigma} f$ where $f - \widehat{\sigma} f \in \mathrm{Ker}(\lambda)$ and $\widehat{\sigma} f \in \mathrm{Ker}(\lambda)^\perp$. Hence we have the orthogonal decomposition

$$\mathrm{Ker}(\lambda) \oplus \mathrm{Ker}(\lambda)^\perp = \mathfrak{X}(T_mM) \tag{3.6}$$

where orthogonality is understood pointwise. We call the map $\widehat{\sigma}$ a λ-*symmetrization*. Note that $\mathrm{Ker}(\lambda) = \mathrm{Ker}(\sigma)$ and $\mathrm{Ker}(\lambda)^\perp = S_mM$ by proposition 3.1.

Another way to view λ-symmetric tensor fields is to take the quotient space $\mathrm{Coim}(\lambda) = \mathfrak{X}(T_mM)/\mathrm{Ker}(\lambda)$ which identifies all tensor fields which differ by an element of $\mathrm{Ker}(\lambda)$. This definition is natural for the geodesic ray transform in the sense that $If = Ih$ whenever $f \sim h$. It follows that if V is any algebraic complement of $\mathrm{Ker}(\lambda)$, i.e. $\mathrm{Ker}(\lambda) \oplus V = \mathfrak{X}(T_mM)$, then $V \cong \mathrm{Coim}(\lambda)$ via the map $v \mapsto [v]$ where $[v]$ is the equivalence class of v. This shows that one can realize the abstract quotient space $\mathrm{Coim}(\lambda)$ as a complementary subspace of $\mathrm{Ker}(\lambda)$ and that all complementary subspaces are isomorphic.

More generally, let $\Omega = \bigcup_{x\in M} \Omega_x \subset TM$ where $\Omega_x \subset T_xM$. Let r_x be the restriction of a multilinear map on T_xM to Ω_x. As before we can decompose $(T_x^*M)^{\otimes m} = \mathrm{Ker}(\lambda_{r,x}) \oplus \mathrm{Ker}(\lambda_{r,x})^\perp$ where $\lambda_{r,x} = r_x \circ \lambda_x$. This splitting can be done globally as follows. Denote by $r\colon \mathcal{F}(TM) \to \mathcal{F}(\Omega)$ the restriction to Ω and define $\lambda_r = r \circ \lambda$. Then we have the decomposition

$$\mathrm{Ker}(\lambda_r) \oplus \mathrm{Ker}(\lambda_r)^\perp = \mathfrak{X}(T_m M) \tag{3.7}$$

by writing $f = (f - \widehat{\sigma}_r f) + \widehat{\sigma}_r f$ where $\widehat{\sigma}_r \colon \mathfrak{X}(T_m M) \to \mathrm{Ker}(\lambda_r)^\perp$ is defined as $(\widehat{\sigma}_r f)_x = P_{\mathrm{Ker}(\lambda_{r,x})^\perp} f_x$ and the space $\mathrm{Ker}(\lambda_r)^\perp$ is defined pointwise as earlier. We call the projection $\widehat{\sigma}_r$ a λ_r-*symmetrization*. As above $\mathrm{Ker}(\lambda_r)^\perp$ can be seen as a realization of the quotient space $\mathrm{Coim}(\lambda_r) = \mathfrak{X}(T_m M)/\mathrm{Ker}(\lambda_r)$. It follows that $\mathrm{Ker}(\lambda) \subset \mathrm{Ker}(\lambda_r)$ and $\mathrm{Ker}(\lambda_r)^\perp \subset \mathrm{Ker}(\lambda)^\perp \subset S_m M$ by proposition 3.1.

Note that if r is the restriction to SM, then $\mathrm{Ker}(\lambda_r) = \mathrm{Ker}(\lambda)$. The geodesic ray transform can then be seen as a composition $I = I_{SM} \circ \lambda_r$. We will generalize this approach in the next subsection.

3.2. **The mixing ray transform.** Let $\mathrm{Aut}(TM)$ be the automorphism bundle of the tangent bundle. A section B of this bundle, called an automorphism field, is a field whose value $B(x)$ at any $x \in M$ is an automorphism (a linear self-bijection) of $T_x M$. In local coordinates B can be expressed as

$$B(x) = B^j_k(x) \mathrm{d}x^k \otimes \partial_j \tag{3.8}$$

where $B^j_k(x)$ is an invertible matrix at every point x.

Let A_i, $i = 1, \dots, m$, be smooth automorphism fields. Their tensor product $A = A_1 \otimes \cdots \otimes A_m$ is a mapping of tensor fields, $A \colon \mathfrak{X}(T_m M) \to \mathfrak{X}(T_m M)$. From an invariant point of view it operates on a tensor field $f \in \mathfrak{X}(T_m M)$ as

$$(Af)_x(v_1, \dots, v_m) = f_x(A_1(x) v_1, \dots, A_m(x) v_m), \tag{3.9}$$

and it can be written in local coordinates as

$$(Af)_{i_1 \dots i_m}(x) = (A_1)^{j_1}_{i_1}(x) \dots (A_m)^{j_m}_{i_m}(x) f_{j_1 \dots j_m}(x). \tag{3.10}$$

Since each $A_i(x)$ (or $(A_i)^j_k(x)$) is invertible and A_i is smooth also A is invertible and smooth. We call such map A an *admissible mixing of degree* m.

Let $r \colon \mathcal{F}(TM) \to \mathcal{F}(\Omega)$ be the restriction to $\Omega = \bigcup_{x \in M} \Omega_x \subset TM$ where $\Omega_x \subset T_x M$ and $\lambda_r = r \circ \lambda$. Let Z be a vector space and $J \colon \mathcal{F}(\Omega) \to Z$ a linear mapping. We define the abstract ray transform $I_{A,r} \colon \mathfrak{X}(T_m M) \to Z$ as

$$I_{A,r} = J \circ \lambda_r \circ A. \tag{3.11}$$

Usually r is the restriction to SM and J is the geodesic ray transform on SM. We call $I_{A,r}$ *the mixing ray transform* when these assumptions hold and write $\lambda \coloneqq \lambda_r$ and $I_A \coloneqq I_{A,r}$ to simplify our notation. Next, we decompose the space $\mathfrak{X}(T_m M)$ into symmetric and antisymmetric parts with respect to $\lambda_r \circ A$. Assume we have the decomposition $\mathrm{Ker}(\lambda_r) \oplus \mathrm{Ker}(\lambda_r)^\perp = \mathfrak{X}(T_m M)$. Since A is bijective linear map, we have $\mathrm{Ker}(\lambda_r \circ A) = A^{-1}(\mathrm{Ker}(\lambda_r))$ and

$$\begin{aligned} \mathfrak{X}(T_m M) &= A^{-1}(\mathrm{Ker}(\lambda_r) \oplus \mathrm{Ker}(\lambda_r)^\perp) \\ &= \mathrm{Ker}(\lambda_r \circ A) \oplus A^{-1}(\mathrm{Ker}(\lambda_r)^\perp). \end{aligned} \tag{3.12}$$

Hence we choose $A^{-1}(\mathrm{Ker}(\lambda_r)^\perp)$ as the space of $(\lambda_r \circ A)$-symmetric tensor fields. The symmetrization map $\widehat{\sigma}_{A,r}$ is a projection onto $A^{-1}(\mathrm{Ker}(\lambda_r)^\perp)$

and it has the expression $\widehat{\sigma}_{A,r} = A^{-1} \circ \widehat{\sigma}_r \circ A$ where $\widehat{\sigma}_r$ is a projection onto $\operatorname{Ker}(\lambda_r)^\perp$.

One can also naturally define the mixing ray transform on a quotient space as a mapping

$$I^q_{A,r} \colon \mathfrak{X}(T_m M)/\operatorname{Ker}(\lambda_r \circ A) \to Z \tag{3.13}$$

such that

$$I^q_{A,r}[f] = I_{A,r} f, \tag{3.14}$$

where $[f] \in \mathfrak{X}(T_m M)/\operatorname{Ker}(\lambda_r \circ A)$ is the equivalence class of $f \in \mathfrak{X}(T_m M)$. The transform $I^q_{A,r}$ is well-defined, i.e. it does not depend on the representative.

We conclude this subsection with the following theorem which basically says that it is enough to know the properties of one mixing ray transform since any other mixing ray transform can be reduced to the known case.

Theorem 3.3. *Let $E_1, E_2, E_3 \subset \mathfrak{X}(T_m M)$ be subspaces and $m \geq 1$. Assume that A and $\widetilde{A}$ are admissible mixings of degree m and let $\mathcal{D} = A^{-1} \circ \widetilde{A}$. Then the following properties hold:*

(a) ***Kernel characterization:*** *Let $\mathcal{H} = \operatorname{Id} - \widehat{\sigma}_{A,r}$ and $Y = \operatorname{Ker}(I_{\widetilde{A},r}) \cap \widetilde{A}^{-1}(\operatorname{Ker}(\lambda_r)^\perp)$. Then $f \in \operatorname{Ker}(I_{A,r})$ if and only if $f = \mathcal{H}f + \mathcal{D}w$ for some $w \in Y$. We have the decomposition*

$$\operatorname{Ker}(I_{A,r}) = \operatorname{Im}(\mathcal{H}) \oplus \operatorname{Im}(\mathcal{D}|_Y) = \operatorname{Ker}(\lambda_r \circ A) \oplus \operatorname{Im}(\mathcal{D}|_Y) \tag{3.15}$$

where $\operatorname{Ker}(I_{A,r}), \operatorname{Im}(\mathcal{H}), \operatorname{Im}(\mathcal{D}|_Y) \subset \mathfrak{X}(T_m M)$.

(b) ***Reconstruction:*** *Let $\mathcal{R}_{\widetilde{A},r} \colon Z \to S$ be a left inverse of $I_{\widetilde{A},r} \colon S \to Z$, where $S \subset \widetilde{A}^{-1}(\operatorname{Ker}(\lambda_r)^\perp)$. Then $\mathcal{R}_{A,r} = \mathcal{D} \circ \mathcal{R}_{\widetilde{A},r} \colon Z \to \mathcal{D}(S)$ is a left inverse of $I_{A,r} \colon \mathcal{D}(S) \to Z$ where $\mathcal{D}(S) \subset A^{-1}(\operatorname{Ker}(\lambda_r)^\perp)$.*

(c) ***Stability:*** *Let $(Z, \|\cdot\|_Z)$ and $(E_1, \|\cdot\|_{E_1})$ be normed spaces. Also assume that $\mathcal{D}$ is bounded on $(E_1, \|\cdot\|_{E_1})$ and that the estimate $\|f\|_{E_1} \leq C \left\| I_{\widetilde{A},r} f \right\|_Z$ holds for some subset $S' \subset \widetilde{A}^{-1}(\operatorname{Ker}(\lambda_r)^\perp)$. Then the estimate*

$$\|f\|_{E_1} \leq C \|\mathcal{D}\|_{E_1} \|I_{A,r} f\|_Z \tag{3.16}$$

holds for all $f \in \mathcal{D}(S') \subset A^{-1}(\operatorname{Ker}(\lambda_r)^\perp)$.

(d) ***Adjoint and normal operator:*** *Let $(Z, \langle\cdot,\cdot\rangle_Z)$ and $(E_2, \langle\cdot,\cdot\rangle_{E_2})$ be Hilbert spaces. Assume that $\mathcal{D}^{-1}$ is bounded in $(E_2, \langle\cdot,\cdot\rangle_{E_2})$ and that $I_{\widetilde{A},r} \colon E_2 \to Z$ is bounded. Then the adjoints and the normal operators of $I_{A,r}$ and $I_{\widetilde{A},r}$ satisfy the formulas*

$$I^*_{A,r} = (\mathcal{D}^{-1})^* I^*_{\widetilde{A},r}, \quad N_{A,r} = (\mathcal{D}^{-1})^* N_{\widetilde{A},r} \mathcal{D}^{-1}. \tag{3.17}$$

(e) ***Stability with normal operators:*** *Suppose that the assumptions of (d) hold and let $\|\cdot\|_{E_3}$ be a norm on E_3. Assume also that $\mathcal{D}^*$ is bounded in*

$(E_3, \|\cdot\|_{E_3})$ and that the estimate $\|f\|_{E_2} \le C \left\| N_{\widetilde{A},r} f \right\|_{E_3}$ holds for some subset $S'' \subset \widetilde{A}^{-1}(\mathrm{Ker}(\lambda_r)^\perp)$. Then the estimate

$$\|f\|_{E_2} \le C \|\mathcal{D}\|_{E_2} \|\mathcal{D}^*\|_{E_3} \|N_{A,r} f\|_{E_3} \tag{3.18}$$

holds for all $f \in \mathcal{D}(S'') \subset A^{-1}(\mathrm{Ker}(\lambda_r)^\perp)$.

Proof. (a) If f is of the form $f = \mathcal{H}f + \mathcal{D}w$ for some $w \in Y$, then clearly $f \in \mathrm{Ker}(I_{A,r})$. For the converse, let $w = \widehat{\sigma}_{\widetilde{A},r}\mathcal{D}^{-1} f = \mathcal{D}^{-1}\widehat{\sigma}_{A,r} f$. We can write

$$f = (f - \widehat{\sigma}_{A,r} f) + \widehat{\sigma}_{A,r} f = \mathcal{H}f + \mathcal{D}\mathcal{D}^{-1}\widehat{\sigma}_{A,r} f = \mathcal{H}f + \mathcal{D}w. \tag{3.19}$$

Clearly $w \in \widetilde{A}^{-1}(\mathrm{Ker}(\lambda_r)^\perp)$ and

$$I_{\widetilde{A},r} w = I_{\widetilde{A},r}\mathcal{D}^{-1}\widehat{\sigma}_{A,r} f = I_{A,r}\widehat{\sigma}_{A,r} f = I_{A,r} f = 0 \tag{3.20}$$

so $w \in \mathrm{Ker}(I_{\widetilde{A},r})$. Assume then that $f \in \mathrm{Im}(\mathcal{H}) \cap \mathrm{Im}(\mathcal{D}|_Y)$. Now $f \in \mathrm{Ker}(\lambda_r \circ A)$ and $f = \mathcal{D}w$ where $w \in \widetilde{A}^{-1}(\mathrm{Ker}(\lambda_r)^\perp)$. But this implies $w = \mathcal{D}^{-1} f \in \mathrm{Ker}(\lambda_r \circ \widetilde{A})$ and hence $w = 0$. Therefore $\mathrm{Im}(\mathcal{H}) \cap \mathrm{Im}(\mathcal{D}|_Y) = \{0\}$.

(b) Clearly $\mathcal{D}(S) \subset \mathcal{D}(\widetilde{A}^{-1}(\mathrm{Ker}(\lambda_r)^\perp)) = A^{-1}(\mathrm{Ker}(\lambda_r)^\perp)$. Let $f \in \mathcal{D}(S)$. Then

$$\mathcal{D}\mathcal{R}_{\widetilde{A},r} I_{A,r} f = \mathcal{D}\mathcal{R}_{\widetilde{A},r} I_{\widetilde{A},r}\mathcal{D}^{-1} f = \mathcal{D}\mathcal{D}^{-1} f = f \tag{3.21}$$

implying that $\mathcal{D} \circ \mathcal{R}_{\widetilde{A},r}$ is a left inverse of $I_{A,r}$ on $\mathcal{D}(S)$.

(c) For $f \in \mathcal{D}(S')$ we find

$$\|f\|_{E_1} = \left\|\mathcal{D}\mathcal{D}^{-1} f\right\|_{E_1} \le \|\mathcal{D}\|_{E_1} \left\|\mathcal{D}^{-1} f\right\|_{E_1} \le C \|\mathcal{D}\|_{E_1} \left\| I_{\widetilde{A},r}\mathcal{D}^{-1} f \right\|_Z \tag{3.22}$$

$$= C \|\mathcal{D}\|_{E_1} \|I_{A,r} f\|_Z \tag{3.23}$$

as claimed.

(d) Using the definitions of adjoints, we obtain

$$\langle I_{A,r} f, h\rangle_Z = \left\langle I_{\widetilde{A},r}\mathcal{D}^{-1} f, h\right\rangle_Z = \left\langle \mathcal{D}^{-1} f, I^*_{\widetilde{A},r} h\right\rangle_{E_2} = \left\langle f, (\mathcal{D}^{-1})^* I^*_{\widetilde{A},r} h\right\rangle_{E_2}. \tag{3.24}$$

Hence $I^*_{A,r} = (\mathcal{D}^{-1})^* I^*_{\widetilde{A},r}$ and the normal operator becomes

$$N_{A,r} = I^*_{A,r} I_{A,r} = (\mathcal{D}^{-1})^* I^*_{\widetilde{A},r} I_{\widetilde{A},r}\mathcal{D}^{-1} = (\mathcal{D}^{-1})^* N_{\widetilde{A},r}\mathcal{D}^{-1}. \tag{3.25}$$

(e) If $f \in \mathcal{D}(S'')$, then we have

$$\|f\|_{E_2} \le \|\mathcal{D}\|_{E_2} \left\|\mathcal{D}^{-1} f\right\|_{E_2} \le C \|\mathcal{D}\|_{E_2} \left\| N_{\widetilde{A},r}\mathcal{D}^{-1} f \right\|_{E_3} \tag{3.26}$$

$$= C \|\mathcal{D}\|_{E_2} \left\| \mathcal{D}^*(\mathcal{D}^{-1})^* N_{\widetilde{A},r}\mathcal{D}^{-1} f \right\|_{E_3} \tag{3.27}$$

$$\le C \|\mathcal{D}\|_{E_2} \|\mathcal{D}^*\|_{E_3} \|N_{A,r} f\|_{E_3}.$$

□

3.3. **Solenoidal injectivity.** In this section we analyze more closely the kernel characterization given in theorem 3.3(a). Specifically, we apply our methods to show s-injectivity of general mixing ray transforms when s-injectivity of the geodesic ray transform is known. We also use our approach to show that the earlier results about the kernel of the mixed ray transform on compact simple surfaces and the kernel of the light ray transform on static globally hyperbolic Lorentzian manifolds can be seen as s-injectivity results under correct notions of symmetry.

3.3.1. *General results.* Let r be the restriction to SM and $J = I_{SM}$ so that $I = I_{SM} \circ \lambda_r$ and $I_{A,r} = I \circ A$. Since now $\operatorname{Ker}(\lambda_r) = \operatorname{Ker}(\lambda)$ we use an abuse of notation and write $\lambda \coloneqq \lambda_r$. By proposition 3.1 we can choose $\operatorname{Ker}(\lambda)^\perp = S_m M$ so that $\widehat{\sigma}_A = A^{-1} \circ \sigma \circ A$ is a projection onto $A^{-1}(S_m M)$. Further, we define the covariant derivative $\nabla^A = A^{-1} \circ \nabla$. The derivative ∇^A is natural for the transform I_A since if $v|_{\partial M} = 0$, then $I_A(\widehat{\sigma}_A \nabla^A v) = 0$.

