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Extensions and corona decompositions of low-dimensional intrinsic Lipschitz graphs in Heisenberg groups

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Abstract
This note concerns low-dimensional intrinsic Lipschitz graphs, in the sense of Franchi, Serapioni, and Serra Cassano, in the Heisenberg group $\mathbb{H}^n$, $n \in \mathbb{N}$. For $1 \leqslant k \leqslant n$, we show that every intrinsic L-Lipschitz graph over a subset of a k-dimensional horizontal subgroup $\mathbb{V}$ of $\mathbb{H}^n$ can be extended to an intrinsic L'-Lipschitz graph over the entire subgroup $\mathbb{V}$, where L' depends only on L, k, and n. We further prove that 1-dimensional intrinsic 1-Lipschitz graphs in $\mathbb{H}^n$, $n \in \mathbb{N}$, admit corona decompositions by intrinsic Lipschitz graphs with smaller Lipschitz constants. This complements results that were known previously only in the first Heisenberg group $\mathbb{H}^1$. The main difference to this case arises from the fact that for $1 \leqslant k < n$, the complementary vertical subgroups of k-dimensional horizontal subgroups in $\mathbb{H}^n$ are not commutative.

Keywords Heisenberg groups · Lipschitz extension · Corona decomposition · Low-dimensional intrinsic Lipschitz graphs

Mathematics Subject Classification 35R03 · 26A16 · 28A75

1 Introduction

This note deals with low-dimensional intrinsic Lipschitz graphs in Heisenberg groups. The *n-th Heisenberg group* $\mathbb{H}^n$ is the set $\mathbb{R}^{2n+1}$ with the group product "$\cdot$" given by

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$$
\begin{aligned}
&(x_1,\dots,x_{2n},t)\cdot(x_1',\dots,x_{2n}',t')\\
&\quad=\left(x_1+x_1',\dots,x_{2n}+x_{2n}',t+t'+\tfrac{1}{2}\sum_{i=1}^{n}x_ix_{n+i}'-x_i'x_{n+i}\right)
\end{aligned}
$$

for $(x_1,\dots,x_{2n},t),(x_1',\dots,x_{2n}',t')\in\mathbb{R}^{2n+1}$. We equip $\mathbb{H}^n$ with the left-invariant metric

$$
d(p,q):=\|q^{-1}\cdot p\|,\quad p,q\in\mathbb{H}^n, \tag{1.1}
$$

where $\|(x,t)\|:=\max\{|x|,\sqrt{|t|}\}$ and $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^{2n}$.

Intrinsic Lipschitz graphs (iLG) in $\mathbb{H}^n$ were introduced by Franchi, Serapioni, and Serra Cassano in [13]. The definition of *codimension-1* iLG is motivated by their appearance in connection with a structure theorem for sets of finite perimeter [12], see also [21–23] for various applications of such sets. The definition of iLG makes perfect sense also for *low dimensions*, but there are fewer works that study specifically low-dimensional iLG. Recently, they have appeared in [1, 2]. To the best of our knowledge, 1-dimensional iLG in $\mathbb{H}^1$ were first applied by Orponen and the second author in [11] to prove the boundedness of certain singular integral operators on regular curves in $\mathbb{H}^1$. The results of the present paper constitute a first step towards the generalization of [11] to higher-dimensional Heisenberg groups. At the same time, we believe that the results are of independent interest in geometric measure theory, as they complement the list of fundamental properties that low-dimensional iLG share with Euclidean Lipschitz graphs. This is our first main result:

Theorem 1.2 (Intrinsic Lipschitz extension) *Let $n\in\mathbb{N}$, $k\in\{1,\dots,n\}$, and assume that $\mathbb{V}$ is a k-dimensional horizontal subgroup of $\mathbb{H}^n$ with complementary vertical subgroup $\mathbb{W}$. Then, for every $L\geqslant 0$, there exists a constant $L'=L'(L,k,n)\geqslant 0$ such that every intrinsic L-Lipschitz function $\phi:E\to\mathbb{W}$, defined on a subset $E\subset\mathbb{V}$, can be extended to an intrinsic L'-Lipschitz function $\overline{\phi}:\mathbb{V}\to\mathbb{W}$. Moreover, if $k=n=1$, then one can take $L'=C(n)\max\{L,L^2\}$ where $C(n)\geqslant 1$ is a constant depending only on n.*

We defer the definitions to Sect. 2 and more precise statements to Sect. 4, and start by discussing the connection between Theorem 1.2 and other Lipschitz extension results.

A pair (X,Y) of metric spaces has the *Lipschitz extension property* if there exists a constant $C>0$ such that, for every $E\subset X$, every Lipschitz function $f:E\to Y$ can be extended to a Lipschitz function $\overline{f}:X\to Y$ with Lipschitz constant $\mathrm{Lip}(\overline{f})\leqslant C\,\mathrm{Lip}(f)$. It is known that the pair of metric spaces $(\mathbb{R}^k,\mathbb{H}^n)$ has the Lipschitz extension property if and only if $k\leqslant n$, see [5, 10, 17, 20, 26]. Theorem 1.2 is related to this result since every intrinsic Lipschitz function $\phi:\mathbb{V}\to\mathbb{W}$ (as in Theorem 1.2) is in one-to-one correspondence with a (metrically defined) Lipschitz function

$$
\Phi:E\subset(\mathbb{R}^k,|\cdot|)\to(\mathbb{H}^n,d)
$$

for which $\Phi(E)$ is an *intrinsic graph* in the sense of Definition 2.1, see Remark 2.4. So the point of Theorem 1.2 is to extend the Lipschitz function $\Phi:E\to\mathbb{H}^n$ to $\overline{\Phi}:\mathbb{R}^k\to\mathbb{H}^n$ in such a way that the intrinsic graph structure of the image is preserved.

While Theorem 1.2 thus yields a conclusion that does not follow from the general Lipschitz extension property of $(\mathbb{R}^k,\mathbb{H}^n)$ for $k\leqslant n$, our assumption is also stronger in that the image of the initially given, partially defined Lipschitz map Φ is an intrinsic graph. This additional information is very helpful in the construction of Lipschitz extensions, and it led us to a proof for Theorem 1.2 that is different from the extension methods used in [20, 26]. The case $k = n = 1$ was proven before in [11], but it also follows, with a different argument, from our proof of Theorem 1.2. A new phenomenon appears in higher-dimensional Heisenberg groups, where there is a qualitative difference between the condition for k-dimensional iLG in the middle dimension $k = n$ and in smaller dimensions $k < n$. We establish the case $k = n$ of Theorem 1.2 by applying a $C^{1,1}$ version of Whitney's extension theorem by Glaeser [16] to the last component of ϕ. The bridge between [16] and intrinsic Lipschitz graphs is provided by the infinitesimal condition appearing in Proposition 3.12. The extension theorem in the case $k < n$ can be deduced by suitably embedding k-dimensional graphs into n-dimensional graphs and applying the $k = n$ version of Theorem 1.2.

Theorem 1.2 complements extension results for low-*co*dimensional iLG in $\mathbb{H}^n$, proved in [14, 21] for codimension 1, and in [25] for codimension $k \leqslant n$. The proofs in [14, 21] use an argument similar to the classical McShane Lipschitz extension theorem, which is possible since 1-dimensional horizontal subgroups in $\mathbb{H}^n$ can be equipped with an order structure. The extension result in [25] is based on a new level set description of low-codimensional iLG. Neither of these approaches is available for low-dimensional iLG.

Theorem 1.2, the results mentioned in the last paragraph, and the Rademacher-type theorems in [1, 14, 25] show that all intrinsic Lipschitz functions between complementary homogeneous subgroups of $\mathbb{H}^n$ share two fundamental properties with Euclidean Lipschitz functions: the extension property and the almost everywhere differentiability. These are crucial features for applications in geometric measure theory. The second main result of the present paper establishes an additional property for 1-dimensional intrinsic 1-Lipschitz graphs, namely a corona decomposition by intrinsic Lipschitz graphs (possibly over different subgroups) with smaller constants. The corresponding result for Euclidean Lipschitz graphs plays a crucial role in the theory of quantitative rectifiability and singular integrals [8, 9, 24].

Theorem 1.3 (Intrinsic Lipschitz corona decomposition) *For every $n \in \mathbb{N}$ and $\eta \in (0, 1)$, every 1-dimensional intrinsic 1-Lipschitz graph in $\mathbb{H}^n$ admits a corona decomposition by 1-dimensional intrinsic η-Lipschitz graphs.*

A *corona decomposition* of a 1-dimensional (intrinsic) Lipschitz graph Γ is a hierarchical partitioning, called *coronization*, of $\mathbb{R}$ (or a 1-dimensional horizontal subgroup) into "good" and "bad" dyadic intervals, where the bad ones are controlled by a Carleson packing condition, and the good ones can be partitioned into a forest of trees satisfying suitable properties.

In particular, each tree $\mathcal{T}$ comes with an (intrinsic) Lipschitz graph $\Gamma_{\mathcal{T}}$ with smaller Lipschitz constant that approximates Γ well at the resolution of the intervals in the tree. A more precise statement is given in Theorem 5.26. Bearing in mind potential applications to singular integral operators, Theorem 5.26 states the approximation in parametric form, using maps defined on a common domain, rather than intrinsic graphs over possibly different horizontal subgroups.

In $\mathbb{H}^1$, a version of Theorem 5.26 was known before, see [11, Theorem 3.15]. Using related ideas in the context of $\mathbb{H}^n$, we give a proof for the case $n > 1$. Theorem 5.26 yields a corona decomposition for 1-dimensional intrinsic 1-Lipschitz maps. By fixing the trees of dyadic intervals in the coronization, and rescaling the components of the map, we also obtain a corona decomposition for all intrinsic Lipschitz maps with constant greater than 1, as stated in Corollary 5.30. This generalizes [11, Corollary 3.22].

In order to prove Theorem 5.26, we start by recalling the corona decomposition for Euclidean Lipschitz graphs given by David and Semmes in [9], which we then state for convenience in a slightly different form, Theorem 5.3. In a certain sense, it allows to approximate a 1-Lipschitz map by δ-Lipschitz maps for given $\delta \in (0,1)$, up to subtracting linear maps. A common challenge in the proof of the extension result (Theorem 1.2) and the corona decomposition (Theorem 1.3) is the presence of nonlinear terms in the intrinsic Lipschitz condition in dimensions $1 \leqslant k < n$, see Lemma 3.1, which are absent for $k = n$, and in particular for $n = 1$.

Structure of the paper. Most concepts relevant for the paper are introduced in Sect. 2. Section 3.1 contains standard computations related to low-dimensional iLG in Heisenberg groups. Section 3.2 provides an infinitesimal characterization of 1- and n-dimensional entire iLG in $\mathbb{H}^n$. The extension result, Theorem 1.2, is proven in Sect. 4. The corona decomposition, Theorem 1.3, is finally given in Sect. 5.

2 Definitions

2.1 Homogeneous subgroups, projections, and intrinsic Lipschitz graphs

Let $n \in \mathbb{N}$. To introduce the relevant concepts, we first fix a *horizontal subgroup* $\mathbb{V}$ of $\mathbb{H}^n$ of dimension $k \in \{1, \dots, n\}$, which is given by a set of the form

$$\mathbb{V} = V \times \{0\} \subset \mathbb{R}^{2n+1},$$

where V is a k-dimensional isotropic subspace of the standard symplectic space $\mathbb{R}^{2n}$. This is equivalent to say that V is a k-dimensional subspace of $\mathbb{R}^{2n}$ so that $(\mathbb{V}, \cdot)$ is an abelian group isomorphic to $(\mathbb{R}^k, +)$, see for instance [4, Section 2]. Equipped with the metric d defined in (1.1), the subgroup $\mathbb{V}$ is isometric to $(\mathbb{R}^k, |\cdot|)$. The *complementary vertical subgroup* $\mathbb{W}$ is given by the Euclidean orthogonal complement of $\mathbb{V}$, that is, $\mathbb{W} = V^{\perp} \times \mathbb{R}$.

Every point $p \in \mathbb{H}^n$ has a unique decomposition as

$$p = \pi_{\mathbb{V}}(p) \cdot \pi_{\mathbb{W}}(p) \quad \text{with} \quad \pi_{\mathbb{V}}(p) \in \mathbb{V} \text{ and } \pi_{\mathbb{W}}(p) \in \mathbb{W}.$$

Definition 2.1 Assume that $\mathbb{V}$ and $\mathbb{W}$ are homogeneous subgroups of $\mathbb{H}^n$ as above. A map $\phi : E \subset \mathbb{V} \to \mathbb{W}$ is said to be *intrinsic L-Lipschitz* for a constant $L \geqslant 0$ if

$$\|\pi_{\mathbb{W}}(\Phi(v')^{-1} \cdot \Phi(v))\| \leqslant L \|\pi_{\mathbb{V}}(\Phi(v')^{-1} \cdot \Phi(v))\|, \quad v, v' \in E,$$

where $\Phi : E \subset \mathbb{V} \to \mathbb{H}^n$ is the *graph map* defined by $\Phi(v) := v \cdot \phi(v)$. The *intrinsic graph* of ϕ is the set

$$\Gamma := \{v \cdot \phi(v) : \ v \in E\} \subset \mathbb{H}^n,$$

and we say that Γ is an *intrinsic L-Lipschitz graph (over $E \subset \mathbb{V}$)*.

It follows from [4, Lemma 2.1] and the choice of the metric d that for any pair of complementary homogeneous subgroups $(\mathbb{V}, \mathbb{W})$ and $(\mathbb{V}', \mathbb{W}')$ as above there exists an isometric isomorphism $f : (\mathbb{H}^n, d) \to (\mathbb{H}^n, d)$ with the properties that $f(\mathbb{V}) = \mathbb{V}'$, $f(\mathbb{W}) = \mathbb{W}'$ and such that f maps every intrinsic L-Lipschitz graph over a subset in $\mathbb{V}$ to an intrinsic

L-Lipschitz graph over a subset in $\mathbb{V}'$. For this reason, it is not restrictive to assume, as we will in the following unless otherwise stated, that

$$\mathbb{V} = \{(x_1, \dots, x_k, 0, \dots, 0) : (x_1, \dots, x_k) \in \mathbb{R}^k\} \tag{2.2}$$

and

$$\mathbb{W} = \{(0, \dots, 0, x_{k+1}, \dots, x_{2n}, t) : (x_{k+1}, \dots, x_{2n}, t) \in \mathbb{R}^{2n+1-k}\}. \tag{2.3}$$

Remark 2.4 It follows from [15, Proposition 3.7] that if $\phi : E \subset \mathbb{V} \to \mathbb{W}$ is intrinsic Lipschitz, then the associated graph map is a Lipschitz function Φ from (E, d) to $(\mathbb{H}^n, d)$, or from $(E, |\cdot|)$ to $(\mathbb{H}^n, d)$, if we identify E with a subset of $\mathbb{R}^k$, using the map

$$(x_1, \dots, x_k, 0, \dots, 0) \mapsto (x_1, \dots, x_k).$$

Conversely, if $\Phi : (E, |\cdot|) \to (\mathbb{H}^n, d)$ is L-Lipschitz with respect to the given metrics, and we assume in addition that it is of the form $\Phi(v) := v \cdot \phi(v) \in \mathbb{V} \cdot \mathbb{W}$ for a map $\phi : E \subset \mathbb{V} \to \mathbb{W}$, then ϕ is intrinsic Lipschitz since, for all $v, v' \in E$,

$$\begin{aligned}\|\pi_{\mathbb{W}}(\Phi(v')^{-1} \cdot \Phi(v))\| &\leqslant d(\Phi(v), \Phi(v')) + \|\pi_{\mathbb{V}}(\Phi(v')^{-1} \cdot \Phi(v))\| \\ &\leqslant L|v - v'| + \|\pi_{\mathbb{V}}(\Phi(v')^{-1} \cdot \Phi(v))\| \\ &= (L+1)\|\pi_{\mathbb{V}}(\Phi(v')^{-1} \cdot \Phi(v))\|.\end{aligned}$$

Once complementary subgroups as in (2.2) and (2.3) have been fixed, it is convenient to identify $\phi : E \subset \mathbb{V} \to \mathbb{W}$ with a function $\phi : E \subset \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$ in the obvious way. This identification applied to intrinsic Lipschitz functions leads to the notion of *tame maps* which we discuss in the next section, see especially Propositions 3.3 and 3.6.

