VOLATILITY AS AN ASSET CLASS: DIVERSIFICATION BENEFITS OF ADDING VSTOXX FUTURES TO EUROPEAN MULTI-ASSET PORTFOLIOS

University of Jyväskylä
School of Business and Economics

Bachelor's Thesis

2021

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ABSTRACT

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Title: Volatility as an asset class: Diversification benefits of adding VSTOXX futures to European multi-asset portfolios	
Subject: Accounting	Type of work: Bachelor's thesis
Date: 10.5.2021	Number of pages: 64

Abstract

Previous research has suggested that cross correlations of different asset classes tend to increase during market downturns. This in turn leads investors to look for alternative sources of diversification. Placing special emphasis on the 2020 stock market crash triggered by COVID-19, this paper examines whether there are diversification benefits in adding VSTOXX futures of different maturities to portfolios constructed from several European and global asset classes.

By analyzing daily returns of VSTOXX futures and several other asset classes on period from January 2011 to December 2020, we employ finite-sample mean-variance spanning test proposed by Huberman and Kandel (1987) as well as the step-down applications of Kan and Zhou (2001) to get a more precise picture of what causes the rejection or acceptance of the spanning test. Furthermore, to get robust measures under non-normality, we employ GMM Wald test and its step-down applications proposed by Kan and Zhou (2001) to control for non-normal distribution. In addition, to determine the magnitude of diversification benefits, we apply two diversification measures proposed by Petrella (2005) with and without investment policy constraints to reflect realistic as well as theoretical investment choices.

Our empirical results are robust and indicate that all chosen VSTOXX futures improve the investment opportunity set during the full period and the 2020 stock market crash. Contrary to previous research, we found that 5-month VSTOXX futures offer the most significant diversification benefits with and without investment policy constraints during the full period and the 2020 stock market crash. This research sheds new light on the subject of volatility diversification and illustrates that it can be beneficial to add VSTOXX futures to European multi-asset portfolios. However, due to weak long-term performance of VSTOXX futures, a more dynamic investment strategy than holding a constant long exposure in VSTOXX is recommended.

Key words:
VSTOXX futures, mean-variance spanning, asset allocation, volatility, diversification

Place of storage Jyväskylä University School of Business and Economics (JSBE)

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ABBREVIATIONS

CAL	Capital allocation line
CML	Capital market line
GMM	Generalized methods of moments
GMV	Global minimum variance
MPT	Modern portfolio theory
s.t.	Subject to
VRP	Volatility risk premium

1 INTRODUCTION

Literature has shown that the cross correlations between asset classes tend to converge on extreme market conditions (Chow, Jacquier, Kritzman, & Lowry, 1999; Szado, 2009). As a result, diversification benefits of traditional portfolios are found to diminish in the times of market distress, when they are needed the most. Also, Platanakis et al. (2019) show that even the diversification benefits of alternative investments were statistically insignificant during the 2007-2008 financial crisis. This further encourages investors to seek alternative means of diversification. When most of the asset classes experienced significant pullbacks during the 2007-2008 financial crisis, there was one asset class which made remarkable returns: volatility. From October 2007 to November 2008, EURO STOXX 50 index lost 40% of its value, while its short-term implied volatility measured by VSTOXX index had more than tripled its value by returning 203,5%. Same effect could be seen during the COVID-19 crisis: from February 2020 to April 2020 EURO STOXX 50 index lost 20,9% of its value, while the VSTOXX index yielded 214% and tripled its value again.

According to Markowitz (1952), the greatest reduction in portfolio risk is achievable through investing in assets with negative cross correlations. Therefore, the potential diversification benefits of adding volatility derivatives into the investment opportunity set comes from the empirical findings that volatility and equity returns tend to move in opposite directions. Also, since the returns of equity and other asset classes are noted to converge on market downturns, this negative correlation between equity and volatility returns implies that investing in volatility can be used as a way to obtain insurance on the downside risk of the whole portfolio. However, investing in the VSTOXX index is not that simple. Since the VSTOXX index measures the 30-day expected volatility of the EURO STOXX 50 index through near-term put and call option prices, it is extremely hard to directly replicate the VSTOXX index. While the VSTOXX index is not directly tradeable, it is possible to gain exposure in the VSTOXX index through derivatives. In this research, VSTOXX futures of several maturities are considered. However, rolling volatility futures can be problematic, since during times of low volatility obtaining exposure in volatility is expensive as the term structure of volatility futures is mostly in contango (Alexander & Korovilas, 2011).

The goal of this thesis is to get answers to the following questions:

- Are there diversification benefits in adding VSTOXX futures of different maturities to an investment portfolio constructed from European and global asset classes during a "normal" market environment and during the 2020 stock market crash?
- What is the magnitude of diversification benefits when holding a continuous long position in VSTOXX futures in diversified portfolios during the aforementioned periods?

To answer these questions, the empirical part of this study is carried out as follows: Firstly, 2 different portfolios with different combinations of European and global asset classes to represent a traditional portfolio and a more diversified portfolio from the view of a European investor. The time period of this study ranges from 3.1.2011 to 31.12.2020, and in addition, we place special emphasis on the 3-month 2020 stock market crash period from 3.2.2020 to 30.4.2020 to examine diversification benefits of volatility during a recent crisis period. Lastly, we run mean-variance spanning tests proposed by Huberman and Kandel (1987) and Kan and Zhou (2001) to examine the statistical significance of diversification benefits to as well as diversification measures proposed by Petrella (2005) to obtain the magnitude of the diversification benefits when VSTOXX futures with maturities of 1, 3 and 5 months are added to the traditional and diversified portfolios.

Previous research considering the diversification benefits of VSTOXX derivatives has been conducted considering portfolios consisting only of equities, bonds or both. Therefore, this thesis is unique for three compelling reasons. Firstly, differing from previous research concerning diversification with exposure to VSTOXX, this thesis examines portfolios constructed from multiple asset classes such as equity, bonds, commodities, high yield bonds, hedge funds and real estate. Secondly, even though there exist multiple studies concerning the diversification benefits of VSTOXX futures, the literature does not cover the statistical testing of significance of the diversification benefits when VSTOXX derivatives are added to a European investment portfolio. Thirdly, this is the first study to examine diversification benefits of VSTOXX futures during the 2020 stock market crash.

The paper is organized as follows: In chapter 2, we will discuss the implied volatility as well as the VSTOXX index and as its derivatives in more detail, theories related to diversification as well as existing literature concerning diversification with volatility products. Subsequently in chapter 3, we will present the data and the methodologies used to transform our data to gain answers for our research questions. Chapter 4 will focus on the empirical results obtained with our dataset and the methodologies discussed earlier. Finally, in chapter 5 we provide our conclusions and final remarks.

2 THEORETICAL BACKGROUND

2.1 Trading European implied volatility

2.1.1 Volatility and implied volatility

Whereas in statistics standard deviation is defined as a dispersion around the mean, in finance we consider volatility as a dispersion of returns around the mean return for a given security. Generally, there are 2 ways to estimate volatility. The first way is to estimate volatility through historical volatility, and thus this measure is called realized volatility. The second way is to measure the expected future volatility of the underlying asset from the price of an option contract. Since this measure of volatility is defined implicitly from a parameter of a model, it is called implied volatility. The first case, realized volatility, can be estimated from time series data using the following formula with an assumption that the asset prices are lognormally distributed (Bodie, Kane, & Marcus, 2018, p. 718):

$$\hat{\sigma}_r = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2} \tag{2.1}$$

Where $r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$ and $\bar{r} = \frac{1}{T}\sum_{t=1}^{T}\ln\left(\frac{P_t}{P_{t-1}}\right)$ in which P_t refers to a price of an asset at time t. If we let ϑ be the length of a time period expressed with respect to the frequency of r_t, we can express the volatility in a certain time period as:

$$Realized\ volatility\ in\ a\ time\ period\ \vartheta = \hat{\sigma}_r\sqrt{\vartheta} \tag{2.2}$$

For example, if r_t refers to daily returns, then annualized volatility of asset, assuming 252 trading days in a year, can be expressed as $\hat{\sigma}_r\sqrt{252}$, whereas if the r_t refers to monthly returns, then annualized volatility of asset can be expressed as $\hat{\sigma}_r\sqrt{12}$.

Since prices of options are based on the common consensus of the market participants view on the expected future volatility, implied volatility is considered as the market participants' expectation about the future realized volatility of the underlying asset over the option's remaining time to expiration (Badshah, 2009). Traditionally, implied volatility is calculated from Black-Scholes (1973) option pricing model for European plain vanilla options or from Cox-Ross-Rubinstein (1979) binomial model for European and American plain vanilla options. In this thesis, only the Black-Scholes option pricing model will be covered.

The Black-Scholes option pricing model relies on several underlying assumptions (Black & Scholes, 1973): (1) The risk free rate is known and constant, (2) stock prices follow a continuous random walk where the volatility is constant and thus the distribution of possible stock prices is lognormally distributed, (3) the stock pays no dividends or other distributions, (4) the option is European, hence it can be exercised only at expiration, (5) it is possible to borrow any faction of the price of a security at a risk-free rate and there are no penalties in short selling and (6) there are no transaction costs in buying or selling the stock or the option. Under these assumptions, it is possible to define an equilibrium price of an option with the Black-Scholes option pricing model. To calculate the equilibrium price of an option, only five parameters are needed: the price of an underlying at time t (S), the strike price of an option (K), time to maturity expressed in years ($T-t$) where T is the maturity time, the annualized risk-free rate (r), the annualized volatility of the underlying (σ)[1]. The equilibrium price of a call option can be then calculated as follows (Black & Scholes, 1973):

$$C(S,t) = \mathcal{N}(d_1)S - \mathcal{N}(d_2)Ke^{-r(T-t)} \tag{2.3}$$

Where:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\mathcal{N}(x) = cumulative\ normal\ density\ function$$

Of all these parameters, only volatility (σ) is not observable in the market. Therefore, the question becomes: What value of volatility in the Black-Scholes model produces a price that is equal to the market price of an option? Given that all the other parameters than σ are known, we can solve the volatility of an option implicitly from Equation (2.3). Thus, within Black-Scholes framework we can define implied volatility as an implicit function of the parameters C, K, S, r and $T-t$:

$$Implied\ volatility = f(C,K,S,r,T-t) \tag{2.4}$$

Typically, for a given time period, the implied volatility and volatility are not equal. This is due to the fact, that investors accept to pay a premium in order to hedge themselves against the market downturns. This difference between implied volatility and realized volatility is called the volatility risk premium (VRP) (Cheng, 2015). Usually, the implied volatility is higher than the realized volatility, but it is also possible that the implied volatility would be less than the realized volatility. In this case, it is expected that the volatility will decrease to a lower level, thus obtaining long volatility products become more affordable than with positive VRP. Thus, in times of low volatility, the sign of the VRP is usually

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[1] Similarly, the price of a put option is defined as $P(S,t) = \mathcal{N}(-d_2)Ke^{-r(T-t)} - \mathcal{N}(-d_1)S$ (Black & Scholes, 1973).

positive, whereas in times of high volatility, the sign of VRP typically turns to negative.

Volatility has some specific characteristics that are widely documented in the literature. Mandelbrot (1963) was the first to note that volatility has a tendency to cluster. By clustering, it is referred to the notion that large changes tend to be followed by large changes and small changes tend to be followed by small changes. Thus, this property implies that when the market or a specific asset experiences a shock in volatility, more volatility should be expected. Volatility is also considered to be mean-reverting in its nature. With mean-reversion, it is meant that volatility will eventually converge to the long-term average level of volatility. This property implies that current information has no effect on the long-term forecast of volatility, thus the long-term volatility is not affected by any shocks. (Engle & Patton, 2001.) Furthermore, numerous studies have shown that the volatility has a strong negative correlation with equity (Badshah, 2009; Whaley, 2000). Since equity returns are typically skewed to the left, it can be inferred from the negative correlation that volatility returns are skewed to the right. Also, as equity and volatility are strongly negatively correlated, it can be said that an investor with a long position in equity has implicitly a short position in volatility.

Volatility has also another unique feature: it has asymmetric returns with equity (Badshah, 2009). The asymmetry refers to the phenomenon that volatility is higher during periods when equity returns are negative, as compared to periods with positive equity returns. Therefore, obtaining a long position in volatility can be seen as an insurance against market downturns. Two different schools of thought lead the debate on the causes of this phenomenon. Black (1976) was the first to propose a theory for this relationship called the “leverage effect”. He suggested that the negative correlation between implied volatility and stocks is due to the firm’s leverage. When asset prices decline, companies become more leveraged since the relative value of their debt increases in relative to equity. As a result, the value of the company’s stock becomes more riskier, which in turn increases the volatility of the stock. Black’s (1976) theory has been empirically confirmed by Christie (1982) and Cheung and Ng (1992). However, Figlewski and Wang (2001) call this theory’s validity into question. They argue that although a strong “leverage effect” is associated with falling stock prices, there is no apparent change in volatility when leverage alters. They suggest that volatility may have a little direct relationship with the firm leverage and suggest that “leverage effect” should be more properly termed as “down market effect”. Another theory considering the reasons behind the negative correlation of volatility and equity returns is known as “volatility feedback effect”. According to this theory, volatility is assumed to be incorporated in stock prices, and when there is an increase in the volatility, the required rate of return on equity increases and as a result the stock prices will be expected to fall to make the returns attainable. Conversely, when there is a decrease in volatility, the required rate of return on equity decreases and simultaneously the stock prices will increase (Bekaert & Wu, 2000; French, Schwert, & Stambaugh, 1987).

2.1.2 VSTOXX Index

Volatility as an investable product is a relatively new invention. Brenner and Galai (1989) were one of the first researchers to suggest developing a volatility index. A few years later Whaley (1993) introduced the CBOE volatility index (VIX), which at first was based on options of the S&P100 index and after 2003 on the options of the S&P500 index. Similarly, as the volatility of an option captures the market expectation of the implied volatility of the underlying asset, implied volatility indices (often referred only as volatility indices) capture the market expectation of the implied volatility of the underlying indices.

The VSTOXX index, which was launched in 1999, measures the future short-term volatility of the EURO STOXX 50 index, which in turn, is an index that consists of 50 blue-chip companies from 8 Eurozone countries. The VSTOXX index is derived from put and call options prices with the EURO STOXX 50 as an underlying asset. It reflects the market expectations of European equity market volatility for a rolling fixed maturity of 30 days by measuring the square root of the implied variance across all the options within the two nearest expiration months. There also exists other similar indices within the Eurozone: VDAX measures the implied volatility of German stock market index DAX, VCAC measures the implied volatility of the French stock market index CAC 40 and VSMI measures the implied volatility of the Swiss stock market index SMI. However, in practice, the VSTOXX or other volatility indices are not directly investable. Instead, investors can buy derivatives that follow the movements of the volatility indices, which we will discuss more about in the next chapter.

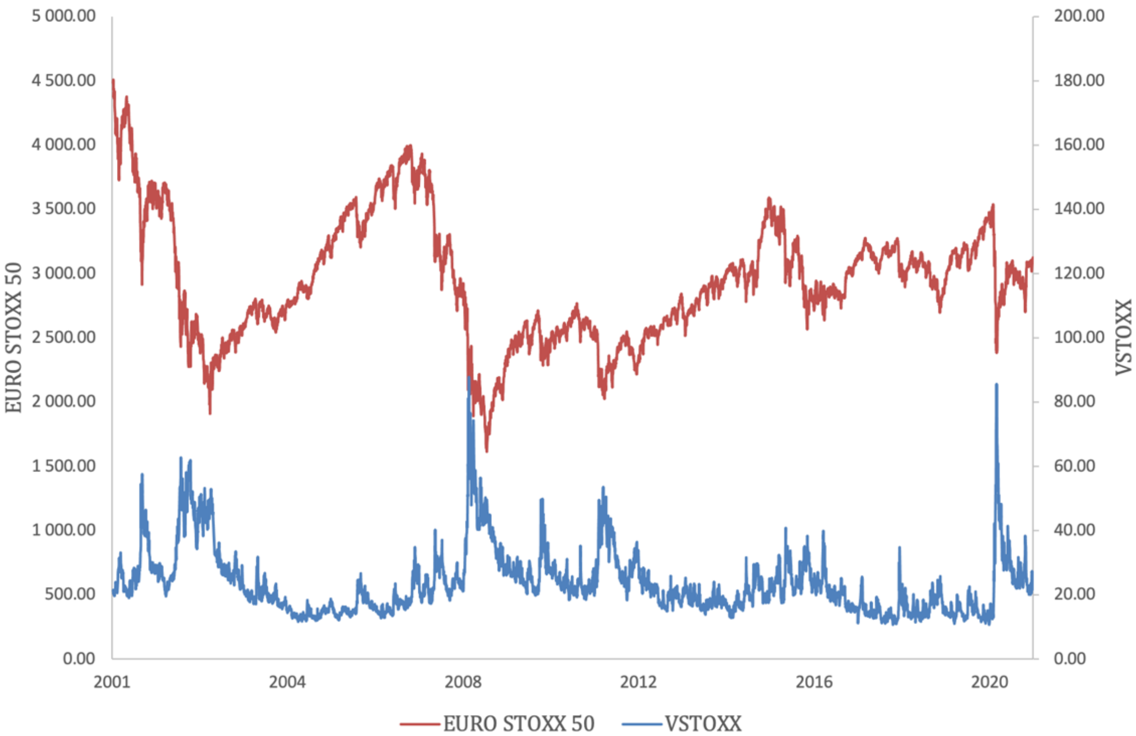

Figure 1 EURO STOXX 50 and VSTOXX daily prices 2001–2020 in EUR

While the VSTOXX index refers to the implied volatility of EURO STOXX 50 index for a rolling fixed maturity of 30 days, the value of VSTOXX and thus the implied volatility is given as an annualized volatility (Dennison, 2021). The value of the VSTOXX can be inferred as follows: Suppose that the VSTOXX index is trading at 15. Then there is a roughly 68% chance (1 standard deviation) that the EURO STOXX 50 index will be within $\pm 15\%$ range within the following year. Using Equation (2.2), we can express the expected volatility in the following 30 days by dividing the value of VSTOXX by $\sqrt{12}$. Thus, when the VSTOXX index is trading at 15, market expects the volatility of the EURO STOXX 50 index to be at $\pm 15\%/\sqrt{12} \approx \pm 4.33\%$ range at roughly 68% chance and at $\pm 4.33\% * 1.96 \approx \pm 8.49\%$ range at roughly 95% (1.96 SD) chance within the following 30 days. From Figure 1 the asymmetric property of volatility and equity returns can be seen clearly: When the EURO STOXX 50 index experiences sharp declines there are also spikes in the value of the VSTOXX index, but when the EURO STOXX 50 index experiences positive returns, the value of the VSTOXX index remains relatively unaffected. Therefore, volatility indices are often called by academics as "investor fear gauges".

The calculation of VSTOXX does not rely on the Black-Scholes or Cox-Ross-Rubinstein models, but rather on a model-free methodology by calculating its price directly from the option prices. The difference between those models is, that model-free volatility accounts for all strike prices, thus obtaining a larger set of implied volatilities, whereas the Black-Scholes or Cox-Ross-Rubinstein models do not account for all strike prices in computation (Badshah, 2009). Also, without solving the implied volatility out of an option pricing formula, it considerably relieves the problems of measurement errors when option pricing formulas are employed. In addition to the most commonly followed 30 days fixed maturity VSTOXX index, there exists another main indices as well for maturities ranging from 30–360 days in increments of 30 days. Whereas the main indices reflect constant rolling maturity and do not expire, there are 8 sub-indices that reflect the actual implied volatility of the next 1, 2, 3, 6, 9, 12, 18 and 24 month expires of the EURO STOXX 50 index option contracts (STOXX, 2021, p. 27–28). First step in the calculation of the VSTOXX main indices is the extraction of implied volatilities for given maturities by determining the VSTOXX sub-indices. The sub-indices can be calculated as follows (STOXX, 2021, p. 29):

$$SubIndex_i = 100 \sqrt{\frac{2}{T_i/T_{365}} \sum_j \frac{\triangle K_{i,j}}{K_{i,j}^2} R_i M(K_{i,j}) - \frac{1}{T_i/T_{365}} \left(\frac{F_i}{K_{i,0}} - 1 \right)^2} \tag{2.5}$$

Where $i = i^{th}$ subindex (i = 1,…,8 months), T_i = seconds to the i^{th} OESX (EURO STOXX 50 option) expiry date, T_{365} = seconds in a standard year of 365 days (31,536,000 sec), F_i = forward ATM price for the i^{th} OESX expiry date, $K_{i,0}$ = highest strike price such that $K_{i,0} < F_i$; $K_{i,j}$ = strike price of the j^{th} OTM option of the i^{th} after sorting the options by their strike price in ascending order, $\triangle K_{i,j}$ =

average distance between $K_{i,j+1}$ and $K_{i,j-1}$, $M(K_{i,j})$ = the most recent trade-, mid- or settlement price of an option with the strike price $K_{i,j}$, $M(K_{i,0})$ = average of put and call option prices of an option with the strike price $K_{i,0}$, R_i = refinancing factor for the i^{th} OESX expiry date such that $R_i = e^{r_i T_i / T_{365}}$; where r_i = interpolated risk-free rate valid for the i^{th} OESX expiry date.