We say that the mixing ray transform I_A is s-injective on a compact Riemannian manifold with boundary if the following property holds for all $f \in C^\infty(T_m M)$: $I_A f = 0$ if and only if $\widehat{\sigma}_A f = \widehat{\sigma}_A \nabla^A u$ for some $u \in C^\infty(S_{m-1} M)$ vanishing on the boundary. S-injectivity allows one to decompose the kernel of I_A as

$$\operatorname{Ker}(I_A|_{C^\infty(T_m M)}) = \operatorname{Im}(\mathcal{H}|_{C^\infty(T_m M)}) \oplus \operatorname{Im}(\widehat{\sigma}_A \nabla^A|_Y) \tag{3.28}$$

where $\mathcal{H} = \operatorname{Id} - \widehat{\sigma}_A$ and

$$Y = \{u \in C^\infty(S_{m-1} M) : u|_{\partial M} = 0\}. \tag{3.29}$$

It follows that s-injectivity of any mixing ray transform implies s-injectivity for all mixing ray transforms.

Corollary 3.4. *Let $m \geq 1$ and (M, g) be a compact Riemannian manifold with boundary so that the transform I_A is s-injective for some A of degree m. Then $I_{\widetilde{A}}$ is s-injective for all $\widetilde{A}$ of degree m.*

Proof. Let us denote $\widehat{\sigma}_A = A^{-1} \sigma A$ and $\widehat{\sigma}_{\widetilde{A}} = \widetilde{A}^{-1} \sigma \widetilde{A}$ for the projections. Using the solenoidal injectivity for I_A we easily obtain

$$\begin{aligned} I_{\widetilde{A}} f = I_A(A^{-1} \widetilde{A} f) = 0 &\Leftrightarrow \exists u \in Y : \widehat{\sigma}_A A^{-1} \widetilde{A} f = \widehat{\sigma}_A \nabla^A u \\ &\Leftrightarrow \exists u \in Y : \widehat{\sigma}_{\widetilde{A}} f = \widehat{\sigma}_{\widetilde{A}} \nabla^{\widetilde{A}} u. \end{aligned} \tag{3.30}$$ □

We immediately obtain the following corollary from the previous corollary.

Corollary 3.5. *Take any $m \geq 1$. Assume that (M, g) is a compact Riemannian manifold with boundary so that the geodesic ray transform is s-injective on m-tensor fields. Then I_A is s-injective for all A of degree m.*

We have similar results for the mixing ray transform in the quotient space $\mathfrak{X}(T_m M)/\operatorname{Ker}(\lambda \circ A)$. We denote by $[\cdot]_A$ the corresponding equivalence classes and say that the quotient transform defined as $I^q_A[f]_A = I_A f$ (see section 3.2) is s-injective if for all $[f]_A \in C^\infty(T_m M)/\operatorname{Ker}(\lambda \circ A)$ we have $I^q_A[f]_A = 0$ if and only if $[f]_A = [\nabla^A u]_A$ for some $u \in C^\infty(S_{m-1} M)$ vanishing on the boundary.

Corollary 3.6. *Let $m \geq 1$ and A be a mixing of degree m. Assume that (M, g) is a compact Riemannian manifold with boundary. Then I_A is s-injective if and only if I_A^q is s-injective.*

Proof. Assume first that I_A is s-injective. We obtain

$$I_A f = I_A^q [f]_A = 0 \Leftrightarrow \widehat{\sigma}_A f = \widehat{\sigma}_A \nabla^A u \tag{3.31}$$

which in turn implies

$$[f]_A = [\widehat{\sigma}_A f]_A = [\widehat{\sigma}_A \nabla^A u]_A = [\nabla^A u]_A. \tag{3.32}$$

Assume then that I_A^q is s-injective. Now

$$I_A^q [f]_A = I_A f = 0 \Leftrightarrow [f]_A = [\nabla^A u]_A \tag{3.33}$$

which implies $f - \nabla^A u = h \in \operatorname{Ker}(\lambda \circ A)$. Hence $\widehat{\sigma}_A f = \widehat{\sigma}_A \nabla^A u$. □

The previous results imply that if I_A^q is s-injective for some A of degree $m \geq 1$, then $I_{\widetilde{A}}^q$ is s-injective for all $\widetilde{A}$ of degree m. We remark that if $A = \mathrm{Id}$, then I_A^q corresponds to the geodesic ray transform in the quotient space $\mathfrak{X}(T_m M)/\operatorname{Ker}(\lambda)$. Especially, s-injectivity of I on m-tensor fields implies s-injectivity for I_A^q where A is any mixing of degree m.

3.3.2. *Mixed ray transform on compact simple surfaces.* Let us then consider the mixed ray transform $L_{k,l} = I_{SM} \circ \lambda \circ A_{k,l}$ on a compact simple surface (M, g) where $A_{k,l}$ is defined via equation (2.19). Define the operators $\lambda' w = \sigma_{k,l}(g \otimes w)$ and $\mathrm{d}' u = \sigma_l \nabla u$ where σ_l is the symmetrization with respect to the last l indices and $\sigma_{k,l}$ is the symmetrization with respect to the first k and the last l indices. In coordinates

$$(\lambda' w)_{i_1 \dots i_k j_1 \dots j_l} = \sigma_{k,l}(g_{i_1 j_1} w_{i_2, \dots i_k j_2 \dots j_l}) \tag{3.34}$$

$$(\mathrm{d}' u)_{i_1 \dots i_k j_1 \dots j_l} = \sigma_l((\nabla_{e_{j_1}} u)_{i_1 \dots i_k j_2 \dots j_l}). \tag{3.35}$$

We compare our approach to the kernel characterization done in [7]. Especially, we obtain the following alternative result for s-injectivity.

Corollary 3.7. *Let (M, g) be two-dimensional compact simple Riemannian manifold and $f \in C^\infty(T_{k+l} M)$. Then $L_{k,l} f = 0$ if and only if $\widehat{\sigma}_{A_{k,l}} f = \widehat{\sigma}_{A_{k,l}} \sigma_l \nabla u$ where $u \in C^\infty(S_k M \otimes S_{l-1} M)$ such that $u|_{\partial M} = 0$.*

Proof. Assume that $L_{k,l} f = 0$. Since

$$\widehat{\sigma}_{A_{k,l}} f \in A_{k,l}^{-1}(C^\infty(S_m M)) \subset C^\infty(S_k M \otimes S_l M) \tag{3.36}$$

we obtain $L_{k,l} \widehat{\sigma}_{A_{k,l}} f = L_{k,l} f = 0$. By [7, Theorem 1] we have $\widehat{\sigma}_{A_{k,l}} f = \sigma_l \nabla u + \sigma_{k,l}(g \otimes w)$ for some $u \in C^\infty(S_k M \otimes S_{l-1} M)$, $u|_{\partial M} = 0$, and $w \in C^\infty(S_{k-1} M \otimes S_{l-1} M)$. Now $\sigma_{k,l}(g \otimes w) \in \operatorname{Ker}(\lambda \circ A_{k,l})$ and hence $\widehat{\sigma}_{A_{k,l}} f = \widehat{\sigma}_{A_{k,l}} \sigma_l \nabla u$.

Then assume that $\widehat{\sigma}_{A_{k,l}} f = \widehat{\sigma}_{A_{k,l}} \sigma_l \nabla u$ for some u vanishing on the boundary. Since $L_{k,l} = I_{SM} \circ \lambda \circ A_{k,l}$ and $\lambda \circ \sigma = \lambda$ we obtain

$$L_{k,l} f = L_{k,l} \widehat{\sigma}_{A_{k,l}} f = (I_{SM} \circ \lambda \circ A_{k,l})(A_{k,l}^{-1} \sigma A_{k,l} \sigma_l \nabla u) = L_{k,l} \sigma_l \nabla u = 0 \tag{3.37}$$

where the last equality follows from the fundamental theorem of calculus. □

Remark 3.8. The previous s-injectivity result is similar to what we obtained earlier, i.e. $I_{A_{k,l}}f = 0$ if and only if $\widehat{\sigma}_{A_{k,l}}f = \widehat{\sigma}_{A_{k,l}}\nabla^{A_{k,l}}u$ for some $u \in C^\infty(S_{m-1}M)$ vanishing on the boundary. We thus have the following alternative characterizations of the kernel of the mixed ray transform

$$\operatorname{Ker}(L_{k,l}|_{C^\infty(T_mM)}) = \operatorname{Im}(\mathcal{H}|_{C^\infty(T_mM)}) \oplus \operatorname{Im}(\widehat{\sigma}_{A_{k,l}}\nabla^A|_Y) \tag{3.38}$$

$$\operatorname{Ker}(L_{k,l}|_{C^\infty(T_mM)}) = \operatorname{Im}(\mathcal{H}|_{C^\infty(T_mM)}) \oplus \operatorname{Im}(\widehat{\sigma}_{A_{k,l}}\mathrm{d}'|_{Y'}) \tag{3.39}$$

where $\mathcal{H} = \operatorname{Id} - \widehat{\sigma}_{A_{k,l}}$ and

$$Y = \{u \in C^\infty(S_{m-1}M) : u|_{\partial M} = 0\} \tag{3.40}$$

$$Y' = \{u \in C^\infty(S_kM \otimes S_{l-1}M) : u|_{\partial M} = 0\}. \tag{3.41}$$

Compare these to the decomposition of the kernel in [7]

$$\operatorname{Ker}(L_{k,l}|_{C^\infty(S_kM\otimes S_lM)}) = \operatorname{Im}(\lambda'|_{C^\infty(S_{k-1}M\otimes S_{l-1}M)}) + \operatorname{Im}(\mathrm{d}'|_{Y'}). \tag{3.42}$$

Our decompositions split any tensor field uniquely into the trivial part and non-trivial part of the kernel. The uniqueness of decomposition (3.42) is not known and it only applies to tensor fields with certain symmetries.

3.3.3. *Light ray transform on Lorentzian manifolds.* We quickly review the relevant definitions for the light ray transform on static globally hyperbolic Lorentzian manifolds. More details can be found in [8].

Let $(\mathcal{N}, \overline{g})$ be a smooth globally hyperbolic Lorentzian manifold of dimension $1 + n$ with signature $(-, +, \dots, +)$. Let β be a maximal light-like geodesic so that

$$\overline{\nabla}_{\dot\beta(s)}\dot\beta(s) = 0, \quad \overline{g}(\dot\beta(s), \dot\beta(s)) = 0 \tag{3.43}$$

where $\overline{\nabla}$ is the covariant derivative with respect to $\overline{g}$. We define the light ray transform of $f \in C_c^\infty(T_m\mathcal{N})$ as

$$\mathcal{L}_\beta f = \int_{-\infty}^{\infty} (\lambda f)(\beta(s), \dot\beta(s))\mathrm{d}s, \tag{3.44}$$

where β ranges over all lightlike geodesics of $\mathcal{N}$. Since $(\mathcal{N}, \overline{g})$ is globally hyperbolic, there exists a Cauchy hypersurface $N \subset \mathcal{N}$, i.e. a hypersurface such that any causal curve intersects N exactly once. We define $g = \overline{g}|_N$; note that (N, g) becomes a Riemannian manifold. We will focus on static Lorentzian manifolds. It follows that if $\mathcal{N}$ is static, then for any Cauchy hypersurface $N \subset \mathcal{N}$ there exists an isometric embedding $\Phi\colon \mathbb{R} \times N \to \mathcal{N}$ so that $\Phi^*\overline{g} = -\kappa \mathrm{d}t^2 + g$ where κ is a smooth positive function on N. We let $g_c = \kappa^{-1}g$.

Let r be the restriction to the set

$$\Omega = \{(x, v) \in T\mathcal{M} : \overline{g}_x(v, v) = 0\} \tag{3.45}$$

where $\mathcal{M} = \Phi(\mathbb{R} \times M)$, $M \subset N$ is a compact submanifold with smooth boundary and $\lambda_r = r \circ \lambda$ as before. We define the quotient light ray transform $\mathcal{L}_\beta^q$ in $C_c^\infty(T_m\mathcal{M})/\operatorname{Ker}(\lambda_r)$ as $\mathcal{L}_\beta^q[f] = \mathcal{L}_\beta f$; note that the definition does not depend on the representative. We obtain the following s-injectivity result for $\mathcal{L}_\beta^q$.

Corollary 3.9. *Let $(\mathcal{N}, \overline{g})$ be static globally hyperbolic Lorentzian manifold of dimension $1+n$ and let $N \subset \mathcal{N}$ be a fixed Cauchy hypersurface. Let $\mathcal{M} = \Phi(\mathbb{R} \times M)$ where $M \subset N$ is a compact n-dimensional submanifold with smooth boundary and Φ is the isometric embedding introduced earlier. Assume that the geodesic ray transform is s-injective on (M, g_c) and let $[f] \in C_c^\infty(T_m\mathcal{M})/\operatorname{Ker}(\lambda_r)$. Then $\mathcal{L}_\beta^q[f] = 0$ for all maximal β in $(\mathcal{M}, \overline{g})$ if and only if $[f] = [\overline{\nabla}T]$ for some $T \in C_c^\infty(S_{m-1}\mathcal{M})$.*

Proof. Assume that $\mathcal{L}_\beta^q[f] = 0$. Then $\mathcal{L}_\beta(\sigma f) = \mathcal{L}_\beta f = 0$ and by [8, Theorem 2] we obtain $\sigma f = \sigma\overline{\nabla}T + \sigma(g \otimes U)$ for some $T \in C_c^\infty(S_{m-1}\mathcal{M})$ and $U \in C_c^\infty(S_{m-2}\mathcal{M})$. Hence

$$[f] = [\sigma f] = [\sigma\overline{\nabla}T] + [\sigma(g \otimes U)] = [\overline{\nabla}T] \tag{3.46}$$

where we used the fact that $\operatorname{Ker}(\lambda) \subset \operatorname{Ker}(\lambda_r)$ and $\sigma(g \otimes U) \in \operatorname{Ker}(\lambda_r)$. This gives the other direction of the claim. Assume then that $[f] = [\overline{\nabla}T]$ for some $T \in C_c^\infty(S_{m-1}\mathcal{M})$. The fundamental theorem of calculus implies that $\mathcal{L}_\beta^q[f] = \mathcal{L}_\beta^q[\overline{\nabla}T] = \mathcal{L}_\beta(\sigma\overline{\nabla}T) = 0$. This concludes the proof. □

Remark 3.10. One can realize the quotient space $\mathfrak{X}(T_m\mathcal{M})/\operatorname{Ker}(\lambda_r)$ as a complementary subspace $V_r \subset \mathfrak{X}(T_m\mathcal{M})$ which satisfies $\operatorname{Ker}(\lambda_r) \oplus V_r = \mathfrak{X}(T_m\mathcal{M})$. This can be done for example by taking the orthogonal complement $V_r = \operatorname{Ker}(\lambda_r)^\perp$ with respect to a Riemannian metric on $\mathcal{M}$ (see section 3.1). Then corollary 3.9 implies that we have the decomposition

$$\operatorname{Ker}(\mathcal{L}_\beta|_{C_c^\infty(T_m\mathcal{M})}) = \operatorname{Im}(\mathcal{H}|_{C_c^\infty(T_m\mathcal{M})}) \oplus \operatorname{Im}(\widehat{\sigma}_r\overline{\nabla}|_{C_c^\infty(S_{m-1}\mathcal{M})}) \tag{3.47}$$

where $\widehat{\sigma}_r$ is the orthogonal projection onto $\operatorname{Ker}(\lambda_r)^\perp$ and $\mathcal{H} = \operatorname{Id} - \widehat{\sigma}_r$.

3.4. **Boundedness and pointwise estimates of mixings.**

In this section we give sufficient conditions which imply pointwise norm estimates and continuity of A in Sobolev spaces. Boundedness and pointwise estimates are used in section 4 to prove stability estimates and injectivity results for the mixed ray transform on two-dimensional orientable Riemannian manifolds.

Lemma 3.11. *Let $f \in C^q(T_mM)$ where $m \geq 1$ and $q \in \mathbb{N}$. Then the following properties hold:*

(a) If A_i satisfy the relation $|A_iv|_{g_x} \leq C_i(x)\,|v|_{g_x}$ for all $v \in T_xM$, then we have the pointwise estimate

$$|Af|_{g_x} \leq n^m C_1(x) \dots C_m(x)\,|f|_{g_x}\,. \tag{3.48}$$

Especially, if $C_i = C_i(x)$ are all bounded, then A extends into a bounded mapping $A\colon L^2(T_mM) \to L^2(T_mM)$.

(b) If in addition $\left|\nabla_{e_j}A_i\right|_{g_x} \leq C_i'(x)$ for any local frame $\{e_j\}$, then we have the pointwise estimate

$$|\nabla(Af)|_{g_x} \leq C''(x)(|f|_{g_x} + |\nabla f|_{g_x}) \tag{3.49}$$

where $C'' = C''(x)$ can be expressed in terms of C_i and C_i'. Especially, if C_i, C_i' are all bounded, then A extends into a bounded mapping $A\colon H^1(T_mM) \to H^1(T_mM)$.

(c) If (M, g) is a two-dimensional orientable Riemannian manifold, then the operator $A_{k,l}$ defined in (2.19) *satisfies*

$$|\nabla^p (A_{k,l} f)|_{g_x} = |\nabla^p f|_{g_x} \tag{3.50}$$

for all $p \in \mathbb{N}$, $p \leq q$. In particular, the mixing $A_{k,l}$ extends into an isometry $A_{k,l} \colon H^p(T_m M) \to H^p(T_m M)$ for all $p \in \mathbb{N}$.

Proof. (a) Choose normal coordinates in a neighborhood of x. The boundedness assumption for A_i implies

$$\left|(A_i)^j_k(x)\right| \leq \left(\sum_{j=1}^n \left|(A_i)^j_k(x)\right|^2 \right)^{1/2} = |A_i e_k|_{g_x} \leq C_i(x) \tag{3.51}$$

where $(A_i)^j_k$ are the components of A_i in these coordinates. Now we can estimate the norm as

$$\begin{aligned} |Af|^2_{g_x} &= \sum_{i_1 \dots i_m = 1}^n ((Af)_{i_1 \dots i_m}(x))^2 \\ &= \sum_{i_1 \dots i_m = 1}^n \left(\sum_{j_1 \dots j_m = 1}^n (A_1)^{j_1}_{i_1}(x) \dots (A_m)^{j_m}_{i_m}(x) f_{j_1 \dots j_m}(x) \right)^2 \\ &\leq n^m C_1^2(x) \dots C_m^2(x) \sum_{i_1 \dots i_m = 1}^n \sum_{j_1 \dots j_m = 1}^n |f_{j_1 \dots j_m}(x)|^2 \\ &\leq n^{2m} C_1^2(x) \dots C_m^2(x) \, |f|^2_{g_x} . \end{aligned} \tag{3.52}$$

If C_i are all bounded, then $A \colon L^2(T_m M) \to L^2(T_m M)$ is bounded by definition and approximation by smooth tensor fields.