2.2 Tame maps

In connection with one-dimensional intrinsic Lipschitz graphs in $\mathbb{H}^1$, *tame maps* from subsets of $\mathbb{R}^k$ to $\mathbb{R}^{2n+1-k}$ for $k = n = 1$ were introduced in [11]. We extend the definition to arbitrary $1 \leqslant k \leqslant n$ with a slight adaptation of the notation. Here, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^k$.

Definition 2.5 Let $k, n \in \mathbb{N}$, $1 \leqslant k \leqslant n$, $E \subset \mathbb{R}^k$, and $L_i \geqslant 0$ for $i \in \{k+1, \dots, 2n+1\}$. We say that a map $\phi = (\phi_{k+1}, \dots, \phi_{2n+1}) : E \to \mathbb{R}^{2n+1-k}$ is $(L_{k+1}, \dots, L_{2n+1})$-*tame* if

(1) ϕ_i is Euclidean L_i-Lipschitz for $i = k+1, \dots, 2n$;
(2) $\psi := (\phi_{n+1}, \dots, \phi_{n+k})$ satisfies the following conditions:

  (a) if $k = n$, then

$$\left|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y - x\rangle\right| + \left|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(x), y - x\rangle\right| \leqslant L_{2n+1}|x-y|^2, \quad x, y \in E,$$

  (b) if $k < n$, then

$$
\begin{aligned}
&\Big|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y-x\rangle \\
&\qquad - \tfrac{1}{2}\sum_{i=k+1}^{n} \phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y)\Big| \\
&\qquad + \Big|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(x), y-x\rangle - \tfrac{1}{2}\sum_{i=k+1}^{n} \phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y)\Big| \\
&\quad \leqslant L_{2n+1}|y-x|^2, \quad x, y \in E.
\end{aligned}
$$

Remark 2.6 Condition (2) in Definition 2.5 is implied (with twice the constant L_{2n+1}) by a one-sided version of itself:

$$
\left|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y-x\rangle\right| \leqslant L_{2n+1}|x-y|^2, \quad x, y \in E, \quad \text{if } k = n,
$$

and

$$
\Big|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y-x\rangle - \tfrac{1}{2}\sum_{i=k+1}^{n} \phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y)\Big| \leqslant L_{2n+1}|y-x|^2,
$$

for $x, y \in E$, if $k < n$.

Remark 2.7 Condition (2) in Definition 2.5 implies by triangle inequality that

$$
\left|\left\langle \psi(y) - \psi(x), \frac{y-x}{|y-x|}\right\rangle\right| \leqslant L_{2n+1}|x-y|, \quad \text{for } x, y \in E, x \neq y. \tag{2.8}
$$

If $k = 1$, then $\psi = \phi_{n+1}$ is a real-valued function and (2.8) shows that ϕ_{n+1} is L_{2n+1} -Lipschitz. In other words, if $k = 1$, then the Lipschitz continuity of ϕ_{n+1} is automatically implied by part (2) of Definition 2.5, and part (1) holds with "L_{n+1}" replaced by "$\min\{L_{n+1}, L_{2n+1}\}$".

Remark 2.9 For all $1 \leqslant k \leqslant n$, Definition 2.5 implies that ϕ_{2n+1} is locally Lipschitz. This is immediate in the case $k = n$, and if $k < n$, it follows easily once one has observed that

$$
\begin{aligned}
&\sum_{i=k+1}^{n} \phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y) \\
&\quad = \sum_{i=k+1}^{n} \big(\phi_i(y) - \phi_i(x)\big)\phi_{n+i}(x) - \phi_i(x)\big(\phi_{n+i}(y) - \phi_{n+i}(x)\big).
\end{aligned} \tag{2.10}
$$

3 Elementary properties of tame maps

3.1 Connection between tame maps and intrinsic Lipschitz functions

In this section, we explore the connection between intrinsic Lipschitz functions (as in Definition 2.1) and tame maps (as in Definition 2.5). It is this connection that initially motivated Definition 2.5. Throughout this section, we assume that $1 \leqslant k \leqslant n$, and $\mathbb{V}$ is a k-dimensional horizontal subgroup of $\mathbb{H}^n$ with complementary vertical subgroup $\mathbb{W}$ with coordinate expressions as in (2.2) and (2.3). Slightly abusing notation, we identify a set $E \subset \mathbb{V}$ with $E \subset \mathbb{R}^k$, and $\phi : E \to \mathbb{W}$ with $\phi : E \to \mathbb{R}^{2n+1-k}$.

Lemma 3.1 *A function $(\phi_{k+1}, \ldots, \phi_{2n+1}) : E \subset \mathbb{V} \to \mathbb{W}$ is intrinsic L-Lipschitz if and only if*

$$\|(0, \ldots, 0, \phi_{k+1}(v') - \phi_{k+1}(v), \ldots, \phi_{2n}(v') - \phi_{2n}(v), H(v, v'))\| \leqslant L|v' - v|, \quad v, v' \in E,$$

where

$$H(v, v') := \phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v \rangle, \quad \text{if } k = n,$$

and, if $k < n$, then

$$H(v, v') := \phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v \rangle + \frac{1}{2} \sum_{i=k+1}^{n} \phi_i(v')\phi_{n+i}(v) - \phi_i(v)\phi_{n+i}(v').$$

Proof We recall from Definition 2.1 that ϕ is intrinsic L-Lipschitz if and only if

$$\|\pi_{\mathbb{W}}(\Phi(v)^{-1} \cdot \Phi(v'))\| \leqslant L\|\pi_{\mathbb{V}}(\Phi(v)^{-1} \cdot \Phi(v'))\|, \quad v, v' \in E. \tag{3.2}$$

The graph map Φ of ϕ is given by

$$\Phi(v) = \Big(v, \phi_{k+1}(v), \ldots, \phi_{2n}(v), \phi_{2n+1}(v) + \tfrac{1}{2} \sum_{i=1}^{k} v_i \phi_{n+i}(v)\Big),$$

for $v = (v_1, \ldots, v_k) \in E$. Recalling that $\psi(v) = (\phi_{n+1}(v), \ldots, \phi_{n+k}(v))$, we observe

$$\Phi(v)^{-1} \cdot \Phi(v') = \big(v' - v, \phi_{k+1}(v') - \phi_{k+1}(v), \ldots, \phi_{2n}(v') - \phi_{2n}(v), h(v, v')\big),$$

where

$$h(v, v') := \phi_{2n+1}(v') - \phi_{2n+1}(v) + \tfrac{1}{2}\langle v' - v, \psi(v) + \psi(v')\rangle, \quad \text{if } k = n,$$

and, if $k < n$, then $h(v, v')$ is equal to

$$\phi_{2n+1}(v') - \phi_{2n+1}(v) + \tfrac{1}{2}\langle v' - v, \psi(v) + \psi(v')\rangle - \tfrac{1}{2} \sum_{i=k+1}^{n} \big(\phi_i(v)\phi_{n+i}(v') - \phi_i(v')\phi_{n+i}(v)\big).$$

Next, since $\pi_{\mathbb{V}}(x_1, \ldots, x_{2n}, t) = (x_1, \ldots, x_k, 0, \ldots, 0)$ and

$$\pi_{\mathbb{W}}(x_1, \ldots, x_{2n}, t) = \Big(0, \ldots, 0, x_{k+1}, \ldots, x_{2n}, t - \tfrac{1}{2} \sum_{i=1}^{k} x_i x_{n+i}\Big),$$

the right-hand side of (3.2) equals $L|v' - v|$, and the left-hand side can be written as

$$\|\pi_{\mathbb{W}}(\Phi(v)^{-1} \cdot \Phi(v'))\| = \|(0, \ldots, 0, \phi_{k+1}(v') - \phi_{k+1}(v), \ldots, \phi_{2n}(v') - \phi_{2n}(v), H(v, v'))\|,$$

for $v, v' \in E$, where $H(v, v')$ is defined as in the statement of the lemma. $\square$

Lemma 3.1 provides a link between intrinsic Lipschitz and tame maps. We formulate this in two separate propositions.

Proposition 3.3 *If* $\phi = (\phi_{k+1}, \dots, \phi_{2n+1}) : E \subset \mathbb{V} \to \mathbb{W}$ *is intrinsic L-Lipschitz, then* $(\phi_{k+1}, \dots, \phi_{2n}, -\phi_{2n+1})$ *is an* $(L_{k+1}, \dots, L_{2n+1})$*-tame map from* $E \subset \mathbb{R}^k$ *to* $\mathbb{R}^{2n+1-k}$ *with*

$$L_i = \begin{cases} L, & \text{for } i = k+1, \dots, 2n, \\ 2L^2, & \text{for } i = 2n+1. \end{cases} \tag{3.4}$$

If $k = 1$, *then one can take* $L_{n+1} = \min\{L, 2L^2\}$.

Proof Once the tameness is established, the improvement for $k = 1$ follows from (3.4) by Remark 2.7. Hence, it remains to prove the first part of the Proposition. Let ϕ be an intrinsic L-Lipschitz function. According to Lemma 3.1, this means that

$$\|(0, \dots, 0, \phi_{k+1}(v') - \phi_{k+1}(v), \dots, \phi_{2n}(v') - \phi_{2n}(v), H(v, v'))\| \leqslant L|v' - v|, \quad v, v' \in E, \tag{3.5}$$

where

$$H(v, v') = \phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v \rangle, \quad \text{if } k = n,$$

and

$$H(v, v') = \phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v \rangle + \tfrac{1}{2} \sum_{i=k+1}^{n} \phi_i(v')\phi_{n+i}(v) - \phi_i(v)\phi_{n+i}(v'),$$

if $k < n$. Recalling that $\|(x, t)\| = \max\{|x|, \sqrt{|t|}\}$ for $(x, t) \in \mathbb{R}^{2n} \times \mathbb{R}$, inequality (3.5) implies first that ϕ_i is a Euclidean L-Lipschitz function for $i = k+1, \dots, 2n$, which is part (1) of the tameness condition in Definition 2.5. Second, we deduce from (3.5) that

$$\left|\phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v \rangle\right|^{1/2} \leqslant L|v' - v|, \quad v, v' \in E, \quad \text{if } k = n,$$

and

$$\left|\phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v \rangle + \tfrac{1}{2} \sum_{i=k+1}^{n} \phi_i(v')\phi_{n+i}(v) - \phi_i(v)\phi_{n+i}(v')\right|^{1/2} \leqslant L|v' - v|,$$

for $v, v' \in E$ if $k < n$. Hence, $(\phi_2, \dots, \phi_{2n}, -\phi_{2n+1})$ is $(L, \dots L, 2L^2)$-tame in both cases.

□

We now consider the converse implication.

Proposition 3.6 *If* $(\phi_{k+1}, \dots, \phi_{2n}, -\phi_{2n+1}) : E \subset \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$ *is an* $(L_{k+1}, \dots, L_{2n+1})$*-tame map, then* $\phi = (\phi_{k+1}, \dots, \phi_{2n+1}) : E \subset \mathbb{V} \to \mathbb{W}$ *is intrinsic L-Lipschitz with*

$$L := \max\left\{|(L_{k+1}, \dots, L_{2n})|, \sqrt{L_{2n+1}}\right\}.$$

Proof If $(\phi_{k+1}, \dots, \phi_{2n}, -\phi_{2n+1})$ is $(L_{k+1}, \dots, L_{2n+1})$-tame, we find by the first condition in Definition 2.5 that for $i = k+1, \dots, 2n$, the function ϕ_i is L_i-Lipschitz on E. Moreover, recalling that

$$\psi(v) = (\phi_{n+1}(v), \dots, \phi_{n+k}(v)),$$

the second condition in the tameness definition for $(\phi_{k+1}, \dots, \phi_{2n}, -\phi_{2n+1})$ reads as follows: if $k = n$,

$$\begin{aligned}&\left|\phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v'), v' - v\rangle\right| + \left|\phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v\rangle\right| \\ &\leqslant L_{2n+1}|v' - v|^2,\end{aligned}$$

and, if $k < n$,

$$\begin{aligned}&|\phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v'), v' - v\rangle + \tfrac{1}{2}\sum_{i=k+1}^{n} \phi_i(v')\phi_{n+i}(v) - \phi_i(v)\phi_{n+i}(v')| \\ &\quad + |\phi_{2n+1}(v') - \phi_{2n+1}(v) + \langle \psi(v), v' - v\rangle + \tfrac{1}{2}\sum_{i=k+1}^{n} \phi_i(v')\phi_{n+i}(v) - \phi_i(v)\phi_{n+i}(v')| \\ &\leqslant L_{2n+1}|v' - v|^2,\end{aligned}$$

for all $v, v' \in E$.

Using Lemma 3.1, we conclude that $\phi := (\phi_{k+1}, \dots, \phi_{2n+1}) : E \to \mathbb{W}$ is an intrinsic L-Lipschitz function since its graph map satisfies

$$\|\pi_{\mathbb{W}}(\Phi(v')^{-1} \cdot \Phi(v))\| \leqslant \max\left\{|(L_{k+1}, \dots, L_{2n})|, \sqrt{L_{2n+1}}\right\}|v - v'| = L|v - v'|, \quad v, v' \in E.$$

□

3.2 Infinitesimal condition for tame maps on open sets

Tame maps defined on open quasiconvex sets can be characterized by an infinitesimal condition. This characterization will be applied in the proofs of the main results of this paper, the extension and the corona decomposition for low-dimensional intrinsic Lipschitz graph. We first discuss the case $k = 1$ and $n > 1$, which will be used in the proof of Theorem 5.26.

Proposition 3.7 *Assume that $n > 1$. Let $I \subset \mathbb{R}$ be an open interval, and let $\phi = (\phi_2, \dots, \phi_{2n+1}) : I \to \mathbb{R}^{2n}$.*

(1) *If ϕ is $(L_2, \dots, L_{2n+1})$-tame, then ϕ_i is L_i-Lipschitz for $i = 2, \dots, 2n$, and ϕ_{2n+1} is differentiable almost everywhere on I, $\dot{\phi}_{2n+1} \in \mathcal{L}^\infty_{loc}(I)$, and*

$$\dot{\phi}_{2n+1} = \phi_{n+1} + \tfrac{1}{2}\sum_{i=2}^{n} \dot{\phi}_i\phi_{n+i} - \phi_i\dot{\phi}_{n+i}, \quad \text{a.e. on } I. \tag{3.8}$$

(2) *Conversely, if ϕ_i is L_i-Lipschitz for $i = 2, \dots, 2n$, ϕ_{2n+1} is locally Lipschitz, and (3.8) holds, then ϕ is $(L'_2, \dots, L'_{2n+1})$-tame with*

$$L'_i := L_i \quad \text{for } i \neq 2n+1 \quad \text{and} \quad L'_{2n+1} := 2\left(L_{n+1} + \sum_{i=2}^{n} L_i L_{n+i} \right).$$

Proof We assume first that ϕ is $(L_2, \dots, L_{2n+1})$-tame, in particular, ϕ_i is a Lipschitz function on I for $i = 2, \dots, 2n$. Rademacher's theorem implies that ϕ_i is differentiable almost everywhere on I with bounded derivative. Condition (2) in Definition 2.5 reads

$$\begin{aligned} &\left| \frac{\phi_{2n+1}(y) - \phi_{2n+1}(x)}{y-x} - \phi_{n+1}(y) - \frac{1}{2}\sum_{i=2}^{n} \frac{\phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y)}{y-x} \right| \\ &+ \left| \frac{\phi_{2n+1}(y) - \phi_{2n+1}(x)}{y-x} - \phi_{n+1}(x) - \frac{1}{2}\sum_{i=2}^{n} \frac{\phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y)}{y-x} \right| \leqslant L_{2n+1}|y-x|, \end{aligned} \tag{3.9}$$

for all $x, y \in I$ with $x \neq y$, and formula (2.10) for $k = 1$ is

$$\phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y) = \phi_i(y)(\phi_{n+i}(x) - \phi_{n+i}(y)) - \phi_{n+i}(y)(\phi_i(x) - \phi_i(y)). \tag{3.10}$$

Using these two facts, it is easy to see that $\dot\phi_{2n+1}$ exists almost everywhere on I and (3.8) holds. In particular, $\dot\phi_{2n+1} \in \mathcal{L}^\infty_{loc}(I)$.