From the calculation of the sub-indices, it can be seen that the closer the option's strike price is to at-the-money value, the higher weight it receives in the calculation. The main indices are then calculated as a time-weighted average of two nearest VSTOXX sub-indices to keep a constant maturity on the implied volatility of the OESX options as follows (STOXX, 2021, p. 28):

$$MainIndex_{tm} = 100\sqrt{\left(\frac{T_{st}}{T_{365}}\left(\frac{SubIndex_{st}}{100}\right)^2 \frac{T_{lt} - T_{tm}}{T_{lt} - T_{st}} + \frac{T_{lt}}{T_{365}}\left(\frac{SubIndex_{lt}}{100}\right)^2 \frac{T_{tm} - T_{st}}{T_{lt} - T_{st}}\right)\frac{T_{365}}{T_{tm}}} \quad (2.6)$$

Where tm = fixed time to maturity expressed in days, $SubIndex_{st}$ = value of VSTOXX sub-index with shorter maturity, $SubIndex_{lt}$ = value of VSTOXX sub-index with longer maturity, T_{st} = seconds to expiry of $SubIndex_{st}$, T_{lt} = seconds to expiry of $SubIndex_{lt}$, T_{tm} = seconds in tm (1 day = 86,400 sec).

2.1.3 Derivatives on VSTOXX

It is essential to note that volatility or the VSTOXX index are not tradeable. However, with derivatives it is possible to add exposure to the movements of a security that is otherwise untradeable, such as VSTOXX index in this research. Important notation with the volatility derivatives is, that they can be seen as a kind of insurance against the extreme downward moves of the underlying asset. Since it has been found that the returns of different asset classes tend to converge during market downturns, derivatives on VSTOXX can be used to hedge the downside risk of the whole portfolio consisting not only of equities, but also other asset classes.

In 2005, Eurex introduced VSTOXX futures, which hold the VSTOXX index as an underlying asset. Due to the low demand on VSTOXX futures, Eurex changed its VSTOXX futures into VSTOXX mini futures in 2009. The only difference between former futures and the new mini futures is the contract value of a one contract, within the former futures contracts the contract value was 1000 EUR per index point of the underlying and within the mini futures the contract value is 100 EUR per index point of underlying. Hence, if a VSTOXX futures contract is trading at 15, then the contract value is 1500 EUR. Thus, by decreasing the value of the futures contracts, Eurex made it possible for a larger audience to obtain VSTOXX futures. Price quotation is in points of two decimals and the minimum price change is 0.05, which corresponds to 5 EUR. The VSTOXX futures are cash settled on the final trading day, which is 30 prior to the expiration day of the underlying options. This is usually the Wednesday prior to the second last Friday of the respective maturity month, if this is a trading day. Otherwise, the settlement is being done on the exchange day immediately preceding that day. (Eurex, 2020.)

While the VSTOXX index represents market's expectations on the 30-day implied volatility of the EURO STOXX 50 index, VSTOXX futures represent market's risk-neutral expectation on the level of the VSTOXX index at the at the futures' expiry date. Within volatility futures, the term structure is in great interest. The term structure of certain futures can be analyzed by plotting futures' prices as a function of time to maturity. If the futures' prices increase as the time to maturity increases, i.e. the term structure has a positive slope, then the term structure is said to be in contango. Correspondingly, when the futures' prices decrease as the time to maturity increases, i.e. the term structure has a negative slope, then the term structure is said to be in backwardation. (Bodie et al., 2018, p. 768–769.) Within VSTOXX futures, contango can be interpreted in a way that the market expects the VSTOXX index to increase from its current level, whereas the market expects the VSTOXX index to decrease from its current level when the term structure is in backwardation. During periods of low volatility, market often expects the volatility to increase in the future i.e. contango, whereas in periods of high volatility the market expects the volatility to decrease in the future, i.e. backwardation. Due to the mean-reverting and skewed nature of volatility, VSTOXX futures tend to be in contango on average. For example, from 2011 to 2014, the term structure of VSTOXX futures has been in contango 70% of the time (Shore, 2018). Thus, long volatility strategies often suffer from the negative cost of carry associated with contango, as the value of a bought future decreases over time. Also, as the maturity of VSTOXX futures increases, their sensitivity to VSTOXX decreases, thus the term structure of VSTOXX futures is steeper with the shorter-term futures. This indicates that the long volatility strategies using longer maturity futures exhibit lower cost of carry, but also a lower hedge against market downturns. On the other hand, when there is uncertainty in the market, i.e. implied volatility is at high levels, then the market typically expects the volatility to settle on lower levels in the future, thus the term structure of the volatility futures is in backwardation making it possible to create positive yield from holding the futures.

Options on VSTOXX were launched in 2012 as European style options with VSTOXX index as the underlying asset. On February 2017, Eurex replaced options on VSTOXX with American style options on VSTOXX futures. This was because the jurisdiction of the SEC allowed trading VSTOXX options only for Qualified Institutional Buyers in the United States. (Shore, 2017.) By changing the underlying asset from VSTOXX index to VSTOXX futures, the options on VSTOXX futures contracts became CFTC-certified, thus allowing for wider market participation. The price quotation, contract value, final settlement and trading day are identical to VSTOXX futures, but instead of cash settlement, the options on VSTOXX futures are settled with VSTOXX futures (Eurex, 2020). Since the settled VSTOXX futures expire on the same day as the options are settled and are cash settled, the settlement of options on VSTOXX futures is in practice cash settled.

In addition to the VSTOXX futures and options on VSTOXX futures, there exists also other derivatives related to VSTOXX. As VSTOXX index is based on the prices of put and call options of the EURO STOXX 50 index, there exists a V-VSTOXX index launched in 2015, which in turn is based on a similar calculation as the VSTOXX index using options as the options on VSTOXX futures. Therefore, the V-VSTOXX measures the implied volatility of the options on VSTOXX futures, reflecting market's expectations on the volatility of the VSTOXX index for a rolling fixed maturity of 30 days. Thus, the V-VSTOXX index is also called the volatility-of-volatility index. However, V-VSTOXX index can be used only as an indicator of the expected volatility of the VSTOXX index, as currently there are no derivatives to obtain exposure to the V-VSTOXX index.

2.1.4 Option strategies and variance swaps

Previously, we discussed about derivatives on the VSTOXX index. However, since VSTOXX futures and options on VSTOXX futures contracts are affected also by other parameters than just the price of the underlying, they do not offer pure exposure to implied volatility of the EURO STOXX 50 index. As we previously noted, it is very hard to replicate the VSTOXX index, but it is still possible to create positions that have exposure to the volatility of the EURO STOXX 50 index.

Besides from the derivatives on VSTOXX index, it is possible construct a position from OESX call and put options that is profitable when the EURO STOXX 50 index experiences large price fluctuations in either direction. This position is called as a long straddle, and it can be established by buying both at-the-money call and put options each with the same underlying asset, the same strike price and the same time to maturity (Bodie et al., 2018, p. 671). Similarly, a short straddle could be constructed from selling put and call options, and thus the position would be a bet on low volatility, as the payoff from a short saddle is positive when the price movement of the underlying remains small. Since large price movements imply large volatility, straddle positions are essentially bets on volatility.

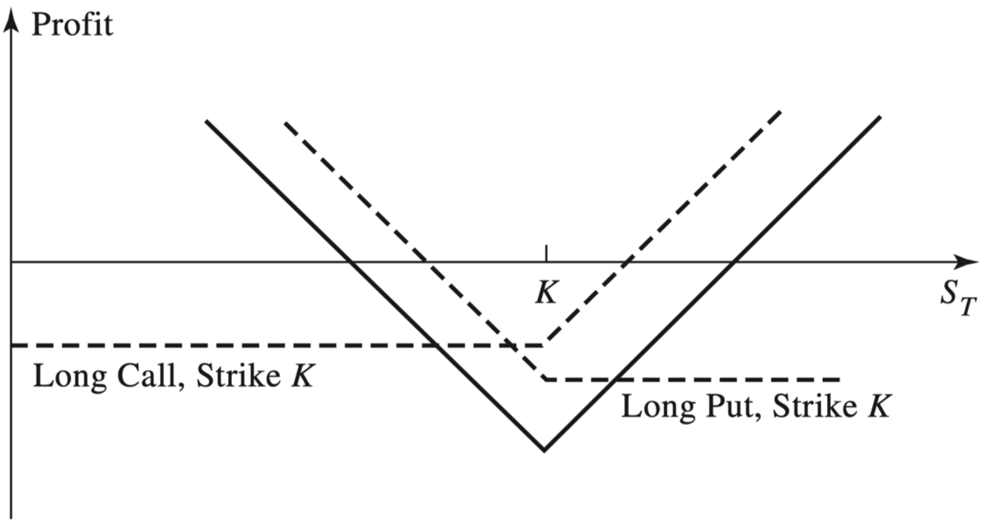

Figure 2 Profit pattern of a long straddle (Hull, 2018, p. 289)

Figure 2 presents the profit pattern of a long straddle, where S_T refers to the price of an underlying asset at time t. The profit pattern of a short straddle looks similar, except it is inverted. However, even though the volatility turns out to be high, it does not guarantee that the price of an underlying asset would significantly differ from the strike price of the options at the expiration date, thus volatility does not guarantee profitability of the long or short straddles. To make the exposure of the straddles to volatility more efficient, it is possible to construct a position that is constantly immune to the price changes of the underlying asset. This strategy, known as delta hedging, allows investors to obtain a purer exposure to the volatility of the underlying by neutralizing the effects of price changes to the option prices.

Delta (Δ) of an option contract is defined as a first partial derivative of a price of an option contract with respect to the value of the underlying (S) (Bodie et al., 2018, p. 730). Thus, Δ represents how much the value of an option contract changes when the price of the underlying changes by one unit. Using the previously defined Black-Scholes model for put and call options and defining V_{option} as a value of an option contract, we can define Δ for European options as:

$$\Delta= \frac{\partial V_{option}}{\partial S} = \begin{cases} \mathcal{N}(d_1), if\ option = call\ option\ (C) \\ -\mathcal{N}(-d_1), if\ option = put\ option\ (P) \end{cases} \tag{2.7}$$

To neutralize the influence of the level of the stock price on the return pattern of the straddle, one can create a delta neutral position ($\Delta= 0$) with offsetting positions in put call options. The total delta of a portfolio can be calculated simply as a weighted sum of individual deltas. Within delta hedging, the objective is then to keep the delta constantly at 0. Since the delta of an option does not remain constant as the price of the underlying changes, the position remains delta hedged for only a relatively short period of time. To obtain a constant delta neutral position, constant rebalancing of options is required to avoid being under- or over-hedged. In contrast, straddles without delta hedging are delta neutral only at the inception, as the deltas of put and call options cancel each other. To gain more insight on how it is possible to trade volatility more efficiently with delta hedging, we need to define vega ($\mathcal{V}$). Vega is defined as the first partial derivative of a price of an option contract with respect to the volatility (Hull, 2018, p. 437). Therefore, vega represents how much the value of an option contract changes when implied volatility of an underlying changes by one unit. Similarly, as we defined delta, we can define vega for European options as:

$$\mathcal{V} = \frac{\partial V_{option}}{\partial \sigma} = S\sqrt{T-t}N'(d_1) > 0 \tag{2.8}$$

Vega is the same and positive with both put and call options. Thus, increase in the implied volatility of the underlying increases the prices of both put and call option contracts by the same amount. It is easy to see that the value of vega depends greatly on the time to maturity, thus option contracts become less sensitive to changes in implied volatility as their expiration day approaches. Vega can be also interpreted in a way, that long positions in options benefit from the increase in volatility, while short positions in options benefit from the decrease in volatility. Another way to achieve delta hedged long (short) position on volatility is by holding a long (short) position in call options and a short (long) position in the underlying asset, by holding of static positions in options and dynamically trade the underlying[2] to hold the delta at 0. (Hull, 2018, p. 424–427.) For example, if an investor considers that the implied volatility of an underlying is too low, he can create a delta hedged long position, thus his payoff is influenced by the difference between realized and implied volatility. Conversely, if an investor believes that the implied volatility is too high, he can create a delta hedged short position to harvest the positive volatility risk premium. Finally, investor can realize his profits by closing the position before expiry of the option.

Even delta hedging strategies do not give pure exposure to volatility, as in practice obtaining a constant delta neutral position is near to impossible, but also costly due transaction costs. However, to obtain a pure volatility exposure, it is possible to trade pure volatility by obtaining variance swap contracts from over-the-counter-market. Variance swap is a forward contract where 2 counterparties agree to exchange the spread between realized variance (σ_r^2) and a specified level (strike) of variance (K_{var}) over a specified period. The payoff of a long position in a variance swap contract can be given as (Demeterfi, Derman, Kamal, & Zou, 1999):

$$N(\sigma_R^2 - K_{var}) \tag{2.9}$$

Where σ_R^2 and K_{var} are quoted in annual terms and N is the notational amount of the swap in currency per annualized variance point. Thus, at expiration the holder of a variance swap contract receives N currency for every point the realized variance exceeds the specified level of variance[3]. (Demeterfi et al., 1999.) For example, a long position in variance swap is profitable if at the end of the period $\sigma_R^2 > K_{var}$. On the other hand, if an investor believes that the future realized variance will be lower than the specified level of variance, he can take a short position in volatility by selling the contract, thus he will make a profit if $\sigma_R^2 < K_{var}$ at the end of the period. It is important to note that variance swaps have a unique property: They are convex in volatility, for example in long positions this means that the magnitude of profits per change in volatility increases as the volatility increases (Brière, Burgues, & Signori, 2010). Therefore, variance swaps are often cited as a way to take the most "aggressive" view in the direction of volatility.

To move forward to discuss the valuation of a variance swap contract, we first need to give a definition of a realized variance in a context of variance swaps. Theoretical definition of realized variance given variance as a function of time $\sigma^2(t, \dots)$ in a continuous time setting can be given as (Demeterfi et al., 1999):

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[2] For example, consider that the Δ of a call option is 0.15, and our long position in call options covers 10.000 shares of the underlying asset. We can create a delta-neutral position by shorting 10.000∗0.15=1500 shares of the underlying. Thus, if we let ϑ denote change in the price of the underlying, the price change of the total position is $0.15\vartheta * 10.000 - 1500\vartheta = 0$ if ϑ is small.

[3] For example, consider that the realized volatility $\sigma_R = 15\%$, the strike volatility $\sqrt{K_{var}} = 10\%$ and the notational principal $N = 1000$ EUR. Then the payoff of a long position at expiration is $1000\text{ EUR} * ((15\%)^2 - (10\%)^2) = 125000\text{ EUR}$.

$$\sigma_R^2 = \frac{1}{T}\int_0^T \sigma^2(t, \dots)dt \tag{2.10}$$

The value of a variance swap contract can be then derived similar as the value of a typical forward contract F. If we let r denote the risk-free discount rate corresponding to the expiration date T, we can define value of a variance swap contract in a risk-neutral world as:

$$F = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(\sigma_R^2 - K_{var})] \tag{2.11}$$

The fair delivery value of a forward contract at inception must be 0 in a risk-neutral world, since no capital is exchanged at the inception of the contract. Thus, the specified level of a variance (K_{var}) at inception is at a level that gives the variance swap contract the net present value of 0. If we set $K_{var} = \mathbb{E}[\sigma_R^2]$, we can define the fair delivery value of future realized volatility as risk-neutral expectation using Equation (2.10) (Demeterfi et al., 1999):

$$K_{var} = \frac{1}{T}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \sigma^2(t, \dots)dt\right] \tag{2.12}$$

However, in reality no one can know with certainty the value of future volatility. To replicate the expected specified level of variance, implied volatilities from option prices need to be taken into account. Thus, the definition of the fair delivery value of future realized volatility can be further extended to take into account put and call option prices in a continuous time setting with option strikes ranging from 0 to ∞. (Demeterfi et al., 1999.) However, in real-world context there are only options with strikes in finite interval, and therefore the K_{var} needs to be estimated with a limited collection of option prices. The method of valuing the fair delivery value of future realized volatility in real-world context is actually very similar to the model-free method used in the pricing of volatility indices (Brière et al., 2010). Thus, the values of the VSTOXX sub-indices given in Equation (2.5) represent the fair delivery values of future realized volatility for given option maturities and correspondingly the values of the VSTOXX main indices given in Equation (2.6) represent the fair delivery values of future realized volatility for fixed time periods (i.e. the squared value of the common 30-day VSTOXX index represents the fair value of K_{var} in a period of next 30 days for the EURO STOXX 50 index).

2.2 Modern Portfolio Theory

The Modern Portfolio Theory (MPT) is a mathematical portfolio selection approach proposed by Harry Markowitz (1952), which establishes the foundation for portfolio optimization faced by any investor and is therefore an important advance in the mathematical modelling of finance. The fundamental concept of MPT is that an investor should not examine the returns and risks of investments in his portfolio separately, but rather consider how the changes in the investment assets' prices contribute to the price changes of the whole portfolio. In this way, the investor can create a portfolio which can achieve a lower risk than any of the individual investment assets alone.

Modern Portfolio Theory, as well as any other theory, stands upon a number of underlying assumptions; (1) investors are rational in nature and behave in manner to maximize their utility, (2) there are no taxes or other transaction costs, (3) returns of the assets are normally distributed, (4) investors can lend or borrow unlimited amount of capital at a risk-free rate, (5) investors are willing to accept more risk only if they are compensated by higher expected return (investors are risk-averse), (6) analysis is based on a single period, (7) all investors have the exact same information of their investments (information is symmetric), and thus any information at any time is incorporated to the prices of securities (Beyhaghi & Hawley, 2013). However, many of these assumptions do not hold in real life context, and thus the MPT has received a lot of criticism since it was first published.

In this chapter, we will discuss topics in the field of MPT that are the most relevant in the context of this research. Firstly, we will analytically discuss the general mean-variance portfolio selection approach proposed by Markowitz (1952). Second, we will analyze the implications when a risk-free asset is added to the investment opportunity set. Lastly, we will examine different types of risk associated with investments and demonstrate the effect of diversification on risk.

2.2.1 Markowitz model of portfolio selection

Next, we will analytically derive the parameters needed for the Markowitz model. Let $P_{i,t}$ denote the price of an asset i at time t and $P_{i,t-1}$ denote the price of an asset i at time $t-1$. We shall define returns as simple returns[4] as follows:

$$r_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \tag{2.13}$$

Now, let $R_i : \Omega_i \to \Omega_i$ with $i = 1, \dots, N$ be discrete random variables and Ω_i be finite sample spaces consisting returns of a single asset i such that $\Omega_i = \{r_{i,1}, r_{i,2}, \dots, r_{i,T}\}$. We define each return as an event and denote the probability of

[4] Why not logarithmic returns? Although logarithmic returns have useful properties, according to Miskolczi (2017) and Dorfleitner (2003) this is a wrong procedure since logarithmic returns work only as an approximation of simple returns and are not asset-additive (i.e. $\mathbb{E}[R_p]$ cannot be represented as a linear combination of expected logarithmic returns). Still, it is common for some research papers to express returns as logarithmic returns in the context of traditional Markowitz portfolio model. If logarithmic returns are preferred, there exists an alternative extension of the Markowitz portfolio model related to the "geometric mean frontier" and the maximization of the geometric mean.

a such event as $\mathbb{P}(R_i = r_{i,t})$. Then the expected return of an asset i can be defined as:

$$\mathbb{E}[R_i] = \sum_{t=1}^{T} r_{i,t}\,\mathbb{P}(R_i = r_{i,t}) \tag{2.14}$$

Since the probabilities of the events in Ω_i are generally unknown, it is essential to create an estimator. If we let $n_{\{r_{i,t}\}}$ represent the frequency of an event $r_{i,t}$, we can set $\left\{\mathbb{P}(R_i = r_{i,t}) = n_{\{r_{i,t}\}}/T\,, \forall r_{i,t} \in \Omega_i : t = 1, \dots, T; i = 1, \dots, N\right\}$. Now, we can estimate Equation (2.14) with an arithmetic average:

$$\mathbb{E}[R_i] = \sum_{r_{i,t} \in \Omega_i} r_{i,t} \frac{n_{\{r_{i,t}\}}}{T} = \frac{1}{T} \sum_{t=1}^{T} r_{i,t} \tag{2.15}$$

Similarly, the expected return of a portfolio is defined as a linear combination of expected returns of the assets in the portfolio. Let ω_i denote the weight of a single asset i in a portfolio of N assets such that $\sum_{i=1}^{N} \omega_i = 1$. We shall define a new random variable $R_p : \Omega \to \mathbb{R}$ from the previously defined random variables $R_1, \dots, R_N$ such that $R_p = \sum_{i=1}^{N} \omega_i R_i$ and Ω is the sample space consisting the expected returns of the assets. Since for random variables $X_1, \dots, X_n$ and for constants $a_1, \dots, a_n \in \mathbb{R}$ it holds $\mathbb{E}[\sum_{t=1}^{n} a_t X_t] = \sum_{t=1}^{n} a_t \mathbb{E}[X_t]$, the expected return of a portfolio of N assets can be defined as:

$$\mathbb{E}[R_p] = \mathbb{E}\left[\sum_{i=1}^{N} \omega_i R_i\right] = \sum_{i=1}^{N} \omega_i \mathbb{E}[R_i] \tag{2.16}$$

For notational simplicity, it is convenient to use a matrix notation. Let[5] $\omega = [\omega_1 \quad \cdots \quad \omega_N]^\top$ and $\mu = [\mathbb{E}[R_1] \quad \cdots \quad \mathbb{E}[R_N]]^\top$ be N-dimensional column vectors of individual asset's weights and expected returns respectively. Then we can define the expected return of a portfolio with matrix notations as follows:

$$\mathbb{E}[R_p] = \omega^\top \mu \tag{2.17}$$

The MPT uses variance as a proxy for risk. For a random variable X, variance is defined as $\sigma_X^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, and thus we can define the portfolio variance as:

$$\sigma_p^2 = \mathbb{E}\left[\mathbb{E}[R_p]^2\right] - \mathbb{E}\left[\mathbb{E}[R_p]\right]^2 \tag{2.18}$$

[5] The symbol $\top$ denotes transpose throughout this research.

