JYU DISSERTATIONS 376

Ville Kivioja

On Metric Relations between Lie Groups

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

JYU DISSERTATIONS 376

Ville Kivioja

On Metric Relations between Lie Groups

Esitetään Jyväskylän yliopiston matemaattis-luonnontieteellisen tiedekunnan suostumuksella julkisesti tarkastettavaksi toukokuun 14. päivänä 2021 kello 14.

Academic dissertation to be publicly discussed, by permission of the Faculty of Mathematics and Science of the University of Jyväskylä,, on May 14, 2020 at 2 p.m. o'clock.

JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2021

Editors
Enrico Le Donne
Department of, University of Jyväskylä
Timo Hautala
Open Science Centre, University of Jyväskylä

This is a printout of the original online publication.
Permanent link to this publication: http://urn.fi/URN:ISBN:978-951-39-8629-2

ISBN 978-951-39-8629-2 (PDF)
URN:ISBN:978-951-39-8629-2
ISSN 2489-9003
Jyväskylä University Printing House, Jyväskylä 2021

Abstract. This thesis approaches the problem of quasi-isometric classification of Lie groups. The point of view is motivated by the known metric properties of Carnot groups, and the strategy to find similar properties in more general settings is thus twofold: First, we ask when a pair of non-isomorphic Lie groups can be made isometric using left-invariant Riemannian distances. Second, we investigate what kind of role the existence of metric dilations plays for quasi-isometry questions. Several new results and viewpoints are found, reducing metric questions to algebraic ones. Examples of the limitations of the theory and the methods to find those examples are studied.

Tiivistelmä. Tämä väitöskirja käsittelee Lien ryhmien kvasi-isometrisen luokittelun ongelmaa. Käytetyt ideat juontavat juurensa Carnot'n ryhmien tunnettuihin metrisiin ominaisuuksiin, ja ongelmaa lähestytäänkin kahtaalta: Ensiksi kysymme millaisia ei-isomorfisia Lien ryhmien pareja voidaan varustaa isometrisillä (vasemmalta) siirtoinvarianteilla Riemannilaisilla etäisyysfunktioilla. Toisekseen tutkimme millainen yhteys on venytyskuvausten olemassaololla ja kvasi-isometrioilla. Nämä kysymykset johtavat moniin uusiin näkökulmiin ja tuloksiin, joissa metriset ongelmat palautuvat algebrallisiksi. Tarkastelemme myös esimerkkien valossa teorian rajoituksia sekä menetelmiä uusien esimerkkien löytämiseksi.

Acknowledgements

Many people have helped me and inspired me either in direct or indirect ways during the journey of the PhD studies. The first thanks go obviously to my advisor Enrico Le Donne: I can thank him not only for all the help and support he has given during all these years, but also for guiding me first to the world of mathematics in general, and to the world of professional mathematicians thereafter. I am also indebted to my collaborators, not only of mathematical collaboration, but also of friendship and kindness. Also, many other colleagues should know how much I admire the inspiring discussions and good time we have had. For the financial support, I am grateful to Emil Aaltonen Foundation and the Department of Mathematics and Statistics in JYU. Regarding the latter, I want to thank all the personnel of the department for being very kind and helpful, also with practical matters. Finally, the patience, support, encouragement and love from my parents and my dear wife Jenni cannot be highlighted too much.

Jyväskylä, April 19, 2021
Department of Mathematics and Statistics at JYU
Ville Kivioja

List of included articles

This dissertation consists of an introductory part and the following articles:

[A] Ville Kivioja and Enrico Le Donne, *Isometries of nilpotent metric groups*, Journal de l'École Polytechnique – Mathématiques, **4** (2017), 473–482.

[B] Michael G. Cowling, Ville Kivioja, Enrico Le Donne, Sebastiano Nicolussi Golo and Alessandro Ottazzi, *From homogeneous metric spaces to Lie groups*, Preprint.

[C] Ville Kivioja, Enrico Le Donne and Sebastiano Nicolussi Golo, *Metric equivalences of Heintze groups and applications to classifications in low dimension*, Preprint.

[D] Eero Hakavuori, Ville Kivioja, Terhi Moisala and Francesca Tripaldi, *Gradings for nilpotent Lie algebras*, Preprint.

The articles are referred as [A], [B], [C] and [D] in the introduction, whereas other references are numbered as [1], [2], etc. The author of this dissertation has actively taken part in the research of the articles.

Contents

1. Lie groups as metric objects

The objects of interest in this thesis are Lie groups. In addition to their structure as differentiable manifolds and groups, we will equip them with distance functions compatible with the two firstly mentioned structures. We will treat such distances both precisely and with respect to a coarse equivalence, and the latter is formalised by the concept of quasi-isometry: We recall that, given two metric spaces (X, d) and (X', d'), a map $f\colon X \to X'$ is said to be a *quasi-isometry* if there are constants $L \geq 1$ and $C \geq 0$ so that all the points $x, y \in X$ satisfy the inequalities

$$\frac{1}{L} d(x, y) - C \leq d'(f(x), f(y)) \leq L d(x, y) + C$$

and in addition for all $x' \in X'$ there is $x \in X$ with $d(f(x), x') \leq C$. If such a map f exists, then the metric spaces X and X' are said to be *quasi-isometric* (or *quasi-isometrically equivalent*).

For the later purposes, we recall immediately also the other relevant equivalence relations of metric spaces: If it is possible to choose above

- $L = 1$, then f is a *rough isometry* and X and X' are *roughly isometric.*
- $C = 0$, then f is a *biLipschitz map* and X and X' are *biLipschitz equivalent.*
- $L = 1$ and $C = 0$, then f is an *isometry* and X and X' are *isometric.*

Intuitively, two isometric metric spaces may be regarded as the same space with different coordinates. Two biLipschitz equivalent metric spaces look like stretched versions of each other, with the factor of stretching allowed to vary but required to stay in some bounds. Two roughly isometric metric spaces may be arbitrarily different on small scales, but they look more and more as the same metric space when zooming out and looking at the space from far away. Finally, a pair of quasi-isometric metric spaces has only the property that the metric spaces look like stretched versions of each other when looking at the space from far away.

Coming back from metric spaces to Lie groups, if a set G has both a Lie group structure and a structure of a metric space, then these structures are regarded to be compatible with each other if the distance function d induces the manifold topology of G (such a distance is said to be *admissible*), and if it is left-invariant, i.e., if the group of left-translations of G acts by isometries for the distance; We call such an object (G, d) a *metric Lie group*. However, we shall not regard Lie groups equipped with distances, i.e., metric Lie groups, as our main objects of interest. Instead, the main objects of interest are some particular relations between Lie groups, and these relations will have a “metric flavour”. We are going to particularly focus on the two relations below.

- Given two Lie groups G and H, we will say they *can be made isometric*, if there exist some left-invariant Riemannian distances d_G and d_H on G and H, respectively, so that the metric spaces (G, d_G) and (H, d_H) are isometric. We will refer to this relation as *the isometry relation.*
- Given two Lie groups G and H, we will say they are *quasi-isometric*, if there exist some left-invariant Riemannian distances d_G and d_H on G and H, respectively, so that the metric spaces (G, d_G) and (H, d_H) are quasi-isometric. We will refer to this relation as *the quasi-isometry relation.*

Traditionally, quasi-isometries have received more attention in mathematical research. They started to become into focus in the early 20th century after the introduction of Cayley graphs (a.k.a. Dehn Gruppenbilds). The research on quasi-isometries accelerated especially after in the 1980s Gromov proposed to study finitely generated groups as large scale geometric objects. For a more in-depth historical account, see [11, Section 1].

Finitely generated groups can be equipped naturally with word distances with respect to finite generating sets, and all the word distances are quasi-isometrically equivalent. Hence by *large scale geometry* of a given finitely generated group G it is meant the equivalence class of those finitely generated groups that are quasi-isometric to G with some, and therefore any, word distances.

One theorem of Gromov that became especially famous is that groups of polynomial growth are virtually nilpotent [23]; this is an algebraic statement detected by large scale geometry. As the groups of polynomial growth are quasi-isometric to nilpotent groups, it is then natural to ask what are the quasi-isometry relations of nilpotent groups?

The conjectural statement, first mentioned by [18], is that *if two torsion-free finitely generated nilpotent groups are quasi-isometric, they have isomorphic Malcev completions.* The Malcev completion (introduced in [35]) of a torsion-free finitely generated nilpotent group is a simply connected nilpotent Lie group on which the finitely generated group isomorphically sits as a lattice. Thus, it is here that Lie groups come to the picture, even if the large scale geometry of finitely generated groups also remains as an active research topic, see for example [15, 16, 17].

The following conjecture can be regarded as very much related to the one about finitely generated groups, although these two conjectures are not completely equivalent since not all nilpotent Lie groups admit lattices.

Conjecture 1.1. *If two simply connected nilpotent Lie groups are quasi-isometric, then they are isomorphic.*

This conjecture is attacked by active research, with the most important steps towards a solution being for now probably [42, 36, 22, 37, 40, 39, 10]. The topic is very well surveyed in [9], together with a more general perspective. It is here that the input of this thesis to the research field starts.

2. Isometries of nilpotent Lie groups

Conjecture 1.1 and the research on isometries of subRiemannian Carnot groups led us to ask the regularity of isometries on nilpotent metric Lie groups with non-Riemannian distances. Isometries of a Euclidean space are affine maps with respect to the vector space structure. If one considers a non-Abelian Lie group, then the corresponding notion would be that a map F between two groups is *affine* if it is a composition of a left-translation and an automorphism. It was established by [25, 30, 34] that isometries between subRiemannian Carnot groups, which are special cases of nilpotent metric Lie groups, are affine.

We realised that, using the Gleason–Yamabe–Montgomery–Zippin structure theory (GYMZ-theory for short), it is possible to establish the smoothness of isometries for general metric Lie groups. This smoothness we used to deduce that the isometry group of a metric Lie group may be extended by passing to Riemannian metrics, and finally we showed how this leads to the affiness of isometries between any connected nilpotent metric Lie groups.

In more detail, we first show that the isometry group I of a metric Lie group (G, d) is a Lie group and that the action $I \curvearrowright G$ is smooth and has compact stabilisers. Notice that a distance of a metric Lie group is not assumed to have precompact balls, which is an issue that needs to be treated to apply GYMZ-theory.

Then, for a connected metric Lie group (G, d), the smoothness of isometries opens up the possibility of averaging a scalar product over the compact stabiliser of the identity element to produce a Riemannian metric g whose isometry group contains the original self-isometries, i.e., $\mathrm{Iso}(G, d) \subset \mathrm{Iso}(G, g)$. As a consequence, isometries between two connected metric Lie groups may be regarded as Riemannian isometries:

Theorem 2.1 (with Le Donne, in [A]). *If (G_1, d_1) and (G_2, d_2) are connected metric Lie groups, then there exist left-invariant Riemannian metrics g_1 and g_2 on G_1 and G_2, respectively, such that $\mathrm{Iso}(G_i, d_i) \subset \mathrm{Iso}(G_i, g_i)$ for $i \in \{1, 2\}$ and for each isometry $F\colon (G_1, d_1) \to (G_2, d_2)$ the map $F\colon (G_1, g_1) \to (G_2, g_2)$ is a Riemannian isometry.*

The fact that isometries between two nilpotent metric Lie groups are affine may now be deduced from the work of Wolf on the Riemannian case. Indeed, Wolf proved in [42, Theorem 4.2] that, for a connected nilpotent Riemannian Lie group G, the group of left-translations of G is the nilradical of the identity component of I; We generalise this result to arbitrary left-invariant admissible distances in [A, Theorem 1.2.iii].

Theorem 2.2 (with Le Donne, in [A]). *Isometries between connected nilpotent metric Lie groups are affine.*

In particular, we find that the following "isometric version" of Conjecture 1.1 is true: *if two connected nilpotent Lie groups admit left-invariant admissible distance functions that make them isometric metric spaces, then they are isomorphic.* Also, we understand from Theorem 2.1 that restricting to Riemannian distances when defining the isometry relation is not important.

In the results above, we only assume that the Lie groups are connected. In almost everything that follows, we will assume more strongly that they are simply connected.

2.1. **Groups isometric to nilpotent groups.** While the regularity of isometries between connected nilpotent metric Lie groups is so high that it forces isometries to induce isomorphisms, this was known to not hold in much higher generality. For a simple example, consider the universal covering group of the isometry group of the Euclidean plane. This group, which we denote by $\widetilde{\mathrm{SE}}(2)$, is a Lie group that is naturally identified with the manifold $\mathbb{R}^3$. Under this identification, the Euclidean distance of $\mathbb{R}^3$ is left-invariant for the group law of $\widetilde{\mathrm{SE}}(2)$. Hence the non-Abelian group $\widetilde{\mathrm{SE}}(2)$ can be made isometric to the Abelian group $\mathbb{R}^3$.

The group $\widetilde{\mathrm{SE}}(2)$ is an example of a simply connected solvable Lie group with polynomial growth, and Breuillard [4] proved in general that every simply connected solvable Lie group with polynomial growth can be made isometric to a nilpotent Lie group. This nilpotent group is its nilshadow, after the work of Auslander and Green and others, see [2, 14]. This result is in relation with Gromov's theorem that finitely generated groups of polynomial growth are quasi-isometric to nilpotent groups. It was natural to ask then, which are exactly the groups that can be made isometric to nilpotent groups, or if some group of polynomial growth might be possible to make isometric to two different nilpotent groups. Notice that the latter was not ruled out by Theorem 2.2 since the transitivity of the isometry relation is not established at this point; we discuss the transitivity more in Section 3.1. These questions are however answered by the following theorem, which we already found in [7]. The article [7] is the first version of the article [B] from 2017, and it was not yet able to treat arbitrary solvable simply connected Lie groups; we will continue discussing those groups in Section 3.

Theorem 2.3 (with Cowling, Le Donne, Nicolussi Golo and Ottazzi, in [B]). *Let G_1 and G_2 be simply connected Lie groups and assume that G_1 is nilpotent. The following are equivalent:*
- *G_1 and G_2 can be made isometric;*
- *G_2 is solvable and of polynomial growth, and G_1 is its nilshadow.*

We next discuss briefly the definition of nilshadow; for more details, see [4, Definition 3.2] or [14]. When $\mathfrak{g}$ is a solvable Lie algebra, one can find a vector subspace $\mathfrak{a}$ complementary to the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ with the property that $\mathrm{ad}_s(X)(Y) = 0$ for all $X, Y \in \mathfrak{a}$. Here $\mathrm{ad}_s(X)$ denotes the semisimple part in the Jordan decomposition of $\mathrm{ad}(X)$. If $\pi_{\mathfrak{a}} \colon \mathfrak{g} \to \mathfrak{a}$ denotes the projection map associated to the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$, the nilshadow of $\mathfrak{g}$ is defined by equipping the vector space $\mathfrak{g}$ with a new bracket law

$$[X, Y]_{\mathrm{nil}} = [X, Y] - \mathrm{ad}_s(\pi_{\mathfrak{a}}(X))(Y) + \mathrm{ad}_s(\pi_{\mathfrak{a}}(Y))(X)$$

for all $X, Y \in \mathfrak{g}$.

Example 2.4. Consider the Lie algebra $\mathfrak{g}$ with a basis $X_1, \ldots, X_6$ and with the non-trivial bracket relations in this basis given by

$$[X_6, X_2] = -X_1\,, \qquad [X_6, X_3] = -X_2\,,$$
$$[X_6, X_4] = X_5\,, \qquad [X_6, X_5] = -X_4\,.$$

This Lie algebra is denoted by $\mathfrak{g}_{6,10}^{\alpha=0,p=0}$ in the classification given in [5]. It may be checked using [29, Theorem 1.4] that the simply connected Lie group with Lie algebra $\mathfrak{g}$ has polynomial growth. The nilradical of $\mathfrak{g}$ is the Abelian subalgebra $\mathrm{span}(X_1, \ldots, X_5)$, and thus we may choose the complementary subspace $\mathfrak{a}$ to be $\mathrm{span}(X_6)$. The non-trivial bracket relations of the nilshadow are therefore given by

$$[X_6, X_i]_{\mathrm{nil}} = [X_6, X_i] - \mathrm{ad}_{\mathrm{s}}(X_6)(X_i) = \begin{cases} 0 & \text{for } i = 1 \\ -X_1 & \text{for } i = 2 \\ -X_2 & \text{for } i = 3 \\ 0 & \text{for } i = 4 \\ 0 & \text{for } i = 5 \end{cases}$$

The Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathrm{nil}})$ is isomorphic to the direct product of Engel Lie algebra and $\mathbb{R}^2$.

3. Isometries of solvable Lie groups

Several points of view then led us to ask if similar results could be achieved with less restrictive algebraic structure. On the one hand, it already follows from the work of Gordon and Wilson [22, Theorems 4.3 and 5.2] that if two simply connected completely solvable Lie groups can be made isometric, then they are isomorphic. A completely solvable Lie group is a solvable Lie group G such that for all $X \in \mathrm{Lie}(G)$ the eigenvalues of the map $\mathrm{ad}(X)$ are real numbers. In particular, nilpotent Lie groups are completely solvable. The above consequence of [22] is the solution of "the isometric version" of the following conjecture.

Conjecture 3.1 (due to Y. Cornulier, see Conjecture 19.113 in [9])**.** *If two simply connected completely solvable Lie groups are quasi-isometric, then they are isomorphic.*

On the other hand, the work of Jablonski [28] allowed us to define the *real-shadow* of a solvable Lie algebra whose construction we outline below. The construction associates to every solvable Lie group a completely solvable Lie group, and it is analogous to the construction of nilshadow.

Let $\mathfrak{g}$ be a solvable Lie algebra and let $\mathfrak{a}$ and $\pi_{\mathfrak{a}}$ be as in the construction of nilshadow. Define a map

$$\varphi_{\mathfrak{a}} \colon \mathfrak{g} \to \operatorname{der}(\mathfrak{g}) \qquad \varphi_{\mathfrak{a}}(X) = -\operatorname{ad}_{\mathrm{si}}(\pi_{\mathfrak{a}}(X)),$$

where $\operatorname{der}(\mathfrak{g})$ denotes the derivation algebra of $\mathfrak{g}$ and $\operatorname{ad}_{\mathrm{si}}$ is the part of $\operatorname{ad}_{\mathrm{s}}$ with purely imaginary eigenvalues, see [33, Section 2.1]. Then

- the graph of $\varphi_{\mathfrak{a}}$, i.e., $\operatorname{Gr}(\varphi_{\mathfrak{a}}) = \{(X, \varphi_{\mathfrak{a}}(X)) : X \in \mathfrak{g}\}$, is a completely solvable subalgebra of $\mathfrak{g} \rtimes \operatorname{der}(\mathfrak{g})$, and
- when the vector space $\mathfrak{g}$ is equipped with the bracket law

  $$[X, Y]_{\mathbb{R}} = [X, Y] + \varphi_{\mathfrak{a}}(X)(Y) - \varphi_{\mathfrak{a}}(Y)(X),$$

  then the map $X \mapsto (X, \varphi_{\mathfrak{a}}(X))$ is a Lie algebra isomorphism from $(\mathfrak{g}, [\cdot,\cdot]_{\mathbb{R}})$ to $\operatorname{Gr}(\varphi_{\mathfrak{a}})$.

We show in [B] that, with respect to the isometry relation, the real-shadow plays a similar role within the family of all simply connected solvable groups as nilshadow does within the family of simply connected solvable groups of polynomial growth. We remark that the real-shadow of a simply connected solvable Lie group of polynomial growth is equal to its nilshadow.

Theorem 3.2 (with Cowling, Le Donne, Nicolussi Golo and Ottazzi, in [B]). *Let G_1 and G_2 be simply connected solvable Lie groups. Then G_1 and G_2 can be made isometric if and only if their real-shadows are isomorphic.*

Within the family of simply connected solvable groups, Theorem 3.2 reduces the isometry relations to algebraic questions. This theorem should be also compared to Theorem 2.3. Together with the added generality, there are two notable differences:

- Theorem 2.3 did not tell that if two solvable groups of polynomial growth can be made isometric, they need to have the same nilshadow. Such a conclusion was possible to make from the later work of Jablonski [28] though.
- Theorem 3.2 does not tell that a group that can be made isometric to a completely solvable group should be solvable. Indeed, such a statement is false, since by [B, Corollary 3.29] the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ can be made isometric to the group $\mathbb{R} \times \mathrm{Aff}^+(\mathbb{R})$, where $\mathrm{Aff}^+(\mathbb{R})$ denotes the unique non-Abelian simply connected Lie group of dimension 2.

We also proved in article [B] that the general study of locally compact connected isometrically homogeneous metric spaces up to quasi-isometry reduces to simply connected solvable Lie groups with left-invariant metrics. In particular, this explains why the title of the thesis does not mention the assumption of solvability while we concentrate discussing only solvable groups.

Theorem 3.3 (with Cowling, Le Donne, Nicolussi Golo and Ottazzi, in [B])**.** *Let (M, d) be a connected locally compact metric space, and assume that the action of the isometry group of (M, d) is transitive. Then there is a solvable Lie group S with a left-invariant admissible distance d_S and a rough isometry $(M, d) \to (S, d_S)$.*

The above result follows when combining Theorem A and Theorem 3.24 of the article [B].

3.1. **Transitivity questions.** It is worth to notice that Theorem 3.2 implies the transitivity of the isometry relation within the family of simply connected solvable Lie groups. It remains open if this relation is transitive for all simply connected Lie groups. Remarkably, there is an example (due to Y. Cornulier, based on a result of Gordon in [21]) of a triple of non-simply connected Lie groups G_1, G_2, G_3 so that the pair (G_1, G_2) can be made isometric, and the pair (G_2, G_3) can be made isometric, while the pair (G_1, G_3) cannot be made isometric. In this example, which is recalled in [C, Proposition 1.9], the group G_2 is solvable and the groups G_1 and G_3 are semisimple.

The quasi-isometry relation instead is transitive because all left-invariant Riemannian distances on a given Lie group are biLipschitz equivalent. It holds more generally that on a given Lie group all left-invariant quasi-geodesic distances that have precompact balls (a property known as *boundedly compact* or *proper*) are quasi-isometric via the identity map. This explains the asymmetry in the definitions of the metric relations: it is common to say two Lie groups "are quasi-isometric" instead of saying that they "can be made quasi-isometric" since this class of quasi-geodesic boundedly compact distances is considered somewhat canonical as it contains the Riemannian distances and also the word distances in case the Lie group admits lattices.

3.2. **Applications to low dimension.** Theorem 3.2 reduces the question of whether two simply connected solvable Lie groups can be made isometric, to an algebraic problem of determining their real-shadows. Since the equivalence classes up to the quasi-isometry relation are unions of the equivalence classes of the isometry relation, there is a potential benefit of this study for the quasi-isometric classification problem of simply connected solvable Lie groups. It might even turn out that these relations completely agree withing this family of groups, but such a statement (being equivalent to Conjecture 3.1) is far from being established for the time being. In order to make some progress, we wanted to concretely determine the equivalence classes of the isometry relation for some families of low dimension for which the algebraic classifications are known. For this task, the following result has practical value.

Theorem 3.4 (with Le Donne and Nicolussi Golo, in [C]). *Let H be a simply connected Lie group and α a derivation of H. Then the Lie group $H \rtimes_\alpha \mathbb{R}$ can be made isometric to the Lie group $H \rtimes_{\alpha_0} \mathbb{R}$, where $\alpha_0 = \alpha - \alpha_{\mathrm{si}}$ where α_{si} is the part of α that is semisimple and has purely imaginary eigenvalues (see* [33, Section 2.1]*)*

We remark that if H is simply connected and completely solvable, then such a result implies that $H \rtimes_{\alpha_0} \mathbb{R}$ is the real-shadow of $H \rtimes_\alpha \mathbb{R}$. Consequently, if $G_1 = H_1 \rtimes_\alpha \mathbb{R}$ and $G_2 = H_2 \rtimes_\beta \mathbb{R}$ are two simply connected solvable Lie groups, and if the groups H_1 and H_2 are completely solvable, then the groups G_1 and G_2 can be made isometric if and only if $H_1 \rtimes_{\alpha_0} \mathbb{R}$ and $H_2 \rtimes_{\beta_0} \mathbb{R}$ are isomorphic.

The classification of simply connected solvable Lie groups of dimension 4 with respect to the isometry relation is given below in the notation of [38]. Similar classification for all Lie groups of dimension 3 and less is surveyed in [19].

Theorem 3.5 (with Le Donne and Nicolussi Golo, in [C]). *Let G and H be simply connected solvable Lie groups of dimension 4. If G and H are both completely solvable, then they can be made isometric if and only if they are isomorphic. Instead, if at least one of them is not completely solvable, then they can be made isometric if and only if they belong to the same set of groups in the following list:*

(I) $\{\mathbb{R}^4,\ \mathbb{R} \times A_{3,6}\}$,
(II) $\{\mathbb{R} \times A_{3,1},\ A_{4,10}\}$,
(III$_\lambda$) $\{A_{4,5}^{\lambda,\lambda}\} \cup \{A_{4,6}^{a,b} : \lambda = \mathrm{sign}(ab)\min(|b/a|, |a/b|)\}$,
(IV) $\{A_{4,9}^{1}\} \cup \{A_{4,11}^{a} : a \in \,]0,\infty[\,\}$,
(V) $\{\mathbb{R} \times A_{3,3},\ A_{4,12}\} \cup \{\mathbb{R} \times A_{3,7}^{a} : a \in \,]0,\infty[\,\}$,
(VI) $\{\mathbb{R}^2 \times A_2\} \cup \{A_{4,6}^{a,0} : a \in \mathbb{R}\}$.

Here (III$_\lambda$) *stands for distinct sets depending on parameter $\lambda \in \mathbb{R}\setminus\{0\}$. Hence the above list contains 5 sets (2 finite and 3 infinite) and one family of sets depending an a parameter.*

In article [C], we also survey the classification of 5-dimensional simply connected solvable Lie groups of polynomial growth with respect to the isometry relation.

4. Self similar groups

We started discussing nilpotent groups and the groups of polynomial growth in Section 2 and moved to more general solvable groups in Section 3, and we always had the viewpoint of the isometry relation. Now we go back to study some subfamilies of nilpotent groups, but making more delicate notes about their metric structures, especially related to the existence of dilations. Eventually, this leads to the discussion of quasi-isometries between Heintze groups.

A Carnot group is a simply connected nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ admits *a Carnot grading* (a.k.a. *a stratification*), i.e., a decomposition $\mathfrak{g} = \bigoplus_{k=1}^{s} V_k$ such that $V_{k+1} = [V_1, V_k]$ for all $1 \leq k \leq s-1$, and $V_s \neq \{0\}$. Here s is the nilpotency step of G. Carnot groups may be naturally equipped with subRiemannian distances, and as such are rich geometric structures with appearances as model spaces (and tangents) of subRiemannian geometry, as asymptotic cones of nilpotent groups and in geometric measure theory and other fields of mathematics; we point the reader to [13] for a list of references and a more in-depth survey of Carnot groups and their subRiemannian (and subFinsler) distances.

A Positive grading is the following relaxation of a Carnot grading: a positive grading is a decomposition $\mathfrak{g} = \bigoplus_{\lambda>0} V_\lambda$, where the subspaces V_λ, called *layers*, are only assumed to satisfy $[V_\lambda, V_{\lambda'}] \subset V_{\lambda+\lambda'}$. A nilpotent Lie algebra may admit several inequivalent positive gradings (for the precise notion of equivalence, see Section 4.1), or none at all; On the contrary, if a Lie algebra admits a Carnot grading, then all its Carnot gradings are conjugate under automorphisms. If the Lie algebra of a simply connected nilpotent Lie group G admits a positive grading, then G is said to be *positively gradable.*

From the metric geometry point of view, the importance of positively gradable Lie groups is in the fact that they admit left-invariant distances with automorphic dilations. In more detail, if G is a positively gradable Lie group and $\mathfrak{g} = \bigoplus_{\lambda>0} V_\lambda$ is a positive grading for its Lie algebra, then there is a left-invariant admissible distance d on G and a homomorphism $\mathbb{R}_+ \to \mathrm{Aut}(G)$, $t \mapsto \delta_t$ so that for all $t \in \mathbb{R}_+$ the map δ_t is *a dilation of factor* t, i.e., for all $x, y \in G$ it holds $d(\delta_t(x), \delta_t(y)) = td(x, y)$. Such a homomorphism is constructed by defining the map δ_t to be the automorphism with the differential that acts by multiplication on the layers so that for all $\lambda > 0$ we have $X \mapsto t^\lambda X$ for all $X \in V_\lambda$. We omit here the general description of the distance, see [26]. We remark however that if a positive grading is indeed a Carnot grading, then the natural subRiemannian distances admit such families of dilations.

In a sense, positively gradable Lie groups equipped with such distances are the most general metric spaces with a dilation structure. This is made precise by the following result that was also proven already in the 2017-version of the article [B]. Its proof is based on a famous theorem of Siebert [41].

Theorem 4.1 (with Cowling, Le Donne, Nicolussi Golo and Ottazzi, in [B]). *Suppose (M, d) is a metric space that is*

- *locally compact,*
- *connected,*
- *isometrically homogeneous, i.e., the group of isometries acts transitively,*
- *self-similar, i.e., admits a dilation δ of factor λ for some $\lambda > 1$.*

Then there is a positively gradable Lie group G with a left-invariant distance d_G and an isometry $F\colon (M, d) \to (G, d_G)$ so that $F \circ \delta \circ F^{-1}$ is an automorphism of G. Moreover, every dilation of (G, d_G) that fixes 1_G is an automorphism.

Pay attention that a metric space with these assumptions may still not admit a one-parameter family of dilations, and in such a situation the distance d_G is not of the type defined in [26], see Example 5.7 in [33] and compare to the discussion after Definition 4.3.

The following structure is more refined than a positive grading in a nilpotent Lie group.

Definition 4.2. A pair (N, α) is a *homogeneous group* if N is a simply connected nilpotent Lie group and α is a derivation of N so that for each eigenvalue λ of α it holds $\mathrm{Re}(\lambda) > 0$. Further, we say that a homogeneous group (N, α) is *purely real* if the eigenvalues of α are real numbers.

As we shall see later in the discussion preceding Conjecture 4.4 (see also Theorem 3.2, Theorem 3.4 and Proposition 4.5), for the metric geometry purposes it is often enough to consider the purely real homogeneous groups. We next discuss what kind of a correspondence for a positively gradable nilpotent Lie group N there is between its positive gradings and its structures as a purely real homogeneous group.

Consider a positive grading $\mathrm{Lie}(N) = \bigoplus_{\lambda>0} V_\lambda$. Equipping N with the derivation α that acts as multiplication by λ on each layer V_λ, the pair (N, α) becomes a purely real homogeneous group. We shall call such α *the derivation associated to the positive grading.* Conversely, given a purely real homogeneous group (N, α), the generalised eigenspaces of α define an associated positive grading (see, for example, [33, Lemma 2.3]). These associations are not inverses of each other since the derivation associated to a purely real homogeneous group is allowed to have a nilpotent part, and this structure is forgotten when passing to the associated positive grading.

For the structures of homogeneous groups, it is preferable not to use Riemannian distances but the distances of the following kind.

Definition 4.3. The triple (N, α, ρ) is a *homogeneous metric group* if ρ is a left-invariant admissible distance function on N, and for all $\lambda > 0$ it holds $\rho(\delta_\lambda x, \delta_\lambda y) = \lambda\rho(x, y)$ for all $x, y \in N$, where δ_λ is the automorphism of N with the differential $(\delta_\lambda)_* = \mathrm{e}^{\log(\lambda)\alpha}$.