Conversely, assume that ϕ_i is an L_i–Lipschitz function for $i = 2, \dots, 2n$ and ϕ_{2n+1} is a locally Lipschitz function satisfying (3.8). Then, the corresponding one-sided version of (3.9) is satisfied for "$L_{n+1} + \sum_{i=2}^{n} L_i L_{n+i}$" instead of "$L_{2n+1}$". Indeed, for $x, y \in I$ with $x < y$, the expression (3.10) can be rewritten as

$$\phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y) = -\phi_i(y)\int_x^y \dot\phi_{n+i}(s)\,ds + \phi_{n+i}(y)\int_x^y \dot\phi_i(s)\,ds, \tag{3.11}$$

and we obtain that

$$\begin{aligned} &\Big|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \phi_{n+1}(y)(y-x) \\ &\quad -\frac{1}{2}\sum_{i=2}^{n} \phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y)\Big| \\ &\overset{(3.11)}{=} \Big| \int_x^y \dot\phi_{2n+1}(s)\,ds - \int_x^y \phi_{n+1}(y)\,ds \\ &\quad + \tfrac{1}{2}\sum_{i=2}^{n} \phi_i(y)\int_x^y \dot\phi_{n+i}(s)\,ds - \phi_{n+i}(y)\int_x^y \dot\phi_i(s)\,ds \Big| \\ &\overset{(3.8)}{=} \Big| \int_x^y \phi_{n+1}(s) - \phi_{n+1}(y) \\ &\quad + \tfrac{1}{2}\sum_{i=2}^{n} \dot\phi_i(s)[\phi_{n+i}(s) - \phi_{n+i}(y)] + \dot\phi_{n+i}(s)[\phi_i(y) - \phi_i(s)]\,ds \Big| \\ &\leqslant \left(L_{n+1} + \sum_{i=2}^{n} L_i L_{n+i} \right)|y-x|^2, \end{aligned}$$

where in the last inequality we used the fact that ϕ_i is L_i–Lipschitz for every $i = 2, \dots, 2n$. $\square$

If $n > 1$, there is a fundamental difference between tame maps $\phi : E \subset \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$ for $k = n$ and for $k < n$. This difference is visible already in part (2) of Definition 2.5, where the expression for $k < n$ contains an additional summand compared to the one for $k = n$. The simple form of tame maps if $k = n$ can be used to characterize them by means of a gradient equation for the last component, at least if E is open and quasiconvex. Recall that a set $U \subset \mathbb{R}^n$ is *C-quasiconvex* for a constant $C \geqslant 1$ (with respect to the Euclidean distance) if for all $x, y \in U$, there is a curve γ connecting x to y inside U of Euclidean length $\mathrm{length}(\gamma) \leqslant C|x-y|$.

Proposition 3.12 *Let $n \in \mathbb{N}$ and assume that U is an open subset of $\mathbb{R}^n$. For a function $\phi = (\phi_{n+1}, \dots, \phi_{2n+1}) : U \to \mathbb{R}^{n+1}$ the following holds:*

(1) *If ϕ is $(L_{n+1}, \dots, L_{2n+1})$-tame, then ϕ_i, $i = n+1, \dots, 2n$, is Euclidean L_i-Lipschitz and ϕ_{2n+1} is differentiable on U with Lipschitz continuous gradient*

$$\nabla \phi_{2n+1} = (\phi_{n+1}, \dots, \phi_{2n}) \quad \text{on } U. \tag{3.13}$$

In particular, $\phi_{2n+1} \in C^{1,1}(U)$.

(2) *If U is additionally assumed to be C-quasiconvex, if $(\phi_{n+1}, \dots, \phi_{2n})$ is L-Lipschitz with respect to the Euclidean metric, and ϕ_{2n+1} satisfies (3.13), then ϕ is $(L'_{n+1}, \dots, L'_{2n+1})$-tame with*

$$L'_i := L \quad \text{for } i \in \{n+1, \dots, 2n\} \quad \text{and} \quad L'_{2n+1} := 2C^2 L.$$

Remark 3.14 If $k < n$, then one can still carry out the argument in the first part of the proof of Proposition 3.12 for tame $\phi : U \subset \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$, but the conclusion is that ϕ_{2n+1} satisfies

$$\nabla \phi_{2n+1} = \begin{pmatrix} \phi_{n+1} + \frac{1}{2} \sum_{i=k+1}^{n} \phi_{n+i}\, \partial_{x_1} \phi_i - \phi_i\, \partial_{x_1} \phi_{n+i} \\ \vdots \\ \phi_{n+k} + \frac{1}{2} \sum_{i=k+1}^{n} \phi_{n+i}\, \partial_{x_k} \phi_i - \phi_i\, \partial_{x_k} \phi_{n+i} \end{pmatrix} \quad \text{almost everywhere on } U,$$

cf. Proposition 3.7 for $k = 1$. Since ϕ_i, ϕ_{n+i}, $i \in \{k+1, \dots, n\}$, are merely Lipschitz functions, they are only almost everywhere differentiable and the derivatives are just bounded measurable functions, so one cannot conclude that ϕ_{2n+1} is $C^{1,1}(U)$.

Remark 3.15 Proposition 3.12 yields a self-improvement phenomenon for the tameness constant L_{2n+1} of a $(L_{n+1}, \dots, L_{2n+1})$-tame map $(\phi_{n+1}, \dots, \phi_{2n+1}) : U \to \mathbb{R}^{n+1}$ defined on an open and C-quasiconvex set $U \subset \mathbb{R}^n$. By assumption, such $(\phi_{n+1}, \dots, \phi_{2n})$ is Euclidean $|(L_{n+1}, \dots, L_{2n})|$-Lipschitz on U, and by Proposition 3.12 (1), the last component ϕ_{2n+1} is differentiable on U with $\nabla \phi_{2n+1} = (\phi_{n+1}, \dots, \phi_{2n})$. It then follows from part (2) of the same proposition that $(\phi_{n+1}, \dots, \phi_{2n+1})$ is in fact tame with constants

$$L_i = \begin{cases} |(L_{n+1}, \dots, L_{2n})|, & i = n+1, \dots, 2n, \\ 2C^2 |(L_{n+1}, \dots, L_{2n})|, & i = 2n+1. \end{cases}$$

Hence, the initially given tameness constant "L_{2n+1}" can be replaced by

$$\min\{L_{2n+1}, 2C^2 |(L_{n+1}, \dots, L_{2n})|\}.$$

In particular, if $U = \mathbb{R}^n$, then this holds with $C = 1$.

Remark 3.16 The correspondence between intrinsic Lipschitz and tame maps relates Proposition 3.12 to earlier results by Magnani [20] and the second author [10], keeping in mind the connection to metric Lipschitz functions explained in Remark 2.4. More precisely, [20, Theorem 1.1] and [20, Theorem 4.5] provide a characterization of (locally) Lipschitz functions Φ from (geodetically convex) subsets of Riemannian manifolds into graded groups through a system of first order PDEs known as *weak contact equations*. This characterization applies in particular in our setting, where the source space is Euclidean space $\mathbb{R}^k$ and the target space is the Heisenberg group $\mathbb{H}^n$. The purpose of Proposition 3.12 is to show that if $\Phi : \mathbb{R}^k \to \mathbb{H}^n$ arises as graph map of an intrinsic Lipschitz function ϕ, and if $k = n$, then this characterization takes a particularly simple form and leads to a gradient equation for the last component of ϕ that holds in the classical sense pointwise everywhere. This generalizes an observation made in [11]: the condition for a curve γ in $\mathbb{H}^1$ to be horizontal (or Lipschitz with respect to d) simplifies if $s \mapsto \gamma(s) = (s, 0, 0) \cdot \phi(s, 0, 0)$ has intrinsic graph form. Indeed, whereas the last component of a Lipschitz curve γ in $\mathbb{H}^1$ need not even be everywhere differentiable, the last component of ϕ is $C^{1,1}$ if γ is Lipschitz.

Proof of Proposition 3.12 We assume first that ϕ is $(L_{n+1}, \dots, L_{2n+1})$-tame, in particular, ϕ_i is a Lipschitz function on U for $i = n+1, \dots, 2n$. Condition (2) (a) in Definition 2.5 and the fact that U is open then imply that $\nabla\phi_{2n+1}$ exists on U and (3.13) holds.

For the converse implication, we assume in addition that U is C-quasiconvex. We claim that if $(\phi_{n+1}, \dots, \phi_{2n})$ is an L-Lipschitz function, and (3.13) holds, then the tameness condition (2) (a) in Definition 2.5 is satisfied with constant

$$L'_{2n+1} := 2C^2L$$

(instead of L_{2n+1}). According to Remark 2.6, it suffices to verify the one-sided version of it (without the constant "2"). To prove the latter, let x and y be arbitrary distinct points in U, and apply the C-quasiconvexity of U to find a curve $\gamma : [0, 1] \to U$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\mathrm{length}(\gamma) \leqslant C|x - y|$. Since γ is a curve of finite length, we may without loss of generality assume that the parametrization is Lipschitz continuous. The fundamental theorem of calculus then yields for $\psi := (\phi_{n+1}, \dots, \phi_{2n})$ that

$$\begin{aligned}
\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y - x\rangle &= \phi_{2n+1}(\gamma(1)) - \phi_{2n+1}(\gamma(0)) - \langle \psi(\gamma(1)), \gamma(1) - \gamma(0)\rangle \\
&= \int_0^1 (\phi_{2n+1} \circ \gamma)'(s) - \langle \psi(\gamma(1)), \dot\gamma(s)\rangle \, ds \\
&= \int_0^1 \langle \nabla\phi_{2n+1}(\gamma(s)), \dot\gamma(s)\rangle - \langle \psi(\gamma(1)), \dot\gamma(s)\rangle \, ds \\
&\overset{(3.13)}{=} \int_0^1 \langle \psi(\gamma(s)) - \psi(\gamma(1)), \dot\gamma(s)\rangle \, ds.
\end{aligned}$$

Taking absolute values on both sides, we conclude that

$$\begin{aligned}
|\phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y - x\rangle| &\leqslant \int_0^1 \mathrm{Lip}(\psi)|\gamma(s) - \gamma(1)|\,|\dot\gamma(s)| \, ds \\
&\leqslant \mathrm{Lip}(\psi)\,\mathrm{length}(\gamma)^2 \leqslant C^2L|x - y|^2.
\end{aligned}$$

$\square$

4 Extension results

The core of this section are extension results for tame maps $\phi : E \subset \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$, first for $k = n$ and then, as a corollary, for $k < n$. In the final subsection, we use Propositions 3.3 and 3.6 to translate these results into an extension theorem for intrinsic Lipschitz functions (Theorem 1.2 in the introduction).

4.1 Extension of tame maps in the case $k = n$

A classical method for extending Lipschitz functions $f : E \to \mathbb{R}^m$ from a closed set $E \subset \mathbb{R}^n$ to the entire space $\mathbb{R}^n$ is based on a Lipschitz partition of unity associated with a Whitney decomposition of the complement $\mathbb{R}^n \setminus E$, see for instance [18, 2.10]. Variants of this approach are also at the core of the Lipschitz extension theorems in [19, 26]. To establish the main result of this section, we apply a version of Whitney's extension theorem, so that the proof is again, albeit indirectly, based on a Whitney decomposition of $\mathbb{R}^n \setminus E$. The key observation is that in our setting it suffices to apply Whitney's construction *to the last component* of the tame map. More precisely, we will use a $C^{1,1}$ version of Whitney's extension theorem due to Glaeser [16], see for instance [6, Definition 2.2, (2.48), and Theorem 2.19], [7, Lemma 10.70], and the references cited in [3]. Here, $C^{1,1}(\mathbb{R}^n)$ is the space of $C^1(\mathbb{R}^n)$ functions with Lipschitz continuous gradients.

Theorem 4.1 (Glaeser's $C^{1,1}$ Whitney extension theorem) *Let $n \in \mathbb{N}$ and assume that E is a subset of $\mathbb{R}^n$. The following conditions for functions $f : E \to \mathbb{R}$ and $\psi : E \to \mathbb{R}^n$ are equivalent:*

(a) *there exists $\overline{f} \in C^{1,1}(\mathbb{R}^n)$ with $\overline{f}|_E = f$ and $(\nabla \overline{f})|_E = \psi$,*
(b) *for a constant $\lambda > 0$ and all $x, y \in E$, the following holds:*

   (1) $|\psi(x) - \psi(y)| \leqslant \lambda |x - y|$,
   (2) $|f(x) - f(y) - \langle \psi(x), x - y\rangle| \leqslant \lambda |x - y|^2$.

Moreover, if (b) holds, then $\overline{f}$ can be constructed so that the Lipschitz constant of $\nabla \overline{f}$ satisfies

$$\inf \lambda \leqslant \mathrm{Lip}(\nabla \overline{f}) \leqslant C(n) \inf \lambda,$$

where the inf *ranges over all λ satisfying (b), and $C(n)$ is a constant depending only on n.*

Remark 4.2 Theorem 4.1 is stated for arbitrary subsets E of $\mathbb{R}^n$, and the same holds true for our application in Theorem 4.3 and the corollaries thereof. As observed in [3, §1], if f and ψ satisfy condition (b) in Theorem 4.1 for a set $E \subset \mathbb{R}^n$, then one can always extend them to the closure $\overline{E}$ of E so that inequalities (1) and (2) in (b) are satisfied on $\overline{E}$ with the same constant λ. (Extending f and ψ as continuous maps to the closure is straightforward since f is locally Lipschitz and ψ is Lipschitz; then, it just remains to verify that the inequalities (1) and (2) continue to hold on $\overline{E}$.) Conversely, if f and ψ defined on E satisfy condition (a), then they can obviously be extended to $\overline{E}$ so that (a) continues to hold. Thus, the proof of Theorem 4.1 is reduced to the case of closed sets.

Theorem 4.3 *Let $n \in \mathbb{N}$. An $(L_{n+1}, \dots, L_{2n+1})$-tame map $\phi : E \subseteq \mathbb{R}^n \to \mathbb{R}^{n+1}$ can be extended to an $(L'_{n+1}, \dots, L'_{2n+1})$-tame map $\overline{\phi} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ such that $\overline{\phi}|_E = \phi$ and*

$$\begin{aligned} L'_i &:= C(n) \max\left\{ |(L_{n+1}, \dots, L_{2n})|, L_{2n+1} \right\}, \quad \text{for } i \in \{n+1, \dots, 2n\}, \\ L'_{2n+1} &:= 2C(n) \max\{ |(L_{n+1}, \dots, L_{2n})|, L_{2n+1} \}. \end{aligned}$$

Proof of Theorem 4.3 Let ϕ be a tame map on $E \subset \mathbb{R}^n$ as in the statement of the theorem. In order to extend ϕ to a tame map defined on all of $\mathbb{R}^n$, we apply the $C^{1,1}$ Whitney extension theorem the function $\phi_{2n+1} : E \to \mathbb{R}$. The tameness conditions in Definition 2.5 ensure that the assumptions of Theorem 4.1 are satisfied with

$$\lambda := \max\left\{ |(L_{n+1}, \dots, L_{2n})|, L_{2n+1} \right\}.$$

Thus we find a $C^{1,1}$ function $\overline{\phi}_{2n+1} : \mathbb{R}^n \to \mathbb{R}$ with $\overline{\phi}_{2n+1}|_E = \phi_{2n+1}$ whose gradient is $C(n)\lambda$ -Lipschitz and agrees with $\psi = (\phi_{n+1}, \dots, \phi_{2n})$ on E. Then, we simply define

$$(\overline{\phi}_{n+1}, \dots, \overline{\phi}_{2n}) := \nabla \overline{\phi}_{2n+1}, \tag{4.4}$$

and observe that this extends $(\phi_{n+1}, \dots, \phi_{2n})$ from E to $\mathbb{R}^n$. Thus,

$$\overline{\phi} := (\nabla \overline{\phi}_{2n+1}, \overline{\phi}_{2n+1}) = (\overline{\phi}_{n+1}, \dots, \overline{\phi}_{2n}, \overline{\phi}_{2n+1})$$

is an extension of ϕ to the entire space $\mathbb{R}^n$.

