

Samuli Ikonen

Efficient Numerical Methods for Pricing American Options







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ABSTRACT

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In this thesis we study efficient numerical methods for pricing American options. We apply option pricing models which are based on the Black and Scholes theory and Heston's stochastic volatility model. Prices for American options are modelled by linear complementarity problems with one-dimensional and two-dimensional parabolic partial differential operators. The use of numerical methods is unavoidable because of the complexity of these option pricing problems. Large scale option trading gives a motivation to develop efficient numerical procedures for solving American option pricing problems.

In this work we apply a finite difference method to the discretization. After discretization, a sequence of discrete linear complementarity problems should be solved in order to obtain prices for American options. This thesis is built around two types of splitting methods. In the articles of this thesis one is referred as the operator splitting method and the other one as the componentwise splitting method.

Operator splitting methods are first applied for solving basic American option pricing models and then they are applied to a solution of a model with a stochastic volatility assumption. The idea in these operator splitting methods is that at each time step a treatment of an obstacle constraint and a solution of a system of linear equations are made in separate fractional steps. Particularly, the advantage of these methods is shown when a stochastic volatility model is used.

Componentwise splitting methods are applied for a solution of the American option pricing problem with a stochastic volatility setting and shown to be highly efficient. In a basic form of this splitting a discrete linear complementarity problem is divided in such a way that three linear complementarity problems with tridiagonal matrices need to be solved. The efficiency of this splitting method is based on the use of a direct solver at each fractional step. Strang symmetrization is used to increase the accuracy of this splitting method.

The efficiency of the proposed numerical techniques is demonstrated with several numerical experiments. This thesis ends with an article considering a numerical solution of the American option pricing problem with the stochastic volatility assumption where an extensive comparison of efficiency of numerical methods are presented.

Keywords: American option pricing, linear complementarity problem, stochastic volatility model, finite difference methods, operator splitting methods

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ABSTRACT

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YHTEENVETO (SUMMARY IN FINNISH)

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1 INTRODUCTION

An option is a financial instrument whose value depends on a value of some underlying asset. Commonly, mathematical models are used when prices for options are evaluated. In many cases numerical methods are required in a solution of option pricing problems because analytical solution formulae do not exist. Options with a complex structure and the large scale of the option trade in general are the motivating factors for developing efficient numerical procedures which can be used in the option pricing as well as in their risk management. Nowadays numerical methods are widely used in the pricing of different kinds of financial option contracts and there are many references to this topic, see for example [42], [68], [74], [80], and a recently appeared [1].

In this thesis we consider numerical methods for pricing American options. We apply parabolic partial differential equation (PDE) models which are based on the famous Black and Scholes theory which was proposed in [8] and afterwards developed in many works. A consequence of an early exercise possibility of the American options is that a time dependent linear complementarity problem (LCP) should be solved when these options are priced. The complexity of these problems requires that numerical methods should be applied in the solution of these LCPs. Methods for pricing such option contracts are considered, for example in the articles [11], [13], [28], [30], [58], and also in the references given above.

The objective of this research is to develop efficient numerical solution methods for two different American option pricing models. In the pricing we apply a model with a classical one-dimensional PDE and a model with a stochastic volatility assumption. The finite difference method is applied to the discretization of PDEs and splitting techniques are proposed to a solution of LCPs. This research focuses on numerical solution methods and hence, many important top-

ics related to the option pricing, like a calibration problem, are omitted.

This introduction to the articles included to this doctoral dissertation is organized as follows. In Section 2, we briefly discuss financial option contracts. We mainly describe European and American options. In Section 3, we consider option pricing models for American options. A basic model and a model with stochastic volatility are introduced. Topics related to a finite difference discretization are briefly considered in Section 4 and in Section 5 a solution of a system of linear equations and a LCP related to a option pricing are discussed. In Section 6, we formulate two numerical solution methods which enable efficient numerical solution of American option pricing problems. An overview of the included articles is given in Section 7, and finally, at the end of this introduction some future prospects are presented.

2 FINANCIAL OPTION CONTRACTS

In this section we give the basics of financial options. We point out some option terminology which helps us to understand the main idea of option pricing. In order to sketch the general view about options and option markets, see for example [42], [81].

An option is a financial contract between two parties and the value of it depends on the value of the underlying asset. The option gives a right for the holder to buy or sell a certain amount of the underlying asset for a specified price at a specified time in the future. The other party in the option contract is called the writer, who has to sell or buy whenever the holder decides to exercise the option. The option is referred to as a call option if it gives a right to buy the underlying asset and as a put option if it gives a right to sell the underlying asset. The underlying asset of the option contract can be, for example, stocks, stock indices, foreign currencies, commodities or some other derivative instruments like future contracts [42].

As said earlier, options can be classified as European and American. The European options can be exercised only at the expiry date while the American options can be exercised at any time prior to the expiry date. This early exercise possibility is the difference between the European and American options. It turns out that the exercise feature of the American option is the essential factor which influences the form of option pricing models. The European option is a simple option contract and that is why we start with a section about European options.

2.1 European option

A European call option is a financial contract which gives a right for the holder to buy the underlying asset for a strike price E at the expiry date T . The option contract between the holder and the writer is made at time $t = 0$ and the exercising decision is made at time $t = T$. Exercising the call option means that the holder decides to buy the underlying asset for the price E . When the holder can choose exercising the option, the writer of the option is obligated to sell the underlying asset if needed. Obviously, exercising the option is rational only if the value of the underlying asset S is higher than the exercise price E at the time T because otherwise, the underlying asset can be bought from the markets without the option contract. The value of the call option at the time $t = T$ is given by a so-called payoff function g and it is

$$g(S) = \max\{S - E, 0\}. \quad (2.1)$$

This shows that the call option is worth of $S - E$, if the asset value is higher than the exercise price and that it is worthless otherwise.

Another basic option contract is a European put option. It gives a right for the holder to sell the underlying asset for the exercise price E at the time T . The payoff function for the European put option is

$$g(S) = \max\{E - S, 0\}. \quad (2.2)$$

In this case the value of the European option is $E - S$, if the exercise price is higher than the asset value and otherwise it is worth of nothing.

In order to get the right to buy or sell the underlying asset for a price E the holder has to pay for the writer of the option. This payment should be adequate for the holder and for the writer of the option contract. The holder should see that this payment is worth of having the right for exercising the option and the writer should see that this payment is sufficient to cover the risk related to the obligation to sell or buy the underlying asset if needed. This payment is the price of the option contract at time $t = 0$.

2.2 American option

An American option is a contract where exercising the option is possible at any time during the life time of the option contract. This early exercise possibility is the difference between the American and the European option contract where exercising is possible only at the time $t = T$. The American options can also be classified to call and put options.

The value of the American option at the exercising moment is determined using a payoff function. Since exercising the option is possible during the time interval $[0, T]$ the payoff function of the American put option, for example, is of the form

$$g(S) = \max\{E - S, 0\}, \quad (2.3)$$

where $t \in [0, T]$ and S is the value of the underlying asset at the time t .

In the case of the European option the exercising decision is simple because it is possible only at the time $t = T$. The early exercise possibility of the American option enables the holder to exercise the option at any time prior to the time $t = T$ and hence, the holder faces a problem to choose the optimal time to exercise. Similarly to the European call and put options, the American options can also become worthless in which case there is no sense for exercising.