Using Equation (2.16), we can expand the above equation to:

$$\sigma_p^2 = \mathbb{E}\left[\left(\sum_{i=1}^{N} \omega_i \mathbb{E}[R_i]\right)^2\right] - \mathbb{E}\left[\sum_{i=1}^{N} \omega_i \mathbb{E}[R_i]\right]^2 \tag{2.19}$$

By using the fact that for random variables $X_1, \ldots, X_n$ and for a constant $a \in \mathbb{R}$ it holds that $\mathbb{E}[\sum_{l=1}^{n} X_l] = \sum_{l=1}^{n} \mathbb{E}[X_l]$ and $\mathbb{E}[aX_l] = a\mathbb{E}[X_l]\ \forall l = 1, \ldots, n$, we have:

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{h=1}^{N} \omega_i \omega_h \big(\mathbb{E}\big[\mathbb{E}[R_i]\mathbb{E}[R_h]\big] - \mathbb{E}\big[\mathbb{E}[R_i]\big]\mathbb{E}\big[\mathbb{E}[R_h]\big]\big) \tag{2.20}$$

By regrouping the terms and using the fact that for random variables X and Y covariance is defined as $\sigma_{X,Y} = \sigma_{Y,X} = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ and variance σ_X^2 is defined as previously, we have:

$$\sigma_p^2 = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{h=1 \\ h\neq i}}^{N} \omega_i \omega_h \sigma_{i,h} \tag{2.21}$$

If we use the fact that $\sigma_X^2 = \sigma_{X,X}$, we can simplify the above equation as:

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{h=1}^{N} \omega_i \omega_h \sigma_{i,h} \tag{2.22}$$

Again, it is convenient to use a matrix notation for simplicity. If we let $\sum_p$ be a positive definite N x N covariance matrix[6] of the asset returns and ω be an N-dimensional column vector of individual asset's weights as described previously, we can define the standard deviation of a portfolio with matrix notations as follows:

$$\sigma_p = \sqrt{\sigma_p^2} = \sqrt{\omega^\top \textstyle\sum_p \omega} \tag{2.23}$$

Where:

$$\textstyle\sum_p = \begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,N} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1} & \cdots & \sigma_{N,N} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1,N} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1} & \cdots & \sigma_N^2 \end{bmatrix}$$

As a result, we can see that the standard deviation (or variance) of a portfolio is a function of covariances and weights of the component assets. Alternatively, we

[6] $\sum_p$ can be estimated as $\widehat{\sum}_p = \frac{1}{N-1}\sum_{t=1}^{T}(R_t - \mu)(R_t - \mu)^\top$, where $R_t = [r_{1,t} \quad \cdots \quad r_{N,t}]^\top$.

could say, instead of covariances, that the portfolio variance is a function of variances and correlations of the component assets, since $\sigma_{X,Y} = \rho_{X,Y}\sigma_X\sigma_Y$, where $\rho_{X,Y} \in [-1,1]$ is the Pearson correlation coefficient of the random variables X and Y. However, using the standard deviation of a portfolio as a proxy for risk is only valid if it is assumed that $R_i \sim \mathcal{N}(\mathbb{E}[R_i], \sigma_i^2)\ \forall i = 1, \dots, N$, i.e. there are no excess skewness and kurtosis (tail-risk). Also, we can now see that by varying the assets' weights in ω, we can obtain different combinations of $\mathbb{E}[R_p]$ and σ_p. Therefore, we can form a subspace with an infinite number of possible portfolios that an investor can achieve with different combinations of weights in ω in the risk-return (σ-$\mathbb{E}[R]$) space (Markowitz, 1952). This subspace is usually called as the feasible set. The frontier of the feasible set can be found by minimizing the variance of a portfolio for given levels of return by varying the weights in ω. When considering a two-asset portfolio given $\omega_i \geq 0$ for $i = 1,2$ all the possible combinations of feasible portfolios are located within the frontier (Markowitz, 1952). As we increase the number of assets in our investment opportunity set to over 2 assets ($N \geq 3$) or allow for short selling, the feasible set becomes a "cloud" of points located inside the frontier. The lower the cross correlations between assets in the portfolio are, the more concave the frontier is, and therefore the payoff from diversification increases from the enlargement of the investment opportunity set (Elton, Gruber, Brown, & Goetzmann, 2013, p. 76–77). Since there is a diminishing marginal return to risk, every increase in risk results in a smaller amount of increase in return, and as a result, the frontier takes a shape of a hyperbola. This frontier is usually called as the minimum-variance frontier, and it consists of all the portfolios in the risk-return space that by varying different values of $\mathbb{E}[R_p]$ solve the following minimization program[7]:

$$\begin{aligned} M\text{–}V\ Frontier = \min_{\omega \in \mathbb{R}^N} \sigma_p &= \sqrt{\omega^\top \Sigma_p\, \omega} \\ s.t.\ \mathbb{E}[R_p] &= \omega^\top \mu \\ \sum_{i=1}^{N} \omega_i &= 1 \end{aligned} \tag{2.24}$$

Graphical representation of the minimum-variance frontier when short selling is allowed is provided in Figure 3. The above program represents a situation where short selling is allowed, although we could add additional constraints on ω, if needed. For example, if we wanted to forbid short selling, we could set $\omega_i \geq 0$ for all $i = 1, \dots, N$. If we allow unlimited short selling, there are no boundaries on the maximum and minimum expected portfolio return, or on the maximal portfolio standard deviation that a portfolio in the feasible set could achieve, as we can always take infinitely large, long and short positions on the condition that the sum of the weights equals 1. On the other hand, if we forbid short selling,

[7] Many studies claim this set as the "efficient frontier". However, following Corner and Mayes (1983, p. 25), this set should not be considered efficient as it does not necessarily fulfill the criteria of maximizing the returns for a given level of risk.

then the maximum and minimum expected returns of the assets define the maximum and minimum expected returns achievable on the feasible set. However, it does not depend on if we forbid short selling or add any other constraints on the weights of the assets, we are always able to find the portfolio included in the feasible set that has the lowest variance of the whole feasible set. This portfolio is usually called the global minimum variance portfolio (GMV), and it is located in farthest to the left on the minimum variance frontier. In contrast to efficient portfolios, the GMV portfolio is completely independent of the expected returns of the assets (Frahm & Memmel, 2010). Graphical representation of the GMV portfolio on the minimum-variance frontier is displayed in Figure 2. The GMV portfolio can be obtained by minimizing the standard deviation[8] of a portfolio as follows:

$$GMV = \min_{\omega \in \mathbb{R}^N} \sigma_p = \sqrt{\omega^\top \Sigma_p \, \omega} \qquad (2.25)$$
$$s.t. \sum_{i=1}^{N} \omega_i = 1$$

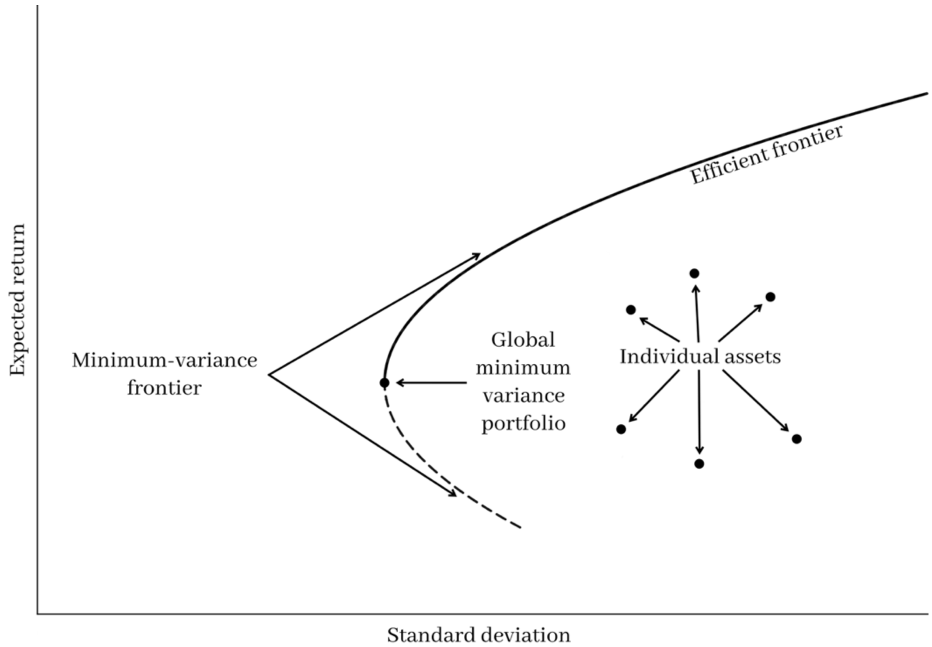

Figure 3 Markowitz minimum-variance frontier of risky assets, short selling allowed

The MPT assumes that investors are both rational and risk averse. Therefore, a rational investor will choose his optimal portfolio from the feasible set, that minimizes the level of risk for a given level of expected return and offers the highest level of expected return for a given level of risk. Portfolios that meet these 2 conditions are said to be efficient (Markowitz, 1959, p. 140). If we observe the Figure 3, we notice that an investor would always obtain the most favorable risk-return relationship by choosing his portfolio from the upper part of the minimum-

[8] One could alternatively minimize the variance of the portfolio. This is just a question of whether one wants to work in $\sigma - \mathbb{E}[R]$ space or in $\sigma^2 - \mathbb{E}[R]$ space.

variance frontier. The upper part of the frontier starting from the GMV portfolio is called the efficient frontier, and it consists of all the efficient portfolios that maximize the expected return for a given level of risk and minimize the risk for a given level of expected return. All portfolios below the efficient frontier are considered to be inefficient, since an investor could achieve a higher expected return for a given level of risk by choosing a portfolio from the efficient frontier (Markowitz, 1959, p. 129). All the efficient portfolios, and therefore the efficient frontier, can be obtained by adding the constraint $\mathbb{E}[R_p] \geq \mathbb{E}[R_{GMV}]$ in with minimization program of Equation (2.24), where $\mathbb{E}[R_{GMV}]$ represents the expected return of the GMV portfolio. Graphical representation of the efficient frontier when short selling is allowed can be seen from Figure 3, where the efficient frontier is the bolded part of the minimum-variance frontier. By looking at the Figure 3, we can easily observe that an increase in risk leads to a smaller increase in expected return

Now, we have defined the optimal set of portfolios a rational investor would choose from, yet we have not focused on which portfolio he would base his final decision to. To gain an answer to this problem, we need to examine investors' preferences. Moving from left to right on the efficient frontier, the expected return increases but so does the risk. Some people might want to chase high returns while consciously taking high risk, whereas some people would want to get rid of all the risks to keep their wealth as safe as possible. Therefore, the investor's portfolio selection problem becomes dependent on his attitude in risk and the relative return. To model the decision making of an investor, it is assumed that a rational investor chooses to maximize his expected utility under uncertainty. In most cases the exact utility function is unknown, but for practical reasons it is convenient to approximate the following quadratic utility function (Bodie et al., 2018, p. 158):

$$U = \mathbb{E}[R_p] - \frac{1}{2} A \sigma_p^2 \tag{2.26}$$

Where A is the risk aversion coefficient. Investors with $A = 0$ are called risk-neutral, which means that they are completely indifferent with the level of risk. $A > 0$ represents a risk-averse investor who, when faced with to similar investments, prefers the one with lower risk. On the other hand, an investor with $A < 0$ is called as risk lover, who in particular enjoys taking more risk. (Bodie et al., 2018, p. 160–162). Equally preferred portfolios associated with the same level of relative utility lie in the σ– $\mathbb{E}[R]$ space on a curve called indifference curve. For investor with $A = 0$, the indifference curve is a straight line, as for investor with $A > 0$ or $A < 0$ the indifference curve is concave or convex respectively. Investors choosing among competitive portfolios will choose the one associated with the highest level of utility and thus the highest indifference curve. The higher the risk aversion of an investor, the greater the requirement for increase in expected return to compensate for the increase in risk. Therefore, more risk-averse investors have steeper indifference curves, and thus their optimal portfolios are located further on the left than the optimal portfolios of the less risk-averse investors.

Graphically, indifference curves located higher corresponds to higher levels of utility. The investor's maximum utility and thus the optimal portfolio is found at the point where the indifference curve is tangent to the efficient frontier.

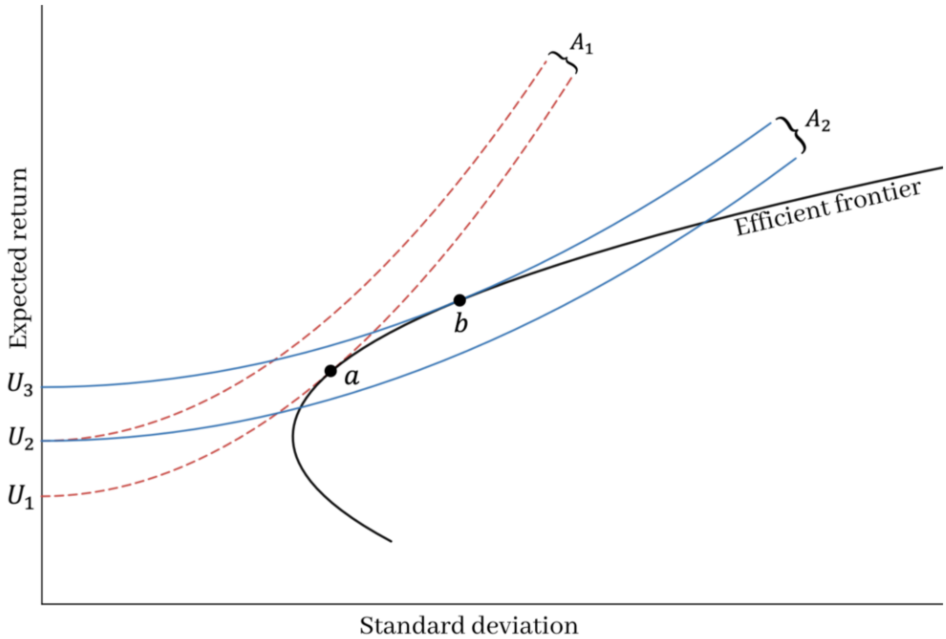

Figure 4 Indifference curves of investors. $A_1, A_2 > 0$ are risk aversion coefficients in which $A_1 > A_2$.

Figure 4 presents the portfolio selection problem for investors with two types of risk aversion. In this case $A_1, A_2 > 0$ and $A_1 > A_2$, therefore both investors are risk averse and the investor with risk aversion of A_1 (investor 1) is more risk averse than the investor with risk aversion of A_2 (investor 2). For investor 1, the indifference curve with utility level U_2 is out of reach, and thus the highest possible utility he can achieve is obtained by choosing the portfolio a with a lower utility level U_1. Since investor 2 is less risk averse, he is willing to hold more risk in order to obtain a higher expected return. This makes his utility curve less steep, and then even the utility level U_2 does not give him the maximum possible satisfaction. By choosing the portfolio b with a higher utility level U_3, the investor 2 maximizes his achievable utility.

2.2.2 Efficient frontier with a risk-free asset

MPT was further improved by Tobin (1958) who examined investors preferences in holding a combination of risk-free and risky assets and Sharpe (1964) who adapted Tobin's idea to the MPT framework. In the previous chapter, we defined efficient frontier for risky assets. Now, we will define the efficient frontier when there is a risk-free asset[9] involved.

[9] In real life, we consider government bonds or treasury bills as risk-free assets since they are backed by governments and therefore considered as safe. However, investor should be cautious

When an investor can invest in a risk-free asset and risky assets, we can denote the expected return as a linear combination expected returns of risk-free asset and risky assets. We assume that an investor has 100% of his wealth invested. Let ω_p denote the percentage of total wealth invested in risky assets and therefore let $(1-\omega_p)$ represent the percentage of total wealth invested in a risk-free asset, of which return we denote as R_f. We can now define the expected return of the combined investments in risky- and risk-free assets as follows:

$$\begin{aligned}\mathbb{E}[R_C] &= (1-\omega_p)R_f + \omega_p\mathbb{E}[R_p] \\ &= R_f + \omega_p(\mathbb{E}[R_p] - R_f)\end{aligned} \tag{2.27}$$

Since the return on risk-free asset is considered to be certain, the risk-free asset has a standard deviation of zero and thus a covariance of zero with risky assets (Elton et al., 2013, p. 82). Therefore, the standard deviation of our combination of risk-free assets and risky assets becomes $\sigma_C = \omega_p\sigma_p$. Now, we can use the fact that $\omega_p = \frac{\sigma_C}{\sigma_p}$ and rewrite the Equation (2.27) as:

$$\mathbb{E}[R_C] = R_f + \sigma_C\frac{\mathbb{E}[R_p] - R_f}{\sigma_p} \tag{2.28}$$

This is known as the Capital Allocation Line (CAL), which graphs all the possible combinations of risky and risk-free assets in the σ– $\mathbb{E}[R]$ space. The slope of the CAL is known as the Sharpe ratio, which represents the additional return for each unit of increase in risk by measuring the ratio of portfolio's excess return and standard deviation. Keeping the risk-free rate constant, we can choose different portfolios from the feasible set and draw CAL to represent all possible combinations of risk-free and risky assets. Portfolios that are located between $(0, R_f)$ and the risky portfolio are achievable by lending money with a risk-free rate, and portfolios which are located after the risky portfolio are achievable by borrowing money with the risk-free rate (which is the same as short selling the risk-free asset). In other words, investors are able to apply leverage in order to increase their expected returns as well as risk (Elton et al., 2013, p. 83–84). Therefore, adding a risk-free asset to the investment opportunity set also changes the shape of the efficient frontier, and therefore it is essential to differentiate between the new efficient frontier and the efficient frontier of risky assets. If there are differences in the lending- and borrowing rates, then the slope of the CAL differs before and after the risky portfolio. Also, if investors are not allowed to borrow at a risk-free rate, then the new efficient frontier is not a straight line after the risky portfolio anymore.

Since there exists a diminishing marginal return to risk, it is possible to find a portfolio where the CAL is tangent to the efficient frontier. The investor cannot rotate the CAL any further, since by the definition of the feasible set there are no

when determining the risk-free rate, as the European debt crisis in 2012 showed that "risk-free" securities issued by some governments may not be as safe as one might had previously thought.

portfolios below the efficient frontier. This point corresponds to a portfolio on the efficient frontier where the Sharpe ratio is at maximum, therefore by choosing this portfolio an investor can obtain the most optimal return/risk combination of risky assets. This portfolio, in which the Sharpe ratio is at maximum, is known as the tangency portfolio. The tangency portfolio depends on the risk-free rate: If the risk-free rate increases, the new tangency portfolio will be further on the right on the efficient frontier, and vice versa. Investors who are more risk averse, would hold portfolios along the segment between the $(0, R_f)$ and the tangency portfolio by lending a part of their portfolio with the risk-free rate. For example, an investor with a very high risk-aversion would hold his all or most of his money on the risk-free asset. On the other hand, investors who want more risk would hold their portfolios on the segment after the tangency portfolio by applying leverage by borrowing money with a risk-free rate (and thus creating a short position) and investing it in the tangency portfolio.

