J. Math. Anal. Appl. 503 (2021) 125329

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A density result on Orlicz-Sobolev spaces in the plane ☆

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ARTICLE INFO

Article history:
Received 19 July 2018
Available online 14 May 2021
Submitted by V. Andriyevskyy

Keywords:
Sobolev space
Orlicz-Sobolev space
Density

ABSTRACT

We show the density of smooth Sobolev functions $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ in the Orlicz-Sobolev spaces $L^{k,\Psi}(\Omega)$ for bounded simply connected planar domains Ω and doubling Young functions Ψ.

1. Introduction

Orlicz-Sobolev spaces appear naturally in analysis as generalizations of the usual Sobolev spaces, for instance when one studies sharp assumptions for mappings of finite distortion [16,18]. Orlicz-Sobolev spaces appear also in many other contexts and have been studied by their own right, see for instance [1,2,5–7,9,8,12,11,14,13,15,24,30,33] for a sample of the literature.

An important basic question in the theory of function spaces is the relation between different spaces. Answers to this question can be given for instance in terms of embeddings and density results. In this paper, we show that in Orlicz-Sobolev spaces on bounded simply connected planar domains we can approximate functions with bounded derivatives if we consider only the highest order derivatives in the norm.

Theorem 1.1. *Let $k \in \mathbb{N}$, Ψ be a doubling Young function, and $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then the subspace $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ is dense in the space $L^{k,\Psi}(\Omega)$.*

Recall that for a domain $\Omega \subset \mathbb{R}^2$ and a Young function Ψ, the version of the Orlicz-Sobolev space $L^{k,\Psi}(\Omega)$ used in Theorem 1.1 is defined as

☆ TR acknowledges the support by the Academy of Finland, projects no. 274372 and 314789. WO acknowledges the support by MINECO Grants MTM201344304P and EEBBI1812876 (Spain).

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https://doi.org/10.1016/j.jmaa.2021.125329
0022-247X/

$$L^{k,\Psi}(\Omega) = \left\{ f \in L^{\Psi}(\Omega) : \nabla^{\alpha} f \in L^{\Psi}(\Omega) \text{ if } |\alpha| = k \right\}.$$

The space $L^{k,\Psi}(\Omega)$ is equipped with the semi-norm $\sum_{|\alpha|=k} \|\nabla^{\alpha} f\|_{L^{\Psi}(\Omega)}$, where $\|\cdot\|_{L^{\Psi}(\Omega)}$ is the Luxemburg norm. If we would consider the size of all the lower order derivatives, we would use the space $W^{k,\Psi}$ defined as

$$W^{k,\Psi}(\Omega) = \left\{ f \in L^{\Psi}(\Omega) : \nabla^{\alpha} f \in L^{\Psi}(\Omega) \text{ if } |\alpha| \leq k \right\}.$$

However, at the moment we are unable to prove the density in this space with the stronger norm $\sum_{|\alpha|\leq k} \|\nabla^{\alpha} f\|_{L^{\Psi}(\Omega)}$. See Section 2 for more basic information on Orlicz and Orlicz-Sobolev spaces. Recall also the notation when we deal with the usual L^p-spaces: by $W^{k,p}(\Omega)$ we mean integrable functions f defined on Ω such that all the partial derivatives $\nabla^{\alpha} f$ for $|\alpha| \leq k$ are L^p-integrable.

We will use a Whitney decomposition of the domain $\Omega \subset \mathbb{R}^2$ and make a polynomial approximation near the boundary $\partial\Omega$. The validity of the approximation is proven using a $\Psi - \Psi$ Poincaré inequality. The form of the polynomial approximation we use here was introduced in [28], where a density result was shown for homogeneous Sobolev spaces on simply connected planar domains. This was then extended to Gromov hyperbolic domains in higher dimensions in [27]. In turn, both of these results were generalizations of density results for first order Sobolev spaces [23,22] which were partly motivated by the recent progress on planar Sobolev extension domains [21,31].