At the time $t = 0$ the holder should pay the price of the American option for the writer. Simply because the American option can be exercised straight away when it is bought, the price of the option should be more than the value of the payoff function. If this were not the case, there would be an obvious arbitrage opportunity. This means that a risk free profit could be made by buying an American option and then immediately exercising it. The early exercise feature makes the American option more valuable than the corresponding European option and it also makes the pricing of the option more difficult. In Section 3 the option pricing models for the American option are considered.

2.3 Use of options

Options and other financial derivative instruments are used in a very large scale nowadays and hence there is a lot of literature on this topic; see for example [42] and references therein. Next we give two examples for the use of the option contracts.

Options can be used to reduce the risk which the investor is facing [42]. This is called hedging. The investor can have for example some amount of stocks and hence, just by owning stocks, the investor faces a risk related to this stock investment. By buying put options the investor reduces the risk in case the stock price decreased unexpectedly. The risk is then hedged by paying the price of the put option. This can be seen as one kind of insurance against decreasing stock values. Similarly, if a company has payments in foreign currency in the future, the company faces a currency risk. This risk can be hedged using currency options which means in the simplest case that the company would buy call or put options where the underlying asset would be the specified foreign currency. The choice between the call and put options depends on the expectation on movements of the exchange rate of the foreign currency.

Options can be used also for speculation. That is, the investor can buy options in order to get profits which are based on movements in stock prices [42], [81]. When a stock price is predicted to rise, an investor can buy some number of stocks and hold them until the stock price has risen as expected. After selling these stocks the investor makes a profit which is the difference between the buying and selling prices. Obviously, there is a possibility to incur losses if the price of the stock falls. One alternative way to try to make a profit is to buy call options. In that case, the investor buys some number of call options where the underlying asset is the specified stock. The price of this option can be much cheaper than the stock price and, hence, the investor does not need to invest the price of the stock

in order to speculate with the rise of the stock price. That is, when stock options are used, the investor can get a profit from stock rise by just investing in the price of a stock option. A profit from this option trade is the difference between the stock price and the exercise price. Losses are limited to the initial premium of the stock trade because the option gives a right, not an obligation to exercise. Hence, the option contract offers a possibility to large profits while limiting the losses to the initial investment.

2.4 Other types of options

At the end of this section, we describe briefly what kinds of other financial options exist. The purpose is to show that there are also other types of options than the call and put options with European or American exercise possibilities. These European and American options are basic options which are sometimes called plain vanilla options or standard options. Nowadays it is typical for the structure of financial options to become more and more complex. In the case of the European put option the exercise date is fixed and the payoff function is simply the positive difference between the price of the underlying asset and the exercise price. Many variants of basic options are obtained by changing the form of the payoff function. Options having complex payoff functions or nonstandard exercising are called exotic options [41], [42], [67], [80]. In the following we explain which kind of options are the so-called barrier options and Asian options.

Barrier options are one class of exotic options [74], [81], [86]. There are a few basic barrier option structures, and next we give a short overview of them. In the case of the so-called up-and-in barrier option the option expires worthless unless the price of the underlying asset has not reached the barrier value before the option contract expires. Here, up refers that the barrier value is above the initial price of the underlying asset and hence, if the asset price reached the barrier value it would reach it from below. Correspondingly, there is an up-and-out barrier option. This option expires worthless if the underlying asset price increased above the barrier value before the expiry. There are also options called down-and-in and down-and-out barrier options. Here, down refers that the barrier value is below the initial asset price. The down-and-out option pays if the barrier value is not reached and the down-and-in option pays off when the underlying asset price drops below the barrier value. These options can be either call or put options as well as European or American. More details on barrier options are given, for example in references [2], [7], [9], [15], [36], [66].

There exist also so-called double barrier options [69], [80]. This type of option has two barriers. An upper barrier is above the initial price of the underlying asset and the other barrier is below the initial asset price. The double out barrier option has such a structure that if either barrier is reached then the option expires worthless. As in the basic barrier options, there are also a double in barrier option. This expires worthless if a price of the underlying asset changes between barrier values.

Barrier options are used, for example, if an investor thinks that the asset

price would not increase a lot. In that case, the investor can purchase an up-and-out barrier call which is cheaper than the basic call option. If the asset price does not reach the upper barrier then the investor gets the desired payoff with a smaller initial investment [80].

Another class of options is known as Asian options [42], [74], [80]. Payoff functions of these options depend on the average value of the underlying asset price. This dependence on the history of the asset values is a reason why this option is called a path-dependence exotic option. In previous sections we have introduced the payoff function for call option. This payoff function depends on the asset value and the exercise price. An average value of underlying asset prices can be used instead of the exercise price in a payoff function. This defines a payoff function for the so-called average strike call option. The payoff function is then the difference between the asset price at expiry and an average of asset prices. When the asset value in the basic call option payoff function is replaced with its average value we have a payoff function for a so-called average rate call option. Asian options can have either European or American style exercise feature. Recently, American Asian options have been studied, for example in [23].

Several Asian options are defined depending on how the average of the underlying asset values is calculated. The arithmetic average or the geometric average can be used. Also, the average depends on the size of the data set applied when the average is computed. Furthermore, an average value can be continuously or discretely sampled. Asian options can be used, for example, related to commodity trading in which case the underlying asset is some commodity. Also, the exchange rate of a foreign currency can be the underlying asset in the Asian option contracts. Methods for pricing Asian options are considered, for example in [4], [6], [46], [51], [59], [84].

Besides the barrier options and the Asian options there is a large range of different kinds of other option contracts which are traded nowadays; see for example [42], [80]. At this point we mention some of them in order to give an idea about these options. An investor can take a position where calls and puts on the same underlying asset are purchased at the same time. Generally, these are referred to as combinations or option strategies. For example in a combination called straddle the investor buys call and put options with the same exercise price and with the same expiration time [42]. The option contracts which have many underlying assets are called basket options. A number of underlying assets can vary from two or three up to hundreds. Options, and generally, derivatives, can be used to reduce different kinds of risks. In [3], a pricing model for weather derivatives is considered and the electricity options are studied in [47]. Option theory can also be applied in the investment planning using so-called real options.

3 OPTION PRICING MODELS

In this section we describe American option pricing models which are applied in the papers of this thesis. The early exercise possibility is the difference between European and American options. In this thesis we use option pricing models which are based on parabolic partial differential equations although there exist other models for a price of American options. We consider models which are based on the Black and Scholes analysis and its extensions. According to these models, the price of the American option is obtained by solving a time dependent linear complementarity problem.

In this thesis we consider two option pricing models which basically differ on how the price movements of the underlying asset are modelled. In the first option pricing model a simple model for the price of the asset is used. This leads to an option pricing model which is based on a one-dimensional PDE. In the second model a volatility of the underlying asset is assumed to follow a stochastic process. This stochastic volatility assumption leads to the option pricing model containing a two-dimensional PDE.

The price for European options can be determined by solving a final value problem. The numerical solution of such problems can be made straightforwardly and even an analytical solution formulae are known for many cases. The price of an American option should be determined by solving an LCP. The numerical solution of these models are more challenging than the solution of European option pricing problems.