When there are homogenous expectations among the investor community and the investment opportunity set is the whole market, the optimal CAL for all investors is called as the Capital Market Line (CML) (Elton et al., 2013, p. 302). The CML can be found by maximizing the slope of the CAL, leading to a tangency portfolio in which we now denote its expected return and standard deviation with $\mathbb{E}[R_M]$ and σ_M respectively to represent it as the market portfolio. Therefore, CML can be defined by modifying Equation (2.28) as:

$$\mathbb{E}[R_C] = R_f + \sigma_C \frac{\mathbb{E}[R_M] - R_f}{\sigma_M} \tag{2.29}$$

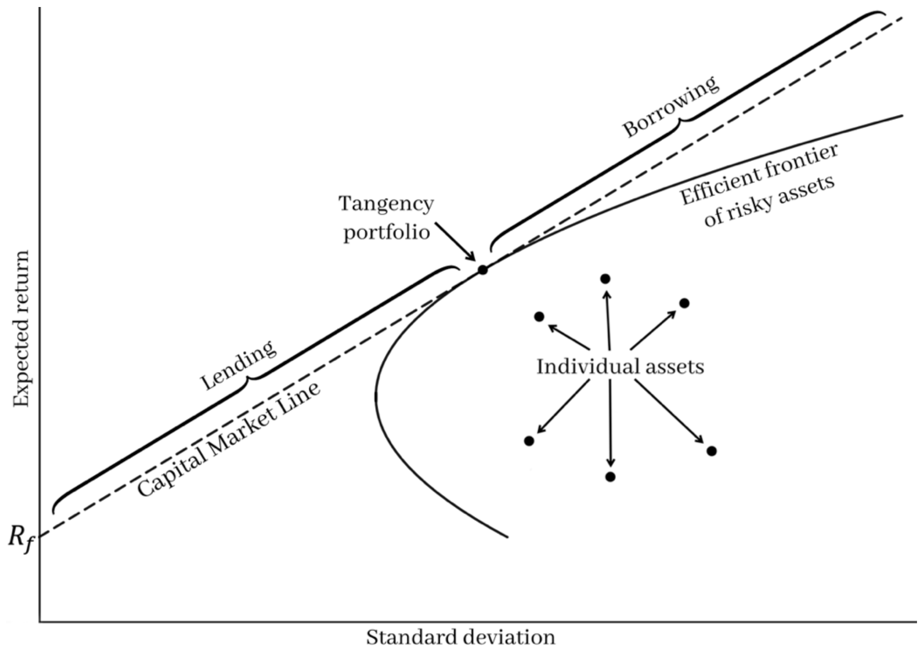

Figure 5 Capital Market Line in σ– $\mathbb{E}[R]$ space

Figure 5 displays the graphical representation of the CML and the tangency portfolio in σ– $\mathbb{E}[R]$ space. To find the tangency portfolio, we can use the following maximization program to find the vector of weights ω that maximizes the Sharpe ratio:

$$Tangency\ portfolio = \max_{\omega \in \mathbb{R}^N} \frac{\mathbb{E}[R_p] - R_f}{\sigma_p} = \frac{\omega^\top \mu - R_f}{\sqrt{\omega^\top \sum_p \omega}} \tag{2.30}$$

$$s.t. \sum_{i=1}^{N} \omega_i = 1$$

2.2.3 Decomposition of portfolio risk

Sharpe (1964) divided the concept of risk into two parts: systematic and unsystematic risk. Systematic risk (i.e. market risk or non-diversifiable risk) consists of factors affecting the economy, such as recessions, inflation, or changes in interest and exchange rates. Unsystematic risk (i.e. asset specific risk or diversifiable risk), on the other hand, is related to specific targets, such as companies or industries. The essential concept of portfolio diversification states that unsystematic risks can be substantially reduced by including non-perfectly correlated risky assets to the portfolio. On average, portfolio risk falls with diversification, but is limited to the common sources of risks (systematic risk). (Bodie et al., 2018, p. 194–195.) Figure 6 presents the graphical relation between the standard deviation and the number of assets included in a portfolio. It is easy to see that adding more assets to the portfolio has diminishing marginal reduction in standard deviation: as the number of assets increases, there are correspondingly lower diversification benefits obtainable measured by the reduction in the standard deviation.

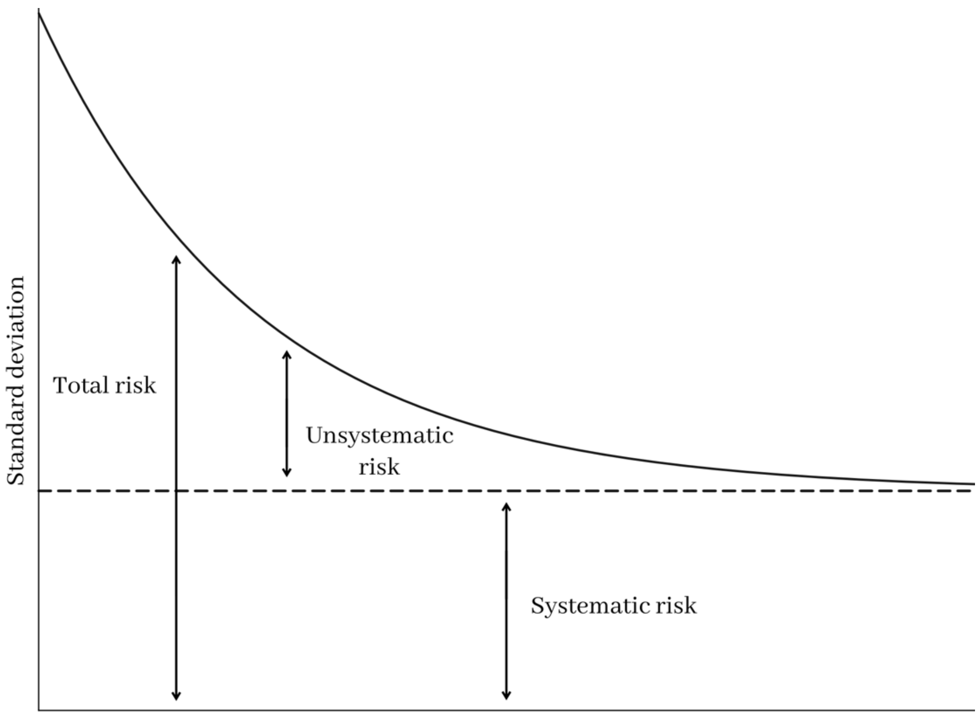

Figure 6 Systematic and unsystematic risk

The elimination of the unsystematic risk as the number of assets grows large can be easily demonstrated using previously defined equations. If we suppose that we have an equally weighted portfolio such that $\omega_i = \frac{1}{N}$ for all $i = 1, \ldots, N$, we can vary the Equation (2.21) and express the portfolio variance as follows:

$$\sigma_p^2 = \sum_{i=1}^{N} \frac{1}{N^2}\,\sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{h=1 \\ h \neq i}}^{N} \frac{1}{N^2}\sigma_{i,h} \tag{2.31}$$

By varying the number of assets and obtaining the corresponding amount of covariances, it can be seen that the amount of covariances follows a sequence of which general term is a second order polynomial $N(N-1)$. Now, we are able to define the average covariance of the assets and rearrange the Equation (2.31) as follows:

$$\begin{aligned} \sigma_p^2 &= \frac{1}{N}\sum_{i=1}^{N}\frac{\sigma_i^2}{N} + \frac{N-1}{N}\sum_{i=1}^{N}\sum_{\substack{h=1 \\ h \neq i}}^{N}\frac{\sigma_{i,h}}{N(N-1)} \\ &= \frac{1}{N}\overline{\sigma^2} + \frac{N-1}{N}\overline{Cov} \end{aligned} \tag{2.32}$$

Where $\overline{\sigma^2}$ and $\overline{Cov}$ represent average variance and covariance of the assets respectively. If we assume that $\sigma_i^2 < \infty$ for all $i = 1, \ldots, N$ and that assets are not necessarily uncorrelated by defining $\{\sigma_{i,h} \in \mathbb{R}\ \forall i, h \in 1, \ldots, N : h \neq i\}$, by evaluating the limit as the number of assets approaches infinity, we obtain:

$$\lim_{N\to\infty} \sigma_p^2 = \lim_{N\to\infty} \frac{1}{N}\overline{\sigma^2} + \left(1 - \frac{1}{N}\right)\overline{Cov} = \overline{Cov} \tag{2.33}$$

It can be seen as the number of assets approaches infinity, the contribution to the portfolio variance of the variance of individual assets converges to zero. However, the contribution of the covariance terms converges to the average covariance as the number of assets approaches infinity. After the point where adding new assets to the portfolio do not have effect on the variance of the portfolio anymore, only the risk caused by the covariance terms of assets remains, which we define as the systematic risk. Thus, we can see that by increasing the number of assets in a portfolio, the variance of the portfolio starts to diminish until all the unsystematic risk is eliminated. It could be possible that $\overline{Cov} \to 0$ as the amount of assets approaches infinity, although it would be very unlikely as it would require there to be a substantial amount of assets with negative covariances[10] which

[10] Could the negative covariances (or correlations) cause the portfolio variance to be negative within this model? It follows from the definition of the covariance matrix that any N x N covariance matrix Σ is positive semi-definite, thus $\varphi^{\top}\Sigma\,\varphi \geq 0\ \forall\varphi \in \mathbb{R}^N$ and by applying the requirement of positive definiteness it holds that $\omega^{\top}\Sigma_p\,\omega = \sigma_p^2 > 0\ \forall\omega \in \mathbb{R}^N$.

are rare in the investment universe. However, we could have assumed otherwise: If we consider the assets in the portfolio to be independent, such that their cross correlations and thus covariances are zero, there would be no risk remaining as the number of assets approach infinity as $\overline{Cov} \to 0$ (Elton et al., 2013, p. 56). Since these additional assumptions are very unlikely to hold in the real market environment, it is rather safe to assume that $\sigma_p^2 \to \overline{Cov} > 0$ as $N \to \infty$.

2.3 Previous research

Previous literature has shown the cross correlations of different asset classes to converge to one on extreme market conditions. Although Gueyie and Amvella (2006) as well as Mull and Soenen (1997) conclude that alternative asset classes such as hedge funds and REITs can complement traditional portfolios, Platanakis et al. (2019) show that the diversification benefits of alternative investments were statistically insignificant during the 2007-2009 credit crisis. Szado (2009) conducted a case study considering portfolio diversification during the 2008 financial crisis. He found that most of the cross correlations between equity, bonds, high yield bonds, hedge funds, private equity and real estate were dramatically higher in 2007 and 2008 than in the 2004 to 2006 period. Also Chow et al. (1999) note that assets classes tend to become more volatile and correlated in turbulent markets. As a result of the increased correlation between asset classes, diversification benefits of traditional portfolios are found to diminish in the times of market distress, when they are needed the most.

The potential diversification benefits of adding volatility derivatives into the investment opportunity set comes from the empirical findings that volatility and equity returns tend to move in opposite directions. Daigler and Rossi (2006) examine the consequences of including volatility as a separate asset class to S&P500 portfolio over the period from 1992 to 2002. They observe that the strong negative correlation between VIX and S&P500 indices offers significant diversification benefits in an equity portfolio. Guobuzaite and Martellini (2012) discovered that cross correlations of the VSTOXX and EURO STOXX 50 indices as well as the VIX and S&P500 indices stays negative regardless of the market conditions. They also found that the cross correlations between volatility and equity indices tends to increase even further when there is high volatility in the market. Although long exposure in volatility typically leads to negative returns in the long term, it may provide significant protection in market downturns (Szado, 2009). There is also evidence that the benefits of volatility diversification seem to be larger with VSTOXX futures than VIX futures within European investment portfolios (Signori, Malongo, Fermanian, & Brière, 2012; Stanescu & Tunaru, 2013).

According to Alexander and Korovilas (2011), the increasing cross correlations of traditional asset classes encourages investors to seek alternative means of diversification. The first study to apply an optimization method in the context of VIX products was by Brière, Signori and Burgues (2010). They examine optimal minimum risk portfolios in the mean-variance framework over the period from 1990 to 2008. Their results indicate that the Sharpe ratio of an equity portfolio can be increased significantly by allocating a part of the portfolio to volatility strategies. However, Alexander and Korovilas (2011) show that the advantages of adding VIX futures to a portfolio are outweighed by high transaction costs and negative cost of carry during the periods of low volatility. Therefore, to obtain positive returns with a long position in volatility futures, an investor should time his volatility trades carefully.

Chen, Chung and Ho (2011) conduct mean-variance spanning tests on VIX-index and tradeable VIX-related derivatives using four Fama-French US portfolios as benchmark indices. Their results demonstrate that investors who include VIX futures in their investment opportunity set are able to enlarge their mean-variance frontier. Additionally, they show that the Sharpe ratios of tangency portfolios increase significantly after adding VIX-related assets to the portfolio. Yet, any mean-variance spanning tests have not been conducted in the VSTOXX framework.

Previous research considering volatility products as a diversification tool has focused mainly on the VIX index and its derivatives. However, in recent years research considering the VSTOXX index and its derivatives as a diversification tool for European investors has embarked. Guobuzaite and Martellini (2012) examine the risk/return characteristics of equity portfolios combined with long exposure to volatility indices and derivatives. They employ a substantially large sample covering the period from January 1999 to April 2011. On the index level, they find that the maximum Sharpe ratio of an equity/volatility portfolio can be achieved by 70% allocation to EURO STOXX 50 index and 30% allocation to VSTOXX index. Implementing the same analysis with VSTOXX futures over the period from April 2008 to April 2011, they find that 3-month futures outperform the 1-month futures in terms of diversification. Additionally, they achieve similar results with VSTOXX futures as they had with the VSTOXX index: the Sharpe ratio maximizing allocation between EURO STOXX 50 and VSTOXX futures is also achieved by allocating 70% to EURO STOXX 50 index and 30% to VSTOXX 3-month futures.

The study of Stanescu and Tunaru (2013) examines the addition of volatility products to EURO STOXX 50 and VIX indices covering the period from April 2009 to February 2012. Similar to the results of the Guobuzaite and Martellini (2012), they discover that the optimal portfolio composition to maximize Sharpe ratio with VSTOXX and EURO STOXX 50 indices could be achieved with 20% allocation to VSTOXX index and 80% allocation to EURO STOXX 50 index. Their results demonstrate that adding 2.5%–10% exposure to VSTOXX futures in a portfolio consisting of EURO STOXX 50 index increases the average portfolio return and decreases the portfolio volatility compared to the EURO STOXX 50 index alone. They also notice that adding EUR-denominated investment grade bonds to the EURO STOXX 50 index/VSTOXX futures portfolio can further increase the Sharpe ratio, and conclude that the best composition for a portfolio would be a mix of equity, bonds and volatility derivatives with first or second maturity. Also, Signori et al. (2012) evaluate the addition of VIX and VSTOXX futures to European equity portfolios constructed from MSCI EMU index and country specific equity indices covering the period from January 1999 to December 2010. Their results indicate that the optimal portfolio allocation to minimize the 95% modified CVaR is achieved with 54,5% allocation to MSCI EMU index and 45,5% allocation to 3-month VSTOXX futures. In addition, they find that the maximum monthly loss is more than halved after introducing 3-month VIX or VSTOXX futures to European equity portfolios.

3 DATA AND METHODOLOGY

3.1 Data

The dataset consists of daily closing prices of different asset classes in EUR collected from Refinitiv Eikon and Bloomberg Terminal. Each asset class covers a ten-year period from 3.1.2011 to 31.12.2020 and is represented by a "Total return" or a "Gross return" index, which does not account for taxes and reinvests any dividends or cash distributions. The dataset is assembled in Microsoft Excel and imported to MATLAB for further processing[11].

Since we cannot include every possible asset in our dataset, we need to choose proxies[12] to represent the investment universes of chosen asset classes. Contrary to previous studies, equities are represented by STOXX Europe 600 Gross Return Index (SXXGR), which consists of 600 European stocks representing large, mid and small capitalizations. This decision is based on the fact, that the STOXX 600 Europe Index represents a larger set of European stocks from multiple capitalizations as compared to the EURO STOXX 50 Index. Also, the aforementioned indices share almost identical cross correlations with the VSTOXX Index, and thus the STOXX 600 Index suits well for diversification with VSTOXX futures Shore, 2018). Bonds are represented by Bloomberg Barclays Pan-European Aggregate Total Return Index (LP06TREU), which tracks fixed-rate, investment grade treasury, corporate and government bonds issued in European currencies with an option adjusted duration of 8,1. Commodities are represented by Bloomberg Commodity Index Euro Total Return (BCOMEUTR), which is composed of futures contracts on different physical commodities in sectors such as energy, grains, industrial metals, precious metals, softs and livestock. High yield bonds are represented by Bloomberg Pan-European High Yield Total Return Index (LP01TREU), which consists of fixed rate, non-investment grade corporate bonds issued in European currencies with an option adjusted duration of 4,1. Hedge funds are represented by HFRX Global Hedge Fund EUR Index (HFRXGLE), which represents the overall composition of the hedge fund universe. Real estate is represented by EURONEXT REIT Europe Gross Return Index (REITG), which is composed of real estate companies listed in European regulated markets. We choose to include three different VSTOXX futures in our research to represent short-, mid- and long-term maturities of futures contracts. Short-, mid- and long-term VSTOXX futures contracts are represented by VSTOXX 1 month, 3 month and 5 month continuous mini-futures contracts (FVSc1, FVSc3, FVSc5). Table 1 shows the proxies discussed above and sources for the asset classes.

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[11] All procedures discussed in this chapter are computed in MATLAB.

[12] It is important to note that these proxies (excluding VSTOXX futures) are in fact not tradeable.

Table 1 Description of asset classes

Asset class	Proxy	Ticker	Source
Equity	STOXX Europe 600 Gross Return Index	SXXGR	Refinitiv Eikon
Bonds	Bloomberg Barclays Pan-European Aggregate Total Return Index	LP06TREU	Bloomberg Terminal
Commodities	Bloomberg Commodity Index Euro Total Return	BCOMEUTR	Refinitiv Eikon
High Yield	Bloomberg Pan-European High Yield Total Return Index	LP01TREU	Bloomberg Terminal
Hedge Funds	HFRX Global Hedge Fund EUR Index	HFRXGLE	Bloomberg Terminal
Real Estate	EURONEXT REIT Europe Gross Return Index	REITG	Refinitiv Eikon
VSTOXX 1M Fut	Eurex VSTOXX MINI Volatility Index Future Continuation 1	FVSc1	Refinitiv Eikon
VSTOXX 3M Fut	Eurex VSTOXX MINI Volatility Index Future Continuation 3	FVSc3	Refinitiv Eikon
VSTOXX 5M Fut	Eurex VSTOXX MINI Volatility Index Future Continuation 5	FVSc5	Refinitiv Eikon

Daily closing prices of SXXGR, BCOMEUTR, REITG, FVSc1, FVSc3 and FVSc5 are retrieved from Refinitiv Eikon. There are no data available on LP06TREU, LP01TREU and HFRXGLE in Refinitiv Eikon, and therefore the daily closing prices of the previously mentioned assets are retrieved from Bloomberg Terminal. Finally, we compute the daily returns from the retrieved asset prices. Let $P_{i,t}$ and $P_{i,t-1}$ represent the daily closing price of an asset i at times t and $t-1$ respectively. Then the daily return of an asset i at time t can be expressed as follows:

$$r_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \tag{3.1}$$

Considering that an individual futures contract is traded only for a limited time, an investor needs to roll over the investment over the series of consecutive futures contracts in order to gain long exposure in the fixed maturity futures contracts. As a result, the continuous futures contracts have a distinct property from the rest of time series of different asset classes: their prices are not continuous. The price series of continuous VSTOXX futures retrieved from Refinitiv Eikon roll into new futures contract in the day after the expiration date, and in order to create a continuous time series from the VSTOXX futures continuous contracts, we propose a following solution: Expiration dates of the VSTOXX futures are downloaded from Eurex's website, and in each month the return of the rolling date (first date after the expiration date) is removed from the VSTOXX futures time series. For the year 2011, we use the Wednesday before the second last Friday of the month as an expiration date for each month, since Eurex's calendar does not cover the previously mentioned period. This procedure suits in our analysis, since we are not interested in the absolute values of asset prices, but rather in the daily percentage changes. Also, in this way the time series of VSTOXX futures represent holding a fixed maturity VSTOXX futures contract for a month, and then rolling to a new futures contract with the same maturity a day after the expiration date. We can also define this procedure in a more formal way. Let $F_{i,t}$ and $F_{i,t-1}$ represent the daily settlement prices of futures contract i at times t and $t-1$ respectively. Then the daily return of a futures contract i at time t can be expressed as follows:

$$r_{i,t}^{futures} = \frac{F_{i,t} - F_{i,t-1}}{F_{i,t-1}} \tag{3.2}$$

$$t = roll\ date \rightarrow r_{i,t}^{futures} = N/A$$

In order to determine whether VSTOXX futures have diversification benefits in European investment portfolio, we construct two benchmark portfolios with different exposures to separate asset classes. This is particularly interesting, since previous studies have reported there to be diversification benefits in adding VSTOXX futures to equity/bond portfolios, yet any mean-variance spanning tests have not been conducted in the VSTOXX framework. Also, the previous studies have not examined the diversification benefits of adding VSTOXX futures to multi-asset portfolios, in which it is reasonable to expect smaller diversification benefits from the VSTOXX futures. The following portfolios, which we will refer simply as Portfolio A and B, will be considered as our benchmark investment opportunity sets which will be compared when testing for mean-variance spanning and diversification measures over the full period from 3.1.2011 to 31.12.2020 and the 3-month 2020 stock market crash period from 3.2.2020 to 30.4.2020.

- **Portfolio A**: Equity and bonds
- **Portfolio B**: Equity, bonds, commodities, high yield bonds, hedge funds and real estate

3.2 Mean-variance spanning

The concept of mean-variance spanning was first introduced by Huberman and Kandel (1987). The idea is to test whether adding a set of N risky assets to an investment universe consisting only of a set of K risky assets expands the minimum-variance frontier. The N risky assets are usually called test assets and the K risky assets are called benchmark assets. We say that a set of K risky assets spans a larger set of K+N risky assets if the mean-variance frontier of the set of K risky assets is identical to the mean-variance frontier of the set of K+N risky assets. In this case, there are no diversification benefits in adding a set of N risky assets to the investment opportunity set consisting only of a set of K risky assets. Equivalently, if a set of K+N risky assets has a larger minimum-variance frontier than the set of K risky assets, then there are no spanning and therefore there are diversification benefits by adding a set of N risky assets to the investment opportunity set consisting the set K risky assets. Figure 6 shows the graphical representation of mean-variance frontiers of the sets of K and K+N risky assets when the set of K+N risky assets has a larger minimum-variance frontier than the set of K risky assets i.e. there are no spanning. In this study, the goal of the spanning tests is to examine whether adding VSTOXX futures to the investment opportunity set consisting of different European and global asset classes improves the portfolio diversification of an investor.