Finally, we apply the infinitesimal characterization from Proposition 3.12 to conclude that $\overline{\phi}$ is a tame map. Since the domain $U = \mathbb{R}^n$ is quasiconvex with constant 1, Proposition 3.12 (2) implies that $\overline{\phi}$ is $(L'_{n+1}, \dots, L'_{2n}, L'_{2n+1})$-tame with

$$L'_i = C(n)\lambda \quad \text{for all } i \in \{n+1, \dots, 2n\} \quad \text{and} \quad L'_{2n+1} = 2C(n)\lambda.$$

□

Remark 4.5 Applying Theorem 4.1 to the last component ϕ_{2n+1} of a tame map ϕ and extending $\psi = (\phi_{n+1}, \dots, \phi_{2n})$ by formula (4.4) ensures that

$$\nabla \overline{\phi}_{2n+1} = (\overline{\phi}_{n+1}, \dots, \overline{\phi}_{2n+1})$$

is satisfied *by definition*. If, on the other hand, one tried to extend $(\phi_{n+1}, \dots, \phi_{2n})$ first, then one would have to make sure that the extension can arise as gradient, and this would entail a further differential constraint for $\overline{\phi}_i$, $i \in \{n+1, \dots, 2n\}$, cf. the related *isotropic mappings* appearing in [10, 20].

4.2 Extension of tame maps in the case $k < n$

In this section, we prove the extension result for tame maps $\phi : E \subset \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$ in the case $k < n$. The situation is qualitatively different from the middle-dimensional case $k = n$ discussed in the previous section. Indeed, recall from Remark 3.14 that the last component of an entire tame map $\phi : \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$ satisfies almost everywhere the nonlinear gradient equation

$$\nabla\phi_{2n+1} = \begin{pmatrix} \phi_{n+1} + \frac{1}{2}\sum_{i=k+1}^{n} \phi_{n+i}\,\partial_{x_1}\phi_i - \phi_i\,\partial_{x_1}\phi_{n+i} \\ \vdots \\ \phi_{n+k} + \frac{1}{2}\sum_{i=k+1}^{n} \phi_{n+i}\,\partial_{x_k}\phi_i - \phi_i\,\partial_{x_k}\phi_{n+i} \end{pmatrix} \quad \text{almost everywhere,}$$

so we are no longer in a setting where Whitney's extension theorem is directly applicable. However, it turns out that the extension in case $k < n$ can be reduced to the case $k = n$. This is best understood if one thinks of intrinsic Lipschitz graphs instead of tame maps. The idea is essentially that a k-dimensional intrinsic Lipschitz graph in $\mathbb{H}^n$ for $k < n$ can be embedded in an n-dimensional intrinsic Lipschitz graph. The latter can be extended using Theorem 4.3, and then it remains to show that one can select a suitable k-dimensional subset of it in order to obtain an extension of the original graph.

Theorem 4.6 *Let $k, n \in \mathbb{N}$ with $1 \leqslant k < n$. An $(L_{k+1}, \dots, L_{2n+1})$-tame map*

$$\phi : E \subset \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$$

can be extended to an $(L_{k+1}, \dots, L_n, L'_{n+1}, \dots, L'_{2n+1})$-tame map $\overline{\phi} : \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$ with $\overline{\phi}|_E = \phi$ and

$$L'_i = c_n\Big(1 + \sum_{j=k+1}^{n} L_j^2\Big)^{\frac{1}{2}} \max\left\{ |(L_{n+1}, \dots, L_{2n})|, L_{2n+1} + \sum_{i=k+1}^{n} L_{n+i}\min\{1, L_i\} \right\}$$

for $i = n+1, \dots, 2n$, and

$$L'_{2n+1} = c_n\Big(1 + \sum_{j=k+1}^{n} L_j^2\Big) \max\left\{ |(L_{n+1}, \dots, L_{2n})|, L_{2n+1} + \sum_{i=k+1}^{n} L_{n+i}\min\{1, L_i\} \right\}$$

for a constant c_n that depends only on n.

Proof Since $k < n$, the Lipschitz map $(\phi_{k+1}, \dots, \phi_n)$ has at least one component, and we can consider the associated k-dimensional Lipschitz graph

$$\Gamma^{(\phi_{k+1},\dots,\phi_n)}(E) := \{(x, \phi_{k+1}(x), \dots, \phi_n(x)) : x \in E\} \subset \mathbb{R}^n.$$

The remaining components of ϕ are used to define $f : \Gamma^{(\phi_{k+1},\dots,\phi_n)}(E) \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ by

$$\begin{aligned} f(\eta_1, \dots, \eta_n) &:= (f_{n+1}(\eta), \dots, f_{2n}(\eta), f_{2n+1}(\eta)) \\ &:= \left(\phi_{n+1}(\eta_1, \dots, \eta_k), \dots, \phi_{2n}(\eta_1, \dots, \eta_k), \phi_{2n+1}(\eta_1, \dots, \eta_k) + \frac{1}{2}\sum_{i=k+1}^{n} \eta_i\phi_{n+i}(\eta_1, \dots, \eta_k) \right). \end{aligned} \tag{4.7}$$

Firstly, we show that f is a tame map and so we can apply Theorem 4.3 to find an extension $\bar{f} = (\bar{f}_{n+1}, \dots, \bar{f}_{2n+1}) : \mathbb{R}^n \to \mathbb{R}^{n+1}$ of f with the corresponding tameness assumption satisfied. Second, if $(\bar{\phi}_{k+1}, \dots, \bar{\phi}_n)$ denotes a suitable Euclidean Lipschitz extension of $(\phi_{k+1}, \dots, \phi_n)$, we prove that the map $\bar{\phi} : \mathbb{R}^k \to \mathbb{R}^{2n+1-k}$ given by

$$\bar{\phi}(x) := \Big(\bar{\phi}_{k+1}(x), \dots, \bar{\phi}_n(x), \bar{f}_{n+1}(x, \bar{\phi}_{k+1}(x), \dots, \bar{\phi}_n(x)), \dots, \bar{f}_{2n}(x, \bar{\phi}_{k+1}(x), \dots, \bar{\phi}_n(x)),$$
$$\bar{f}_{2n+1}(x, \bar{\phi}_{k+1}(x), \dots, \bar{\phi}_n(x)) - \tfrac{1}{2} \sum_{i=k+1}^{n} \bar{\phi}_i(x) \bar{f}_{n+i}(x, \bar{\phi}_{k+1}(x), \dots, \bar{\phi}_n(x)) \Big), \tag{4.8}$$

is the tame extension of ϕ to $\mathbb{R}^k$ we are looking for. We begin by proving that the map f defined in (4.7) is $(L''_{n+1}, \dots, L''_{2n+1})$-tame with

$$L''_i := L_i, \quad i \in \{n+1, \dots, 2n\}, \quad \text{and} \quad L''_{2n+1} := L_{2n+1} + \sum_{i=k+1}^{n} L_{n+i} \min\{1, L_i\}. \tag{4.9}$$

To see this, fix two points

$$p = (x, \phi_{k+1}(x), \dots, \phi_n(x)), q = (y, \phi_{k+1}(y), \dots, \phi_n(y)) \in \Gamma^{(\phi_{k+1}, \dots, \phi_n)}(E). \tag{4.10}$$

The components $f_{n+1}, \dots, f_{2n}$ are clearly Lipschitz since, for $i = n+1, \dots, 2n$, we have by the L_i-Lipschitz continuity of ϕ_i that

$$|f_i(p) - f_i(q)| = \Big| \phi_i\big(x, \phi_{k+1}(x), \dots, \phi_n(x)\big) - \phi_i\big(y, \phi_{k+1}(y), \dots, \phi_n(y)\big) \Big| \leqslant L_i |p - q|.$$

It remains to check that the tameness condition (2) in Definition 2.5 holds with constant L'''_{2n+1}. Using the notation $\psi = (\phi_{n+1}, \dots, \phi_{n+k})$ and recalling the expressions for p and q given in (4.10), we have that

$$\begin{aligned} I(p,q) :=& \Big| f_{2n+1}(q) - f_{2n+1}(p) - \big\langle (f_{n+1}(q), \dots, f_{2n}(q)), (q_1 - p_1, \dots, q_n - p_n) \big\rangle \Big| \\ =& \Big| \phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y - x \rangle \\ & + \tfrac{1}{2} \sum_{i=k+1}^{n} \phi_i(y)\phi_{n+i}(y) - \phi_i(x)\phi_{n+i}(x) - 2\phi_{n+i}(y)(\phi_i(y) - \phi_i(x)) \Big|. \\ \leqslant& \Big| \phi_{2n+1}(y) - \phi_{2n+1}(x) - \langle \psi(y), y - x \rangle - \tfrac{1}{2} \sum_{i=k+1}^{n} \phi_i(y)\phi_{n+i}(x) - \phi_i(x)\phi_{n+i}(y) \Big| \\ & + \tfrac{1}{2} \Big| \sum_{i=k+1}^{n} \phi_{n+i}(y)(\phi_i(x) - \phi_i(y)) - \phi_{n+i}(x)(\phi_i(x) - \phi_i(y)) \Big|, \end{aligned}$$

and analogously with the roles of p and q reverted. Summing up the two expressions, we obtain by the $(L_{k+1}, \dots, L_{2n+1})$-tameness of ϕ that

$$\begin{aligned} I(p,q) + I(q,p) &\leqslant L_{2n+1} |y - x|^2 + \Big| \sum_{i=k+1}^{n} (\phi_{n+i}(y) - \phi_{n+i}(x))(\phi_i(x) - \phi_i(y)) \Big| \\ &\leqslant L_{2n+1} |y - x|^2 + \sum_{i=k+1}^{n} L_{n+i} |y - x| \min\{1, L_i\} |p - q| \\ &\leqslant \Big(L_{2n+1} + \sum_{i=k+1}^{n} L_{n+i} \min\{1, L_i\} \Big) |q - p|^2. \end{aligned}$$

Hence, condition (2) in Definition 2.5 holds, and we have shown that f is $(L''_{n+1},\dots,L''_{2n+1})$-tame with the constants defined in (4.9). By Theorem 4.3 applied to f, it then follows that there exists an extension $\bar f=(\bar f_{n+1},\dots,\bar f_{2n+1}) : \mathbb{R}^n\to\mathbb{R}^{n+1}$ of f which is $(L'''_{n+1},\dots,L'''_{2n+1})$-tame with

$$\begin{aligned} L'''_i &= 2C(n)\max\left\{|(L''_{n+1},\dots,L''_{2n})|, L''_{2n+1}\right\}\\ &= 2C(n)\max\left\{|(L_{n+1},\dots,L_{2n})|, L_{2n+1}+\sum_{i=k+1}^{n} L_{n+i}\min\{1,L_i\}\right\}. \end{aligned} \tag{4.11}$$

To construct an extension $\overline{\phi}$ for the given map ϕ, we first extend independently the components $\phi_{k+1},\dots,\phi_n$. For $i=k+1,\dots,n$, we simply apply McShane's extension theorem to extend the L_i-Lipschitz function $\phi_i : E\to\mathbb{R}$ to an L_i-Lipschitz function $\overline{\phi}_i : \mathbb{R}^k\to\mathbb{R}$. With the extensions $\overline{f}$ and $\overline{\phi}_{k+1},\dots,\overline{\phi}_n$ at hand, we are now able to prove that the map $\bar\phi : \mathbb{R}^k\to\mathbb{R}^{2n+1-k}$ defined in (4.8) is the desired tame extension of ϕ.

Recalling that $\overline{f}|_{\Gamma^{\phi_{k+1},\dots,\phi_n}(E)}=f$ and keeping in mind expression (4.7) for f, it is clear that $\bar\phi$ is an extension of ϕ. Moreover, for every $i=n+1,\dots,2n$, the function

$$\bar\phi_i : \mathbb{R}^k\to\mathbb{R},\quad \bar\phi_i(x)=\bar f_i(x,\bar\phi_{k+1}(x),\dots,\bar\phi_n(x))$$

is Lipschitz:

$$|\overline{\phi}_i(x)-\overline{\phi}_i(y)|\leqslant \mathrm{Lip}(\overline{f}_i)\Big(1+\sum_{j=k+1}^{n}L_j^2\Big)^{\frac12}|x-y|.$$

Recalling formula (4.11) for $L'''_i=\mathrm{Lip}(\overline{f}_i)$, we conclude that $\overline{\phi}_i$ is L'_i-Lipschitz with

$$L'_i=c_n\Big(1+\sum_{j=k+1}^{n}L_j^2\Big)^{\frac12}\max\left\{|(L_{n+1},\dots,L_{2n})|, L_{2n+1}+\sum_{i=k+1}^{n}L_{n+i}\min\{1,L_i\}\right\}$$

for $i=n+1,\dots,2n$ and a constant c_n that depends only on n.

As a consequence, the only nontrivial condition to check for the map $\bar\phi$ is the second part of the tameness condition, namely (2) in Definition 2.5. Let $x,y\in\mathbb{R}^k$, and, for simplicity, put $p:=(x,\bar\phi_{k+1}(x),\dots,\bar\phi_n(x)), q:=(y,\bar\phi_{k+1}(y),\dots,\bar\phi_n(y))$ and

$$\bar\psi(y):=(\bar\phi_{n+1}(y),\dots,\bar\phi_{n+k}(y))=(\bar f_{n+1}(q),\dots,\bar f_{n+k}(q)).$$

We have that

$$\begin{aligned} J(x,y) &:= \left|\bar\phi_{2n+1}(y)-\bar\phi_{2n+1}(x)-\langle\bar\psi(y),y-x\rangle-\tfrac12\sum_{i=k+1}^{n}\bar\phi_i(y)\bar\phi_{n+i}(x)-\bar\phi_i(x)\bar\phi_{n+i}(y)\right|\\ &\overset{(4.8)}{\leqslant} \left|\bar f_{2n+1}(q)-\bar f_{2n+1}(p)-\big\langle(\bar f_{n+1}(q),\dots,\bar f_{2n}(q)),(q_1-p_1,\dots,q_n-p_n)\big\rangle\right|\\ &\quad+\tfrac12\left|\sum_{i=k+1}^{n}\bar\phi_i(y)(\bar f_{n+i}(q)-\bar f_{n+i}(p))-\bar\phi_i(x)(\bar f_{n+i}(q)-\bar f_{n+i}(p))\right|\\ &\leqslant \left|\bar f_{2n+1}(q)-\bar f_{2n+1}(p)-\big\langle(\bar f_{n+1}(q),\dots,\bar f_{2n}(q)),(q_1-p_1,\dots,q_n-p_n)\big\rangle\right|\\ &\quad+\tfrac12\sum_{i=k+1}^{n}|\bar\phi_i(y)-\bar\phi_i(x)||\bar f_{n+i}(q)-\bar f_{n+i}(p)|. \end{aligned}$$

Summing this term and the corresponding expression with the roles of x and y reverted, we find by the tameness condition of $\overline{f}$ and the Lipschitz continuity of $\bar{\phi}_{k+1}, \ldots, \bar{\phi}_n$ that

$$J(x,y) + J(y,x) \leqslant L'''_{2n+1}|p-q|^2 + \sum_{i=k+1}^{n} L_i L'''_{n+i}|x-y||p-q|$$
$$\leqslant \left(L'''_{2n+1}\left(1 + \sum_{j=k+1}^{n} L_j^2\right) + \sum_{i=k+1}^{n} L_i L'''_{n+i}\left(1 + \sum_{j=k+1}^{n} L_j^2\right)^{\frac{1}{2}}\right)|x-y|^2,$$

for arbitrary $x, y \in \mathbb{R}^k$. Now we only have to recall the expressions for L'''_j, $j = n+1, \ldots, 2n$, given in (4.11). Then, for a constant c_n that depends only on n, the last tameness constant of $\overline{\phi}$ can be chosen as

$$L'_{2n+1} = c_n\left(1 + \sum_{j=k+1}^{n} L_j^2\right) \max\left\{ |(L_{n+1}, \ldots, L_{2n})|, L_{2n+1} + \sum_{i=k+1}^{n} L_{n+i}\min\{1, L_i\}\right\}.$$

□

4.3 Extension result for low-dimensional intrinsic Lipschitz graphs

Combining the previous results, we establish the extension theorem for low-dimensional intrinsic Lipschitz graphs in $\mathbb{H}^n$, Theorem 1.2 from the introduction.