All the distances that make a given homogeneous group into a homogeneous metric group are biLipschitz-equivalent (see [C, Remark 1.4]).

In [33, Proposition 4.5] it is proven, based on Theorem 2.2, that if a nilpotent metric Lie group admits dilations of all factors, then it is a homogeneous metric group. This result, combined with Theorem 4.1, characterises the homogeneous metric groups as the only metric spaces that are connected, locally compact, isometrically homogeneous, and admit dilations of every factor. This characterisation and Theorem 4.1 may both be compared to the characterisations given in [32, Theorem 1.1] or also [3, Theorem 2]. Regarding these characterisations, observe that a homogeneous metric group (N, α, ρ) is geodesic if and only if α is the derivation associated to a Carnot grading of N and ρ is a subFinsler distance associated to the Carnot grading.

We recall that a Heintze group is a semidirect product $G = N \rtimes_\alpha \mathbb{R}$, where N is a simply connected nilpotent Lie group and the structure of a semidirect product is defined by a derivation α of N so that for each eigenvalue λ of α it holds $\mathrm{Re}(\lambda) > 0$. Thus, Heintze groups and homogeneous groups are in a natural one-to-one correspondence: their defining data coincide. However, there is actually a deeper geometric correspondence, and we are going to discuss in Section 4.2 how homogeneous metric groups appear as (parabolic) visual boundaries of Heintze groups and thus the first are related to the quasi-isometric classification of the latter. First we discuss a bit how to find homogeneous groups, or at least positive gradings.

4.1. **Finding the examples of low dimension.**

Already when studying examples of Carnot groups, one encounters the problem of identifying Carnot groups from some given list of nilpotent groups. For this problem, a solution is described by Y. Cornulier [8, Lemma 3.10]. Using it, we wrote down in article [D] a description of an explicit procedure to find a Carnot grading for a Lie algebra given in terms of its bracket relations in some basis, or to find that no Carnot grading exists. The algorithm, described as Algorithm 3.5 in [D] is written in a form to be easily implemented by a computer software and we did such an implementation in [24].

The ability to find Carnot gradings or their non-existence led us to ask if the same can be done for positive gradings. It turns out to be possible, but one needs to pay attention what is meant by "finding all positive gradings". In particular, one needs to consider a grading either as a set of layers, or take into account the indexing of the layers over some group (that in the above was always $\mathbb{R}$). These two notions of "a grading" give then rise to corresponding notions of equivalence. A clear study of these concepts is carried out by M. Kochetov in [31] and we recall it in [D, Section 2.1].

As people from the field of metric geometry Carnot groups and homogeneous groups might be not familiar with the terminology of [31], we present the main concepts briefly here to be able to state our main results clearly. We actually slightly alter the terminology from [31] and [D], hopefully to achieve added clarity.

By a *weak grading* (just *grading* in [31] and in [D]) of a Lie algebra $\mathfrak{g}$ we mean a direct sum decomposition $\mathfrak{g} = \bigoplus_{s\in S} V_s$, where S is a set and we assume that for all $s_1, s_2 \in S$ there is $s_3 \in S$ so that $[V_{s_1}, V_{s_2}] \subset V_{s_3}$. The *weights* of the weak grading are the elements of the set S corresponding to the non-trivial subspaces V_s. If S can be embedded into an Abelian group A so that the condition

$$[V_{s_1}, V_{s_2}] \subset V_{s_3} \neq \{0\}$$

implies $s_1 + s_2 = s_3$, then $\mathfrak{g} = \bigoplus_{s\in A} V_s$ is called a *group grading* or an *A-grading*, and it is *a realisation* of the original weak grading. Colloquially speaking, for a weak grading the focus is only on the subspaces, while for a group grading the indexing is considered important. This slightly vague statement is made precise by defining that two weak gradings are *equivalent* if they have the same layers up to automorphism of $\mathfrak{g}$. Instead, two group gradings over A_1 and A_2 are *group equivalent* if the automorphism induces a map on the weights that is a restriction of a group isomorphism between A_1 and A_2.

When one considers a realisation of a weak grading, one adds the information about the indexing group. A weak grading usually admits several realisations. It may also admit none at all (see [31, p. 5]), but we are not interested in these cases. Actually, we will pay attention only to weak gradings that admit some realisation over a torsion-free Abelian group.

A weak grading (that admits a realisation over a torsion-free Abelian group) admits always *a universal realisation*. It is a $\mathbb{Z}^k$-grading for some $k \geq 1$, and it has the property that all other realisations of the weak grading can be projected from the universal realisation, colloquially speaking. This projection is made precise by the concept of *push-forward grading*, for which see [D, Definition 2.5]. All universal realisations of a given weak grading are group equivalent, hence we may speak about *the* universal realisation. A group grading is said to be a *universal grading* if it is group equivalent to the universal realisation of its corresponding weak grading.

Finally, we need to discuss the *maximal grading* of a given Lie algebra $\mathfrak{g}$: It is a group grading that has such a fine layer-structure that all the universal gradings of $\mathfrak{g}$ can be found by "combining" some layers of the maximal grading. The above is again formalised by push-forward gradings, and one may see [D, Section 2.4] for the precise statements. Every Lie algebra admits a maximal grading and it is unique up to group equivalence. Maximal grading is also a universal grading.

It is perhaps time to see some examples to illustrate the concepts above. Let $\mathfrak{g}$ be the Lie algebra of the Heisenberg group with the only non-trivial bracket relation $[X, Y] = Z$. There are two obvious weak gradings:

$$\mathfrak{g} = \langle X, Y \rangle \oplus \langle Z \rangle \qquad \text{and} \qquad \mathfrak{g} = \langle X \rangle \oplus \langle Y \rangle \oplus \langle Z \rangle \,,$$

where angle brackets denote the linear spans. The first one of these has a realisation over $\mathbb{Z}$ by $V_1 = \langle X, Y \rangle$ and $V_2 = \langle Z \rangle$, and this group grading is actually the universal realisation of the weak grading. For the latter one of the weak gradings we might consider the realisation

(I) over $\mathbb{Z}$ by $V_1 = \langle X \rangle$, $V_2 = \langle Y \rangle$ and $V_3 = \langle Z \rangle$.
(II) over $\mathbb{Z}$ by $V_2 = \langle X \rangle$, $V_4 = \langle Y \rangle$ and $V_6 = \langle Z \rangle$.
(III) over $\mathbb{Z}$ by $V_1 = \langle X \rangle$, $V_{-1} = \langle Y \rangle$ and $V_0 = \langle Z \rangle$.
(IV) over $\mathbb{Z}^2$ by $V_{(1,0)} = \langle X \rangle$, $V_{(0,1)} = \langle Y \rangle$ and $V_{(1,1)} = \langle Z \rangle$.

The group gradings (I) and (II) are group equivalent. The group grading (IV) is the universal realisation of the weak grading under consideration, and the group grading (IV) is actually a maximal grading of $\mathfrak{g}$. The group gradings (I) and (II) are positive gradings, while (III) and (IV) are not. In this terminology, a positive grading is a grading over a subgroup of $\mathbb{R}$ with all the weights required to lay on the open interval $]0, \infty[$. The positive gradings (I) and (II) can be "projected" (using push-forwards) from the group grading (IV).

If one wants to find all the positive gradings of a given Lie algebra $\mathfrak{g}$ up to group equivalence, there are three main steps to take in practice: (1) find all the weak gradings of $\mathfrak{g}$, (2) determine which of the weak gradings admit realisations as positive gradings, and (3) find all inequivalent positive realisations for those weak gradings.

A strategy for the task of finding all weak gradings is outlined in [31]: if one is able to find a maximal grading for the Lie algebra, then it is possible to extract a complete list of universal gradings (up to group equivalence) for a given Lie algebra and such a list will be finite. Notice that weak gradings and universal gradings are in one-to-one correspondence. In turn, maximal gradings can be constructed using maximal split tori, this is again shown by Cornulier, see [8, Proposition 3.20], and thus step (1) may be preformed. From such a list of universal gradings, one can use the observations in [8, Proposition 3.22] to find which of the weak gradings in the finite list admit realisations as positive gradings, thus solving step (2). In step (3), given a weak grading that admits some realisation as a positive grading, one needs to parametrise all the projections to $\mathbb{R}$ from the corresponding universal grading and figure out which, if any, of them produce positive gradings. Finally, one needs to find out which of the positive gradings are group equivalent.

Using all this previous work, in [D] we described explicitly the full procedure of executing the steps (1) and (2) above. Namely, we described the construction of a maximal grading and the process of finding then all universal gradings that a given Lie algebra has, and finally we made it explicit how to find those universal gradings with realisations as positive gradings. We refer to Section 3.2, Section 3.3 and Appendix A of [D]. A computer implementation is written in [24], and using the implementation we explicitly list all universal realisations of gradings of all nilpotent Lie algebras of dimension 6 using the classification given in [12]. We do the same for a representative list of nilpotent Lie algebras up to dimension 7 using the classification of [20]. Because in dimension 7 there are one parameter families of nilpotent Lie algebras, it was not convenient to treat all of them, but we considered some representative set of parameter values, see more in [D, Section 4.2]. All these lists of gradings are presented in [24].

The above results make it possible to directly find all positive gradings for a given nilpotent Lie algebra $\mathfrak{g}$. We leave open the question of finding all the structures of $\mathfrak{g}$ as a homogeneous group; This would mean describing not only all possible positive gradings of $\mathfrak{g}$, but also all the possible derivations that have a particular positive grading as its decomposition to generalised eigenspaces, i.e., finding all the possible nilpotent parts.

4.2. **Boundaries of Heintze groups.** The quasi-isometric classification of Heintze groups is one of the most active subareas of the quasi-isometric classification of Lie groups. From the research presented above, we arrive naturally to the quasi-isometric classification problem of Heintze groups for two reasons.

First, it has been known for a while (see [9]) that every Heintze group is quasi-isometric to a purely real Heintze group. Our results about isometries between solvable groups introduced in Section 3 prove the stronger statement that every Heintze group can be made isometric to a purely real Heintze group, although this seems to be known due to the results of [1]. The important remaining conjecture on the area of quasi-isometric classification of Heintze groups is the following.

Conjecture 4.4. *If two purely real Heintze groups are quasi-isometric, then they are isomorphic.*

Second, the parabolic visual boundaries of Heintze groups are homogeneous groups; We shall present this well known relation explicitly below. It is also well known that the quasi-isometries between Heintze groups induce biLipschitz-maps to the homogeneous groups on the boundary, see Proposition 4.5. Thus, we end up considering again distances of nilpotent groups, as we did in Section 2. One needs to pay attention that as the homogeneous metric groups are not usually geodesic, their distances do not belong to the quasi-isometry equivalence class of Riemannian distances. However, as we show in Theorem 4.6, if a biLipschitz map exists between some homogeneous distances, then a quasi-isometry exists for all left-invariant Riemannian distances.

There is a link here to the Section 2, namely the analogy between Conjecture 4.4 and Conjecture 1.1. Remark that a possible answer to the validity of 4.4 does not tell the validity of Conjecture 1.1, or vice versa. However, an immediate consequence of Theorem 4.6 is that the validity of Conjecture 1.1 would imply that two quasi-isometric purely real Heintze groups have isomorphic nilradicals. Recall also that both Conjecture 1.1 and Conjecture 4.4 are special cases of the more general Conjecture 3.1.

Homogeneous distances on the boundary. A Heintze group $G = N \rtimes_\alpha \mathbb{R}$ may always be equipped with a left-invariant Riemannian metric g for which $N \times \{0\}$ and $\{1_N\} \times \mathbb{R}$ are orthogonal and the maximum of sectional curvatures is -1. Denoting by d_g the distance function induced by such a metric tensor g, we shall show that the parabolic visual boundary of (G, d_g) may be identified with the Lie group N and that, under such an identification, N inherits a homogeneous distance. The result is well known, we only record it here for completeness.

The vertical line with the support $\{1_N\} \times \mathbb{R}$ is length minimising between any of its points. Indeed, by orthogonality of $\mathbb{R}$ and N, every path with a non-trivial component of its derivative on $T_{1_N}N \times \{0\}$ will have a non-zero contribution to its length-integral coming from this component. Hence, using also the left-invariance of the distance, we deduce that there is some $\lambda > 0$ so that all the curves of the form $s \mapsto (n, \lambda s)$, where $n \in N$, are isometric embeddings. Let $\xi\colon [0,\infty[\, \to G$ be the curve $s \mapsto (1_N, \lambda s)$. We may follow [27, p. 384] and define the *parabolic visual boundary* of (G, d_g), denoted by $\partial_\infty(G, d_g)$, to be the set of isometric embeddings $\gamma\colon \mathbb{R} \to (G, d_g)$ that satisfy

$$\lim_{s\to\infty} d_g(\gamma(s), \xi(s)) = 0\,. \tag{1}$$

The parabolic visual boundary is then equipped with so called Hamenstädt distance

$$d(\sigma, \gamma) = \exp(-\tfrac{1}{2} \lim_{s\to\infty} (2s - d_g(\sigma(-s), \gamma(-s))))\,.$$

Next we argue how the Lie group N may be identified with $\partial_\infty(G, d_g)$. For one direction, to each element $n \in N$ we associate the infinite geodesic $\gamma(s) = (n, \lambda s)$. For this to be well defined, one needs to verify the condition (1). Using the group law of the semidirect product and the left-invariance of g, and denoting by φ_s the automorphism of N with differential $\mathrm{e}^{s\alpha}$, we may calculate for any $s_0 \in \mathbb{R}$ that

$$\begin{aligned}
\lim_{s\to\infty} d_g((n, \lambda s + s_0), (1_N, \lambda s)) &= \lim_{s\to\infty} d_g((1_N, \lambda s) * (\varphi_{-\lambda s} n, s_0), (1_N, \lambda s)) \\
&= \lim_{s\to\infty} d_g(((\varphi_{-\lambda s} n, s_0), (1_N, 0)) \\
&= d_g((1_N, s_0), (1_N, 0)) = s_0/\lambda\,.
\end{aligned} \tag{2}$$

In particular, putting $s_0 = 0$, the curve γ satisfies (1).

For the other direction, we consider $\gamma \in \partial_\infty(G, d_g)$ and denote by $(n, s_0) \in N \rtimes_\alpha \mathbb{R}$ the point $\gamma(0)$. If we can show $s_0 = 0$, i.e., $\gamma(0) \in N \times \{0\}$, then we have a natural map $\partial_\infty(G, d_g) \to N$. Let σ be the infinite geodesic $s \mapsto (n, \lambda s + s_0)$. First, the computation (2) shows that

$$\lim_{s\to\infty} d(\sigma(s), \xi(s)) = s_0/\lambda\,.$$

We find

$$0 \le \lim_{s\to\infty} d(\sigma(s), \gamma(s)) \le \lim_{s\to\infty} d(\sigma(s), \xi(s)) + \lim_{s\to\infty} d(\xi(s), \gamma(s)) \le s_0/\lambda\,.$$

Thus the triangle formed by the points $\gamma(0) = \sigma(0)$, $\sigma(s)$ and $\gamma(s)$ has two sides of length s and one at most s_0/λ. Because the space (G, d_g) is CAT(-1) and s can be taken arbitrarily large, the triangle has to be degenerate, i.e., $\gamma = \sigma$.

Using the correspondence above, we see d as an admissible left-invariant distance function on N (for admissibility, [33, Theorem A] can be used). A computation analogous to (2) gives

$$\begin{aligned}
&d(\varphi_t n, \varphi_t n') = \exp(-\tfrac{1}{2}\lim_{s\to\infty}(2s - d_g((\varphi_t n, -\lambda s), (\varphi_t n', -\lambda s))))\\
&= \exp(-\tfrac{1}{2}\lim_{s\to\infty}(2s - d_g((1_N, t) * (n, -\lambda s - t), (1_N, t) * (n', -\lambda s - t))))\\
&= \exp(-\tfrac{1}{2}\lim_{s\to\infty}(2s - d_g((n, -\lambda s - t), (n', -\lambda s - t))))\\
&= \exp(-\tfrac{1}{2}\lim_{h\to\infty}(2(h - t/\lambda) - d_g((n, -\lambda h), (n', -\lambda h))))\\
&= \mathrm{e}^{t/\lambda}\exp(-\tfrac{1}{2}\lim_{h\to\infty}(2h - d_g((n, -\lambda h), (n', -\lambda h))))\\
&= \mathrm{e}^{t/\lambda} d(n, n')\,.
\end{aligned}$$

Notice that by writing $\psi_t = \varphi_{\log(t)}$ for $t > 0$ we got the formula $d(\psi_t n, \psi_t n') = t^{1/\lambda} d(n, n')$ for all $n, n' \in N$. Moreover, defining $\delta_t = \psi_{t^\lambda}$ the differentials satisfy $(\delta_t)_* = (\psi_{t^\lambda})_* = \mathrm{e}^{\lambda \log(t)\alpha}$, and thus δ_t is the one-parameter subgroup of automorphisms associated to the derivation $\lambda\alpha$ and we have $d(\delta_t n, \delta_t n') = t d(n, n')$ for all $n, n' \in N$. We thus proved that under the identification of N with the parabolic visual boundary, the Hamenstädt distance makes $(N, \lambda\alpha)$ into a homogeneous metric group.

4.3. **Quasi-isometric invariants for Heintze groups.** When attacking Conjecture 4.4, one may always do the analysis in the level of parabolic visual boundaries; This is demonstrated by the following well known fact for which the references are recorded in [6, p. 6], see also [C, Proposition 1.5].

Proposition 4.5. *Let (N_1, α) and (N_2, β) be homogeneous groups. Then Heintze groups $N_1 \rtimes_\alpha \mathbb{R}$ and $N_2 \rtimes_\beta \mathbb{R}$ are quasi-isometric if and only if there exists $\lambda_1, \lambda_2 > 0$ so that $(N_1, \lambda_1\alpha)$ and $(N_2, \lambda_2\beta)$ are biLipschitz equivalent.*

Inspired by the results and ideas of [6], we proved in article [C] the following theorem.

Theorem 4.6 (with Le Donne and Nicolussi Golo, in [C])**.** *Let (N_1, α) and (N_2, β) be purely real homogeneous groups that are biLipschitz equivalent. Then N_1 and N_2 are quasi-isometric as Riemannian Lie groups. As a consequence, the associated Carnot groups of N_1 and N_2 are isomorphic by* [37].

The last conclusion in the above theorem is a strong algebraic similarity between N_1 and N_2. It implies for example that the nilpotency steps of N_1 and N_2 agree. In particular, if N_1 is Abelian, then N_2 must also be Abelian; This observation completes, in a sense, the quasi-isometric classification of Heintze groups with Abelian nilradicals due to X. Xie. Namely, it is proven in [43] that if two purely real Heintze groups with Abelian nilradicals are quasi-isometric, then the Jordan-forms of the associated derivations must agree up to scalar multiple, and hence the two Heintze groups are isomorphic. With Theorem 4.6 we rule out the possibility that there could exist a quasi-isometry between two Heintze groups out of which only one has Abelian nilradical.

Since quasi-isometries of Heintze groups correspond to biLipschitz maps of homogeneous group as in Proposition 4.5, then biLipschitz invariants can be used to get information about quasi-isometries. However, it is a priori non-trivial which biLipschitz invariants are practically computable and hence useful. One invariant that we investigated is the set of those points that can be reached by curves starting from the identity element and having Hausdorff-dimension less or equal to s, for some fixed $s \geq 1$ (notice that curves have always Hausdorff-dimension at least 1). Formally, when (N, α) is a homogeneous group, we denote

$$R(s) = \{\gamma(1) : \gamma \in \mathcal{C}^0([0,1], N),\ \gamma(0) = 1_N,\ \mathcal{H}\text{-dim}(\gamma([0,1])) \leq s\}.$$

The method to compute this set algebraically, and hence practically, should not come as a surprise, although it is likely a new result. Such a set may be computed as the subgroup of N, denoted by $(N, \alpha)^{(s)}$, corresponding to the subalgebra $\mathrm{LieSpan}(\bigoplus_{0<\lambda\leq s} V_\lambda)$. Here $\mathrm{Lie}(N) = \bigoplus_{\lambda>0} V_\lambda$ is the decomposition of $\mathrm{Lie}(N)$ by the generalised eigenspaces of the derivation α.

Theorem 4.7 (with Le Donne and Nicolussi Golo, in [C])**.** *Let (N, α) be a purely real homogeneous group. Then $R(s) = (N, \alpha)^{(s)}$ for every $s \geq 1$.*

We give examples of the usefulness of this result in Section 2.3 of article [C]. The Examples 2.3 and 2.6 in [C] illustrate the situation where these new invariants are not enough but more refined invariants need to be found.

References

[1] D. V. Alekseevskiĭ. Homogeneous Riemannian spaces of negative curvature. *Mat. Sb. (N.S.)*, 96(138):93–117, 168, 1975.

[2] L. Auslander and L. W. Green. G-induced flows. *Amer. J. Math.*, 88:43–60, 1966.

[3] V.N. Berestovskii. Homogeneous manifolds with intrinsic metric. i. *Siberian Mathematical Journal*, 29(6):887–897, 1988.

[4] Emmanuel Breuillard. Geometry of locally compact groups of polynomial growth and shape of large balls. *Groups Geom. Dyn.*, 8(3):669–732, 2014.

[5] Rutwig Campoamor-Stursberg. Generalized casimir invariants of six dimensional solvable real lie algebras with five dimensional nilradical. *Preprint, arXiv:math/0206210v2*, 2002.

[6] Matias Carrasco Piaggio and Emiliano Sequeira. On quasi-isometry invariants associated to a Heintze group. *Geom. Dedicata*, 189:1–16, 2017.

[7] Michael G. Cowling, Ville Kivioja, Enrico Le Donne, Sebastiano Nicolussi Golo, and Alessandro Ottazzi. From homogeneous metric spaces to lie groups. *arXiv preprint arXiv:1705.09648v2*, 2017.

[8] Yves de Cornulier. Gradings on Lie algebras, systolic growth, and cohopfian properties of nilpotent groups. *Bull. Soc. Math. France*, 144(4):693–744, 2016.

[9] Yves de Cornulier. On the quasi-isometric classification of locally compact groups. In *New directions in locally compact groups*, volume 447 of *London Math. Soc. Lecture Note Ser.*, pages 275–342. Cambridge Univ. Press, Cambridge, 2018.

[10] Yves de Cornulier. On sublinear bilipschitz equivalence of groups. *Ann. ENS*, 52(5):1201–1242, 2019.

[11] Yves de Cornulier and Pierre de la Harpe. *Metric geometry of locally compact groups*, volume 25 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2016. Winner of the 2016 EMS Monograph Award.

[12] Willem A. de Graaf. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. *J. Algebra*, 309(2):640–653, 2007.

[13] Enrico Le Donne. A primer on carnot groups: Homogenous groups, carnot-carathéodory spaces, and regularity of their isometries. *Analysis and Geometry in Metric Spaces*, 5(1):116–137, 2017.

[14] Nick Dungey, A. F. M. ter Elst, and Derek W. Robinson. *Analysis on Lie groups with polynomial growth*, volume 214 of *Progress in Mathematics*. Birkhäuser Boston, 2003.

[15] Tullia Dymarz. Large scale geometry of certain solvable groups. *Geom. Funct. Anal.*, 19(6):1650–1687, 2010.

[16] Alex Eskin, David Fisher, and Kevin Whyte. Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs. *Ann. of Math. (2)*, 176(1):221–260, 2012.

[17] Alex Eskin, David Fisher, and Kevin Whyte. Coarse differentiation of quasi-isometries II: Rigidity for Sol and lamplighter groups. *Ann. of Math. (2)*, 177(3):869–910, 2013.

[18] Benson Farb and Lee Mosher. Problems on the geometry of finitely generated solvable groups. In *Crystallographic groups and their generalizations (Kortrijk, 1999)*, volume 262 of *Contemp. Math.*, pages 121–134. Amer. Math. Soc., Providence, RI, 2000.

[19] Katrin Fässler and Enrico Le Donne. On the quasi-isometric and bilipschitz classification of 3d riemannian lie groups. *Geometriae Dedicata*, 210:27–42, 2021.

[20] Ming-Peng Gong. *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and R)*. ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–University of Waterloo (Canada).

[21] C. Gordon. Transitive riemannian isometry groups with nilpotent radicals. *Annales de l'Institut Fourier*, 31(2):193–204, 1981.

[22] Carolyn S. Gordon and Edward N. Wilson. Isometry groups of Riemannian solvmanifolds. *Trans. Amer. Math. Soc.*, 307(1):245–269, 1988.

[23] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, 53:53–73, 1981.
[24] Eero Hakavuori, Ville Kivioja, Terhi Moisala, and Francesca Tripaldi. ehaka/lie-algebra-gradings: v1.0, November 2020. https://doi.org/10.5281/zenodo.4267884.
[25] Ursula Hamenstädt. Some regularity theorems for carnot-carathéodory metrics. *Journal of Differential Geometry*, 32(3):819–850, 1990.
[26] Waldemar Hebisch and Adam Sikora. A smooth subadditive homogeneous norm on a homogeneous group. *Studia Math.*, 96(3):231–236, 1990.
[27] Sa'ar Hersonsky and Frédéric Paulin. On the rigidity of discrete isometry groups of negatively curved spaces. *Comment. Math. Helv.*, 72(3):349–388, 1997.
[28] Michael Jablonski. Maximal symmetry and unimodular solvmanifolds. *Pacific J. Math.*, 298(2):417–427, 2019.
[29] J. W. Jenkins. Growth of connected locally compact groups. *J. Functional Analysis*, 12:113–127, 1973.
[30] Iwao Kishimoto. Geodesics and isometries of carnot groups. *Journal of Mathematics of Kyoto University*, 43(3):509–522, 2003.
[31] Mikhail Kochetov. Gradings on finite-dimensional simple Lie algebras. *Acta Appl. Math.*, 108(1):101–127, 2009.
[32] Enrico Le Donne. A metric characterization of Carnot groups. *Proc. Amer. Math. Soc.*, 143(2):845–849, 2015.
[33] Enrico Le Donne and Sebastiano Nicolussi Golo. Metric lie groups admitting dilations. *Preprint, arXiv:1901.02559, To appear in Arkiv för Matematik*, 2019.
[34] Enrico Le Donne and Alessandro Ottazzi. Isometries of carnot groups and sub-finsler homogeneous manifolds. *The Journal of Geometric Analysis*, pages 1–16, 2014.
[35] A. I. Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Translation*, 1951(39):33, 1951.
[36] Pierre Pansu. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems*, 3(3):415–445, 1983.
[37] Pierre Pansu. Métriques de carnot-carathéodory et quasiisométries des espaces symétriques de rang un. *Annals of Mathematics*, pages 1–60, 1989.
[38] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus. Invariants of real low dimension Lie algebras. *J. Mathematical Phys.*, 17(6):986–994, 1976.
[39] R. Sauer. Homological invariants and quasi-isometry. *Geometric & Functional Analysis GAFA*, 16(2):476–515, Apr 2006.
[40] Yehuda Shalom. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.*, 192(2):119–185, 2004.
[41] Eberhard Siebert. Contractive automorphisms on locally compact groups. *Math. Z.*, 191(1):73–90, 1986.
[42] Joseph A. Wolf. On locally symmetric spaces of non-negative curvature and certain other locally homogeneous spaces. *Comment. Math. Helv.*, 37:266–295, 1962/1963.
[43] Xiangdong Xie. Large scale geometry of negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$. *Geom. Topol.*, 18(2):831–872, 2014.

Included articles

[A]

Isometries of nilpotent metric groups

Ville Kivioja and Enrico Le Donne

First published in
Journal de l'École Polytechnique – Mathématiques, **4** (2017), 473–482

ISOMETRIES OF NILPOTENT METRIC GROUPS

VILLE KIVIOJA AND ENRICO LE DONNE

Abstract. We consider Lie groups equipped with arbitrary distances. We only assume that the distances are left-invariant and induce the manifold topology. For brevity, we call such objects metric Lie groups. Apart from Riemannian Lie groups, distinguished examples are sub-Riemannian Lie groups, homogeneous groups, and, in particular, Carnot groups equipped with Carnot–Carathéodory distances. We study the regularity of isometries, i.e., distance-preserving homeomorphisms. Our first result is the analyticity of such maps between metric Lie groups. The second result is that if two metric Lie groups are connected and nilpotent then every isometry between the groups is the composition of a left translation and an isomorphism. There are counterexamples if one does not assume the groups to be either connected or nilpotent. The first result is based on a solution of the Hilbert's fifth problem by Montgomery and Zippin. The second result is proved, via the first result, reducing the problem to the Riemannian case, which was essentially solved by Wolf.

Contents

Date: March 23, 2017.
2010 *Mathematics Subject Classification.* 22E25, 53C30, 22F30.
Key words and phrases. Isometries, nilpotent groups, affine transformations, nilradical.

1. Introduction

In this paper, with the term *metric Lie group* we mean a Lie group equipped with a left-invariant distance that induces the manifold topology. An *isometry* is a distance-preserving bijection. Hence, a priori it is only a homeomorphism. As a general fact we show the following regularity result.

Theorem 1.1. *Isometries between metric Lie groups are analytic maps.*

We say that a map between groups is *affine* if it is the composition of a left translation and a group homomorphism. For nilpotent groups we have the following stronger result.

Theorem 1.2. *Isometries between nilpotent connected metric Lie groups are affine.*

In particular we have that

[1.2.i] two isometric nilpotent connected metric Lie groups are isomorphic;
[1.2.ii] given a connected metric Lie group N, its isometry group $\mathrm{Isom}(N)$, which always is a Lie group, is a semidirect product if N is nilpotent. Namely,

$$\mathrm{Isom}(N) = N \rtimes \mathrm{AutIsom}(N),$$

where N is seen inside $\mathrm{Isom}(N)$ as left translations and $\mathrm{AutIsom}(N)$ denotes the group of automorphisms of N that are isometries.

Moreover, with the above notation, we have

[1.2.iii] N is a maximal connected nilpotent subgroup of $\mathrm{Isom}(N)$ and the Lie algebra of N is the nilradical of the Lie algebra of $\mathrm{Isom}(N)$, see Section 3.2.

Theorem 1.2 is a generalization of previous results. On the one hand, in the case of nilpotent Lie groups equipped with left-invariant *Riemannian* distances the result is essentially known from the work of Wolf, see [Wol63, Wil82] and Remark 3.3. On the other hand, Theorem 1.2 has been shown in the case of Carnot groups equipped with Carnot–Carathéodory distances, see [Pan89, Ham90, Kis03, LO16]. In fact our strategy of proofs is built on both [Wol63] and [LO16].

Examples of groups not considered before are sub-Riemannian, and more generally sub-Finsler, groups that are not Carnot groups (i.e., the sub-Riemannian structure is not coming from the first layer of a stratification), together with their subgroups, and their snowflakes. Other examples are given by the Heisenberg group equipped with the Korányi gauge and, more generally, by any other homogeneous group (in the sense of Folland and Stein), i.e., a graded group equipped with a homogeneous norm, see more in [LN16, LR17].

We remark that both assumptions 'connectedness' and 'nilpotency' are necessary for Theorem 1.2 to hold. In this respect in Section 4 we provide some counterexamples.