Although for every domain smooth functions are dense in $W^{k,p}(\Omega)$ [26] and, more generally, in $W^{k,\Psi}(\Omega)$ for doubling Ψ [10], the derivatives of the approximating smooth functions might blow up near the boundary. Therefore, the density of other function spaces, such as $W^{k,q}(\Omega)$ in $W^{k,p}(\Omega)$ might be false, see [20,22] for this, and for instance [3,19,29] for earlier counter examples on other function spaces. The density of global smooth functions in $W^{1,p}(\Omega)$ is known for instance for Jordan domains [25,23], but the case for higher order Sobolev spaces is still open. Similarly, in the case $k \geq 2$, the density result presented in Theorem 1.1 remains still open for the full Orlicz-Sobolev space $W^{k,\Psi}(\Omega)$ as well as for the usual Sobolev space $W^{k,p}(\Omega)$.

In the same way as for the first order Sobolev spaces [23], for $W^{1,\Psi}(\Omega)$ we get a better density result as a corollary of Theorem 1.1. This is simply because we may first cut a function in $W^{1,\Psi}(\Omega)$ from above and below introducing a small error in the norm, so that the function becomes an $L^{\infty}(\Omega)$ function. The remaining approximations do not change the fact that the function is in $L^{\infty}(\Omega)$.

Corollary 1.2. *Let Ψ be a doubling Young function and $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then the subspace $W^{1,\infty}(\Omega) \cap C^{\infty}(\Omega)$ is dense in the space $W^{1,\Psi}(\Omega)$.*

The paper is organized as follows. In Section 2 we recall the Whitney decomposition and list the required prerequisites from the Orlicz theory. In Section 3 we give a partition of unity for the domain using a Whitney type decomposition. Finally, in Section 4 we show the proof of Theorem 1.1.

2. Preliminaries

In this paper, we will usually denote constants by C. The value of the constant might change between appearances, even in a chain of inequalities, but the dependence of the constant on a set of fixed parameters is always stated. Sometimes, to clarify the dependence, the parameters are written inside parentheses $C(\cdot)$.

2.1. Whitney decomposition

In this section we recall the Whitney decomposition of a domain in $\mathbb{R}^d$. Such decomposition is standard in analysis, see for instance Whitney [34] or the book of Stein [32, Chapter VI]. We will use a version of the decomposition that was used in [28].

We denote the sidelength of a square Q by $\ell(Q)$. For notational convenience we start the Whitney decomposition below from squares with sidelength 2^{-1}. Formally, since we are working with doubling Young functions, by rescaling, we may consider all bounded domains $\Omega \subset \mathbb{R}^2$ to have $\operatorname{diam}(\Omega) \leq 1$ in which case no Whitney decomposition would have squares larger than the ones used below regardless of the starting scale. A Whitney decomposition in the plane consists of dyadic squares. Let us first recall those.

Definition 2.1 *(Dyadic squares).* A dyadic interval in $\mathbb{R}$ is an interval of the form $[m2^{-k}, (m+1)2^{-k}]$ where $m, k \in \mathbb{Z}$. A dyadic square in $\mathbb{R}^2$ is a product of dyadic intervals of the same length. That is, a dyadic square is a set of the form

$$[m_1 2^{-k}, (m_1+1)2^{-k}] \times [m_2 2^{-k}, (m_2+1)2^{-k}]$$

for some integers m_1 and $, m_2$.

Let us now define a Whitney decomposition following Lemma 2.3 in [28].

Definition 2.2 *(Whitney decomposition).* Let Ω be a bounded open subset of $\mathbb{R}^2$. A Whitney decomposition is a collection $\tilde{\mathcal{F}}$ of dyadic squares inside Ω satisfying the following properties.

(W1) $\Omega = \bigcup_{Q \in \tilde{\mathcal{F}}} Q$
(W2) $\ell(Q) < \operatorname{dist}(Q, \Omega^c) \leq 3\sqrt{2}\ell(Q) = 3\operatorname{diam}(Q)$ for all $Q \in \tilde{\mathcal{F}}$
(W3) $\operatorname{int} Q_1 \cap \operatorname{int} Q_2 = \emptyset$ for all $Q_1, Q_2 \in \tilde{\mathcal{F}}, Q_1 \neq Q_2$
(W4) If $Q_1, Q_2 \in \tilde{\mathcal{F}}$ and $Q_1 \cap Q_2 \neq \emptyset$, then $\frac{\ell(Q_1)}{\ell(Q_2)} \leq 2$.