Proof of Theorem 1.2 Let $1 \leqslant k \leqslant n$. First, if $\phi = (\phi_{k+1}, \ldots, \phi_{2n+1})$ is intrinsic L-Lipschitz on $E \subset \mathbb{R}^k$, Proposition 3.3 implies that $(\phi_{k+1}, \ldots, \phi_{2n}, -\phi_{2n+1})$ is $(L_{k+1}, \ldots, L_{2n+1})$-tame with

$$L_i = L \quad \text{for } i \neq 2n+1, \quad \text{and} \quad L_{2n+1} = 2L^2.$$

Applying the extension result from Theorem 4.3, if $k = n$, and Theorem 4.6, if $k < n$, to this tame map yields an $(L'_{k+1}, \ldots, L'_{2n+1})$-tame extension $(\overline{\phi}_{k+1}, \ldots, \overline{\phi}_{2n}, -\overline{\phi}_{2n+1})$, where the tameness constants depend only on $L_{k+1}, \ldots, L_{2n+1}$ (thus on L), k, and n. Finally, we use Proposition 3.6 to conclude that $\overline{\phi} := (\overline{\phi}_{k+1}, \ldots, \overline{\phi}_{2n}, \overline{\phi}_{2n+1})$ is an intrinsic L'-Lipschitz function on $\mathbb{V}$ with

$$L' := \max\{|(L'_{k+1}, \ldots, L'_{2n})|, \sqrt{L'_{2n+1}}\}.$$

The better quantitative control over the intrinsic Lipschitz constant if $k = n = 1$ follows since Proposition 3.3 yields $L_2 = \min\{L, 2L^2\}$ in this case. Then, L'_2 and L'_3 in Theorem 4.3 can be bounded from above by a constant times L^2, and it follows that we can take $L' = C\max\{L^2, L\}$ for a suitable constant C. □

5 Corona decomposition for 1-dimensional intrinsic Lipschitz graphs

The main result of this section is a corona decomposition of 1-dimensional iLG in $\mathbb{H}^n$, $n > 1$, by iLG with smaller Lipschitz constant (Theorem 5.26). The corresponding result for $n = 1$ (and tame maps) was proven in [11, Theorem 3.15], motivated by an application to singular integral operators on 1-dimensional iLG in $\mathbb{H}^1$. As was the case for [11], our argument is ultimately based on a corona decomposition for Euclidean Lipschitz graphs. The version that we will employ looks a little different from the formulations in the literature, so we state it in Section 5.1 and explain how to deduce it from the "standard" corona decomposition for Euclidean Lipschitz graphs given in [9, p.57, Definition 3.19 and p.61, (3.33)]. Based on these preparations, we prove the result for iLG in Sect. 5.2.

5.1 Corona decomposition for 1-dimensional Euclidean Lipschitz graphs

Definition 5.1 **(Dyadic intervals and trees)** The family of standard *dyadic intervals* of $\mathbb{R}$ is called "$\mathcal{D}$". For $j \in \mathbb{Z}$, we write $\mathcal{D}_j \subset \mathcal{D}$ for the dyadic intervals Q of length $|Q| = 2^{-j}$. A collection $\mathcal{T} \subset \mathcal{D}$ is a *tree* if

(T1) $\mathcal{T}$ contains a *top interval* $Q(\mathcal{T})$, that is, a unique maximal element.
(T2) $\mathcal{T}$ is *coherent*: if $Q \in \mathcal{T}$, then $Q' \in \mathcal{T}$ for all dyadic intervals $Q \subset Q' \subset Q(\mathcal{T})$.
(T3) If $Q \in \mathcal{T}$, then either both, or neither, of the children of Q lie in $\mathcal{T}$.

Definition 5.2 **(Coronization)** A decomposition $\mathcal{D} = \mathcal{G} \dot\cup \mathcal{B}$ of $\mathcal{D}$ into *good* intervals $\mathcal{G}$ and *bad* intervals $\mathcal{B}$ (with $\mathcal{G} \cap \mathcal{B} = \emptyset$) is called a *coronization* if there exists a constant C such that the following conditions are satisfied:

(1) The intervals in $\mathcal{B}$ satisfy a Carleson packing condition:

$$\sum_{\substack{Q \in \mathcal{B} \\ Q \subset Q_0}} |Q| \leqslant C|Q_0|, \quad \text{for all } Q_0 \in \mathcal{D}.$$

(2) The intervals in $\mathcal{G}$ can be decomposed into a *forest* $\mathcal{F}$ of disjoint trees $\mathcal{T}$

$$\mathcal{G} = \dot{\bigcup}_{\mathcal{T} \in \mathcal{F}} \mathcal{T}$$

whose top intervals satisfy a Carleson packing condition:

$$\sum_{\substack{\mathcal{T} \in \mathcal{F} \\ Q(\mathcal{T}) \subset Q_0}} |Q(\mathcal{T})| \leqslant C|Q_0|, \quad \text{for all } Q_0 \in \mathcal{D}.$$

Theorem 5.3 (Corona decomposition for Lipschitz maps) *For every $n \in \mathbb{N}$ and $\delta \in (0, 1)$, there exists a constant $C \geqslant 1$ with the following property. Let $\psi : \mathbb{R} \to \mathbb{R}^{2n-1}$ be 1-Lipschitz.*

Then, there exists a coronization $\mathcal{D} = \mathcal{B}\dot{\cup}\mathcal{G}$ and a forest $\mathcal{F}$, satisfying the conditions in Definition 5.2 with constant C, such that the following holds. For every $\mathcal{T} \in \mathcal{F}$ there is a 2-Lipschitz linear function $L_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ and a δ-Lipschitz function $\psi_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ such that

$$|\psi(s) - (\psi_{\mathcal{T}} + L_{\mathcal{T}})(s)| \leqslant \delta|Q|, \qquad s \in 2Q,\ Q \in \mathcal{T}, \tag{5.4}$$

where 2Q is the interval with the same midpoint as Q but twice its length.

Proof For $n = 1$, the result was deduced in [11, Theorem 3.20] from the corona decomposition in [9, p.61, (3.33)]. The same approach works for $n > 1$, although the reduction to [9, p.61, (3.33)] is now a bit more involved.

To start the proof, let us fix $n > 1$. By [9, p.61, (3.33) and p.328, §2.2] we know that for every $\delta' \in (0, 1)$, there exists a constant $C = C(n, \delta')$ such that, for every 1-Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}^{2n-1}$, there is a coronization $\mathcal{D} = \mathcal{B}\dot{\cup}\mathcal{G}$ with constant C that satisfies the following property. For every tree $\mathcal{T}$ in the associated forest $\mathcal{F}$ there is a 1-dimensional δ' -Lipschitz graph $\Gamma_{\mathcal{T}}$ such that

$$\operatorname{dist}\big((s, \psi(s)), \Gamma_{\mathcal{T}}\big) \leqslant \delta'|Q|, \quad \text{for all } s \in 2Q,\ Q \in \mathcal{T}. \tag{5.5}$$

In other words, there exists a δ'-Lipschitz function $\varphi_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ and $R \in O(2n)$ such that (5.5) holds for

$$\Gamma_{\mathcal{T}} = \{R(x, \varphi_{\mathcal{T}}(x)) : \ x \in \mathbb{R}\}. \tag{5.6}$$

To be precise, [9, p.61, (3.33)] refers to a system of dyadic cubes on the graph of ψ, rather than in the domain $\mathbb{R}$, but (5.5) is easily deduced. The thus given coronization is the same that appears in the statement that we are about to prove, so the only challenge is to find $\delta'(\delta, n)$ so that we can deduce from (5.5) that (5.4) holds for suitable $\psi_{\mathcal{T}}$ and $L_{\mathcal{T}}$. As it will be convenient to work in coordinates, we represent R as a matrix with respect to the standard basis of $\mathbb{R}^{2n}$,

$$R = \begin{pmatrix} b_{1,1} & \cdots & b_{1,2n} \\ & \ddots & \\ b_{2n,1} & \cdots & b_{2n,2n} \end{pmatrix}, \tag{5.7}$$

so that for $\varphi_{\mathcal{T}} = (\varphi_{\mathcal{T},2}, \ldots, \varphi_{\mathcal{T},2n})$ the identity (5.6) reads

$$\Gamma_{\mathcal{T}} = \left\{ \begin{pmatrix} b_{1,1}x + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x) b_{1,l} \\ \vdots \\ b_{2n,1}x + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x) b_{2n,l} \end{pmatrix} : \ x \in \mathbb{R} \right\}. \tag{5.8}$$

Then, the approximation property (5.5) means exactly that for every $Q \in \mathcal{T}$, and for every $s \in 2Q$, there exists $x_s \in \mathbb{R}$ such that

$$\left| \begin{pmatrix} s \\ \psi_2(s) \\ \vdots \\ \psi_{2n}(s) \end{pmatrix} - \begin{pmatrix} b_{1,1}x_s + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x_s) b_{1,l} \\ b_{2,1}x_s + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x_s) b_{2,l} \\ \vdots \\ b_{2n,1}x_s + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x_s) b_{2n,l} \end{pmatrix} \right| \leqslant \delta'|Q|. \tag{5.9}$$

For a given point s, there can be several points with this property, but we just choose one of them and call it x_s. We will apply (5.9) for small enough $\delta' \in (0,1)$ depending on n and the parameter δ in the statement of the theorem. The precise condition will appear in (5.19), but the bound on δ' has to be chosen such that $\Gamma_{\mathcal{T}}$ in (5.8) can be written as graph *over the* x_1*-axis* of a function of the form $\psi_{\mathcal{T}} + L_{\mathcal{T}}$, where $\psi_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ is δ-Lipschitz, and $L_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ is linear with Lipschitz constant 2. For this purpose, we define

$$z(x) := b_{1,1}x + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x) b_{1,l}$$

and, recalling (5.8), our goal is to write

$$\begin{pmatrix} b_{1,1}x + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x) b_{1,l} \\ b_{2,1}x + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x) b_{2,l} \\ \vdots \\ b_{2n,1}x + \sum_{l=2}^{2n} \varphi_{\mathcal{T},l}(x) b_{2n,l} \end{pmatrix} = \begin{pmatrix} z(x) \\ \psi_{\mathcal{T},2}(z(x)) + L_{\mathcal{T},2}(z(x)) \\ \vdots \\ \psi_{\mathcal{T},2n}(z(x)) + L_{\mathcal{T},2n}(z(x)) \end{pmatrix}, \quad x \in \mathbb{R} \tag{5.10}$$

for $\psi_{\mathcal{T}}$ and $L_{\mathcal{T}}$ as mentioned above. Assume for a moment that we know that $b_{1,1} \neq 0$. Then, we can define

$$L_{\mathcal{T},l}(x) := \frac{b_{l,1}}{b_{1,1}} x, \quad \text{for } l = 2, \ldots, 2n \tag{5.11}$$

and solving (5.10) for $\psi_{\mathcal{T}}(z(x))$ yields

$$\psi_{\mathcal{T},l}(z(x)) = \sum_{i=2}^{2n} \left(b_{l,i} - \tfrac{b_{l,1}}{b_{1,1}} b_{1,i} \right) \varphi_{\mathcal{T},i}(x), \quad \text{for } l = 2, \ldots, 2n. \tag{5.12}$$

To establish that the thus defined functions $L_{\mathcal{T}}$ and $\psi_{\mathcal{T}}$ are Lipschitz with the claimed constants will require us to prove a suitable uniform upper bound for $|(b_{2,1}, \ldots, b_{2n,1})|/|b_{1,1}|$ assuming an upper bound for δ' (this will show in particular that $b_{1,1} \neq 0$). Supposing for a moment that this can be done, we will then prove that $z : \mathbb{R} \to \mathbb{R}$ is a bi-Lipschitz homeomorphism if δ' is chosen small enough (it is obviously always Lipschitz continuous). This will finally also yield that $\Gamma_{\mathcal{T}}$ is indeed the graph of $\psi_{\mathcal{T}} + L_{\mathcal{T}}$. Before entering the computations, we recall that R is an orthogonal matrix, so all its rows and columns have length 1, and in particular we obtain that

$$1 \geqslant \left|(b_{1,2}, \cdots, b_{1,2n})\right|. \tag{5.13}$$

Combined with the δ'-Lipschitz continuity of $\varphi_{\mathcal{T}}$, this yields that

$$\begin{aligned} |z(x_1) - z(x_2)| &= \left| b_{1,1}(x_1 - x_2) + \sum_{l=2}^{2n} b_{1,l}(\varphi_{\mathcal{T},l}(x_1) - \varphi_{\mathcal{T},l}(x_2)) \right| \\ &\geqslant \left(|b_{1,1}| - \delta' \left|(b_{1,2}, \ldots, b_{1,2n})\right| \right) |x_1 - x_2| \\ &\overset{(5.13)}{\geqslant} \left(|b_{1,1}| - \delta' \right) |x_1 - x_2| \end{aligned} \tag{5.14}$$

for all $x_1, x_2 \in \mathbb{R}$. Once again, a universal positive lower bound on $b_{1,1}$ will conclude the proof if δ' is chosen small enough.

Motivated by these considerations, we now concentrate our efforts on proving that

$$\delta' \leqslant \tfrac{1}{100\sqrt{2n-1}} \quad \Rightarrow \quad |b_{1,1}| \geqslant 1/\sqrt{5} \quad \big(\Leftrightarrow \quad |(b_{2,1},\dots,b_{2n,1})|/|b_{1,1}| < 2.\big) \tag{5.15}$$

To see why such a statement is plausible, it may help the reader to picture the case $n=1$. Then, (5.15) essentially says that if the graph of the 1-Lipschitz function ψ is well-approximated by $\Gamma_{\mathcal{T}}$, which is obtained by rotating the graph of a δ'-Lipschitz function $\varphi_{\mathcal{T}}$ over the x_1-axis by angle θ, then $|\tan(\theta)| = |b_{2,1}/b_{1,1}| < 2$ if δ' is small enough.