Next we give a short description of two American option pricing models. First, in Section 3.1 we sketch out the derivation of the Black-Scholes equation and we give the model for the pricing of the American options. In Section 3.2 the model with the stochastic volatility assumption is considered. In the last subsection we briefly mention other option pricing models. The purpose of this last

section is to give an idea of other types of approaches that have been considered in the field of option pricing. As said earlier, this thesis focuses on developing numerical methods for solving already existing models and hence, the modelling issues are discussed only in a general level.

3.1 Basic model for American option

The price of the option depends on the price of its underlying asset. That is why the price of the underlying asset should be modelled when forming option pricing models. Although asset prices and hence also returns are known for the preceding time, it is not obvious how to forecast asset prices for the near future. A simple model for the price of the underlying asset is used in a derivation of the one-dimensional parabolic PDE option pricing model [29], [80], [81]. This leads to the famous and widely used Black-Scholes equation.

A simple asset price model contains a deterministic and a stochastic part. In the deterministic part the risk free return is modelled. This part is given in the form μdt , where μ is the constant rate of growth of a capital and dt describes an infinitesimal time interval. The stochastic part models the randomness related to the return on an asset. The purpose of this part is to model unexpected events in the market. A normal distribution is used to generate random samples σdw , where the σ is the standard deviation (volatility) of the returns, and dw is the infinitesimal increment of Brownian motion. By summing up these two parts we get a simple asset price model which is of the form

$$dx_t = \mu x_t dt + \sigma x_t dw_t. \quad (3.1)$$

Although this model simplifies the price movements a lot, it is commonly used. The Black and Scholes analysis leads us to a PDE which has become very popular in the field of finance. After the famous article [8], there have been many publications in the field. Next, we outline the derivation of this PDE.

The derivation begins by forming a portfolio consisting one option contract and a specific number of underlying assets. At a given discrete time t , the value of this portfolio is denoted by Π and its change in the unit time interval by $d\Pi$. The change of the value of this portfolio contains randomness because the underlying asset is modelled by (3.1). This randomness can be eliminated by applying the so-called Ito's lemma and by choosing the number of the underlying asset appropriately at each discrete time moment. The value of the portfolio at the starting moment is fixed and its value changes in time depending on the price changes of the underlying asset. After making the appropriate choices we have a portfolio such that the change of its value does not have any stochastic part and hence, the formula for $d\Pi$ is deterministic.

Based on the arbitrage theory, if the portfolio value Π is invested in a risk-free bank account with the interest rate r it can be assumed that it gives the same profit as $d\Pi$. That is, because the change $d\Pi$ does not contain any randomness. Otherwise, if $d\Pi$ makes more profit than the riskless investment there exist an arbitrage opportunity which is against the market assumptions. Using the above

mentioned derivation steps, the Black-Scholes equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} + rx \frac{\partial \phi}{\partial x} - r\phi = 0, \quad (3.2)$$

can be derived, where σ is the volatility of the underlying asset and r is the risk free interest rate. The variable x is the value of the underlying asset and t is the time.

In order to derive the PDE (3.2), some assumptions have to be made about financial markets. Firstly, the value of the underlying asset is assumed to follow a lognormal random walk which appears in the simple asset price model given in (3.1). Secondly, it is assumed that the risk free interest rate is known beforehand, and that this interest rate is constant in time. Furthermore, it is assumed that the underlying asset does not pay dividends and that there are no transaction costs in the market. This latter assumption allows us to do as many transactions in a certain time interval as we like without any additional payments. Also, it is assumed that the so-called delta hedging can be made continuously, and also without any additional costs, as assumed before. An important assumption on the market is that there are no arbitrage opportunities. This says that, if the value of the expected return increases, also the risk related to this profit increases. These assumptions simplify financial markets and hence, option prices do not necessarily correspond to prices observed in the real option markets. Although the above mentioned assumptions simplify option markets, perhaps even too much, the Black-Scholes equation is quite popular in practical option pricing.

The price for the European option based on the Black-Scholes theory can be achieved by solving the final value problem with appropriate boundary conditions. In basic cases there exist an analytical solution for the European option price, and this formula has become very popular in the field of finance.

The American put option gives a right to sell the underlying asset for the exercise price E at any time prior to the expiry date. The consequence of this early exercise possibility of the American option is that the price of the option can never be below the payoff function and this feature should be included in the option pricing model. The option pricing problem is final value problem where the final value is given by the payoff function. The purpose in the option pricing is to define the price of the option at the present time which is $t = 0$. However, we use the common practice to transform this final value problem into an initial value problem. Furthermore, the unbounded domain $[0, \infty)$ is truncated to be a bounded interval $[0, X]$ where X is sufficiently large. Thus, the price of the American put option is obtained by solving the LCP

$$\begin{cases} \frac{\partial \phi}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} - rx \frac{\partial \phi}{\partial x} + r\phi \geq 0, \\ \phi(x, t) \geq g(x), \\ \left(\frac{\partial \phi}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} - rx \frac{\partial \phi}{\partial x} + r\phi \right) (\phi - g) = 0, \end{cases} \quad (3.3)$$

with the boundary conditions

$$\phi(0, t) = E, \quad \phi(X, t) = 0, \quad (3.4)$$

and with the initial value

$$\phi(x, 0) = g(x) := \max(E - x, 0). \quad (3.5)$$

According to this model, the value of the American option depends on the form of the payoff function, the volatility σ of the underlying asset, the exercise price E , time to expiry T , and the risk free interest rate r .

In the case of the pricing of the European option the exercise moment is fixed. This implies that the next decision after buying the option contract is to decide about exercising the option at the time $t = T$. This is not the case when the American option is under consideration. In addition to the pricing problem of the American option there is also a problem related to the optimal exercise time. At each time it should be decided whatever the option should be exercised or not. Of course, this exercising the option depends on the value of the underlying asset and hence, there are asset values when exercising the option is rational and values when it is not. Due to this the American option pricing problem is a free boundary problem where the free boundary can be seen as a boundary which distinguishes the asset values into two separate parts. In one of these parts exercising the option is rational whereas in the other it is not. This boundary is not known a priori and it should be determined by solving the free boundary problem. It is worth of noticing that this boundary varies in time.

3.2 Stochastic volatility model for American option

In this section we describe an American option pricing model where the volatility is assumed to be stochastic [5], [25], [37], [40], [79], [80], [83]. This kind of option pricing model is the other model that is used in the articles of this thesis. In the following we apply the Heston's stochastic volatility model which is proposed in [37].

In (3.1) we gave the simple asset price model. The risk free interest rate and the volatility of the underlying asset were assumed to be constants. These assumptions simplify the asset price process significantly. One way to improve this asset price model is by assuming that the volatility of the asset price follows a stochastic process instead being constant or time-varying. Volatility models have been considered in many references; see for example [29], [74], [83], and references therein. The stochastic volatility assumption leads to a two-dimensional parabolic PDE which is one of the generalizations of the basic Black-Scholes equation. The derivation of this two-dimensional PDE is omitted in this introduction.