(b) By choosing normal coordinates, the covariant derivative at the point $x \in M$ reduces to the ordinary derivative. Now

$$\left|\nabla_{e_j} A_i\right|^2_{g_x} = \sum_{k,l=1}^n (\partial_j (A_i)^k_l(x))^2, \tag{3.53}$$

which implies $\left|\partial_j (A_i)^k_l(x)\right| \leq C_i'(x)$. Using the Leibniz rule we obtain

$$\begin{aligned} |\nabla (Af)|^2_{g_x} &= \sum_{k, i_1 \dots i_m = 1}^n (\partial_k (Af)_{i_1 \dots i_m}(x))^2 \\ &\leq n^{2m+1} ((C_1')^2(x) \dots C_m^2(x) + \dots + C_1^2(x) \dots (C_m')^2(x)) \, |f|^2_{g_x} \\ &\quad + n^{2m} C_1^2(x) \dots C_m^2(x) \, |\nabla f|^2_{g_x} \\ &= \widehat{C}(x) \, |f|^2_{g_x} + \widetilde{C}(x) \, |\nabla f|^2_{g_x} . \end{aligned} \tag{3.54}$$

By taking $C''(x) = \sqrt{2} \max\{\widehat{C}^{1/2}(x), \widetilde{C}^{1/2}(x)\}$ we get the desired inequality. If C_i, C_i' are all bounded, then $A \colon H^1(T_m M) \to H^1(T_m M)$ is bounded.

(c) Again using normal coordinates, one can calculate that

$$g_x(A_{k,l} f, A_{k,l} f) = g_x(f, A_{k,l}^{-1} A_{k,l} f) = g_x(f, f), \tag{3.55}$$

where we used the relations $(A_i)^j_m = -(A_i)^m_j$ for $i = 1, \dots k$, $(A_i)^j_m = \delta^j_m$ for $i = k+1, \dots k+l$ and $(-1)^k A_{k,l} = A^{-1}_{k,l}$. For the derivatives we get

$$g_x(\nabla^p(Af), \nabla^p(Af)) = g_x(\nabla^p f, \nabla^p f) \tag{3.56}$$

using the fact that $\sum_{j=1}^n (A_i)^m_j (A_i)^q_j = \delta^m_q$ for all $i = 1, \dots k+l$. □

Remark 3.12. In a similar fashion as in part (b) one obtains the boundedness of $A\colon H^k(T_mM) \to H^k(T_mM)$ if one assumes boundedness of the derivatives up to order $k \in \mathbb{N}$, i.e. $|\nabla_\alpha A_i|_{g_x} \leq C_i^\alpha(x)$ for all $|\alpha| \leq k$ where $C_i^\alpha = C_i^\alpha(x)$ is bounded, $\nabla_\alpha = \nabla^{\alpha_1}_{e_1} \dots \nabla^{\alpha_n}_{e_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

4. The mixed and transverse ray transforms on two-dimensional orientable Riemannian manifolds

4.1. Solenoidal injectivity on compact and non-compact surfaces.

First we state a general result on s-injectivity of the mixed ray transform on compact orientable surfaces with boundary. This follows from corollary 3.5. We use the notation introduced in section 3.3.1. Note that $A_{k,l}f \in C^p(T_mM)$ whenever $f \in C^p(T_mM)$ where $p \in \mathbb{N}$.

Corollary 4.1. *Let $m \geq 1$. Let (M, g) be compact two-dimensional orientable Riemannian manifold with boundary such that the geodesic ray transform is s-injective on $C^\infty(S_mM)$ and let $f \in C^\infty(T_mM)$. Then $L_{k,l}f = 0$ if and only if $\widehat{\sigma}_{A_{k,l}} f = \widehat{\sigma}_{A_{k,l}} \nabla^{A_{k,l}} h$ for some $h \in C^\infty(S_{m-1}M)$ vanishing on the boundary ∂M.*

We note that the previous result holds on a wide class of two-dimensional orientable manifolds. These include for example compact simple surfaces [21] and simply connected compact surfaces with strictly convex boundary and non-positive sectional curvature [20, 26]. See [12, 22] for more manifolds with s-injective geodesic ray transform.

We have the following corollary for the mixed ray transform on Cartan–Hadamard manifolds which is a simple consequence of the pointwise estimates for $A_{k,l}$ and the results in [16]. We denote by $K(x)$ the Gaussian curvature of (M, g) at $x \in M$.

Corollary 4.2. *Let (M, g) be a two-dimensional Cartan–Hadamard manifold and let $m \geq 1$. The following claims are true:*

(a) Let $-K_0 \leq K \leq 0$ for some $K_0 > 0$ and $f \in E^1_\eta(T_mM)$ for some $\eta > \frac{3}{2}\sqrt{K_0}$. Then $L_{k,l}f = 0$ if and only if $\widehat{\sigma}_{A_{k,l}} f = \widehat{\sigma}_{A_{k,l}} \nabla^{A_{k,l}} h$ for some $h \in S_{m-1}M$ such that $h \in E_{\eta-\varepsilon}(T_{m-1}M)$ for all $\varepsilon > 0$.

(b) Let $K \in P_\kappa(M)$ for some $\kappa > 2$ and $f \in P^1_\eta(T_mM)$ for some $\eta > 2$. Then $L_{k,l}f = 0$ if and only if $\widehat{\sigma}_{A_{k,l}} f = \widehat{\sigma}_{A_{k,l}} \nabla^{A_{k,l}} h$ for some $h \in S_{m-1}M \cap P_{\eta-1}(T_{m-1}M)$.

Proof. (a) If $f \in E^1_\eta(T_mM)$, then from the pointwise estimates for the transform $A_{k,l}$ we obtain that

$$|\sigma A_{k,l} f|_{g_x} \leq (m!)^{1/2} |A_{k,l} f|_{g_x} = (m!)^{1/2} |f|_{g_x} \leq C e^{-\eta d(x,o)} \tag{4.1}$$

for some $C > 0$ and

$$|\nabla(\sigma A_{k,l} f)|_{g_x} \leq (m!)^{1/2} |\nabla(A_{k,l} f)|_{g_x} = (m!)^{1/2} |\nabla f|_{g_x} \leq C' e^{-\eta d(x,o)} \tag{4.2}$$

for some $C' > 0$. Hence $\sigma A_{k,l}f \in S_mM \cap E^1_\eta(T_mM)$ for some $\eta > \frac{3}{2}\sqrt{K_0}$. Since $I(\sigma A_{k,l}f) = L_{k,l}f = 0$, we must have $\sigma A_{k,l}f = \sigma\nabla h$ for some $h \in S_{m-1}M$ where $h \in E_{\eta-\varepsilon}(T_{m-1}M)$ for all $\varepsilon > 0$ by [16, Theorem 1.1]. This gives the claim for the first part.

(b) Similarly using the pointwise estimates one obtains that $\sigma A_{k,l}f \in S_mM \cap P^1_\eta(T_mM)$ for some $\eta > 2$. Now [16, Theorem 1.2] implies that $\sigma A_{k,l}f = \sigma\nabla h$ for some $h \in S_{m-1}M \cap P_{\eta-1}(T_{m-1}M)$. This proves the second part. □

Remark 4.3. One can study the mixed ray transform on asymptotically hyperbolic surfaces [9]. Let (M, g) be an asymptotically hyperbolic surface, $\overline{M}$ the compactification of M and ρ a geodesic boundary defining function as defined in [9]. One usually assumes that $f \in \rho^{1-m}C^\infty(S_m\overline{M})$ to obtain s-injectivity results for the geodesic ray transform. It then follows that $\sigma A_{k,l}f \in \rho^{1-m}C^\infty(S_m\overline{M})$ and similar s-injectivity result as in corollary 4.1 holds under certain assumptions on (M, g); we refer to [9] for a more detailed discussion. One can also study the mixed ray transform on asymptotically conic surfaces (M', g'). One obtains s-injectivity for tensor fields $f \in A^{-1}_{k,l}\rho'^r C^\infty(S^{sc}_m\overline{M'})$ where ρ' is the boundary defining function, $r > n/2 + 1$ and $S^{sc}_m\overline{M'} \subset S_m\overline{M'}$ is the set of scattering tensor fields on the compactification $\overline{M'}$. See [10] for more details.

4.2. **Stability results on compact surfaces.** In this section we obtain stability estimates for the mixed ray transform. We begin with the following lemma.

Lemma 4.4. *Let (M, g) be a compact simple surface. Then the normal operator of the mixed ray transform $L_{k,l}$ is $N_{k,l} = (-1)^k A_{k,l}NA_{k,l}$ where N is the normal operator of the geodesic ray transform I on $(k+l)$-tensor fields.*

Proof. By theorem 3.3 part (d) we only need to calculate $(\mathcal{D}^{-1})^* = A^*_{k,l}$. Now for the matrix representations of A_i we have that $(A_i)^j_m = -(A_i)^m_j$ for $i = 1, \dots k$ and $(A_i)^j_m = \delta^j_m$ for $i = k+1, \dots k+l$. Using this one obtains

$$g_x(A_{k,l}f, h) = g_x(f, (-1)^k A_{k,l}h) \tag{4.3}$$

and thus

$$\langle A_{k,l}f, h\rangle_{L^2(T_mM)} = \left\langle f, (-1)^k A_{k,l}h\right\rangle_{L^2(T_mM)}. \tag{4.4}$$

Hence $A^*_{k,l} = (-1)^k A_{k,l}$ which gives the claim. □

The next estimates are direct consequences of the results in [20, 28, 29]. We denote by $\mathrm{Sol}(T_mM)$ the set of solenoidal tensor fields. For the definition of the tangential norm $\|\cdot\|_{H^{1/2}_T(\partial_{\mathrm{in}}SM)}$ see [20].

Corollary 4.5. *For any compact simple surface (M, g) and nonnegative integers k and l there is a constant $C > 0$ so that:*

(a) Let $k + l = 1$. Let g be extended to a simple metric in $M_1 \supset\supset M$. Then the estimate

$$\|f\|_{L^2(T_1M)}/C \le \|N_{k,l}f\|_{H^1(T_1M_1)} \le C\,\|f\|_{L^2(T_1M)} \tag{4.5}$$

holds for all $f \in A_{k,l}^{-1}(\mathrm{Sol}(T_1M)) \cap L^2(T_1M)$.

(b) *Let* $k+l=2$. *Let* g *be extended to a simple metric in* $M_1 \supset\supset M$. *Then the estimate*

$$\|f\|_{L^2(T_2M)}/C \leq \|N_{k,l}f\|_{H^1(T_2M_1)} \leq C\|f\|_{L^2(T_2M)} \tag{4.6}$$

holds for all $f \in A_{k,l}^{-1}(\mathrm{Sol}(T_2M) \cap H^1(S_2M))$.

(c) *Let* $m := k+l \geq 1$. *Assume further that* (M,g) *has non-positive sectional curvature. Then the estimate*

$$\|f\|_{L^2(T_mM)} \leq C\|L_{k,l}f\|_{H_T^{1/2}(\partial_{\mathrm{in}}SM)} \tag{4.7}$$

holds for all $f \in A_{k,l}^{-1}(\mathrm{Sol}(T_mM) \cap H^1(S_mM))$.

Proof. (a) We know that the stability estimate holds for the geodesic ray transform [29, Theorem 4]. Now $A_{k,l}\colon L^2(T_1M) \to L^2(T_1M)$ and $A_{k,l}^* = A_{k,l}^{-1}\colon H^1(T_1M_1) \to H^1(T_1M_1)$ are isometries by lemma 3.11 part (c). By theorem 3.3 part (e) we obtain

$$\|f\|_{L^2(T_1M)}/C \leq \|N_{k,l}f\|_{H^1(T_1M_1)} \leq C\|f\|_{L^2(T_1M)}. \tag{4.8}$$

(b) By [28, Theorem 1] the stability estimate holds for the geodesic ray transform if we know s-injectivity. But s-injectivity holds on two-dimensional simple manifolds for tensor fields of all order [21, Theorem 1.1]. Using the fact that $A_{k,l}\colon L^2(T_2M) \to L^2(T_2M)$ and $A_{k,l}^*\colon H^1(T_2M) \to H^1(T_2M)$ are isometries we obtain the stability estimate as in part (a) above.

(c) We know that the stability estimate is true for the geodesic ray transform [20, Theorem 1.3]. Since $A_{k,l}\colon L^2(T_mM) \to L^2(T_mM)$ is an isometry theorem 3.3 part (c) implies

$$\|f\|_{L^2(T_mM)} \leq C\|L_{k,l}f\|_{H_T^{1/2}(\partial_{\mathrm{in}}SM)}. \tag{4.9}$$

This concludes the proof. □

Remark 4.6. Note that for example the estimate

$$\|f\|_{L^2}/C \leq \|N_{k,l}f\|_{H^1} \leq C\|f\|_{L^2} \tag{4.10}$$

holds for all $f \in A_{k,l}^{-1}(S'')$ if and only if the estimate

$$\|h\|_{L^2}/C \leq \|Nh\|_{H^1} \leq C\|h\|_{L^2} \tag{4.11}$$

holds for all $h \in S''$. This follows since $A_{k,l}\colon H^p(T_mM) \to H^p(T_mM)$ is an isometry for all $p \in \mathbb{N}$ and $N_{k,l} = (-1)^k A_{k,l} N A_{k,l}$. Therefore the sets defined in corollary 4.5 are in a sense largest sets where such stability estimates can hold. A similar sharp stability estimate as in part (c) of corollary 4.5 can be proved on compact simple surfaces when $m = 1, 2$ [2, Theorem 1.1] (see also [4] for the Euclidean case).

4.3. **Transverse ray transform of one-forms.** Next we study the kernel of the transverse ray transform on one-forms in two dimensions. The result which we obtain is previously known in $\mathbb{R}^2$ [19]. We recall that in our notation the transverse ray transform is $I_\perp f = I_A f$ where $A_i = \star$ for all $i \in \{1, \dots, m\}$. For a scalar field ϕ, we define $\mathrm{curl}(\phi) = e_2(\phi)e^1 - e_1(\phi)e^2$ where $\{e_1, e_2\}$ is any positively oriented local orthonormal frame and $\{e^1, e^2\}$ its coframe.

Corollary 4.7. *Let (M, g) be two-dimensional orientable Riemannian manifold with boundary such that the geodesic ray transform is s-injective on smooth one-forms and let $f \in C^\infty(T_1M)$. Then $I_\perp f = 0$ if and only if $f = \mathrm{curl}(\phi)$ for some smooth function ϕ vanishing on the boundary.*

Proof. If $f = \mathrm{curl}(\phi)$ where ϕ vanishes on the boundary, then $Af = \mathrm{d}\phi$ and $I_\perp f = I(Af) = 0$ by the fundamental theorem of calculus. For the converse, if $I_\perp f = 0$, then $I(Af) = 0$. By solenoidal injectivity we have that $Af = \mathrm{d}\phi$ for some smooth scalar function ϕ vanishing on the boundary ∂M. This implies that $f = A^{-1}\mathrm{d}\phi$ which in local positively oriented orthonormal frame $\{e_1, e_2\}$ means $f_1 = e_2(\phi)$ and $f_2 = -e_1(\phi)$, i.e. $f = \mathrm{curl}(\phi)$. □

Remark 4.8. We note that on two-dimensional Cartan–Hadamard manifolds one can also deduce from $I_\perp f = 0$ that $f = \mathrm{curl}(\phi)$ if one of the following assumptions holds

(a) $-K_0 \le K \le 0$ for some $K_0 > 0$ and $f \in E^1_\eta(T_1M)$ for some $\eta > \frac{3}{2}\sqrt{K_0}$
(b) $K \in P_\kappa(M)$ for some $\kappa > 2$ and $f \in P^1_\eta(T_1M)$ for some $\eta > 2$.

If we combine the data from the geodesic ray transform If and the transverse ray transform $I_\perp f$, we can uniquely reconstruct any smooth one-form on two-dimensional compact simple manifolds. This result is also known previously in $\mathbb{R}^2$ [5]. Recall that $\Delta_g u = \mathrm{div}((\mathrm{grad}(u))$ where $\mathrm{grad}(u) = (\mathrm{d}u)^\sharp$.

Corollary 4.9. *Let (M, g) be a compact simple surface. Then the geodesic ray transform and the transverse ray transform together determine $f \in C^\infty(T_1M)$ uniquely, i.e. if both $If = 0$ and $I_\perp f = 0$, then $f = 0$.*

Proof. Since (M, g) is simple, the solenoidal injectivity of I (see [21]) implies that $f = \mathrm{d}u$ for some smooth function u vanishing on the boundary. On the other hand, $I_\perp f = 0$ gives that $f = \mathrm{curl}(\phi)$ for some smooth scalar function ϕ by corollary 4.7. But this implies that $\mathrm{div}(f) = 0$. Therefore $\Delta_g u = \mathrm{div}(f) = 0$ so u is a harmonic function vanishing on the boundary. We obtain $u = 0$ and hence $f = 0$. □

Remark 4.10. One could also use solenoidal decomposition to prove the previous corollary. By the solenoidal decomposition $f = f^{\mathrm{s}} + \mathrm{d}u$. Now $If = 0$ implies that $f^{\mathrm{s}} = 0$. On the other hand, $I_\perp f = 0$ implies that f is solenoidal, i.e. $f = f^{\mathrm{s}} = 0$.

The previous corollary holds also on two-dimensional Cartan–Hadamard manifolds as we will prove next. We first state and prove a version of Liouville's theorem on Cartan–Hadamard manifolds.

Lemma 4.11. *Let (M, g) be a two-dimensional Cartan–Hadamard manifold and u harmonic function on M, i.e. $\Delta_g u = 0$. Fix any point $o \in M$. Assume that one of the following conditions hold:*

(a) $-K_0 \leq K \leq 0$ *for some* $K_0 > 0$ *and*

$$|u(x)|\,|\mathrm{d}u(x)| \leq Ce^{-\eta d(x,o)} \tag{4.12}$$

for some $C > 0$ *and some* $\eta > \sqrt{K_0}$.

(b) The curvature satisfies

$$|K(x)| \leq C(1 + d(x,o))^{-\kappa} \tag{4.13}$$

for some $C > 0$ *and* $\kappa > 2$ *and the function satisfies*

$$|u(x)|\,|\mathrm{d}u(x)| \leq C(1 + d(x,o))^{-\eta} \tag{4.14}$$

for some $\eta > 1$.

Then u *is constant.*

We point out that the conditions above are independent of the choice of $o \in M$ as in the definition of the spaces in (2.11). Moving the point will only change the constants.

Proof. Assume first that (a) holds. Let $B_r(o)$ be the geodesic ball of radius $r > 0$ centered at o. Using the integration by parts formula (see [14]) we obtain

$$\begin{aligned} 0 &= \int_M u\Delta u \mathrm{d}V_g \\ &= \lim_{r\to\infty} \int_{B_r(o)} u\Delta u \mathrm{d}V_g \\ &= \lim_{r\to\infty} \left(-\int_{B_r(o)} |\mathrm{grad}(u)|^2_{g_x} \mathrm{d}V_g + \int_{S_r(o)} uN(u)\mathrm{d}\widehat{V}_g \right) \end{aligned} \tag{4.15}$$

where $\mathrm{d}\widehat{V}_g$ is the induced volume form on the geodesic sphere $S_r(o) = \partial B_r(o)$ and N is the outward unit normal vector field. We focus on the second term. Since $N(u) = g_x(\mathrm{grad}(u), N)$ and $|\mathrm{grad}(u)|_{g_x} = |\mathrm{d}u|_{g_x}$, we can estimate that $|uN(u)| \leq |u|\,|\mathrm{grad}(u)|_{g_x} = |u|\,|\mathrm{d}u|_{g_x}$. The volume form can be expressed in polar coordinates as $\mathrm{d}\widehat{V}_g = J_o(r,\theta)\mathrm{d}\theta$ where $|J_o(r,\theta)| \leq Ce^{\sqrt{K_0}r}$ [15, Lemma 4.7]. Therefore we obtain

$$\left| \int_{S_r(o)} uN(u)\mathrm{d}\widehat{V}_g \right| \leq C'e^{(-\eta+\sqrt{K_0})r} \xrightarrow{r\to\infty} 0. \tag{4.16}$$

This implies $|\mathrm{d}u|_{g_x} = |\mathrm{grad}(u)|_{g_x} = 0$ and hence $\mathrm{d}u = 0$. Connectedness of M implies that u is constant.