Suppose $Q_1, \ldots, Q_m$ are Whitney squares such that Q_j and Q_{j+1} touch and $\frac{1}{4} \leq \frac{\ell(Q_j)}{\ell(Q_{j+1})} \leq 4$ for all $j, 1 \leq j \leq m-1$. We say then $\{Q_1, \ldots, Q_m\}$ is a *chain* connecting Q_1 to Q_m and define the length of that chain to be the integer m.

2.2. Orlicz spaces

Definition 2.3. A function $\Psi\colon [0, \infty) \to [0, \infty]$ is a *Young function* if

$$\Psi(s) = \int_0^s \psi(t)\, \mathrm{d}t,$$

where $\psi : [0, \infty] \to [0, \infty]$ is an increasing, left continuous function which is neither identically zero nor identically infinite on $(0, \infty)$ and which satisfies $\psi(0) = 0$.

A Young function Ψ is convex, increasing, left continuous, $\Psi(0) = 0$ and $\Psi(t) \to \infty$ as $t \to \infty$. A continuous Young function with the properties $\Psi(t) = 0$, only if $t = 0$, $\Psi(t)/t \to \infty$ as $t \to \infty$ and $\Psi(t)/t \to 0$ as $t \to 0$ is called an N-function.

It follows easily from the convexity and $\Psi(0) = 0$, that the function $t \to \Psi(t)/t$ is increasing. This implies that if Ψ is strictly increasing, then the function $\Psi^{-1}(t)/t$ is decreasing. A Young function Ψ is doubling if there is a constant $C > 0$ such that

$$\Psi(2t) \leq C\Psi(t) \tag{2.1}$$

for each $t \geq 0$. The smallest constant C satisfying (2.1) is called the doubling constant of Ψ.

Definition 2.4. Given a doubling Young function Ψ and an open set $\Omega \subset \mathbb{R}^2$, we denote by $L^\Psi(\Omega)$, the *Orlicz space* associated to Ψ, defined by

$$L^\Psi(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} : \int_\Omega \Psi(|u(x)|)\,\mathrm{d}x < \infty \right\}.$$

$L^\Psi(\Omega)$ is a Banach space, when equipped with the *Luxemburg norm*

$$\|u\|_{L^\Psi(\Omega)} = \inf \left\{ k > 0 \,:\, \int_\Omega \Psi(k^{-1}|u(x)|)\,\mathrm{d}x \leq 1 \right\}.$$

We will not use the Luxemburg norm in this paper, but work with the integrals. This is justified by the following fact.

Lemma 2.5. *Let Ψ be a doubling Young function. Then*

$$\|u_i - u\|_{L^\Psi(\Omega)} \to 0 \qquad \text{as } i \to \infty$$

if and only if

$$\int_\Omega \Psi(|u_i(x) - u(x)|)\,\mathrm{d}x \to 0 \qquad \text{as } i \to \infty.$$

A direct consequence of Jensen's inequality is the following.

Lemma 2.6. *Let Ψ be a Young function, $u \in L^1_{loc}(\mathbb{R}^2)$ and $A \subset \mathbb{R}^2$ of positive and finite measure, then*

$$\Psi\left(\fint_A |u(x)|\,\mathrm{d}x \right) \leq \fint_A \Psi(|u(x)|)\,\mathrm{d}x,$$

where $u_A = \fint_A |u(x)|\,\mathrm{d}x$ is the average integral.

2.3. Poincaré inequalities and polynomial approximation

From now on, Ψ always refers to a doubling Young function. We will construct an approximation by replacing the original function by approximating polynomials near the boundary of the domain. For this purpose, we will need a few lemmas regarding the polynomials. Here and later on by $|E|$ we denote the Lebesgue measure of a set $E \subset \mathbb{R}^2$.

Lemma 2.7. *Let Q be any square in $\mathbb{R}^2$ and P be a polynomial of degree k defined in $\mathbb{R}^2$. Let $F \subset Q$ be such that $|F| > \eta|Q|$ where $\eta > 0$. Then*

$$\int_Q \Psi(|P(x)|)\,\mathrm{d}x \leq C \int_F \Psi(|P(x)|)\,\mathrm{d}x,$$

where the constant C depends only on η, k and the doubling constant of Ψ.