It is possible that there exists such a portfolio that is mean-variance efficient for both investment opportunity sets K and K+N. In this case, there exists a single mean-variance utility function in which there are no benefits in adding new risky assets to the investment opportunity set. A graphical representation of this is expressed in the Figure 2 where Portfolio w* lies on the mean-variance frontier but remains unchanged after expanding the set K with a set of N risky assets.

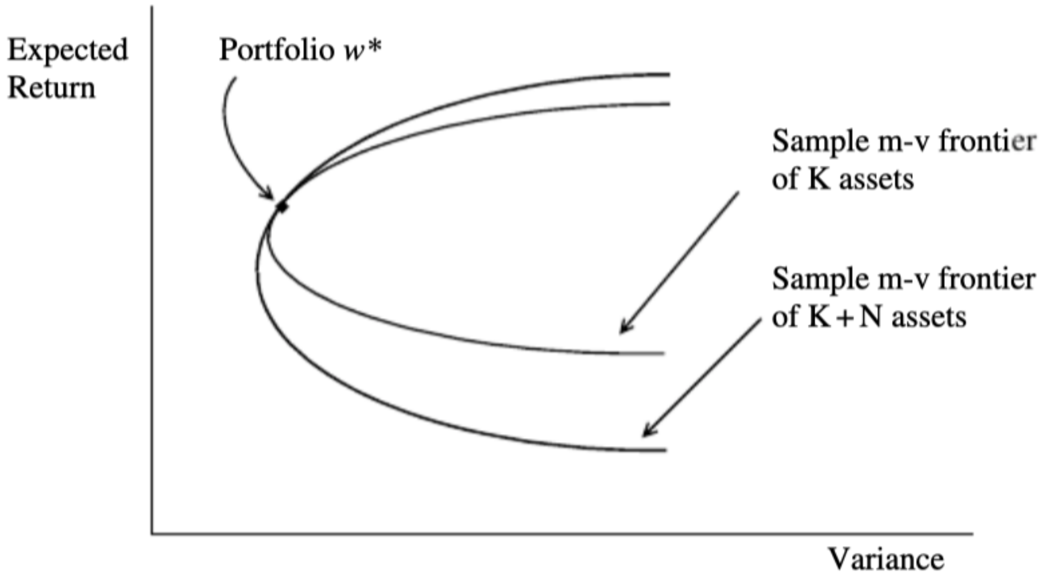

Figure 7 Spanning of mean-variance frontiers (Petrella, 2005)

According to Huberman and Kandel (1987) the OLS based method of mean-variance spanning tests can be described in the following way. Let $R_{2,t}$ be an N-dimensional column vector[13] of the returns on the set of the test assets N and $R_{1,t}$ be a K-dimensional column vector of the returns on the set of benchmark assets K at time t. In addition, let the intercepts α and residuals ε_t at time t be N-dimensional column vectors and the coefficients β be an N x K matrix. Returns of the set of test assets N are regressed on the returns of the set of the benchmark assets K with a multivariate linear regression model as follows:

$$R_{2,t} = \alpha + \beta R_{1,t} + \varepsilon_t,\ t = 1, \ldots, T \tag{3.3}$$

When N=1 as in our case, we can use the following univariate multiple linear regression model, where V refers to one of the test assets (VSTOXX 1M, 3M or 5M Future) and K to the number of assets in the set of benchmark assets (Portfolio A or B):

$$r_{V,t}^{futures} = \alpha + \sum_{i=1}^{K} \beta_i r_{i,t} + \varepsilon_t,\ t = 1, \ldots, T; V = 1M, 3M\ or\ 5M\ Fut \tag{3.4}$$

[13] Throughout this chapter, we refer N and K as the amount of test and benchmark assets respectively.

The null hypothesis of spanning is a joint hypothesis that α is equal to zero and the sum of coefficients β is 1. We can define δ as $\delta = 1_N - \beta 1_K$ where β is a N x K matrix of beta coefficients and 1_N and 1_K are N-and K-dimensional column vectors of ones respectively. Then 0_N represents an N-dimensional column vector and the null hypothesis can be defined as:

$$H_0: \alpha = 0_N \; and \; \delta = 0_N \tag{3.5}$$

The null hypothesis can be interpreted in the following way: $\alpha = 0_N$ tests if the distance between the two tangency portfolios formed from the sets K and K+N assets differ, whereas $\delta = 0_N$ tests whether there is a difference in the standard deviations of the two global minimum variance portfolios of the sets of K and K+N assets. If we are able to reject the null hypothesis, then the mean-variance frontier of the benchmark assets is unlikely to be larger than the mean-variance frontier of benchmark assets plus test assets. In other words, if we are able to reject the null hypothesis, then it is likely that the mean-variance frontier of the benchmark assets does not span the mean-variance frontier of the benchmark assets plus the test assets.

Next, we will present three different types of spanning tests to test our null hypothesis of Equation (3.5). All tests and their corresponding p-values will be obtained using MATLAB. When considering a regression setting, all dates with missing values will be removed from the dataset to reduce bias separately for both Portfolios A and B, whereas considering covariance matrices, all available data will be used by performing the covariance estimation using a "pairwise" method. Since ready packages to perform mean-variance spanning tests cease to exist, the following tests will be constructed by the author of this paper.

3.2.1 Finite sample test

For notational brevity, equation (3.3) can be expressed in a matrix form the in following way:

$$Y = XB + E \tag{3.6}$$

Where:

$Y = T$ x N matrix of $R_{2,t}$
$X = T$ x (K+1) matrix with its typical row as $[1 \quad R_{1,t}^{\mathsf{T}}]$
$B =$ (K+1) x N matrix with its typical column as $[\alpha \quad \beta]^{\mathsf{T}}$
$E = \; T$ -dimensional column vector of ε_t

Let $\hat{\Sigma}$ be the unconstrained maximum likelihood estimator of the covariance matrix of the residuals Σ and let $\hat{B}$ be the maximum likelihood estimator of B. Then according to Kan and Zhou (2001), $\hat{\Sigma}$ can be defined as:

$$\hat{\Sigma} = \frac{1}{T}\left(Y - X\hat{B}\right)^{\top}\left(Y - X\hat{B}\right) \tag{3.7}$$

Let $\tilde{\Sigma}$ be the constrained maximum likelihood estimator[14] of the covariance matrix of the residuals Σ by imposing the constraints $\alpha = 0_N$ and $\delta = 0_N$. In practice, we use the MATLAB function "lsqlin" to estimate constrained parameters for matrix B. Then we can define the parameter U as[15] (Huberman & Kandel, 1987):

$$U = \frac{det(\hat{\Sigma})}{det(\tilde{\Sigma})} \tag{3.8}$$

Kan and Zhou (2001) examine three widely used asymptotic spanning tests to test the null hypothesis in Equation (3.5): Likelihood ratio (LR), Lagrange multiplier (LM) and Wald test (W). They however note that asymptotic tests can be misleading in finite samples. As an alternative, they propose two F-tests for finite samples for the cases when N=1 and N≥2. According to Kan and Zhou (2001), the corrected F-test and distribution to test our null hypothesis under a finite sample for N=1 should be:

$$F = \left(\frac{1}{U} - 1\right)\left(\frac{T - K - 1}{2}\right) \sim F_{2,T-K-1} \tag{3.9}$$

3.2.2 Step-down test

Kan and Zhou (2001) suggest that researchers should break the joint hypothesis in Equation (3.5) into two separate components and test them individually. This is because the distance between the two tangency portfolios is relatively unimportant in determining the power of the spanning test, whereas the distance between the standard deviations of the two global minimum variance portfolios is the primary determinant in rejecting the null hypothesis (Kan & Zhou, 2001). A small difference in standard deviations of the global minimum variance portfolios may not necessarily be economically significant, but on the other hand a large distance between the tangency portfolios can have a great economic significance. Therefore, the benefit of using a step-down test is to be able to identify the possible source of rejection on the joint hypothesis and thereby to yield more useful information. With a step-down test, we can test whether the two tangency portfolios and global minimum variance portfolios are very different. The first test (F_1) tests if $\alpha = 0_N$ and the second test (F_2) tests whether $\delta = 0_N$ conditional on $\alpha = 0_N$. If the null hypothesis of F_1 is not rejected, then the distance between the two tangency portfolios (or the difference between maximum Sharpe ratios) does

[14] It should be noted that all constrained maximum likelihood estimators of Σ can be obtained with the same method as in equation (3.7) by estimating the constrained parameters for matrix $\hat{B}$.
[15] In the case when N=1, Σ is a scalar and therefore $det(\Sigma) = \Sigma$.

not differ significantly. If the null hypothesis of F_2 is not rejected, then the standard deviations (or distance) of the two GMV portfolios do not differ significantly. Then according to Kan and Zhou (2001), the test statistics F_1 and F_2 and their distributions can be defined as:

$$F_1 = \left(\frac{T-K-N}{N}\right)\left(\frac{det(\bar{\Sigma})}{det(\hat{\Sigma})} - 1\right) \sim F_{N,T-K-N} \tag{3.10}$$

$$F_2 = \left(\frac{T-K-N+1}{N}\right)\left(\frac{det(\tilde{\Sigma})}{det(\bar{\Sigma})} - 1\right) \sim F_{N,T-K-N-1} \tag{3.11}$$

Where $\bar{\Sigma}$ is defined as a constrained maximum likelihood estimate of Σ by imposing only the constraint of $\alpha = 0_N$.

3.2.3 Spanning tests under non-normality

The tests described previously assume that the returns of the assets are normally distributed. Due to the nature of financial data, it is not surprising for the returns of the assets and the error term ε_t of Equation (3.3) to exhibit non-normality. There are two types of non-normality to consider: conditional homoskedasticity and conditional heteroskedasticity. In the former case, the error term ε_t is non-normal, but still independent and identically distributed, conditional on $R_{1,t}$. In the latter case, the variance of error term ε_t can be time-varying as a function of $R_{1,t}$ (Kan & Zhou, 2001).

When the error term ε_t exhibits conditional homoskedasticity, the asymptotic tests (LR, LM, W) are still asymptotically distributed. However, Kan and Zhou (2001) note that then the finite sample distributions of asymptotic tests will not be the same anymore, although they can still provide a good approximation. In the case of conditional heteroskedasticity, the asymptotic tests are not valid anymore. In this case, Hansen's (1982) Generalized method of moments (GMM) Wald test can be used as an alternative. The benefit of using a GMM Wald test comes from the fact that the GMM Wald test is valid for all distributions and does not rely on the normality assumption. We define the test statistic (W_a) with the following approach used by Kan and Zhou (2001) and Ferson, Foerster and Keim (1993).

We shall define $x_t = [1 \quad R_{1,t}^\top]^\top$, $\varepsilon_t = R_{2,t} - B^\top x_t$ and denote Kronecker product[16] as $\otimes$. The moment conditions for the GMM estimation of B are:

$$\mathbb{E}[g_t] = \mathbb{E}[x_t \otimes \varepsilon_t] = 0_{(K+1)N} \tag{3.12}$$

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[16] Definition of Kronecker product: Let $A \in \mathbb{R}^{NxM}$ and $B \in \mathbb{R}^{PxK}$. Then
$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1M}B \\ \vdots & \ddots & \vdots \\ a_{N1}B & \cdots & a_{NM}B \end{bmatrix} \in \mathbb{R}^{NPxMK}$.

The sample moments can be expressed as follows, assuming that $R_t = [R_{2,t}^{\top} \quad R_{1,t}^{\top}]^{\top}$ is stationary with finite fourth moments:

$$\bar{g}_t(B) = \frac{1}{T}\sum_{t=1}^{T} x_t \otimes \left(R_{2,t} - B^{\top}x_t\right) \tag{3.13}$$

The GMM estimation of B is obtained by minimizing $\bar{g}_t(B)'S_T^{-1}\bar{g}_t(B)$, where S_T is a consistent estimate of $S_0 = \mathbb{E}[g_t g_t^{\top}]$ where g_t is assumed to be serial uncorrelated. To obtain the GMM estimate of S_T^{-1}, we prefer to use Mike Cliff's (2003) GMM package in MATLAB. Since the system is exactly identified, the GMM estimate of B (and thus $\widehat{\Theta}$) remains the same as the OLS estimate described in previous chapter. Kan and Zhou (2001) define the GMM Wald test statistic and its distribution as follows:

$$W_a = Tvec\left(\widehat{\Theta}^{\top}\right)^{\top}[(A_T \otimes I_N)S_T(A_T^{\top} \otimes I_N)]^{-1}vec(\widehat{\Theta}^{\top})\overset{A}{\sim}\chi_{2N}^2 \tag{3.14}$$

Where:

[17] $\hat{V}_{11} = \frac{1}{T}\sum_{t=1}^{T}\left(R_{1,t} - \hat{\mu}_1\right)\left(R_{1,t} - \hat{\mu}_1\right)^{\top}$

[19] $\hat{\mu}_1 = \frac{1}{T}\sum_{t=1}^{T} R_{1,t}$

$\hat{a}_1 = \hat{\mu}_1^{\top}\hat{V}_{11}^{-1}\hat{\mu}_1$

$\hat{b}_1 = \hat{\mu}_1^{\top}\hat{V}_{11}^{-1}1_K$

vec = vectorization[18]

I_N = N x N identity matrix

$A_T = \begin{bmatrix} 1 + \hat{a}_1 & -\hat{\mu}_1^{\top}\hat{V}_{11}^{-1} \\ \hat{b}_1 & -1_K^{\top}\hat{V}_{11}^{-1} \end{bmatrix}$

$\widehat{\Theta} = [\hat{\alpha} \quad \hat{\delta}]^{\top}$

We will also conduct step-down GMM Wald tests similar to the step-down tests defined in chapter 3.2.2. We derive the parameters for the step-down GMM Wald tests by following the procedure used by Lee (2010, 174). To test for $\alpha = 0_N$ (W_{a1}), we ignore the null hypothesis of $\delta = 0_N$. In practice, this means that the parameter $\widehat{\Theta}$ includes only the GMM estimated intercept term $\hat{\alpha}$. We can still use the Equation (3.14), but due to the change in the null hypothesis, we need to change A_T to $A_T = [1 + \hat{a}_1 \quad -\hat{\mu}_1^{\top}\hat{V}_{11}^{-1}]$. To test for $\delta = 0_N$ conditional on $\alpha = 0_N$ (W_{a2}), we are actually testing if $\delta = 0_N$ without an intercept term $\hat{\alpha}$. Again, in practice this means that we perform the GMM estimation assuming that $\alpha = 0_N$ by excluding the first row from x_t (row of 1). We can then again use the Equation (3.14) by switching A_T to $A_T = \left[-1_K^{\top}\left(\hat{V}_{11} + \hat{\mu}_1\hat{\mu}_1^{\top}\right)^{-1}\right]$. One should also note that omitting the intercept from GMM estimation causes a change in the dimensions of the

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[17] This is actually a definition of a covariance matrix of independent variables (the set of K assets). If we have defined R_{1_t} by removing dates with missing values, we will get a better estimate of V_{11} by defining $\hat{V}_{11}$ with the full dataset using a "pairwise" method.

[18] Definition of vectorization: Let $A \in \mathbb{R}^{NxM}$ and let $a_{ij}, i = 1, \ldots, N; j = 1, \ldots, M$ represent the elements of A. Then $vec(A) = [a_{11} \cdots a_{N1} \cdots a_{1M} \cdots a_{NM}]^{\top} \in \mathbb{R}^{NMx1}$.

[19] This represents the same as μ defined in chapter 2.2 for the set of K assets. As well as in the case of $\hat{V}_{11}$, we will get a better estimate of μ_1 by using the full dataset.

weighting matrix S_T. Both step-down GMM Wald test statistics follow chi-squared distribution with N degrees of freedom.

3.3 Measurement of diversification benefits

When considering performance measures and Markowitz efficient frontiers, the return notation matters. In the crisis period, yearly returns are not meaningful, and in the full period, daily returns are hard to interpret. As a tradeoff to keep our research consistent, we consider monthly returns and variances when examining mean-variance frontiers and diversification benefits. Let T represent the number of days and r_t the daily return on day t. We assume that the average number of trading days per month is 21 and therefore convert the average daily returns to average monthly returns as follows:

$$Average\ monthly\ return = \left(\frac{1}{T}\sum_{t=1}^{T} r_t + 1\right)^{21} - 1 \tag{3.15}$$

When converting covariance matrices and variances constructed from daily data to represent their monthly values, they are multiplied by 21. Similarly, individual assets' daily standard deviations are converted to monthly values by multiplying with $\sqrt{21}$ as described in Equation (2.2). In order to assess the magnitude of diversification benefits in the mean-variance frontier, we examine two distinct measures. In addition, we define two constraints on our investment policy to consider diversification benefits in a more realistic setting.

3.3.1 Risk reduction

To evaluate the change in the portfolio risk in global minimum variance portfolios after introducing new assets to the opportunity set, we measure the difference in the standard deviations of the global minimum variance portfolios that are formed by benchmark assets K (σ_K) and benchmark assets K plus test assets N (σ_{K+N}). Let ω_{K+N} and ω_K represent the individual asset's weights in the sets $K+N$ and K such that $\omega_{K+N} = [\omega_1 \cdots \omega_{K+N}]^\top$ and $\omega_K = [\omega_1 \cdots \omega_K]^\top$. Let ω_i denote the weight of a single asset i in the set of K+N assets and ω_j denote the weight of a single asset j in the set of K assets. The covariance matrices of the sets $K+N$ and K are denoted by $\sum_{K+N}$ and $\sum_K$ respectively. Then according to Petrella (2005) and by using Equation (2.25), $\triangle\, GMV$ can be formally defined as:

$$\triangle\, GMV = \min_{\omega \in \mathbb{R}^K} \sqrt{\omega_K{}^\top \textstyle\sum_K \omega_K} - \min_{\omega \in \mathbb{R}^{K+N}} \sqrt{\omega_{K+N}{}^\top \textstyle\sum_{K+N} \omega_{K+N}} \tag{3.16}$$
$$s.t. \sum_{i=1}^{K+N} \omega_i = 1\ and \sum_{j=1}^{K} \omega_j = 1$$

Also, Golosnoy, Hildebrandt and Köhler (2019) as well as Frahm and Memmel (2010) argue that the distance between σ_K and σ_{K+N} is of great interest. It should be noted that $\triangle\, GMV \geq 0$. When $\triangle\, GMV = 0$, global minimum variance portfolio remains unchanged after introducing the test asset N to the investment opportunity set. $\triangle\, GMV > 0$ indicates that adding the set of N test assets to the investment opportunity set lowers the variance of the global minimum variance portfolio. When $\triangle\, GMV > 0$, investor can construct a global minimum variance portfolio with a lower risk and equal or greater expected return by utilizing the diversification benefits of adding a set of N test assets to the investment opportunity set.

3.3.2 Increase in portfolio efficiency

To examine the change in the portfolio efficiency after introducing the set of N assets to a set of K assets, we measure the difference in the Sharpe ratios of the tangency portfolios formed from the sets of K+N and K assets. We assume that the returns of the assets in the sets of K+N and K assets have multivariate normal distributions with expected returns $\mu_{K+N} = \left[\mathbb{E}[R_1] \cdots \mathbb{E}[R_{K+N}]\right]^{\top}$ and $\mu_K = \left[\mathbb{E}[R_1] \cdots \mathbb{E}[R_K]\right]^{\top}$ respectively. Following Petrella (2005) and assuming the risk-free rare as zero as advised by Jorion (1985) to reduce any undesirable characteristics of the tangency portfolio, we call this measure $\triangle\, SP$ and define it using Equation (2.30) formally as:

$$\triangle\, SP = \max_{\omega \in \mathbb{R}^{K+N}} \frac{{\omega_{K+N}}^{\top} \mu_{K+N}}{\sqrt{{\omega_{K+N}}^{\top} \sum_{K+N} \omega_{K+N}}} - \max_{\omega \in \mathbb{R}^{K}} \frac{{\omega_K}^{\top} \mu_K}{\sqrt{{\omega_K}^{\top} \sum_K \omega_K}} \tag{3.17}$$

$$s.t. \sum_{i=1}^{K+N} \omega_i = 1 \; and \; \sum_{j=1}^{K} \omega_j = 1$$

$\triangle\, SP$ is always greater than or equal to zero. When $\triangle\, SP = 0$, then the largest mean return per unit of standard deviation remains unchanged. $\triangle\, SP > 0$ indicates that we can increase our return/risk ratio of the tangency portfolio by introducing a set of N test assets to the investment opportunity set. When $\triangle\, SP > 0$, investor can obtain a better risk/return profile in his in his tangency portfolio by utilizing the diversification benefits of adding a set of N test assets to the investment opportunity set.

3.3.3 Constraints on investment policy

Typically, investors face constraints on their investment decisions. For example, some investors may want to avoid short selling or too large positions on assets. Also, the traditional unconstrained mean-variance frontier can include portfolios that consist of substantially large short and long positions in assets, which would be near to impossible to accomplish in the real world.