If (b) holds, then $|J_o(r,\theta)| \leq Cr$ [15, Lemma 4.7]. The claim is proved identically as in part (a). ☐

Remark 4.12. One can prove the previous lemma in the exact same way for Cartan–Hadamard manifolds of dimension $n > 2$ using the growth estimates for the Jacobi fields proved in [16]. In the condition (a) one requires $\eta > (n-1)\sqrt{K_0}$ and in the condition (b) one requires $\eta > n - 1$.

Corollary 4.13. *Let* (M, g) *be two-dimensional Cartan–Hadamard manifold. Assume that one of the following conditions holds:*

(a) $-K_0 \leq K \leq 0$ *for some* $K_0 > 0$ *and* $f \in E^1_\eta(T_1M)$ *for some* $\eta > \frac{3}{2}\sqrt{K_0}$.

(b) The curvature satisfies the estimate (4.13) *for some* $C > 0$ *and* $\kappa > 2$ *and* $f \in P^1_\eta(T_1M)$ *for some* $\eta > 2$.

Then the geodesic ray transform and the transverse ray transform together determine the one-form f *uniquely, i.e. if both* $If = 0$ *and* $I_\perp f = 0$*, then* $f = 0$.

Proof. Assume that (a) holds. The condition $If = 0$ implies that $f = \mathrm{d}h$ for some $h \in E_{\eta-\varepsilon}(M)$ where $\varepsilon > 0$ is arbitrary (see [16]). On the other hand, $I_\perp f = 0$ implies that $\Delta_g h = \mathrm{div}(f) = 0$. Hence h is harmonic and satisfies the decay estimate in lemma 4.11. Thus h is constant and $f = 0$. The proof under the assumption (b) is identical. □

Appendix A. Notation

A.1. Integral transforms.

- If, the geodesic X-ray transform of a tensor field f of order m. See section 2.4 and equations (2.7) and (2.10).
- $I_{SM}h$, the geodesic ray transform of a function $h\colon SM \to \mathbb{R}$. See section 2.4 and equation (2.8).
- $I_{A,r}f$, the (abstract) mixing ray transform with a mixing A of degree m, operating on a tensor field f of order m. See section 3.2 and equation (3.11).
- $L_{k,l}f = I_{A_{k,l}}f$, the mixed ray transform of a tensor field f of order $k + l$ on two-dimensional orientable Riemannian manifold. See section 2.5 and equations 2.19 and 2.20.
- $I_\perp f$, the transverse ray transform of a tensor field f of order k, corresponding to the mixed ray transform with $l = 0$. See section 2.5 and equation 2.19.
- $I^q_{A,r}[f] = I_{A,r}f$, the quotient transform of an equivalence class of tensor field f of degree m. See section 3.2.
- $\mathcal{L}_\beta f$, the light ray transform of a (compactly supported) tensor field of order m. See section 3.3.3 and equation (3.44).
- $\mathcal{L}^q_\beta[f] = \mathcal{L}_\beta f$, the quotient light ray transform of an equivalence class of a (compactly supported) tensor field f of degree m. See section 3.3.3.

A.2. Other operators on tensor fields.

- A, a mixing composed of automorphisms of the tangent bundle. See section 3.2 and equation (3.9).
- A_i, automorphisms (fiberwise linear bijections) of the tangent bundle. See the beginning of section 3.2.
- λ and λ_x, operators converting m-tensor field and m-tensor into a function on the tangent bundle and tangent space. See section 3.1 and equation (3.3).
- $\lambda_r = r \circ \lambda$ and $\lambda_{r,x} = r_x \circ \lambda_x$, where r and r_x are the restriction operators on the tangent bundle and tangent space. See section 3.1.
- $A_{k,l}$, the mixing corresponding to the mixed ray transform $L_{k,l}$. See section 2.5 and equation 2.19.
- σ, the usual symmetrization operator of tensor fields. See section 3.1 and equation (3.1).

- $\widehat{\sigma}_{A,r}$, the projection operator onto $A^{-1}(\mathrm{Ker}(\lambda_r)^{\perp})$, related to the mixing ray transform $I_{A,r}$. See sections 3.1 and 3.2, and equations (3.7) and (3.12).
- $\mathcal{H} = \mathrm{Id} - \widehat{\sigma}_{A,r}$, an operator projecting m-tensor field onto $\mathrm{Ker}(\lambda_r \circ A)$. See sections 3.2 and 3.3, and theorem 3.3.
- $\mathcal{D} = A^{-1} \circ \widetilde{A}$, an auxiliary operator related to two admissible mixings A and $\widetilde{A}$ of degree m. See section 3.2 and theorem 3.3.
- $\nabla^A = A^{-1} \circ \nabla$, the weighted covariant derivative of a m-tensor field where A is an admissible mixing of degree m. See section 3.3.1.
- $N_{k,l}$, the normal operator of the mixed ray transform $L_{k,l}$ on compact simple surfaces. See section 4.2 and lemma 4.4.

A.3. **Other.**

- $\mathcal{F}(X)$, the set of all functions $X \to \mathbb{C}$.
- M or (M, g), a connected (pseudo-) Riemannian manifold of dimension $n \geq 2$.
- SM, the sphere bundle whose fibers are unit spheres of the tangent spaces. See section 2.4.
- $\mathfrak{X}(T_m M)$, the space of all covariant m-tensor fields. See section 2.1.
- $S_m M$, the space of symmetric m-tensor fields. See sections 2.1 and 3.1.
- $C^q(T_m M)$ and $C^q(S_m M)$, the set of C^q-smooth (symmetric) m-tensor fields where $q \in \mathbb{N}$. See section 2.1.
- $H^k(T_m M)$ and $H^k(S_m M)$, the L^2-Sobolev space of (symmetric) m-tensor field where $k \in \mathbb{N}$. See section 2.2.
- $P_\eta(T_m M)$ and $P^1_\eta(T_m M)$, the spaces of polynomially decaying m-tensor fields on Cartan–Hadamard manifolds. See section 2.4 and equation (2.11).
- $E_\eta(T_m M)$ and $E^1_\eta(T_m M)$, the spaces of exponentially decaying m-tensor fields on Cartan–Hadamard manifolds. See section 2.4 and equation (2.11).
- $[f]$ and $[f]_A$, the equivalence class of the tensor field f, under the relation $f \sim h$ if and only if $f - h \in \mathrm{Ker}(\lambda_r \circ A)$. See sections 3.1, 3.2, 3.3.1 and 3.3.3.

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Boundary rigidity for Randers metrics

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BOUNDARY RIGIDITY FOR RANDERS METRICS

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Abstract. If a non-reversible Finsler norm is the sum of a reversible Finsler norm and a closed 1-form, then one can uniquely recover the 1-form up to potential fields from the boundary distance data. We also show a boundary rigidity result for Randers metrics where the reversible Finsler norm is induced by a Riemannian metric which is boundary rigid. Our theorems generalize Riemannian boundary rigidity results to some non-reversible Finsler manifolds. We provide an application to seismology where the seismic wave propagates in a moving medium.

1. Introduction

In this article we study a certain type of inverse problem for a special class of Finsler norms. The inverse problem we consider is known as the boundary rigidity problem: does the boundary distance data determine the Finsler norm uniquely up to the natural gauge in question? Here we present the problem and our results in a general level; more detailed information can be found in sections 1.1, 1.2 and 2.

Let M be a smooth manifold with boundary ∂M. A Finsler norm F on M is a non-negative function on the tangent bundle $F\colon TM \to [0,\infty)$ such that for each $x \in M$ the map $y \mapsto F(x,y)$ defines a positively homogeneous norm in T_xM. In general, Finsler norms are homogeneous only in positive scalings and they induce a distance function on M which is not necessarily symmetric in contrast to the Riemannian distance function.

Let β be a smooth 1-form on M and F_r a reversible Finsler norm, i.e. $F_r(x,-y) = F_r(x,y)$ for all $x \in M$ and $y \in T_xM$. If the norm of β with respect to F_r is small enough, we can define the non-reversible Finsler norm $F = F_r + \beta$. The Finsler norm F is non-reversible in the sense that $F(x,-y) = F(x,y)$ for all $x \in M$ and $y \in T_xM$ if and only if $\beta = 0$. We can thus think that β is an anisotropic perturbation to the reversible Finsler norm F_r. We further assume that β is closed ($\mathrm{d}\beta = 0$) which implies that F and F_r have the same geodesics as point sets and that F has reversible geodesics.

Suppose we know the boundary distance data of $F = F_r + \beta$, i.e. we know the lengths of all geodesics of F connecting two points on the boundary ∂M. The question is: can we say something about β and F_r from this information? We prove that if M is simply connected, then one can uniquely

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recover the 1-form β (up to potential fields) and the boundary distance data of F_r from the boundary distance data of F (see theorem 1.3 for a precise statement).

Riemannian metrics form a special class of reversible Finsler norms. Suppose that F_r is induced by a Riemannian metric g and write $F_r = F_g$. If $\|\beta\|_g < 1$, then $F = F_g + \beta$ defines a non-reversible Finsler norm called Randers metric. We say that the Riemannian manifold (M, g) is boundary rigid, if the boundary distance data determines the metric g uniquely up to boundary preserving diffeomorphism. We prove that if M is simply connected and (M, g) is boundary rigid, then (M, F) is also boundary rigid in the sense that one can uniquely recover the 1-form β up to potential fields and the Riemannian metric g up to boundary preserving diffeomorphism from the boundary distance data of F. See theorem 1.5 for a precise statement.

Our proofs are mainly based on the following two facts. First, if two Finsler norms differ only by a closed 1-form, then they are projectively equivalent (they have the same geodesics modulo orientation preserving reparametrizations). Second, since $F = F_r + \beta$, we can express the length of any curve γ with respect to F_r in terms of the symmetric part of the length functional $L_F(\gamma)$. Similarly, the integral $\int_\gamma \beta$ can be expressed in terms of the antisymmetric part of $L_F(\gamma)$. This allows us to reduce the boundary rigidity problem of F to the boundary rigidity problem of F_r.

Boundary rigidity has been studied earlier mainly on Riemannian manifolds. Boundary rigidity is known for example for simple subspaces of Euclidean space [34], simple subspaces of symmetric spaces of constant negative curvature [10], conformal simple metrics which agree on the boundary [26, 55] and for certain two-dimensional manifolds including compact simple surfaces [25, 43, 45, 50]. It is also conjectured that compact simple manifolds of any dimension are boundary rigid [43]. Our results generalize the boundary rigidity results to certain Randers metrics whenever the boundary rigidity of the unperturbed Riemannian manifold is known (see theorem 1.5). For a more comprehensive treatment of the boundary rigidity problem in Riemannian geometry, see the review [55].

Closest to our main theorems are rigidity results for magnetic geodesics on Riemannian manifolds. In [27] the authors prove boundary rigidity in the presence of a magnetic field (see also [7] for a generalization). Magnetic geodesics can be seen as geodesics of a Randers metric under additional assumptions for the vector potential which induces the magnetic field (the magnetic field has to be "weak") [36, 56]. There is also a correspondence between Randers metrics and stationary Lorentzian metrics [16, 17, 40] (see [57] for a boundary rigidity result on stationary Lorentzian manifolds). We note that projectively flat Finsler norms (geodesics of the Finsler norm are segments of straight lines) on compact convex domains in $\mathbb{R}^2$ are completely determined by their boundary distance data [2, 3, 41]. In fact, this holds for a more general class of projective metrics in the plane [41].

Some geometric results similar to the boundary rigidity are known on Finsler manifolds. It was shown in [30] that the collection of boundary distance maps, which measure distances from the interior to the boundary, determines the topological and differential structures of the Finsler manifold.

Further, it was shown in [31] that the broken scattering relation (lengths of all geodesics with endpoints on the boundary and reflecting once in the interior) determines the isometry class of reversible Finsler manifolds admitting a strictly convex foliation.

The boundary rigidity problem is known in seismology as the travel time tomography problem where one tries to recover the speed of sound inside the Earth by measuring travel times of seismic waves on the surface. The ray paths of the seismic waves correspond to geodesics and the travel times correspond to lengths of the geodesics. The travel time tomography problem was solved in the beginning of 20th century for spherically symmetric metrics $g = c^{-2}(r)e$ where e is the Euclidean metric and $c = c(r)$ is a radial sound speed satisfying the Herglotz condition (see equation (1)) [35, 58]. Our results apply to the situation where the seismic wave propagates in a moving medium: one can uniquely recover both the sound speed and the velocity of the medium up to potential fields from travel time measurements (see theorem 1.5 and section 1.2). The linearization of the boundary rigidity or travel time tomography problem leads to tensor tomography where one wants to characterize the kernel of the geodesic ray transform on symmetric 2-tensor fields [52]. For results in this direction and a general overview of tensor tomography, see the reviews [37, 48].

1.1. **The main results.** Before stating our main results let us briefly introduce some notation; more details can be found in section 2. The proofs of the main theorems can be found in section 3.

We denote by M an n-dimensional smooth manifold with boundary ∂M where $n \geq 2$. We let F be a Finsler norm and F_r refers to a reversible Finsler norm, i.e. $F_r(x, -y) = F_r(x, y)$ for all $x \in M$ and $y \in T_xM$. Riemannian metrics are a special case of Finsler norms: if g is a Riemannian metric, then it induces a reversible Finsler norm F_g as $F_g(x, y) = \sqrt{g_{ij}(x)y^i y^j}$. We denote by β a smooth closed 1-form ($\mathrm{d}\beta = 0$) and $\|\beta\|_{F^*} = \sup_{x\in M} F^*(x, \beta(x))$ is the dual norm of β with respect to the co-Finsler norm F^* in T^*M.

We say that the Finsler norm F is admissible, if for every two boundary points $x, x' \in \partial M$ there is unique geodesic γ of F with finite length going from x to x'. If F is admissible, then we define the (not necessarily symmetric) map $d_F(\cdot,\cdot)\colon \partial M \times \partial M \to [0,\infty)$ by setting $d_F(x, x') = L_F(\gamma)$ where $L_F(\gamma)$ denotes the length of the curve γ with respect to F. We call the map $d_F(\cdot,\cdot)$ the boundary distance data of F. Finally, we say that the Riemannian manifolds (M, g_1) and (M, g_2) are boundary rigid, if $d_{g_1}(x, x') = d_{g_2}(x, x')$ for all $x, x' \in \partial M$ implies that $g_2 = \Psi^* g_1$ where $\Psi\colon M \to M$ is a diffeomorphism such that $\Psi|_{\partial M} = \mathrm{Id}$. In other words, g_1 and g_2 are isometric as Riemannian metrics.

We recall that a diffeomorphism $\Psi\colon (M, F_2) \to (M, F_1)$ is an isometry between Finsler manifolds if $\Psi^* F_1 = F_2$, or equivalently Ψ preserves the Finslerian distance [6]. We make the following observations before giving our first theorem.

Remark 1.1. *We note that Finsler norms are very flexible with respect to the boundary distance data, i.e. they are not usually boundary rigid in the same sense as Riemannian metrics. Let* $\Psi\colon M \to M$ *be a diffeomorphism which is identity on the boundary. If* F_1 *is an admissible*

Finsler norm and ϕ *is a scalar field which is constant on the boundary and its differential* $\mathrm{d}\phi$ *has sufficiently small norm with respect to* Ψ^*F_1*, then* F_1 *and* $F_2 = \Psi^*F_1 + \mathrm{d}\phi$ *give the same boundary distance data (Finslerian isometries preserve geodesics* [6] *and addition of* $\mathrm{d}\phi$ *only changes parametrizations of geodesics* [21]*). Especially, if* F_1 *is reversible, then* $\{\Psi^*F_1 + \mathrm{d}\phi : \Psi|_{\partial M} = \mathrm{Id}$ *and* $\phi|_{\partial M} = constant\}$ *provides a large family of Finsler norms which give the same boundary distances but are not isometric to* F_1 *(since* $\Psi^*F_1 + \mathrm{d}\phi$ *is non-reversible whenever* ϕ *is not constant). See also* [12, 22, 23, 38] *for results and constructions of non-isometric Finsler norms giving the same boundary distances.*

Remark 1.2. *Finsler norms* F_1 *and* F_2 *which satisfy* $F_2 = \Psi^*F_1 + \mathrm{d}\phi$ *for some scalar field* ϕ *and diffeomorphism* Ψ *are sometimes called almost isometric Finsler norms and the map* $\Psi\colon (M, F_2) \to (M, F_1)$ *is called almost isometry* [14, 28, 36, 39]*. We show in theorem 1.5 that under certain assumptions the boundary distance data determines Randers metrics up to an almost isometry (see also remark 1.6). Almost isometries have many good properties: they for example are projective transformations which preserve (minimizing) geodesics up to reparametrization* [39]*. Almost isometries can also be defined on general quasi-metric spaces* (X, d)*. It follows that if* $\Psi\colon (X_1, d_1) \to (X_2, d_2)$ *is an almost isometry between quasi-metric spaces, then* Ψ *is an isometry between the metric spaces* $(X_1, \widetilde{d_1}) \to (X_2, \widetilde{d_2})$ *where* $\widetilde{d_i}(p, q) = \frac{1}{2}(d_i(p, q) + d_i(q, p))$ *is the symmetrized metric* [14, 39]*. Especially, in the case of metric spaces almost isometries are isometries.*

Our first theorem says that one can uniquely recover (up to potential fields) the perturbation β and the boundary distance data of F_r from the boundary distance data of $F = F_r + \beta$.

Theorem 1.3. *Let* M *be a compact and simply connected smooth manifold with boundary. For* $i \in \{1, 2\}$ *let* $F_i = F_{r,i} + \beta_i$ *be admissible Finsler norms where* $F_{r,i}$ *is an admissible and reversible Finsler norm and* β_i *is a smooth closed 1-form such that* $\|\beta_i\|_{F^*_{r,i}} < 1$*. Then the following are equivalent:*

(i) $d_{F_1}(x, x') = d_{F_2}(x, x')$ *for all* $x, x' \in \partial M$*.*

(ii) There is unique scalar field ϕ *vanishing on the boundary such that* $\beta_2 = \beta_1 + \mathrm{d}\phi$*, and* $d_{F_{r,1}}(x, x') = d_{F_{r,2}}(x, x')$ *for all* $x, x' \in \partial M$*.*

Remark 1.4. *Since* β_i *is closed and* M *is simply connected, it follows that* $\beta_i = \mathrm{d}\phi_i$ *for some scalar field* ϕ_i*. Thus* $F_{r,i}$ *and* $F_i = F_{r,i} + \beta_i = F_{r,i} + \mathrm{d}\phi_i$ *are almost isometric (but not isometric) Finsler norms (see remark 1.2). Trivially one can define* $\phi = \phi_2 - \phi_1$ *so that* $\mathrm{d}\phi = \beta_2 - \beta_1$*. The assumption* $d_{F_1}(x, x') = d_{F_2}(x, x')$ *for all* $x, x' \in \partial M$ *is then used to show that* ϕ *is constant on the boundary (and one can choose this constant to be zero).*

Let us clarify some of our assumptions in theorem 1.3. We need the assumption $\|\beta_i\|_{F^*_{r,i}} < 1$ to guarantee that the sum $F_{r,i} + \beta_i$ defines a Finsler norm. Reversibility of $F_{r,i}$ is needed so that any curve has the same length with respect to $F_{r,i}$ as any of its reversed reparametrizations. The condition that β_i is closed is used in three places. First, it is equivalent to that F_i

and $F_{r,i}$ have the same geodesics up to orientation preserving reparametrizations (F_i and $F_{r,i}$ are projectively equivalent, see lemma 2.2). Second, closedness of β_i is also equivalent to that F_i has reversible geodesics (F_i is projectively reversible, see lemma 2.1). Third, $\mathrm{d}\beta_i = 0$ implies that β_i is exact since M is assumed to be simply connected. All these properties are in a crucial role in our proofs.