Proof. Since the claim of the lemma is invariant under scalings and translations of the square Q, we may assume it to be fixed. Write the space of degree k polynomials as

$$\mathcal{P}_k = \left\{ \sum_{i+j\leq k, i,j\geq 0} a_{i,j} x_1^i x_2^j \, : \, a_{i,j} \in \mathbb{R} \right\}$$

and consider it equipped with the norm $\| \sum_{i+j\leq k,i,j\geq 0} a_{i,j} x_1^i x_2^j \| = \sum |a_{i,j}|$. Observe that the function

$$S \colon \{P \in \mathcal{P}_k \, : \, \|P\| = 1\} \to \mathbb{R} \colon P \mapsto \inf_{|E|\geq \frac{\eta}{2}|Q|} \sup_{x\in E} |P(x)|$$

is strictly positive. Moreover, since the function $P \mapsto \sup_{x\in Q} |P(x)|$ is continuous, so is the function S. Hence, it obtains its strictly positive minimum. Let us call the minimum $\varepsilon > 0$. This implies that

$$\inf_{|E|\geq \frac{\eta}{2}|Q|} \sup_{x\in E} |P(x)| \geq \varepsilon ||P|| \tag{2.2}$$

for every $P \in \mathcal{P}_k$.

Since the space $\mathcal{P}_k$ is finite dimensional, the norms $\|P\|$ and $\max_{y\in Q} |P(y)|$ are comparable. Hence, by (2.2), there exists a constant $\delta > 0$ so that

$$\inf_{|E|\geq \frac{\eta}{2}|Q|} \sup_{x\in E} |P(x)| > \delta \max_{y\in Q} |P(y)| \tag{2.3}$$

for every $P \in \mathcal{P}_k$ with $P \neq 0$. We claim that the set

$$\tilde{F} := \left\{ x \in F : |P(x)| \geq \delta \max_{y\in Q} |P(y)| \right\}$$

satisfies $|\tilde{F}| \geq \frac{1}{2}|F|$. If this were not the case, then $|F \setminus \tilde{F}| > \frac{1}{2}|F| \geq \frac{\eta}{2}|Q|$, and in particular $P \neq 0$. Thus, by (2.3), and the fact that $|P(x)| < \delta \max_{y\in Q} |P(y)|$ for all $x \in F \setminus \tilde{F}$, we get

$$\delta \max_{y\in Q} |P(y)| < \sup_{x\in F\setminus \tilde{F}} |P(x)| \leq \delta \max_{y\in Q} |P(y)|,$$

giving a contradiction.

Now, by applying monotonicity and doubling properties on Ψ we obtain

$$\begin{aligned}
\int_F \Psi(|P(x)|) \, \mathrm{d}x &\geq \int_{\tilde{F}} \Psi(|P(x)|) \, \mathrm{d}x \geq \int_{\tilde{F}} \Psi(\delta \max_{y\in Q} |P(y)|) \, \mathrm{d}x \\
&\geq C \int_{\tilde{F}} \Psi(\max_{y\in Q} |P(y)|) \, \mathrm{d}x = C|\tilde{F}|\Psi(\max_{y\in Q} |P(y)|) \\
&\geq \frac{C|\tilde{F}|}{|Q|} \int_Q \Psi(|P(x)|) \, \mathrm{d}x \geq C \int_Q \Psi(|P(x)|) \, \mathrm{d}x,
\end{aligned}$$

which gives the claim. □

Given a function $u \in C^\infty(\Omega)$, degree $k \in \mathbb{N}$, and a bounded set $E \subset \Omega$ with $|E| > 0$, we define (see [17]) the polynomial approximation of u in E, $P_k(u, E)$ to be the polynomial of order $k - 1$ which satisfies

$$\int_E \nabla^\alpha (u - P_k(u,E)) = 0$$

for each $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| = \alpha_1 + \alpha_2 \leq k-1$. Once k is fixed, we denote the polynomial approximation of u in a bounded set E as P_E.

Proposition 2.8 *($\Psi - \Psi$ Poincaré inequality). Let $k, m \in \mathbb{N}$. There exists a constant C depending only on k, m and the doubling constant of Ψ such that for any domain $\Omega \subset \mathbb{R}^2$, a chain $\{Q_i\}_{i=1}^m$ of dyadic squares in Ω, and a function $u \in L^{k,\Psi}(\Omega)$ we have*

$$\fint_E \Psi\left(\frac{|u(x) - P_E(x)|}{\ell(Q_1)^k}\right) \mathrm{d}x \leq C \fint_E \Psi(|\nabla^k u(x)|)\,\mathrm{d}x,$$

where we have abbreviated $E = \bigcup_{i=1}^m Q_i$.