In Heston's model, stochastic differential equations

$$dx_t = \mu x_t dt + \sqrt{y_t} x_t dw_1, \quad (3.6)$$

$$dy_t = \alpha(\beta - y_t)dt + \gamma\sqrt{y_t}dw_2, \quad (3.7)$$

define the stock price process x_t and the variance process y_t . Equation (3.6) models the value of the underlying asset, where the parameter μ is the deterministic growth rate of the value of the underlying asset and $\sqrt{y_t}$ is the standard deviation

(the volatility) of the returns of the underlying asset. The model for the variance process y_t is given by (3.7). The volatility of the variance process y_t is denoted by γ and the variance will drift back to a mean value $\beta > 0$ at a rate $\alpha > 0$. These two processes contain randomness, that is, w_1 and w_2 are Brownian motions with a correlation factor $\rho \in [-1, 1]$. This type of underlying asset price model makes the option pricing model more realistic.

In a general level, the derivation of a two-dimensional PDE follows somewhat similar steps to those in the case of the derivation of the basic Black-Scholes equation. A generalized Black-Scholes operator is defined by

$$L\phi = \frac{\partial\phi}{\partial t} - \frac{1}{2}yx^2\frac{\partial^2\phi}{\partial x^2} - \rho\gamma yx\frac{\partial^2\phi}{\partial x\partial y} - \frac{1}{2}\gamma^2y\frac{\partial^2\phi}{\partial y^2} - rx\frac{\partial\phi}{\partial x} - \{\alpha(\beta - y) - \vartheta\gamma\sqrt{y}\}\frac{\partial\phi}{\partial y} + r\phi, \quad (3.8)$$

where the r is the risk free interest rate, β is the mean level of the variance, α is the rate of reversion to the mean level, and γ is the volatility of the variance. The correlation between the price of the asset and its variance is ρ . Furthermore, ϑ is the so-called market price of the risk. Typical parameter values can be seen for example in [17], [58], [83].

Again, American options can exercised at any time before the expiry and hence, option prices can not be below the payoff function. Similarly, as in the basic case, the model is a time dependent LCP

$$\begin{cases} L\phi \geq 0, & \phi \geq g, \\ (\phi - g)L\phi = 0, \end{cases} \quad (3.9)$$

where g is the payoff function. An initial value for this problem is the payoff function, and on other boundaries appropriate boundary conditions are applied. Originally this problem is given in an unbounded domain but we truncate this to a computational domain defined by

$$(x, y, t) \in [0, X] \times [0, Y] \times [0, T] = \Omega \times [0, T], \quad (3.10)$$

where X and Y are large enough.

3.3 Other models

In this section we have described briefly the model for the American option pricing as well as the model for a price of the European option. In both cases one can make extensions for these basic models. Although this is not a topic of the thesis we mention some of the extensions. In a derivation of the Black-Scholes equation it was assumed that the underlying asset does not pay dividends at all. If the underlying asset pays dividends it has obvious effect also for the price of the option on that asset. There are several dividend payment structures. In [80], for example, a constant dividend yield and discrete dividend payments are considered. Other extensions are achieved if the interest rate and the volatility is assumed to be time-varying [80]. Use of jump processes in the financial modelling is discussed in [19].

4 DISCRETIZATION OF PARABOLIC PDE

As we have seen in the previous section the American option pricing models which are applied in this thesis are based on parabolic partial differential equations. These models can not be solved analytically and hence, the use of numerical approximation methods is unavoidable. A finite difference method offers a straightforward and quite a simple way to approximate solutions of these models numerically. To be more exact, the finite difference method is used to approximate derivatives in PDEs leading to a sequence of discrete linear complementarity problems.

This section considers briefly the space and time discretizations. These topics are considered in a case of an initial value problem without the obstacle constraint due to the early exercise possibility of the American option because the finite difference discretization can be performed in the same way as with the obstacle constraint. The purpose of this section is to give a general overview of topics from a viewpoint of discretization of the PDEs used in the American option pricing. This section ends with a sequence of systems of linear equations which can be used in pricing European options considered in Section 2.1. A solution of discrete linear complementarity problems arising from the American option pricing is considered in the next section.

4.1 Space discretization

In the model (3.3) the PDE has one space variable and in the model (3.9) the PDE has two space variables. In both cases derivatives have non-constant coefficients. We apply finite difference methods in the articles of this thesis because

they are well suited for rectangular computational domains. In the finite difference method partial derivatives are replaced by approximations [38], [56], [57], [65], [70], [76]. These approximations are based on Taylor series expansions at some specific points. Basically, the form of the finite discretization scheme depends on the order of the partial differential derivative and on the accuracy of the finite difference approximation. Finite difference methods have been applied already for many decades and for many applications in the field of scientific computing.

Other discretization methods such as the finite element method and the finite volume method can be used for discretization of the above mentioned PDEs. The finite element method is more flexible than the finite difference method and it is easier to use for example when the geometry of the computational domain is more complex than rectangular. Finite element methods are applied also in the option pricing problem; see for example [61], [68]. Also finite volume methods are used in pricing problems; see [62], [83], [85].

It is possible to construct many different kinds of finite difference approximations, see for example [55]. Forward and backward finite difference schemes are the most simplest. They give first-order accurate approximations with respect to the grid step size for the first-order derivative. A second-order derivative can be approximated using symmetric central-difference scheme having second-order accuracy. There are also more complicated approximations. For example, higher order compact discretization schemes are applied to time dependent convection diffusion problems in [71]. Similar higher order scheme is applied to the option pricing in [43]. In general, there exist many finite difference schemes and in a context of the option pricing some are presented in [74].

The simplest way to use finite difference schemes is to apply them using grids which are formed with constant step lengths. It is also possible to apply more sophisticated grids. For example, often it is useful to have more accurate approximations in some specific part of the computational domain. This can be achieved efficiently by using nonuniform grids where the most dense discretization grid is located in that specific part of the domain. In the case of the option pricing the most dense grid can be located near to the exercise price.

The finite difference discretization leads to a semi-discrete equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\mathbf{u} = 0, \quad (4.1)$$

where the structure of the matrix \mathbf{A} depends on the form of the finite difference approximation. The central finite difference schemes lead to a tridiagonal matrix when the one-dimensional Black-Scholes equation is discretized while discretization of the two-dimensional PDE (3.8) gives a sparse block tridiagonal matrix \mathbf{A} .

4.2 Time discretization

The parabolic PDEs (3.2) and (3.8) have a first-order time derivative which also appears in the semi-discrete form (4.1). A common way to approximate the time

derivative is to use the first-order accurate explicit or implicit Euler schemes. More accurate and also well-known scheme is the Crank-Nicolson method. This method leads to a sequence of systems of linear equations

$$\left(\mathbf{I} + \frac{1}{2}\Delta t \mathbf{A}\right) \mathbf{u}^{(k+1)} = \left(\mathbf{I} - \frac{1}{2}\Delta t \mathbf{A}\right) \mathbf{u}^{(k)}, \quad (4.2)$$

where $k = 0, \dots, l-1$ and Δt is the length of the time step. The basics of discretization of initial value problems are considered, for example, in [56], [65].