In order to examine realistic investments decisions, we consider two constraints on the weights of the assets in addition to the unconstrained[20] GMV and tangency portfolios. Firstly, we limit our portfolios to include only assets with long positions. Although short selling can be profitable, in practice it comes with high risks. Sharpe (1991) reports that many institutional investors are prohibited from taking short positions either by law or explicit rules. Also, allowing short selling can lead to unrealistically large weights on assets, which can be near to impossible accomplish in practice. Therefore, we consider prohibiting short selling to give a more realistic representation of the whole investor segments preferences. Let N be the number of assets in a portfolio and denote ω_i as a weight of a single asset. Then we can define the "no short selling" policy mathematically as:

$$0 \leq w_i \leq 1 \; \forall i = 1, \dots, N \qquad (3.18)$$
$$s.t \sum_{i=1}^{N} w_i = 1$$

Institutions are often restricted by law from placing more than a certain percentage of their funds on a single asset. Therefore, we consider it reasonable to limit the absolute weight of a single asset to be 60% at maximum in addition to the previously defined "no short selling" policy. We can denote this as a "upper/lower bound" policy and define it mathematically as:

$$|w_i| \leq 0.6 \; \forall i = 1, \dots, N \qquad (3.19)$$
$$s.t. \sum_{i=1}^{N} w_i = 1$$

[20] In an unconstrained investment policy, weights of the assets can be any real numbers on the condition that sum of the weights equals 1, that is $w_i \in \mathbb{R} \; \forall i = 1, \dots, N \; subject \; to \; \sum_{i=1}^{N} w_i = 1$.

4 RESULTS AND ANALYSIS

4.1 Summary statistics

In this chapter, we will start by discussing the main statistical features of the asset classes included in the research by focusing on the full and crisis period. Furthermore, in chapter 4.2 we will analyze the empirical results associated with the mean-variance spanning tests and the magnitude of diversification benefits.

Table 2 Descriptive statistics of returns and standard deviations

Asset Class	Obs.	Arithmetic monthly mean return	Geometric monthly mean return	Max daily return	Min daily return	Average monthly SD	Total return
Full period: 1/2011 – 12/2020							
Equity	2565	0.70 %	0.57 %	8.42 %	-11.47 %	4.94 %	101.22 %
Bonds	2534	0.36 %	0.35 %	1.56 %	-1.72 %	1.02 %	52.72 %
Commodities	2520	-0.41 %	-0.49 %	4.81 %	-5.07 %	3.99 %	-44.24 %
High Yield	2534	0.54 %	0.53 %	1.99 %	-3.77 %	1.30 %	89.27 %
Hedge Funds	2569	-0.04 %	-0.05 %	1.15 %	-2.00 %	0.99 %	-5.46 %
Real Estate	2557	0.60 %	0.04 %	10.62 %	-13.70 %	5.85 %	68.66 %
VSTOXX 1M Fut	2422	-2.67 %	-4.98 %	40.23 %	-22.19 %	22.23 %	-99.72 %
VSTOXX 3M Fut	2422	-1.34 %	-2.04 %	19.70 %	-22.57 %	11.99 %	-90.71 %
VSTOXX 5M Fut	2422	-0.74 %	-1.09 %	19.53 %	-14.81 %	8.53 %	-71.88 %
Crisis period: 2/2020 – 4/2020							
Equity	62	-5.03 %	-5.87 %	8.42 %	-11.47 %	13.28 %	-16.35 %
Bonds	62	-0.50 %	-0.53 %	1.56 %	-1.72 %	2.37 %	-1.56 %
Commodities	62	-6.02 %	-6.31 %	4.44 %	-5.07 %	7.88 %	-17.51 %
High Yield	62	-3.42 %	-3.54 %	1.99 %	-3.77 %	5.05 %	-10.10 %
Hedge Funds	63	-1.95 %	-1.98 %	1.15 %	-2.00 %	2.47 %	-5.83 %
Real Estate	62	-9.25 %	-10.54 %	10.62 %	-12.66 %	16.93 %	-28.03 %
VSTOXX 1M Fut	59	67.92 %	50.25 %	32.20 %	-22.19 %	49.56 %	213.86 %
VSTOXX 3M Fut	59	39.81 %	32.50 %	19.35 %	-22.57 %	33.37 %	120.48 %
VSTOXX 5M Fut	59	39.79 %	35.48 %	19.53 %	-14.81 %	25.73 %	134.71 %

Table 2 displays the major descriptive statistics for each of the asset classes. From looking at the returns of the VSTOXX futures during the full period, it is clear that taking a long position into the implied volatility of an equity index is not the best solution to generate profits in the long term. The most extreme case being the VSTOXX 1-month futures, which have returned -99.72% in the full period, it is clear that as the maturity of the VSTOXX futures increases, so does their returns. This is consistent with the finding that rolling volatility futures during times of low volatility is expensive (Alexander & Korovilas, 2011). Further explanation on the behavior of VSTOXX futures can be observed from Figure 8.

It is clear that during the crisis period asset classes become more volatile as well as their returns turn into negative. The most negative returns are associated with the most volatile asset classes equity, commodities and real estate, whereas the least negative returns are associated with the least volatile asset classes hedge funds and bonds measured by arithmetic and geometric means. However, the only asset class to exhibit positive returns during the crisis period is volatility: As compared to the other asset classes, VSTOXX futures have had extreme positive returns in the crisis period. Although VSTOXX 1-month futures have exhibited the largest returns, they are also more volatile than the corresponding 3- and 5-month futures. All asset classes except real estate have had their minimum daily returns during the crisis period as the market plummeted. It is also interesting, as the VSTOXX futures have experienced their all-time low returns during the crisis period, but not their all-time high return, excluding VSTOXX 5-month futures. Respectively, many asset classes have had their maximum return during the crisis period as many asset classes experienced large positive returns in the post-crash market.

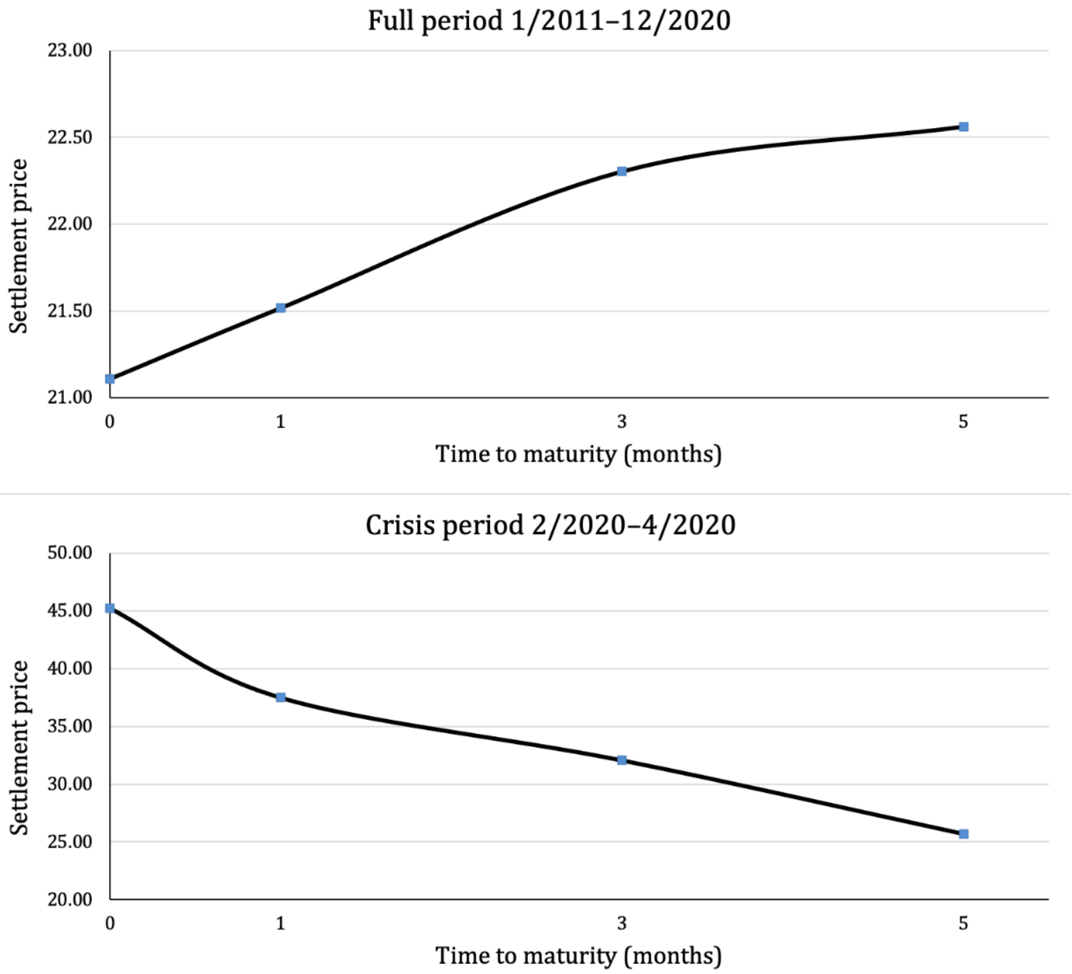

Figure 8 Average term structure of VSTOXX futures

Figure 8 displays the average term structure of the 1-,3- and 5-month VSTOXX futures on the full and crisis periods. It can be clearly seen that during the full period the term structure of VSTOXX futures has been in contango on average. Therefore, while the VSTOXX futures have had large returns, they cannot compensate for the negative cost of carry of the futures contracts. Because the slope of the average term structure is steeper with short-term contracts than with longer-term contracts, the cost of carry is more negative with the short-term contracts. This further explains the extremely negative returns of the VSTOXX futures during the full period observed in Table 2, and especially why the shorter-term contracts have more negative returns than the longer-term contracts. On the other hand, during the crisis period the term structure of VSTOXX futures has been in backwardation on average, thus an investor could have benefit from the positive roll yield. Again, the steepness of the curve decreases as maturity increases, which further explains the positive returns of the VSTOXX futures on the crisis period and why they are larger within shorter-term contracts.

Distributions can be described on the basis of two parameters: kurtosis and skewness. The former measures the degree of heavy tails. The distribution is assumed to exhibit no heavy tails when the kurtosis parameter equals 3. When the kurtosis parameter is higher (lower) than 3, then the distribution tends to have heavy (light) tails. In other words, kurtosis higher (lower) than 3 indicates that extreme values are more likely (unlikely) than in the normal distribution. The latter measures the degree of symmetry. If the skewness parameter equals 0, then distribution is assumed to be symmetrical. When the skewness parameter is higher (lower) than 0, then the distribution is more skewed to right (left). In other words, positive (negative) skewness indicates that the extreme values on the right (left) side of the distribution are more likely than the extreme values on the left (right) side of the distribution. To test whether the asset classes' returns are normally distributed, the Jarque-Bera test which is based on a null hypothesis that the data is normally distributed is used.

Table 3 Kurtosis, Skewness and Jarque-Bera tests for normality

	Full period 1/2011 – 12/2020				Crisis period 2/2020 – 4/2020			
Asset Class	Kurtosis	Skewness	Jarque-Bera	p-value	Kurtosis	Skewness	Jarque-Bera	p-value
Equity	12.85	-0.79	10633.61	<0.001	6.33	-0.90	37.02	<0.001
Bonds	8.70	-0.52	3540.69	<0.001	6.34	-0.69	33.78	<0.001
Commodities	5.69	-0.16	767.86	<0.001	4.06	-0.35	4.23	0.069
High Yield	43.60	-3.23	178460.80	<0.001	5.66	-1.31	36.18	<0.001
Hedge Funds	10.95	-1.21	7400.92	<0.001	5.45	-1.01	26.40	0.002
Real Estate	19.68	-0.72	29856.81	<0.001	5.79	-0.18	20.44	0.003
VSTOXX 1M Fut	9.42	1.30	4844.61	<0.001	3.57	0.72	5.91	0.039
VSTOXX 3M Fut	12.10	0.90	8685.82	<0.001	4.80	0.00	7.96	0.024
VSTOXX 5M Fut	17.13	1.20	20734.66	<0.001	5.51	0.45	17.50	0.005

Table 3 reports the kurtosis and skewness parameters of assets' returns as well Jarque-Bera test statistics and their corresponding p-values. In the full period, the null hypothesis of Jarque-Bera test is rejected for all asset classes with 0.01% significance level. In the crisis period, the null hypothesis is rejected at 1% and 5% significance levels for most asset classes, except for commodities the null hypothesis can be rejected at 10% significance level. The kurtosis and skewness parameters of asset classes demonstrate typical features of financial assets' returns in which it is common for financial assets to exhibit heavy tails and negative skewness. All returns of the asset classes exhibit leptokurtic distribution, which indicates that the probability of extreme events is significantly higher than predicted in the normal distribution. Most of the asset classes except VSTOXX futures are negatively skewed and thus they possess more downside risk than normally distributed returns. The VSTOXX futures yield positive skewness as expected, which prompts that an investor investing in VSTOXX futures may expect frequent small losses but also large gains. However, from Table 2 and Figure 8 it can be seen that in the full period the occasional extremely large returns are not enough to compensate for the negative cost of carry as the term structure of VSTOXX futures is in contango on average. On the other hand, in the crisis period the positive skewness of VSTOXX futures is rewarding, since the term structure of VSTOXX futures is in backwardation on average. It is worth noticing that on the full period the kurtosis and skewness parameters of high yield bonds is significantly higher than with other asset classes, and thus high yield bonds have the highest tail-risk compared to other asset classes.

Table 4 Correlation matrix of asset classes. Period from January 2011 to December 2020 is depicted in the bottom left corner, and the crisis period from February 2020 to April 2020 is depicted in the top right corner with bolded figures. ***, ** and * indicate statistical significance at the 1%, 5% and 10% levels respectively.

Asset Class	Equity	Bonds	Commo-dities	High Yield	Hedge Funds	Real Estate	VSTOXX 1M Fut	VSTOXX 3M Fut	VSTOXX 5M Fut
Equity	1.000	**0.320****	**0.590*****	**0.667**	**0.665*****	**0.833*****	**-0.757*****	**-0.716*****	**-0.718*****
Bonds	0.025	1.000	**0.191**	**0.636*****	**0.574*****	**0.547*****	**-0.382*****	**-0.453*****	**-0.471*****
Commo-dities	0.350***	0.092***	1.000	**0.417*****	**0.471*****	**0.481*****	**-0.614*****	**-0.575*****	**-0.588*****
High Yield	0.559***	0.324***	0.230***	1.000	**0.858*****	**0.710*****	**-0.542*****	**-0.542*****	**-0.537*****
Hedge Funds	0.664***	0.070***	0.342***	0.530***	1.000	**0.619*****	**-0.600*****	**-0.632*****	**-0.600*****
Real Estate	0.774***	0.179***	0.252***	0.568***	0.527***	1.000	**-0.647*****	**-0.699*****	**-0.711*****
VSTOXX 1M Fut	-0.770***	0.035*	-0.237***	-0.441***	-0.575***	-0.564***	1.000	**0.921*****	**0.883*****
VSTOXX 3M Fut	-0.766***	0.018	-0.229***	-0.460***	-0.576***	-0.585***	0.888***	1.000	**0.967*****
VSTOXX 5M Fut	-0.757***	0.003	-0.236***	-0.476***	-0.558***	-0.596***	0.825***	0.910***	1.000

Table 4 presents the cross correlations of asset classes and their statistical significance measured by t-tests on the full and crisis period. As expected, in the full period the cross correlations between VSTOXX futures and equity are strongly negative, from which the strongest negative correlation is found between VSTOXX 1-month futures. Also, most of the cross correlations between VSTOXX futures and other asset classes are negative as well, excluding bonds. On the full period, most of the cross correlations are significant at the 1% significance level. The only exception comes from cross correlations of almost zero with VSTOXX futures and bonds, from which only the cross correlation with VSTOXX 1-month futures and bonds is significant at 10% significance level.

In the crisis period, it is clear that the asset classes become more correlated. Subsequently, also the cross correlations between VSTOXX futures and asset classes become more negative than in the full period. Interestingly, the cross correlations between equity and VSTOXX futures are not as strongly negative in the crisis period as compared to the full period, although they still exhibit the strongest negative correlations on both periods. Majority of the correlations are significant at the 1% significance level. It is also interesting, that correlation between hedge funds and equity remains almost unchanged.

These results on the increasing cross correlations during the 2020 stock market crash are similar to the findings of Szado (2009). This insight gives further evidence on the fact that diversification benefits of traditional asset classes seem to diminish in the times of extreme market conditions. The strong negative cross correlations between VSTOXX futures and other examined asset classes provide optimal conditions to lower the volatilities of benchmark portfolios when VSTOXX futures are added to the portfolios. Furthermore, in addition to the fact that VSTOXX futures exhibit large returns during the crisis period, this confirms the fact that VSTOXX futures are suitable for hedging purposes during market downturns.

4.2 Empirical results

4.2.1 Mean-variance spanning analysis

Mean-variance spanning analysis is carried out as described in chapter 3. Table 5 shows the test statistics and p-values for finite sample (F) and GMM Wald (W_a) tests as well as for their step-down applications (F_1, F_2, W_{a1} and W_{a2}).

When examining the full period from January 2011 to December 2020, the H_0 of spanning is rejected at 1% significance level for all VSTOXX futures within both Portfolio A and B by the finite sample and GMM Wald tests. However, when looking at their step-down applications, we observe that the H_0 of F_1 and W_{a1} is not rejected for any test asset in either portfolio, but H_0 of F_2 and W_{a2} is rejected at 1% significance level for all test assets in both portfolios. We can conclude that during the full period, the differences in the maximum Sharpe ratios between the opportunity sets (Portfolio A and B) and VSTOXX futures are not statistically significant, and therefore there is a weak evidence that the tangency portfolios can be improved. Also, the differences in the variances of the global minimum variance portfolios between the opportunity sets and VSTOXX futures are statistically significant, and hence there is a strong evidence that the global minimum variance portfolio can be improved. Graphical representations[21] of the changes in efficient frontiers for the full period for both Portfolios A and B when VSTOXX futures are added in the opportunity sets are presented in Appendices 1 and 2, from where it can be clearly seen that the efficient frontiers shift left after introducing VSTOXX futures to the opportunity set.

Table 5 Mean-variance spanning tests. P-values lower than 0.05 have been bolded in order to make interpreting easier.

Opportunity set	Test asset	$\hat{\alpha}$	$\hat{\delta}$	F-test (p-value)	F_1-test (p-value)	F_2-test (p-value)	W_a-test (p-value)	W_{a1}-test (p-value)	W_{a2}-test (p-value)
Full period: 1/2011 – 12/2020									
Portfolio A	VSTOXX 1M Fut	-0.000	3.345	67.603	0.041	135.220	59.589	0.037	58.199
				(<0.001)	(0.839)	**(<0.001)**	**(<0.001)**	(0.982)	**(<0.001)**
	VSTOXX 3M Fut	-0.000	2.461	123.474	0.000	247.052	27.206	0.000	22.470
				(<0.001)	(0.986)	**(<0.001)**	**(<0.001)**	(0.999)	**(<0.001)**
	VSTOXX 5M Fut	0.000	2.145	178.079	0.358	355.897	28.186	0.204	21.258
				(<0.001)	(0.550)	**(<0.001)**	**(<0.001)**	(0.903)	**(<0.001)**
Portfolio B	VSTOXX 1M Fut	-0.000	5.926	90.999	0.091	181.977	69.785	0.088	69.827
				(<0.001)	(0.763)	**(<0.001)**	**(<0.001)**	(0.957)	**(<0.001)**
	VSTOXX 3M Fut	0.000	3.878	132.440	0.002	264.991	36.605	0.002	35.745
				(<0.001)	(0.966)	**(<0.001)**	**(<0.001)**	(0.999)	**(<0.001)**
	VSTOXX 5M Fut	0.000	2.952	145.027	0.548	289.563	27.609	0.425	26.621
				(<0.001)	(0.459)	**(<0.001)**	**(<0.001)**	(0.809)	**(<0.001)**
Crisis period: 2/2020 – 4/2020									
Portfolio A	VSTOXX 1M Fut	0.016	6.905	9.612	3.027	15.641	35.471	3.680	27.754
				(<0.001)	(0.087)	**(<0.001)**	**(<0.001)**	(0.159)	**(<0.001)**
	VSTOXX 3M Fut	0.010	6.085	14.179	2.522	25.164	26.474	3.306	15.105
				(<0.001)	(0.118)	**(<0.001)**	**(<0.001)**	(0.191)	**(<0.001)**
	VSTOXX 5M Fut	0.011	5.125	18.805	5.578	29.651	33.279	6.752	10.431
				(<0.001)	**(0.022)**	**(<0.001)**	**(<0.001)**	**(0.034)**	**(0.001)**
Portfolio B	VSTOXX 1M Fut	0.012	10.850	10.936	1.718	19.881	39.904	2.283	40.380
				(<0.001)	(0.196)	**(<0.001)**	**(<0.001)**	(0.319)	**(<0.001)**
	VSTOXX 3M Fut	0.006	8.810	14.490	1.063	27.884	75.109	1.414	55.741
				(<0.001)	(0.308)	**(<0.001)**	**(<0.001)**	(0.493)	**(<0.001)**
	VSTOXX 5M Fut	0.009	6.680	16.284	3.512	27.718	26.269	5.233	14.143
				(<0.001)	(0.067)	**(<0.001)**	**(<0.001)**	(0.073)	**(<0.001)**

During the crisis period from February 2020 to April 2020, we are able to find similar results as in the full period. The H_0 of spanning is rejected at 1% significance level for all VSTOXX futures within the Portfolio A and B by the finite sample and GMM Wald test. Also, the H_0 of F_2 and W_{a2} is rejected at 1% significance

[21] It is important to keep in mind that visual changes in efficient frontiers do not imply statistical significance.

level for all test assets in both portfolios. However, when examining F_1 and W_{a1}, we are able to find new interesting observations. Firstly, in at the case when VSTOXX 1-month futures are added to Portfolio A, the H_0 of F_1 can be rejected at 10% significance level, but when correcting for the effects of conditional heteroskedasticity, the H_0 of W_{a1} cannot be rejected at 10% significance level anymore. Secondly, there is a strong evidence that VSTOXX 5-month futures improve the Sharpe ratios of Portfolios A and B. In the case when VSTOXX 5-month futures are added to Portfolio A, the H_0 of F_1 and W_{a1} can be rejected at 5% significance level. In addition, when examining VSTOXX 5-month futures added to Portfolio B, the H_0 of F_1 and W_{a1} can be rejected at 10% significance level. Graphical representations of the changes in efficient frontiers for the crisis period for both portfolios A and B when VSTOXX futures are added in the opportunity sets are presented in Appendices 5 and 6, from where it can be seen that the efficient frontiers move significantly upwards after introducing VSTOXX futures to the opportunity set.