Proof. By [4, Lemma 1] the claim is true for $k = 1$ in the case where E is convex. By a change of variables, the claim extends to the case $k = 1$ and $E = \bigcup_{i=1}^m Q_i$, for the chain $\{Q_i\}_{i=1}^m$. What remains to show is the case $k > 1$.

We do this by induction. Suppose the claim is true for the order $k-1$. Then, using the Poincaré inequality first for the $k-1$ orders for the function $v = \frac{1}{\ell(Q_1)}(u - P_k(u,E))$, for which $P_{k-1}(v,E) = 0$, and then for the first order for the function $\nabla^{k-1}(u - P_k(u,E))$, we obtain

$$\begin{aligned}
\fint_E \Psi\left(\frac{|u(x) - P_k(u,E)(x)|}{\ell(Q_1)^k}\right) \mathrm{d}x &\leq C \fint_E \Psi\left(\frac{|\nabla^{k-1}(u(x) - P_k(u,E)(x))|}{\ell(Q_1)}\right) \mathrm{d}x \\
&\leq C \fint_E \Psi\left(|\nabla^k (u(x) - P_k(u,E)(x))|\right) \mathrm{d}x \\
&= C \fint_E \Psi\left(|\nabla^k u(x)|\right) \mathrm{d}x. \quad \square
\end{aligned}$$

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain and $\tilde{\mathcal{F}}$ a Whitney decomposition of Ω. Let $\{Q_i\}_{i=1}^m$ be a chain of dyadic squares in $\tilde{\mathcal{F}}$. Then there exists a constant C depending only on k, m and the doubling constant of Ψ such that if $|\alpha| \leq k$, we have*

$$\int_{Q_1} \Psi\left(\frac{\nabla^\alpha P_{Q_1} - \nabla^\alpha P_{Q_m}}{\ell(Q_1)^{k-|\alpha|}}\right) \leq C \int_{\bigcup_{i=1}^m Q_i} \Psi(|\nabla^k u|).$$

Proof. Let us abbreviate $E = \bigcup_{i=1}^m Q_i$. Now, using the triangle inequality, Lemma 2.7, the doubling property of Ψ, then triangle inequality again and Proposition 2.8 we obtain

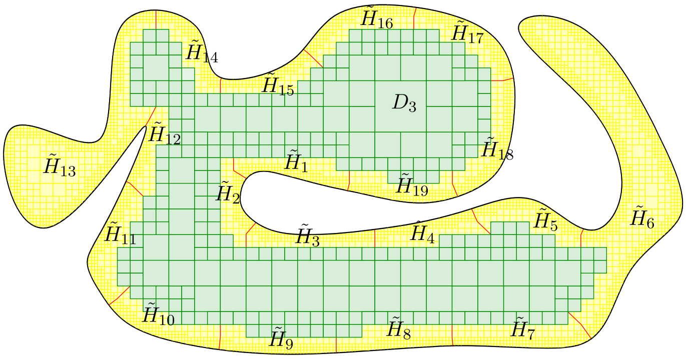

Fig. 1. The domain Ω is decomposed into a core part D_3 (obtained as a connected component of a union of Whitney squares) and boundary parts $\tilde{H}_i$. A partition of unity is made using this decomposition.

$$
\begin{aligned}
&\int_{Q_1} \Psi\left(\frac{|\nabla^\alpha P_{Q_1} - \nabla^\alpha P_{Q_m}|}{\ell(Q_1)^{k-|\alpha|}}\right) \\
&\quad\le \int_{Q_1} \Psi\left(\frac{|\nabla^\alpha P_{Q_1} - \nabla^\alpha P_{Q_m} + \nabla^\alpha P_E - \nabla^\alpha P_E|}{\ell(Q_1)^{k-|\alpha|}}\right) \\
&\quad\le C\int_{Q_1} \Psi\left(\frac{|\nabla^\alpha P_{Q_1} - \nabla^\alpha P_E|}{\ell(Q_1)^{k-|\alpha|}}\right) + C\int_{Q_m} \Psi\left(\frac{|\nabla^\alpha P_{Q_m} - \nabla^\alpha P_E|}{\ell(Q_m)^{k-|\alpha|}}\right) \\
&\quad\le C\int_{Q_1} \Psi\left(\frac{|\nabla^\alpha (u - P_{Q_1})|}{\ell(Q_1)^{k-|\alpha|}}\right) + C\int_{Q_m} \Psi\left(\frac{|\nabla^\alpha (u - P_{Q_m})|}{\ell(Q_m)^{k-|\alpha|}}\right) \\
&\qquad + C\int_{E} \Psi\left(\frac{|\nabla^\alpha (u - P_E)|}{\ell(Q_1)^{k-|\alpha|}}\right) \\
&\quad\le C\int_{\bigcup_{i=1}^m Q_i} \Psi(|\nabla^k u|). \quad \square
\end{aligned}
$$