The initial value in the option pricing problems is the payoff function of the option contract. In the case of call and put options this payoff function is non-smooth because its first derivative is discontinuous. Although the Crank-Nicolson method is popular in the option pricing it can create undesired oscillations to numerical solutions [33], [44], [49], [62], [86]. A reason for this is that the Crank-Nicolson method is not stable enough. A Runge-Kutta scheme and backward difference formula (BDF2) are also second-order accurate in time and they both are also L -stable [14], [35], [58]. This property is suitable in the option pricing problems because, it prevents oscillations arising from non-smooth initial values.

The simple and accurate Rannacher scheme is proposed in [63]. This time discretization scheme is based on the combination of the implicit Euler scheme and the Crank-Nicolson method. An oscillation problem of the Crank-Nicolson method is removed in a way that the first few time steps are made using the implicit Euler scheme and the rest of time steps are made using the Crank-Nicolson method. This Rannacher time stepping is applied in the option pricing, for example in [62]. In order to simplify the implementation of the Rannacher time stepping the implicit Euler steps can be made applying a step length which is half of the step length of the Crank-Nicolson time stepping. A consequence of this choice is that only one coefficient matrix arising from the Crank-Nicolson discretization needs to be formed, although two different time discretizations are used.

5 NUMERICAL SOLUTION OF DISCRETEZED PROBLEMS

In this section we discuss briefly the numerical solution of a system of linear equations and a discrete linear complementarity problem. Such problems arise when partial differential equation models are discretized using, for example, a finite difference method or a finite element method. In the following, we only mention some basic methods, as the aim of this section is not to give any extensive and detailed description of solution methods for systems of linear equations and LCPs. Moreover, this section is written from the point of view of computational option pricing.

5.1 Numerical solution of a system of linear equations

Although the topic of this thesis is the numerical solution of American option pricing problems it is natural first to explain some topics related to the numerical pricing of European options. That is also the case when considering numerical solution of discretezed option pricing problems. In addition, it should be noticed that also in some algorithms for American option pricing the solution of a system of linear equations is required. The discretization of European option pricing problems leads to a sequence of systems of linear equations and, hence, linear equation solvers are needed.

Basically, direct and iterative solvers can be used in the solution of a system of linear equations. The most basic way to solve a system of linear equations is to use an LU decomposition. This is an efficient solution technique, for example in the case of tridiagonal matrices. However, when modelling leads to a large and

sparse matrix this basic direct solution method can be inefficient.

Iterative methods are often more efficient when solving a system of linear equations which arises from discretization of a PDE with more than one dimension. The idea in the iterative methods is to generate a sequence of approximate solutions starting from an initial guess until a desired accuracy is achieved. The efficiency of an iterative method depends on how fast the iterates converge. The most basic iterative methods are Jacobi, Gauss-Seidel, and successive over-relaxation (SOR) methods [32].

A system of linear equations can also be solved using more sophisticated methods such as multigrid methods. The basic idea in these methods is to reduce low frequency error using coarser grids while high frequency error can be typically reduced using a standard iterative method on the fine grid. A multigrid method uses a restriction operation to transfer the solution or residual to a coarser grid. The solution or correction to it is transferred to a finer grid using a prolongation (interpolation) operation. Multigrid methods are considered, for example in [12], [34], [75], [78].

At this stage it is also relevant to mention splitting methods related to the solution of systems of linear equations. The Peaceman-Rachford formula and the Douglas-Rachford method are introduced [60], [24]. These types of methods are referred to as alternating direction implicit (ADI) methods. Moreover, a large number of different kind of splitting methods are considered, for example, in [31], [50], [53], [54], [82], and references therein. The basic idea in such splitting methods is to decompose a complicated operator into a sequence of operators which have simpler structure. Hence, after splitting it remains to solve a sequence of systems of linear equations. The advantage of a suitable splitting is that the remaining systems are much easier to solve than the original one. Symmetrization proposed by Strang in [72] can be used to increase the accuracy of the splitting methods. Related to option pricing, the ADI method is applied in [18]. They transform the pricing equation to the form of standard diffusion equation where the cross-derivative term does not appear, which enables straightforward use of the ADI method. Moreover, in books [74], [80], the use of the ADI method related to the options pricing problems is mentioned briefly. Similarly, in [26] the two-dimensional option pricing problem is transformed to the diffusion form after which the ADI method is applied. Furthermore, in [48], the splitting method is applied to the pricing of European style options under the stochastic volatility assumption.

5.2 Numerical solution of linear complementarity problem

A finite difference discretization of the time dependent problems (3.3) and (3.9) leads to a sequence of discrete LCPs

$$\begin{cases} B\mathbf{u}^{(k+1)} \geq C\mathbf{u}^{(k)}, & \mathbf{u}^{(k+1)} \geq \mathbf{g}, \\ \left(B\mathbf{u}^{(k+1)} - C\mathbf{u}^{(k)} \right)^T \left(\mathbf{u}^{(k+1)} - \mathbf{g} \right) = 0, \end{cases} \quad (5.1)$$

for $k = 0, \dots, l - 1$, where a structure of matrices B and C arises from the discretization of a one-dimensional or two-dimensional parabolic PDE. This shows that the pricing of American options requires a solution of a discrete LCP at each time step [39], [80]. A compact notation for the problem (5.1) is given by

$$\text{LCP}(B, u^{(k+1)}, Cu^{(k)}, g), \quad (5.2)$$

for $k = 0, \dots, l - 1$. In addition to the option pricing, LCPs arise in many different fields. Applications of LCPs in engineering and economics are described, for example, in [27]. This article clearly shows that LCPs arise also from other application areas than the field of financial option pricing. LCPs are studied extensively in [20]. Different kinds of pivoting methods are described for LCPs under certain assumptions for a matrix. These pivoting methods are useful when problems are not large. Iterative methods offer a more flexible way to solve LCPs. The most well-known iterative method is the projected SOR (PSOR) method [21] which is based on the SOR method for solving a system of linear equations. Efficient domain decomposition methods for obstacle problems with an M -matrix is given in [73].

The articles [13] and [30] give an overview about numerical methods used in pricing financial derivatives. In [11], Brennan and Schwartz propose a direct solution method for solving the American option pricing problem. The price of an American option is modelled by a LCP although it is not mentioned and hence, the proposed numerical method solves a LCP. Furthermore, this method is analysed in detail in [45]. This method is used also in this doctoral thesis in the case of the American option pricing under stochastic volatility assumption. The consequence of the use of a componentwise splitting method is that the direct Brennan and Schwartz method can be applied in the solution of subproblems. A direct method for the American option pricing problem is also applied in [52].

American option pricing from the point of view of LCPs is considered in [39]. This article considers several American option pricing models based on PDEs which are discretized using finite difference methods. Properties of LCPs which are essential for a convergence and stability of the numerical methods are considered. For example, a pivoting algorithm is applied for solving an American option pricing model with transaction costs. An American option pricing problem with a one-dimensional PDE is solved using a linear programming formulation in [22].

Numerical methods based on the use of multigrid methods are applied to LCPs and a projected full approximation scheme (PFAS) is introduced in [10]. Multigrid methods are also applied to the pricing of American style option contracts. Fast multigrid iteration with coordinate transformation is considered in the case of American option pricing with the stochastic volatility model in [16], [17]. Multigrid methods with several smoothers were studied in [58]. Moreover, recently in [64] multigrid methods were considered for high-dimensional European and American option pricing problems.