The large-scale analogue of Theorem 1.2 is a challenging open problem that has raised a lot of attention since the papers of Pansu and Shalom [Pan89, Sha04]. What is expected is that if two finitely generated nilpotent groups are torsion-free, then every quasi-isometry between them induces an isomorphism between their Malcev completions. The quasi-isometric classification of locally compact groups is also a very active area, see the (quasi-)survey [Cor15].

We spend the rest of the introduction to explain the strategy of the proofs of the two theorems and the structure of the paper. To study isometries between two metric Lie groups, we first treat the case when the two groups are the same, i.e., they are isometric via a Lie group isomorphism. If M is a connected metric Lie group, we consider its isometry group G, that is, the set of self-isometries of M equipped with the composition rule and the compact-open topology. Hence, the group G acts continuously, transitively and by isometries on M. It is crucial that G is a locally compact group. This latter fact follows from Ascoli–Arzelà Theorem but it needs some argument since closed balls are not necessarily assumed to be compact. At this point, the theory of locally compact groups, [MZ74], provides a Lie group structure on G such that the action $G \curvearrowright M$ is smooth, see Section 2.1.

Assume that M_1, M_2 are metric Lie groups and $F\colon M_1 \to M_2$ is an isometry. We consider the above-mentioned Lie group structures on the respective isometry groups G_1, G_2. The conjugation by F provides a map from G_1 to G_2 that is a continuous homomorphism between Lie groups, hence it is analytic. This observation will give the conclusion of the proof of Theorem 1.1, see Section 2.2.

An important consequence of Theorem 1.1 is that every isometry between metric Lie groups can be seen as a Riemannian isometry. Namely, for every map $F\colon M_1 \to M_2$ as above there are Riemannian left-invariant structures g_1, g_2 such that $F\colon (M_1, g_1) \to (M_2, g_2)$ is a Riemannian isometry, see Proposition 2.4. Of a separate interest is the fact that the Riemannian structures can be chosen independently of F. Together with Wolf's study of nilpotent Riemannian Lie groups, Theorem 1.2 and the other statements now follow.

We also show that if M is a group equipped with a left-invariant distance, then its isometries are affine if and only if its isometry group G splits as semi-direct product

$$G = M \rtimes \mathrm{Stab}_1(G),$$

where $\mathrm{Stab}_1(G)$ is the set of isometries fixing the identity element 1 of M. We provide the simple proof in Lemma 3.2.

Acknowledgement. The authors thank M. Jablonski, A. Ottazzi, P. Petersen, and the referee for helpful discussions. In particular, Remark 3.3 is due to the referee. E.L.D. acknowledges the support of the Academy of Finland project no. 288501.

2. Regularity of isometries

2.1. **Lie group structure of isometry groups.** The first aim of this section is to show that the isometry group of a metric Lie group is a Lie group. Such a fact is a consequence of the solution of the Hilbert's fifth problem by Montgomery–Zippin, together with the observation that the isometry group is locally compact. This latter property follows by Ascoli–Arzelà Theorem.

We stress that a metric Lie group (M, d) may not be boundedly compact. Namely, the closed balls $\bar{B}_d(1_M, r) := \{p \in M : d(p, 1_M) \leq r\}$ with respect to d may not be compact. For example, this is the case for the distance $\min\{d_E, 1\}$ on $\mathbb{R}$, where d_E denotes the Euclidean distance.

Remark 2.1. If (M, d) is a connected metric Lie group, then there exists a distance ρ such that (M, ρ) is a metric Lie group that is boundedly compact and $\mathrm{Isom}(M, d) \subseteq \mathrm{Isom}(M, \rho)$. Indeed, since the topology induced by d is the manifold topology, then there exists some $r_0 > 0$ such that $\bar{B}_d(1_M, r_0)$ is compact. Then we can consider the distance

$$\rho(p, q) := \inf\{\sum_{i=1}^{k} d(p_{i-1}, p_i) \, : \, k \in \mathbb{N}, \, p_i \in M, \, p_0 = p, \, p_k = q, \, d(p_{i-1}, p_i) \leq r_0\}.$$

Once can check that (M, ρ) is a metric Lie group, for all $r > 0$ the set $\bar{B}_\rho(1_M, r)$ is compact, and $\mathrm{Isom}(M, d) \subseteq \mathrm{Isom}(M, \rho)$.

Let us clarify now why the isometry group of a connected metric Lie group is locally compact, which was not justified in [LO16]. With the terminology of Remark 2.1 the stabilizer S of 1 in $\mathrm{Isom}(M, d)$ is a closed subgroup of the stabilizer S_ρ of 1 in $\mathrm{Isom}(M, \rho)$. Furthermore, for any $r > 0$ and $f \in S_\rho$ we have that $f(\bar{B}_\rho(1, r)) = \bar{B}_\rho(1, r)$, which is compact. Hence, the maps from S restricted to $\bar{B}(1, r)$ form an equi-uniformly continuous and pointwise precompact family. Ascoli–Arzelà Theorem implies that S_ρ is compact, being also closed in $C^0(M, M)$. Consequently, S is compact and because M is locally compact, then also $\mathrm{Isom}(G, d)$ is locally compact. At this point we are allowed to use the theory of locally compact groups after Gleason–Montgomery–Zippin [MZ74]. In fact, the argument in [LO16, Proposition 4.5] concludes the proof of the following result.

Proposition 2.2. *Let M be a metric Lie group with isometry group G. Assume that M is connected.*

(1) *The stabilizers of the action $G \curvearrowright M$ are compact.*
(2) *The topological group G is a Lie group (finite dimensional and with finitely many connected components) acting analytically on M.*

Remark 2.3. The assumption of M being connected in Proposition 2.2 is necessary. Indeed, one can take as a counterexample the group $\mathbb{Z}$ with the discrete distance.

2.2. **Proof of smoothness.** With the use of Proposition 2.2, we give the proof of the analyticity of isometries (Theorem 1.1). We remark that in the Riemannian setting the classical result of Myers and Steenrod gives smoothness of isometries, see [MS39], and more generally [CL16]. However, the following proof is different in spirit and, nonetheless, it will imply (see Proposition 2.4) that such metric isometries are Riemannian isometries for some Riemannian structures.

Proof of Theorem 1.1. Let $F\colon M_1 \to M_2$ be an isometry between metric Lie groups. Without loss of generality we may assume that $F(1_{M_1}) = 1_{M_2}$ and that both M_1 and M_2 are connected, since left translations are analytic isometries and connected components of identity elements are open. By Proposition 2.2, for $i \in \{1, 2\}$, the space $G_i := \mathrm{Isom}(M_i)$ is a Lie group. The map $\mathrm{C}_F\colon G_1 \to G_2$ defined as $I \mapsto F \circ I \circ F^{-1}$ is a group isomorphism that is continuous, see [Are46, Theorem 4]. Hence, the map C_F is analytic, see [Hel01, p. 117, Theorem 2.6].

Consider also the inclusion $\iota\colon M_1 \to G_1$, $m \mapsto L_m$, which is analytic being a continuous homomorphism, and the orbit map $\sigma\colon G_2 \to M_2$, $I \mapsto I(1_{M_2})$, which is analytic since the action is analytic (Proposition 2.2). We deduce that $\sigma \circ \mathrm{C}_F \circ \iota$ is analytic. We claim that this map is F. Indeed, for any $m \in M_1$ it holds

$$(\sigma \circ \mathrm{C}_F \circ \iota)(m) = \sigma(F \circ L_m \circ F^{-1}) = (F \circ L_m \circ F^{-1})(1_{M_2}) = F(m). \qquad \square$$

2.3. **Isometries as Riemannian isometries.** We show next that isometries between metric Lie groups are actually Riemannian isometries for some left-invariant structures. Let us point out that when M is a Lie group and g is a left-invariant Riemannian metric tensor on g, then one has an induced Riemannian distance d_g and, by the theorem of Myers and Steenrod [MS39], the group $\mathrm{Isom}(M, d_g)$ of distance-preserving bijections coincides with the group $\mathrm{Isom}(M, g)$ of tensor-preserving diffeomorphisms. In what follows we shall write (M, g) to denote the metric Lie group (M, d_g).

Proposition 2.4. *If (M_1, d_1) and (M_2, d_2) are connected metric Lie groups, then there exists left-invariant Riemannian metrics g_1 and g_2 on M_1 and M_2, respectively, such that $\mathrm{Isom}(M_i, d_i) \subseteq \mathrm{Isom}(M_i, g_i)$ for $i \in \{1, 2\}$ and for all isometries $F\colon (M_1, d_1) \to (M_2, d_2)$ the map $F\colon (M_1, g_1) \to (M_2, g_2)$ is a Riemannian isometry.*

Let us first deal with the case $(M_1, d_1) = (M_2, d_2)$.

Lemma 2.5. *If (M, d) is a connected metric Lie group, then there is a Riemannian metric g such that $\mathrm{Isom}(M, d) \subseteq \mathrm{Isom}(M, g)$.*

Proof of Lemma 2.5. Fix a scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ on the tangent space T_1M at the identity 1 of M. From Proposition 2.2, the stabilizer S of 1 in $\mathrm{Isom}(M, d)$ is compact and acts smoothly on M. Let μ_S be the probability Haar measure on S. Consider for $v, w \in T_1 M$

$$\langle v, w\rangle := \int_S \langle\langle \mathrm{d}Fv, \mathrm{d}Fw\rangle\rangle \mathrm{d}\mu_S(F).$$

Then $\langle\cdot,\cdot\rangle$ defines an S-invariant scalar product on T_1M, and one can take g as the left-invariant Riemannian metric that coincides with $\langle\cdot,\cdot\rangle$ at the identity. $\square$

Proof of Proposition 2.4. By Lemma 2.5 let g_2 be a Riemannian metric on M_2 with

$$\mathrm{Isom}(M_2, d_2) \subseteq \mathrm{Isom}(M_2, g_2). \tag{2.6}$$

Fix $F\colon (M_1, d_1) \to (M_2, d_2)$ an isometry. By Theorem 1.1 the map F is smooth, and we may define a Riemannian metric on M_1 by $g_1 := F^*g_2$. There are two things to check: a) $\mathrm{Isom}(M_1, d_1) \subseteq \mathrm{Isom}(M_1, g_1)$, which in particular gives that g_1 is left-invariant and b) every isometry $H\colon (M_1, d_1) \to (M_2, d_2)$ is an isometry of Riemannian manifolds.

For part a, since by construction F is also a Riemannian isometry, the map $I \mapsto F \circ I \circ F^{-1}$ is a bijection between $\mathrm{Isom}(M_1, d_1)$ and $\mathrm{Isom}(M_2, d_2)$ and between $\mathrm{Isom}(M_1, g_1)$ and $\mathrm{Isom}(M_2, g_2)$. Therefore the inclusion (2.6) implies the inclusion $\mathrm{Isom}(M_1, d_1) \subseteq \mathrm{Isom}(M_1, g_1)$.

For part b, since $H \circ F^{-1} \in \mathrm{Isom}(M_2, d_2) \subseteq \mathrm{Isom}(M_2, g_2)$, then $(H \circ F^{-1})^*g_2 = g_2$. Consequently, we get $H^*g_2 = F^*(H \circ F^{-1})^*g_2 = g_1$. $\square$

3. Affine decomposition

3.1. **Preliminary lemmas.** Given a group M we denote by M^L the group of left translations on M. The following two results make sense in the settings of groups equipped with left-invariant distances. We call such groups *metric groups.*

Lemma 3.1. *Let M_1 and M_2 be metric groups. Suppose $F\colon M_1 \to M_2$ is an isometry and $F \circ M_1^L \circ F^{-1} = M_2^L$. Then F is affine.*

Proof. Up to precomposing with a translation, we assume that $F(1_{M_1}) = 1_{M_2}$. So we want to prove that F is an isomorphism. The map $\mathrm{C}_F\colon \mathrm{Isom}(M_1) \to \mathrm{Isom}(M_2)$, $I \mapsto F \circ I \circ F^{-1}$, is an isomorphism and by assumption it gives an isomorphism between M_1^L and M_2^L. We claim that F is the same isomorphism when identifying M_i with M_i^L. Namely, we want to show that for all $m \in M_1$ we have $L_{F(m)} = \mathrm{C}_F(L_m)$. By assumption for all $m_1 \in M_1$ exists $m_2 \in M_2$ such that $L_{m_2} = \mathrm{C}_F(L_{m_1})$. Evaluating at 1_{M_2}, we get

$$m_2 = L_{m_2}(1_{M_2}) = \mathrm{C}_F(L_{m_1})(1_{M_2}) = F(L_{m_1}(F^{-1}(1_{M_2}))) = F(m_1). \qquad \square$$

With the next result we clarify that the condition of self-isometries being affine is equivalent to left translations being a normal subgroup of the group of isometries. Equivalently, we have a semi-direct product decomposition of the isometry group. Namely, given a metric group M and denoting by G the isometry group and by

$\mathrm{Stab}_1(G)$ the stabilizer of the identity element, we have that M has affine isometries if and only if $G = M^L \rtimes \mathrm{Stab}_1(G)$. We denote by $\mathrm{Aff}(M)$ the group of affine maps from M to M and by $\mathrm{Aut}(M)$ the group of automorphisms of M.

Lemma 3.2. *Let M be a metric group with isometry group G. Then the following are equivalent:*

(a) $M^L \lhd G$*, i.e.,* $F \circ M^L \circ F^{-1} = M^L$*, for all* $F \in G$*;*
(b) $G < \mathrm{Aff}(M)$*;*
(c) $\mathrm{Stab}_1(G) < \mathrm{Aut}(M)$*;*
(d) $G = M^L \rtimes \mathrm{Stab}_1(G)$*;*
(e) $G = M^L \rtimes (G \cap \mathrm{Aut}(M))$*.*

Proof. Property (a) implies (b) by Lemma 3.1. Regarding the fact that (b) implies (a), consider a map $F \in G$, which we know is of the form $F = \tau \circ \Phi$ with $\tau \in M^L$ and $\Phi \in \mathrm{Aut}(M)$. For all $p \in M$ we get

$$\begin{aligned} F \circ L_p \circ F^{-1} &= (\tau \circ \Phi) \circ L_p \circ (\tau \circ \Phi)^{-1} \\ &= \tau \circ \Phi \circ L_p \circ \Phi^{-1} \circ \tau^{-1} \\ &= \tau \circ L_{\Phi(p)} \circ \Phi \circ \Phi^{-1} \circ \tau^{-1} \\ &= \tau \circ L_{\Phi(p)} \circ \tau^{-1} \in M^L, \end{aligned}$$

which gives $M^L \lhd G$.

The equivalence of (b) with (c) is trivial. The equivalence of (a) with (d) follows from the facts $M^L \cdot \mathrm{Stab}_1(G) = G$ and $M^L \cap \mathrm{Stab}_1(G) = \{\mathrm{id}\}$. Finally, (e) implies (d), and (e) is implied by (d) together with (c). ☐

Remark 3.3. As said in the introduction, Theorem 1.2 is essentially due to Wolf in the Riemannian setting. Indeed, in [Wol63, p. 278, Theorem 4.2] he proved the semi-direct product decomposition of the isometry group of a Riemannian nilpotent Lie group, which is equivalent to self-isometries being affine, as in the lemma above. To conclude that an isometry $F\colon N_1 \to N_2$ between Riemannian nilpotent Lie groups is affine one considers the self-isometry of the product $N_1 \times N_2$ given by $(n, m) \mapsto (F^{-1}(m), F(n))$. Also, one can check that the proof of [Wil82, Theorem 3] gives the same result.

3.2. **Theorem 1.2 from Proposition 2.4.** For every Riemannian nilpotent Lie group Wolf proved a characterization of the group inside its isometry group. In fact, he described the nilpotent group as the nilradical of its isometry group. We shall give the same characterization in the general setting. We introduce some terminology inspired by [Wol63, Wil82, GW88].

Definition 3.4 (Nilradical condition)**.** Let $\mathfrak{g}$ be a Lie algebra. The *nilradical* of $\mathfrak{g}$, denoted by $\mathrm{nil}(\mathfrak{g})$, is the largest nilpotent ideal of $\mathfrak{g}$. We say that a connected metric Lie group N with isometry group $\mathrm{Isom}(N)$ satisfies *the nilradical condition* if it holds

$$\mathrm{Lie}(N^L) = \mathrm{nil}(\mathrm{Lie}(\mathrm{Isom}(N))). \tag{3.5}$$

Clearly, a metric Lie group N can satisfy the nilradical condition only if it is nilpotent. The nilradical of a Lie algebra $\mathfrak{g}$ can also be defined as the sum of all nilpotent ideals of $\mathfrak{g}$, see [HN12, Definition 5.2.10].

Remark 3.6. The nilradical condition is satisfied by Riemannian nilpotent Lie groups, where the distance is induced by a left-invariant metric tensor. Such a result was proved by Wolf [Wol63, p. 278, Theorem 4.2], see also [Wil82, p. 341 Theorem 2]. Actually, Wolf proved the stronger statement that such a group N is a maximal connected nilpotent subgroup inside $\mathrm{Isom}(N)$, which implies the nilradical condition since $N^L \lhd \mathrm{Isom}(N)$. Clearly, there may be several maximal connected nilpotent subgroups inside $\mathrm{Isom}(N)$.

The nilradical condition is an algebraic characterization of the Lie algebra of a nilpotent metric Lie group inside the Lie algebra of its isometry group. Hence, by Lemma 3.1 it is clear that if two connected metric Lie groups N_1 and N_2 satisfy the nilradical condition (3.5), then any isometry $F\colon N_1 \to N_2$ is affine. Indeed, the map $I \mapsto F \circ I \circ F^{-1}$ induces a Lie algebra isomorphism between $\mathrm{Lie}(\mathrm{Isom}(N_1))$ and $\mathrm{Lie}(\mathrm{Isom}(N_2))$, and therefore, since the exponential map is surjective, one concludes that the map sends N_1^L to N_2^L.

We also mention that the work of Wolf, together with the work of Gordon and Wilson, is one of the initial steps in the study of (Riemannian) nilmanifolds, solvmanifolds, and homogeneous Ricci solitons, see [GW88, Jab15a, Jab15b].

Proof of Theorem 1.2. Let $F\colon (N_1, d_1) \to (N_2, d_2)$ be an isometry between two nilpotent connected metric Lie groups. By Proposition 2.4 for $i \in \{1, 2\}$ there exist left-invariant metric tensors g_i on N_i such that $F\colon (N_1, g_1) \to (N_2, g_2)$ is a Riemannian isometry. By Remark 3.3, the map F is affine. In particular, we have [1.2.i].

Because of Lemma 3.2 we also deduce that the isometry group of a nilpotent connected metric Lie group N has the semi-direct product decomposition [1.2.ii]. Regarding [1.2.iii], given such a group N we use again Proposition 2.4 and have that $N \subseteq \mathrm{Isom}(N) \subseteq \mathrm{Isom}(N, g)$, for some left-invariant metric tensor g on N. By Remark 3.6, the group N^L is a maximal connected nilpotent subgroup inside $\mathrm{Isom}(N, g)$, thus also inside $\mathrm{Isom}(N)$. Since from [1.2.ii] we have $N^L \lhd \mathrm{Isom}(N)$, so $\mathrm{Lie}(N^L)$ is an ideal of $\mathrm{Lie}(\mathrm{Isom}(N))$. Thus, by the maximalitity of N, we deduce the nilradical condition (3.5). □

4. Examples for the sharpness of the assumptions

In this section we provide several examples to illustrate the sharpness of the assumptions in Theorem 1.2. Namely, we show that if one of the groups is not assumed connected and nilpotent then there may be isometries that are not affine.

Regarding the connectedness assumption, there are examples of Abelian metric Lie groups with finitely many components for which some isometries are not affine. One of the simplest examples is the subgroup of $\mathbb{C}$ consisting of the four points $\{1, \mathrm{i}, -1, -\mathrm{i}\}$ equipped with the discrete distance. Here every permutation is an isometry. However, any automorphism needs to fix -1, since it is the only point of order 2.

Regarding the nilpotent assumption, there are both compact and non-compact examples. We remark that in any group equipped with a bi-invariant distance the involution is an isometry. Consequently, every compact group admits a distance for which the involution is an isometry. Such a map is a group isomorphism only if the group is Abelian. Nonetheless, we point out the following fact which is a consequence of the work of Baum–Browder and Ochiai–Takahashi, see [BB65, OT76] and also [Sch68, HK85].

Corollary 4.1. *Let G_1, G_2 be connected compact simple metric Lie groups. If $F\colon G_1 \to G_2$ is an isometry, then G_1 and G_2 are isomorphic as Lie groups. If, moreover, G_1, G_2 are the same metric Lie group and F is homotopic to the identity map via isometries, then F is affine.*

We point out that there exist examples of pairs of metric Lie groups that are isomorphic as Lie groups and are isometric, but are not isomorphic as metric Lie groups: an example is the rototranslation group (see below) with different Euclidean distances.

Other interesting results for isometries between compact groups can be found in [Oze77] and [Gor80].

The conclusion of Corollary 4.1 may not hold for arbitrary connected metric Lie groups. In fact, we recall the following example, due to Milnor [Mil76, Corollary 4.8], of a group that is solvable and isometric to the Euclidean 3-space. Let G be the universal cover of the group of orientation-preserving isometries of the Euclidean plane, which is also called the rototranslation group. Such a group admits coordinates making it diffeomorphic to $\mathbb{R}^3$ with the product

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In these coordinates, the Euclidean metric is left-invariant. On the one hand, one can check that the isometries that are also automorphisms of G form a 1-dimensional space. On the other hand, the isometries fixing the identity element and homotopic to the identity map form a group isomorphic to $\mathrm{SO}(3)$. Hence, we conclude that not all such isometries are affine. Moreover, this group gives an example of a non-nilpotent metric Lie group isometric (but not isomorphic) to a nilpotent connected metric Lie group, namely the Euclidean 3-space.

Notice that also the Riemannian metric with orthonormal frame $\partial_x, \partial_y, 2\partial_z$ gives a left-invariant structure on G, which is isometric to the previous one, but there is no isometric automorphism between the two structures. Hence, these spaces are not isomorphic as metric Lie groups.

A further study of metric Lie groups isometric to nilpotent metric Lie groups can be found in [CKL+]. In the simply connected case, such groups are exactly the solvable groups of type R.

We finally recall another example. The unit disc in the plane admits a group structure that makes the hyperbolic distance left-invariant. In this metric Lie group not all isometries are affine.

References

[Are46] Richard Arens, *Topologies for homeomorphism groups*, Amer. J. Math. **68** (1946), 593–610.

[BB65] Paul F. Baum and William Browder, *The cohomology of quotients of classical groups*, Topology **3** (1965), 305–336.

[CKL+] Michael Cowling, Ville Kivioja, Enrico Le Donne, Sebastiano Nicolussi Golo, and Alessandro Ottazzi, *From homogeneous metric spaces to Lie groups*, In preparation.

[CL16] Luca Capogna and Enrico Le Donne, *Smoothness of subRiemannian isometries*, Amer. J. Math. **138** (2016), no. 5, 1439–1454.

[Cor15] Yves Cornulier, *On the quasi-isometric classification of locally compact groups*, Preprint, submitted, arXiv:1212.2229 (2015).

[Gor80] Carolyn Gordon, *Riemannian isometry groups containing transitive reductive subgroups*, Math. Ann. **248** (1980), no. 2, 185–192.

[GW88] Carolyn S. Gordon and Edward N. Wilson, *Isometry groups of Riemannian solvmanifolds*, Trans. Amer. Math. Soc. **307** (1988), no. 1, 245–269.

[Ham90] Ursula Hamenstädt, *Some regularity theorems for Carnot-Carathéodory metrics*, J. Differential Geom. **32** (1990), no. 3, 819–850.

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.

[HK85] J. R. Hubbuck and R. M. Kane, *The homotopy types of compact Lie groups*, Israel J. Math. **51** (1985), no. 1-2, 20–26.

[HN12] Joachim Hilgert and Karl-Hermann Neeb, *Structure and geometry of Lie groups*, Springer Monographs in Mathematics, Springer, New York, 2012. MR 3025417

[Jab15a] Michael Jablonski, *Homogeneous Ricci solitons*, J. Reine Angew. Math. **699** (2015), 159–182.

[Jab15b] ———, *Strongly solvable spaces*, Duke Math. J. **164** (2015), no. 2, 361–402.

[Kis03] Iwao Kishimoto, *Geodesics and isometries of Carnot groups*, J. Math. Kyoto Univ. **43** (2003), no. 3, 509–522.

[LN16] Enrico Le Donne and Sebastiano Nicolussi Golo, *Regularity properties of spheres in homogeneous groups*, Trans. Amer. Math. Soc. (2016).

[LO16] Enrico Le Donne and Alessandro Ottazzi, *Isometries of Carnot Groups and Sub-Finsler Homogeneous Manifolds*, J. Geom. Anal. **26** (2016), no. 1, 330–345.

[LR17] Enrico Le Donne and Séverine Rigot, *Besicovitch Covering Property on graded groups and applications to measure differentiation*, J. Reine Angew. Math. (2017).

[Mil76] John Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.

[MS39] S. B. Myers and N. E. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. (2) **40** (1939), no. 2, 400–416.

[MZ74] Deane Montgomery and Leo Zippin, *Topological transformation groups*, Robert E. Krieger Publishing Co., Huntington, N.Y., 1974, Reprint of the 1955 original.

[OT76] Takushiro Ochiai and Tsunero Takahashi, *The group of isometries of a left invariant Riemannian metric on a Lie group*, Math. Ann. **223** (1976), no. 1, 91–96.

[Oze77] Hideki Ozeki, *On a transitive transformation group of a compact group manifold*, Osaka J. Math. **14** (1977), no. 3, 519–531.

[Pan89] Pierre Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), no. 1, 1–60.

[Sch68] Hans Scheerer, *Homotopieäquivalente kompakte Liesche Gruppen*, Topology **7** (1968), 227–232.

[Sha04] Yehuda Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), no. 2, 119–185.

[Wil82] Edward N. Wilson, *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata **12** (1982), no. 3, 337–346.

[Wol63] Joseph A. Wolf, *On locally symmetric spaces of non-negative curvature and certain other locally homogeneous spaces*, Comment. Math. Helv. **37** (1962/1963), 266–295.

E-mail address: kivioja.ville@gmail.com

E-mail address: enrico.ledonne@gmail.com

(Le Donne and Kivioja) Department of Mathematics and Statistics, University of Jyväskylä, 40014 Jyväskylä, Finland

[B]

From homogeneous metric spaces to Lie groups

Michael G. Cowling, Ville Kivioja, Enrico Le Donne,
Sebastiano Nicolussi Golo and Alessandro Ottazzi

Preprint

FROM HOMOGENEOUS METRIC SPACES TO LIE GROUPS

MICHAEL G. COWLING, VILLE KIVIOJA, ENRICO LE DONNE, SEBASTIANO NICOLUSSI, AND ALESSANDRO OTTAZZI

Abstract. We study homogeneous metric spaces, by which we mean connected, locally compact metric spaces whose isometry group acts transitively.

After a review of some classical results, we use the Gleason–Iwasawa–Montgomery–Yamabe–Zippin structure theory to show that for all positive ε, each such space is $(1, \varepsilon)$-quasi-isometric to a connected metric Lie group.

Next, we develop the structure theory of Lie groups to show that every homogeneous metric manifold is homeomorphically roughly isometric to a quotient space of a connected amenable Lie group, and roughly isometric to a simply connected solvable metric Lie group.

Third, we investigate solvable metric Lie groups in more detail, and expound on and extend work of Gordon and Wilson [28, 29] and Jablonski [40] on these, showing, for instance, that connected, simply connected solvable Lie groups may be made isometric if and only if they have the same real-shadow.

Finally, we extend [44] to show that homogeneous metric spaces that admit a metric dilation are all metric Lie groups with an automorphic dilation.

Contents

Date: March 30, 2021.
2010 *Mathematics Subject Classification.* 53C30, 22F30, 20F69, 22E25.

1. Introduction

In this paper we present some links between Lie theory and metric geometry. We consider *homogeneous metric spaces*, that is, metric spaces whose isometry groups act transitively, subject to a number of standing assumptions:

(a) homogeneous metric spaces are *connected and locally compact*, unless explicitly stated otherwise;
(b) a metric means a distance function unless it is preceded by *infinitesimal*; and
(c) metrics are *admissible*, that is, compatible with the topology of the underlying space.

However, we do *not* assume that they are riemannian, or geodesic, or quasigeodesic, or even proper. If the metric space is also a topological manifold, and the metric topology and manifold topology coincide, then we write *metric manifold*. We consider locally compact groups and Lie groups equipped with admissible left-invariant metrics, which we call *metric groups* and *metric Lie groups*.

1.1. **Background.** Geometry and topology on Lie groups and their quotients have a very long history, which we cannot even begin to survey here; rather, we refer the reader to Helgason [34], Kobayashi and Nomizu [46, 47] and Samelson [64]. Nevertheless, there are a few milestones that are specially relevant for this paper, namely Milnor [53], Wolf [72], Alekseevskiĭ [2], Wilson [71] and Gordon and Wilson [28, 29]; in these papers and the texts cited previously, Lie groups and their quotients are considered as models for *riemannian* manifolds.

There are very good reasons to consider Lie groups and their quotients with more general metrics. These appear naturally in studying rigidity of symmetric spaces (see Mostow [58] and Pansu [61]), regularity of subelliptic operators (see Folland and Stein [25] and Rothschild and Stein [63]), and asymptotic properties of nilpotent groups (see Gromov [30, 31] and Pansu [60]). Negatively curved homogeneous riemannian manifolds, classified by Heintze [35], have parabolic visual boundaries that are self-similar Lie groups with metrics that are not always riemannian. The restriction to a connected closed subgroup of a riemannian metric need not be riemannian, or even geodesic. For more information on these developments, see Montgomery [55], Cornulier and de la Harpe [19], and Dungey, ter Elst and Robinson [21].

The prototypical examples of homogeneous metric spaces are connected locally compact groups with left-invariant metrics. Solvable and nilpotent Lie groups, including the stratified groups of Folland [24] or Carnot groups of Pansu [59], are particularly nice examples. Starting with these, one may obtain new examples by considering ℓ^p products, passing to subgroups and quotients, and composing the metric with concave functions, as in the "snowflake" construction.

1.2. **Main results and contents.** Section 2 reviews the basic facts about homogeneous metric spaces and their isometry groups. In more detail, we consider the realisation of homogeneous metric spaces as coset spaces of almost connected locally compact isometry groups, we describe various constructions to produce new metric spaces from old, and we discuss polynomial growth and doubling properties. Because we allow metrics that are not proper or quasigeodesic, we observe some paradoxical phenomena, such as metric groups that are of polynomial growth as groups but not as metric spaces. The introduction to Section 2 provides more information.

Section 3 focusses on the use of Lie theory. In dealing with general rather than riemannian metrics on Lie groups, what happens at the Lie algebra level may not determine what happens at the group level, and so the global approach is to be preferred. That being said, however, the theory is similar in the riemannian and in the general cases.

Our first theorem is a consequence of the Gleason–Iwasawa–Montgomery–Yamabe–Zippin structure theory of almost connected locally compact groups.

Theorem A. *Let M be a homogeneous metric space. Then M is*

(i) $(1, \varepsilon)$-quasi-isometric to a connected metric Lie group G_ε, for all positive ε, and
(ii) roughly isometric to a contractible metric manifold.

We prove an extended version of Theorem A as Theorem 3.7.