3. Decomposition and partition of unity

In this section we recall the decomposition of the domain Ω and the associated partition of unity that was obtained in [28]. In order to make the comparison between this paper and [28] easy, we use here the notation from [28].

3.1. Decomposition of the domain

We fix a square Q_0, which is one of the largest Whitney squares in Ω. For each $n \in \mathbb{N}$, the domain Ω is then divided into a core part D_n, and a boundary layer, which is the union of sets $\tilde{H}_i$, see Fig. 1. The core part D_n is the connected component containing Q_0 of the interior of the union of Whitney squares of side-length at least 2^{-n}. The construction of the boundary parts $\{\tilde{H}_i\}_{i=1}^l$ is more involved. The sets $\tilde{H}_i$ are labelled so that $\tilde{H}_i \cap \tilde{H}_j \neq \emptyset$ if and only if $|i-j| \le 1$ in a cyclical manner.

The sets $\tilde{H}_i$ are expanded by taking a connected component H_i of a 2^{-n-3} neighbourhood of $\tilde{H}_i$. The main property of the decomposition is that these expanded sets still satisfy $H_i \cap H_j \neq \emptyset$ if and only if $|i-j| \le 1$ in a cyclical manner (Lemma 3.4 in [28]). Moreover, since the neighbourhoods are taken in the Euclidean distance, we may use an Euclidean partition of unity.

For each i we associate a Whitney square $Q_i \subset D_n$ of side length 2^{-n} so that $H_i \cap Q_i \neq \emptyset$. We have that for every i and every n the cardinality of

$$\mathcal{B}_i = \{Q \text{ Whitney square of } \Omega \,:\, Q \cap \partial H_i \cap \partial D_n \neq \emptyset\}$$

is bounded by a universal constant. Notice that $Q_i \in \mathcal{B}_i$ and that the set $\bigcup_{Q\in\mathcal{B}_i} Q$ is connected. Moreover, if $|i-j| \le 1$, we have that

$$\bigcup_{Q\in\mathcal{B}_i} Q \cap \bigcup_{Q\in\mathcal{B}_j} Q \neq \emptyset,$$

and so there is a chain of Whitney squares $\mathcal{Q}_{i,j} \subset \mathcal{B}_i \cup \mathcal{B}_i$ with length bounded by a universal constant connecting Q_i and Q_j.

3.2. Partition of unity

Using the decomposition of the domain Ω introduced above, we make a partition of unity for the domain. The partition of unity consists of functions φ_i, $i \in \{0, \dots, l\}$ with the following properties:

(1) The function φ_0 is supported in $\{x \in \Omega \,:\, \mathrm{dist}(x, D_n < \frac{2^{-n}}{10}\}$.
(2) For $i \ge 1$ the function φ_i is supported in H_i.
(3) For all i, $0 \le \varphi_i \le 1$.
(4) $\sum \varphi_i \equiv 1$ on Ω.
(5) For all i, $|\nabla^\alpha \varphi_i| \le C(\alpha)2^{n|\alpha|}$ for all multi-indices α.

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1, using the results of Section 2 and the partition of unity from [28] that was recalled in Section 3. The polynomial approximation is exactly the same as in [28]. What is different is the way the estimates are carried out using Poincaré inequalities. Since the usual Poincaré inequality is replaced by a $\Psi - \Psi$ Poincaré inequality (Proposition 2.8), we need to be more careful with the chains of inequalities.