The existence of unique geodesics connecting boundary points is used in the proof as well and for this reason we assume that the Finsler norms are admissible. We note that since F_i and $F_{r,i}$ are projectively equivalent, the admissibility of $F_{r,i}$ implies the admissibility of F_i, and vice versa. We also note that $\mathrm{d}\phi$ is closed so the conclusion $\beta_2 = \beta_1 + \mathrm{d}\phi$ is compatible with the assumptions on β_i. The conclusion that β_i differ only by a potential is similar to the solenoidal injectivity result for the geodesic ray transform of 1-forms [4, 48].

As an application of theorem 1.3 we have the following boundary rigidity result for Randers metrics (see [27, Theorem 6.4] for a similar result).

Theorem 1.5. *Let M be a compact and simply connected smooth manifold with boundary. For $i \in \{1,2\}$ let $F_i = F_{g_i} + \beta_i$ be admissible Finsler norms where g_i is an admissible Riemannian metric and β_i is a smooth closed 1-form such that $\|\beta_i\|_{g_i} < 1$. Assume that (M, g_i) is boundary rigid. Then the following are equivalent:*

- *(a) $d_{F_1}(x, x') = d_{F_2}(x, x')$ for all $x, x' \in \partial M$.*
- *(b) There is unique scalar field ϕ vanishing on the boundary and a diffeomorphism Ψ which is identity on the boundary such that $\beta_2 = \beta_1 + \mathrm{d}\phi$ and $g_2 = \Psi^* g_1$.*
- *(c) There is unique scalar field ϕ vanishing on the boundary and a diffeomorphism Ψ which is identity on the boundary such that $\beta_2 = \Psi^*\beta_1 + \mathrm{d}\phi$ and $g_2 = \Psi^* g_1$.*

Remark 1.6. *Theorem 1.5 part (c) implies that $F_2 = \Psi^* F_1 + \mathrm{d}\phi$, i.e. the Randers metrics F_1 and F_2 are almost isometric (see remark 1.2). Hence we obtain a boundary rigidity result for Randers metrics in the special case when the 1-form β is closed and the Riemannian metric g is boundary rigid. This generalizes earlier boundary rigidity results to non-reversible (and hence non-Riemannian) Finsler norms. Note that the diffeomorphism $\Psi\colon (M, F_2) \to (M, F_1)$ in part (c) is an almost isometry but not an isometry since this would require that $\Psi^*\beta_1 = \beta_2$ [9]. Also note that if $\beta_1 = 0$ and $\beta_2 \neq 0$, then F_1 and F_2 can not be isometric since F_1 is reversible and F_2 is non-reversible.*

The assumptions of theorem 1.5 are the same as in theorem 1.3 except that we also assume the boundary rigidity of (M, g). We can simultaneously recover the metric g and the 1-form β from the boundary distance data $d_F(\cdot,\cdot)$ since the reversibility of F_g implies that the data for β_i and g_i "decouple": for any curve γ one can obtain $\int_\gamma \beta$ from the antisymmetric part and $L_g(\gamma)$ from the symmetric part of the length functional $L_F(\gamma)$. We note that in theorems 1.3 and 1.5 we only use the lengths of geodesics connecting boundary points as data.

Admissible Finsler norms as we have defined are closely related to simple Finsler norms and simple Riemannian metrics. A Riemannian metric g on a smooth manifold M with boundary is simple if it is non-trapping (geodesics have finite length), geodesics have no conjugate points and the boundary ∂M is strictly convex with respect to g (the second fundamental form on ∂M is positive definite). See [47, Section 3.7] for many equivalent definitions of simple Riemannian metrics. The concept of a simple Finsler norm can be defined analogously [13, 38]. The simplicity of the Finsler norm or Riemannian metric implies that there exists unique minimizing geodesic between any two points of the manifold [13, 38, 47]. More generally, if the manifold admits a convex function which has a minimum point, then there is a finite number of geodesics between any two non-conjugate points [15, 32] (see also [49]).

We remark that one can take (M, g_i) to be a compact simple surface in theorem 1.5 since simple Riemannian metrics are admissible and in two dimensions they are boundary rigid [50]. If g_1 and g_2 are simple metrics which are conformal and agree on the boundary, then they are boundary rigid in any dimension $n \geq 2$ [26, 45, 55].

Theorem 1.5 has an application to Randers metrics arising in seismology (see section 1.2 for more details). Let $M = \overline{B}(0, R)$ be a closed ball of radius $R > 0$ and $g = c^{-2}(r)e$ where e is the Euclidean metric and $c = c(r)$ is a radial sound speed satisfying the Herglotz condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right) > 0, \quad r \in [0, R]. \tag{1}$$

It follows that (M, g) is a non-trapping Riemannian manifold with strictly convex boundary [44, 55]. Let us further assume that g has no conjugate points, i.e. $g = c^{-2}(r)e$ is a simple Riemannian metric. Then g is admissible and one can recover c and hence g uniquely in theorem 1.5 (see [47, Remark 2.10] and [52, 55]). Especially, the diffeomorphism Ψ becomes identity in this case ($\Psi = \mathrm{Id}$ also for general conformal simple metrics which agree on the boundary). However, Ψ can be a nontrivial diffeomorphism for general spherically symmetric Riemannian metrics g (see [29, Appendix C]). We also note that there are sound speeds c satisfying the Herglotz condition (1) such that g has conjugate points (and g is not admissible anymore, see [44, Section 3.3.2 and figure 6]). In section 1.2 we give a physical interpretation for the 1-form β in theorem 1.5 (β corresponds to the flow field of a moving medium).

1.2. **Application in seismology.** Here we give an application of theorems 1.3 and 1.5 to seismology where the seismic wave propagates in a moving medium. Assume that we have an object moving on a Riemannian manifold (M, g) with constant speed $\|U\|_g = 1$. The speed is fixed, but the object can change the direction of the velocity vector U arbitrarily. Let W be a vector field which can be interpreted as the additional velocity resulting from a time-independent external force field acting on the object. The net velocity is $U + W$ and we assume $\|W\|_g < 1$ so that the object can move freely in any direction.

Given any two points $p, q \in M$ we would like to know which path gives the least time when traveling from p to q taking the drift W into account.

This is known as the Zermelo's navigation problem (see [9, 20, 54]). It turns out that the unique solution is given by a geodesic of the Randers metric $F = F_\alpha + \beta$ where (see [20, Section 2.2])

$$\alpha_{ij} = \frac{g_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda}, \quad \beta_i = -\frac{W_i}{\lambda}$$
$$W_i = g_{ij}W^j, \quad \lambda = 1 - \|W\|_g^2$$

and we have left the dependence on $x \in M$ implicit. Especially, if the parameter of a piecewise smooth curve $\gamma\colon [0,T] \to M$ represents time, then (see [54, Lemma 3.1] and [21, Lemma 1.4.1])

$$T = L_F(\gamma). \tag{2}$$

Let us interpret the object as a seismic wave (or ray) propagating in a moving medium. The manifold M corresponds to the Earth which can be modelled as a compact and simply connected smooth manifold with boundary (a ball). By the Fermat's principle the path of the ray is a critical point of the travel time functional [5, 11, 19]. But since this functional equals to the length functional $L_F(\gamma)$ of the Randers metric $F = F_\alpha + \beta$ by equation (2), the ray paths of seismic waves correspond to geodesics of F which is the unique solution to the Zermelo's navigation problem.

If our Riemannian metric is of the form $g = c^{-2}e$ where e is the Euclidean metric and $c = c(x)$ is the sound speed, then $\|U\|_g = 1$ is equivalent to $\|U\|_e = c$ where $\|\cdot\|_e$ is the Euclidean norm of vectors. Thus U corresponds to the velocity of the propagating wave and the medium moves with velocity W for which $\|W\|_e < c$. The components of the Randers metric take the form

$$\alpha_{ij} = \frac{c^{-2}\delta_{ij}}{1 - c^{-2}\|W\|_e^2} + \frac{c^{-4}W^iW^j}{(1 - c^{-2}\|W\|_e^2)^2}$$
$$\beta_i = -\frac{c^{-2}W^i}{1 - c^{-2}\|W\|_e^2}.$$

Note that here we have identified $W^i = \delta_{ij}W^j$. Now if the 1-form β is closed, then theorem 1.3 implies that one can uniquely recover β up to potential fields from travel time measurements of seismic waves (assuming admissibility of α). In addition, if the Riemannian manifold (M, α) is boundary rigid, then by theorem 1.5 one can also uniquely recover the Riemannian metric α up to boundary preserving diffeomorphism from the travel time data.

Let us do the following approximation. If we assume that $\|W\|_e / c \ll 1$, then

$$\alpha_{ij} \approx c^{-2}\delta_{ij} + \frac{W^i}{c^2}\frac{W^j}{c^2}$$
$$\beta_i \approx -\frac{W^i}{c^2}.$$

When we only work to first order in $\|W\|_e / c$, the Riemannian metric α reduces to

$$\alpha_{ij} \approx c^{-2}\delta_{ij} = g_{ij}$$

and the ray paths of seismic waves correspond to geodesics of the Randers metric $F = F_g + \beta$. Similar linearization result is obtained in [33] for sound waves propagating in air under the influence of wind. We also note that the same result can be obtained from the linearization of travel time measurements [46].

If the sound speed $c = c(r)$ is radial, c satisfies the Herglotz condition (1) and $g = c^{-2}(r)e$ has no conjugate points, then theorem 1.5 implies that in the first order approximation (with respect to $\|W\|_e/c$) one can uniquely recover the sound speed c and the velocity of the medium W up to potential fields from travel time measurements. If the speed of sound c is constant, then the condition $\mathrm{d}(W/c^2) = 0$ reduces to $\mathrm{d}W = 0$, which in the case of a fluid flow means that W is irrotational (or curl-free). Note that in the approximation we identify $W^i = \delta_{ij}W^j$. In general, if c is not constant, then the condition $\mathrm{d}(W/c^2) = 0$ only means that the scaled flow field W/c^2 is irrotational.

To summarize this section: our results (theorems 1.3 and 1.5) apply to the propagation of seismic waves in a moving medium. Under certain assumptions one can recover the velocity of the medium from travel time measurements, and at the same time one reduces the travel time tomography problem in moving medium to the case where no flow field is involved. This allows one to recover the speed of sound as well in the first order approximation.

2. Finsler manifolds

In this section we give a brief introduction to Finsler geometry. We only go through definitions and results which are needed in proving our main theorems. Basic theory of Finsler geometry can be found for example in [1, 8, 21, 53]. We use the Einstein summation convention, i.e. indices which appear both as a subscript and superscript are implicitly summed over.

Let M be a smooth manifold. We denote by $x \in M$ the base point on the manifold and by $y \in T_xM$ the tangent vectors. A Finsler norm F on M is a non-negative function on the tangent bundle $F\colon TM \to [0,\infty)$ such that

(F1) F is smooth in $TM \setminus \{0\}$ (smoothness outside zero section)
(F2) $F(x,y) = 0$ if and only if $y = 0$ (positivity)
(F3) $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda \geq 0$ (positive homogeneity of degree 1)
(F4) $\frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$ is positive definite whenever $y \neq 0$ (convexity).

The pair (M,F) is called a Finsler manifold. If F is a Finsler norm, then one can define the reversed Finsler norm $\overleftarrow{F}$ by setting $\overleftarrow{F}(x,y) = F(x,-y)$. It follows that $\overleftarrow{F}$ also satisfies the properties (F1)–(F4).

The conditions (F1)–(F4) imply that for every $x \in M$ the map $y \mapsto F(x,y)$ defines a positively homogeneous norm in T_xM. If $F(x,-y) = F(x,y)$ for all $x \in M$ and $y \in T_xM$, we say that the Finsler norm F is reversible (or absolutely homogeneous). If F is reversible, then the map $y \mapsto F(x,y)$ defines a norm in T_xM. Every Riemannian metric $g = g(x)$ on M induces a reversible Finsler norm F_g on M by setting

$$F_g(x,y) = \sqrt{g_{ij}(x)y^iy^j}.$$

The condition (F4) allows us to define the local metric $g_{ij} = g_{ij}(x,y)$ as

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}.$$

One can then define the Legendre transformation $L\colon TM \to T^*M$ using the local metric g_{ij} (see for example [53, Chapter 3.1]). If $F\colon TM \to [0,\infty)$ is a Finsler norm, then by using the Legendre transformation one obtains the dual norm (or co-Finsler norm) $F^*\colon T^*M \to [0,\infty)$ satisfying the properties (F1)–(F4) in T^*M. The dual norm of a covector $\omega \in T^*_x M$ becomes

$$F^*(x,\omega) = \sup_{\substack{y \in T_x M \\ F(x,y)=1}} \omega(y).$$

If $F = F_g$ where $g = g(x)$ is a Riemannian metric, then $g_{ij}(x,y) = g_{ij}(x)y^i y^j$ and the Legendre transformation L and its inverse correspond to the musical isomorphisms.

In this article we study a class of non-reversible Finsler norms. Let F_1 be a Finsler norm on M and β a smooth nonzero 1-form on M. Assume that the dual norm of β satisfies

$$\|\beta\|_{F_1^*} := \sup_{x\in M} F_1^*(x,\beta(x)) < 1.$$

Then $F = F_1 + \beta$ defines also a Finsler norm on M (see [53, Example 6.3.1] and [8, Chapter 11.1]). We study the special case $F = F_r + \beta$ where F_r is a reversible Finsler norm. It follows that Finsler norms of this kind are non-reversible since $F(x,-y) = F(x,y)$ for all $x \in M$ and $y \in T_x M$ if and only if $\beta \equiv 0$. If $F_r = F_g$ where g is a Riemannian metric, then $F = F_g + \beta$ is called a Randers metric (see [51] for the original definition of a Randers metric). Randers metrics are examples of Finsler norms which are not induced by any Riemannian metric (since Riemannian metrics are always reversible).

The length of a piecewise smooth curve $\gamma\colon [a,b] \to M$ is defined to be

$$L_F(\gamma) = \int_a^b F(\gamma(t),\dot\gamma(t))\mathrm{d}t.$$

In general, $L_F(\gamma)$ is invariant only in orientation preserving reparametrizations. If in addition F is reversible, then $L_F(\gamma)$ is also invariant in orientation reversing reparametrizations. When F is induced by a Riemannian metric g, then we simply write $L_g := L_{F_g}$. If F is a Finsler norm such that $F = F_1 + \beta$ where F_1 is a Finsler norm and β is a 1-form, then for any piecewise smooth curve γ we have

$$L_F(\gamma) = L_{F_1}(\gamma) + \int_\gamma \beta.$$

Note that for the term coming from the 1-form β we have

$$\int_{\widetilde\gamma} \beta = \pm \int_\gamma \beta$$

where the plus sign corresponds to reparametrizations $\widetilde\gamma$ of γ preserving the orientation and the minus sign corresponds to reparametrizations reversing the orientation.

A smooth curve γ on (M, F) is a geodesic, if it satisfies the geodesic equation

$$\ddot{\gamma}^i(t) + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0$$

where the spray coefficients $G^i = G^i(x, y)$ are given by

$$G^i(x, y) = \frac{1}{4} g^{il}(x, y) \left(y^k \frac{\partial^2 F^2(x, y)}{\partial x^k \partial y^l} - \frac{\partial F^2(x, y)}{\partial x^l} \right).$$

Here $g^{ij}(x, y)$ are the components of the inverse matrix of $g_{ij}(x, y)$. Geodesics correspond to straightest possible paths in (M, F) and they are locally minimizing. Geodesics are also critical points of the length functional $L_F(\gamma)$.

We say that two Finsler norms F_1 and F_2 on a smooth manifold M are projectively equivalent, if F_1 and F_2 have the same geodesics as point sets. More precisely, F_1 and F_2 are projectively equivalent, if for any geodesic γ of F_1 there is an orientation preserving reparametrization η of γ such that η is a geodesic of F_2, and vice versa. We also say that a Finsler norm F has reversible geodesics (or is projectively reversible), if for any geodesic γ of F the reversed curve $\overleftarrow{\gamma}$ is also a geodesic of F up to orientation preserving reparametrization. In other words, F has reversible geodesics if F and $\overleftarrow{F}$ are projectively equivalent.

In general, if γ is a geodesic of F, then the reversed curve $\overleftarrow{\gamma}$ is not necessarily a geodesic of F. If F is reversible, then $\overleftarrow{\gamma}$ is also a geodesic. The following lemma says that the same holds (modulo orientation preserving reparametrization) if we perturb a reversible Finsler norm with a closed 1-form (see also [42, Theorem 7.1] for a more general version of the lemma).

Lemma 2.1 ([24, p. 406])**.** *Let $F = F_r + \beta$ be a Finsler norm where F_r is a reversible Finsler norm and β is a 1-form such that $\|\beta\|_{F_r^*} < 1$. Then F has reversible geodesics (is projectively equivalent to $\overleftarrow{F}$) if and only if β is closed ($\mathrm{d}\beta = 0$).*

The next lemma has a central role in the proofs of our main theorems. It says that if we perturb a Finsler norm with a closed 1-form, then the geodesics change only by an orientation preserving reparametrization (see also [18, Theorem 3.3] and [1, Example 2.11]).

Lemma 2.2 ([21, Theorem 3.3.1 and Example 3.3.2])**.** *Let F_1 be a Finsler norm on a smooth manifold M. Let $F_2 = F_1 + \beta$ be another Finsler norm where β is a 1-form such that $\|\beta\|_{F_1^*} < 1$. Then (M, F_1) and (M, F_2) are projectively equivalent if and only if β is closed ($\mathrm{d}\beta$=0).*

3. Proofs of the main theorems

In this section we prove our main results. The proofs are based on lemmas 2.1 and 2.2 which imply that F_i and $F_{r,i}$ have the same geodesics up to orientation preserving reparametrizations and that F_i has reversible geodesics. This allows us to express the integrals of the 1-forms β_i in terms of the boundary distance data of F_i. Similarly we can express the boundary distance data of g_i in terms of the boundary distance data of F_i, which implies that g_1 and g_2 differ only by a boundary preserving diffeomorphism since the underlying manifolds (M, g_i) are assumed to be boundary rigid.