Our contributions here are the observations that quasi-isometry may be sharpened to rough isometry and the additive constant in (a) may be made arbitrarily small. For fundamental groups of compact riemannian manifolds, part (b) was shown by Švarc [67] and rediscovered by Milnor [53]. More recently it has been extended, with quasi-isometry rather than rough isometry, to the case of quasigeodesic metrics and to spaces of polynomial growth: see [19, Theorem 4.C.5.] and [14, Proposition 1.3].

One of our aims is to study the following relation between topological groups. Given two topological groups G and H, we say that G *may be made isometric to* H if there exist admissible left-invariant metrics d_G and d_H such that the metric spaces (G, d_G) and (H, d_H) are isometric. Moreover, if G is already a metric group, then we may impose the extra condition that the new metric is roughly isometric to the initial one; in this case, the Gromov–Hausdorff distance of the new metric space from the original one is finite.

Our next theorem, which relies heavily on the Levi and Iwasawa decompositions, shows that every homogeneous metric manifold may be made isometric to a compact quotient of a direct product of a solvable and a compact Lie group.

Theorem B. *Let (M, d) be a homogeneous metric manifold. Then there is a metric d' on M such that the identity mapping from (M, d) to (M, d') is a homeomorphic rough isometry, and there is a transitive closed connected amenable subgroup A of* $\mathrm{Iso}(M, d')$*; hence M is homeomorphic to A/K, where K is a compact subgroup of A.*

We prove an extended version of Theorem B as Theorem 3.24.

We believe that Theorem B is new, though it may have been known but not published. Much is known about the isometry of riemannian symmetric spaces and riemannian solvmanifolds; but we are not aware of a complete treatment of the general case. Gordon and Wilson [28, 29] certainly came close to this, and promised a solution to the general case at the end of [29], but as far as we know this proposed paper did not eventuate.

In various special cases, we obtain simpler and more explicit results; see Corollaries 3.26, 3.28, and 3.29. Corollary 3.26 is of particular interest: there we consider riemannian homogeneous spaces and riemannian metrics. In this case, the result of Theorem B holds with rough isometry replaced by bi-Lipschitz equivalence. Bi-Lipschitz equivalence is stronger locally, but weaker globally, than rough isometry, and our Theorem B provides more information about the large scale behaviour of homogeneous spaces than the strictly riemannian version. This is further evidence that consideration of more general metrics can unlock information that is not accessible in the riemannian framework.

In Section 4, we examine solvable metric Lie groups. We need more background, which we discuss in more detail later. Auslander and Green [5] discovered that a connected simply connected solvable Lie group G of polynomial growth could be embedded in a connected solvable Lie group H (the *hull* of G), in such a way that

$$H = G \rtimes T \qquad \text{and} \qquad H = N \rtimes T,$$

where T is a torus (a compact connected abelian Lie group) in H, and N is the nilradical (the largest connected normal nilpotent subgroup) of H. Then G is homeomorphic to N, since both may be identified with H/T, and G and N enjoy various similarities (see [5, 3]); N is known as the nilshadow of G. Gordon and Wilson [28, 29] considered this from a Lie algebraic point of view, and described G and N as modifications of each other; they considered general solvable Lie groups. Recently, Cornulier [16], and very recently, Jablonski [40] showed that every connected, simply connected solvable Lie group is homeomorphic to a split-solvable Lie group, which we call its real-shadow, in the same way as a connected, simply connected solvable group of polynomial growth is homeomorphic to its nilshadow.

We give a complete and coherent treatment of this recent development. We then proceed to describe when simply connected solvable Lie groups may be made isometric. Here is our third main theorem.

Theorem C. *Let G_0 be a connected simply connected split-solvable Lie group, T be a maximal torus in $\mathrm{Aut}(G_0)$, and d_0 be a T-invariant metric on G_0. Let G_1 be a connected simply connected solvable Lie group. Then the following are equivalent:*

- *(i) G_1 may be made isometric to G_0;*
- *(ii) G_1 may be made isometric to (G_0, d_0);*
- *(iii) G_0 is the real-shadow of G_1; and*
- *(iv) G_1 may be embedded in $H \coloneqq G_0 \rtimes T$ in such a way that every element of h has a unique factorisation gt, where $g \in G_1$ and $t \in T$.*

We prove an extended version of Theorem C as Theorem 4.21.

While the results here are mostly known, our proofs are often different to and sometimes simpler than those of previous authors, and we believe that the reader will find it useful to have a clear account of this development.

Theorem C has various corollaries and extensions, some of which are due to Gordon and Wilson [29] (for riemannian metrics) and Breuillard [14] (for the polynomial growth case). First, the metric d_0 on a connected, simply connected split-solvable Lie group considered in Theorem C may be taken to be riemannian. Next, if G_1 and G_2 are connected, simply connected solvable Lie groups, then they may be made isometric if and only if they have the same real-shadow G_0, and in this case they may both be made isometric to (G_0, d_0). In the special case in which G_0 is of polynomial growth, then G_0 is necessarily nilpotent, and so we obtain a characterisation of groups which may be made isometric to nilpotent Lie groups.

The classification of nilpotent groups up to quasi-isometry is an important unsolved problem. Our result shows that if a connected simply connected Lie group admits one metric for which it is isometric to a nilpotent metric Lie group (N_1, d_1) and another for which it is isometric to another nilpotent metric Lie group (N_2, d_2), then necessarily N_1 and N_2 are isomorphic.

For more details and other results, see the discussion following the proof of Theorem C in Section 4.

Finally, in Section 5, we discuss homogeneous metric spaces that admit metric dilations. A map $\delta : X \to Y$ between metric spaces is called a *metric dilation* if δ is bijective and $d(\delta(x), \delta(x')) = \lambda d(x, x')$ for all $x, x' \in X$, for some $\lambda \in (1, +\infty)$, and a *metrically self-similar group* is a metric group (G, d) that admits a map $\delta : G \to G$ that is both a metric dilation and an automorphism. The stratified groups of Folland and Stein [26] with the Hebisch–Sikora metric [33], the Carnot groups of Pansu [61] and finite dimensional normed vector spaces are examples of metrically self-similar groups; so are the parabolic visual boundaries of the negatively curved connected homogeneous riemannian spaces described by Heintze [35]. Our fourth main theorem described homogeneous metric spaces with dilations.

Theorem D. *If a homogeneous metric space admits a metric dilation, then it is isometric to a metrically self-similar Lie group. Moreover, all metric dilations of a metrically self-similar Lie group are automorphisms.*

Theorem D appears later as Theorem 5.5. It generalises a result of [51], where it is shown that a space is a sub-Finsler Carnot group if and only if the conditions in Theorem D hold and the metric is geodesic.

As a consequence of [65, Proposition 2.2] and [44], if a metric space M is isometric to a metrically self-similar Lie group (G, d'), then G is a gradable, connected simply connected nilpotent Lie group isomorphic to the nilradical of $\mathrm{Iso}(M)$. However, M may also be isometric to a Lie group that is not nilpotent. As discussed after Theorem C, there are metric groups that are not nilpotent but which are isometric to metrically self-similar metric Lie groups; it follows from Theorem D that if M is a metric Lie group and δ is a metric dilation, then δ is an automorphism if and only if M is nilpotent.

While many of the results will be familiar to the experts, we included proofs if we could not find an explicit proof in the literature or if we could give an easier one. We have not attempted to provide a full bibliography of all the areas that we touch on, but rather refer mainly to those papers that we use. At the end of Sections 2 to 5, the reader will find some discussion of who did what and when, and of related results. The reader may wish to consult some other works in this area, in particular, the books of Cornulier and de la Harpe [19] for more information. Recent papers, such as [22], refer to other relevant recent works.

1.3. **Notation and conventions.** We remind the reader of our convention that *homogeneous metric spaces are connected and locally compact, unless explicitly stated otherwise.* Metric manifolds, metric groups and metric Lie groups are examples of these. Some of our results may be proved in greater generality, but this assumption will save space.

A set that is a *neighbourhood* need not be open. Locally compact groups are always locally compact Hausdorff topological groups.

The expression *the isometry group* means the full isometry group, while *an isometry group* means a closed subgroup of the full isometry group.

Constants are always nonnegative real numbers that may vary from one occurrence to the next. These are often denoted by C or ε, and we do not specify that these letters denote constants when they occur.

We denote by e_G, or more simply e, the *identity element* of a group G; the identity of G_1 may be denoted by e_1.

1.4. **Thanks.** We thank the referee of an earlier version of this work for very many helpful comments that led to substantial improvements.

2. Preliminaries

In this section, we recall some more or less familiar facts. First, we discuss homogeneous metric spaces, then changes of metrics. Third, we consider when there are simply transitive isometry groups, and finally, we discuss invariant measures, polynomial growth, and the doubling property. While these are very closely related in the case of *proper quasigeodesic* metrics (see [18]), this is not the case for more general metrics, as we are going to see.

2.1. **Notation.** When (M, d) is a metric space, we sometimes write just M, leaving the metric d implicit. We denote by $B(x, r)$ or $B_d(x, r)$ the open ball $\{y \in M : d(x, y) < r\}$, and by $\breve{B}(x, r)$ or $\breve{B}_d(x, r)$ the closed ball $\{y \in M : d(x, y) \leq r\}$, which need not be the closure of the open ball $B(x, r)$; set closure is denoted with a bar. The metric space is said to be *proper* if closed bounded sets are compact, or equivalently, if all balls $\breve{B}_d(x, r)$ are compact, and is said to be *geodesic* if every pair of points may be joined by a curve whose (rectifiable) length is equal to the distance between the points. Berestovskiĭ [8] showed that a homogeneous metric manifold is geodesic if and only if it is equipped with an invariant infinitesimal sub-Finsler metric.

A function $f : (M_1, d_1) \to (M_2, d_2)$ is an (L, C)-*quasi-isometry* if

$$L^{-1} d_1(x, y) - C \leq d_2(f(x), f(y)) \leq L d_1(x, y) + C$$

for all $x, y \in M_1$, and for every $z \in M_2$ there is $x \in M_1$ such that $d_2(f(x), z) \leq C$. If such a function exists between two metric spaces, then we say that they are (L, C)-*quasi-isometric*, or more simply *quasi-isometric*.

There is a zoo of equivalences of metric spaces that we might consider. Quasi-isometry (for some choice of the constants L and C, possibly depending on the function) is an equivalence relation. If $C = 0$, then f is called *bi-Lipschitz*; bi-Lipschitz gives us another equivalence relation, which, in contrast to quasi-isometry, implies homeomorphism. A third equivalence relation is *rough isometry*, which is defined to be $(1, C)$-quasi-isometry for a suitable choice of C, which may depend on f; we sometimes call C the implicit constant of a rough isometry. This is finer than general quasi-isometry and more restrictive at large scales than bi-Lipschitz. Yet another equivalence relation that we consider is homeomorphic rough isometry. A fifth relation that we consider applies to topological rather than metric groups: we say that G_1 and G_2 *may be made isometric* provided that there exist admissible left-invariant metrics d_1 and d_2 such that (G_1, d_1) and (G_2, d_2) are isometric.

2.2. **Homogeneous metric spaces.** We define an *isometry* of a metric space (M, d) to be a *surjective* map f on M such that

$$d(f(x), f(y)) = d(x, y) \qquad \forall x, y \in M. \tag{2.1}$$

We denote by $\mathrm{Iso}(M, d)$ the set of all isometries of (M, d); given the surjectivity, it is evident that $\mathrm{Iso}(M, d)$ is a group under composition. We recall that a metric space (M, d)

is said to be *homogeneous* if its isometry group acts transitively, and our convention that a homogeneous metric space (M, d) is connected and locally compact, but not necessarily proper, unless explicitly stated otherwise.

Changing the metric on a space (without changing its topology) may change its isometry group. For instance, we may equip $\mathbb{R}^2$ with any one of the bi-Lipschitz equivalent translation-invariant metrics

$$d((x_1, y_1), (x_2, y_2)) = \left(|x_1 - x_2|^p + a|y_1 - y_2|^p\right)^{1/p},$$

where $1 \leq p < +\infty$ and $0 < a < +\infty$. When $p = 2$, the isometry group includes rotations, but otherwise it does not. And when $p = 2$, the rotation group depends on the parameter a. However, in this example, each of the isometry groups act by bi-Lipschitz transformations with respect to all the other metrics.

We prove that $\mathrm{Iso}(M, d)$ is a metrisable, locally compact and σ-compact topological group that acts with compact stabilisers (Theorem 2.6), and whose identity component acts transitively (Corollary 2.8). In Theorem 2.7, we also prove a more quantitative and precise statement about the metrisability, namely that for every $\varepsilon \in \mathbb{R}^+$, the group $\mathrm{Iso}(M, d)$ may be metrised so that it is $(1, \varepsilon)$-quasi-isometric to (M, d).

Proposition 2.1. *Let (M, d) be a metric space, not necessarily connected or locally compact. Then the compact-open topology and the topologies of uniform convergence on compacta and of pointwise convergence agree on $\mathrm{Iso}(M, d)$, and the group $\mathrm{Iso}(M, d)$, endowed with any of these topologies, is a topological group.*

Proof. For the fact that these topologies agree on $\mathrm{Iso}(M, d)$, see [19, Lemmas 5.B.1 and 5.B.2]. That this structure makes the isometry group a topological group is well known; van Dantzig and van der Waerden [20] show this in the case where M is connected, locally compact and separable, and a proof of the general case may be found in [19, Lemma 5.B.3]. $\square$

We now equip $\mathrm{Iso}(M, d)$ with any of the topologies above.

We are not assuming that our metric spaces are proper, but we still need some substitute for a proper metric, and this construction (and some other useful facts) will be the subject of the next two lemmas. Much of this is “folklore”, but we do not know a reference and so we include proofs. We first choose $\ell \in \mathbb{R}^+$ small enough so that $\breve{B}(p, 2\ell)$ is compact for one and hence every p in M by homogeneity. Then there exists a positive integer L for which the compact set $\breve{B}(p, 2\ell)$ may be covered by L open balls of radius ℓ, for one and hence all p in M by homogeneity.

Given a point $o \in M$, we define sets $V_n(o, \ell)$ inductively: first, $V_0(o, \ell) \coloneqq \{o\}$, then

$$V_n(o, \ell) \coloneqq \bigcup_{p \in V_{n-1}(o, \ell)} \breve{B}(p, \ell) \tag{2.2}$$

when $n \in \mathbb{Z}^+$. Further, we define $U_o \coloneqq \{g \in \mathrm{Iso}(M, d) : g(o) \in \breve{B}(o, \ell)\}$.

Lemma 2.2. *Let G be the isometry group of a homogeneous metric space (M, d), and o be any point of M. Then*

(i) $V_n(o, \ell)$ may be covered by at most L^n open balls $B(p, \ell)$ for all $n \in \mathbb{Z}^+$;
(ii) $M = \bigcup_{n \in \mathbb{N}} V_n(o, \ell)$, whence (M, d) is σ-compact and second countable;
(iii) a subset A of M is precompact if and only if $A \subseteq V_n(o, \ell)$ for some $n \in \mathbb{N}$;
(iv) the n-fold product U_o^n is equal to $\{g \in G : g(o) \in V_n(o, \ell)\}$ for all $n \in \mathbb{N}$;
(v) U_o is compact in G, whence U_o^n is compact in G and so $V_n(o, \ell)$ is compact in M for all $n \in \mathbb{N}$.

Proof. First, if $x \in \bigcup_{q\in \breve{B}(p,\ell)} \breve{B}(q,\ell)$, then

$$d(x,p) \le d(x,q) + d(q,p) \le 2\ell.$$

Hence $\bigcup_{q\in \breve{B}(p,\ell)} \breve{B}(q,\ell)$ may be covered by L balls of radius ℓ, by our choice of L. Now (i) may be proved by induction.

From (i), we see that $V_n(o,\ell)$ is precompact. Now $\bigcup_{n\in\mathbb{N}} V_n(o,\ell)$ is both open and closed in M and hence coincides with M. It follows that M is σ-compact and hence second countable, which completes the proof of (ii).

To prove (iii), note that $\{\bigcup_{p\in V_n(o,\ell)} B(p,\ell) : n \in \mathbb{N}\}$ is an increasing open cover of M, and hence if A is a precompact subset of M, then for some n,

$$A \subseteq \bar{A} \subseteq \bigcup_{p\in V_n(o,\ell)} B(p,\ell) \subseteq V_{n+1}(o,\ell).$$

Conversely, if $A \subseteq V_{n+1}(o,\ell)$ then A is precompact.

For (iv), we must show that

$$U_o^n = \{g \in G : g(o) \in V_n(o,\ell)\}. \tag{2.3}$$

If $n = 1$, then (2.3) holds by definition. Assume that (2.3) holds when $n = k$. On the one hand, if $f \in U_o^{k+1}$, then $f = gh$ where $g \in U_o^k$ and $h \in U_o$, so

$$f(o) \in g(\breve{B}(o,\ell)) = \breve{B}(g(o),\ell) \subseteq V_{k+1}(o,\ell).$$

On the other hand, suppose that $f(o) \in V_{k+1}(o,\ell)$. By definition, there exists $q \in V_k(o,\ell)$ such that $f(o) \in \breve{B}(q,\ell)$, and by transitivity and the inductive hypothesis, there exists $g \in U_o^k$ such that $q = g(o)$. Now $g^{-1}f(o) \in \breve{B}(o,\ell)$, that is, $g^{-1}f \in U_o$, since $g^{-1}(\breve{B}(g(o),\ell)) = \breve{B}(o,\ell)$, and we may conclude that $f \in U_o^{k+1}$. By induction, (2.3) holds for all n.

For (v), the Arzelà–Ascoli theorem implies that U_o is precompact in the compact-open topology. Moreover, if $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of U_o that converges to $f \in G$, then $f_n(o)$ converges to $f(o) \in M$ and $d(f_n(o),o) \le \ell$ for all n, whence $d(f(o),0) \le \ell$ and $f \in U_o$. Thus U_o is compact.

Since G is a topological group, U_o^n is compact for each $n \in \mathbb{N}$, and so $V_n(o,\ell)$ is compact from (iv) and the continuity of the map $g \mapsto g(o)$ from G to M. □

We now construct two proper metrics on M; the first has the advantage that it is closely related to the sets $V_n(o,\ell)$ and the second that it is admissible. We define the *Busemann gauge* $\rho_{[\ell]}$ on M by

$$\rho_{[\ell]}(o,p) = \ell \min\{n \in \mathbb{N} : p \in V_n(o,\ell)\} \tag{2.4}$$

and the *derived semi-intrinsic metric* $d_{[\ell]}$ by

$$d_{[\ell]}(p,q) = \inf\Big\{\sum_{j=1}^{k} d(x_j,x_{j-1}) : x_0,\dots,x_k \in M, x_0 = p, x_k = q, d(x_j,x_{j-1}) \le \ell\Big\}. \tag{2.5}$$

We note that $\rho_{[\ell]}$ takes discrete values. Observe that, in the case where the metric space is $\mathbb{R}$ and the metric is given by $d(x,y) = |x-y|^\theta$, where $\theta \in (0,1)$ and $\ell = 1$, the derived semi-intrinsic metric is given by $d_{[\ell]}(x,y) = \lfloor |x-y| \rfloor + (|x-y| - \lfloor |x-y| \rfloor)^\theta$, and is somewhat bizarre; here $\lfloor x \rfloor$ denotes the integer part of x.

Lemma 2.3. *The Busemann gauge $\rho_{[\ell]}$ and the derived semi-intrinsic metric $d_{[\ell]}$ are both metrics on the set M. Further,*

$$d(p,q) \le d_{[\ell]}(p,q) \le \rho_{[\ell]}(p,q) \le 2d_{[\ell]}(p,q) + \ell \qquad \forall p,q \in M.$$

Hence $d_{[\ell]}$ is proper since $\rho_{[\ell]}$ is proper. In addition, if $d(x,y) \le \ell$, then $d_{[\ell]}(x,y) = d(x,y)$, for all $x,y \in M$, whence $d_{[\ell]}$ is admissible.

Proof. It is easy to see that both $\rho_{[\ell]}$ and $d_{[\ell]}$ are metrics.

Take $p, q \in M$. On the one hand, if $q \in V_n(p,\ell)$, then by definition there are points $x_j \in M$, where $0 \le j \le n$, such that $x_0 = p$, $x_n = x$ and $x_j \in \breve{B}(x_{j-1},\ell)$. It follows immediately that $d_{[\ell]}(p,q) \le n\ell$.

On the other hand, for any positive ε, we can find points $x_0, \dots, x_k$ such that $x_0 = p$, $x_k = q$ and $\sum_{j=1}^{k} d(x_j, x_{j-1}) \le d_{[\ell]}(p,q) + \varepsilon$. Observe that we may omit points x_j if $d(x_{j+1}, x_j) + d(x_j, x_{j-1}) \le \ell$, for in this case

$$d(x_{j+1}, x_{j-1}) \le d(x_{j+1}, x_j) + d(x_j, x_{j-1}) \le \ell.$$

We omit such points recursively until this is no longer possible. Now we may not only assume that $\sum_{j=1}^{k} d(x_j, x_{j-1}) \le d_{[\ell]}(p,q) + \varepsilon$, but also that $d(x_{j+1}, x_j) + d(x_j, x_{j-1}) > \ell$. It follows that

$$d_{[\ell]}(p,q) + \varepsilon \ge \sum_{j=1}^{k} d(x_j, x_{j-1}) > \lfloor k/2 \rfloor \ell,$$

and this implies that

$$\rho_{[\ell]}(p,q) \le k\ell \le \ell + 2d_{[\ell]}(p,q).$$

The rest of the proof is evident. □

It is easy to see that, if d is a geodesic metric, then $d_{[\ell]}$ coincides with d. Moreover, if we start with arbitrary admissible metrics d_1 and d_2 with a common transitive isometry group, and construct the Busemann gauges $\rho_{1,[\ell]}$ and $\rho_{2,[\ell]}$ or the derived semi-intrinsic metrics $d_{1,[\ell]}$ and $d_{2,[\ell]}$ (still with the assumption that the balls $B_{d_1}(p, 2\ell_1)$ and $B_{d_2}(p, 2\ell_2)$ are relatively compact) then $\rho_{1,[\ell]}$ and $\rho_{2,[\ell]}$ are quasi-isometric. Hence by Lemma 2.3 all the metrics $\rho_{1,[\ell]}$, $\rho_{2,[\ell]}$, $d_{1,[\ell]}$ and $d_{2,[\ell]}$ are quasi-isometric. It is also straightforward to see that the derived semi-intrinsic metrics $d_{[\ell_1]}$ and $d_{[\ell_2]}$ are quasi-isometric (again, provided that the balls $B_{d_1}(p, 2\ell_1)$ and $B_{d_2}(p, 2\ell_2)$ are relatively compact).

We now introduce an important class of metrics.

Definition 2.4. A metric on a homogeneous metric space (M,d) is called *proper quasi-geodesic* if the identity map is a quasi-isometry from (M,d) to $(M, \rho_{[\ell]})$, where $\rho_{[\ell]}$ is the Busemann gauge defined in (2.4).

This definition is not standard, but coincides with the usual versions. Two distinct proper quasigeodesic metrics on M are quasi-isometric.

2.3. **Metric spaces and coset spaces.** We begin by clarifying notation. A group H *acts* on a set M if there is a homomorphism α from H to $\mathrm{Trans}(M)$, the group of all invertible transformations of M. If the action is *effective*, that is, if $\alpha(h)p = p$ for all $p \in M$ only if $h = e$, then H may be identified with a subgroup of $\mathrm{Trans}(M)$.

Remark 2.5. If a group H acts transitively on a set M, then all the stabilisers of points in M are conjugate. Hence a normal subgroup of H that is contained in one stabiliser is contained in all stabilisers, that is, it fixes all points. Thus if H acts effectively and transitively on a set, then no nontrivial compact normal subgroups of H are contained in a stabiliser. In general, if H acts transitively but not effectively, and K is the stabiliser of a point, then $N \coloneqq \bigcap_{h \in H} hKh^{-1}$ is a normal subgroup of H that may be factored out to obtain a effective action of H/N, since H/K may be identified with $(H/N)/(K/N)$.

We write $Z(H)$ for the centre of a group H; then what we have just shown implies in particular that if H acts effectively on a set, and K is the stabiliser of a point, then $K \cap Z(H) = \{e\}$.

An action α of a group H on a metric space (M,d) is *isometric* or *by isometries* if $\alpha(H) \subseteq \mathrm{Iso}(M,d)$.

Theorem 2.6. *Let (M,d) be a homogeneous metric space, o be a point of M, $\rho_{[\ell]}$ be the Busemann gauge of (2.4), and H be the isometry group of (M,d). Then*

(i) H is locally compact, σ-compact and second countable;
(ii) the stabiliser K of o is compact;
(iii) H is metrisable, and for each $\varepsilon \in \mathbb{R}^+$, the Busemann metric d_H on H, given by

$$d_H(g,h) \coloneqq \sup\{d(g(q),h(q))\mathrm{e}^{-\rho_{[\ell]}(o,q)/\varepsilon} : q \in M\},$$

is an admissible left-invariant metric on H;
(iv) the map $\pi : g \mapsto g(o)$ from (H,d_H) to (M,d) is 1-Lipschitz and $(1, 2\varepsilon/\mathrm{e})$-quasi-isometric; more precisely,

$$d_H(g,h) - 2\varepsilon/\mathrm{e} \le d(g(o),h(o)) \le d_H(g,h) \qquad \forall g,h \in H.$$

(v) $\operatorname{diam}_H(K) \le 2\varepsilon/\mathrm{e}$, and d_H is right-K-invariant, that is, $d_H(gk,hk) = d_H(g,h)$ for all $g,h \in H$ and all $k \in K$.

Proof. The local compactness of H was shown by van Dantzig and van der Waerden [20].

By Lemma 2.2 (v), (ii) and (iv) and Proposition 2.1, the set U_o and hence the sets U_o^n are compact in H when $n \in \mathbb{N}$, and $H = \bigcup_{n\in\mathbb{N}} U_o^n$, whence H is σ-compact. The second countability of H follows from that of M.

Next, van Dantzig and van der Waerden proved (ii), which also follows from the fact that the stabiliser of o is a closed subset of the compact set U_o of Lemma 2.2.

Clearly d_H is left-invariant; we need to show that it is admissible. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in H. On the one hand, if $g_n \to g$ in (H,d_H), then

$$d(g_n(p),g(p)) \le \mathrm{e}^{\rho_{[\ell]}(o,p)/\varepsilon} d_H(g_n,g),$$

for all $p \in M$, and hence g_n converges to g pointwise, and so in H.

On the other hand, if $g_n \to g$ in H, then the convergence is uniform on compacta, by Proposition 2.1. Fix $\eta \in (0,1)$, and take $R \in \mathbb{R}^+$ such that $t\mathrm{e}^{-t/\varepsilon} < \eta$ whenever $t > R$. Define A to be the closure of $\{p \in M : \rho_{[\ell]}(o,p) \le R\}$. Then A contains o and is compact in M by definition and part (v) of Lemma 2.2. Hence there is $n_0 \in \mathbb{N}$ such that $d(g_n(p),g(p)) \le \eta$ for all $p \in A$ and all $n \ge n_0$. Therefore

$$d(g_n(p),g(p))\mathrm{e}^{-\rho_{[\ell]}(o,p)/\varepsilon} \le \eta,$$

if $n \ge n_0$ and $p \in A$, while if $n \ge n_0$ and $p \notin A$, then

$$\begin{aligned}
&d(g_n(p),g(p))\mathrm{e}^{-\rho_{[\ell]}(o,p)/\varepsilon} \\
&\quad\le \big(d(g_n(p),g_n(o)) + d(g_n(o),g(o)) + d(g(o),g(p))\big)\mathrm{e}^{-\rho_{[\ell]}(o,p)/\varepsilon} \\
&\quad\le (2d(o,p)+\eta)\mathrm{e}^{-\rho_{[\ell]}(o,p)/\varepsilon} \\
&\quad\le 3\eta.
\end{aligned}$$

We conclude that $d_H(g_n,g) \le 3\eta$ for all $n \ge n_0$. As η may be chosen to be arbitrarily small, $g_n \to g$ in (H,d_H).

By definition, $d(\pi(g),\pi(h)) = d(g(o),h(o)) \le d_H(g,h)$ for all $g,h \in H$, so π is 1-Lipschitz. Moreover, π is surjective by the homogeneity assumption, and

$$\begin{aligned}
d_H(g,h) &\le \sup\{(d(g(p),g(o)) + d(g(o),h(o)) + d(h(o),h(p)))\mathrm{e}^{-\rho_{[\ell]}(o,p)/\varepsilon} : p \in M\} \\
&\le d(g(o),h(o))\sup\{\mathrm{e}^{-\rho_{[\ell]}(o,p)/\varepsilon} : p \in M\} + 2\sup\{d(o,p)\mathrm{e}^{-\rho_{[\ell]}(o,p)/\varepsilon} : p \in M\} \\
&\le d(\pi(g),\pi(h)) + 2\varepsilon/\mathrm{e}
\end{aligned} \tag{2.6}$$

for all $g,h \in H$, whence π is a $(1,2\varepsilon/\mathrm{e})$-quasi-isometry.

Finally, for $g, h \in H$ and $k \in K$,

$$\begin{aligned} d_H(gk, hk) &= \sup\{d(gk(q), hk(q))\mathrm{e}^{-\rho_{[\ell]}(o,q)/\varepsilon} : q \in M\} \\ &= \sup\{d(g(k(q)), h(k(q)))\mathrm{e}^{-\rho_{[\ell]}(o,k(q))/\varepsilon} : q \in M\} = d_H(g,h), \end{aligned}$$

as required. Further, from (2.6),

$$\operatorname{diam}(K) = \sup\{d_H(k,e) : k \in K\} \le 2\varepsilon/\mathrm{e} + \sup\{d(k(o), e(o)) : k \in K\} = 2\varepsilon/\mathrm{e},$$

and the proof is complete. □

Observe that we could define the Busemann metric in the statement of the theorem using $d_{[\ell]}$ rather than $\rho_{[\ell]}$, and the proof above would work with minor modifications. Observe also that $\operatorname{Iso}(M,d) \subseteq \operatorname{Iso}(M, d_{[\ell]})$, where d is a derived semi-intrinsic metric as defined just before Lemma 2.3.

We now consider closed subgroups of the isometry group in more detail.

Theorem 2.7. *Let (M,d) be a homogeneous metric space, G be a closed subgroup of $\operatorname{Iso}(M,d)$, and S be the stabiliser in G of a point o in M. Then*

(i) G is locally compact and S is compact;
(ii) if G acts transitively on M, then the map $gS \mapsto g(o)$ is a homeomorphism from G/S to M;
(iii) if $\breve{B}(o,\ell) \subseteq G(o)$ for some choice of $\ell \in \mathbb{R}^+$ and $o \in M$, then G acts transitively on M;
(iv) if G is open in $\operatorname{Iso}(M,d)$, then it acts transitively on M;
(v) if G acts transitively on M, then for each $\varepsilon \in \mathbb{R}^+$ and $o \in M$, we may equip G with an admissible left-invariant metric in such a way that the map $g \mapsto g(o)$ is 1-Lipschitz and a $(1,\varepsilon)$-quasi-isometry;
(vi) if G acts transitively on M, then for each $n \in \mathbb{N}$ and $o \in M$,

$$\{g \in G : g(o) \in \breve{B}(o,\ell)\}^n = \{g \in G : g(o) \in V_n(o,\ell)\}.$$

Proof. Part (i) is standard: closed subspaces of locally compact or compact spaces are locally compact or compact.