We will now prove (5.15) for $n>1$. To this end, pick an arbitrary interval $S \in \mathcal{T}$ and denote by s_1 and s_2 the endpoints of S. Moreover, let $x := x_{s_1} \in \mathbb{R}$ be such that (5.9) holds for $s=s_1$ and $Q=S$, and in the same way, associate to the other endpoint s_2 a point $x' := x_{s_2} \in \mathbb{R}$ such that (5.9) holds for $s=s_2$ and $Q=S$. Using this property, combined with the 1-Lipschitz continuity of ψ, the matrix representation (5.7) of $R \in O(2n)$, and the δ'-Lipschitz continuity of $\varphi_{\mathcal{T}}$, we conclude that

$$\begin{aligned}
&\left|\begin{pmatrix} b_{2,1}\\ \vdots \\ b_{2n,1}\end{pmatrix}(x-x') + \begin{pmatrix} \langle (b_{2,2},\dots,b_{2,2n}), \varphi_{\mathcal{T}}(x)-\varphi_{\mathcal{T}}(x')\rangle \\ \vdots \\ \langle (b_{2n,2},\dots,b_{2n,2n}), \varphi_{\mathcal{T}}(x)-\varphi_{\mathcal{T}}(x')\rangle\end{pmatrix}\right|\\
&\quad\overset{(5.9)}{\leqslant} |\psi(s_1)-\psi(s_2)| + 2\delta'|S| \leqslant |s_1-s_2| + 2\delta'|S|\\
&\quad\overset{(5.9)}{\leqslant} \left| b_{1,1}(x-x') + \sum_{l=2}^{2n} b_{1,l}(\varphi_{\mathcal{T},l}(x)-\varphi_{\mathcal{T},l}(x'))\right| + 4\delta'|S|\\
&\quad\leqslant |b_{1,1}||x-x'| + |\varphi_{\mathcal{T}}(x)-\varphi_{\mathcal{T}}(x')| + 4\delta'|S| \leqslant \big(|b_{1,1}|+\delta'\big)|x-x'| + 4\delta'|S|.
\end{aligned}$$

This implies that

$$\left|\begin{pmatrix} b_{2,1}\\ \vdots \\ b_{2n,1}\end{pmatrix}\right| |x-x'| \leqslant \Big(|b_{1,1}| + (\sqrt{2n-1}+1)\delta'\Big)|x-x'| + 4\delta'|S|. \tag{5.16}$$

The above estimates hold for arbitrary points in $2S$, but the fact that s_1 and s_2 are the endpoints of S, allow us to show that $|S| \lesssim |x-x'|$. More precisely, since $\varphi_{\mathcal{T}}$ is δ'-Lipschitz, we find

$$\begin{aligned}
|S| = |s_1-s_2| &\leqslant \left| s_1 - b_{1,1}x - \sum_{l=2}^{2n} b_{1,l}\varphi_{\mathcal{T},l}(x)\right| + \left| s_2 - b_{1,1}x' - \sum_{l=2}^{2n} b_{1,l}\varphi_{\mathcal{T},l}(x')\right|\\
&\quad + |b_{1,1}||x-x'| + \delta'|x-x'|\\
&\overset{(5.9)}{\leqslant} 2\delta'|S| + |b_{1,1}||x-x'| + \delta'|x-x'|.
\end{aligned}$$

Moving the terms with $|S|$ to the left-hand side, we conclude for small enough δ' as in (5.15) that

$$|S| \leqslant \left(\frac{|b_{1,1}|+\delta'}{1-2\delta'}\right)|x-x'| \leqslant \frac{5}{2}|x-x'|.$$

Inserting this estimate in (5.16) and dividing both sides by $|x-x'|$, we observe that

$$|(b_{2,1},\dots,b_{2n,1})| \leqslant |b_{1,1}| + \delta'\left(11+\sqrt{2n-1}\right).$$

As the columns of the orthogonal matrix R are unit vectors, we have

$$1-b_{1,1}^2 = |(b_{2,1},\dots,b_{2n,1})|^2 \leqslant b_{1,1}^2 + 2\left(11+\sqrt{2n-1}\right)\delta' + \left(11+\sqrt{2n-1}\right)^2 \delta'^2. \tag{5.17}$$

If δ' is small enough, say as in (5.15), then we can deduce from (5.17) that $b_{1,1}^2 > 1/5$, and hence

$$\frac{\left|b_{2,1},\dots,b_{2n,1}\right|^2}{b_{1,1}^2} = \frac{1-b_{1,1}^2}{b_{1,1}^2} < 4, \tag{5.18}$$

which concludes the proof of (5.15).

It is now immediate from formula (5.11) that $L_{\mathcal{T}}$ is 2-Lipschitz. Moreover, it follows from the formula for $\psi_{\mathcal{T}}$ in (5.12), the δ'-Lipschitz continuity of $\varphi_{\mathcal{T}}$, the Cauchy–Schwarz inequality, and (5.18) that

$$|\psi_{\mathcal{T}}(z(x_1)) - \psi_{\mathcal{T}}(z(x_2))| \leqslant 3\sqrt{2n-1}|\varphi_{\mathcal{T}}(x_1) - \varphi_{\mathcal{T}}(x_2)| \leqslant 3\sqrt{2n-1}\,\delta'\,|x_1-x_2|$$

for all $x_1, x_2 \in \mathbb{R}$. Since $|b_{1,1}| \geqslant 1/\sqrt{5}$ and (5.14) holds, it is clear that for

$$\delta' \leqslant \frac{\delta}{100\sqrt{2n-1}} \tag{5.19}$$

the function $\psi_{\mathcal{T}}$ is δ-Lipschitz, as required.

It remains to verify the approximation condition (5.4). This follows easily from (5.9), which can now be rewritten as

$$\left|\begin{pmatrix} s \\ \psi(s)\end{pmatrix} - \begin{pmatrix} z(x_s) \\ \psi_{\mathcal{T}}(z(x_s)) + L_{\mathcal{T}}(z(x_s))\end{pmatrix}\right| \leqslant \delta'|Q|, \quad Q \in \mathcal{T},\ s \in 2Q. \tag{5.20}$$

Since $|s - z(x_s)| \leqslant \delta'|Q|$, and $\psi_{\mathcal{T}} + L_{\mathcal{T}}$ is 3-Lipschitz, it follows that (5.20) holds with $z(x_s)$ replaced by s, and δ' replaced by δ, recalling the bound (5.19). This concludes the proof of Theorem 5.3. □

We make one more modification in the construction of $\psi_{\mathcal{T}}$, see Corollary 5.22. This is an additional step compared to the proof for $n = 1$ in [11], necessitated by the noncommutativity of codimension-1 vertical subgroups in $\mathbb{H}^n$ for $n > 1$.

In the setting of Theorem 5.3, if $\mathcal{T} \in \mathcal{F}$ is a tree with top interval $Q(\mathcal{T})$, we denote by $\mathcal{S}(\mathcal{T})$ the (possibly empty) collection of minimal intervals in $\mathcal{T}$. Moreover, we write

$$E := Q(\mathcal{T}) \setminus \bigcup_{S \in \mathcal{S}(\mathcal{T})} S$$

for the set of points in $Q(\mathcal{T})$ in infinite branches of $\mathcal{T}$. By the approximation condition (5.5) in the corona decomposition, we have

$$\psi(s) = [L_{\mathcal{T}} + \psi_{\mathcal{T}}](s), \quad \text{for all } s \in E. \tag{5.21}$$

For the application to intrinsic Lipschitz graphs in $\mathbb{H}^n$, $n > 1$, in Sect. 5.2, it will be beneficial to have the identity (5.21) also in all points $s \in \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S$. Otherwise, error terms will appear depending on the position of the intrinsic graph and caused by the noncommutativity of codimension-1 vertical subgroups.

Corollary 5.22 *For $\mathcal{T} \in \mathcal{F}$, the function $\psi_{\mathcal{T}}$ in Theorem 5.3 can be constructed so that*

$$\psi(s) = [L_{\mathcal{T}} + \psi_{\mathcal{T}}](s), \quad \text{for all } s \in \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S. \tag{5.23}$$

Proof Fix an arbitrary tree $\mathcal{T}$ in the forest $\mathcal{F}$ associated with the corona decomposition given by Theorem 5.3 for the 1-Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}^{2n-1}$ and parameter "$\delta/(8\sqrt{2n-1})$". Assume that $\mathcal{S}(\mathcal{T})$ is nonempty, otherwise there is nothing to prove. Let $\psi_{\mathcal{T}}$ and $L_{\mathcal{T}}$ be the associated functions provided by Theorem 5.3. We will now slightly modify $\psi_{\mathcal{T}}$ inside each minimal interval $S \in \mathcal{S}(\mathcal{T})$ so that (5.23) holds for the modified function $\widetilde{\psi}_{\mathcal{T}}$, which will be δ-Lipschitz and satisfy (5.5) (for $\psi, L_{\mathcal{T}}$ and "δ"). Set $\widetilde{\psi}_{\mathcal{T}}(s) := \psi_{\mathcal{T}}(s) = \psi(s) - L_{\mathcal{T}}(s)$ for $s \in E$. It is possible to define the modified function $\psi_{\mathcal{T}}$ on $Q(\mathcal{T}) \setminus E$ by considering one minimal interval S at the time, since the boundary points of S either belong to E, where (5.21) already holds, or they are boundary points of two adjacent minimal intervals, so that the modification will be well-defined. Assuming that $\psi_{\mathcal{T}}$ is defined on the entire real line, we set $\widetilde{\psi}_{\mathcal{T}} = \psi_{\mathcal{T}}$ outside $Q(\mathcal{T})$.

Fix $S = [a, b] \in \mathcal{S}(\mathcal{T})$. The components of $\widetilde{\psi}_{\mathcal{T},l}$, $l = 2, \ldots, 2n$, can be taken of the form

$$(\widetilde{\psi}_{\mathcal{T},l})|_S(s) := \psi_{\mathcal{T},l}(s) + (\psi_l(a) - L_{\mathcal{T},l}(a) - \psi_{\mathcal{T},l}(a)) + c_{S,l}(s-a), \quad s \in S,$$

for suitable constants $c_{S,l} \in \mathbb{R}$ with

$$|(c_{S,2}, \ldots, c_{S,2n})| \leqslant \delta/4. \tag{5.24}$$

Since we have merely added an affine function, the modified map $\widetilde{\psi}_{\mathcal{T}}$ is clearly Lipschitz with constant $\delta/(8\sqrt{2n-1}) + |(c_{S,2}, \ldots, c_{S,2n})| \leqslant (3/8)\delta$.

Moreover,

$$\widetilde{\psi}_{\mathcal{T},l}(a) = \psi_l(a) - L_{\mathcal{T},l}(a),$$

by construction, and

$$\widetilde{\psi}_{\mathcal{T},l}(s) + L_{\mathcal{T},l}(s) = \psi_{\mathcal{T},l}(s) + L_{\mathcal{T},l}(s) = \psi_l(s), \quad \text{for all } s \in E. \tag{5.25}$$

So we only have to show that $c_{S,2}, \ldots, c_{S,2n}$ can be chosen such that the two functions match also at the other endpoint of S, that is $\widetilde{\psi}_{\mathcal{T},l}(b) = \psi_l(b) - L_{\mathcal{T},l}(b)$ for $l = 2, \ldots, 2n$. This forces us to take

$$c_{S,l} := \frac{1}{b-a}\left[\psi_l(b) - L_{\mathcal{T},l}(b) - \psi_{\mathcal{T},l}(b) - (\psi_l(a) - L_{\mathcal{T},l}(a) - \psi_{\mathcal{T},l}(a))\right].$$

Then, the approximation condition (5.5) provides the desired bound for $c_{S,l}$, $l = 2, \ldots, 2n$, namely

$$\begin{aligned}|c_{S,l}| &\leqslant \frac{1}{b-a}|\psi_l(b) - L_{\mathcal{T},l}(b) - \psi_{\mathcal{T},l}(b)| + \frac{1}{b-a}|\psi_l(a) - L_{\mathcal{T},l}(a) - \psi_{\mathcal{T},l}(a)|\\ &\leqslant \frac{\delta|S|}{8\sqrt{2n-1}|S|} + \frac{\delta|S|}{8\sqrt{2n-1}|S|} = \frac{\delta}{4\sqrt{2n-1}},\end{aligned}$$

which yields (5.24). We apply the same procedure for every minimal interval S. This yields the modified function $\widetilde{\psi}_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$, which is $\frac{3}{8}\delta$-Lipschitz. This follows from

$$\widetilde{\psi}_{\mathcal{T}}(s) = \psi(s) - L_{\mathcal{T}}(s) \text{ for } s \in E \cup \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S \quad \text{and} \quad \widetilde{\psi}_{\mathcal{T}}(s) = \psi_{\mathcal{T}}(s) \text{ for } s \in E,$$

and the $\frac{\delta}{8\sqrt{2n-1}}$-Lipschitz continuity of $\psi_{\mathcal{T}}$ as well as the $\frac{3}{8}\delta$-Lipschitz continuity of $\widetilde{\psi}_{\mathcal{T}_{|S}}$ for all $S \in \mathcal{S}(\mathcal{T})$.

Finally, we verify that $\widetilde{\psi}_{\mathcal{T}}$ has the desired approximation property. For every $Q \in \mathcal{T}$, $s \in 2Q$, there exists a point

$$s_1 \in Q \cap \Big(E \cup \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S\Big)$$

with $|s - s_1| \leqslant |Q|$. This point satisfies by construction

$$\psi(s_1) = \widetilde{\psi}_{\mathcal{T}}(s_1) + L_{\mathcal{T}}(s_1).$$

Then, for $s \in 2Q$ as above, we find that

$$\begin{aligned}|\psi(s) - [\widetilde{\psi}_{\mathcal{T}} + L_{\mathcal{T}}](s)| &\leqslant |\psi(s) - [\psi_{\mathcal{T}} + L_{\mathcal{T}}](s)| + |\psi_{\mathcal{T}}(s) - \widetilde{\psi}_{\mathcal{T}}(s)|\\ &\leqslant \frac{\delta|Q|}{8\sqrt{2n-1}} + |\psi_{\mathcal{T}}(s) - \psi_{\mathcal{T}}(s_1)| + |\psi_{\mathcal{T}}(s_1) - \widetilde{\psi}_{\mathcal{T}}(s_1)| + |\widetilde{\psi}_{\mathcal{T}}(s_1) - \widetilde{\psi}_{\mathcal{T}}(s)|\\ &\leqslant \frac{\delta|Q|}{4\sqrt{2n-1}} + |\psi_{\mathcal{T}}(s_1) - \widetilde{\psi}_{\mathcal{T}}(s_1)| + \frac{3\delta}{8}|Q|,\end{aligned}$$

where in the last inequality we used the facts that $\psi_{\mathcal{T}}$ and $\widetilde{\psi}_{\mathcal{T}_{|S}}$ are $\frac{\delta}{8\sqrt{2n-1}}$-Lipschitz and $\frac{3}{8}\delta$ -Lipschitz, respectively. Hence, it remains to estimate the term $|\psi_{\mathcal{T}}(s_1) - \widetilde{\psi}_{\mathcal{T}}(s_1)|$. There are two cases: $s_1 \in E$ or $s_1 \in \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S$. In the first case, $|\psi_{\mathcal{T}}(s_1) - \widetilde{\psi}_{\mathcal{T}}(s_1)| = 0$ because of (5.25). On the other hand, if $s_1 \in Q \cap \partial S$ for some $S \in \mathcal{S}(\mathcal{T})$, then $\widetilde{\psi}_{\mathcal{T}}(s_1) = \psi(s_1) - L_{\mathcal{T}}(s_1)$ by construction, and so

$$|\psi_{\mathcal{T}}(s_1) - \widetilde{\psi}_{\mathcal{T}}(s_1)| = |\psi_{\mathcal{T}}(s_1) - (\psi(s_1) - L_{\mathcal{T}}(s_1))| \leqslant \frac{\delta}{8\sqrt{2n-1}}|Q|.$$

Inserting the bound for $|\psi_{\mathcal{T}}(s_1) - \widetilde{\psi}_{\mathcal{T}}(s_1)|$ in the previous estimate, we deduce that

$$|\psi(s) - [\widetilde{\psi}_{\mathcal{T}} + L_{\mathcal{T}}](s)| \leqslant \frac{\delta|Q|}{4\sqrt{2n-1}} + \frac{\delta|Q|}{8\sqrt{2n-1}} + \frac{3\delta|Q|}{8} \leqslant \delta|Q|, \quad Q \in \mathcal{T}, s \in 2Q,$$

which shows that the approximation condition (5.4) holds for $\widetilde{\psi}_{\mathcal{T}}$, and thus concludes the proof of Corollary 5.22. □

5.2 Corona decomposition for 1-dimensional intrinsic Lipschitz graphs

With Corollary 5.22 at hand, we are now ready to establish the corona decomposition for 1-dimensional intrinsic Lipschitz graphs. Given such a graph in $\mathbb{H}^n$, the idea is to apply Corollary 5.22 to the 1-dimensional Euclidean Lipschitz graph in $\mathbb{R}^{2n}$ which is obtained by projecting the intrinsic graph to the horizontal coordinate plane $\{t = 0\}$. We obtain a corona decomposition of this graph with approximating Lipschitz functions

$$\psi_{\mathcal{T}} + L_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}.$$

While it is straightforward to lift these approximating functions to intrinsic Lipschitz functions, these lifts may not approximate the initially given iLG well enough in the t-variable. Analogously to the approach in [11], the following theorem is therefore based on a modification of $\psi_{\mathcal{T}}$ which will ensure that the lift of $\psi_{\mathcal{T}} + L_{\mathcal{T}}$ has the desired approximation properties.