We are now ready to prove our main theorems. Recall that a Finsler norm F is admissible if every two boundary points can be joined by unique geodesic of F with finite length.

Proof of theorem 1.3. Let us first prove the direction (i)⇒(ii). We note that if γ is any curve on M and $\overleftarrow{\gamma}$ any of its reversed reparametrizations, then reversibility of $F_{r,i}$ implies that $L_{F_{r,i}}(\gamma) = L_{F_{r,i}}(\overleftarrow{\gamma})$ and

$$\int_\gamma \beta_i = \frac{L_{F_i}(\gamma) - L_{F_i}(\overleftarrow{\gamma})}{2}.$$

Let $x, x' \in \partial M$ and γ_i be the unique geodesic of $F_{r,i}$ connecting x to x' (see figure 1). By lemma 2.2 there is an orientation preserving reparametrization η_i of γ_i such that η_i is a geodesic of F_i. Note that since η_i connects x to x' we have $L_{F_i}(\eta_i) = d_{F_i}(x, x')$ by admissibility of F_i. Using lemma 2.1 let $\overleftarrow{\eta_i}$ be the reversed curve which is a geodesic of F_i. Now again by admissibility of F_i we have $L_{F_i}(\overleftarrow{\eta_i}) = d_{F_i}(x', x)$ since $\overleftarrow{\eta_i}$ connects x' to x. We obtain

$$\begin{aligned}\int_{\gamma_1} \beta_1 = \int_{\eta_1} \beta_1 = \frac{L_{F_1}(\eta_1) - L_{F_1}(\overleftarrow{\eta_1})}{2} = \frac{d_{F_1}(x, x') - d_{F_1}(x', x)}{2} \\ = \frac{d_{F_2}(x, x') - d_{F_2}(x', x)}{2} = \frac{L_{F_2}(\eta_2) - L_{F_2}(\overleftarrow{\eta_2})}{2} = \int_{\eta_2} \beta_2 = \int_{\gamma_2} \beta_2.\end{aligned}$$

The closed 1-form β_i is exact because M is simply connected, i.e. $\beta_i = \mathrm{d}\phi_i$ for some scalar field ϕ_i. Since γ_1 and γ_2 both connect x to x', we obtain that

$$\phi_1(x') - \phi_1(x) = \int_{\gamma_1} \beta_1 = \int_{\gamma_2} \beta_2 = \phi_2(x') - \phi_2(x).$$

It follows that $\phi_2 - \phi_1$ is constant on the boundary. Let this constant be $c \in \mathbb{R}$ and define the scalar field $\phi = \phi_2 - \phi_1 - c$. Then ϕ satisfies $\mathrm{d}\phi = \beta_2 - \beta_1$ and $\phi|_{\partial M} = 0$. If there is another scalar field ϕ' such that $\mathrm{d}\phi' = \beta_2 - \beta_1$ and $\phi'|_{\partial M} = 0$, then $\mathrm{d}(\phi - \phi') = 0$ and $\phi - \phi' = \text{constant} = 0$ since M is connected and both scalar fields vanish on the boundary. This proves the first claim of the first implication.

For the second claim we note that for any curve γ and any of its reversed reparametrization $\overleftarrow{\gamma}$ it holds that

$$L_{F_{r,i}}(\gamma) = \frac{L_{F_i}(\gamma) + L_{F_i}(\overleftarrow{\gamma})}{2}.$$

Now let γ_i and η_i be as in the beginning of the proof. It follows that

$$\begin{aligned}d_{F_{r,1}}(x, x') = L_{F_{r,1}}(\gamma_1) = L_{F_{r,1}}(\eta_1) = \frac{L_{F_1}(\eta_1) + L_{F_1}(\overleftarrow{\eta_1})}{2} \\ = \frac{d_{F_1}(x, x') + d_{F_1}(x', x)}{2} = \frac{d_{F_2}(x, x') + d_{F_2}(x', x)}{2} \\ = \frac{L_{F_2}(\eta_2) + L_{F_2}(\overleftarrow{\eta_2})}{2} = L_{F_{r,2}}(\eta_2) = L_{F_{r,2}}(\gamma_2) = d_{F_{r,2}}(x, x')\end{aligned}$$

proving the second claim of the first implication.

Let us then prove the implication (ii)⇒(i). First we note that for any curve γ it holds that

$$L_{F_i}(\gamma) = L_{F_{r,i}}(\gamma) + \int_\gamma \beta_i.$$

Let $x, x' \in \partial M$ and η_i be the unique geodesic of F_i connecting x to x'. By lemma 2.2 there is an orientation preserving reparametrization σ_i of η_i such that σ_i is a geodesic of $F_{r,i}$. Simply connectedness of M and the assumptions on β_i imply that $\int_{\eta_2} \beta_2 = \int_{\eta_1} \beta_1$. Using the assumption $d_{F_{r,1}}(x, x') = d_{F_{r,2}}(x, x')$ and the admissibility of $F_{r,i}$ it follows that

$$\begin{aligned} d_{F_2}(x, x') = L_{F_2}(\eta_2) = L_{F_{r,2}}(\eta_2) + \int_{\eta_2} \beta_2 = L_{F_{r,2}}(\sigma_2) + \int_{\eta_1} \beta_1 \\ = d_{F_{r,2}}(x, x') + \int_{\eta_1} \beta_1 = d_{F_{r,1}}(x, x') + \int_{\eta_1} \beta_1 \\ = L_{F_{r,1}}(\sigma_1) + \int_{\eta_1} \beta_1 = L_{F_{r,1}}(\eta_1) + \int_{\eta_1} \beta_1 \\ = L_{F_1}(\eta_1) = d_{F_1}(x, x'). \end{aligned}$$

This concludes the proof. □

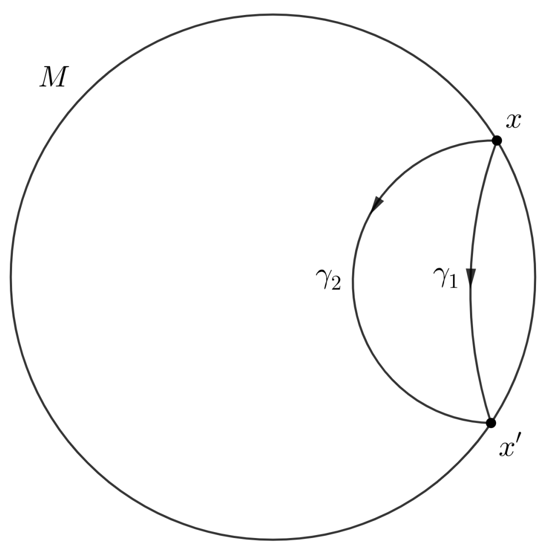

FIGURE 1. A picture illustrating the proof of theorem 1.3. Here γ_i is a geodesic of $F_{r,i}$ connecting the boundary point $x \in \partial M$ to the boundary point $x' \in \partial M$. The curves γ_i are also geodesics of F_i up to orientation preserving reparametrization by lemma 2.2. The picture is highly simplified; in reality the curves γ_1 and γ_2 can for example cross each other.

Proof of theorem 1.5. If $d_{F_1}(x,x') = d_{F_2}(x,x')$ for all $x,x' \in \partial M$, then by theorem 1.3 there is unique scalar field ϕ vanishing on the boundary such that $\beta_2 = \beta_1 + \mathrm{d}\phi$, and $d_{g_1}(x,x') = d_{g_2}(x,x')$ for all $x,x' \in \partial M$. Since we assume that (M,g_i) are boundary rigid, there is a diffeomorphism $\Psi\colon M \to M$ which is identity on the boundary such that $g_2 = \Psi^* g_1$. This proves the implication (a)⇒(b). The implication (b)⇒(a) is proved in the same way as the implication (ii)⇒(i) in theorem 1.3 using the fact that Ψ is a Riemannian isometry fixing boundary points.

Let us then prove the equivalence (b)⇔(c). Let $\Psi\colon M \to M$ be a diffeomorphism which is identity on the boundary. Since β_1 is closed and the pullback commutes with the differential, we have that $\Psi^*\beta_1$ is also closed. This implies that $\beta_1 - \Psi^*\beta_1$ is closed and hence exact because M is simply connected, i.e. there is a scalar field $\widetilde{\phi}$ such that $\beta_1 - \Psi^*\beta_1 = \mathrm{d}\widetilde{\phi}$. Let $x,x' \in \partial M$ be any two boundary points and γ any curve connecting x to x'. Since Ψ is identity on the boundary and β_1 is exact we have

$$0 = \int_\gamma (\beta_1 - \Psi^*\beta_1) = \int_\gamma \mathrm{d}\widetilde{\phi} = \widetilde{\phi}(x') - \widetilde{\phi}(x).$$

Therefore $\widetilde{\phi}$ is constant on the boundary and we can subtract this constant to obtain a scalar field ϕ' such that $\beta_1 - \Psi^*\beta_1 = \mathrm{d}\phi'$ and $\phi'|_{\partial M} = 0$. Thus β_1 and $\Psi^*\beta_1$ differ only by a potential which vanishes on the boundary, concluding the proof. ☐

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Partial data problems and unique continuation in scalar and vector field tomography

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PARTIAL DATA PROBLEMS AND UNIQUE CONTINUATION IN SCALAR AND VECTOR FIELD TOMOGRAPHY

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ABSTRACT. We prove that if $P(D)$ is some constant coefficient partial differential operator and f is a scalar field such that $P(D)f$ vanishes in a given open set, then the integrals of f over all lines intersecting that open set determine the scalar field uniquely everywhere. This is done by proving a unique continuation property of fractional Laplacians which implies uniqueness for the partial data problem. We also apply our results to partial data problems of vector fields.

1. Introduction

Let f be a scalar field and $V \subset \mathbb{R}^n$ a nonempty open set where $n \geq 2$. We study the following partial data problem in X-ray tomography: can we say something about f if we know the integrals of f over all lines intersecting V? Especially, we are interested in the uniqueness problem which can be formulated in terms of the X-ray transform X_0 as follows: if $X_0 f = 0$ on all lines which intersect V, does it follow that $f = 0$? In general, the answer is "no" [29] and one has to put some conditions on $f|_V$. We prove that if there is some constant coefficient partial differential operator $P(D)$ such that $P(D)f|_V = 0$ and $X_0 f = 0$ on all lines intersecting V, then $f = 0$. This generalizes a recent partial data result in [20]. As a special case we obtain that if f is for example polynomial or (poly)harmonic in V, then f is uniquely determined by its partial X-ray data.

The partial data result is proved by using a unique continuation property of fractional Laplacian $(-\Delta)^s$. We prove that if $s \in (-n/2, \infty) \setminus \mathbb{Z}$ and there is some constant coefficient partial differential operator $P(D)$ such that $P(D)f|_V = (-\Delta)^s f|_V = 0$, then $f = 0$. This generalizes earlier results about unique continuation of fractional Laplacians [6, 14]. The unique continuation of $(-\Delta)^s$ implies a unique continuation result for the normal operator N_0 of the X-ray transform X_0, and the uniqueness for the partial data problem follows directly from the unique continuation of N_0. This approach which uses the unique continuation of the normal operator in proving uniqueness for partial data problems was also used in [6, 20, 21].

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We also study partial data problems of vector fields. Let F be a vector field and denote by $\mathrm{d}F$ its exterior derivative or curl which components are $(\mathrm{d}F)_{ij} = \partial_i F_j - \partial_j F_i$. We prove that if there are some constant coefficient partial differential operators $P_{ij}(D)$ such that $P_{ij}(D)(\mathrm{d}F)_{ij}|_V = 0$ and the integrals of F over all lines intersecting V vanish, then F must be a potential field (F is the gradient of some scalar field). This is a generalization of a recent result in [21]. The partial data result is proved by using a relation between the normal operator of the X-ray transform of scalar fields and the normal operator of the X-ray transform of vector fields (see lemma 4.4). This allows one to reduce the partial data problem for the vector field F to partial data problems for the scalar fields $(\mathrm{d}F)_{ij}$. As a special case we obtain that if F is for example componentwise polynomial or (poly)harmonic in V, then the solenoidal part of F is uniquely determined by the partial X-ray data of F.

The partial data problems we study have a relation to the region of interest (ROI) tomography [4, 23, 24, 29, 46]. The main goal in such imaging problems is to determine the attenuation inside a small part of a human body (region of interest) by using only the X-ray data on lines which go through the ROI. This for example reduces the needed X-ray dose which is given to the patient. Our results imply that if the attenuation f satisfies $P(D)f|_V = 0$ for some open subset V of the ROI and some constant coefficient partial differential operator $P(D)$, then f is uniquely determined by its partial X-ray data on lines which intersect the ROI. Note that f is uniquely determined not only in the ROI but also outside the ROI. This holds for example if the attenuation is polynomial or (poly)harmonic in a small subregion of the ROI. In general, f does not have to be smooth and it can have singularities in the ROI. We also note that our proof for uniqueness does not give stability for the partial data problem. Especially, outside the ROI we have invisible singularities which cannot be seen by the X-ray data and the reconstruction of such singularities is not stable (see remark 1.5 and [25, 34, 35]).

Similar ROI tomography problems can be studied in the case of vector fields. In vector field tomography the usual objective is to determine the velocity field of a fluid flow using acoustic travel time or Doppler backscattering measurements [30, 31, 39]. Assuming that the velocity of the fluid flow is much smaller than the speed of the propagating signal one can linearize the problem. Linearization then leads to the X-ray transform of the velocity field. Our results imply that if the velocity field F satisfies $P_{ij}(D)(\mathrm{d}F)_{ij}|_V = 0$ for some open subset V of the ROI and some constant coefficient partial differential operators $P_{ij}(D)$, then the solenoidal part of F is uniquely determined everywhere by the partial X-ray data of F on lines intersecting the ROI. Examples of such velocity fields are those which are componentwise polynomial or (poly)harmonic in a small subregion of the ROI. As in the scalar case, F can have singularities in the ROI, and our proof does not give stability for the partial data problem (since it is based on reduction to the scalar case).

The article is organized as follows. In section 1.1 we introduce our notation, in section 1.2 we give our main theorems and in section 1.3 we discuss some related results. We go through the theory of distributions and the X-ray transform in section 2, and study the space of admissible functions in section 3. Finally, we prove our main results in section 4.

1.1. **Notation.** We quickly go through the notation used in our main theorems. More detailed information about distributions and the X-ray transform of scalar and vector fields can be found in section 2.

We denote by f a scalar field. The set $\mathscr{O}_C'(\mathbb{R}^n)$ is the space of rapidly decreasing distributions and the space $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$ consists of compactly supported distributions. The subset $C_\infty(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$ is the set of all continuous functions which decay faster than any polynomial at infinity. We let X_0 be the X-ray transform of scalar fields and it maps a function to its line integrals. The normal operator is $N_0 = X_0^* X_0$ where X_0^* is the adjoint of X_0.

We let $H^r(\mathbb{R}^n)$ be the fractional L^2-Sobolev space of order $r \in \mathbb{R}$ and $H^{-\infty}(\mathbb{R}^n) = \bigcup_{r\in\mathbb{R}} H^r(\mathbb{R}^n)$. We define the fractional Laplacian as $(-\Delta)^s f = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{f})$ where $\hat{f} = \mathcal{F}(f)$ is the Fourier transform of f and $\mathcal{F}^{-1}$ is the inverse Fourier transform. The fractional Laplacian $(-\Delta)^s$ is well-defined in $\mathscr{O}_C'(\mathbb{R}^n)$ for all $s \in (-n/2, \infty) \setminus \mathbb{Z}$ and in $H^r(\mathbb{R}^n)$ for all $s \in (-n/4, \infty) \setminus \mathbb{Z}$.

We denote by $\mathcal{P}$ the set of all polynomials in $\mathbb{R}^n$ with complex coefficients with the convention that the zero polynomial $P \equiv 0$ does not belong to $\mathcal{P}$. A polynomial $P \in \mathcal{P}$ of degree $m \in \mathbb{N}$ induces a constant coefficient partial differential operator $P(D)$ of order $m \in \mathbb{N}$ by setting $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ where $a_\alpha \in \mathbb{C}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial_j$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index such that $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The set of admissible functions $\mathcal{A}_V$ is defined as

$$\mathcal{A}_V = \{f \in H^{-\infty}(\mathbb{R}^n) : P(D)f|_V = 0 \text{ for some } P \in \mathcal{P}\} \tag{1}$$

where $V \subset \mathbb{R}^n$ is some nonempty open set.

We denote by F a vector field. The notation $F \in (\mathcal{E}'(\mathbb{R}^n))^n$ means that $F = (F_1, \ldots, F_n)$ where $F_i \in \mathcal{E}'(\mathbb{R}^n)$ for all $i = 1, \ldots, n$. The exterior derivative of F is written in components as $(\mathrm{d}F)_{ij} = \partial_i F_j - \partial_j F_i$. For scalar fields ϕ the notation $\mathrm{d}\phi$ denotes the gradient of ϕ. We let X_1 be the X-ray transform of vector fields which maps a vector field to its line integrals. The normal operator is $N_1 = X_1^* X_1$ where X_1^* is the adjoint of X_1.

1.2. **Main results.** In this section we give our main theorems. The proofs of the results can be found in section 4.

Our main theorem is the following unique continuation result for the fractional Laplacian.

Theorem 1.1. *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $f \in \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $(-\Delta)^s f|_V = 0$, then $f = 0$. If $f \in \mathscr{O}_C'(\mathbb{R}^n) \cap \mathcal{A}_V$, then the claim holds for $s \in (-n/2, \infty) \setminus \mathbb{Z}$.*

Theorem 1.1 generalizes the result in [6] (see lemma 4.1) where one assumes that $(-\Delta)^s f|_V = f|_V = 0$. In fact, theorem 1.1 is proved by reducing the claim to the case treated in [6, Theorem 1.1] (see section 4). The meaning of the condition $f \in \mathcal{A}_V$ is discussed in section 3 (see remark 3.3). When $s \in (-n/2, -n/4] \setminus \mathbb{Z}$, we need to have $f \in \mathscr{O}_C'(\mathbb{R}^n)$ so that $(-\Delta)^s f$ is well-defined and we can use lemma 4.1 in the proof of theorem 1.1.

Remark 1.2. *If $f \in \mathcal{E}'(\mathbb{R}^n)$, then instead of assuming $(-\Delta)^s f|_V = 0$ in theorem 1.1 we could only require that $(-\Delta)^s f$ vanishes to infinite order at some point $x_0 \in V$, i.e. $\partial^\beta((-\Delta)^s f)(x_0) = 0$ for all $\beta \in \mathbb{N}^n$. This follows since a corresponding unique continuation result is known for $f \in \mathcal{E}'(\mathbb{R}^n)$ under the assumptions $f|_V = 0$ and $\partial^\beta((-\Delta)^s f)(x_0) = 0$ for all $\beta \in \mathbb{N}^n$ (see corollary 4 on page 12 in [6]), and constant coefficient partial differential operators $P(D)$ commute with fractional Laplacians and ordinary derivatives. Therefore we can use the same proof to prove this slightly stronger result (see the proof of theorem 1.1).*

From the unique continuation of fractional Laplacians we immediately obtain the following unique continuation result for the normal operator of the X-ray transform of scalar fields. The reason is that the normal operator can be written as $N_0 = (-\Delta)^{-1/2}$ up to a constant factor (see section 2.2).