Part (ii) follows from [34, Theorem 3.2, page 121].

For part (iii), the orbit $G(o)$ is nonempty, open and closed. As M is connected, by our standing assumption, $M = G(o)$.

For part (iv), it follows from part (ii) that the map $g \mapsto g(o)$ from G to M is open. Consequently $G(o)$ is open and G acts transitively by part (iii).

The proof of part (v) is similar to the proof of part (iii) in Theorem 2.6, and the proof of part (vi) is similar to the proof of part (iv) in Lemma 2.2. □

Corollary 2.8. *Let (M,d) be a homogeneous metric space. The connected component H of the identity in $\operatorname{Iso}(M,d)$ is locally compact and acts transitively on M, and the quotient $\operatorname{Iso}(M,d)/H$ is compact.*

Proof. The subgroup H is closed in $\operatorname{Iso}(M,d)$, and hence is locally compact. It is also normal, and the totally disconnected locally compact group $\operatorname{Iso}(M,d)/H$ has a neighbourhood base N of the identity consisting of open and closed subgroups, ordered by reverse inclusion; see [66, Proposition 4.13]. For each $\nu \in \mathrm{N}$, let H_ν be the preimage of ν in $\operatorname{Iso}(M,d)$. Then $(H_\nu)_{\nu\in\mathrm{N}}$ is a net of open and closed subgroups of $\operatorname{Iso}(M,d)$ such that $H = \bigcap_{\nu\in\mathrm{N}} H_\nu$, and H_ν acts transitively on M for every $\nu \in \mathrm{N}$ by Theorem 2.7.

Take $o, p \in M$. For each $\nu \in \mathrm{N}$, there is $g_\nu \in H_\nu$ such that $g_\nu(o) = p$. By the Arzelà–Ascoli theorem, $\{g \in \operatorname{Iso}(M,d) : g(o) = p\}$ is compact; since each g_ν lies in this set, we may assume that g_ν converges to $g \in \operatorname{Iso}(M,d)$ by passing to a subnet if necessary. For each $\nu \in \mathrm{N}$, $g_{\nu'} \in H_\nu$ when $\nu' \geq \nu$, and hence $g \in H_\nu$. In conclusion, $g \in \bigcap_{\nu \in \mathrm{N}} H_\nu = H$ and $g(o) = p$.

Let K be the stabiliser in $\mathrm{Iso}(M,d)$ of the point o in M; then K is compact. Since H acts transitively, for every $g \in \mathrm{Iso}(M,d)$, there exists $h \in H$ such that $h^{-1}g(o) = o$, that is, $h^{-1}g \in K$. It follows that $\mathrm{Iso}(M,d) \subseteq HK$. □

The next definition summarises and extends the structure that we have seen in the last theorems.

Definition 2.9. A *homogeneous metric projection* is a pair of homogeneous metric spaces (M_1, d_1) and (M_2, d_2), with a group H acting isometrically, continuously and transitively on both M_1 and M_2, and an H-equivariant projection $\pi : M_1 \to M_2$ such that

$$d_2(x_2, y_2) = \inf\{d_1(x_1, y_1) : \pi x_1 = x_2, \pi y_1 = y_2\} \qquad \forall x_2, y_2 \in M_2.$$

The set $\{x_1 \in M_1 : \pi x_1 = x_2\}$ is called the *fibre above* x_2 in M_2.

Because H acts continuously on both M_1 and M_2, the stabilisers K_1 of a point x in M_1 and K_2 of πx in M_2 are closed, and it is clear that $K_1 \subseteq K_2$. There is then a natural identification of the fibre above x with the quotient space K_2/K_1, and all the fibres are isometric to each other because H acts transitively. As noted in the remark above, the subgroup of H of elements that act trivially on M_1 (and a fortiori on M_2) is a closed normal subgroup that may be factored out.

With K_1 and K_2 as above, if the set K_2/K_1 is compact, then the diameter of each fibre is bounded; hence there exists a constant C such that

$$d_1(x, y) - C \leq d_2(\pi x, \pi y) \leq d_1(x, y) \qquad \forall x, y \in M_1,$$

that is, π is 1-Lipschitz and a rough isometry. The constant C is called the implicit constant of the projection π and may be identified with the diameter of K_2/K_1.

Let π be the projection from a locally compact group H onto a quotient space H/K. We recall that a *section* σ for H/K in H is a mapping such that $\pi \circ \sigma$ is equal to $\mathrm{Id}_{H/K}$, the identity map on H/K. It is well-known that sections exist: they may be taken to be Borel or even Baire (see, for instance, [43]). It is evident that if π is a homogeneous metric projection from (M_1, d_1) onto (M_2, d_2) and H is a common transitive isometry group, then M_2 may be identified with H/K_2, where K_2 is a compact subgroup of H, and a section from M_2 to H composed with the projection from H to M_1 is a section from M_2 to M_1. If

$$d_1(x, y) - C \leq d_2(\pi x, \pi y) \leq d_1(x, y) \qquad \forall x, y \in M_1,$$

and if σ is a section for M_2 in M_1, then

$$d_2(p, q) \leq d_1(\sigma(p), \sigma(q)) \leq d_2(p, q) + C \qquad \forall p, q \in M_2.$$

We conclude this section with a remark.

Remark 2.10. Let (M, d) be a homogeneous metric space, and let H be a subgroup of $\mathrm{Iso}(M, d)$ that acts transitively on M. Equip $\mathrm{Iso}(M, d)$ with the topology of Proposition 2.1, H with the relative topology, and M with the topology induced by d. Take an arbitrary point o of M.

Then the relative topology on H is also the only topology on H such that the mapping $\pi : h \mapsto ho$ is continuous and open. Indeed, the sets $\{g \in H : d(hx, x) < \varepsilon\}$, where $x \in M$ and $\varepsilon \in \mathbb{R}^+$ form a subbase for any topology on H such that π is continuous and open, and also for the topology of pointwise convergence.

This implies that if $U \subset H$ and $U = UK$, then U is open in H if and only if Uo is open in M. It follows that if we change the metric on M to a new metric that induces a different topology and is such that H is still an isometry group, then the topology of H as an isometry group with the new metric must also change.

2.4. **Modifying metrics.** In dealing with homogeneous metric spaces, a useful technique is the use of pseudometrics on groups; we show how to use these to modify metrics.

Pseudometrics are functions that satisfy all the conditions required of a metric, except perhaps the condition that $d(x,y) = 0$ implies that $x = y$. Let $\dot{d}$ be a left-invariant pseudometric on a topological group G. We define the *kernel* of $\dot{d}$ to be the subset $\{x \in G : d(x,e) = 0\}$, and say that $\dot{d}$ on G is *continuous* if $\dot{d}(x_n, y) \to \dot{d}(x,y)$ for all $y \in G$ whenever $x_n \to x$ in G, *semiproper* if $\{x \in G : \dot{d}(x,e) = 0\}$ is compact, and *proper* if $\{x \in G : \dot{d}(x,e) < C\}$ is relatively compact for all $C \in \mathbb{R}^+$.

Given a pseudometric space $(M, \dot{d})$, we define the *ball* $B_{\dot{d}}(x,r) \coloneqq \{y \in M : \dot{d}(x,y) < r\}$; then $B_{\dot{d}}(x,r)$ is open if $\dot{d}$ is continuous. Further, given pseudometric spaces $(M_1, \dot{d}_1)$ and $(M_2, \dot{d}_2)$, we say that a bijection $f : M_1 \to M_2$ is an *isometry* if $\dot{d}_2(fx_1, fy_1) = \dot{d}_1(x_1, y_1)$ for all $x_1, y_1 \in M_1$.

Lemma 2.11. *Suppose that (M,d) is a homogeneous metric space, that G is a transitive closed subgroup of $\operatorname{Iso}(M,d)$, and that K is the stabiliser in G of a point o in M. Then $\dot{d} : G \times G \to [0, +\infty)$, defined by*

$$\dot{d}(x,y) \coloneqq d(xo, yo) \qquad \forall x, y \in G,$$

is a continuous left-invariant pseudometric on G, and

(i) *$\bigcap_{x \in G} xKx^{-1} = \{e\}$;*
(ii) *$\dot{d}(x,e) = 0$ if and only if $x \in K$;*
(iii) *$\dot{d}(x,y) = \dot{d}(xk, yk')$ for all $x, y \in G$ and $k, k' \in K$;*
(iv) *the topology induced by d on G/K coincides with the quotient topology on G/K.*

Conversely, if $\dot{d}$ is a continuous left-invariant pseudometric on a connected metrisable topological group G, then $K \coloneqq \{x \in G : \dot{d}(x,e) = 0\}$ is a closed subgroup of G, and $\{x \in G : \dot{d}(x,y) = 0\} = yK$; further, (iii) holds. The function $d : G/K \times G/K \to [0, +\infty)$, defined by

$$d(xK, yK) \coloneqq \dot{d}(x,y) \qquad \forall x, y \in G, \tag{2.7}$$

is a metric on the set G/K, and G acts continuously and transitively by isometries on $(G/K, d)$. Further, the subgroup $N \coloneqq \bigcap_{x \in G} xKx^{-1}$ is closed and normal in G, and acts trivially on G/K, so that G/N may be identified with a transitive subgroup of $\operatorname{Iso}(G/K, d)$. Finally, suppose that the topology induced by d on G/K coincides with the quotient topology on G/K. Then

(v) *the Busemann metric d_ε on G/N, given by*

$$d_\varepsilon(g,h) \coloneqq \sup\{d(g(q), h(q)) \mathrm{e}^{-\rho_{[\ell]}(o,q)/\varepsilon} : q \in G/N\},$$

is admissible on G/N; and

(vi) *the subgroup K/N of G/N is compact.*

Proof. Take $x, y, z \in G$. Then $\dot{d}(x,y) \geq 0$ and $\dot{d}(x,y) = \dot{d}(y,x)$ by definition; further,

$$\dot{d}(x,z) = d(xo, zo) \leq d(xo, yo) + d(yo, zo) = \dot{d}(x,y) + \dot{d}(y,z),$$

and

$$\dot{d}(x,y) = d(xo, yo) = d(zxo, zyo) = \dot{d}(zx, zy).$$

Hence $\dot{d}$ is a left-invariant pseudometric on G.

The compactness of K and items (i) and (iv) are proved in Section 2.3; items (ii) and (iii) follow immediately from the definitions.

Conversely, if $\dot{d}$ is a continuous left-invariant pseudometric on a topological group G, and $K = \{x \in G : \dot{d}(x,e) = 0\}$, then

$$\dot{d}(x^{-1}y, e) = \dot{d}(y,x) \le \dot{d}(y,e) + \dot{d}(e,x) = 0,$$

for all $x, y \in K$ whence K is a subgroup of G, which is closed since $\dot{d}$ is continuous. Observe that

$$\dot{d}(x,y) = 0 \iff \dot{d}(y^{-1}x, e) = 0 \iff y^{-1}x \in K \iff x \in yK.$$

Moreover,

$$\dot{d}(xk, yk') \le \dot{d}(xk, x) + \dot{d}(x,y) + \dot{d}(y, yk') = \dot{d}(x,y)$$

and

$$\dot{d}(x,y) \le \dot{d}(x, xk) + \dot{d}(xk, yk') + \dot{d}(yk', y) = \dot{d}(xk, yk'),$$

so (iii) holds. It follows immediately that $\dot{d}$ induces a metric d on G/K, by the formula

$$d(xK, yK) = \dot{d}(x,y) \qquad \forall x, y \in G,$$

and G acts transitively and continuously by isometries on $(G/K, d)$. It is evident that N is closed and normal, and is precisely the subgroup of G that stabilises every point of G/K, hence G/N acts effectively, transitively and isometrically on G/K, which we may identify with $(G/N)/(K/N)$ by a standard isomorphism theorem.

Now we suppose that the topology induced by d on G/K coincides with the quotient topology on G/K, that is, that d is admissible, and prove (v) and (vi). We may and shall suppose that N is trivial, otherwise we just divide it out. By Remark 2.10, the topology on G coincides with the relative topology as a subgroup of $\mathrm{Iso}(G/K, d)$, and Theorem 2.7 implies (v) and (vi). □

The reader may wish to check that, in the first part of the preceding lemma, if d is proper on G/K, then $\dot{d}$ is proper on G, while in the second part, $\dot{d}$ is semiproper if and only if d is proper.

Definition 2.12. A left-invariant continuous pseudometric $\dot{d}$ on a topological group G with kernel K is said to be *admissible* if the topology of the induced metric on G/K coincides with the quotient topology on G/K. Equivalently, the sets $B_{\dot{d}}(x,r)K$, where $x \in G$ and $r \in \mathbb{R}^+$ form a base for the topology of G/K, or the sets $B_{\dot{d}}(x,r)$, where $x \in G$ and $r \in \mathbb{R}^+$ form a base for the subtopology of G of all right-K-invariant sets of the topology.

By the proof of the previous lemma and the continuity of $\dot{d}$, the sets $B_{\dot{d}}(x,r)$ satisfy $B_{\dot{d}}(x,r) = B_{\dot{d}}(x,r)K$ and are open in G. Hence the key to showing admissibility is to show that if U is an open neighbourhood of x in G and $U = UK$, then $B_{\dot{d}}(x,r) \subseteq U$ when r is small enough.

Corollary 2.13. *If $\dot{d}$ is a left-invariant continuous admissible pseudometric on G, and $x_n \to x$ in G as $n \to \infty$, then $\sup_{y \in K_c} \dot{d}(x_n y, xy) \to 0$ for all compact subsets K_c of G.*

Proof. Let K be the kernel of $\dot{d}$, and d be the corresponding metric on G/K. Convergence of a sequence in G implies pointwise convergence and hence locally uniform convergence of the corresponding sequence of elements of $\mathrm{Iso}(G/K, d)$, by Proposition 2.1. □

We show now that if G is a locally compact group and d_G is an admissible left-invariant metric on G that is also right-K-invariant, where K is a closed bounded subgroup of G, then the quotient space G/K may be equipped with a metric in a natural way.

Lemma 2.14. *Let K_0 and K be compact subgroups of a locally compact group G such that $K_0 \subseteq K$. Suppose that $\dot{d}$ is a left-invariant right-K-invariant continuous admissible pseudometric on G with kernel K_0, and take $C \coloneqq \sup\{\dot{d}(x,y) : x, y \in K\}$ (which is finite). Then*

$$\ddot{d}(x,y) \coloneqq \min\{\dot{d}(xk, yk') : k, k' \in K\} \qquad \forall x, y \in G$$

defines a left-invariant continuous admissible pseudometric on G with kernel K, and

$$\dot{d}(x,y) - C \le \ddot{d}(x,y) \le \dot{d}(x,y) \qquad \forall x, y \in G.$$

Proof. Since $\dot{d}$ is continuous and right-K-invariant and K is compact, we may write

$$\ddot{d}(x,y) = \min\{\dot{d}(xk, y) : k \in K\} = \min\{\dot{d}(x, yk') : k' \in K\}. \tag{2.8}$$

Clearly $\ddot{d}$ is left-invariant and $\ddot{d}(x,y) \ge 0$ and $\ddot{d}(x,y) = \ddot{d}(y,x)$ for all $x, y \in G$. Further,

$$\dot{d}(xk, zk') \le \dot{d}(xk, y) + \dot{d}(y, zk'),$$

and taking minima shows that $\ddot{d}(x,z) \le \ddot{d}(x,y) + \ddot{d}(y,z)$ for all $x, y, z \in G$. Suppose that $\ddot{d}(x,y) = 0$; then there exists $k \in K$ such that $\dot{d}(x, yk) = 0$. Hence $x \in yK_0$ and $xK = yK$.

We now show that the pseudometric $\ddot{d}$ is admissible. By the remark following Definition 2.12, it suffices to consider $x \in G$ and an open neighbourhood U of x in G such that $U = UK$, and show that some $B_{\ddot{d}}(x,r) \subseteq U$. Clearly $U = UK_0$, and since $\dot{d}$ is admissible, there exists $r \in \mathbb{R}^+$ such that $x \in B_{\dot{d}}(x,r) \subseteq U$. From (2.8),

$$B_{\ddot{d}}(x,r) = \bigcup_{k \in K} B_{\dot{d}}(xk, r) = B_{\dot{d}}(xk, r)K \subseteq UK = U,$$

so $\ddot{d}$ is admissible. □

Corollary 2.15. *Let K_0 and K be compact subgroups of a locally compact group G such that $K_0 \subseteq K$. If d_0 is a G-invariant admissible metric on G/K_0 such that*

$$d_0(xkK_0, ykK_0) = d_0(xK_0, yK_0) \qquad \forall x, y \in G \quad \forall k \in K,$$

then d, defined by

$$d(xK, yK) = \min\{d_0(xkK_0, yk'K_0) : k, k' \in K\} \qquad \forall x, y \in G,$$

is a G-invariant admissible metric on G/K, and the projection $\pi : G/K_0 \to G/K$ is a G-equivariant rough isometry; more precisely,

$$d_0(xK_0, yK_0) - C \le d(xK, yK) \le d_0(xK_0, yK_0)$$

for all $x, y \in G$.

Proof. This follows from the preceding lemma, translated into the language of metrics using Lemma 2.11. Indeed, the metric d_0 induces a pseudometric $\dot{d}$ on G which satisfies the conditions required in the previous lemma; the previous lemma constructs another pseudometric $\ddot{d}$ on G; finally d is the metric on G/K induced by $\ddot{d}$. □

A locally compact topological group G is said to be metrisable if there is a metric d_G on G that induces the topology of G; it is known that d_G may be taken to be left-invariant (see [37, Theorem 8.3]), and we shall always do so. Conversely, it is easy to check that if d_G is a left-invariant metric on G, then G with the topology induced by d_G is a topological group (that is, multiplication and inversion are continuous) if and only if d_G satisfies the condition $d_G(x_n, x) \to 0$ as $n \to +\infty$ implies that $d_G(x_n z, xz) \to 0$ as $n \to +\infty$ for all $z \in G$.

Lemma 2.14 suggests the question whether, given a pseudometric group (G, d) and a closed d-bounded subgroup K of G, it is possible to adjust d on G to obtain a pseudometric that is both left-invariant and right-K-invariant. This is the point of the next lemma. We say that a closed subgroup K of G is compact modulo a closed central subgroup Z of G provided that $K/(K \cap Z)$ is compact.

Lemma 2.16. *Let Z be a closed central subgroup of a locally compact group G, and let $\dot{d}$ be a left-invariant continuous admissible pseudometric on G. Suppose that K is a subgroup of G that is compact modulo Z, and set*

$$C \coloneqq \sup_{k \in K} \inf_{z \in Z} \dot{d}(kz, e). \tag{2.9}$$

Then C is finite. Further, $\dot{d}_K$, defined by

$$\dot{d}_K(g,h) = \sup_{k \in K} \dot{d}(gk, hk) \qquad \forall g, h \in G,$$

is a left-invariant, right-K-invariant, continuous, admissible pseudometric on G, and

$$\dot{d}(g,h) \leq \dot{d}_K(g,h) \leq \dot{d}(g,h) + 2C \qquad \forall g, h \in G. \tag{2.10}$$

Proof. In light of the existence of suitable sections for quotients of locally compact groups (see, for instance, [43]), there is a compact subset K_c of K such that $K \subseteq K_c Z$. Then

$$\sup_{k \in K} \dot{d}(gk, hk) \leq \sup_{k \in K_c} \sup_{z \in Z} \dot{d}(gkz, hkz) = \sup_{k \in K_c} \dot{d}(gk, hk) \leq \sup_{k \in K} \dot{d}(gk, hk),$$

and so

$$\dot{d}_K(g,h) = \sup_{k \in K_c} \dot{d}(gk, hk) \qquad \forall g, h \in G. \tag{2.11}$$

Similarly,

$$C = \sup_{k \in K} \inf_{z \in Z} \dot{d}(kz, e) = \sup_{k \in K_c} \inf_{z \in Z} \dot{d}(kz, e) \leq \sup_{k \in K_c} \dot{d}(k, e) < +\infty.$$

By definition, given $k \in K$ and $z \in Z$,

$$\dot{d}(gk, hk) = \dot{d}(gkz, hkz) \leq \dot{d}(gkz, g) + \dot{d}(g,h) + \dot{d}(h, hkz) \leq \dot{d}(g,h) + 2\dot{d}(kz, e)$$

for all $g, h \in G$; we obtain (2.10) for $\dot{d}_K$ by optimising in z. In particular, we see that $\dot{d}_K$ is finite. We may easily check that $\dot{d}_K$ is a pseudometric on G. It remains to show that $\dot{d}_K$ is admissible and continuous.

The continuity of $\dot{d}_K$ follows immediately from (2.11) and Corollary 2.13.

To check admissibility, we suppose that $x \in G$ and V is an open neighbourhood of x in G, and take $U = VK$. We need to show that $B_{\dot{d}_K}(x,r) \subseteq U$ when r is small enough. But $B_{\dot{d}_K}(x,r) \subseteq B_{\dot{d}}(x,r)$ and the admissibility of $\dot{d}$ implies that $B_{\dot{d}}(x,r) \subseteq U$ when r is small enough. □

The next result follows immediately from Lemmas 2.11 and 2.16.

Corollary 2.17. *Suppose that K_o is a compact subgroup of a locally compact group G, and K is a subgroup of G that contains K_o and is compact modulo the centre of G. If d is a G-invariant metric on G/K_o, then there is a metric d' on G/K_o such that the identity mapping on G/K_o is a rough isometry from $(G/K_o, d)$ to $(G/K_o, d')$ and d' is left-invariant and right-K-invariant, in the sense that*

$$d'(gg'kK_o, gg''kK_o) = d'(g'K_o, g''K_o) \qquad \forall g, g', g'' \in G \quad \forall k \in K.$$

We have seen that, starting from a homogeneous metric space (M,d), we may construct various transitive isometry groups H, which are metrisable locally compact groups, and realise M as H/K, where K is the stabiliser of a point o in M. Conversely, given a quotient space H/K of a metrisable locally compact group, it is natural to ask whether H/K may be given the structure of a metric space on which H acts isometrically. The following corollary answers this question.

Corollary 2.18. *Given a compact subgroup K of a connected metrisable locally compact group H, there exists an admissible metric d on H/K such that H acts isometrically on $(H/K, d)$. Hence there also exists a left-invariant admissible metric d_H on H such that (H, d_H) is $(1, \varepsilon)$-quasi-isometric to $(H/K, d)$.*

Proof. First, if H is metrisable, then, as noted above, there is a left-invariant admissible metric d_1 on H. We modify d_1 if necessary so that it is right-K-invariant, by defining d_2 by

$$d_2(x, y) \coloneqq \max\{d_1(xk, yk) : k \in K\} \qquad \forall x, y \in H.$$

Lemma 2.16 shows that d_2 is a metric. By Lemma 2.14, d, defined by

$$d(xK, yK) = \inf\{d_2(xk, yk') : k, k' \in K\} \qquad \forall xK, yK \in H/K,$$

is an admissible metric on H/K, and H acts isometrically on $(H/K, d)$.

Finally, we may find an admissible metric d_H on H such that (H, d_H) is $(1, \varepsilon)$-quasi-isometric to $(H/K, d)$ using Theorem 2.7. □

Now we discuss covering maps of homogeneous metric spaces. If $M^\sharp$ and M are connected topological spaces, then a continuous surjection $\pi : M^\sharp \to M$ is said to be a *covering map* provided that, for all sufficiently small neighbourhoods U in M, there are disjoint neighbourhoods V_z in $M^\sharp$, where $z \in Z$, such that $\pi^{-1}(U) = \bigsqcup_{z \in Z} V_z$ and the restriction of π to V_z is a homeomorphism onto U.

In the case of connected topological groups, which we write $H^\sharp$ and H, we take π to be a homomorphism, with kernel Z. In this case, Z is discrete and normal in $H^\sharp$, which implies that Z is central, since $\{x \in G : xzx^{-1} = z\}$ is both open and closed in G for each $z \in \ker \pi$. For such π, for all sufficiently small neighbourhoods U in H, there is a neighbourhood V in $H^\sharp$ such that the restriction of π to V is a homeomorphism onto U and $\pi^{-1}(U) = \bigsqcup_{z \in \ker \pi} zV$.

When we deal with homogeneous metric spaces, universal covering spaces need not exist; consider, for example, an infinite product of circles.

Lemma 2.19. *Suppose that $\pi : G^\sharp \to G$ is a covering map of connected locally compact topological groups and that $K^\sharp$ and K are closed subgroups of $G^\sharp$ and G such that $K^\sharp$ is an open subgroup of $\pi^{-1}K$. Then the canonical projection $\pi^\sharp : G^\sharp/K^\sharp \to G/K$ is a covering map. Suppose that d is a G-invariant metric on G/K. Then for all $\varepsilon \in \mathbb{R}^+$, there exists a $G^\sharp$-invariant metric $d^\sharp$ on $G^\sharp/K^\sharp$ such that*

$$d^\sharp(x, y) - \varepsilon \le d(\pi x, \pi y) \le d^\sharp(x, y) \qquad \forall x, y \in G^\sharp/K^\sharp.$$

If K_1 is a connected subgroup of G that contains K and d is right-K_1-invariant, then $d^\sharp$ may be taken to be right-$\pi^{-1}K_1$-invariant.

Proof. The mapping $\pi^\sharp$ is the composition of two mappings: the canonical projection from $G^\sharp/K^\sharp$ to $G^\sharp/\pi^{-1}K$ and the canonical isomorphism of $G^\sharp/\pi^{-1}K$ with G/K, which is a homeomorphism. It is obvious that we can use the latter map to transfer the metric from G/K to $G^\sharp/\pi^{-1}K$ so that the homeomorphic isomorphism is also an isometry, so it suffices to deal with the canonical projection. To simplify the notation, we replace $G^\sharp$, $K^\sharp$, $\pi^\sharp$ and $\pi^{-1}K$ by G, K, π and $K^\flat$. Thus K is an open subgroup of $K^\flat$, which is a closed subgroup of G, and we consider the projection $\pi : G/K \to G/K^\flat$; we need to prove that π is a covering map and show how to lift a metric on $G/K^\flat$ to G/K.

From the hypotheses, we may find points $z_j \in K^\flat$ such that $K^\flat = \bigsqcup_j z_j K$. Moreover, there is a symmetric (that is, $U = U^{-1}$) open set U in G such that $U^2 \cap K^\flat = K$. Then the sets Uz_jK are open in G and disjoint, and the mapping $uz_jK \mapsto uK^\flat$ is a homeomorphism from Uz_jK to $UK^\flat$, and then by the G-equivariance of π, the restriction of π to a set gUz_jK, where $g \in G$, is a homeomorphism to $gUK^\flat$. It follows that π is a covering map.

Next, a metric d on $G/K^\flat$ gives rise to a pseudometric $\dot{d}$ on G with kernel $K^\flat$. We may define a (not necessarily proper) metric d_1 on G/K by choosing ε small enough that $B(eK^\flat, \varepsilon) \subseteq UK^\flat$, and then setting

$$d_1(xK, yK) \coloneqq \begin{cases} \min\{d(\pi x, \pi y), \varepsilon) & \text{if } x, y \in gUK \text{ for some } g \in G \\ \varepsilon & \text{otherwise.} \end{cases}$$

We leave to the reader the task of checking that a suitable linear combination $d_{G/K}$ of $\dot{d}$ and d_1 has the required properties. $\square$

Lemma 2.20. *Let H be a locally compact group with closed subgroups S_1 and S_2 such that $H = S_1 \cdot S_2$, and let $H_\times = S_1 \times S_2$. Let $\omega : H_\times \to H$ be the mapping $(s_1, t) \mapsto s_1 t^{-1}$. Then ω is a homeomorphism. Further, if $\dot{d}$ is a left-invariant and right-S_2-invariant continuous admissible pseudometric on H, then $\dot{d}_\times$, given by*

$$\dot{d}_\times((s_1, s_2), (s_1', s_2')) = \dot{d}(s_1 s_2^{-1}, s_1' s_2'^{-1}) \qquad \forall s_1, s_1' \in S_1 \quad \forall s_2, s_2' \in S_2,$$

is a left-invariant continuous admissible pseudometric on $S_1 \times S_2$.

Proof. Since $s_2 \mapsto s_2^{-1}$ is a homeomorphism of S_2 and $\psi : (s_1, s_2) \to s_1 s_2$ is a homeomorphism, ω is a homeomorphism from $H_\times$ to H. Since $\dot{d}$ is a left-S_1-invariant and right-S_2-invariant pseudometric, $\dot{d}_\times$ is a left-$(S_1 \times S_2)$-invariant pseudometric. Since $\dot{d}$ is continuous, so is $\dot{d}_\times$.

Let K be the kernel of $\dot{d}$ and $K_\times$ be the kernel of $\dot{d}_\times$. From Lemma 2.11,

$$p_\times K_\times = \{q_\times \in H_\times : \dot{d}_\times(p_\times, q_\times) = 0\}$$

and

$$pK = \{q \in H : \dot{d}(p, q) = 0\}.$$

for all $p_\times \in H_\times$ and all $p \in H$. The definition of $\dot{d}_\times$ then implies that

$$q \in pK_\times \iff \dot{d}_\times(p, q) = 0 \iff \dot{d}(\omega(p), \omega(q)) = 0 \iff \omega(q) \in \omega(p)K.$$

It follows that ω induces a homeomorphism from $H_\times/K_\times$ to H/K, which is an isometry by construction. The admissibility of $\dot{d}$ and that of $\dot{d}_\times$ are therefore equivalent. $\square$

We note conversely that if the map $\omega : S_1 \times S_2 \to H$, given by $\omega(s_1, s_2) = s_1 s_2^{-1}$ is an isometry from the pseudometric group $S_1 \times S_2$ to the pseudometric group H, then the pseudometric on H must be right-S_2-invariant.

2.5. **Simply transitive isometry groups.** Here we are interested in the question whether a homogeneous metric space admits a simply transitive isometry group.

Theorem 2.21. *Let (M, d) be a homogeneous metric space, H denote* $\operatorname{Iso}(M, d)$ *and K denote the stabiliser of a base point o in M; let G be a group. Then the following are equivalent:*

(i) there is a simply transitive action of G on M by isometries;
(ii) there is a left-invariant metric d_G on G such that (G, d_G) is isometric to (M, d);
(iii) there is a monomorphism $\alpha : G \to H$ such that $H = \alpha(G)K$ and $\alpha(G) \cap K = \{e_H\}$.

In addition, if (i), (ii) and (iii) hold, and G is a topological group, then the following are equivalent:

(iv) the metric d_G of (ii) is admissible;

(v) α is a homeomorphism from G to $\alpha(G)$, equipped with the relative topology as a subset of H.

Finally if (i) to (v) all hold, then $\alpha(G)$ is closed in H.

Proof. Suppose that (i) holds, and denote the action by α. We define the left-invariant pull-back metric d_G on G by

$$d_G(g, g') = d(\alpha(g)o, \alpha(g')o) \qquad \forall g, g' \in G;$$

then the map $g \mapsto \alpha(g)o$ is an isometry from (G, d_G) to (M, d), so (ii) holds.

Assume that (ii) holds, and that $F\colon (G, d_G) \to (M, d)$ is an isometry. By composing with a translation of G if necessary, we may suppose that $F(e) = o$. For $g \in G$, define the mapping $\alpha(g) : M \to M$ by the formula

$$\alpha(g)(p) = F(gF^{-1}(p)) \qquad \forall p \in M.$$

It is straightforward to check that (iii) holds.