Theorem 5.26 (Corona decomposition for intrinsic 1-Lipschitz maps) *Let $n > 1$, and assume that $\mathbb{V}$ is a 1-dimensional horizontal subgroup in $\mathbb{H}^n$ with complementary vertical subgroup $\mathbb{W}$. For every $\eta \in (0,1)$, there exists a constant $C \geqslant 1$ such that the following holds. Let*

$$\phi = (\phi_2, \dots, \phi_{2n+1}) : \mathbb{V} \to \mathbb{W}$$

be intrinsic 1-Lipschitz. Then, there exists a coronization $\mathcal{D} = \mathcal{G} \dot\cup \mathcal{B}$ satisfying the conditions in Definition 5.2 with constant C such that, for every $\mathcal{T} \in \mathcal{F}$, there is an intrinsic Lipschitz map $\phi_{\mathcal{T}} = (L_{\mathcal{T}} + \psi_{\mathcal{T}}, \phi_{\mathcal{T},2n+1}) : \mathbb{V} \to \mathbb{W}$ where $L_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ is a linear 2-Lipschitz map and $\psi_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ is a η-Lipschitz map such that $\phi_{\mathcal{T}}$ approximates ϕ well at the resolution of the intervals in $\mathcal{T}$:

$$d(\phi(s), \phi_{\mathcal{T}}(s)) \leqslant \eta |Q|, \quad s \in 2Q,\ Q \in \mathcal{T}. \tag{5.27}$$

In (5.27), d refers to left invariant metric on $\mathbb{H}^n$ as defined in (1.1):

$$\begin{aligned} d(\phi(s), \phi_{\mathcal{T}}(s)) = \max \Big\{ & |(\phi_2, \dots, \phi_{2n})(s) - [L_{\mathcal{T}} + \psi_{\mathcal{T}}](s)|, \\ & |\phi_{2n+1}(s) - \phi_{\mathcal{T},2n+1}(s) + \frac{1}{2} \sum_{i=2}^{n} -\phi_{\mathcal{T},i}(s)\phi_{n+i}(s) + \phi_i(s)\phi_{\mathcal{T},n+i}(s)|^{1/2} \Big\}. \end{aligned}$$

If Φ and $\Phi_{\mathcal{T}}$ denote the intrinsic graph maps of ϕ and $\phi_{\mathcal{T}}$, respectively, then $d(\phi(s), \phi_{\mathcal{T}}(s)) = d(\Phi(s), \Phi_{\mathcal{T}}(s))$, so that (5.27) really gives an estimate on how well the intrinsic graph of $\phi_{\mathcal{T}}$ approximates the intrinsic graph of ϕ.

Theorem 5.26 looks a little different from the corona decomposition stated in Theorem 1.3 in the introduction. The reason is simply that the graph of ϕ and all the approximating intrinsic graphs in Theorem 5.26 are written as intrinsic graphs of functions over the same horizontal subgroup, say they are all graphs over the x_1-axis. Consequently, $\phi_{\mathcal{T}}$ itself need not have small intrinsic Lipschitz constant, but its first components have small Lipschitz constants, up to subtracting the linear term $L_{\mathcal{T}}$. The following lemma can be used to deduce Theorem 1.3 from Theorem 5.26 applied with constant "η^2/c_n^2".

Lemma 5.28 *Let $n \in \mathbb{N}$, and consider the horizontal subgroup $\mathbb{V} = \{(x_1, 0, \dots, 0) : x_1 \in \mathbb{R}\}$ with complementary vertical subgroup $\mathbb{W}$. There exists a constant $1 \leqslant c_n < \infty$, such that* *the following holds for every* $\eta \in (0,1)$. *If* $\phi = (\phi_2, \dots, \phi_{2n+1}) : \mathbb{V} \to \mathbb{W}$ *is an intrinsic Lipschitz function with the property that*

$$\mathbb{R} \to \mathbb{R}^{2n-1}, \quad x \mapsto (\phi_2, \dots, \phi_{2n})(x, 0, \dots, 0) = \psi(x) + L(x)$$

is the sum of an η*-Lipschitz map* $\psi : \mathbb{R} \to \mathbb{R}^{2n-1}$ *and a linear map* $L : \mathbb{R} \to \mathbb{R}^{2n-1}$, *then* $\{v \cdot \phi(v) : v \in \mathbb{V}\}$ *is an intrinsic* $c_n\sqrt{\eta}$*-Lipschitz graph over the horizontal subgroup*

$$\mathbb{V}_L := \{(x, L(x), 0) \in \mathbb{R} \times \mathbb{R}^{2n-1} \times \mathbb{R} : x \in \mathbb{R}\}.$$

Proof We write $\mathbb{V}_L$ as the span of a unit vector $v_1 := (b_{1,1}, \dots, b_{1,2n})$ in the horizontal plane $\{t = 0\}$. Now we can use similar arguments as in the proof of Theorem 5.3. For arbitrary points $(x, \psi(x) + L(x))$ and $(x', \psi(x') + L(x'))$, we compute

$$\begin{aligned}
&|\langle (x - x', [\psi + L](x) - [\psi + L](x')), v_1\rangle| \\
&\quad \geqslant |(x - x', L(x - x'))| - |\psi(x) - \psi(x')|\,|(b_{2,1}, \dots, b_{2n,1})| \\
&\quad \geqslant |x - x'|\left(\sqrt{1 + |(b_{2,1}, \dots, b_{2n,1})|^2} - |(b_{2,1}, \dots, b_{2n,1})|\right) \\
&\quad \geqslant (\sqrt{2} - 1)|x - x'|
\end{aligned}$$

Here, the first inequality can be deduced by triangle inequality and the fact that the vector $(x - x', L(x - x'))$ is parallel to v_1. The remaining inequalities use that $v_1 = (b_{1,1}, \dots, b_{2n,1})$ is a unit vector, and ψ is η-Lipschitz for some $\eta \in (0,1)$. Denoting

$$z(x) := \langle (x, [\psi + L](x)), v_1\rangle,$$

the previous computations show

$$|z(x) - z(x')| \geqslant (\sqrt{2} - 1)|x - x'|, \tag{5.29}$$

and $z : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Now we complete v_1 to an orthonormal basis $\{v_1, \dots, v_{2n}\}$ of $\mathbb{R}^{2n}$, and we define a map $\varphi : \mathrm{span}(v_1) \to \mathrm{span}\{v_2, \dots, v_{2n}\}$ by setting

$$\varphi(z(x)v_1) = \sum_{j=2}^{2n} \left\langle \begin{pmatrix} x \\ [\psi + L](x) \end{pmatrix}, v_j \right\rangle v_j$$

so that the graph of φ (over $\mathbb{V}_L$) as a set in $\mathbb{R}^{2n}$ coincides with graph of $\psi + L$ (over $\mathbb{V}$). Then, there exists a constant λ_n, depending only on n, such that

$$|\varphi(z(x)v_1) - \varphi(z(x')v_1)| \leqslant \lambda_n |\psi(x) - \psi(x')| \leqslant \lambda_n \eta |x - x'| \leqslant \frac{\lambda_n \eta}{\sqrt{2} - 1}|z(x) - z(x')|.$$

This shows that the projection of the intrinsic graph Γ of ϕ to the horizontal plane $\{t = 0\}$ is the graph of the Euclidean Lipschitz function φ over $\mathrm{span}(v_1)$. Then, it is easy to see that there exists a unique real-valued function φ_{2n+1} so that Γ is the intrinsic graph of $(\varphi, \varphi_{2n+1})$ over $\mathbb{V}_L$, and the graph map of ϕ at x equals the graph map of $(\varphi, \varphi_{2n+1})$ at $z(x)v_1$. It remains to show that $(\varphi, \varphi_{2n+1})$ is intrinsic $c_n\sqrt{\eta}$-Lipschitz for a suitable constant c_n. It is easy to deduce from the intrinsic Lipschitz property of ϕ, Remark 2.4 applied to ϕ, and (5.29) that the graph map of $(\varphi, \varphi_{2n+1})$ is a Lipschitz function with respect to the Heisenberg metric. Applying again Remark 2.4, but now to $(\varphi, \varphi_{2n+1})$, we conclude that this function is intrinsic Lipschitz. Finally, it follows from the Euclidean $(\lambda_n \eta)/(\sqrt{2} - 1)$-Lipschitz continuity and the arguments in Sect. 2 that the intrinsic Lipschitz constant of $(\varphi, \varphi_{2n+1})$ can be taken to be $c_n\sqrt{\eta}$. $\square$

Before proving Theorem 5.26, we give another version for intrinsic N-Lipschitz maps $\phi = (\phi_2, \dots, \phi_{2n+1})$ with $N \geqslant 1$, similarly as in [11]. We can consider

$$\hat{\phi} = (\hat{\phi}_2, \dots, \hat{\phi}_{2n+1}) = \left(\tfrac{1}{N}\phi_2, \dots, \tfrac{1}{N}\phi_n, \tfrac{1}{N^2}\phi_{n+1}, \tfrac{1}{N}\phi_{n+2}, \dots, \tfrac{1}{N}\phi_{2n}, \tfrac{1}{N^2}\phi_{2n+1}\right),$$

which is an intrinsic 1-Lipschitz map. Hence, we apply Theorem 5.26 to $\hat{\phi}$ and constant η. This yields a coronization with Carleson packing constants independent of N, and for every associated tree $\mathcal{T}$ an approximating map $\hat{\phi}_{\mathcal{T}} = (\hat{\psi}_{\mathcal{T}} + \hat{L}_{\mathcal{T}}, \hat{\phi}_{\mathcal{T},2n+1})$ as stated in Theorem 5.26. Then,

$$\phi_{\mathcal{T}} := \left(N\hat{\phi}_{\mathcal{T},2}, \dots, N\hat{\phi}_{\mathcal{T},n}, N^2\hat{\phi}_{\mathcal{T},n+1}, N\hat{\phi}_{\mathcal{T},n+2}, \dots, N\hat{\phi}_{\mathcal{T},2n}, N^2\hat{\phi}_{\mathcal{T},2n+1}\right)$$

is intrinsic Lipschitz and its projection to the horizontal plane $\{t = 0\}$ is the sum of a ηN^2-Lipschitz map $\psi_{\mathcal{T}}$ and a linear $2N^2$-Lipschitz map $L_{\mathcal{T}}$ with the properties stated in the following corollary. The appearance of the constant N^2 (instead of N) is related to the fact that intrinsic N-Lipschitz maps correspond essentially to $(N, \dots, N, N^2)$-tame maps by Propositions 3.3 and 3.6.

Corollary 5.30 (Corona decomposition for intrinsic N-Lipschitz maps) *For every $n \in \mathbb{N}$, $n > 1$, and $\eta \in (0, 1)$, there exists a constant $C \geqslant 1$ such that the following holds. Let $N \geqslant 1$ be arbitrary and let $\phi = (\phi_2, \dots, \phi_{2n+1}) : \mathbb{V} \to \mathbb{W}$ be intrinsic N-Lipschitz. Then, there exists a coronization $\mathcal{D} = \mathcal{G}\dot{\cup}\mathcal{B}$ satisfying the conditions in Definition 5.2 with constant C such that, for every $\mathcal{T} \in \mathcal{F}$, there is an intrinsic Lipschitz map $\phi_{\mathcal{T}} = (L_{\mathcal{T}} + \psi_{\mathcal{T}}, \phi_{\mathcal{T},2n+1}) : \mathbb{V} \to \mathbb{W}$ where $L_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ is a linear $2N^2$-Lipschitz map and $\psi_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{2n-1}$ is a ηN^2-Lipschitz map such that $\phi_{\mathcal{T}}$ approximates ϕ well at the resolution of the intervals in $\mathcal{T}$:*

$$d(\phi(s), \phi_{\mathcal{T}}(s)) \leqslant (\eta N^2)|Q|, \quad s \in 2Q,\ Q \in \mathcal{T}.$$

Proof of Theorem 5.26 We apply the Lipschitz corona decomposition stated in Theorem 5.3 and Corollary 5.22 with parameter $\delta := \eta^2/(100n)$ to the 1-Lipschitz map $\psi := (\phi_2, \dots, \phi_{2n}) : \mathbb{R} \to \mathbb{R}^{2n-1}$. Hence, there are a coronization with Carleson packing constant depending on n and η, and an associated forest $\mathcal{F}$ of trees. We fix $\mathcal{T} \in \mathcal{F}$, and consider the top interval $Q(\mathcal{T}) = [x, y]$ with $x < y$. Then, there exists a δ-Lipschitz map $\psi_{\mathcal{T}} = (\psi_{\mathcal{T},2}, \dots, \psi_{\mathcal{T},2n}) : \mathbb{R} \to \mathbb{R}^{2n-1}$ and a linear 2-Lipschitz map $L_{\mathcal{T}} = (L_{\mathcal{T},2}, \dots, L_{\mathcal{T},2n}) : \mathbb{R} \to \mathbb{R}^{2n-1}$ such that

$$|(\phi_2, \dots, \phi_{2n})(s) - [L_{\mathcal{T}} + \psi_{\mathcal{T}}](s)| \leqslant \delta|Q|, \quad s \in 2Q,\ Q \in \mathcal{T}. \tag{5.31}$$

In addition, we may assume by Corollary 5.22 that

$$\phi_i(s) = [L_{\mathcal{T},i} + \psi_{\mathcal{T},i}](s), \quad \text{for all } s \in E \cup \bigcup_{S \in \mathcal{S}(\mathcal{T})} \partial S, \quad i = 2, \dots, 2n. \tag{5.32}$$

Here, as before, $\mathcal{S}(\mathcal{T})$ is the collection of minimal intervals in $\mathcal{T}$ (possibly an empty collection) and

$$E = Q(\mathcal{T}) \setminus \bigcup_{S \in \mathcal{S}(\mathcal{T})} S.$$

Now we would like to produce an intrinsic Lipschitz function

$$\phi_{\mathcal{T}} = (L_{\mathcal{T},2} + \psi_{\mathcal{T},2}, \dots, L_{\mathcal{T},2n} + \psi_{\mathcal{T},2n}, \phi_{\mathcal{T},2n+1}) : \mathbb{V} \to \mathbb{W} \tag{5.33}$$

satisfying the claims stated in Theorem 5.26. The challenge is to find the last component of $\phi_{\mathcal{T}}$ so that the intrinsic Lipschitz and approximation property hold, and this will require some changes in the terms $L_{\mathcal{T},n+1} + \psi_{\mathcal{T},n+1}$ (but not in the other components).