Theorem 1.3. *Let $n \geq 2$ and $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $N_0 f|_V = 0$, then $f = 0$.*

Theorem 1.3 is a generalization of the result in [20] where one assumes $N_0 f|_V = f|_V = 0$. When $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$, we could replace the assumption $N_0 f|_V = 0$ with the requirement that $N_0 f$ vanishes to infinite order at some point $x_0 \in V$ (see remark 1.2). In order to use theorem 1.1 in the case $s = -1/2$ and $n \geq 2$, and to guarantee that $N_0 f$ is well-defined, we need to have $f \in \mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n)$ or $f \in C_\infty(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n)$ in theorem 1.3.

The unique continuation of N_0 implies uniqueness for the following partial data problem.

Theorem 1.4. *Let $n \geq 2$ and $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $X_0 f = 0$ on all lines intersecting V, then $f = 0$.*

Theorem 1.4 generalizes theorem 1.2 in [20], where one assumes $f|_V = 0$, to the case $P(D)f|_V = 0$ for some $P \in \mathcal{P}$. We note that if f is polynomial in V, then there is $P \in \mathcal{P}$ such that $P(D)f|_V = 0$. Hence those scalar fields which are polynomial in V can be uniquely determined from their X-ray data on lines intersecting V. This special case of theorem 1.4 is previously known in two dimensions [23, 46]. We also note that theorem 1.4 includes much larger class of functions than just polynomials. The scalar field f can be (poly)harmonic in V and f can also have singularities in V if f is for example a non-smooth solution to the wave equation (see section 3 for more examples of admissible functions).

It is important to notice that from the vector space structure of admissible functions $\mathcal{A}_V$ it follows that theorem 1.4 is indeed a uniqueness result: if f_1 and f_2 satisfy $P_1(D)f_1|_V = P_2(D)f_2|_V = 0$ for some $P_1, P_2 \in \mathcal{P}$ and $X_0 f_1 = X_0 f_2$ on all lines intersecting V, then $f_1 = f_2$ (see proposition 3.4 and remark 3.5 for more details). Especially, the equality of the X-ray data on all lines intersecting V implies that the scalar fields are equal everywhere even though f_1 and f_2 a priori can have very different behaviour in V since $P_1(D)$ can be different from $P_2(D)$.

Remark 1.5. *We note that our proof for theorem 1.4 gives only uniqueness but not stability for the partial data problem. In theorem 1.4 we eventually* *have to assume that f is not supported in V since otherwise we would have $P(D)f = 0$ everywhere and therefore $f = 0$ without assuming anything about the X-ray data (see the proof of theorem 1.1). When f is supported outside V we do not have access to all singularities of f via the X-ray data, i.e. we have invisible singularities outside V. It is known that the recovery of such invisible singularities is not stable* [25, 34, 35].

Remark 1.6. *We note that similar results as in theorems 1.3 and 1.4 also hold for the d-plane transform when d is odd (see corollaries 1 and 2 on page 6 in* [6]*). The d-plane transform takes a scalar field and integrates it over d-dimensional affine planes where $0 < d < n$. The case $d = 1$ corresponds to the X-ray transform. The normal operator N_d of the d-plane transform can be expressed as $N_d = (-\Delta)^{-d/2}$ up to a constant factor (see* [6, 16]*). Hence N_d admits the same unique continuation property as in theorem 1.1 for functions in $\mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ provided d is odd. The unique continuation of N_d then implies a similar uniqueness result as in theorem 1.4 for a partial data problem of the d-plane transform when d is odd.*

From the unique continuation of fractional Laplacians we also obtain a partial data result for the X-ray transform of vector fields. The normal operators satisfy the relationship $\mathrm{d}(N_1 F) = N_0(\mathrm{d}F)$ up to a constant factor (see lemma 4.4). Hence the unique continuation and partial data problems of vector fields can be reduced to the corresponding problems for scalar fields, namely the components $(\mathrm{d}F)_{ij}$.

The next theorem generalizes the result in [21] where the authors assume that $\mathrm{d}F|_V = 0$ instead of $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$.

Theorem 1.7. *Let $n \geq 2$ and $F \in (\mathcal{E}'(\mathbb{R}^n))^n$ such that $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$ for all $i, j = 1, \ldots, n$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $X_1 F = 0$ on all lines intersecting V, then $\mathrm{d}F = 0$. Especially, $F = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

The conclusion $F = \mathrm{d}\phi$ in theorem 1.7 is equivalent to that the solenoidal part F^{s} vanishes in the solenoidal decomposition $F = F^{\mathrm{s}} + \mathrm{d}\phi$ (see e.g. [41]). Therefore theorem 1.7 can be seen as a solenoidal injectivity result in terms of partial data (see [21] and [33, 41]). Theorem 1.7 holds also for vector fields $F \in (\mathscr{S}(\mathbb{R}^n))^n$ which components are Schwartz functions since in that case $(\mathrm{d}F)_{ij} \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$.

We note that if the components F_i are all polynomial in V, then also $(\mathrm{d}F)_{ij}$ are all polynomial in V. Hence there are some $P_{ij} \in \mathcal{P}$ such that $P_{ij}(D)(\mathrm{d}F)_{ij}|_V = 0$ and therefore $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$. This means that solenoidal vector fields which are polynomial in V can be uniquely determined from their X-ray data on lines intersecting V. However, this is only a small subset of admissible vector fields: F can be for example componentwise (poly)harmonic in V and more generally F can also have singularities in V.

Remark 1.8. *Theorems 1.1 and 1.3 imply a unique continuation result for N_1: if $F \in (\mathcal{E}'(\mathbb{R}^n))^n$ satisfies $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$ for all $i, j = 1, \ldots, n$ and $N_1 F|_V = 0$, then $\mathrm{d}F = 0$. This follows since $\mathrm{d}(N_1 F) = N_0(\mathrm{d}F)$ up to a constant factor (see lemma 4.4) and one can use theorem 1.3 for the components $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n)$. One also obtains a stronger version where one can replace the assumption $N_1 F|_V = 0$ with the requirement that $\mathrm{d}(N_1 F)$ vanishes componentwise to infinite order at some point $x_0 \in V$ (see remark 1.2).*

1.3. **Related results.** There are some earlier unique continuation and partial data results for scalar and vector fields. The partial data problem for scalar fields has a unique solution if $f|_V$ vanishes [4, 20, 24], $f|_V$ is polynomial or piecewise polynomial [23, 24, 46] or $f|_V$ is real analytic [23]. A complementary result is the Helgason support theorem: if the integrals of f vanish on all lines not intersecting a given compact and convex set, then f has to vanish outside that set [16, 42]. The normal operator of the X-ray transform of scalar fields has a unique continuation property under the assumptions $N_0 f|_V = f|_V = 0$ [20]. This is a special case of a more general unique continuation property of fractional Laplacians [6, 14]. There are also partial data and unique continuation results for the d-plane transform of scalar fields when d is odd, including the X-ray transform as a special case $d = 1$ (see [6] and remark 1.6).

The partial data problem of vector fields is known to be uniquely solvable up to potential fields, if $\mathrm{d}F|_V = 0$ [21]. Similarly, the normal operator of the X-ray transform of vector fields has a unique continuation property under the assumptions $N_1 F|_V = \mathrm{d}F|_V = 0$ [21]. There are other partial data results for vector fields where one knows the integrals of F over lines which are parallel to a finite set of planes [22, 38, 40] or which intersect a certain type of curve [9, 36, 44]. There is also a Helgason-type support theorem for vector fields: if the integrals of F vanish on all lines not intersecting a given compact and convex set, then $\mathrm{d}F$ vanishes outside that set [21, 42].

The normal operator of scalar fields, the normal operator of vector fields and the fractional Laplacian all admit stronger versions of the unique continuation property (see [6, 11, 12, 13, 20, 21, 37, 47] and remarks 1.2 and 1.8). Other applications of unique continuation of fractional Laplacians include fractional inverse problems. Especially, the unique continuation of $(-\Delta)^s$ is used to prove uniqueness for different versions of the fractional Calderón problem (see e.g. [1, 2, 5, 6, 7, 14]).

2. The X-ray transform and distributions

In this section we define the X-ray transform of scalar and vector fields, and introduce the distribution spaces we use in our main theorems. The basic theory of distributions and Sobolev spaces can be found in [15, 17, 27, 28, 43] and the X-ray transform is treated for example in [29, 41, 42].

2.1. **Distributions and Sobolev spaces.** We let $\mathcal{E}(\mathbb{R}^n)$ be the space of smooth functions, $\mathscr{S}(\mathbb{R}^n)$ is the Schwartz space and $\mathcal{D}(\mathbb{R}^n)$ is the space of compactly supported smooth functions. We equip all these spaces with their standard topologies. The corresponding duals are denoted by $\mathcal{E}'(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$. Elements in $\mathcal{E}'(\mathbb{R}^n)$ are identified with distributions of compact support and elements in $\mathscr{S}'(\mathbb{R}^n)$ are called tempered distributions.

We define the space of rapidly decreasing distributions $\mathscr{O}'_C(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ as follows: $f \in \mathscr{O}'_C(\mathbb{R}^n)$ if and only if $\hat{f} \in \mathscr{O}_M(\mathbb{R}^n)$ where $\hat{f} = \mathcal{F}(f)$ is the Fourier transform of tempered distributions. Here $\mathscr{O}_M(\mathbb{R}^n)$ is the space of polynomially growing smooth functions, i.e. $f \in \mathscr{O}_M(\mathbb{R}^n)$ if f and all its derivatives are polynomially bounded. We note that the Fourier transform is an isomorphism $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ and also extends to an isomorphism $\mathcal{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. We have the inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n) \subset$

$\mathscr{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. As a special case we have $\mathscr{S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n)$ where $C_\infty(\mathbb{R}^n)$ is the set of all continuous functions which decay faster than any polynomial at infinity.

The fractional L^2-Sobolev space of order $r \in \mathbb{R}$ is defined as

$$H^r(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) : \langle\cdot\rangle^r \hat{f} \in L^2(\mathbb{R}^n)\} \tag{2}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. The space $H^r(\mathbb{R}^n)$ is equipped with the norm

$$\|f\|_{H^r(\mathbb{R}^n)} = \left\| \langle\cdot\rangle^r \hat{f} \right\|_{L^2(\mathbb{R}^n)} \tag{3}$$

and $H^r(\mathbb{R}^n)$ becomes a separable Hilbert space for every $r \in \mathbb{R}$. It follows that the spaces are nested, i.e. $H^r(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ continuously when $r \geq t$. One can isomorphically identify $H^{-r}(\mathbb{R}^n)$ with the dual $(H^r(\mathbb{R}^n))^*$ for all $r \in \mathbb{R}$. We define the following spaces

$$H^\infty(\mathbb{R}^n) = \bigcap_{r\in\mathbb{R}} H^r(\mathbb{R}^n), \quad H^{-\infty}(\mathbb{R}^n) = \bigcup_{r\in\mathbb{R}} H^r(\mathbb{R}^n). \tag{4}$$

It holds that $\mathscr{O}'_C(\mathbb{R}^n) \subset H^{-\infty}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n) \subset H^\infty(\mathbb{R}^n)$. Further, using the Sobolev embedding one can see that $H^\infty(\mathbb{R}^n) = C^\infty_{L^2}(\mathbb{R}^n)$ where $f \in C^\infty_{L^2}(\mathbb{R}^n)$ if f is smooth and f and all its derivatives belong to $L^2(\mathbb{R}^n)$ (see [15, Theorem 6.12]).

The fractional Laplacian is defined as

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{f}) \tag{5}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform of tempered distributions. It follows that $(-\Delta)^s f$ is well-defined as a tempered distribution for $f \in \mathscr{O}'_C(\mathbb{R}^n)$ when $s \in (-n/2, \infty) \setminus \mathbb{Z}$, and for $f \in H^r(\mathbb{R}^n)$ when $s \in (-n/4, \infty) \setminus \mathbb{Z}$ (see [6, Section 2.2]). We have that $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is continuous whenever $s \in (0, \infty) \setminus \mathbb{Z}$ and $(-\Delta)^s$ also admits a Poincaré-type inequality for $s \in (0, \infty) \setminus \mathbb{Z}$ (see [6]). We note that $(-\Delta)^s$ is a non-local operator in contrast to the ordinary Laplacian $(-\Delta)$. The non-locality implies a unique continuation property (see theorem 1.1 and lemma 4.1) which cannot hold for local operators.

We also use local versions of distributions and fractional Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be an open set and $r \in \mathbb{R}$. We denote by $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ etc. the test function and distribution spaces defined in Ω. We define the local Sobolev space $H^r(\Omega)$ as

$$H^r(\Omega) = \{g \in \mathcal{D}'(\Omega) : g = f|_\Omega \text{ for some } f \in H^r(\mathbb{R}^n)\}. \tag{6}$$

In other words, the space $H^r(\Omega)$ consists of restrictions of distributions $f \in H^r(\mathbb{R}^n)$. The local Sobolev space is equipped with the quotient norm

$$\|g\|_{H^r(\Omega)} = \inf\{\|f\|_{H^r(\mathbb{R}^n)} : f \in H^r(\mathbb{R}^n) \text{ such that } f|_\Omega = g\}. \tag{7}$$

Then $H^r(\Omega)$ becomes a separable Hilbert space and the restriction map $|_\Omega \colon H^r(\mathbb{R}^n) \to H^r(\Omega)$ is continuous. If $r \geq t$, then $H^r(\Omega) \hookrightarrow H^t(\Omega)$ continuously. One can also isomorphically identify $H^{-r}(\Omega)$ as the dual $(\widetilde{H}^r(\Omega))^*$ for every $r \in \mathbb{R}$ where $\widetilde{H}^r(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^r(\mathbb{R}^n)$ (see [3] and [27]). If $r \geq 0$, then $H^r(\Omega) \subset W^r(\Omega)$ where $W^r(\Omega)$ is the Sobolev-Slobodeckij space which is defined by using weak derivatives of L^2-functions (see [27] for a precise definition). If Ω is a Lipschitz domain, then we have the equality $H^r(\Omega) = W^r(\Omega)$ for all $r \geq 0$.

More generally, we define the vector-valued test function space $(\mathcal{D}(\mathbb{R}^n))^n$ by saying that $\varphi \in (\mathcal{D}(\mathbb{R}^n))^n$ if and only if $\varphi = (\varphi_1, \dots, \varphi_n)$ and $\varphi_i \in \mathcal{D}(\mathbb{R}^n)$ for all $i = 1, \dots, n$. A sequence converges to zero in $(\mathcal{D}(\mathbb{R}^n))^n$ if and only if all its components converge to zero in $\mathcal{D}(\mathbb{R}^n)$. We then define the space of vector-valued distributions $(\mathcal{D}'(\mathbb{R}^n))^n$ by saying that $F \in (\mathcal{D}'(\mathbb{R}^n))^n$ if and only if $F = (F_1, \dots, F_n)$ where $F_i \in \mathcal{D}'(\mathbb{R}^n)$ for all $i = 1, \dots, n$. The duality pairing is defined as $\langle F, \varphi\rangle = \sum_{i=1}^n \langle F_i, \varphi_i\rangle$. The test function spaces $(\mathcal{E}(\mathbb{R}^n))^n$ and $(\mathscr{S}(\mathbb{R}^n))^n$, and the corresponding distribution spaces $(\mathcal{E}'(\mathbb{R}^n))^n$ and $(\mathscr{S}'(\mathbb{R}^n))^n$ are defined analogously. The elements in $(\mathcal{E}'(\mathbb{R}^n))^n$ are called compactly supported vector-valued distributions. Vector-valued distributions are a special case of currents (continuous linear functionals in the space of differential forms, see [8, Section III]).

For $F \in (\mathcal{D}'(\mathbb{R}^n))^n$ we define the exterior derivative or curl of F as a matrix which components are $(\mathrm{d}F)_{ij} = \partial_i F_j - \partial_j F_i$. It follows from the Poincaré lemma (see e.g. [26, Theorem 2.1] and lemma 4.2) that if $\mathrm{d}F = 0$, then $F = \mathrm{d}\phi$ for some $\phi \in \mathcal{D}'(\mathbb{R}^n)$ where $\mathrm{d}\phi$ is the distributional gradient of ϕ.

2.2. **The X-ray transform of scalar fields.** Let $f \in \mathcal{D}(\mathbb{R}^n)$ be a scalar field. The X-ray transform X_0 is defined as

$$X_0 f(\gamma) = \int_\gamma f \mathrm{d}s \tag{8}$$

where γ is an oriented line in $\mathbb{R}^n$. When we parameterize the set of all oriented lines with the set

$$\Gamma = \{(z, \theta) : \theta \in S^{n-1},\ z \in \theta^\perp\} \tag{9}$$

the X-ray transform becomes

$$X_0 f(z, \theta) = \int_{\mathbb{R}} f(z + s\theta)\mathrm{d}s. \tag{10}$$

The adjoint or back-projection X_0^* is defined as

$$X_0^*\psi(x) = \int_{S^{n-1}} \psi(x - (x\cdot\theta)\theta, \theta)\mathrm{d}\theta \tag{11}$$

where $\psi \in \mathcal{E}(\Gamma)$. One then sees that $X_0 \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\Gamma)$ and $X_0^* \colon \mathcal{E}(\Gamma) \to \mathcal{E}(\mathbb{R}^n)$ are continuous maps. Using duality we can define $X_0 \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\Gamma)$ and $X_0^* \colon \mathcal{D}'(\Gamma) \to \mathcal{D}'(\mathbb{R}^n)$ as

$$\langle X_0 f, \varphi\rangle = \langle f, X_0^*\varphi\rangle \tag{12}$$

$$\langle X_0^*\psi, \eta\rangle = \langle \psi, X_0\eta\rangle \tag{13}$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing.

The normal operator is $N_0 = X_0^* X_0$ and it can be expressed as the convolution

$$N_0 f(x) = 2(f * |\cdot|^{1-n})(x). \tag{14}$$

Using duality the normal operator extends to a map $N_0 \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ and the convolution formula holds in the sense of distributions. The normal operator can be seen as the fractional Laplacian $(-\Delta)^{-1/2}$ up to a constant factor and we have the reconstruction formula

$$f = c_{0,n}(-\Delta)^{1/2} N_0 f \tag{15}$$

where $c_{0,n}$ is a constant which depends on dimension. Both X_0 and N_0 are also defined for functions $f \in C_\infty(\mathbb{R}^n)$.