Now assume that (iii) holds. Then $\alpha(G)$ is transitive since every element of H may be written as $\alpha(g)k$ where $g \in G$ and $k \in K$, and $\alpha(G)$ is simply transitive since $\alpha(G) \cap K = \{e_H\}$. So G acts simply transitively by isometries on (M, d), and (i) is proved.

Now assume that (i), (ii) and (iii) hold, and that G is a topological group. Consider, for g and a net of elements g_ν in G, the following statements:

(a) $g_\nu \to g$ in G as $\nu \to \infty$;
(b) $g_\nu g' \to gg'$ in G as $\nu \to \infty$ for all $g' \in G$;
(c) $d_G(g_\nu g', gg') \to 0$ as $\nu \to \infty$ for all $g' \in G$;
(d) $d(\alpha(g_\nu g')(o), \alpha(gg')(o)) \to 0$ as $\nu \to \infty$ for all $g' \in G$;
(e) $\alpha(g_\nu)(p) \to \alpha(g)(p)$ in M as $\nu \to \infty$ for all $p \in M$;
(f) $\alpha(g_\nu) \to \alpha(g)$ in H.

Since G is a topological group, (a) and (b) are equivalent, while (c) and (d) are equivalent by definition, (d) and (e) are equivalent by writing $p = g'(o)$, and (e) and (f) are equivalent by definition of the topology on H. Further, (b) and (c) are equivalent if and only if d_G is admissible.

If d_G is admissible, then (a) and (f) are equivalent, so α is a homeomorphism of G onto its image in H. Conversely, if the topology of $\alpha(G)$ induced by that of G coincides with that induced by H, then (a) and (f) are equivalent, and so d_G is admissible.

We now suppose that if (i) to (v) all hold, and show that $\alpha(G)$ is closed in H. We take a net (g_ν) in G such that $\alpha(g_\nu) \to h$ in H, and need to prove that $h \in \alpha(G)$. Now $h = \alpha(g)k$, where $g \in G$ and $k \in K$; by replacing g_ν by $g^{-1}g_\nu$ if necessary, we may assume that $\alpha(g_\nu) \to k$ in H, and must prove that $k = e$. Now

$$d_G(g_\nu, e_G) = d(\alpha(g_\nu)o, o) \to d(ko, o) = 0,$$

so $g_\nu \to e_G$, as required. □

The theorem above shows that, if we are looking for metric groups that are isometric to a given homogeneous space, and whose topology is related to that of the homogeneous space, it will suffice to look for closed subgroups of the isometry group. Actually, since our homogeneous spaces are assumed to be connected, it will suffice to look for closed subgroups of the connected component of the identity in the isometry group. The conditions in the theorem will appear quite often, and so it is useful to have some additional notation.

Definition 2.22. If G and K are subgroups of a group H, then GK denotes the subset $\{gk : g \in G, k \in K\}$ of H.

We write $H = G \cdot K$ to indicate that G and K are closed subgroups of a locally compact group H, such that the mapping $(g, k) \mapsto gk$ from the set $G \times K$ with the product topology to H is a homeomorphism.

If $H = G \cdot K$ and moreover G is normal in H, then we write $H = G \rtimes K$ and call H the semidirect product of G and K.

Remark 2.23. If $H = G \cdot K$, then G is homeomorphic to H/K. Further, if H is connected, so are G and K.

The subgroup K is not required to be compact in Definition 2.22. However, if K is compact, then the condition that the mapping is a homeomorphism in the definition of the expression $H = G \cdot K$ is satisfied provided only that the mapping is a bijection. Indeed, if (g_ν) and $(k_{\nu'})$ are nets such that $g_\nu \to g$ in G and $k_{\nu'} \to k$ in K, then $g_\nu k_{\nu'} \to gk$ in H since multiplication is continuous. Conversely if G is closed and K is compact, and $g_\nu k_\nu \to h$ in H, then, by passing to a subnet, we may assume that $k_\nu \to k$ in K, and then $g_\nu \to hk^{-1}$ in H and so in G since G is closed; if the net k_ν had two limits, then we could factorise h as a product gk in two distinct ways, which contradicts bijectivity.

The next lemma is about groups that nearly act simply transitively.

Lemma 2.24. *Suppose that α is a continuous monomorphism of a connected locally compact group G into a connected metrisable locally compact group H, and that K is a compact subgroup of H. Let $\omega : G \times K \to H$ be the continuous mapping $(g, k) \mapsto \alpha(g)k^{-1}$. Suppose also that there are neighbourhoods U_0 of e_G in G and V_0 of e_K in K such that, if $e_G \in U \subseteq U_0$ and $e_K \in V \subseteq V_0$, then the restricted mapping $\omega|_{U\times V}$ is a bijection onto a neighbourhood of e_H in H. Then*

(i) $H = \alpha(G)K$,
(ii) there is an open set U_1 in G containing e_G such that the restriction $\omega|_{U_1\times K}$ is a homeomorphism onto its image, with the relative topology;
(iii) $\alpha^{-1}(K)$ is discrete in G and $G/\alpha^{-1}(K)$ is homeomorphic to H/K;
(iv) $\alpha(G) \cap K$ is finite if and only if $\alpha(G)$ is closed in H; and
(v) if $\alpha(G) \cap K = \{e_H\}$, then $H = \alpha(G) \cdot K$.

Proof. To prove (i), we equip the connected space H/K with an H-invariant metric, by using Corollary 2.18, so that G acts isometrically on H/K. By assumption, $\omega(G \times K)$ contains a neighbourhood of e_H, so the image of the base point K in H/K under $\alpha(G)$ contains a neighbourhood of the base point, whence G acts transitively on H/K by part (ii) of Theorem 2.7, and $H = \alpha(G)K$. Hence (i) holds.

Now we prove (ii). By compactness, there exist finitely many points $k_1, \dots, k_I$ in K such that $K = \bigcup_i k_i V_0$. Suppose that i in $\{1, \dots, I\}$. If $\alpha(U_0) \cap k_i V_0 \neq \varnothing$, then there exist $u_i \in U_0$ and $v_i \in V_0$ such that $\alpha(u_i) = k_i v_i$. Now if $u \in U_0 \cap \alpha^{-1}(K)$, then there exist j in $\{1, \dots, I\}$ and $v \in V_0$ such that $\alpha(u) = k_j v$. We deduce that

$$\alpha(u)v^{-1} = k_j = \alpha(u_j)v_j^{-1},$$

whence $u = u_j$. Thus

$$U_0 \cap \alpha^{-1}(K) = \{u_j : \alpha(U_0) \cap k_j V_0 \neq \varnothing,\ k_j = \alpha(u_j)v_j^{-1}\},$$

which is a finite set. It follows that there exists a neighbourhood U_0' of e_G in G such that $\alpha(U_0') \cap K = \{e_H\}$. We take a neighbourhood U_1 of e_G in G such that $U_1^{-1}U_1 \subseteq U_0'$. Now if $g_1, g_2 \in U_1$ and $k_1, k_2 \in K$ are such that $\alpha(g_1)k_1^{-1} = \alpha(g_2)k_2^{-1}$, then $\alpha(g_2^{-1}g_1) = k_2^{-1}k_1$ and $g_2^{-1}g_1 \in U_0'$ and $k_2^{-1}k_1 \in K$. It follows that $g_1 = g_2$ and $k_1 = k_2$, and $\omega|_{U_1\times K}$ is a bijection. The hypothesis on ω implies that $\omega|_{U_1\times K}$ is open, and it is continuous by definition.

Part (iii) follows immediately from (ii). Indeed, $\alpha^{-1}(K) \cap U_1 = \{e_G\}$, so the point e_G is isolated in $\alpha^{-1}(K)$. By a translation argument, every point of $\alpha^{-1}(K)$ is isolated, and $\alpha^{-1}(K)$ is discrete. Further, standard isomorphism theorems show that α induces a continuous bijection, $\dot{\alpha}$ say, of $G/\alpha^{-1}(K)$ onto H/K. The hypothesis on ω implies that $\dot{\alpha}$ is open, so $\dot{\alpha}$ is indeed a homeomorphism.

We now prove one implication of (iv). If $\alpha(G)$ is closed in H, then $\alpha(G) \cap K$ is a closed subgroup of K, so is compact. Now G is connected and locally compact by hypothesis, and so is σ-compact; further, $\alpha^{-1}(K)$ is a discrete subgroup of G, and hence there is a neighbourhood W of e_G such that the sets xW, as x ranges over $\alpha^{-1}(K)$, are disjoint. It follows that $\alpha^{-1}(K)$ is countable, whence $\alpha(G) \cap K$ is a countable compact group, hence finite (see the notes and remarks at the end of this section).

Conversely, to complete the proof of (iv), we assume that $\alpha(G) \cap K$ is finite, and take a net (g_ν) in G such that $\alpha(g_\nu) \to h$ in H; we must show that $h = \alpha(g^*)$ for some g^* in G, and $g_\nu \to g^*$ in G. By the transitivity of the G action on H/K, proved in (i), there exists g in G such that $h \in \alpha(g)K$; then $\alpha(g^{-1}g_\nu) \to \alpha(g^{-1})h$ in H, and, by replacing g_ν and h by $g^{-1}g_\nu$ and $\alpha(g^{-1})h$, we may assume that $h \in K$. Next, from (ii), if ν is large enough, there exists $\tilde{g}_\nu$ in U_1 such that $\alpha(\tilde{g}_\nu)K = \alpha(g_\nu)K$, and $\tilde{g}_\nu \to e$ in G; by replacing g_ν by $\tilde{g}_\nu^{-1}g_\nu$, we may assume that $\alpha(g_\nu) \in K$. Since $\alpha(G) \cap K$ is finite, the convergent net g_ν is eventually constant, so the limit is in G.

Finally, if $\alpha(G) \cap K = \{e_H\}$, then $\alpha(G)$ is closed in H from part (iv). By Remark 2.23, $H = \alpha(G) \cdot K$. □

We now clarify when two connected locally compact groups may be made isometric.

Corollary 2.25. *Suppose that G_1 and G_2 are connected locally compact groups. Then G_1 and G_2 may be made isometric if and only if there exists a metrisable locally compact group H with a compact subgroup K such that $H = G_1 \cdot K = G_2 \cdot K$.*

Proof. If G_1 and G_2 may be made isometric, then we may assume that the isometry sends e_1 to e_2, and that they have a common isometry group, H say. Then we may take K to be the stabiliser of e_1 in G_1 or e_2 in G_2.

Conversely, given H and K, Corollary 2.18 constructs a metric d on H/K so that H acts isometrically on $(H/K, d)$. Since G_j acts simply transitively on H/K, we may transport the metric d on H/K to G_j by the formula

$$d_j(x,y) = d(xK, yK) \qquad \forall x, y \in G_j,$$

and obtain left-invariant metrics on G_j, when j is 1 or 2. Now (G_1, d_1) and (G_2, d_2) are both isometric to $(H/K, d)$, and so are isometric to each other. □

2.6. **Invariant measure and growth.** Every locally compact group G admits a Haar measure μ, that is, a left-invariant Radon measure that gives positive mass to all nonempty open sets; the Haar measure is unique up to a multiplicative constant.

If K is a compact subgroup of a locally compact group G, with a left-invariant Haar measure μ, and $\pi : G \to G/K$ is the quotient map, then there is a unique G-invariant Radon measure m on G/K such that

$$m(U) = \mu(\pi^{-1}(U)) \tag{2.12}$$

for all Borel subsets U of G/K; see [37, §15]. From Theorem 2.7 and Corollary 2.8, if (M, d) is a homogeneous metric space and G is $\mathrm{Iso}(M, d)$, then M may be identified with G/K for some compact subgroup K of G. Thus every homogeneous metric space (M, d) admits a unique (up to scalar multiplication) Radon measure that is invariant under $\mathrm{Iso}(M, d)$.

A compactly generated locally compact group G with Haar measure μ is said to be *of polynomial growth* if there is a compact generating neighbourhood U of the identity in G such that

$$\mu(U^n) \le Cn^Q \qquad \forall n \in \mathbb{Z}^+. \tag{2.13}$$

If G is of polynomial growth and V is another compact generating neighbourhood of the identity in G, then the same equation holds but with a possibly different constant C. From part (i) of Lemma 2.2, $m(V_n(o,\ell))$ grows no faster than exponentially in n; however, it may grow only polynomially, or even be bounded.

The following definition is standard, at least for quasigeodesic metrics.

Definition 2.26. Let (M,d) be a homogeneous metric space. We say that (M,d) is of *polynomial growth* if for a given point and hence for an arbitrary point o in M,

$$m(B(o,r)) \le Cr^Q \tag{2.14}$$

for all sufficiently large r.

At this point, for a metric Lie group we have two notions of polynomial growth, which in general are not equivalent. For instance, $\mathbb{R}$ is a group of polynomial growth, but if we define the metric d on $\mathbb{R}$ by

$$d(x,y) := \log(|x-y|+1) \qquad \forall x,y \in \mathbb{R},$$

then $(\mathbb{R},d)$ is not of polynomial growth. More generally, $m(B(o,r))$ may grow much faster in r than $m(V_n(o,\ell))$ grows in n.

A proper quasigeodesic homogeneous metric space is of polynomial growth if and only if its isometry group is of polynomial growth. For general metric spaces, only one implication may be proved, as follows.

Lemma 2.27. *If M is a homogeneous metric space of polynomial growth, and G is a subgroup of* $\mathrm{Iso}(M,d)$ *that acts transitively on M, then G is of polynomial growth.*

Proof. By part (v) of Theorem 2.7, we may fix $o \in M$ and $\ell \in \mathbb{R}^+$ such that the set $U := \{f \in G : f(o) \in \breve{B}(o,\ell)\}$ is a compact neighbourhood of the identity element in G and

$$U^n = \{f \in G : f(o) \in V_n(o,\ell)\}.$$

Let μ be a Haar measure on G and m be a G-invariant measure on M such that (2.12) holds, as discussed at the beginning of this section, and suppose that $m(B(o,r)) \le Cr^Q$ for all sufficiently large r. Then

$$\mu(U^n) = m(V_n(o,\ell)) \le C\ell^Q(n+1)^Q$$

since $V_n(o,\ell) \subseteq B(o,(n+1)\ell)$. □

We now connect growth to the doubling property.

Definition 2.28. Let (M,d) be a homogeneous metric space. We say that (M,d) is *doubling* if there is a constant N such that each ball of radius $2r$ may be covered by at most N balls of radius r for all $r \in \mathbb{R}^+$. We say that (M,d) is doubling *at small scale* or *at large scale* if the covering property holds for all sufficiently small r or sufficiently large r.

Polynomial growth is often linked with the property of being doubling at large scale. Indeed, if (M,d) is proper quasigeodesic, then it is of polynomial growth if and only if it is doubling at large scale; see, for instance, [18]. However, these two notions are not equivalent in our setting. More precisely, if a metric space (M,d) is doubling at large scale, it may fail to be of polynomial growth; see Remark 2.29. However, if (M,d) is doubling at large scale and proper, then it is of polynomial growth; see Remark 2.30. Conversely, if (M,d) is of polynomial growth, then it is proper, but it does not need to be doubling at large scale; see Remarks 2.31 and 2.32. This paradoxical behaviour reflects the fact that polynomial growth and properness are not quasi-isometric invariants when metrics are not proper quasigeodesic.

Remark 2.29. The space $(\mathbb{R},d)$, where the metric d is given by $d(x,y) = \min\{|x-y|,1\}$, is trivially doubling at large scale, but is evidently not of polynomial growth.

Remark 2.30. If a homogeneous metric space is proper and doubling, then it is of polynomial growth. Indeed, if one and hence every ball of radius $2r$ may be covered by N balls of radius r, then it may be seen that

$$m(B(o,r)) \leq N m(B(o,1)) r^{\log_2(N)}$$

when $r > 1$.

Remark 2.31. It is easy to construct homogeneous metric spaces of polynomial growth that are not locally doubling (consider the product $\prod_{n\in\mathbb{N}}(\mathbb{R}/2^{-n}\mathbb{Z})$, where each factor has the metric induced from the euclidean metric on $\mathbb{R}$ and the product has the ℓ^∞-metric) and to construct nonhomogeneous metric spaces of polynomial growth that are not doubling at large scale (consider sparsely branching $\mathbb{R}$-trees of unbounded degree). The next example shows that having polynomial growth does not even imply being doubling at large scale for proper connected homogeneous metric spaces.

Consider the piecewise linear function $D : [0,+\infty) \to [0,+\infty)$ with nodes at $(0,0)$, $(1,1)$, and (x_n,y_n), where $n \in \mathbb{N}$, given by $x_n = 2^{2^{n+1}}$ and $y_n = 2^{2^n}$. The nodes all lie on the graph $y = x^{1/2}$, so D is evidently increasing and concave. Hence $d(x,y) \coloneqq D(|x-y|)$ is a translation-invariant metric on $\mathbb{R}$, and $|B(x_0,r)| = 2D^{-1}(r)$ for all $r \in [0,+\infty)$.

Take $r = y_n$, and consider the ratio

$$\frac{|B(0,2r)|}{|B(0,r)|} = \frac{D^{-1}(2y_n)}{D^{-1}(y_n)} = \frac{D^{-1}(2y_n)}{x_n}.$$

We shall now show that the right-hand fraction is unbounded in n, which shows that d is not a doubling metric.

If (x,y) lies on the line segment between (x_n,y_n) and (x_{n+1},y_{n+1}), then

$$\frac{y-y_n}{x-x_n} = \frac{y_{n+1}-y_n}{x_{n+1}-x_n} = \frac{y_n^2-y_n}{y_n^4-y_n^2} = \frac{1}{y_n(y_n+1)},$$

so

$$x = x_n + y_n(y_n+1)(y-y_n).$$

Since $2y_n \leq y_{n+1}$, if $D(x) = 2y_n$, then $(x,2y_n)$ lies on the line segment, and so $x = x_n + x_n(y_n+1)$ and

$$\frac{D^{-1}(2y_n)}{x_n} = \frac{x}{x_n} = y_n + 2,$$

which tends to infinity as n increases.

The same argument also shows that if (x,y) lies on this line segment, then

$$\begin{aligned} |B(0,y)| = 2x &= 2x_n + 2y_n(y_n+1)(y-y_n) \\ &\leq 2y_n^2 + 2y_n y(y_n+1) \leq 2y^2 + 2y^2(y+1), \end{aligned}$$

and it follows that d is of polynomial growth.

Remark 2.32. If (M,d) is a homogeneous metric space of polynomial growth, then it is proper. Indeed, if there were a noncompact closed ball $\breve{B}(p,r)$, then there would be $\varepsilon \in \mathbb{R}^+$ and points x_i in $\breve{B}(p,r)$, where $i \in \mathbb{N}$, such that $d(x_i,x_j) > 2\varepsilon$ if $i \neq j$. But then it would follow that

$$C(r+\varepsilon)^Q \geq m(\breve{B}(p,r+\varepsilon)) \geq \sum_{i\in\mathbb{N}} m(B(x_i,\varepsilon)) = +\infty,$$

which would be a contradiction.

2.7. **Notes and remarks.** Here we include some additional comments on the results established above.

2.2. Homogeneous metric spaces. If f is a metric preserving mapping of a homogeneous metric space (M,d), in the sense that condition (2.1) holds, then f is surjective; this need not be true for metric preserving mappings of general metric spaces. The proof involves first composing with an isometry, so that $f(o) = o$, then using compactness to show that f is bijective on closed balls (defined relative to the Busemann gauge), and finally letting the radius of the balls go to infinity.

2.3. Metric spaces and coset spaces. The simple observations of this section raise further questions about isometry groups. Given a metric space (M,d) and a transitive isometry group G of M, let o be a point in M and K be the stabiliser of o in G. Is there a left-invariant metric d_G on G such that

$$d(p,q) = \min\{d_G(g,h) : g(o) = p, h(o) = q\}?$$

Under what circumstances do the stabilisers of all points in M have the same diameter? And if we equip M with the metric d' that is defined by the right-hand side of the above formula, is it true that $G = \mathrm{Iso}(M,d')$?

2.4. Modifying metrics. The use of pseudometrics leads to another interpretation of Theorem 2.6. Given a metric on a homogeneous metric space (M,d), we may define a family of pseudometrics $\dot{d}_x$, where x runs over M, on the isometry group H, by setting $\dot{d}_x(g,h) = d(gx,hx)$ for all $g,h \in H$. If $g,h \in H$ and $\dot{d}_x(g,h) = 0$ for all x in M, then $g^{-1}h$ acts trivially on M, so $g = h$. Thus expressions such as $\sup_{x\in M} \dot{d}_x(g,h)$, where x runs over M, only vanish when $g = h$. The pseudometrics $\dot{d}_x$ satisfy the inequality

$$\begin{aligned}\dot{d}_x(g,h) = d(gx,hx) &\leq d(gx,gy) + d(gy,hy) + d(hy,hx)\\ &\leq d(gy,hy) + 2d(x,y) = \dot{d}_y(g,h) + 2d(x,y)\end{aligned}$$

for all $g,h \in H$, and if M is unbounded, then $\sup_{x\in M} \dot{d}_x(g,h)$ might well be infinite. However, the formula given in Theorems 2.6 and 2.7 is but one of many ways of combining these pseudometrics to get a metric on H.

We will use Corollary 2.17 later. For future purposes, we note that if K_o and K are compact subgroups of a Lie group G and $K_o \subset K$, then there exists a riemannian metric d on G/K_o such that

$$d(gg'kK_o, gg''kK_o) = d(g'K_o, g''K_o) \qquad \forall g,g',g'' \in G \quad \forall k \in K.$$

All riemannian metrics are bi-Lipschitz equivalent.

The reader may wish to check whether the new metrics produced in Corollary 2.15 or Lemma 2.15 are proper or derived semi-intrinsic (as defined just before Lemma 2.3) or proper quasigeodesic or geodesic if the initial metric has this property.

2.5. Simply transitive isometry groups. Definition 2.22 deserves a further comment.

In the definition of a semidirect product, it suffices to suppose that G and K are closed subgroups and G is normal, and the mapping $(g, k) \mapsto gk$ is a bijection. Indeed, if $g_\nu \to g$ in G and $k_\nu \to k$ in K, then $g_\nu k_\nu \to gk$ in H by definition. Conversely, if $g_\nu k_\nu \to gk$ in H, then $Gk_\nu \to Gk$ in the quotient group $G\backslash H$, which is homeomorphic to K by [37, Theorem 5.26], that is, $k_\nu \to k$ in K, and hence also $g_\nu \to g$ in G.

The results of this and the previous section offers us an alternative viewpoint on homogeneous metric spaces and their isometry groups. We begin by taking the basic object to be a metric space (M, d) with a topology that is compatible with the metric, and showed that a closed subgroup H of the isometry group that acts transitively is a topological group with a metric compatible with the topology, and that the projection from H to M is both a metric projection and a topological projection (that is, it is continuous and open). However, we might also take the basic object to be a metrisable topological group H, and consider various quotient spaces H/K with the quotient topologies and quotient metrics, or even just a topological group H acting on a quotient space H/K that may be endowed with a metric that is compatible with the quotient topology.

2.6. Invariant measure and growth. Suppose that M is the coset space G/K, where G is a (not necessarily connected) locally compact group and K is a compact subgroup. We claim that if M is compact and countable, then M is finite. Indeed, M admits a G-invariant Radon measure m, and the regularity of M implies that there is an open set U of positive but finite measure. Since M is compact, $m(M)$ must be finite, since it may be covered by finitely many translates of U. All points of M have the same measure. If points had measure 0, then M would have measure 0; hence points have positive measure and the cardinality of M is $m(M)/m(\{p\})$ for any point p.

3. Lie theory and metric spaces

This section is concerned with homogeneous metric manifolds, which for us are locally euclidean, but not *a priori* smooth. However, as a consequence of the solution of Hilbert's fifth problem, they are quotient spaces of Lie groups, and hence may be given analytic structures such that the connected component of the identity in the isometry group acts analytically.

In this section we first review the Gleason–Iwasawa–Montgomery–Yamabe–Zippin structure theorem of almost connected locally compact groups, and then discuss some variants and consequences thereof, which include our first main theorem, that homogeneous metric spaces may be approximated by homogeneous metric manifolds.

We then look at more Lie theory, such as the Levi decomposition, and see how this enables us to prove our second main theorem, on the finer structure of homogeneous metric manifolds. We should mention that there have been exhaustive investigations into the homogeneous spaces of semisimple Lie groups and those of solvable Lie groups, but the general case seems less well known.

Many of the results here may be proved by a reduction to the riemannian case and then appealing to the appropriate classical result. Indeed, as we shall see in Corollary 3.4, if two homogeneous metric manifolds are isometric, then they admit riemannian structures for which they are isometric. However, classical riemannian geometers did not consider quasi-isometries, and at least some of our theorems are not true in the context of isometries, and are certainly not in the literature (at least in forms that we are able to recognise).

3.1. **The main structure theorem.** A locally compact group G is said to be *almost connected* if G/G_0 is compact, where G_0 is the connected component of the identity in G; this is closed and normal. The isometry groups of homogeneous metric spaces are almost connected, by Theorem 2.6.

We recall without proof one version of the solution to Hilbert's fifth problem by Gleason, Iwasawa, Yamabe, Montgomery and Zippin. See, for instance, [54, Section 4.6] or [69, Theorem 1.6.1].

Theorem 3.1. *Let G be an almost connected locally compact group. Then every neighbourhood U of the identity in G contains a compact normal subgroup N such that G/N is locally euclidean. If G is locally euclidean, then G may be given a unique analytic structure for which it is a Lie group.*

The following related result was first stated by Szenthe [68]. Unfortunately, there was a mistake in his argument, discovered by Antonyan, but the gap was filled independently by Antonyan and Dobrowolski and by George Michael. See Glockner's review [27] for the history and location of the proof.

Theorem 3.2. *If K is a compact subgroup of an almost connected locally compact group H, and $\bigcap_{h\in H} hKh^{-1} = \{e\}$, then the following are equivalent:*

(i) H is a Lie group and H/K is a manifold;
(ii) H/K is locally contractible.

Corollary 3.3. *Let K be a compact subgroup of an almost connected locally compact group H such that H/K is connected and $\bigcap_{h\in H} hKh^{-1} = \{e\}$. Suppose also that H/K is locally euclidean or that H is locally euclidean. Then H and hence H/K may be given analytic structures, compatible with their topologies, such that H is a Lie group and the action of H on H/K is analytic.*

Proof. If H/K is locally euclidean, then so is H, by Theorem 3.2, so we may assume that H is locally euclidean.

By Theorem 3.1, we may endow H with an analytic structure so that H becomes a Lie group, and this analytic structure on H induces an analytic structure on H/K. These analytic structures are compatible with the topologies of H and H/K. Further, H acts analytically on H/K. □

In particular, if (M,d) is a homogeneous metric manifold, H is its isometry group, and K is the stabiliser of a point o in M in H, then we may identify M with H/K and apply this corollary to deduce that H and M have analytic structures such that H acts analytically on M.

In light of Theorems 3.1 and 3.2 and Lemma 3.6 below, there are several criteria which ensure that H is locally euclidean or H/K is locally euclidean.

Corollary 3.4. *Let (M_1,d_1) and (M_2,d_2) be homogeneous metric manifolds. Then there exist analytic structures and left-invariant analytic infinitesimal riemannian metrics g_1 and g_2 on M_1 and M_2 such that*

(i) $\mathrm{Iso}(M_1,d_1) \subseteq \mathrm{Iso}(M_1,g_1)$ *and* $\mathrm{Iso}(M_2,d_2) \subseteq \mathrm{Iso}(M_2,g_2)$*; and*
(ii) each isometry f from (M_1,d_1) to (M_2,d_2) is also an isometry from (M_1,g_1) to (M_2,g_2).

Proof. Write H_1 and H_2 for $\mathrm{Iso}(M_1,d_1)$ and $\mathrm{Iso}(M_2,d_2)$, and let K_1 and K_2 be the stabilisers in H_1 and H_2 of points o_1 in M_1 and o_2 in M_2; we may and shall identify M_1 and M_2 with H_1/K_1 and H_2/K_2. By the previous result, H_1 and H_2 are Lie groups and act analytically on H_1/K_1 and H_2/K_2.

The action of K_1 on H_1/K_1 induces an action of K_1 on the tangent space to H_1/K_1 at the point K. Take an inner product on this tangent space; then by averaging over the action of K_1 using the Haar measure of K_1, we may assume that the inner product is K_1-invariant. We may extend this inner product to an analytic left-invariant infinitesimal riemannian metric g_1 on H_1/K_1; the key is that if h and h' in H_1 both map K_1 to hK_1, then $h' = hk$ for some $k \in K_1$, and the K_1-invariance of the inner product at the point K_1 implies that h and h' induce the same inner product at hK_1. It follows immediately that H_1 acts on $(H_1/K_1, g_1)$ by riemannian isometries, and we conclude that $\mathrm{Iso}(M_1, d_1) \subseteq \mathrm{Iso}(H_1/K_1, g_1)$.

If there are no isometries from (M_1, d_1) to (M_2, d_2), we repeat this argument to put a riemannian metric on M_2, and there is nothing more to prove.

Otherwise, we take one isometry f from (M_1, d_1) to (M_2, d_2); we may and shall suppose that $f(o_1) = o_2$. Conjugation with f induces a homeomorphic isomorphism F of the isometry groups $\mathrm{Iso}(M_1, d_1)$ and $\mathrm{Iso}(M_2, d_2)$, and $F(K_1) = K_2$. Hence we may identify f with the map $xK_1 \mapsto F(x)K_2$ from H_1/K_1 to H_2/K_2. The groups H_1 and H_2 are Lie groups, and continuous homomorphisms of Lie groups are automatically analytic, so F is analytic.

We transport the infinitesimal riemannian metric g_1 on H_1/K_1 to an infinitesimal riemannian metric g_2 on H_2/K_2, and then f is also an analytic riemannian isometry from (M_1, g_1) to (M_2, g_2); further, $\mathrm{Iso}(M_2, d_2) \subseteq \mathrm{Iso}(M_2, g_2)$.

Finally, if f' is any isometry from (M_1, d_1) to (M_2, d_2), then $f^{-1} \circ f' \in H_1$. It follows that f' is also a riemannian isometry from (M_1, g_1) to (M_2, g_2). □

This result was proved for metric Lie groups in [44, Proposition 2.4].

3.2. **Compact subgroups.**

We summarise some results about compact subgroups of connected locally compact groups, and establish some corollaries of the structure theorems above.

Lemma 3.5 (After Iwasawa [39])**.** *Let G be a connected locally compact group. Then every compact subgroup of G is contained in a maximal compact subgroup of G, and all maximal compact subgroups are conjugate to each other.*

If N is a connected normal subgroup of G and K is a maximal compact subgroup of G, then $N \cap K$ is a maximal compact subgroup of N and KN/N is a maximal compact subgroup of G/N; conversely, if K_N is a maximal compact subgroup of N and $K_{G/N}$ is a maximal compact subgroup of G/N, then there exists a maximal compact subgroup K of G such that $K \cap N = K_N$ and $KN/N = K_{G/N}$.

Proof. The first result is [39, Theorem 13], and the second is [39, Lemma 4.10]. In both cases, the results are first proved for Lie groups and then for groups that admit approximations by Lie groups, as in Theorem 3.1. □

It follows that the intersection of all maximal compact subgroups is the unique maximal compact normal subgroup of a connected locally compact group.