Finally, we point out that the ADI type method is also applied in the American option pricing. An option written on two stocks is priced numerically using the ADI type method in [77]. The pricing equation in this model is a two-dimensional parabolic PDE. Before using the ADI type method the convection

diffusion type equation is transformed to the basic heat equation. An early exercise constraint is then taken into account using the direct Brennan and Schwartz method.

6 MAIN RESULTS

At this stage we would like to mention briefly the results of this thesis. In the following we formulate two numerical methods which proved to be efficient for solving American option pricing problems with the stochastic volatility model. These numerical methods can be seen as the main results of this thesis and they are referred to as the operator splitting method and the componentwise splitting method. Next, we give an overview of them.

The proposed operator splitting method is of the form

$$B\tilde{\mathbf{u}}^{(k+1)} = C\mathbf{u}^{(k)} + \Delta t\boldsymbol{\lambda}^{(k)}, \quad (6.1)$$

$$\begin{cases} \mathbf{u}^{(k+1)} - \tilde{\mathbf{u}}^{(k+1)} - \Delta t(\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\lambda}^{(k)}) = 0, \\ \left(\boldsymbol{\lambda}^{(k+1)}\right)^T \left(\mathbf{u}^{(k+1)} - \mathbf{g}\right) = 0, \quad \mathbf{u}^{(k+1)} \geq \mathbf{g} \quad \text{and,} \quad \boldsymbol{\lambda}^{(k+1)} \geq 0, \end{cases} \quad (6.2)$$

for $k = 0, \dots, l-1$, where the matrices are the same as in (5.1) and the additional $\boldsymbol{\lambda}$ is a Lagrange multiplier. At the first step (6.1) a system of linear equations is solved and then the obstacle constraint is taken into account using a simple update step (6.2). The efficiency of this method depends on the computational efficiency of the solution of a system of linear equations because the update step (6.2) is computationally inexpensive. An efficient solution algorithm is achieved when the multigrid method is applied at the first fractional step (6.1). This method is considered in the articles [I], [III], and [V].

In this doctoral thesis the use of the componentwise splitting method for the pricing of American options under the stochastic volatility assumption is proposed and shown to be computationally efficient. The Strang symmetrized

componentwise splitting method reads

$$\left\{ \begin{array}{l} \text{LCP}(B_{x/2}, u^{(k+1/5)}, C_{x/2}u^{(k)}, g), \\ \text{LCP}(B_{y/2}, u^{(k+2/5)}, C_{y/2}u^{(k+1/5)}, g), \\ \text{LCP}(B_{xy}, u^{(k+3/5)}, C_{xy}u^{(k+2/5)}, g), \\ \text{LCP}(B_{y/2}, u^{(k+4/5)}, C_{y/2}u^{(k+3/5)}, g), \\ \text{LCP}(B_{x/2}, u^{(k+1)}, C_{x/2}u^{(k+4/5)}, g), \end{array} \right. \quad (6.3)$$

for $k = 0, \dots, l-1$. In this splitting the discrete LCP with sparse matrix is split into five subproblems at each time step. These subproblems are then solved using the direct Brennan and Schwartz algorithm, because the matrices in these subproblems are tridiagonal. In order to have a stable method, appropriate nonuniform discretization grids should be used. This method is considered in the articles [IV] and [V].

In addition to these splittings, we have also some other results related to the numerical American option pricing. In the case of the option pricing model with stochastic volatility, we propose a space discretization scheme for a two-dimensional PDE which leads to a coefficient matrix with an M -matrix property. Also, accurate time discretization schemes for the option pricing problems are considered and numerically compared. Finally, the computational efficiency of several numerical methods proposed by other authors for solving American option pricing problem under stochastic volatility are compared numerically with the splitting methods proposed by us in [V].

7 OUTLINE OF THIS THESIS

This doctoral thesis considers efficient numerical solution methods for the American option pricing problems and it contains the research articles [I], [II], [III], [IV], and [V]. In the following subsections we give an overview of these articles. For more detailed information about problems, discretizations, numerical methods, and numerical experiments we refer to the articles themselves.

Basically, this thesis focuses on the study of numerical methods for pricing American option contracts using two different option pricing models. We start by considering numerical methods for a basic model where the volatility is assumed to be constant. The other model is based on the stochastic volatility assumption leading to a more complicated option pricing problem. As described in the Section 4, the solution of a discrete linear complementarity problem is required at each time step when solving these option pricing problems. From the point of view of numerical mathematics the main difference between these models arises from the form of the partial differential equations involved. The basic model is based on the one-dimensional parabolic PDE while the model with the stochastic volatility assumption is based on the two-dimensional parabolic PDE. The main topic of this thesis is to study the efficient numerical treatment of the early exercise constraint of American options. In other words, we aim to find out how to solve a discrete LCP efficiently at each time step.

We propose two different kinds of splitting techniques for solving the American option pricing problems. The operator splitting methods are considered in the articles [I] and [III]. The idea in this operator splitting approach is that at each time step the solution of a discrete LCP is divided into two simple fractional steps. At the first fractional step a system of linear equations is solved and at the following step a simple update related to the early exercise constraint is performed. This type of approach allows the use of efficient solvers in the solution

of a system of linear equations. Secondly, componentwise splitting methods are proposed for the numerical solution when the stochastic volatility model is used [IV]. In the basic form of this splitting the LCP with a large and sparse coefficient matrix is divided into three subproblems with tridiagonal matrices. This kind of splitting enables us to use direct methods in the solution which makes the componentwise splitting algorithm efficient. These direct solution methods are applied in [II] to the American option pricing with the basic one-dimensional model. In the componentwise splitting algorithm each subproblem can be seen as a one-dimensional LCP.

In the numerical experiments of the articles we demonstrate the usability of the proposed methods to the option pricing problems. In the last article we compare numerically the proposed operator splitting method and the componentwise splitting method with other numerical methods presented in the literature. Our research demonstrates that the componentwise splitting method in particular is an efficient way to solve the American option pricing problems when the volatility is assumed to follow a stochastic process.

The modelling of the price of the underlying asset or the price of the American option is not the objective of this doctoral thesis as was pointed out at the beginning of this introduction section. The main purpose of this research is to develop efficient numerical methods for solving existing American option pricing models. The models for the prices of American option contracts used in this thesis are presented in the cited literature.

This thesis is divided into five articles. The first two articles consider the numerical solution of the one-dimensional option pricing models. The option pricing problem under the stochastic volatility assumption is studied in the last three articles. In the following, the contents of these articles are described briefly.

Author's contribution. The included papers have been done in cooperation with my supervisor Dr. J. Toivanen. Next, I report my personal contribution to these papers. First, the idea of applying splitting techniques to the solution of the American option pricing problems came from Dr. J. Toivanen. The proposed discretizations and numerical algorithms, except the multigrid solver, were derived and implemented by me. Also, I designed and conducted the numerical experiments in [I], [II], [III], and [IV] while the numerical experiments in [V] were jointly performed by me and Dr. J. Toivanen. I wrote the first versions of the papers and then Dr. J. Toivanen helped to finalize them.