For $S = [a, b] \in \mathcal{S}(\mathcal{T})$ fixed, we will modify the restriction of $\psi_{\mathcal{T},n+1}$ to $\frac{1}{2}S = [s_1, s_2]$ with $s_1 \leqslant s_2$, which is the interval with the same centre but half the length as S. The property of $\frac{1}{2}S$ needed in the future is that if $Q \in \mathcal{T}$ with $|Q| < |S|$, then

$$2Q \cap \tfrac{1}{2}S = \emptyset. \tag{5.34}$$

Analogously as in the proof of [11, Theorem 3.15], we add to $\psi_{\mathcal{T},n+1}$ a suitable "correction term" $\xi_S : S \to \mathbb{R}$ in order that

(1) $\xi_S(t) = 0, \quad \forall t \neq [s_1, s_2]$;
(2) it holds

$$\begin{aligned}\int_a^b -\xi_S(r)\, dr = \int_a^b &-\phi_{n+1}(r) + \psi_{\mathcal{T},n+1}(r) + L_{\mathcal{T},n+1}(r) \\ &+ \frac{1}{2}\sum_{i=2}^{n} \phi_i(r)\dot{\phi}_{n+i}(r) - \dot{\phi}_i(r)\phi_{n+i}(r) - \phi_{\mathcal{T},i}(r)\dot{\phi}_{\mathcal{T},n+i}(r) - \dot{\phi}_{\mathcal{T},i}(r)\phi_{\mathcal{T},n+i}(r)\, dr.\end{aligned} \tag{5.35}$$

The idea behind (5.35) is the following. As suggested by (5.33), we will define

$$\phi_{\mathcal{T},i} := \psi_{\mathcal{T},i} + L_{\mathcal{T},i}, \quad \text{for } i = 2, \dots, n, n+2, \dots, 2n,$$

but for $i = n + 1$ and $S \in \mathcal{S}(\mathcal{T})$, we set

$$\phi_{\mathcal{T},n+1}|_S := \psi_{\mathcal{T},n+1}|_S + \xi_S + L_{\mathcal{T},n+1}|_S,$$

while $\phi_{\mathcal{T},n+1}|_E := \psi_{\mathcal{T},n+1}|_E + L_{\mathcal{T},n+1}|_E$. The function ξ_S allows us to match $\phi_{\mathcal{T},2n+1}$ with ϕ_{2n+1} in endpoints of minimal intervals. Up to a sign change, the desired intrinsic Lipschitz property of $\phi_{\mathcal{T}}$ is equivalent to the tameness condition. Tame maps on intervals can be characterized as in Proposition 3.7, so we will obtain $\phi_{\mathcal{T},2n+1}$ by integrating an expression involving the components $\phi_{\mathcal{T},2}, \dots, \psi_{\mathcal{T},2n}$. Then, (5.35) ensures that the thus defined $\phi_{\mathcal{T},2n+1}$ agrees with ϕ_{2n+1} in endpoints of the minimal intervals $S \in \mathcal{S}(\mathcal{T})$.

To obtain (5.35), we define $\xi_S : S \to \mathbb{R}$ as

$$\xi_S(t) := \begin{cases} 4c(t - s_1), & \text{for } t \in [s_1, \frac{s_1+s_2}{2}], \\ 4c(s_2 - t), & \text{for } t \in (\frac{s_1+s_2}{2}, s_2], \\ 0, & \text{otherwise} \end{cases}$$

where $c \in \mathbb{R}$ is such that (5.35) holds. Since

$$\int_S \xi_S(r)\,dr = \frac{c|S|^2}{4},$$

and $S = [a,b]$, the requirement (5.35) means that

$$\begin{aligned} -c =& \frac{4}{(b-a)^2}\int_a^b -\phi_{n+1}(r) + \psi_{\mathcal{T},n+1}(r) + L_{\mathcal{T},n+1}(r) \\ &+ \frac{1}{2}\sum_{i=2}^{n} \phi_i(r)\dot{\phi}_{n+i}(r) - \dot{\phi}_i(r)\phi_{n+i}(r) - \phi_{\mathcal{T},i}(r)\dot{\phi}_{\mathcal{T},n+i}(r) + \dot{\phi}_{\mathcal{T},i}(r)\phi_{\mathcal{T},n+i}(r)\,dr. \end{aligned} \tag{5.36}$$

The η-Lipschitz continuity of

$$(\psi_{\mathcal{T},2},\dots,\psi_{\mathcal{T},n},\psi_{\mathcal{T},n+1}+\xi_S,\psi_{\mathcal{T},n+2},\dots,\psi_{\mathcal{T},2n}) \tag{5.37}$$

on S will follow from a bound on $|c|$. We claim that $|c| \leqslant 24n\delta$; indeed, by (5.36),

$$\begin{aligned} -c =& \frac{4}{(b-a)^2}\int_a^b -\phi_{n+1}(r) + \psi_{\mathcal{T},n+1}(r) + L_{\mathcal{T},n+1}(r)\,dr \\ &+ \frac{2}{(b-a)^2}\sum_{i=2}^{n}\int_a^b (\phi_i(r)-\phi_{\mathcal{T},i}(r))\dot{\phi}_{n+i}(r) - (\phi_{n+i}(r)-\phi_{\mathcal{T},n+i}(r))\dot{\phi}_i(r)\,dr \\ &+ \frac{2}{(b-a)^2}\sum_{i=2}^{n}\int_a^b (\phi_i(r)-\phi_{\mathcal{T},i}(r))\dot{\phi}_{\mathcal{T},n+i}(r) - (\phi_{n+i}(r)-\phi_{\mathcal{T},n+i}(r))\dot{\phi}_{\mathcal{T},i}(r)\,dr \\ &+ \frac{2}{(b-a)^2}\sum_{i=2}^{n}\int_a^b \phi_{\mathcal{T},i}(r)\dot{\phi}_{n+i}(r) - \phi_{\mathcal{T},n+i}(r)\dot{\phi}_i(r) - \phi_i(r)\dot{\phi}_{\mathcal{T},n+i}(r) + \phi_{n+i}(r)\dot{\phi}_{\mathcal{T},i}(r)\,dr \\ =:& \; I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.38}$$

Using (5.31), we obtain that $|I_1| \leqslant 4\delta$; moreover, using again (5.31) and recalling that $\psi_{\mathcal{T}}$ is a δ-Lipschitz map with $\delta < 1$ and $L_{\mathcal{T}}$ is a linear 2-Lipschitz map, we have that $|I_i| \leqslant 12(n-1)\delta$ for $i = 2,3$. Finally, integrating by parts, and using $\phi_i(s) = \phi_{\mathcal{T},i}(s)$ for $s \in \{a,b\}$ and $i = 2,\dots,2n$, we get that $I_4 = 0$.

Hence, $|c| \leqslant 24n\delta$, as desired. Consequently, we get that ξ_S is $96n\delta$-Lipschitz with $\|\xi_S\|_{L^\infty} \leqslant 24n\delta|S|$.

We make analogous modifications inside all intervals $S \in \mathcal{S}(\mathcal{T})$. Recalling that we have chosen δ so that $100n\delta = \eta^2 \leqslant \eta$, we obtain an η-Lipschitz map on $\mathbb{R}$, piecewise defined on S as in (5.37).

We next show that the modified map still satisfies (5.31), albeit with a larger constant than δ. Indeed, for $Q \in \mathcal{T}$ it suffices to check the condition for $s \in 2Q$ that belong to $\frac{1}{2}S$ for some $S \in \mathcal{S}(\mathcal{T})$, as this is the only place where we have done a modification. So assume $s \in 2Q \cap \frac{1}{2}S$. Then, $|S| \leqslant |Q|$ by (5.34), and (5.31) yields

$$|(\phi_2,\dots,\phi_{2n})(s) - (\phi_2,\dots,\phi_{2n})(s)| \leqslant 25n\delta|Q| \leqslant \eta|Q|. \tag{5.39}$$

Now we consider the last component of the approximation map $\phi_{\mathcal{T}}$ of ϕ. For $Q(\mathcal{T}) = [x,y]$, we define

$$\phi_{\mathcal{T},2n+1}(s) := \phi_{2n+1}(x) + \int_x^s -\phi_{\mathcal{T},n+1}(r) + \frac{1}{2}\sum_{i=2}^{n} \phi_{\mathcal{T},i}(r)\dot{\phi}_{\mathcal{T},n+i}(r) - \dot{\phi}_{\mathcal{T},i}(r)\phi_{\mathcal{T},n+i}(r)\,dr,$$

for all $s \in [x, y]$. By Proposition 3.7, $\phi_{\mathcal{T}} : \mathbb{V} \to \mathbb{W}$ is an intrinsic Lipschitz map. The next step is to show

$$\phi_{2n+1}(s) = \phi_{\mathcal{T},2n+1}(s), \tag{5.40}$$

for $s \in E \cup \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S$. By construction, this is equivalent to verifying that for $s \in E \cup \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S$ and $Q(\mathcal{T}) = [x, y]$ we have

$$\int_x^s \dot{\phi}_{2n+1}(r)\,dr = \int_x^s \dot{\phi}_{\mathcal{T},2n+1}(r)\,dr.$$

We recall that E is a measurable set because $E = Q(\mathcal{T}) \setminus \bigcup_{S\in\mathcal{S}(\mathcal{T})} S$ and $\mathcal{S}(\mathcal{T})$ is a countable family of intervals. Moreover,

$$\phi_{\mathcal{T},2n+1}(s) = \phi_{2n+1}(x) + \int_{E\cap[x,s]} \dot{\phi}_{\mathcal{T},2n+1}(r)\,dr + \sum_{S\in\mathcal{S}(\mathcal{T})} \int_{S\cap[x,s]} \dot{\phi}_{\mathcal{T},2n+1}(r)\,dr.$$

If E is a Lebesgue null set, then the integral over E is not relevant. On the other hand, the derivatives of $\phi_i, \phi_{\mathcal{T},i}$, $i = 2, \dots, 2n+1$ exist almost everywhere, and if E has positive measure, then almost every point in E is a Lebesgue density point of E. Since $\phi_i(s) = \phi_{\mathcal{T},i}(s)$ for all $s \in E$ and $i = 2, \dots, 2n$, it follows that $\dot{\phi}_i(s) = \dot{\phi}_{\mathcal{T},i}(s)$ for almost every $s \in E$ and so $\dot{\phi}_{\mathcal{T},2n+1}(s) = \dot{\phi}_{2n+1}(s)$ for almost every $s \in E$. Moreover,

$$\sum_{S\in\mathcal{S}(\mathcal{T})} \int_{S\cap[x,s]} \dot{\phi}_{\mathcal{T},2n+1}(r)\,dr = \sum_{S\in\mathcal{S}(\mathcal{T})} \int_{S\cap[x,s]} \dot{\phi}_{2n+1}(r)\,dr. \tag{5.41}$$

Indeed, by the choice of $s \in E \cup \bigcup \partial S$ we have two cases to consider: $S \cap [x, s] = S$ or $S \cap [x, s] = \emptyset$. The latter intervals S can be ignored, and for the former, the integrals on the left and on the right-hand side of (5.41) agree, by (5.35). This proves (5.40).

Finally, it remains to check the approximation condition (5.27). Having already established (5.39), the only nontrivial inequality is

$$A := \left| \phi_{2n+1}(s) - \phi_{\mathcal{T},2n+1}(s) + \frac{1}{2}\sum_{i=2}^{n} -\phi_{\mathcal{T},i}(s)\phi_{n+i}(s) + \phi_i(s)\phi_{\mathcal{T},n+i}(s) \right| \leqslant \eta^2 |Q|^2, \tag{5.42}$$

for every $Q \in \mathcal{T}$, $s \in 2Q$. We have two different cases: $s \in E$ and $s \notin E$. Firstly, for $s \in E$, the left-hand side of (5.42) vanishes by (5.32) and (5.40), and so the inequality holds trivially true. Secondly, for $s \notin E$, as in the proof of [11, Proposition 3.6], we know that there is $\tilde{s} \in Q \cap (E \cup \bigcup_{S\in\mathcal{S}(\mathcal{T})} \partial S)$ such that $|s - \tilde{s}| \leqslant |Q|$. Since $\phi_{\mathcal{T},n+1}(\tilde{s}) = \phi_{n+1}(\tilde{s})$, we can estimate as follows:

$$
\begin{aligned}
A &\leqslant \Bigg| \phi_{2n+1}(s) - \phi_{2n+1}(\tilde{s}) - \phi_{\mathcal{T},2n+1}(s) + \phi_{\mathcal{T},2n+1}(\tilde{s}) \\
&\quad + \tfrac{1}{2}\sum_{i=2}^{n} -\phi_{\mathcal{T},i}(s)\phi_{n+i}(s) + \phi_i(s)\phi_{\mathcal{T},n+i}(s) \Bigg| \\
&= \Bigg| \int_{\tilde{s}}^{s} -\phi_{n+1}(r) + \phi_{\mathcal{T},n+1}(r) + \tfrac{1}{2}\sum_{i=2}^{n} \phi_i(r)\dot{\phi}_{n+i}(r) - \dot{\phi}_i(r)\phi_{n+i}(r) \\
&\quad - \tfrac{1}{2}\sum_{i=2}^{n} \phi_{\mathcal{T},i}(r)\dot{\phi}_{\mathcal{T},n+i}(r) - \phi_{\mathcal{T},n+i}(r)\dot{\phi}_{\mathcal{T},i}(r)\,dr \\
&\quad + \tfrac{1}{2}\sum_{i=2}^{n} \phi_{\mathcal{T},n+i}(s)\phi_i(s) - \phi_{\mathcal{T},i}(s)\phi_{n+i}(s) \Bigg|.
\end{aligned}
$$

For convenience, let us denote the integral in the above expression by "I", so that

$$
A \leqslant \left| I + \tfrac{1}{2}\sum_{i=2}^{n} \phi_{\mathcal{T},n+i}(s)\phi_i(s) - \phi_{\mathcal{T},i}(s)\phi_{n+i}(s) \right|. \tag{5.43}
$$

Using similar computations as in the bound for c, see (5.38), we find

$$
\begin{aligned}
I &= \int_{\tilde{s}}^{s} -\phi_{n+1}(r) + \psi_{\mathcal{T},n+1}(r)\,dr \\
&\quad + \tfrac{1}{2}\sum_{i=2}^{n} \int_{\tilde{s}}^{s} (\phi_i(r) - \phi_{\mathcal{T},i}(r))\dot{\phi}_{n+i}(r) - (\phi_{n+i}(r) - \phi_{\mathcal{T},n+i}(r))\dot{\phi}_i(r)\,dr \\
&\quad + \tfrac{1}{2}\sum_{i=2}^{n} \int_{\tilde{s}}^{s} (\phi_i(r) - \phi_{\mathcal{T},i}(r))\dot{\phi}_{\mathcal{T},n+i}(r) - (\phi_{n+i}(r) - \phi_{\mathcal{T},n+i}(r))\dot{\phi}_{\mathcal{T},i}(r)\,dr \\
&\quad + \tfrac{1}{2}\sum_{i=2}^{n} \int_{\tilde{s}}^{s} \phi_{\mathcal{T},i}(r)\dot{\phi}_{n+i}(r) - \phi_{\mathcal{T},n+i}(r)\dot{\phi}_i(r) - \phi_i(r)\dot{\phi}_{\mathcal{T},n+i}(r) + \phi_{n+i}(r)\dot{\phi}_{\mathcal{T},i}(r)\,dr \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

Notice that $[\tilde{s}, s] \subset 2Q$ (or $[s, \tilde{s}] \subset 2Q$), so $|\phi_{\mathcal{T},n+1}(r) - \phi_{n+1}(r)| \leqslant 25n\delta|Q|$ for all $r \in [\tilde{s}, s]$ by (5.39), and $|\phi_{\mathcal{T},i}(r) - \phi_i(r)| \leqslant \delta|Q|$ for $i \in \{2, \dots, 2n\} \setminus \{n+1\}$ by the property coming from Theorem 5.3. Using also that $|s - \tilde{s}| \leqslant |Q|$, we obtain the desired estimates for the first three summands: $|J_1| + |J_2| + |J_3| \leqslant 50n\delta|Q|^2$. The term "$J_4$" might look problematic at first since $\Phi(s)$ does not necessarily agree with $\Phi_{\mathcal{T}}(s)$. However, if we combine it first with the second summand in (5.43), then cancellation occurs by partial integration:

$$
J_4 + \tfrac{1}{2}\sum_{i=2}^{n} \phi_{\mathcal{T},n+i}(s)\phi_i(s) - \phi_{\mathcal{T},i}(s)\phi_{n+i}(s) = \tfrac{1}{2}\sum_{i=2}^{n} -\phi_{\mathcal{T},i}(\tilde{s})\phi_{n+i}(\tilde{s}) + \phi_{\mathcal{T},n+i}(\tilde{s})\phi_i(\tilde{s}).
$$

Since $\Phi(\tilde{s}) = \Phi_{\mathcal{T}}(\tilde{s})$, the expression on the right vanishes. Hence, we obtain that

$$
A \leqslant 50n\delta|Q|^2.
$$

Finally, we recall that $100n\delta = \eta^2$, so (5.42) holds, as desired. This concludes the proof. $\square$

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