To summarize, the results of the mean-variance spanning tests indicate that there are statistically significant diversification benefits obtainable when including VSTOXX futures to Portfolio A and B. However, these benefits of diversification limit mostly on the decrease on the variance of the global minimum variance portfolio. On the other hand, there is evidence that the difference in tangency portfolios is significant during the crisis period for both Portfolios A and B when VSTOXX 5-month futures are added to the investment opportunity set. It is also important to keep in mind that these tests allow for short selling. If we were to prohibit short selling in this case, it is reasonable to expect the test statistics to perform even better: When short selling is allowed, investors can benefit from all high positive cross correlations between asset classes, but when short selling is prohibited, investors can have major benefits from the negative cross correlations between asset classes and VSTOXX futures in terms of diversification.

4.2.2 Magnitude of diversification benefits

If one prohibits short selling on assets with negative returns, the combination of those assets will have a negative Sharpe ratio. Since negative Sharpe ratios do not have any useful meaning and can even reward for taking more risk while further decreasing returns, we choose to ignore them. As a result, when our benchmark's tangency portfolio has a negative Sharpe ratio, it will be replaced by zero. This procedure will be done for constrained Portfolios A and B during the crisis period. By looking at Appendices 7 and 8, it can be already seen that their efficient frontiers have negative expected returns. The constrained efficient frontier of Portfolio A during the crisis period is just a single point, all other possible portfolios would require more risk for a lower expected return. In the case of the constrained Portfolio B during the crisis period, the efficient frontier is more than a single point, but still very insignificantly small and hard to see that we choose to replace it with a marker.

Table 6 presents decreases in standard deviations of global minimum variance portfolios ($\triangle\, GMV$) and increases in Sharpe ratios ($\triangle\, SP$) in Portfolios A and B when VSTOXX futures of different maturities are added to the investment opportunity set during the full and crisis period. The benchmark, which is the standard deviation of the global minimum variance portfolio or maximum Sharpe ratio of Portfolio A or B is marked with parentheses. Since the returns of the asset classes exhibit unsymmetrical leptokurtic distributions, the traditional performance measures of standard deviation and Sharpe ratio do not fully describe the risk profiles of different portfolios, since they do not account for tail-risk given by higher moments. This is an issue especially with high yield bonds, since their returns have a very large downside risk as noted previously. Therefore, the following diversification measures are valid only if it is assumed that an investor is concerned only on the first 2 moments of the return distributions.

Table 6 Diversification measures

Opportunity set	Test asset	Without constraints		With constraints	
		ΔGMV (Benchmark)	ΔSP (Benchmark)	ΔGMV (Benchmark)	ΔSP (Benchmark)
Full period: 1/2011 – 12/2020					
	Portfolio A	(1.008%)	(0.373)	(2.082%)	(0.237)
Portfolio A	VSTOXX 1M Fut	0.027 %	0.003	0.809 %	0.019
	VSTOXX 3M Fut	0.048 %	0.001	0.925 %	0.032
	VSTOXX 5M Fut	0.068 %	0.001	1.002 %	0.056
	Portfolio B	(0.655%)	(0.605)	(0.737%)	(0.472)
Portfolio B	VSTOXX 1M Fut	0.024 %	0.001	0.081 %	0.001
	VSTOXX 3M Fut	0.033 %	0.000	0.096 %	0.002
	VSTOXX 5M Fut	0.037 %	0.002	0.102 %	0.008
Crisis period: 2/2020 – 4/2020					
	Portfolio A	(2.344%)	(0.356)	(5.921%)	(0)
Portfolio A	VSTOXX 1M Fut	0.276 %	1.398	2.860 %	1.753
	VSTOXX 3M Fut	0.406 %	1.089	3.029 %	1.445
	VSTOXX 5M Fut	0.472 %	1.648	3.276 %	2.003
	Portfolio B	(1.555%)	(0.807)	(2.146%)	(0)
Portfolio B	VSTOXX 1M Fut	0.194 %	1.195	0.472 %	1.753
	VSTOXX 3M Fut	0.284 %	0.880	0.582 %	1.445
	VSTOXX 5M Fut	0.272 %	1.408	0.590 %	2.003

Looking at the Table 6, we can see that results are consistent with results of the step-down tests discussed earlier. In the full period, $\triangle SP$ measures are small for the unconstrained portfolios, which is consistent with the fact that none of the F_1 and W_{a1} statistics were statistically significant. In addition, the $\triangle GMV$

measures are small as well for the unconstrained portfolios in the full period, although all F_2 and W_{a2} were statistically significant. In the crisis period, the $\triangle SP$ and $\triangle GMV$ measures increase significantly compared to the full period. The highest $\triangle SP$ measures are obtained within 5-month VSTOXX futures, which is again consistent with our previous results since F_1 and W_{a1} were statistically significant in the case when 5-month VSTOXX futures are added to the Portfolios A and B during the crisis period. Also, the second largest $\triangle SP$ during the crisis period within unconstrained portfolios can be achieved with 1-month VSTOXX futures contracts within both Portfolios A and B, which is again consistent with the fact that F_1 was statistically significant in the case when 5-month VSTOXX futures are added to the Portfolio A. For the Portfolio A, the highest $\triangle GMV$ measure during the crisis period within unconstrained portfolios are achieved with 3-month VSTOXX futures contracts, as for the Portfolio B the highest $\triangle GMV$ measure is achieved with 3-month VSTOXX futures contracts. Again, the large gains in $\triangle GMV$ measures are consistent with our previous results as F_2 and W_{a2} were statistically significant for both Portfolios A and B.

Most importantly, the constraints on the weights of the asset classes have large impacts on the diversification gains on both periods. By comparing $\triangle GMV$ and $\triangle SP$ with and without constraints on both periods, it is clear that imposing constraints increase gains in $\triangle GMV$ and $\triangle SP$ in every scenario as compared to their constrained counterparts. This is logical, since VSTOXX futures have large negative cross correlations with other asset classes which can be fully exploited when short selling is restricted. Also, the $\triangle GMV$ and $\triangle SP$ measures are the higher the larger the maturity of VSTOXX futures is. We can conclude, that the 5-month VSTOXX futures contracts offer the largest diversification measures measured by both $\triangle GMV$ and $\triangle SP$. It is also interesting, that $\triangle SP$ measures are the same for both constrained Portfolios A and B. This is due to the fact that they both include the same combination assets with the same weights. It can be seen from Appendix 9, that these portfolios consist of equities, bonds and VSTOXX futures. Clearly, during a crisis period an investor is better off on holding only a traditional portfolio of equities and bonds complemented with VSTOXX futures.

Lastly, it is equally important to pay attention on how the constrained GMV and tangency portfolios are constructed. Our results are in line with the findings of Stanescu and Tunaru (2013), and show that in the full period the optimal weights for constrained tangency portfolios can be obtained by small allocations to VSTOXX futures ranging from 3.5%–10.5% for Portfolio A and 0.3%–2.3% for Portfolio B. In addition to VSTOXX futures, the Portfolio A holds only equities, bonds and VSTOXX futures, while the Portfolio B is composed of bonds and high yield bonds. Since the returns of the VSTOXX futures increase significantly on the crisis period, their allocations in tangency portfolios increase as well. As noted previously, the constrained tangency portfolios are identical for both Portfolios A and B consisting only of equities and bonds, in which the allocations to VSTOXX futures range from 17.3%–21.4%. In the case of constrained GMV portfolios, in the full period the smallest standard deviation of a portfolio can be achieved by allocations to VSTOXX futures ranging from 6.3%–14% for Portfolio A and 1.7%–4.8% for Portfolio B. In contrast to the tangency portfolios for

Portfolio B during the full period, the Portfolio B is composed of bonds, high yield bonds and hedge funds. Again, when we examine the crisis period, the percentage allocated to VSTOXX futures increases. This is logical and expected, since during the crisis period the cross correlations of asset classes excluding VSTOXX futures increased. The lowest standard deviation of a portfolio can be achieved by allocations to VSTOXX futures ranging from 8.4%–14.5% for Portfolio A and 3%–5.9% for Portfolio B. The benchmark Portfolio B consists of bonds and hedge funds, but when VSTOXX futures are added, the composition of Portfolio B includes bonds, commodities and hedge funds.

Another interesting finding is that the percentage allocated to VSTOXX futures increases more the longer the maturity of the VSTOXX futures contract. Since all VSTOXX futures have similar cross correlations with other asset classes, in the case of tangency portfolios this is due to the fact that VSTOXX futures of longer maturity have better reward-risk ratios (Sharpe ratios) as can be inferred from Table 2. However, as discussed earlier the GMV portfolios are independent of assets' returns but the same phenomenon can be seen with GMV portfolios. This is due to the fact, in addition to the similar cross correlations between asset classes, that standard deviations of VSTOXX futures decreases as their maturity increases.

5 CONCLUSIONS

As stated in chapter 1, the main goal of this research was to investigate (1) diversification benefits in adding VSTOXX futures of different maturities to investment portfolio constructed from European and global asset classes during a "normal" market environment and during the 2020 stock market crash and (2) their magnitude when holding a continuous long position in VSTOXX futures in diversified portfolios during the aforementioned periods.

This study has some interesting findings to conclude this paper with. First of all, the cross correlations of different asset classes were significantly higher during the 2020 stock market crash than during the full 10-year period as suggested by studies of Szado (2009) and Chow et al. (1999) in the times of market downturns. This further confirms the fact that cross correlations of asset classes tend to be higher in absolute terms during bear markets than during bull markets. Interestingly, the cross correlations between equity and VSTOXX futures remain relatively unchanged in the crisis period as compared to the full period.

Furthermore, this research included various mean-variance spanning tests in order to gain information about the significance of diversification benefits. To gain robust measures, one must use mean-variance spanning tests that correct for non-normality. Our results indicate, that adding VSTOXX futures to traditional equity/bond and more diversified portfolios is beneficial, but when disentangling the sources of rejection, the rejection is mostly due to a significant change in the global minimum variance portfolios. In the full period, the increases in Sharpe ratios of tangency portfolios were insignificant, whereas in the crisis period the corresponding significant improvements were associated with VSTOXX 5-month futures for both portfolios. Finally, one must note that no short selling constraints were added to the mean-variance spanning analysis.

In addition, we analyzed the magnitude of diversification by measuring the decrease in portfolio risk and the increase in the Sharpe ratios of portfolios after VSTOXX futures of several maturities were added to the investment opportunity set. The diversification measures are consistent with the findings of the mean-variance spanning tests and show that the largest diversification gains are associated with VSTOXX 5-month futures. Also, these diversification gains increased significantly in the crisis period, indicating that VSTOXX futures is a suitable hedge in market downturns. Adding short selling and upper bound constraints to the analysis on the magnitude of diversification benefits did not change the main findings, but it did show that adding constraints on the weights of the assets indeed improves the diversification gains as compared to unconstrained portfolios. However, the measures on the magnitude of diversification benefits remain biased, as all asset classes exhibit non-normal distributions on both periods, thus the concept of tail-risk is ignored in the measures.

Contrary to previous research made by Signori et al. (2012) and Guobuzaite and Martellini (2012), we found that the most significant diversification benefits are associated with VSTOXX 5-month futures during the crisis and full period.

Since this research included only futures with maturities of 1-, 3- and 5-months, it would be interesting to expand the analysis to include VSTOXX futures with 2-, 4-, and longer maturities as well. As the VSTOXX futures have been in contango on average in the period from January 2011 to December 2020, obtaining long positions in VSTOXX futures have large negative cost of carry. We also found that the negative cost of carry is significantly larger with VSTOXX futures with shorter maturity due to the steepness of the term structure. This is in line with the findings of Alexander and Korovilas (2011), and suggests that a more dynamic trading strategy than holding a constant long exposure in VSTOXX is recommended. With a more active trading approach, it would be possible to minimize or completely eliminate the negative cost of carry in the long run.

One of the biggest shortcomings of this research is that the mean-variance spanning tests did not take into account any short selling constraints. Future research could therefore expand this paper's analysis to include short selling constraints. Also, in this study we have not taken into account transaction costs. As a result, the calculated portfolio returns are slightly overestimated. However, Guobuzaite and Martellini (2012) have found the diversification benefits to be robust even after including transaction costs associated with rolling over volatility derivatives.

For further research, it is recommended incorporate short selling constraints to the mean-variance spanning analysis to gain more robust information on the significance of diversification benefits associated with VSTOXX futures. Fahling (2019) has shown that movements of the VSTOXX index are predictable to an extent. Thus, research could be implemented further to derive a dynamic asset allocation strategy that utilizes this anomaly and its possibilities to create excess returns. Also, since in this research we ignored the tail-risks when measuring the magnitude of diversification benefits, further research could implement the same analysis with multiple asset classes using Value-at-Risk and Conditional Value-at-Risk optimization.

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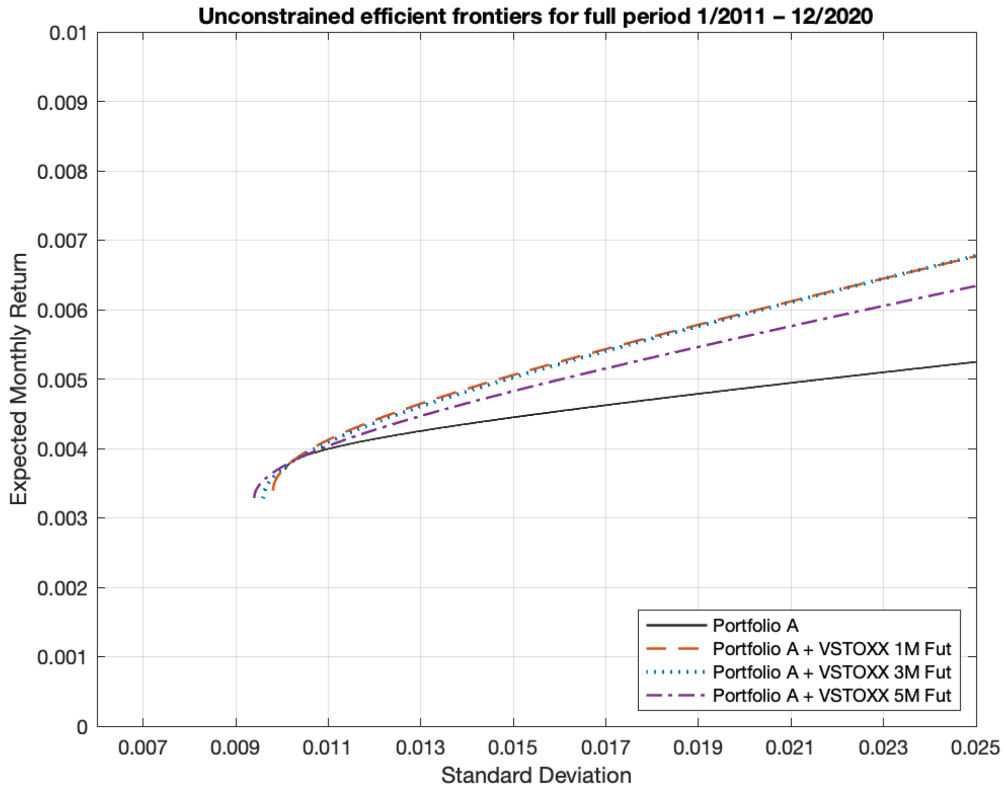
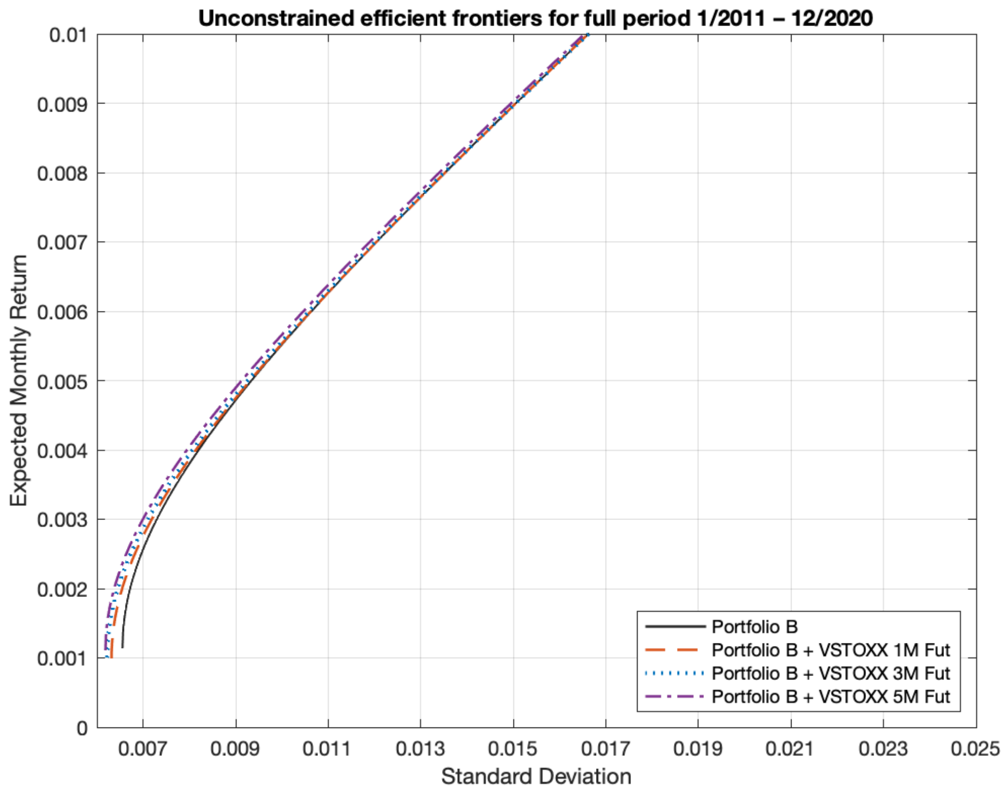
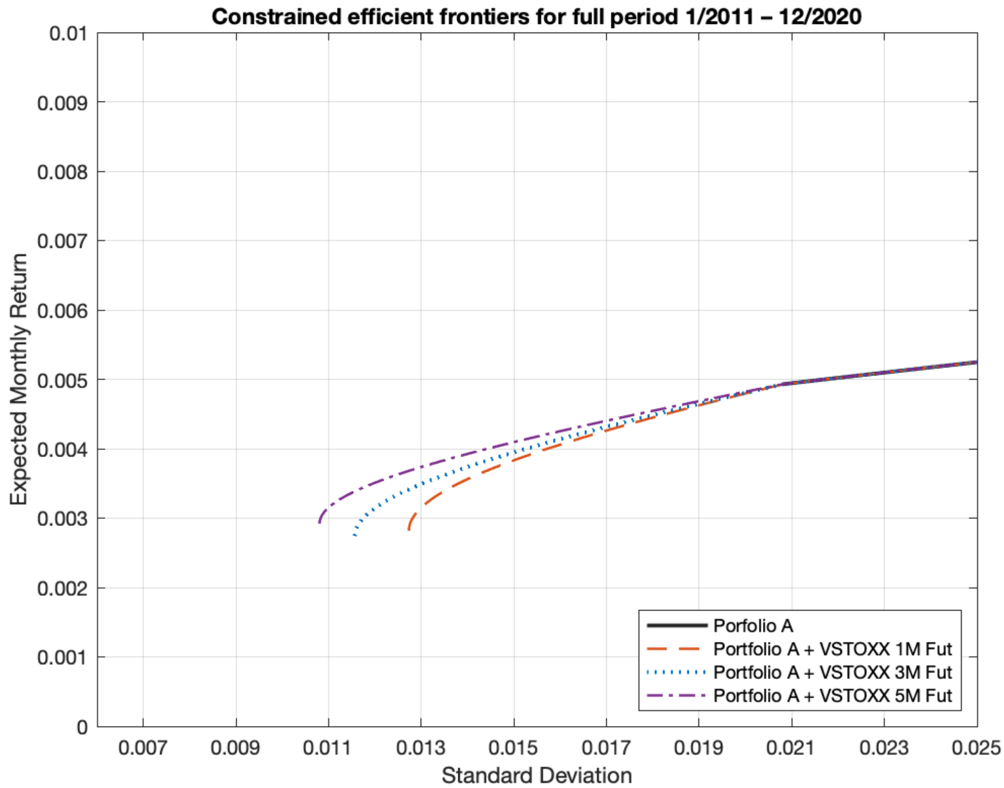
APPENDICES