7.1 Article I

In paper [I], operator splitting methods for an American option pricing model are presented. The goal of this paper is to develop an efficient numerical solution method that can be further applied for solving more complicated American style option pricing problems.

The Black-Scholes partial differential equation has variable coefficients for the first-order and the second-order spatial partial derivatives. We use the central finite difference schemes to approximate these partial derivatives and we give a

condition when this discretization leads to a coefficient matrix with an M -matrix property. The initial value of the American option pricing problem is given by the payoff function of the option contract. This non-smooth initial value requires that a time discretization scheme has good damping properties for higher frequencies. The Crank-Nicolson method and a backward difference formula (BDF2) are applied for the time discretization. Both methods are second-order accurate in time but the latter one has better stability properties and hence, it proved to be more suitable. The discretizations of the spatial derivatives and the first-order time derivative were performed using grids with constant step lengths. After the discretization, the prices for the American options are obtained by solving a sequence of discrete LCPs.

The operator splitting method proposed in this paper is used to solve an LCP at each time step. The basic idea in our operator splitting method is to divide the solution of the discrete LCP into two separate fractional steps. At the first step a system of linear equations is solved, while in the second fractional step the early exercise constraint is taken into account using a simple projection. The form of the operator splitting method depends on the time discretization scheme and that is why the operator splitting method is presented in the cases of the Crank-Nicolson method and the two-step backward difference formula.

The computational efficiency of the proposed methods depends on the solution algorithm of a system of linear equations. The central finite difference discretization of the Black-Scholes equation leads to a tridiagonal coefficient matrix and hence, the system of linear equations can be solved efficiently by using an LU decomposition. In the numerical experiments the efficiency of the operator splitting method is compared with the well-known PSOR method. The operator splitting method is shown to produce accurate option prices with much less CPU time than the PSOR method.

7.2 Article II

In paper [II], we continue to consider numerical solution methods for the one-dimensional American option pricing problems. As in the previous paper the option pricing models are based on the Black-Scholes framework. The main topic in this paper is to study two existing direct solution methods in detail. That is, the well-known Brennan and Schwartz method and a method presented by Elliott and Ockendon are studied closely and the similarities of these algorithms are considered. Moreover, we reformulate these methods into simple forms.

Again, we apply the central finite difference schemes for the discretization of the spatial derivatives in the Black-Scholes equation. The implicit Euler scheme, the Crank-Nicolson method, a backward difference formula (BDF2), and a second-order implicit Runge-Kutta scheme are applied for the time discretization. The discretization schemes are used with uniform grids. It is essential that the discretization of the Black-Scholes equation is made in a way that the coefficient matrix is tridiagonal. This allows the efficient use of an LU decomposition in the solution algorithms. Moreover, it is required that the form of the payoff

function of the option contract is as for the basic call or put options.

First, we show that the Brennan and Schwartz method can be reformulated into a very simple form. That is, after computing the LU decomposition of the discretization matrix, the solution of the discrete LCP at each time step is obtained by performing the standard forward substitution and a modified backward substitution. In the modified backward substitution the maximum between the value given by the backward substitution and the corresponding value of the payoff function is taken. Computationally this method is as expensive as the solution of a tridiagonal system of linear equations based on the LU decomposition.

We show that the method proposed by Elliott and Ockendon can also be reformulated into a form where the LU decomposition is applied. This shows that there are many similarities with the Brennan and Schwartz algorithm. Different LU decompositions are given in order to make a reasonable comparison between these algorithms.

In the case of four time discretization schemes, time convergence rates are considered numerically. Also, it is shown that the Runge-Kutta scheme and the BDF2 formula lead to more accurate numerical solutions than the Crank-Nicolson method with a small number of time steps. The required CPU times are compared between the direct Brennan and Schwartz algorithm and the iterative PSOR method in order to demonstrate the efficiency of the direct algorithm. Moreover, we compute the first-order space derivatives, i.e. delta functions, for the American call and put options. We show that the Crank-Nicolson method creates oscillations to the computed delta functions while the Runge-Kutta leads to smooth delta functions.

7.3 Article III

In paper [III], the numerical solution for the American option pricing problem where the volatility is assumed to follow a stochastic process is considered. The price of the option is modelled using a two-dimensional convection diffusion type parabolic PDE with varying coefficients. This PDE is a generalization of the one-dimensional Black-Scholes equation which was used in the previous two papers. The main purpose of this paper is to develop an efficient numerical method for the LCP with a large and sparse coefficient matrix. In this paper an operator splitting method is proposed for the option pricing problem with stochastic volatility.

The operator splitting method proposed in [I] is applied here to the American option pricing problem with a stochastic volatility assumption. The advantage of this splitting method is that the solution of a system of linear equations can be made separately from the treatment of the early exercise constraint of the American option. The early exercise constraint, i.e. the obstacle constraint, is taken into account in a simple step and hence, the efficiency of this splitting method depends on the efficiency of the solution method of a system of linear equations. In order to achieve an efficient numerical method we apply a special multigrid method in the solution of a large and sparse system of linear equations

at each time step.

In the discretization we use one space discretization scheme and several time discretizations with constant step sizes. Generally, this discretization does not lead to a matrix with M -matrix property which would prevent the oscillation arising from the space discretization. The choice of the discretization is considered in more detailed manner in the next two papers. In this paper, we prove that the operator splitting method with the Crank-Nicolson time discretization method has the same order of accuracy as the Crank-Nicolson method with the implicit treatment of the complementarity condition.

In the numerical experiments we compare computed option prices to the ones presented in literature. The option prices computed by the proposed operator splitting method are in good agreement with the reference prices. The convergence rates of time discretization schemes are studied numerically. Moreover, the required CPU times are compared when the PSOR method is used to solve the same option pricing problem. Numerical experiments show that the operator splitting method with the multigrid method is more efficient than the PSOR method when the option prices are required with high accuracy. It should be noted that the multigrid iteration is quite expensive and that is why the advantage of the multigrid method arises when the size of the problem is increased.

7.4 Article IV

In paper [IV], we apply a componentwise splitting method to the American option pricing problem with the stochastic volatility assumption. The main idea in this algorithm is to divide the solution of the original discrete problem at each time step in such a way that three or five LCPs with tridiagonal matrices are required to be solved instead of one LCP with a large and sparse matrix. These sub LCPs are then solved using the direct Brennan and Schwartz method.

In its basic form the componentwise splitting method has three fractional steps at each time step and the method is only first-order accurate respect in to the size of time step. In order to increase the accuracy, we apply the so-called Strang symmetrization. This symmetrization increases the number of fractional steps from three to five while more accurate solutions are achieved. The space discretization matrix is decomposed into three matrices which, after reordering rows and columns, are tridiagonal. The sub LCPs are then solved efficiently using the direct Brennan and Schwartz method.

The componentwise splitting method with constant grid steps proved to be unstable. In this paper we apply a nonuniform grid in the x -direction in order to obtain matrices with an M -matrix property. At each point in the x -direction we compute the upper and lower bounds for the step length. These bounds guarantee that the M -matrix property is achieved, and hence the undesired oscillation problem is avoided. Moreover, our primary purpose is to apply such a grid generating function in the x -direction that produces the most dense grid near the exercise price. We apply parabola type grid generating functions. Finally, the actual step lengths in the x -direction are chosen by using first the grid generating func-

tion and then restricting step lengths according to the lower and upper bounds if needed. Using this kind of grid generation procedure the decomposition leads to tridiagonal matrices which have an M -matrix property.