The following result is almost standard and may be extended (see [4]); compact contractibility is the only new ingredient. We say that a topological space M is *compactly contractible* if, for each compact subset S of M, there are $x \in M$ and a continuous map $F : [0,1] \times S \to M$ such that $F(0, s) = s$ and $F(1, s) = x$ for all $s \in S$.

Lemma 3.6. *If K is a compact subgroup of a connected locally compact group H, then the following are equivalent:*

(i) K is a maximal compact subgroup of H;
(ii) H/K is homeomorphic to a euclidean space;

(iii) H/K is contractible;
(iv) H/K is compactly contractible.

Proof. By [54, page 188], (i) implies (ii). It is trivial that (ii) implies (iii) and (iii) implies (iv). We prove that (iv) implies (i) by modifying the argument of [4, Theorem 1.3] that shows that (iii) implies (i).

Suppose that (iv) holds. By [7], there is a maximal compact subgroup K_0 of H that contains K, and then by [54, page 188], there is a map $\Phi : \mathbb{R}^n \to H$ such that the map $(x, y) \mapsto \Phi(x)y$ is a homeomorphism from $\mathbb{R}^n \times K_0$ to H. Hence H/K is homeomorphic to $\mathbb{R}^n \times K_0/K$. The contraction of the compact set K_0/K in H/K composed with the projection onto K_0/K is a contraction of K_0/K. From Antonyan [4], K_0/K is contractible if and only if $K = K_0$, so K is maximal. ☐

3.3. **Proof of Theorem A.** In this section, we prove our first main theorem, which we restate in more detailed form.

Theorem 3.7. *Let (M, d) be a homogeneous metric space, and H be the connected component of the identity in* $\mathrm{Iso}(M, d)$.

(i) For all positive ε, there is a connected metric Lie group $(H_\varepsilon, d_\varepsilon)$ and a $(1, \varepsilon)$-quasi-isometry $\varphi : M \to H_\varepsilon$.
(ii) There are an H-invariant metric d_0 of M, a contractible metric manifold (M', d') and an H-equivariant projection π from (M, d_0) to (M', d'), such that the identity mapping is a homeomorphic rough isometry from (M, d) to (M, d_0), and π is a homogeneous metric projection with compact fibre, and hence a rough isometry.

Proof. Let K_o be the stabiliser of a point o in M, so that M may be identified with H/K_o.

To prove part (a), take a compact normal subgroup N of H such that H/N is a Lie group and No has diameter less that ε. Define

$$\dot{d}(g, h) = \sup_{k \in N} d(gko, hko).$$

By Lemma 2.16, $\dot{d}$ is a continuous admissible left-invariant and right-K_oN-invariant pseudometric on H, and

$$d(go, ho) \le \dot{d}(g, h) \le d(go, ho) + 2\,\mathrm{diam}(No) \qquad \forall g, h \in H.$$

By the second part of Lemma 2.14, there is an admissible metric d' on $M' \coloneqq H/K_oN$ such that

$$d'(gK_oN, hK_oN) = \dot{d}(g, h) \qquad \forall g, h \in H.$$

Hence (M, d) is $(1, \varepsilon)$-quasi-isometric to the homogeneous metric manifold (M', d'). By Theorem 2.7, (M', d') is itself $(1, \varepsilon)$-quasi-isometric to the metric Lie group $(H/N, d'_\varepsilon)$.

The proof of part (b) is similar. Let K be a maximal compact subgroup of H such that $K_o \subseteq K$, whence $K_oN \subseteq K$, and take M' to be G/K. As before, we lift the metric d on M to a pseudometric $\dot{d}$ on H with kernel K_o, using Lemma 2.11, and then using Lemma 2.16, we define a left-invariant, right-K-invariant pseudometric $\ddot{d}$ on H by

$$\ddot{d}(g, h) = \max\{\dot{d}(gk, hk) : h \in K\}.$$

This then induces H-invariant metrics d_0 on G/K_o and d' on $M' \coloneqq G/K$ by Lemma 2.14, and the projection from G/K_o to G/K has the required properties by construction. ☐

3.4. **Lie groups and algebras.** To say more about homogeneous metric spaces, we need more background on Lie theory; we review some aspects thereof in this section. We begin with some standard definitions and results.

Recall that the adjoint group of a Lie algebra $\mathfrak{h}$ is the Lie group of linear transformations of $\mathfrak{h}$ generated by the elements $\exp(\mathrm{ad}(X))$, where $X \in \mathfrak{h}$. Recall also that if H is a Lie group with Lie algebra $\mathfrak{h}$, and $\mathfrak{g}$ is a subalgebra of $\mathfrak{h}$, then there is a Lie subgroup G of H whose Lie algebra is $\mathfrak{g}$, but G need not be closed. Next, if G is a Lie subgroup of H, then G with its own Lie structure is analytically immersed, but not necessarily embedded, in H. Of course, G is embedded if and only if it is closed. In light of this correspondence between Lie groups and algebras, we denote the Lie algebra of a Lie group G by the corresponding fraktur letter $\mathfrak{g}$.

We recall also that a discrete normal subgroup Γ of a connected Lie group G is central, since $\{x \in G : x\gamma x^{-1} = \gamma\}$ is both open and closed in G for each $\gamma \in \Gamma$. This implies that if G is connected and Γ is a discrete central subgroup, then a discrete subgroup Δ of G that contains Γ is central in G if and only if Δ/Γ is central in G/Γ.

Definition 3.8. A *torus* or *toral group* is a connected compact abelian Lie group, that is, a finite power of the multiplicative group of complex numbers of modulus 1. A subalgebra $\mathfrak{t}$ of a Lie algebra $\mathfrak{h}$ is *compact* if $\mathrm{ad}(U)$ is semisimple and has purely imaginary eigenvalues on $\mathfrak{h}$ for all $U \in \mathfrak{t}$ and is *toral* if it is abelian and compact. The subgroup T corresponding to a compact subalgebra need not be compact, but $\mathrm{Ad}(T)$ is a compact subgroup of $\mathrm{End}(\mathfrak{h})$, and is a torus if $\mathfrak{t}$ is toral.

If K is a compact subgroup of a connected Lie group H, then $\mathfrak{k}$ is a subalgebra of $\mathfrak{h}$, and $\mathrm{ad}(U)$ is semisimple and has purely imaginary eigenvalues for all $U \in \mathfrak{k}$. Indeed, by averaging any inner product over K, using the Haar measure, we may produce an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{h}$; then $\mathrm{Ad}(K)$ is a group of orthogonal mappings of $\mathfrak{h}$. Hence if U in $\mathfrak{k}$, then $\exp(t\,\mathrm{ad}(U))$ is semisimple with eigenvalues of modulus 1 for all $t \in \mathbb{R}$, and $\mathrm{ad}(U)$ is semisimple with purely imaginary eigenvalues. If moreover K is a torus, then K is abelian and $\mathfrak{k}$ is abelian; in this case we may simultaneously diagonalise $\mathrm{ad}(K)$ acting on the complexification of $\mathfrak{g}$. (For information about complexifications of Lie algebras, see, for example, [70, p. 47].)

In general, the implicit use of an inner product to construct complements of subspaces that are invariant under the action of a compact group K, or to decompose a space into a direct sum of minimal invariant subspaces, or to show that $\mathrm{ad}(U)$ acts semisimply with purely imaginary eigenvalues for all U in its Lie algebra $\mathfrak{k}$ will be referred to here as "Weyl's unitarian trick", though for Weyl this was just the starting point. See [70, p. 342] for more information. Quite often the compact group K will be a torus, and we usually write T rather than K in this case.

Finally we recall that, if G is a connected Lie group with Lie algebra $\mathfrak{g}$, then the radical R of G is the maximal connected solvable normal subgroup of G, while the nilradical N is the maximal connected nilpotent normal subgroup of G; both are closed. Their Lie algebras $\mathfrak{r}$ and $\mathfrak{n}$ are the maximal ideals of $\mathfrak{g}$ that are respectively solvable and nilpotent. (The existence of these ideals may be established by showing that the sum of nilpotent or solvable ideals is a nilpotent or solvable ideal respectively, whence the sum of all nilpotent or solvable ideals is the largest nilpotent or solvable ideal respectively.) Sometimes we write $R = \mathrm{rad}(G)$ and $N = \mathrm{nil}(G)$, or $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$ and $\mathfrak{n} = \mathrm{nil}(\mathfrak{g})$.

Remark 3.9. It is well-known that $[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}$.

For these results and much more, see Bourbaki [12, pp. 44–47 and p. 354] or Varadarajan [70, pp. 204–207 and 244–245].

We will need a structural result concerning tori in a connected Lie group H; this illustrates the power of Lie theory in establishing results that are of interest in our study of homogeneous metric spaces.

Lemma 3.10. *Let H be a connected Lie group with nilradical N. If T is a normal torus in H, then $T \subseteq N$. If K is a compact subgroup of N, then K is central in H.*

Proof. Let $\mathfrak{h}$, $\mathfrak{t}$, $\mathfrak{k}$ and $\mathfrak{n}$ be the Lie algebras of H, T, K and N.

Since $\mathfrak{t}$ and $\mathfrak{n}$ are nilpotent ideals, so is $\mathfrak{t} + \mathfrak{n}$, and consequently $\mathfrak{t} \subseteq \mathfrak{n}$, that is, $T \subseteq N$.

We now take a compact subgroup K of N, and show that K is central in H. We suppose without loss of generality that K is a maximal compact subgroup of N, so that K is connected; as K is also compact and nilpotent, K is abelian.

Let Z be the centre of N, which is closed and connected [70, Corollary 3.6.4], and so of the form $T \times V$, where T is a torus and V is a vector space. Since KZ/Z is a compact subgroup of the simply connected nilpotent group N/Z, whose only compact subgroup is trivial, $K \subseteq Z$, and hence $K = T$.

We have now shown that T is central in N, but not that T is central in H. To complete the proof, we write Λ for the set of all elements $U \in \mathfrak{n}$ such that $\exp(U) = e$. Then Λ is a lattice in $\mathfrak{t}$ (see [70, Theorem 3.6.1]), and is contained in the centre $Z(\mathfrak{n})$ of $\mathfrak{n}$, which is a characteristic ideal in $\mathfrak{h}$. Hence for each $U \in \Lambda$, $\mathrm{Ad}(H)U$ is a connected subset of Λ that contains U, and hence coincides with $\{U\}$.

Thus, if $h \in H$, then $\mathrm{Ad}(h)_{Z(\mathfrak{n})}$ is a linear mapping that fixes all U in Λ, and hence acts trivially on the linear span of Λ, that is, on $\mathfrak{t}$; exponentiating, T is central. □

3.5. **Lie theory and metric spaces.** We return to the situation that arises in the context of isometry groups.

The main result of this section, Corollary 3.12, is an algebraic criterion for when a Lie group G_2 may be made isometric to a metric Lie group (G_1, d_1).

The material in this section is largely an extension to the case of more general metrics of ideas that go back many years to deal with riemannian Lie groups, which may be found in Helgason [34] or Kobayashi and Nomizu [46, 47].

Lemma 3.11. *Suppose that K is a compact subgroup of a connected Lie group H and denote by π the quotient map from H to H/K. Let G be a Lie subgroup of H (not necessarily closed) such that $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{k}$ as vector spaces. Then*

(i) $H = GK$,

(ii) the map $\pi|_G : G \to H/K$ is a covering map,

(iii) G is closed in H if and only if $G \cap K$ is finite, and

(iv) if H/K is simply connected, then $H = G \cdot K$.

Proof. The derivative of the product of exponential maps $(X, Y) \mapsto \exp(X)\exp(Y)$ from $\mathfrak{g} \oplus \mathfrak{k}$ to H is nonsingular at 0, whence H, G and K satisfy the hypotheses of Lemma 2.24 (with α taken to be the identity mapping). Part (i) follows from Lemma 2.24 (i).

Lemma 2.24 (iii) implies that $\pi|_G : G \to H/K$ is a covering map, which proves (ii); part (iii) is just Lemma 2.24 (iv).

Finally, if H/K is simply connected, then the covering map $\pi|_G$ is a homeomorphism, whence $G \cap K = \{e_H\}$. Part (iv) now follows from Lemma 2.24 (v). □

We remind the reader that when $H = G \cdot K$, the spaces G and H/K are homeomorphic, and if H is connected, so is K.

Corollary 3.12. *Let G_1 and G_2 be connected simply connected Lie groups, let d_1 be an admissible left-invariant metric on G_1, let $H \coloneqq \mathrm{Iso}(G_1, d_1)$, and let K be the stabiliser of e_1 in H. The following are equivalent:*

(i) *G_2 may be made isometric to (G_1, d_1);*
(ii) *there is a Lie group monomorphism $\alpha : G_2 \to H$ such that we may write $H = G_1 \cdot K = \alpha(G_2) \cdot K$;*
(iii) *there is a Lie algebra monomorphism $\tau : \mathfrak{g}_2 \to \mathfrak{h}$ such that $\tau(\mathfrak{g}_2) \oplus \mathfrak{k} = \mathfrak{h}$.*

Proof. This follows by combining Lemma 3.11 above with Theorem 2.21 (we may write $H = G_2 \cdot K$) and Corollary 3.3 (the isometry group of a metric Lie group is a Lie group). □

In the context of riemannian metrics, this result was well-known.

3.6. **Decompositions of Lie groups.** We are going to deal with semidirect products $R \rtimes L$, and refer the reader to Definition 2.22 for the details. We shall also use the following nomenclature.

Definition 3.13. Suppose that Γ is a subgroup of the semidirect product $R \rtimes L$. We say that Γ is *strongly central* if both (r, e) and (e, l) are central in $R \rtimes L$ whenever $(r, l) \in \Gamma$.

It will be useful to recall some features of the Levi decomposition of a connected Lie group G. Write $\mathfrak{g}$ for the Lie algebra of G. The Lie algebra of the universal covering group $\tilde{G}$ of G is also $\mathfrak{g}$, and G is a quotient of $\tilde{G}$ by a discrete central subgroup Γ. The Levi decomposition writes $\mathfrak{g}$ as the sum $\mathfrak{r} \oplus \mathfrak{l}$, where $\mathfrak{r}$ is the radical and $\mathfrak{l}$ is a semisimple subalgebra of $\mathfrak{g}$, known as a Levi subalgebra. While $\mathfrak{r}$ is uniquely determined, $\mathfrak{l}$ need not be, but all choices of $\mathfrak{l}$ are conjugate under the adjoint group of $\mathfrak{g}$.

Let $\tilde{R}$ and $\tilde{L}$ be the analytic subgroups of $\tilde{G}$ and R and L be the analytic subgroups of G corresponding to $\mathfrak{r}$ and $\mathfrak{l}$; $\tilde{R}$ and R are the radicals of $\tilde{G}$ and G, while $\tilde{L}$ and L are called Levi subgroups. The subgroup $\tilde{L}$ is closed in $\tilde{G}$ but L need not be closed in G. Denote $\Gamma \cap \tilde{R}$ and $\Gamma \cap \tilde{L}$ by Γ_R and Γ_L.

The centre $Z(\tilde{L})$ of the simply connected semisimple group $\tilde{L}$ is discrete and contains a finite index subgroup $Z^+(\tilde{L})$ which is the intersection of the kernels of all finite-dimensional representations of $\tilde{L}$; in particular, $Z^+(\tilde{L})$ is contained in the kernel of the restriction of the adjoint representation of $\tilde{G}$ to $\tilde{L}$, and hence $Z^+(\tilde{L}) \subseteq Z(\tilde{G}) \cap \tilde{L}$. Hence $Z(\tilde{G}) \cap \tilde{L}$ is of finite index in $Z(\tilde{L})$. Similarly we consider $Z^+(L)$, the intersection of the kernels of all finite-dimensional representations of L, and show that $Z(G) \cap L$ is of finite index in $Z(L)$. The subgroups $Z^+(\tilde{L})$ and $Z^+(L)$ do not depend on the choice of $\tilde{L}$ and L in the Levi decomposition, since all Levi subgroups are conjugate to each other.

The next lemma summarises many properties of the Levi decomposition.

Lemma 3.14. *Let G, $Z(G)$, R, L, $Z^+(L)$, $\tilde{G}$, $\tilde{R}$, $\tilde{L}$, $Z(\tilde{L})$, $Z^+(\tilde{L})$, Γ, Γ_R and Γ_L be as defined above. Then the following hold.*

(i) *$\tilde{R}$ and $\tilde{L}$ are simply connected and closed in $\tilde{G}$, and $\tilde{R}$ is normal; further, $\tilde{G}$ is the semidirect product $\tilde{R} \rtimes \tilde{L}$ of these subgroups.*
(ii) *$\tilde{R}$ and $\tilde{L}$ are the universal covering groups of R and L, and R and L may be identified with $\tilde{R}/\Gamma_R$ and $\tilde{L}/\Gamma_L$.*
(iii) *R is closed and normal in G, but L need not be closed. However, $Z^+(L)^- L$ is closed in G.*
(iv) *G may be identified with $(R \rtimes L)/\Gamma_0$, where $\Gamma_0 = \Gamma/(\Gamma_R \times \Gamma_L)$, and $|\Gamma_0| = |R \cap L|$.*
(v) *G is a semidirect product of its radical and a Levi subgroup if and only if $R \cap L = \{e\}$ if and only if $\Gamma_0 = \{e\}$ if and only if $\Gamma = \Gamma_R \Gamma_L$.*
(vi) *$R \rtimes L$ is the smallest covering group of G that is a semidirect product of its radical and a Levi subgroup, in the sense that every covering group that is a semidirect product of its radical and a Levi subgroup also covers $R \rtimes L$.*

(vii) *L is closed in G if and only if $\Gamma_0 L$ is closed in $R \rtimes L$ if and only if the projection of Γ_0 onto R is closed in R.*

(viii) *Γ_0 has a largest strongly central subgroup Γ_1, whose index in Γ_0 is bounded by $|Z(\tilde{L})/Z^+(\tilde{L})|$. We may identify G with the finite quotient $(R \rtimes L/\Gamma_1)/(\Gamma_0/\Gamma_1)$.*

(ix) *the subgroup $R \cap L$ is discrete and central in L, and so is finite if L has finite centre. The connected component of the identity in its closure $(R \cap L)^-$ in G is central in G. If Γ_0 is strongly central in $R \rtimes L$, then $R \cap L$ is central in G.*

Proof. Item (i), the structure of $\tilde{G}$, is well-known; see, for instance, [70, p. 244]. Item (ii) and the first part of item (iii) are also standard; we prove the second part of (iii) below. Item (iv) is a consequence of a standard isomorphism theorem. Items (v), (vi) and (vii) are trivial.

To prove item (viii), observe that if $(r_0, l_0) \in \Gamma$ and (e, l_0) lies in the centre of $\tilde{G}$, then so does (r_0, e). We define $\Gamma_1 = \{(r_0, l_0) \in \Gamma_0 : (e, l_0) \in Z(\tilde{G})\}$; then Γ_1 is a subgroup of Γ_0.

In the semisimple group $\tilde{L}$, the set $Z^+(\tilde{L})$ of elements that lie in the kernel of every finite dimensional representation of $\tilde{L}$ is a subgroup of finite index in the centre $Z(\tilde{L})$ of $\tilde{L}$. The index of Γ_1 in Γ_0 is bounded by $Z(\tilde{L})/Z^+(\tilde{L})$.

Now we prove (ix). Since $\mathfrak{r} \cap \mathfrak{l} = \{0\}$ and R is closed and normal in G, $R \cap L$ is a closed normal zero-dimensional subgroup of L, so it is discrete and central in L, but it may not be closed in R. Obviously $R \cap L$ is finite if L has finite centre (e.g., if L is compact).

As noted before the statement of this lemma, $L \cap Z(G)$ is a subgroup of finite index of $Z(L)$. Hence $R \cap L \cap Z(G)$ is of finite index in $R \cap L$. Thus the closures of $R \cap L \cap Z(G)$ and of $R \cap L$ in G have the same connected component of the identity, and the closure of $R \cap L \cap Z(G)$ in G is of finite index in the closure of $R \cap L$ in G. Since the closure of a central subgroup is central, the closure of $R \cap L \cap Z(G)$ in G is central. We conclude that the connected component of the identity in $(R \cap L)^-$ is central, as required.

If moreover Γ_0 is strongly central in $R \rtimes L$ and $h \in R \cap L$, then both (h, e) and (e, h) in $R \rtimes L$ map to h under the canonical quotient mapping, and so $(h, h^{-1}) \in \Gamma_0$, whence h is central in G.

Finally, we prove the second part of (iii). We repeat the above proofs for the quotient group $G/Z^+(L)^-$. The semisimple subgroup L' in the Levi decomposition $R'L'$ of $G/Z^+(L)^-$ is such that $Z^+(L')$ is trivial, and hence $Z(L')$ is finite, so that $R' \cap L'$ is finite and L' is closed in $G/Z^+(L)^-$, whence $Z^+(L)^- L$ is closed in G. □

Note in particular that (iv) and (vii) of the lemma imply that if $R \cap L$ is finite, then L is closed in G, while if $R \cap L$ is infinite, and L may or may not be closed. Note also that every connected Lie group G has a covering group that is a semidirect product of its radical and a Levi subgroup, and the number of leaves in the cover is equal to the cardinality of $R \cap L$, or equivalently, the cardinality of Γ_0. By contrast, to obtain a quotient that is a semidirect product of its radical and a Levi subgroup, it may be necessary to factor out a subgroup of positive dimension: this is illustrated by the following example.

Example 3.15. Consider the connected, simply connected Lie group $\tilde{G}$ that is the semidirect product $\mathbb{C}^n \rtimes (\mathrm{SU}(n) \times \mathbb{R})$, where $\mathrm{SU}(n) \times \mathbb{R}$ acts on $\mathbb{C}^n$ by $\alpha(u, t)v = e^{it}uv$. The centre of this group may be identified with the subgroup of $(\mathrm{SU}(n) \times \mathbb{R})$ of elements (u, t) such that $e^{it}u$ is the identity matrix.

The centre Γ of $\tilde{G}$ is discrete but is not the product of the groups of central elements of the Levi subgroup L (which is $\mathrm{SU}(n)$) and of the central elements of the radical R (which is $\mathbb{C}^n \rtimes \mathbb{R}$); hence the group $G \coloneqq \tilde{G}/\Gamma$ has trivial centre and is not a semidirect product of the form $R \rtimes L$, and has no quotient of the same dimension that is a semidirect product of its radical and a Levi factor. The group Γ is central, but unless $n = 2$, it is not strongly central, though by Lemma 3.14, it has a subgroup Γ_1 of finite index that is strongly central.

We recall the Iwasawa decomposition of a semisimple Lie algebra $\mathfrak{l}$ and of a corresponding connected semisimple Lie group L. The Lie algebra $\mathfrak{l}$ may always be decomposed as a direct sum of three subalgebras:

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k},$$

where $\mathrm{ad}(X)$ is semisimple with real eigenvalues for all elements X in $\mathfrak{a}$, is semisimple with purely imaginary eigenvalues for all elements X of $\mathfrak{k}$, and is nilpotent for all elements X in $\mathfrak{n}$. Further, $\mathfrak{a}$ is abelian and $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$, so $\mathfrak{a} \oplus \mathfrak{n}$ is also a subalgebra. The subalgebra $\mathfrak{k}$ is in turn a direct sum $\mathfrak{t} \oplus \mathfrak{k}'$, where $\mathfrak{t}$, the centre of $\mathfrak{k}$, is a toral subalgebra and $\mathfrak{k}'$, the commutator subalgebra of $\mathfrak{k}$, is a compact semisimple subalgebra.

The analytic subgroups A and N corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ are closed in L, and simply connected; further, AN is solvable, closed, simply connected, and exponential, that is, the exponential mapping is a homeomorphism from $\mathfrak{a} \oplus \mathfrak{n}$ to AN. The analytic subgroup K of L corresponding to $\mathfrak{k}$ is also closed, and is a covering group of a compact Lie group; thus it may or may not be compact. We may always write K as $V \times K_c$, where V is a vector subgroup and K_c is a compact subgroup, and $Z(L)$ is a discrete subgroup of K. The Iwasawa decomposition of L is the statement that

$$L = A \cdot N \cdot K. \tag{3.1}$$

All Iwasawa decompositions of L or of $\mathfrak{l}$ are conjugate to each other by an inner automorphism of L or under the adjoint group of $\mathfrak{l}$.

Remark 3.16. If L is a connected semisimple Lie group with Iwasawa decomposition $A{\cdot}N{\cdot}K$, then K is a deformation refract of the semisimple Lie group L, so L is contractible or simply connected if and only if K is. From the classification of semisimple Lie groups (see, e.g., [34, Chapter X]), L is contractible if and only if it is a product of copies of the universal covering group of $\mathrm{SL}(2, \mathbb{R})$. Other simple Lie groups have compact subgroups that are not contractible.

Thus if G is a contractible Lie group, then $G = R \rtimes L$, where R is its radical and L a Levi subgroup; both R and L are contractible. For connected solvable Lie groups, it is known that contractibility and simple connectedness coincide, while the contractible Levi factor is as just described.

It is worth pointing out that, for a simply connected semisimple Lie group L, the Lie algebra of V is $\mathfrak{t}$ and that of K_c is $\mathfrak{k}'$. For a general semisimple Lie group L, there is a projection π from its universal covering group $\tilde{L}$ onto L, and $\pi(V)$ is the product of a torus (which is absorbed into K_c) and a vector subgroup of V.

Lemma 3.17. *Let G be a connected Lie group, $\mathfrak{r} \oplus \mathfrak{l}$ be a Levi decomposition of $\mathfrak{g}$ and $\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$ be an Iwasawa decomposition of $\mathfrak{l}$. Let L, AN and K be the analytic subgroups of G corresponding to $\mathfrak{l}$, $\mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{k}$. Then AN is a closed, solvable, connected and simply connected subgroup of G. Further, $Z(L)$ and $Z^+(L)$ are subgroups of K, $\bar{K} = ZK$ and $\bar{L} = ZL$, where Z is the connected component of the identity in $(Z^+(L))^-$.*

Proof. Let π_R be the canonical projection of G onto G/R, where R is the radical of G, which coincides with $L^\flat \coloneqq LR/R \simeq L/(R \cap L)$. Let $K^\flat$, $A^\flat$ and $N^\flat$ be the subgroups $\pi_R(K)$, $\pi_R(A)$ and $\pi_R(N)$ of $L^\flat$; then $A^\flat \cdot N^\flat \cdot K^\flat$ is an Iwasawa decomposition of $L^\flat$. Thus AN and $A^\flat N^\flat$ are simply connected exponential solvable Lie groups whose Lie algebras may be identified, and the restriction $\pi_R|_{AN}$ of π_R to AN is a homeomorphic isomorphism onto $A^\flat N^\flat$.

It follows immediately that AN is closed in G. Take $a_j \in A$, $n_j \in N$ such that $a_j n_j \to g \in G$ as $j \to \infty$; we must show that $g \in AN$. Now $\pi_R(a_j n_j) \to \pi_R(g)$ in G/R, and the Iwasawa decomposition of $L^\flat$ implies that there exist $an \in AN$ such that $\pi_R(g) = \pi_R(an)$. The identification of AN and $A^\flat N^\flat$ in the first paragraph of this proof implies that $a_j n_j \to an$ in AN and hence $a_j n_j \to an$ in G.

We have already noted that $Z(L)$ is a discrete subgroup of L; *a fortiori* $Z^+(L)$ is a discrete subgroup of L, central in G. The closure $(Z^+(L))^-$ is a central Lie subgroup of G, and it is immediate that $(Z^+(L))^- = Z^+(L)Z$, where Z is the connected component of the identity in $(Z^+(L))^-$. We have noted that $K/Z^+(L)$ is compact, and so there is a compact subset S of K such that every element of K may be written as zs where $z \in Z^+(L)$ and $s \in S$. It follows that $\bar{K} \subseteq (Z^+(L))^- S \subseteq Z^+(L)ZS = ZK$; it is obvious that $ZK \subseteq \bar{K}$, and so equality holds.

Finally to identify $\bar{L}$, we observe that if $a_j \in A$, $n_j \in N$, $k_j \in K$, and $a_j n_j k_j \to g$ in G, then $\pi_R(a_j nj)\pi_R(k_j) \to \pi_R(g)$ in $L^\flat$, whence $\pi_R(a_j n_j) \to \pi_R(an)$ for some $a \in A$ and $n \in N$ from the properties of the Iwasawa decomposition of $L^\flat$, and hence $a_j n_j \to an$ from the identification of AN and $A^\flat N^\flat$. It is now immediate that k_j converges in G to some element of ZK. □

Our next lemma links maximal compact subgroups to the Levi and Iwasawa decompositions.

Lemma 3.18. *Suppose that G is a connected Lie group with radical R. Then the following hold.*

(i) Given a Levi subgroup L with Iwasawa decomposition ANK, there exists a maximal compact subgroup K_R of R such that K commutes with K_R; if K is compact then $K_R K$ is a maximal compact subgroup of G.
(ii) Given a maximal compact subgroup K'_R of R, there exists a Levi subgroup L' of G with Iwasawa decomposition $A'N'K'$ such that K' commutes with K'_R; if K' is compact then $K'_R K'$ is a maximal compact subgroup of G.
(iii) Given a maximal compact subgroup K_0 of G, there exists a Levi subgroup L of G with Iwasawa decomposition ANK such that K commutes with K_R and $K_0 \subseteq K_R \bar{K}$, where $K_R = K_0 \cap R$.

Proof. To prove (i), take any Levi subgroup L of G; then the group $R \ltimes L$ is a covering group of G by Lemma 3.14. It is also a covering group of $R \ltimes L/Z^+(L)$. Hence G is locally isomorphic to $R \ltimes L/Z^+(L)$. Observe that two connected closed subgroups of G commute if and only if the two connected closed subgroups of $R \ltimes L/Z^+(L)$ with the same Lie algebras commute.

Let ANK be an Iwasawa decomposition of L; then $AN(K/Z^+(L))$ is an Iwasawa decomposition of $L/Z^+(L)$, and $K/Z^+(L)$ is a maximal compact subgroup of $L/Z^+(L)$. Extend $K/Z^+(L)$ to a maximal compact subgroup K_m of $R \ltimes L/Z^+(L)$. Then $K_R \coloneqq K_m \cap R$ is a maximal compact subgroup of R, and $K_m R/R$, which is naturally isomorphic to K_m/K_R, is a maximal compact subgroup of $(R \ltimes L/Z^+(L))/R$, which is naturally isomorphic to $L/Z^+(L)$. Under this isomorphism, the image of $K_m R/R$ is a maximal compact subgroup of $L/Z^+(L)$ that contains $K/Z^+(L)$, and hence these subgroups coincide. Thus $K_m = (K/Z^+(L))K_R$, and K_R is a connected compact solvable normal subgroup of the connected compact Lie group K_m, and hence is a central torus. It follows that $K/Z^+(L)$ and K_R commute, and hence K and K_R commute.

If K is compact, then $K_R K$ is a compact subgroup of G; further, $K_R K \cap R \geq K_R$, but K_R is a maximal compact subgroup of R and so equality holds. It follows that $K_R K$ is a maximal compact subgroup of G from the fact that K_R and K are maximal compact subgroups of R and L.

Now we prove (ii). Given another maximal compact subgroup K_R' of R, there exists $r \in R$ such that $K_R' = rK_R r^{-1}$; then rLr^{-1} is a Levi subgroup of G with Iwasawa decomposition $rAr^{-1}rNr^{-1}rKr^{-1}$ and K_R' commutes with rKr^{-1}, as required.