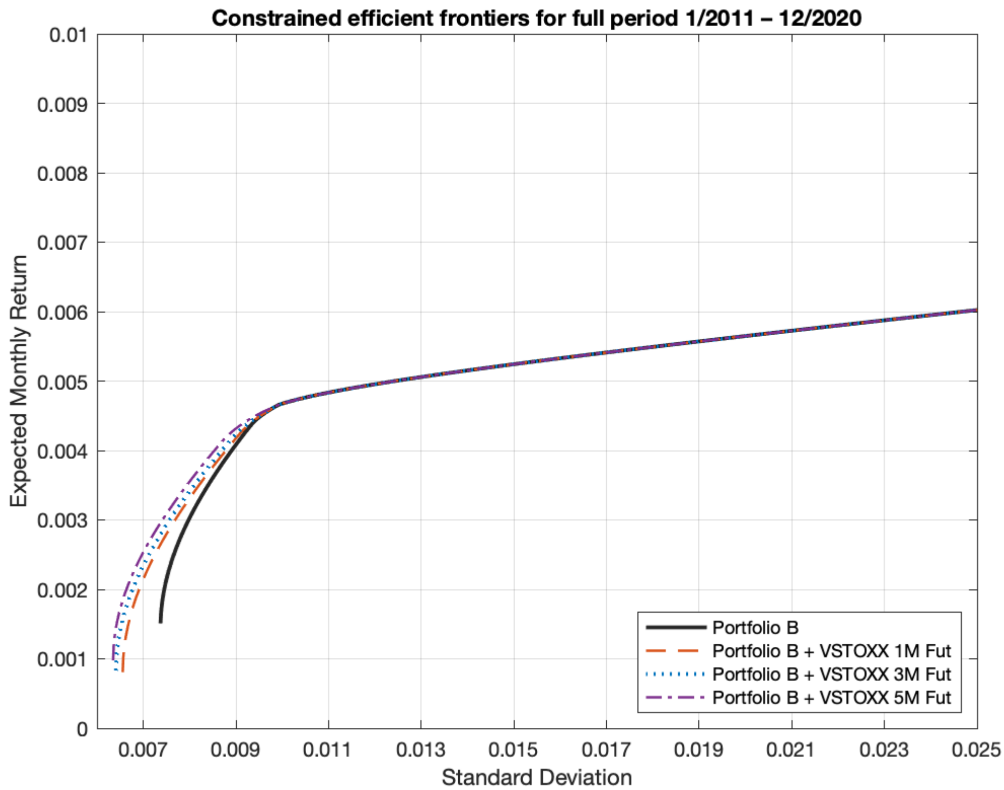
Appendix 1 Unconstrained efficient frontiers for full period Portfolio A (0.001=0.1%)



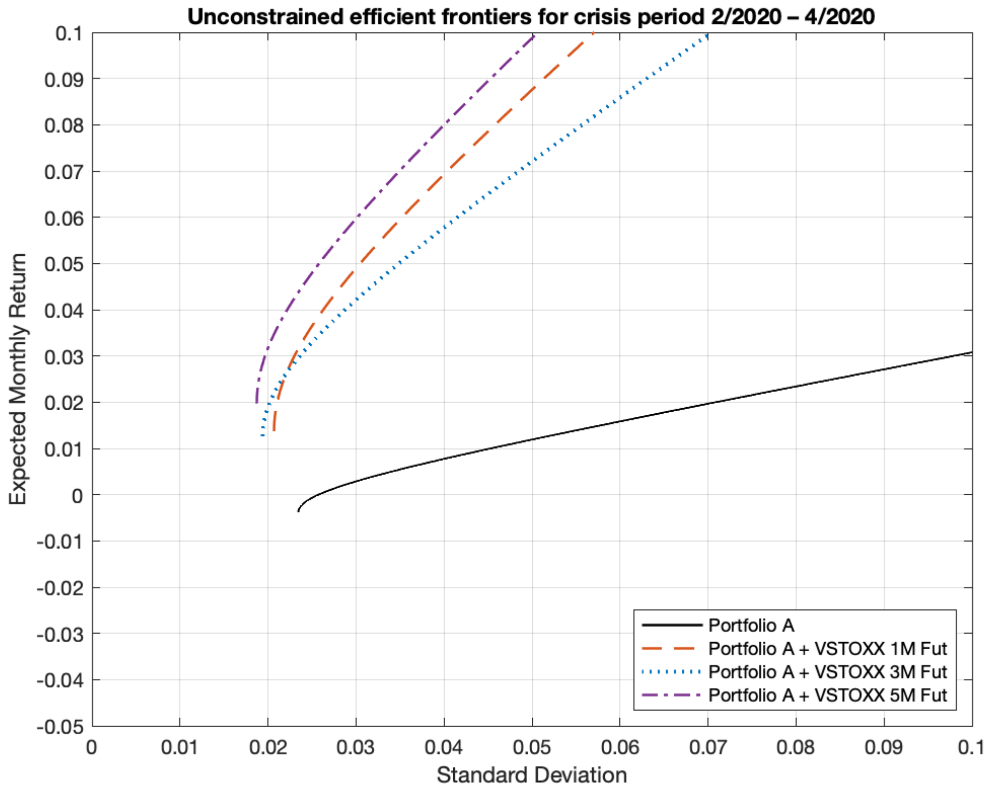
Appendix 2 Unconstrained efficient frontiers for full period Portfolio B (0.001=0.1%)



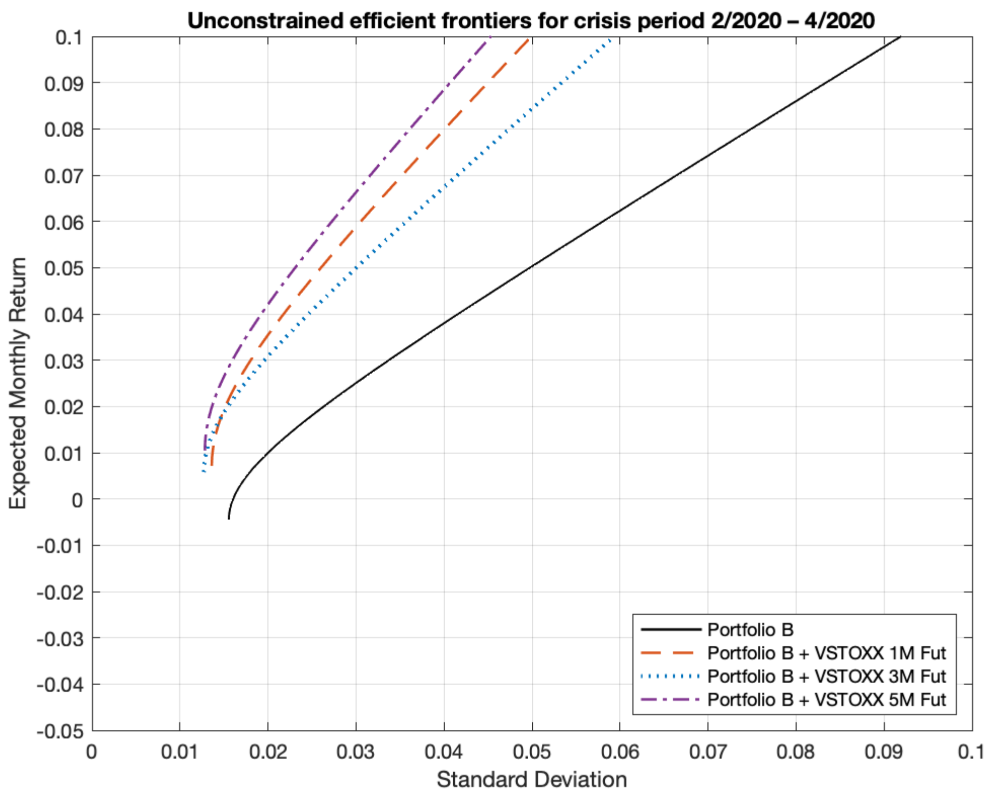
Appendix 3 Constrained efficient frontiers for full period Portfolio A (0.001=0.1%)



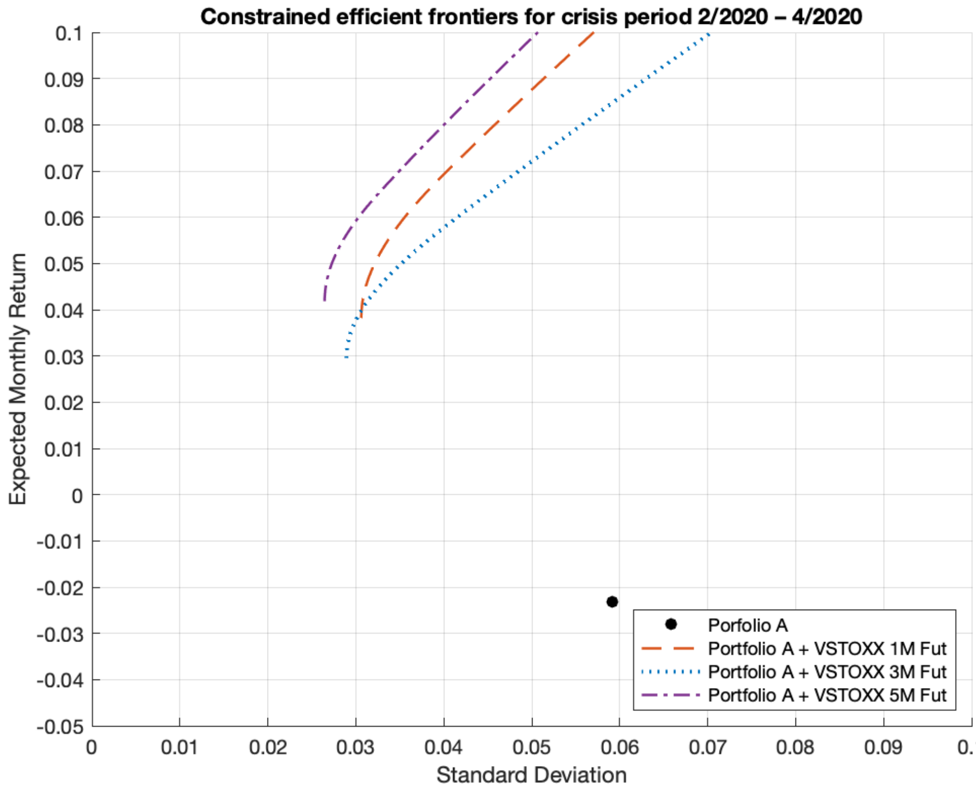
Appendix 4 Constrained efficient frontiers for full period Portfolio B (0.001=0.1%)



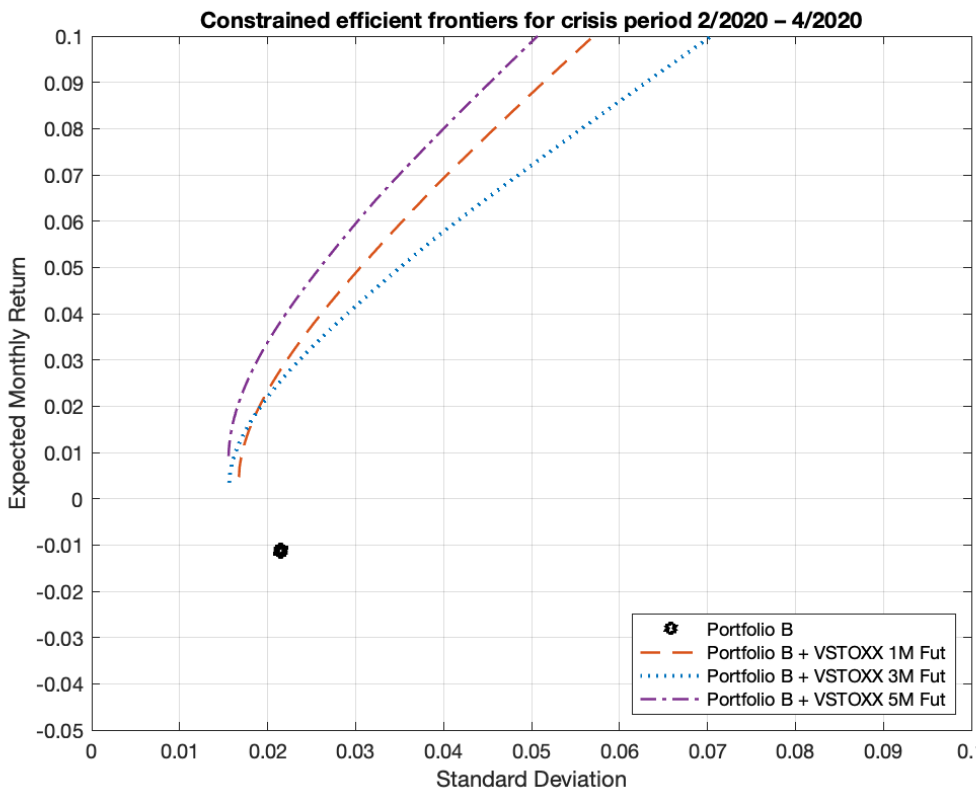
Appendix 5 Unconstrained efficient frontiers for crisis period Portfolio A (0.01=1%)



Appendix 6 Unconstrained efficient frontiers for crisis period Portfolio B (0.01=1%)



Appendix 7 Constrained efficient frontiers for crisis period Portfolio A (0.01=1%)



Appendix 8 Constrained efficient frontiers for crisis period Portfolio B (0.01=1%)

Opportunity set	Equity	Bonds	Commodities	High Yield	Hedge Funds	Real Estate	VSTOXX 1M Fut	VSTOXX 3M Fut	VSTOXX 5M Fut
Unconstrained GMV portfolios full period									
Portfolio A	3.7 %	96.3 %							
Portfolio A + VSTOXX 1M Fut	9.1 %	89.3 %					1.6 %		
Portfolio A + VSTOXX 3M Fut	10.5 %	85.7 %						3.8 %	
Portfolio A + VSTOXX 5M Fut	11.2 %	82.7 %							6.1 %
Portfolio B	-4.7 %	36.8 %	-0.9 %	11.0 %	61.5 %	-3.6 %			
Portfolio B + VSTOXX 1M Fut	-0.3 %	32.7 %	-1.2 %	10.9 %	60.3 %	-3.7 %	1.2 %		
Portfolio B + VSTOXX 3M Fut	0.1 %	31.6 %	-1.3 %	11.2 %	59.1 %	-3.3 %		2.6 %	
Portfolio B + VSTOXX 5M Fut	0.1 %	31.2 %	-1.1 %	11.9 %	57.4 %	-3.1 %			3.7 %
Constrained GMV portfolios full period									
Portfolio A	40.0 %	60.0 %							
Portfolio A + VSTOXX 1M Fut	33.7 %	60.0 %					6.3 %		
Portfolio A + VSTOXX 3M Fut	29.2 %	60.0 %						10.8 %	
Portfolio A + VSTOXX 5M Fut	26.0 %	60.0 %							14.0 %
Portfolio B	0.0 %	48.3 %	0.0 %	0.0 %	51.7 %	0.0 %			
Portfolio B + VSTOXX 1M Fut	0.0 %	33.2 %	0.0 %	6.0 %	59.1 %	0.0 %	1.7 %		
Portfolio B + VSTOXX 3M Fut	0.0 %	31.5 %	0.0 %	7.3 %	57.8 %	0.0 %		3.4 %	
Portfolio B + VSTOXX 5M Fut	0.0 %	31.0 %	0.0 %	8.4 %	55.8 %	0.0 %			4.8 %
Unconstrained tangency portfolios full period									
Portfolio A	7.4 %	92.6 %							
Portfolio A + VSTOXX 1M Fut	4.3 %	96.6 %					-1.0 %		
Portfolio A + VSTOXX 3M Fut	5.9 %	95.1 %						-0.9 %	
Portfolio A + VSTOXX 5M Fut	8.5 %	90.5 %							1.0 %
Portfolio B	29.3 %	97.6 %	-16.7 %	151.2 %	-142.2 %	-19.2 %			
Portfolio B + VSTOXX 1M Fut	N/A	N/A	N/A	N/A	N/A	N/A	N/A		
Portfolio B + VSTOXX 3M Fut	28.9 %	101.0 %	-17.0 %	155.2 %	-147.5 %	-19.7 %		-0.8 %	
Portfolio B + VSTOXX 5M Fut	30.1 %	88.7 %	-15.6 %	140.7 %	-128.9 %	-17.6 %			2.7 %
Constrained tangency portfolios full period									
Portfolio A	40.0 %	60.0 %							
Portfolio A + VSTOXX 1M Fut	36.5 %	60.0 %					3.5 %		
Portfolio A + VSTOXX 3M Fut	33.1 %	60.0 %						6.9 %	
Portfolio A + VSTOXX 5M Fut	29.5 %	60.0 %							10.5 %
Portfolio B	0.0 %	47.5 %	0.0 %	52.5 %	0.0 %	0.0 %			
Portfolio B + VSTOXX 1M Fut	0.0 %	45.7 %	0.0 %	54.0 %	0.0 %	0.0 %	0.3 %		
Portfolio B + VSTOXX 3M Fut	0.0 %	44.0 %	0.0 %	55.1 %	0.0 %	0.0 %		0.9 %	
Portfolio B + VSTOXX 5M Fut	0.0 %	40.9 %	0.0 %	56.9 %	0.0 %	0.0 %			2.3 %
Unconstrained GMV portfolios crisis period									
Portfolio A	-2.7 %	102.7 %							
Portfolio A + VSTOXX 1M Fut	6.3 %	90.6 %					3.1 %		
Portfolio A + VSTOXX 3M Fut	6.3 %	88.6 %						5.1 %	
Portfolio A + VSTOXX 5M Fut	6.8 %	86.3 %							6.9 %
Portfolio B	0.9 %	53.6 %	1.5 %	-31.5 %	80.3 %	-4.8 %			
Portfolio B + VSTOXX 1M Fut	5.9 %	50.0 %	3.9 %	-29.3 %	71.0 %	-3.8 %	2.3 %		
Portfolio B + VSTOXX 3M Fut	3.4 %	45.6 %	3.6 %	-32.4 %	76.7 %	-0.7 %		3.8 %	
Portfolio B + VSTOXX 5M Fut	3.7 %	49.1 %	4.1 %	-31.5 %	70.5 %	-0.8 %			4.9 %
Constrained GMV portfolios crisis period									
Portfolio A	40.0 %	60.0 %							
Portfolio A + VSTOXX 1M Fut	31.6 %	60.0 %					8.4 %		
Portfolio A + VSTOXX 3M Fut	28.2 %	60.0 %						11.8 %	
Portfolio A + VSTOXX 5M Fut	25.5 %	60.0 %							14.5 %
Portfolio B	0.0 %	54.9 %	0.0 %	0.0 %	45.1 %	0.0 %			
Portfolio B + VSTOXX 1M Fut	0.0 %	38.9 %	5.5 %	0.0 %	52.5 %	0.0 %	3.0 %		
Portfolio B + VSTOXX 3M Fut	0.0 %	39.9 %	4.9 %	0.0 %	50.6 %	0.0 %		4.6 %	
Portfolio B + VSTOXX 5M Fut	0.0 %	43.5 %	5.7 %	0.0 %	44.8 %	0.0 %			5.9 %
Unconstrained tangency portfolios crisis period									
Portfolio A	N/A	N/A							
Portfolio A + VSTOXX 1M Fut	38.1 %	44.6 %					17.3 %		
Portfolio A + VSTOXX 3M Fut	23.6 %	57.0 %						19.4 %	
Portfolio A + VSTOXX 5M Fut	21.9 %	56.7 %							21.4 %
Portfolio B	N/A	N/A	N/A	N/A	N/A	N/A			
Portfolio B + VSTOXX 1M Fut	65.6 %	145.9 %	-6.3 %	-41.5 %	-59.1 %	-19.0 %	14.3 %		
Portfolio B + VSTOXX 3M Fut	48.4 %	141.8 %	-14.0 %	-62.0 %	-23.2 %	-8.4 %		17.5 %	
Portfolio B + VSTOXX 5M Fut	32.2 %	105.1 %	-2.0 %	-46.0 %	-8.0 %	-1.3 %			19.8 %
Constrained tangency portfolios crisis period									
Portfolio A	60.0 %	40.0 %							
Portfolio A + VSTOXX 1M Fut	38.1 %	44.6 %					17.3 %		
Portfolio A + VSTOXX 3M Fut	23.6 %	57.0 %						19.4 %	
Portfolio A + VSTOXX 5M Fut	21.9 %	56.7 %							21.4 %
Portfolio B	60.0 %	40.0 %	0.0 %	0.0 %	0.0 %	0.0 %			
Portfolio B + VSTOXX 1M Fut	38.1 %	44.6 %	0.0 %	0.0 %	0.0 %	0.0 %	17.3 %		
Portfolio B + VSTOXX 3M Fut	23.6 %	57.0 %	0.0 %	0.0 %	0.0 %	0.0 %		19.4 %	
Portfolio B + VSTOXX 5M Fut	21.9 %	56.7 %	0.0 %	0.0 %	0.0 %	0.0 %			21.4 %

Appendix 9 Mean-variance optimal weights. Asset weights over ±1000% has been replaced with N/A