2.3. **The X-ray transform of vector fields.** Let $F \in (\mathcal{D}(\mathbb{R}^n))^n$ be a vector field. The X-ray transform X_1 is defined as

$$X_1 F(\gamma) = \int_\gamma F \cdot \mathrm{d}\overline{s} \tag{16}$$

where γ is an oriented line. Using the parametrization Γ for oriented lines (see equation (9)) we have

$$X_1 F(z,\theta) = \int_{\mathbb{R}} F(z + s\theta) \cdot \theta \mathrm{d}s. \tag{17}$$

We define the adjoint X_1^* as the vector-valued operator

$$(X_1^* \psi)_i(x) = \int_{S^{n-1}} \theta_i \psi(x - (x \cdot \theta)\theta, \theta) \mathrm{d}\theta \tag{18}$$

where $\psi \in \mathcal{E}(\Gamma)$ is a scalar field in the space of oriented lines. One sees that $X_1 \colon (\mathcal{D}(\mathbb{R}^n))^n \to \mathcal{D}(\Gamma)$ and $X_1^* \colon \mathcal{E}(\Gamma) \to (\mathcal{E}(\mathbb{R}^n))^n$ are continuous and by duality we can define $X_1 \colon (\mathcal{E}'(\mathbb{R}^n))^n \to \mathcal{E}'(\Gamma)$ and $X_1^* \colon \mathcal{D}'(\Gamma) \to (\mathcal{D}'(\mathbb{R}^n))^n$ by setting

$$\langle X_1 F, \varphi \rangle = \langle F, X_1^* \varphi \rangle \tag{19}$$

$$\langle X_1^* \psi, \eta \rangle = \langle \psi, X_1 \eta \rangle . \tag{20}$$

We define the normal operator as $N_1 = X_1^* X_1$ and it can be expressed in terms of convolution

$$(N_1 F)_i = \sum_{j=1}^n \frac{2x_i x_j}{|x|^{n+1}} * F_j. \tag{21}$$

The normal operator extends to a map $N_1 \colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{D}'(\mathbb{R}^n))^n$ by duality and the convolution formula holds in the sense of distributions. One has the reconstruction formula for the solenoidal part F^{s} in the solenoidal decomposition $F = F^{\mathrm{s}} + \mathrm{d}\phi$ (see for example [41, 42])

$$F^{\mathrm{s}} = c_{1,n}(-\Delta)^{1/2} N_1 F \tag{22}$$

where $c_{1,n}$ is a constant depending on dimension and $(-\Delta)^{1/2}$ operates componentwise on $N_1 F$. Both X_1 and N_1 are also defined for vector fields $F \in (\mathscr{S}(\mathbb{R}^n))^n$.

3. Partial differential operators and admissible functions

In this section we introduce constant coefficient partial differential operators and also study the space of admissible functions $\mathcal{A}_V$ in more detail. A comprehensive treatment of constant coefficient partial differential operators can be found in Hörmander's book [18].

Let us denote by $\mathcal{P}$ the set of all polynomials in $\mathbb{R}^n$ with complex coefficients excluding the zero polynomial $P \equiv 0$. A polynomial $P \in \mathcal{P}$ of degree

$m \in \mathbb{N}$ can be identified with the constant coefficient partial differential operator $P(D)$ of order $m \in \mathbb{N}$ as

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{C}, \tag{23}$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial_j$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index such that $|\alpha| = \alpha_1 + \ldots + \alpha_n$. In fact, using the Fourier transform one sees that

$$\widehat{P(D)} = P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha \tag{24}$$

where $\xi \in \mathbb{R}^n$ and $\xi^\alpha = \xi^{\alpha_1} \cdots \xi^{\alpha_n}$. The polynomial $P(\xi)$ is also known as the full symbol of $P(D)$. If $g \in \mathcal{D}'(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set, then one can define the distributional derivative $P(D)g \in \mathcal{D}'(\Omega)$ by duality. Further, it holds that $P(D)\colon H^r(\Omega) \to H^{r-m}(\Omega)$ is continuous with respect to the quotient norm [28, Theorem 12.15] (see equation (7)).

The set of admissible functions $\mathcal{A}_V$ which we use in our main theorems can be written as the union

$$\mathcal{A}_V = \bigcup_{\substack{P \in \mathcal{P} \\ r \in \mathbb{R}}} \mathcal{H}^r_{P,V}(\mathbb{R}^n) = \bigcup_{\substack{P \in \mathcal{P} \\ r \in \mathbb{R}}} \{f \in H^r(\mathbb{R}^n) : P(D)f|_V = 0\} \tag{25}$$

where $V \subset \mathbb{R}^n$ is some nonempty open set and $\mathcal{H}^r_{P,V}(\mathbb{R}^n) = \{f \in H^r(\mathbb{R}^n) : P(D)f|_V = 0\}$. We note that $\mathcal{A}_V \subset H^{-\infty}(\mathbb{R}^n)$. The following proposition implies that the sets $\mathcal{H}^r_{P,V}(\mathbb{R}^n)$ in the union (25) are also Hilbert spaces.

Proposition 3.1. *The subset $\mathcal{H}^r_{P,V}(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ is a separable Hilbert space for all $r \in \mathbb{R}$, $P \in \mathcal{P}$ and nonempty open set $V \subset \mathbb{R}^n$.*

Proof. Clearly $\mathcal{H}^r_{P,V}(\mathbb{R}^n)$ is a linear subspace of $H^r(\mathbb{R}^n)$. Let $f_k \in \mathcal{H}^r_{P,V}(\mathbb{R}^n)$ be a sequence such that $f_k \to f$ in $H^r(\mathbb{R}^n)$. Then by the continuity of the restriction map $|_V\colon H^r(\mathbb{R}^n) \to H^r(V)$ we have that $f_k|_V \to f|_V$ in $H^r(V)$. From the continuity of $P(D)\colon H^r(V) \to H^{r-m}(V)$ we obtain that $0 = P(D)f_k|_V \to P(D)f|_V$ in $H^{r-m}(V)$, implying that $f \in \mathcal{H}^r_{P,V}(\mathbb{R}^n)$. Therefore $\mathcal{H}^r_{P,V}(\mathbb{R}^n)$ is a closed subspace of the separable Hilbert space $H^r(\mathbb{R}^n)$ and hence itself a separable Hilbert space. ☐

Remark 3.2. *We note that in the smooth case we have that $\mathcal{E}_{P,V}(\mathbb{R}^n) = \{f \in \mathcal{E}(\mathbb{R}^n) : P(D)f|_V = 0\} \subset \mathcal{E}(\mathbb{R}^n)$ is a closed subspace of $\mathcal{E}(\mathbb{R}^n)$ and hence a Fréchet space. More generally, $\mathcal{D}'_{P,V}(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : P(D)f|_V = 0\} \subset \mathcal{D}'(\mathbb{R}^n)$ is sequentially closed in $\mathcal{D}'(\mathbb{R}^n)$ under the weak* convergence. These two facts follow from the continuity of $P(D)\colon \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ and $P(D)\colon \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ with respect to the standard topologies. More topological properties of kernels of constant coeffiecient partial differential operators can be found in* [45].

Remark 3.3. *The interpretation of the condition $f \in \mathcal{A}_V$ is the following. If $f \in \mathcal{A}_V$, then there is some $r \in \mathbb{R}$ and some $P \in \mathcal{P}$ such that $f \in H^r(\mathbb{R}^n)$ and $P(D)f|_V = 0$. The distributional derivatives commute with restrictions, i.e. $P(D)f|_V = P(D)(f|_V)$ where $f|_V \in \mathcal{D}'(V)$. Since $f \in H^r(\mathbb{R}^n)$ we see that $f|_V$ is not only a distribution but in addition $f|_V \in H^r(V)$ for some*

$r \in \mathbb{R}$. Therefore the existence of $r \in \mathbb{R}$ and $P \in \mathcal{P}$ for which $P(D)f|_V = 0$ means that $f|_V \in H^r(V)$ and $f|_V$ is a weak solution to some homogeneous constant coefficient partial differential equation. In other words, $f|_V$ satisfies

$$\sum_{|\alpha|\leq m} a_\alpha D^\alpha(f|_V) = 0, \quad f|_V \in \bigcup_{r\in\mathbb{R}} H^r(V), \tag{26}$$

for some coefficients $a_\alpha \in \mathbb{C}$ and some integer $m \in \mathbb{N}$.

The following proposition is important in the uniqueness of the partial data problem.

Proposition 3.4. *The set $\mathcal{A}_V \subset H^{-\infty}(\mathbb{R}^n)$ is a vector space for every nonempty open set $V \subset \mathbb{R}^n$.*

Proof. Let $f_1, f_2 \in \mathcal{A}_V$ and $\lambda \in \mathbb{C}$. This means that $f_1 \in H^{r_1}(\mathbb{R}^n)$, $f_2 \in H^{r_2}(\mathbb{R}^n)$ and $P_1(D)f_1|_V = P_2(D)f_2|_V = 0$ for some $r_1, r_2 \in \mathbb{R}$ and $P_1, P_2 \in \mathcal{P}$. It follows that $f_1 + \lambda f_2 \in H^r(\mathbb{R}^n)$ where $r = \min\{r_1, r_2\}$ since the spaces $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$, are nested vector spaces. We also have that $P_1(D)P_2(D)(f_1 + \lambda f_2)|_V = 0$ since the distributional derivatives commute $P_1(D)P_2(D) = P_2(D)P_1(D)$. This implies that $f_1 + \lambda f_2 \in \mathcal{A}_V$, i.e. $\mathcal{A}_V$ is a linear subspace of the vector space $H^{-\infty}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. □

Remark 3.5. *The vector space structure of $\mathcal{A}_V$ is important since it implies that the partial data results we have proved in this article are indeed uniqueness results. Namely, if $f_1, f_2 \in \mathcal{E}'(\mathbb{R}^n)\cap\mathcal{A}_V$ (or $f_1, f_2 \in C_\infty(\mathbb{R}^n)\cap\mathcal{A}_V$) such that $X_0 f_1 = X_0 f_2$ on all lines intersecting V, then $f_1 - f_2 \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ (or $f_1 - f_2 \in C_\infty(\mathbb{R}^n)\cap\mathcal{A}_V$) and $X_0(f_1 - f_2) = 0$ on all lines intersecting V. Theorem 1.4 then implies that $f_1 - f_2 = 0$, i.e. the solution to the partial data problem is unique.*

We list some examples of admissible functions. We have that the function $f \in H^{-\infty}(\mathbb{R}^n)$ belongs to $\mathcal{A}_V$, if

- f is polyharmonic in V, i.e. $(-\Delta)^k f|_V = 0$ for some $k \in \mathbb{N}$.
- f is polynomial in V.
- f is independent of one of the variables $x_1, \dots, x_n$ in V.
- $f(x) = q(x)e^{ix\cdot\zeta}$ in V where $q(x)$ is a suitable polynomial and $\zeta \in \mathbb{C}^n$ is a generalized frequency. Especially, if f is of the form $f(x) = e^{ix\cdot\xi_0}$ in V where $\xi_0 \in \mathbb{C}^n$ is a zero of $P \in \mathcal{P}$, then $P(D)f|_V = 0$.

Further, it holds that for convex sets V and a fixed $P \in \mathcal{P}$ the linear span of solutions of the form $q(x)e^{ix\cdot\zeta}$ is dense in the space of all smooth solutions of $P(D)g = 0$ in V (see [17, Theorem 7.3.6] and a more general result [18, Theorem 10.5.1]).

We note that if $P(D)$ is a hypoelliptic operator, then the condition $P(D)f|_V = 0$ already implies that f is smooth in V (see [18, 28]). Basic examples of hypoelliptic operators are elliptic operators such as integer powers of Laplacians ($(-\Delta)^k$ where $k \in \mathbb{N}$) and also the non-elliptic heat operator $\partial_t - \Delta$. However, there are non-smooth distributions $f|_V$ which satisfy the condition $P(D)f|_V = 0$ for some $P \in \mathcal{P}$ and therefore f can have singularities in V. For example, the wave operator $\partial_t^2 - \Delta$ is not hypoelliptic and has non-smooth weak solutions.

4. Proofs of the main theorems

In this section we prove our main theorems. We need a few auxiliary results. The first one is a unique continuation result for fractional Laplacians and the second one is the Poincaré lemma for compactly supported vector-valued distributions.

Lemma 4.1 ([6, Theorem 1.1])**.** *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $u \in H^t(\mathbb{R}^n)$ where $t \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$. The claim holds also for $s \in (-n/2, -n/4] \setminus \mathbb{Z}$ if $u \in \mathscr{O}'_C(\mathbb{R}^n)$.*

Lemma 4.2 (Poincaré lemma)**.** *Let $U \in (\mathcal{E}'(\mathbb{R}^n))^n$ such that $\mathrm{d}U = 0$. Then there is $\phi \in \mathcal{E}'(\mathbb{R}^n)$ such that $U = \mathrm{d}\phi$.*

The proof of lemma 4.2 can be found for example in [19, 26]. The third lemma is a known result about the zero set of multivariate polynomials.

Lemma 4.3 ([32, Lemma on p.1])**.** *Let $Q = Q(x)$ be a non-zero multivariate polynomial of order $m \in \mathbb{N}$*

$$Q(x) = \sum_{|\alpha| \leq m} b_\alpha x^\alpha = \sum_{|\alpha| \leq m} b_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad b_\alpha \in \mathbb{C}, \tag{27}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index such that $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Then the set $Z_Q = \{x \in \mathbb{R}^n : Q(x) = 0\}$ has Lebesgue measure zero.

Lemma 4.3 is proved in [32] for real coefficients but the result holds also for complex coefficients by splitting $b_\alpha \in \mathbb{C}$ to its real and imaginary parts. We note that the set Z_Q is Zariski closed but not the whole space $\mathbb{R}^n$. From the coarseness of the Zariski topology (i.e. there are relatively few closed sets) one can already deduce that the set Z_Q must be small in topological sense (see e.g. [10, Chapter 15.2]).

The next lemma shows how the normal operator of the X-ray transform of vector fields is related to the normal operator of scalar fields (see also [21, Proof of theorem 1.1]).

Lemma 4.4. *Let $F \in (\mathcal{E}'(\mathbb{R}^n))^n$. Then $\mathrm{d}(N_1 F) = (n-1)^{-1} N_0(\mathrm{d}F)$ holds componentwise where N_0 acts on the components $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n)$.*

Proof. The normal operator has the expression

$$(N_1 F)_i = \sum_{j=1}^n \frac{2 x_i x_j}{|x|^{n+1}} * F_j. \tag{28}$$

Rewrite the kernel as

$$\frac{2 x_i x_j}{|x|^{n+1}} = \frac{2}{n-1}\left(\delta_{ij} |x|^{1-n} - \partial_i (x_j |x|^{1-n})\right) \tag{29}$$

which implies that

$$(N_1 F)_i = \frac{2}{n-1}\left(\frac{1}{2} N_0 F_i - \sum_{j=1}^n x_j |x|^{1-n} * \partial_i F_j\right). \tag{30}$$

Calculating the components of $\mathrm{d}(N_1F)$ we obtain

$$\partial_k(N_1F)_i - \partial_i(N_1F)_k = \frac{1}{n-1}N_0(\partial_kF_i - \partial_iF_k). \tag{31}$$

This means that $\mathrm{d}(N_1F) = (n-1)^{-1}N_0(\mathrm{d}F)$ where N_0 acts componentwise on $\mathrm{d}F$, giving the claim □

Now we are ready to prove our results. We start with the main theorem.

Proof of theorem 1.1. Let $f \in \mathcal{A}_V$ and $s \in (-n/4, \infty) \setminus \mathbb{Z}$. This means that $f \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$ and $P(D)f|_V = 0$ for some constant coefficient partial differential operator $P(D)$ of order $m \in \mathbb{N}$ and nonempty open set $V \subset \mathbb{R}^n$. In particular, we have $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\hat{f} = \langle\cdot\rangle^r g$ where $g \in L^2(\mathbb{R}^n)$. Using the properties of the Fourier transform we see that $P(D)((-\Delta)^s f) = (-\Delta)^s(P(D)f)$ because $P(D)$ has constant coefficients. Since $P(D)$ is a local operator we obtain the conditions $P(D)f|_V = (-\Delta)^s(P(D)f)|_V = 0$. Now $P(D)\colon H^r(\mathbb{R}^n) \to H^{r-m}(\mathbb{R}^n)$ is continuous (see e.g. [28, Theorem 12.7]) and we have $P(D)f \in H^{r-m}(\mathbb{R}^n)$. We can use lemma 4.1 for $P(D)f$ to obtain that $P(D)f = 0$ as a tempered distribution. Taking the Fourier transform this is equivalent to that $P(\xi)\hat{f}(\xi) = P(\xi)\langle\xi\rangle^{-r}g(\xi) = 0$ almost everywhere where $P(\xi)$ is a multivariate polynomial of order $m \in \mathbb{N}$. Since $\langle\cdot\rangle^{-r} \neq 0$ everywhere and $P(\xi) \neq 0$ almost everywhere by lemma 4.3, we have that $g = 0$ almost everywhere. This implies that $\hat{f} = 0$ as a tempered distribution and hence $f = 0$.

Let then $f \in \mathscr{O}'_C(\mathbb{R}^n) \cap \mathcal{A}_V$ and $s \in (-n/2, \infty) \setminus \mathbb{Z}$. Using the same arguments as above we obtain that $P(D)f|_V = (-\Delta)^s(P(D)f)|_V = 0$ for some constant coefficient partial differential operator $P(D)$ and nonempty open set $V \subset \mathbb{R}^n$. We know that $f \in \mathscr{O}'_C(\mathbb{R}^n)$ is equivalent to that $\hat{f} \in \mathscr{O}_M(\mathbb{R}^n)$. Now $\mathcal{F}(P(D)f)(\xi) = P(\xi)\hat{f}(\xi)$ where $P(\xi)$ is a multivariate polynomial of order $m \in \mathbb{N}$. It follows from the Leibnitz product rule for multivariable functions that $\mathcal{F}(P(D)f) \in \mathscr{O}_M(\mathbb{R}^n)$ since $P(\xi)$ is polynomial and the derivatives of $\hat{f}$ are polynomially growing. This is equivalent to that $P(D)f \in \mathscr{O}'_C(\mathbb{R}^n)$ and we can use lemma 4.1 to deduce that $P(D)f = 0$ as a tempered distribution. Taking the Fourier transform this is equivalent to that $P(\xi)\hat{f}(\xi) = 0$ almost everywhere. As a polynomial $P(\xi) \neq 0$ almost everywhere and we obtain that $\hat{f} = 0$ almost everywhere. But $\hat{f}$ is continuous and hence $\hat{f} = 0$, implying $f = 0$. □

The rest of the results are then direct consequences of theorem 1.1.

Proof of theorem 1.3. If $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$, then also $f \in \mathscr{O}'_C(\mathbb{R}^n) \cap \mathcal{A}_V$. Since $N_0 = (-\Delta)^{-1/2}$ up to a constant factor and $n \geq 2$ we have that $-1/2 \in (-n/2, \infty) \setminus \mathbb{Z}$ and we can use theorem 1.1 to obtain that $f = 0$. □

Proof of theorem 1.4. The assumption $X_0f = 0$ on all lines intersecting V implies that $N_0f|_V = 0$. Since we also assume that $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ we obtain $f = 0$ by theorem 1.3. □

Proof of theorem 1.7. The assumption $X_1 F = 0$ on all lines intersecting V implies that $N_1 F|_V = 0$. By lemma 4.4 we have $\mathrm{d}(N_1 F) = N_0(\mathrm{d}F)$ componentwise up to a constant factor. The locality of the exterior derivative implies that $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n)$ and $N_0(\mathrm{d}F)_{ij}|_V = 0$. Since $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ we can use theorem 1.3 for the components $(\mathrm{d}F)_{ij}$ to obtain that $\mathrm{d}F = 0$. Lemma 4.2 implies that $F = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$. □

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