Again, in the numerical experiments we compare the computed option prices with the ones given in the references. Moreover, we demonstrate that the componentwise splitting method leads to CPU times which are much shorter than the CPU times with the PSOR method although the overrelaxation parameter in the PSOR method was optimized for each discretization grid. By comparing the CPU times for the operator splitting method and componentwise splitting method we noticed that the splitting method proposed in this paper is very efficient.

7.5 Article V

In paper [V], we study the efficiency of the operator splitting method and the componentwise splitting method. These methods are used to solve the American option pricing problem with stochastic volatility. In addition, we also solve this option pricing problem using three different numerical methods given in the literature. One of the conclusions of this paper is that the proposed componentwise splitting method is very efficient when solving the American option pricing problem. Moreover, we propose some improvements for the existing methods.

The space discretization is a generalization of the one used in paper [IV]. In order to develop more efficient method we apply a nonuniform grid also in the y -direction. The grid in this direction was generated using a linear grid generating function. A finite difference approximation was applied in a way that the arising coefficient matrix fulfil an M -matrix property. The Rannacher time stepping was used in the time discretization.

In the numerical experiments of this paper we solve the option pricing problem using several discretization grids and we report the required CPU times for each numerical method. Errors respect to the reference solution are also reported.

8 FUTURE PROSPECTS

This doctoral research has focused on developing efficient numerical methods for pricing the basic American option contract when two pricing models based on PDEs have been used. There are several future prospects related to the research on the numerical option pricing. It will be interesting to study how useful the proposed splitting methods are in the pricing of option contracts with more complex payoff functions and also to consider how these methods can be made more efficient.

The space discretization is one possible research topic in future. The proposed componentwise splitting method requires that special a grid should be used in order to have a stable numerical solution. In some cases this leads to nonuniformly clustered nodes. A special space discretization could solve this problem. An advantage of such a discretization might be that also the number of required nodes could be reduced. Another interesting research topic would be calibration problems related to the parameters of the option pricing models. The value of the volatility parameter is particularly difficult to determine. Option prices from option exchanges can be used to determine the actual volatility value for the underlying asset. Using this kind of approach the option pricing models can be calibrated related to some specific underlying asset. The solution of such calibration problem requires that American option pricing problem is solved repeatedly and, hence, it would be useful to apply the proposed efficient splitting methods. Moreover, European and American option can be priced using jump-diffusion pricing models. In these cases partial integro-differential equations should be solved numerically. Developing efficient solution methods for such problems needs further research. Finally, it would be interesting to test how useful the proposed operator splitting method is when applied to linear complementarity problems arising from different application fields.

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YHTEENVETO (SUMMARY IN FINNISH)

Tämä väitöskirjatutkimus käsittelee Amerikkalaisten optioiden numeerista hinnoittelua. Väitöskirja koostuu viidestä artikkelista sekä johdannosta. Tutkimuksessa on keskitytty kehittämään tehokkaita numeerisia laskentamenetelmiä option hintaa mallittavien yhtälöiden ratkaisemiseksi. Numeeristen menetelmien käyttö on Amerikkalaisten optioiden hinnoittelussa välttämätöntä, koska analyyttisiä ratkaisukaavoja ei ole olemassa. Työssä käytetään kahta hinnoittelumallia, jotka pohjautuvat laajasti tunnettuun Black-Scholes -teoriaan sekä Hestonin kehittämään volatilititeettimalliin. Nämä osittaisdifferentiaaliyhtälöihin perustuvat mallit ovat tyypiltään ajasta riippuvia estetehtäviä. Näistä yksinkertaisempaan malliin liittyy yksi paikkamuuttuja, kun taas stokastinen volatilititeetti johtaa malliin, jossa on kaksi paikkamuuttujaa. Tutkimuksessa on rajoitettu laskentamenetelmien kehittämiseen, ja siten esimerkiksi option kohde-etuuden arvon mallinnus on rajattu tutkimuksen ulkopuolelle.

Osittaisdifferentiaaliyhtälöiden diskretisointiin on käytetty differenssimenetelmää. Menetelmän stabiilisuuteen on kiinnitetty huomiota, jotta välttyttiin oskillointeja sisältäviltä numeerisilta ratkaisuilla. Numeerinen approksimaatio Amerikkalaisen option hinnalle saadaan ratkaisemalla jono diskreettejä estetehtäviä. Erityisesti kaksiulotteisessa tapauksessa nämä tehtävät ovat suuria ja laskennallisesti vaativia. Tämän takia tehokkaiden ratkaisumenetelmien kehittäminen on hyödyllistä.

Ajasta riippuvien estetehtävien numeerinen ratkaiseminen on väitöstutkimuksen pääaihe. Tässä tutkimuksessa esitetään kaksi numeerista menetelmää, joiden kehittämisessä menetelmän laskennallinen tehokkuus on ollut avainasemassa. Kehitetyt menetelmät perustuvat operaattorin jakotekniikoihin. Ensimmäisessä menetelmässä yhtälöryhmän ratkaiseminen ja este-ehdon huomioiminen on erotettu kullakin aika-askeleella toisistaan. Tämä mahdollistaa tehokkaiden ratkaisualgoritmien käyttämisen yhtälöryhmien ratkaisemisessa. Menetelmässä este-ehdon huomioiminen on pystytty toteuttamaan hyvin yksinkertaisesti, jolloin menetelmän tehokkuus riippuu lähes täysin yhtälöryhmän ratkaisemisen tehokkuudesta.

Toinen kehitetty menetelmä liittyy stokastisen volatilititeetin sisältävän hinnoittelumallin ratkaisemiseen. Esitetystä operaattorin jakotekniikassa alkuperäinen estetehtävä ratkaistaan usean tridiagonaalimatriisin sisältävän estetehtävän avulla. Näiden yksinkertaisempien estetehtävien ratkaisemiseen voidaan käyttää tehokasta suoraa ratkaisualgoritmia, minkä ansiosta esitetystä ratkaisumenetelmästä muodostuu tehokas.

Väitöstutkimuksessa esitettyjen menetelmien tehokkuutta tarkastellaan numeeristen esimerkkien avulla. Viimeisessä väitöskirjaan kuuluvassa artikkelissa vertaillaan kahden kehitetyn menetelmän sekä kolmen kirjallisuudessa aiemmin esitetyn menetelmän tehokkuuksia. Tämä vertailu osoittaa, että tutkimuksessa on onnistuttu kehittämään kaksi tehokasta laskentamenetelmää.

I

OPERATOR SPLITTING METHODS FOR AMERICAN OPTION PRICING

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II

PRICING AMERICAN OPTIONS USING LU DECOMPOSITION

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III

OPERATOR SPLITTING METHODS FOR PRICING AMERICAN OPTIONS WITH STOCHASTIC VOLATILITY

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IV

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V

**EFFICIENT NUMERICAL METHODS FOR PRICING
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