Given a function $u \in L^{k,\Psi}(\Omega)$ and $\varepsilon > 0$, our aim is to find a function $u_\varepsilon \in W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ satisfying $\left\|\nabla^k u_\varepsilon - \nabla^k u\right\|_{L^\Psi(\Omega)} \lesssim \varepsilon$. We start by noting that we may assume $u \in L^{k,\Psi}(\Omega) \cap C^\infty(\Omega)$, since smooth functions are dense in $L^{k,\Psi}(\Omega)$, see [10].

For $n \in \mathbb{N}$ fixed, we let D_n and $\{H_i\}_{i=1}^l$ be as in Section 3. With these we define a function $u_n \in W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ by setting for all $x \in \Omega$

$$u_n(x) = \varphi_0(x)u(x) + \sum_{i=1}^{l} \varphi_i(x)P_i(x),$$

where we have abbreviated $P_i := P_{Q_i}$ with the choice of squares Q_i done in Section 3.2. Clearly, $u_n \in W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$. Therefore, what remains to show is that

$$\left\|\nabla^k u_n - \nabla^k u\right\|_{L^\Psi(\Omega)} \to 0, \qquad \text{as } n \to \infty.$$

First of all, by the definition of u_n, we have

$$u_n(x) = u(x) \qquad \text{for all } x \in \{z \in \Omega \,:\, \varphi_0(z) = 1\} \subset D_{n-1}.$$

Also, for all $i \in \{1 \dots, l\}$, since P_i is a degree $k-1$ polynomial, we have

$$\nabla^k u_n(x) = 0 \qquad \text{for all } x \in \{z \in \Omega : \varphi_i(z) = 1\}.$$

Therefore,

$$\begin{aligned}\left\|\nabla^k u_n - \nabla^k u\right\|_{L^\Psi(\Omega)} &= \left\|\nabla^k u_n - \nabla^k u\right\|_{L^\Psi(\{\varphi_0 \neq 1\})} \\ &\leq \left\|\nabla^k u\right\|_{L^\Psi(\Omega \setminus D_{n-1})} + \left\|\nabla^k u_n\right\|_{L^\Psi(\bigcup_{i=1}^l A_i)},\end{aligned}$$

where we have written $A_i := \{x \in \Omega : 0 < \varphi_i(x) < 1\}$ for $i \in \{1, \dots, l\}$. Since the sets D_n increasingly exhaust the domain Ω, we have

$$\left\|\nabla^k u\right\|_{L^\Psi(\Omega \setminus D_{n-1})} \to 0 \qquad \text{as } n \to \infty.$$

Thus, it remains to show that

$$\left\|\nabla^k u_n\right\|_{L^\Psi(\bigcup_{i=1}^l A_i)} \to 0 \qquad \text{as } n \to \infty,$$

or equivalently, via Lemma 2.5, that for each multi-index α with $|\alpha| = k$ we have

$$\int_{\bigcup_{i=1}^l A_i} \Psi(|\nabla^\alpha u_n(x)|) \to 0 \qquad \text{as } n \to \infty. \tag{4.1}$$

In order to show (4.1), we write for each $i \in \{1, \dots, l\}$ and multi-index α with $|\alpha| = k$,

$$\begin{aligned} u_n(x) &= \varphi_0(x)u(x) + \sum_{j=1}^l \varphi_j P_j(x) \\ &= \varphi_0(x)u(x) - \varphi_0 P_i(x) + \varphi_0 P_i(x) + \sum_{j=1}^l \varphi_j P_j(x) \\ &= \varphi_0(x)(u(x) - P_i(x)) + P_i(x) + \sum_{j=1}^l \varphi_j (P_j(x) - P_i(x)) \end{aligned}$$

and estimate, by using the fact that $\nabla^\alpha P_i(x) = 0$, the triangle inequality, and Jensen's inequality

$$\begin{aligned} \int_{A_i} \Psi(|\nabla^\alpha u_n(x)|)\,\mathrm{d}x &= \int_{A_i} \Psi\left(\left|\nabla^\alpha \left(\varphi_0(x)u(x) + \sum_{j=1}^l \varphi_j(x)P_j(x)\right)\right|\right)\mathrm{d}x \\ &\leq C \sum_{\beta \leq \alpha} \int_{A_i} \Psi(|\nabla^\beta u(x) - \nabla^\beta P_i(x)||\nabla^{\alpha-\beta}\varphi_0(x)|)\,\mathrm{d}x \\ &\quad + C \sum_{\beta \leq \alpha} \int_{A_i} \sum_{j=1}^l \Psi(|\nabla^\beta P_j(x) - \nabla^\beta P_i(x)||\nabla^{\alpha-\beta}\varphi_j(x)|)\,\mathrm{d}x. \end{aligned}$$

We estimate the above two terms separately.

Let us take $\beta \leq \alpha$ and write

$$\begin{aligned}\int_{A_i} &\Psi(|\nabla^\beta u(x) - \nabla^\beta P_i(x)||\nabla^{\alpha-\beta}\varphi_0(x)|)\,\mathrm{d}x \\ &= \sum_{Q\in\mathcal{B}_i}\int_Q \Psi(|\nabla^\beta u(x) - \nabla^\beta P_i(x)|C2^{n(|\alpha|-|\beta|)})\,\mathrm{d}x.\end{aligned}$$

Notice that the integral is split into Whitney squares $Q \in \mathcal{B}_i$. These are exactly the Whitney squares Q for which $Q \cap A_i \cap A_0 \neq \emptyset$ and hence the squares that intersect A_i where $|\nabla^{\alpha-\beta}\varphi_0(x)|$ does not vanish. There are only a uniformly bounded amount of squares in $\mathcal{B}_i$, and for each $Q \in \mathcal{B}_i$ we have

$$\begin{aligned}\int_Q &\Psi(|\nabla^\beta u(x) - \nabla^\beta P_i(x)|C2^{n(|\alpha|-|\beta|)})\,\mathrm{d}x \\ &\leq C\int_Q \Psi\left(\frac{|\nabla^\beta u(x) - \nabla^\beta P_Q(x)|}{2^{-n(|\alpha|-|\beta|)}}\right)\mathrm{d}x + C\int_Q \Psi\left(\frac{|\nabla^\beta P_Q(x) - \nabla^\beta P_i(x)|}{2^{-n(|\alpha|-|\beta|)}}\right)\mathrm{d}x \\ &\leq C\int_Q \Psi(|\nabla^k u(x)|)\,\mathrm{d}x + C\int_{\bigcup_{Q'\in\mathcal{B}_i} Q'} \Psi(|\nabla^k u(x)|)\,\mathrm{d}x,\end{aligned}$$

using the doubling property of Ψ, triangle inequality, Proposition 2.8 and Lemma 2.9 for a chain of cubes inside $\mathcal{B}_i$.

Next we estimate for $\beta \leq \alpha$,

$$\begin{aligned}\int_{A_i} &\sum_{j=1}^{l}\Psi(|\nabla^\beta P_j(x) - \nabla^\beta P_i(x)||\nabla^{\alpha-\beta}\varphi_j(x)|)\,\mathrm{d}x \\ &= \sum_{j=i-1}^{i+1}\int_{A_i\cap A_j} \Psi(|\nabla^\beta P_j(x) - \nabla^\beta P_i(x)|C2^{n(|\alpha|-|\beta|)})\,\mathrm{d}x \\ &\leq C\sum_{j=i-1}^{i+1}\int_{Q_i} \Psi\left(\frac{|\nabla^\beta P_j(x) - \nabla^\beta P_i(x)|}{2^{-n(|\alpha|-|\beta|)}}\right)\mathrm{d}x \\ &\leq C\sum_{j=i-1}^{i+1}\int_{\bigcup_{Q'\in\mathcal{B}_i\cup\mathcal{B}_j} Q'} \Psi(|\nabla^k u(x)|)\,\mathrm{d}x\end{aligned}$$

using the fact that $H_i \cap H_j \neq \emptyset$ if and only if $|i-j| \leq 1$, Lemma 2.7 (with the smallest square Q containing $(A_i \cap A_j) \cup Q_i$ and with $F = Q_i$) and Lemma 2.9 for the chain $\mathcal{Q}_{i,j}$ contained in $\mathcal{B}_i \cup \mathcal{B}_j$.

Combining the above estimates and using the fact that there is only a uniform number of overlaps for the estimates we have

$$\int_{\bigcup_{i=1}^{l} A_i} \Psi(|\nabla^\alpha u_n(x)|) \leq C \int_{\bigcup_{i=1}^{l}\bigcup_{Q\in\mathcal{B}_i} Q} \Psi(|\nabla^\alpha u(x)|),$$

giving (4.1). This proves the theorem.

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