We prove part (iii) by induction on the dimension of R, the radical of G. Suppose that the result holds whenever $\dim(R) < r$, and suppose that $\dim(R) = r$. We consider two cases, according to the properties of $Z^+(L)$.

If $Z^+(L)^-$ has dimension 0, then $Z^+(L)$ is discrete in G. We write $G^\flat$ for $G/Z^+(L)$ and consider the local isomorphism $\pi : G \to G^\flat$. Take a Levi decomposition $R^\flat L^\flat$ of $G^\flat$; then $R^\flat$ and $L^\flat$ are locally isomorphic to the subgroups R and L that arise in a Levi decomposition of G, but $Z^+(L^\flat) = \{e\}$, which means that the subgroup K in an Iwasawa decomposition of L is compact. It is evident $\pi(K_0)$ is contained in a maximal compact subgroup of $G^\flat$, and maximal compact subgroups of $G^\flat$ are of the form $K_R K^\flat$. In this case, the desired result follows.

If $Z^+(L)^-$ has positive dimension, then Z, the connected component of the identity in $Z^+(L)^-$, is a nontrivial closed connected normal subgroup of R. We let $\pi : G \to G/Z$ be the canonical projection; the radical of the quotient group G/Z has dimension less than r, while a Levi factor $L^\flat$ of the quotient is locally isomorphic to a Levi factor of G; the main difference is that $Z^+(L^\flat)$ is trivial. The result follows from the inductive hypothesis applied to $G^\flat$. □

We also need some information about maximal solvable subalgebras of a Lie algebra which follows from the Levi decomposition and an argument of Mostow.

Lemma 3.19 (After Mostow [57]). *Suppose that $\mathfrak{h}$ is a Lie algebra. There exist finitely many maximal solvable subalgebras $\mathfrak{g}_j$ of $\mathfrak{h}$ such that every maximal solvable subalgebra $\mathfrak{g}$ is conjugate under the adjoint group to exactly one of the $\mathfrak{g}_j$. Exactly one of these subalgebras, $\mathfrak{g}_0$ say, has the property that there is a compact subalgebra $\mathfrak{k}$ of $\mathfrak{h}$ such that $\mathfrak{g}_0 + \mathfrak{k} = \mathfrak{h}$.*

Proof. Let $\mathfrak{r}$ be the radical of $\mathfrak{h}$ and $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{h}$, so that $\mathfrak{h} = \mathfrak{r} \oplus \mathfrak{l}$. Denote by π the canonical projection of $\mathfrak{h}$ onto the quotient $\mathfrak{q} := \mathfrak{h}/\mathfrak{r}$, which may be identified with $\mathfrak{l}$.

If $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{h}$, then $\mathfrak{r} \subseteq \mathfrak{g}$, since otherwise $\mathfrak{g} + \mathfrak{r}$ would be a larger solvable subalgebra than $\mathfrak{g}$. Further, for subalgebras $\mathfrak{g}$ of $\mathfrak{h}$ that contain $\mathfrak{r}$, $\mathfrak{g}$ is solvable if and only if $\pi(\mathfrak{g})$ is solvable (this relies on that fact that if $\mathfrak{s}_1$ and $\mathfrak{s}_2/\mathfrak{s}_1$ are solvable, so is $\mathfrak{s}_2$). Consequently, $\pi(\mathfrak{g})$ is a maximal solvable subalgebra of $\mathfrak{q}$ if and only if $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{h}$.

Mostow [57] classified the maximal solvable subalgebras of the semisimple Lie algebra $\mathfrak{q}$ (showing that they correspond to Cartan subalgebras of $\mathfrak{q}$), and described finitely many maximal solvable subalgebras $\mathfrak{s}_j$ of $\mathfrak{q}$ with the property that every maximal solvable subalgebra is conjugate to exactly one of these. The maximal solvable subalgebras $\mathfrak{s}$ of $\mathfrak{q}$ for which there exists a compact subalgebra $\mathfrak{k}$ of $\mathfrak{q}$ such that $\mathfrak{s} + \mathfrak{k} = \mathfrak{q}$ are all conjugates under the adjoint group of $\mathfrak{q}$ of a particular subalgebra $\mathfrak{s}_0$, which is a toral extension of the subalgebra $\mathfrak{a} + \mathfrak{n}$ of $\mathfrak{q}$ arising from an Iwasawa decomposition of $\mathfrak{q}$.

We define $\mathfrak{g}_j := \pi^{-1}(\mathfrak{s}_j)$ and $\mathfrak{k}'$ to be the compact subalgebra of $\mathfrak{l}$ that corresponds to $\mathfrak{k}$ under the identification of $\mathfrak{l}$ and $\mathfrak{q}$. Then $\mathfrak{g}_j$ is a maximal solvable subalgebra of $\mathfrak{h}$ (containing $\mathfrak{r}$), and every maximal solvable subalgebra of $\mathfrak{h}$ is conjugate to one of these.

Further, $\pi(\mathfrak{g}_0) + \pi(\mathfrak{k}') = \mathfrak{s}_0 + \pi(\mathfrak{k}') = \mathfrak{q}$, and $\pi(\mathfrak{k}')$ is compact, whence $\mathfrak{g}_0 + \mathfrak{k}' = \mathfrak{h}$. If $\mathfrak{s}$ is a maximal solvable subalgebra of $\mathfrak{h}$ and $\mathfrak{s} + \mathfrak{k}'' = \mathfrak{h}$ for some compact subalgebra $\mathfrak{k}''$ of $\mathfrak{h}$, then $\pi(\mathfrak{s})$ is a maximal solvable subalgebra of $\mathfrak{q}$ and $\pi(\mathfrak{s}) + \pi(\mathfrak{k}'') = \mathfrak{q}$ for some compact subalgebra $\pi(\mathfrak{k}'')$ of $\mathfrak{q}$, whence $\pi(\mathfrak{s})$ is conjugate to $\mathfrak{s}_0$ under the adjoint group of $\mathfrak{q}$ and hence $\mathfrak{s}$ is conjugate to $\mathfrak{g}_0$. □

Suppose that H is a connected Lie group with centre $Z(H)$. The above result implies that there exist finitely many maximal connected solvable subgroups G_j of H such that every maximal solvable subgroup G is conjugate to exactly one of the G_j. Since the closure of a connected solvable group is connected and solvable, these maximal connected solvable subgroups are closed. Exactly one of these subgroups, G_0 say, has the property that $H/G_0Z(H)$ is compact.

While we are focussing on solvable Lie groups, we mention that for solvable groups, simply connected and contractible coincide.

3.7. **Polynomial growth and amenability.** We now look at the structure of two particular types of Lie group in more detail. If G is a connected Lie group, then it is of polynomial growth if and only if its Lie algebra $\mathfrak{g}$ is of type (R), that is, the eigenvalues of $\operatorname{ad} X$ are purely imaginary for each $X \in \mathfrak{g}$. For instance, nilpotent Lie groups and euclidean motion groups are of polynomial growth. For more on this, see [32, 42].

Lemma 3.20. *Let G be a connected Lie group with radical R and a Levi subgroup L. Then G is of polynomial growth if and only if R is of polynomial growth and L is compact. If G is of polynomial growth and contractible, then G is solvable.*

Proof. Both Guivarc'h [32, p. 345] and Jenkins [42, p. 123] showed that Lie groups are of polynomial growth if and only if their radicals are of polynomial growth and their Levi subgroups are compact. See also [21, Theorem II.4.8].

If G is contractible, then G is simply connected, so $G = R \rtimes L$ (see, for example, [21, II.1.17]) and thus R and L are contractible. A contractible compact Lie group is trivial, by Lemma 3.6, so G coincides with R and is solvable. □

Note that the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ is contractible but not of polynomial growth.

Definition 3.21. A connected Lie group with a compact Levi factor is said to be *amenable.*

The standard definition of amenability of a group G involves the existence of a left-invariant mean on $L^\infty(G)$. The fact that for connected Lie groups this amounts to the definition above is well known (see, for instance, [74, Corollary 4.1.9]. It is also well known (and follows from the standard definition or from ours) that connected closed subgroups and quotients of amenable groups are amenable.

It is clear that connected Lie groups of polynomial growth are amenable, but examples such as the "$ax + b$-group", which is solvable but not of polynomial growth, show that the converse is false.

Lemma 3.22. *Suppose that K is a maximal compact subgroup of a connected amenable Lie group H, and that $\bigcap_{h\in H} hKh^{-1} = \{e\}$. Then there is a closed connected solvable normal subgroup G of H such that*

(i) $H = G \cdot K$, whence G acts simply transitively on H/K;
(ii) $TG = G$ whenever T is an automorphism of H and $TK = K$.

Proof. Let N and R be the nilradical of and radical of H; then $N \subseteq R$. Write H as RL, where L is a necessarily compact Levi subgroup; in light of Lemma 3.18, we may assume without loss of generality that $L \subseteq K$. The assumption on K implies that $K \cap Z(H) = \{e\}$. We write $\mathfrak{n}$, $\mathfrak{r}$ and so on for the Lie algebras of these groups; then $\mathfrak{k} \cap Z(\mathfrak{h}) = \{0\}$.

We are going to use the *Killing form*, a bilinear form on $\mathfrak{h}$ defined by

$$B(X,Y) = \operatorname{trace}(\operatorname{ad}(X)\operatorname{ad}(Y)) \qquad \forall X, Y \in \mathfrak{h}.$$

This has many important properties, for which see, for instance, [12, pp. 33–50]; we will use the following:

(a) if T_* is an automorphism of $\mathfrak{h}$, then $B(T_*X, T_*Y) = B(X,Y)$ for all $X, Y \in \mathfrak{h}$;
(b) $B(X,X) < 0$ for all $X \in \mathfrak{k} \setminus \{0\}$ (because $\mathfrak{k}$ is compact and $\mathfrak{k} \cap Z(\mathfrak{h}) = \{0\}$);
(c) $B(X,Y) = 0$ for all $X \in \mathfrak{h}$ and all $Y \in \mathfrak{n}$;
(d) $B([X,Y],Z) = 0$ for all $X \in \mathfrak{h}$ if and only if $Z \in \mathfrak{r}$;

We denote by $\mathfrak{g}$ the subspace $\{X \in \mathfrak{h} : B(X,Y) = 0 \ \forall Y \in \mathfrak{k}\}$. Because $\mathfrak{k}$ is semisimple, $[\mathfrak{h}, \mathfrak{h}] \supseteq [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$, and from (c) and (d) it follows that $\mathfrak{n} \subseteq \mathfrak{g} \subseteq \mathfrak{r}$. Then $\mathfrak{g}$ is an ideal in $\mathfrak{h}$, from Remark 3.9, and $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{k}$ from (b) and linear algebra.

Write K_R for $K \cap R$, which is connected since it is a maximal compact subgroup of R, and abelian, since it is both compact and solvable, and so is a torus; further, $K_R \cap N = \{e\}$ by assumption and Lemma 3.10.

Let T be an automorphism of H that fixes K. Then $T_*\mathfrak{k} = \mathfrak{k}$, and from (a), $T_*\mathfrak{g} = \mathfrak{g}$.

Since H/K is simply connected, Lemma 3.11 implies that the connected analytic subgroup G of R with Lie algebra $\mathfrak{g}$ is closed, $H = G \cdot K$, and G acts simply transitively on H/K. Further, if T is an automorphism of H and $TK = K$, then its infinitesimal version T_* is an automorphism of $\mathfrak{h}$ and $T_*\mathfrak{k} = \mathfrak{k}$, whence $T_*\mathfrak{g} = \mathfrak{g}$ and $TG = G$. □

Remark 3.23. This lemma may be extended to more general connected Lie groups H at the cost of relaxing the requirement that G be normal. However, the following example shows that no such result can hold for all connected Lie groups.

Let H be the simply connected covering group of $\mathrm{SU}(n,1)$, where $n \geq 1$, and K be a maximal compact subgroup. Then H/K is contractible, but there is no solvable subgroup G of H that acts transitively on H/K.

3.8. **Proof of Theorem B.** We are now ready to prove our next main theorem, which we restate, in a longer version.

Theorem 3.24. *Let (M,d) be a homogeneous metric manifold. Then there is a metric d' on M such that the identity mapping on M from (M,d) to (M,d') is a homeomorphic rough isometry, and there is a transitive closed connected amenable subgroup $H_\times$ of $\mathrm{Iso}(M,d')$; hence M is homeomorphic to $H_\times/K_\times$, where $K_\times$ is a compact subgroup of $H_\times$.*

If M is a metric Lie group, then we may take $K_\times$ to be a finite group; if M is a simply connected metric Lie group, then we may take $K_\times$ to be trivial.

If M is a contractible metric space, then we may take $K_\times$ to be trivial and $H_\times$ to be solvable, so that M is homeomorphically roughly isometric to a connected, simply connected solvable metric Lie group.

Proof. Let M be a homogeneous metric manifold, and suppose that H is a connected transitive isometry group of M, so that we may identify M with H/K_o, where K_o is the compact stabiliser of a point o in M. We may take H to be a Lie group that acts on M by analytic maps, by Theorem 3.2.

We begin with a short outline of the proof. Up to local isomorphism, the connected Lie group H is a semidirect product $R \rtimes L$, where R is the solvable radical and L

is a semisimple Levi subgroup. Further, up to local isomorphism, L has an Iwasawa decomposition $AN \cdot K$, where K is compact and AN is solvable. Then $H = S \cdot K$, where S is the closed solvable subgroup $R \rtimes AN$ of H. If H has a left-invariant, right-K-invariant metric d, then H is isometric to the group $S \times K$, equipped with a left-invariant metric $d_\times$, as described in Lemma 2.20. We need to deal with two additional complications: first, we need to deal with groups H that are not semidirect products, but quotients thereof, and second, we need to deal with the quotient H/K_o. Now we provide the details.

We recall from Lemma 3.14 that, in general, there is a continuous open projection $\pi : R \rtimes L \to H$, with discrete kernel, Γ say, and Γ has a subgroup of finite index Γ_1 that is strongly central, that is, if $(r, l) \in \Gamma_1$, then both (r, e) and (e, l) are central in $R \rtimes L$. In particular, this implies that l lies in the subgroup K for any Iwasawa decomposition ANK of L.

Now L has an Iwasawa decomposition (see (3.1)) ANK, in which $K = V \times K_c$, where V is a vector group which is compact modulo $V \cap Z^+(L)$, and K_c is a maximal compact subgroup of L, while AN is solvable; as above, we write S for the solvable group $R \rtimes AN$, and then $R \rtimes L = S \cdot (V \times K_c)$. Let K_o be the stabiliser of a point o in M in H, let K_m be a maximal compact subgroup of H that contains K_o and let $K_R = K_m \cap R$. Then $K_o \subseteq (Z(L)^+)^- K_R K_c$, by Lemmas 3.18 and 3.17. The subgroup $K_R K_c \times V$ of H is compact modulo the centre of H, so that we may apply Corollary 2.17 to modify d and obtain a new admissible metric d' on M, such that the identity map on M is a rough isometry from (M, d) to (M, d'), and

$$d'(gg'kK_o, gg''kK_o) = d'(g'K_o, g''K_o) \qquad \forall g, g', g'' \in H \quad \forall k \in K_R K_c V. \tag{3.2}$$

For simplicity of notation, we replace d' by d and assume that d has the invariance property (3.2).

We define $\omega : S \times (V \times K_c) \to S \cdot (V \times K_c)$ by

$$\omega(s, k) = sk^{-1} \qquad \forall s \in S \quad \forall k \in (V \times K_c).$$

Then ω is a homeomorphism. We lift the metric d on the space H/K_o, first to a pseudometric on H with kernel K_o, and then to a pseudometric $\dot{d}$ on the covering group $S \cdot (V \times K_c)$ with kernel $\pi^{-1}K_o$: more precisely, we define

$$\dot{d}(x, y) \coloneqq d(\pi x K_o, \pi y K_o) \qquad \forall x, y \in S \cdot (V \times K_c).$$

By construction, $\dot{d}$ is continuous, admissible, and left-invariant and right-$\pi^{-1}(K_R K_c V)$-invariant. By Lemma 2.20, $\dot{d}_\times \coloneqq \dot{d} \circ (\omega \otimes \omega)$ is a continuous admissible left-invariant pseudometric on $S \times (V \times K_c)$, whose kernel is a closed subgroup of $S \times (V \times K_c)$, by Lemma 2.11. When we identify points at $\dot{d}_\times$-distance 0, we obtain a $S \times (V \times K_c)$-invariant admissible metric $d_\times$ on the quotient $M_\times$. Since the mapping ω from $S \times (V \times K_c)$ to $S \cdot (V \times K_c)$ is an isometry of pseudometric spaces, the quotient metric space $(M_\times, d_\times)$ is isometric to (M, d).

Trivially, the amenable Lie group $S \times V \times K_c$ acts transitively and isometrically on $(M_\times, d_\times)$, so there is a continuous homomorphism $\alpha : S \times V \times K_c \to \mathrm{Iso}(M_\times, d_\times)$. The image of α is the product of the compact group $\alpha(K_c)$ and the solvable group $\alpha(S \times V)$, and so $H_\times$, the closure of this image in $\mathrm{Iso}(M_\times, d_\times)$, is the commuting product of the compact group $\alpha(K_c)$ and the closed solvable group $(\alpha(S \times V))^-$. The intersection of these subgroups may be nontrivial, but $H_\times$ is still amenable. We may identify $M_\times$ with the space $H_\times / K_\times$, where $K_\times$ is the compact stabiliser in $H_\times$ of a point in $M_\times$.

This proves the general part of the theorem. However there are still some particular cases to consider.

First, if M is a metric group, then we may take K_o to be trivial, and trace through the argument above. We see that $M_\times$ is a finite quotient of an amenable metric group, and the order of the group that we factor out is bounded. Indeed, in this case, M may be identified with the covering group $R \rtimes L/\Gamma_0$, and by Lemma 3.14, Γ_0 has a subgroup Γ_1 of finite index such that $\omega^{-1}\Gamma_1$ is central and a fortiori normal in $S \times (V \times K_c)$. Then $M_\times$ may be identified with $(S \times V \times K_c)/\omega^{-1}\Gamma_0$, which is a finite quotient of the amenable Lie group $(S \times V \times K_c)/\omega^{-1}\Gamma_1$. If M is also simply connected, then no factoring out of discrete subgroups is involved, and we may identify $M_\times$ with $S \times V \times K_c$.

Another special case is when M is contractible. In this case, $M_\times$ is contractible, and so is of the form $H_\times/K_\times$ where $K_\times$ is a maximal compact subgroup of $H_\times$, and $H_\times$ is amenable. Then there is a simply connected solvable group that acts simply transitively on M by Lemma 3.22. □

Remark 3.25. We may choose the metric in Theorem 3.7, in such a way that it is not necessary to change the metric at the beginning of this theorem. Moreover, for any $\varepsilon \in \mathbb{R}^+$, there is a homogeneous metric manifold of the form $S \times K/K_0$, where K_0 is a compact subgroup of $S \times K$, that is $(1, \varepsilon)$-quasi-isometric to the original space (M, d_0).

Before we move on, we make a few observations. It is evident that if we start with slightly different hypotheses, we can modify the argument of the proof above to prove slightly different results. For example, if we start with a riemannian metric, we can work throughout with riemannian metrics and bi-Lipschitz mappings rather than general metrics and rough isometries. Or if we start with a semisimple Lie group, we do not need to consider the Levi decomposition. Or if we are allowed to choose the metrics, then we may do so to ensure that we have an isometry rather than a rough isometry. By doing this, we easily obtain the following corollaries, which are really corollaries of the method of proof rather than of the result.

Corollary 3.26. *Let (M, d) be a homogeneous riemannian manifold. Then there is a riemannian metric d' on M (so the identity mapping on M from (M, d) to (M, d') is bi-Lipschitz), such that (M, d') admits a transitive connected isometry group of the form $S \times L$, where S is solvable and L is compact and semisimple; hence M is homeomorphic to $(S \times L)/K$, where K is a compact subgroup of $S \times L$.*

If M is a metric Lie group, then we may take K to be a finite group; if M is a simply connected metric Lie group, then we may take K to be trivial.

If M is a contractible metric space, then we may take L and K to be trivial, so that M is bi-Lipschitz to a connected, simply connected solvable metric Lie group.

Corollary 3.27. *Let G be a connected Lie group. Then the following are equivalent:*

- *(i) G may be made isometric to a connected simply connected solvable Lie group; and*
- *(ii) $G = R \rtimes L$, where R is the solvable radical and L is a Levi subgroup of G; further, R is simply connected and L is a direct product of finitely many (possibly zero) copies of the universal covering group of* $\mathrm{SL}(2, \mathbb{R})$.

Proof. If (i) holds, then G is contractible; by Remark 3.16, (ii) holds.

On the other hand, if (ii) holds, then G may be made isomorphic to a solvable Lie group by Theorem 3.24. □

Corollary 3.28. *Suppose that (G, d) is either a simply connected metric Lie group or a connected semisimple metric Lie group. Then there exist a connected Lie group H that is the product of a solvable and a compact Lie group, and admissible left-invariant metrics d_G and d_H such that (G, d_G) and (H, d_H) are isometric and the identity map is a rough isometry from (G, d) to (G, d_G).*

Corollary 3.29. *Let G be a connected semisimple Lie group with Iwasawa decomposition ANK. Write K as $V \times K_0$, where V is a vector group and K_0 is compact. Then G may be made isometric to the direct product $AN \times V \times K_0$.*

It seems reasonable to ask whether a general connected metric Lie group (G, d) is homeomorphically quasi-isometric to an amenable connected metric Lie group. Example 3.15 provides a counterexample. Indeed, with the notation of that example, we consider the group $G = \tilde{G}/\Gamma$, and observe that the arguments used to prove Theorem 3.24 show that $\tilde{G}$ is homeomorphic to an amenable direct product group $\tilde{G}^*$, and that $\tilde{G}/\Gamma$ is isometric to $\tilde{G}^*/\Gamma^*$, where Γ^* is the group $\{(r, l^{-1}) : (r, l) \in \Gamma\}$. However, unless $n = 2$, the subgroup Γ^* is not normal, but has a subgroup of finite index that is normal. In this case, G^*/Γ^* is not a group but is a finite quotient of a group. Further, $\tilde{G}/\Gamma_1$ is a finite covering group of G, and is isometric to the group $\tilde{G}^*/\Gamma_1^*$. More generally, we state without proof the following variant of Theorem 3.24.

Theorem 3.30. *Let (M, d) be a homogeneous metric manifold. Then there is a metric d' on M such that the identity mapping on M from (M, d) to (M, d') is a homeomorphic rough isometry, and (M, d') has a finite cover that admits a simply transitive connected isometry group of the form $S \times L$, where S is solvable and L is compact and semisimple; hence M is homeomorphic to $(S \times L)/K$.*

3.9. **Notes and remarks.**

3.1. The main structure theorem. We have mentioned some of the contributors to the solution of Hilbert's fifth problem, on the structure of locally euclidean topological groups. It is worth pointing out that earlier the structure of compact groups was elaborated by von Neumann, and that of solvable groups by Chevalley. For much more, see [54].

Apropos of Corollary 3.4, riemannian geometers have known for a long time that spaces H/K, where K is a compact subgroup of a connected Lie group H, may be equipped with a riemannian metric such that H acts by isometries, by choosing a K_*-invariant infinitesimal metric at the point K of H/K and then translating this to the whole space. For instance, this fact is described as well known in a 1954 paper of Nomizu [62].

3.2. Compact subgroups. It is well known that connected compact Lie groups contain maximal connected abelian subgroups, or maximal tori, all of which are conjugate (see, for instance, [45, Corollary 4.35, p. 255]). It is perhaps not so well known that all connected compact groups contain maximal connected abelian subgroups, which are automatically closed, and all of these are conjugate. See [52, Theorem 9.32] for more details.

We have stated Corollaries 3.3 and 3.4 for connected groups for simplicity, and Lemma 3.5 for connected groups since Iwasawa did so. For the almost connected case, see [66, Theorem 32.5] and the references cited there.

For more classical theory of the topology of Lie groups, see [64].

3.3. Proof of Theorem 3.7. Let o be a point in a homogeneous metric space (M, d). Then there is a connected locally compact group H that acts effectively and transitively on (M, d) by isometries, and M may be identified with the space H/K_o, where K_o is the stabiliser of o in H. Let K be a maximal compact subgroup of H that contains K_o, and suppose that d is right-K-invariant, which may always be arranged as in the proof of the theorem.

Then the collection of compact subgroups K_ν of H such that $K_o \subseteq K_\nu \subseteq K$ is a partially ordered set, and in the corresponding collection of quotient spaces H/K_ν, and by extending the construction following Definition 2.9, we may find a family of homogeneous metric space projections $\pi_{\nu,\nu'} : H/K_\nu \to H/K'_\nu$ whenever $K_\nu \subseteq K'_\nu$, and the implicit constants in all these projections are uniformly bounded. This family of projections is an inverse system, and H/K_o is (trivially) the limit of spaces H/K_ν as K_ν shrinks down to K_o. If we restrict to the subgroups K_ν such that H/K_ν is a Lie group, then the limit is no longer trivial if H/K_o is not a manifold.

When the spaces H/K_ν and H/K'_ν are manifolds, then H/K_ν is a fibre bundle over H/K'_ν. However, in general, we cannot assert this: local triviality is a problem.

3.4. Lie groups. Apropos of the exponential mapping on a Lie group, it may be of interest that in some cases, $G = \exp(\mathfrak{g})$, while $G = \exp(\mathfrak{g})\exp(\mathfrak{g})$ always (see [56]).

3.6. Decompositions of Lie groups. Under suitable conditions, a connected locally compact group H has a connected simply connected locally compact universal covering group $\tilde{H}$ (an infinite-dimensional torus is a counter-example). We refer the reader to [9] for more information. Thus it would be possible to extend the Levi decomposition to more general locally compact connected groups, but to discuss this would take us too far from our main goals.

We give two more examples that illustrate what may happen in the Levi decomposition when L is not closed. Let U denote the universal covering group of $\mathrm{SL}(2,\mathbb{R})$, and $\{k_t : t \in \mathbb{R}\}$ be the one-parameter subgroup of U that projects down to the standard rotation subgroup of $\mathrm{SL}(2,\mathbb{R})$, normalised so that k_t projects to the rotation through an angle t; thus the elements $k_{2\pi n}$, where $n \in \mathbb{Z}$, project to the identity of $\mathrm{SL}(2,\mathbb{R})$.

Example 3.31. Let G be the group $(U \times T)/Z$, where $T = \{z \in \mathbb{C} : |z| = 1\}$ and Z is the central discrete subgroup $\{(k_{2\pi n}, e^{in}) : n \in \mathbb{Z}\}$ of $U \times T$. The Levi subgroup of G is an analytic subgroup, which may be identified with U, and the radical is a torus, which may be identified with T; these have an intersection which is dense in the radical. This group cannot be written as a semidirect product of its radical and a Levi factor, and nor can any finite covering group or finite quotient, though a compact quotient of lower dimension is trivially a semidirect product of its radical and a Levi factor.

Example 3.32. Let G be the group $(U \times U \times \mathbb{R})/Z$ and Z be the central discrete subgroup $\{(k_{2\pi m}, k_{2\pi n}, m + \alpha n) : m, n \in \mathbb{Z}\}$, where α is irrational. Then the Levi subgroup of G is an analytic subgroup, which may be identified with $U \times U$, and the radical is a line; these have an intersection which is dense in the radical. This group cannot be written as a semidirect product of its radical and a Levi subgroup, and nor can any finite covering group or compact quotient.

3.7. Polynomial growth and amenability. A propos of Definition 3.21, the term was apparently coined by M.M. Day, to indicate the existence of a left-invariant mean on a group. For us, amenable groups are amenable because they are much more tractable than general Lie groups.

3.8. Proof of Theorem B. Theorem 3.24 shows that the class of solvable Lie groups is not closed under isometries. It was already known (see [1, 53]) that the infinite covering group of $\mathrm{SL}(2,\mathbb{R})$ and the direct product of $\mathbb{R}$ and the "$ax+b$-group" may be made isometric, even though the former group is not solvable and the latter is.

We remark that rough isometry is connected to Cornulier's [17] notion of *commability*; two homogeneous spaces are *commable* if they may be connected by a finite number of projections from a group G onto a quotient G/K, where K is a compact subgroup of G, and cocompact embeddings; the arguments above show that G and G/K may be metrised (subject to some topological separability) in such a way that the projection and section are rough isometries. But when we allow metrics that are not proper quasigeodesic, then rough isometry need not imply commability. For instance, infinite covering projections may be rough isometries, by Lemma 2.19, but a space and its infinite cover are not commable.

Finally, it may be useful to recall that there is significant literature showing that the topology alone comes close to determining compact Lie groups; see [38] and the works cited there. On the other hand, relations such as (L, C)-quasi-isometry do not "see" compact factors at all if C is sufficiently large.

4. Solvable groups

In this section, we restrict our attention to connected simply connected solvable Lie groups. We discuss the classification of connected, simply connected solvable Lie groups up to isometry, due to Gordon and Wilson [28, 29] in the riemannian case, when two such groups may be made isometric, and make some minor contributions to the question of their classification up to quasi-isometry, which has not yet been achieved and seems to be very difficult. We present a different point of view to previous authors and extend some existing definitions and results.

We remind the reader of Definition 2.22: we write $H = G \cdot K$ to mean that G and K are subgroups of H and the map $(g, k) \mapsto gk$ is a homeomorphism from $G \times K$ to H.

Up to now, we have been looking at homogeneous metric spaces of the form H/K, where H is a connected group and K is a compact subgroup. For example, we showed in Corollary 2.25 that if G_1 and G_2 are connected groups that both act simply transitively by isometries on a homogeneous metric space (M, d), and H is the connected component of the identity in $\mathrm{Iso}(M, d)$ and K is the stabiliser in H of a point in M, then it is possible to write $H = G_1 \cdot K = G_2 \cdot K$. However, this does not tell us whether G_1 and G_2 are algebraically similar.

In Section 3.3, we showed that homogeneous metric spaces are roughly isometric to connected simply connected solvable Lie groups. In this section we use the additional information available from Lie theory to discuss when two connected simply connected solvable Lie groups are isometric, or may be made isometric, or even when they are roughly isometric (and here there are many open problems). The first main step in doing this is to show that if G_1 and G_2 are isometric connected, simply connected solvable metric Lie groups, then there is a connected, simply connected solvable metric Lie group H and a toral subgroup T such that $H = G_1 \cdot T = G_2 \cdot T$. Then we proceed to a detailed analysis of solvable Lie groups and their subgroups of this form.

In Section 4.1, we examine derivations of Lie algebras, and particularly solvable Lie algebras, in detail. In Section 4.5, we briefly describe "twisted versions" of solvable Lie groups, and show that two isometric connected, simply connected solvable groups are both twisted versions of the same connected, simply connected solvable group. We connect twisted versions of groups to the normal modifications of Gordon and Wilson [28, 29], and to *hulls* and *real-shadows* of solvable groups in Section 4.6. In Section 4.7, we prove Theorem C and a number of consequences. Much of what we do, or at least something similar, is known; we leave a brief description of the history of this development to Section 4.8.

We end this introductory discussion with a remark; before stating it, we remind the reader that $Z(H)$ and $Z(\mathfrak{h})$ mean the centres of H and $\mathfrak{h}$.

Remark 4.1. Let (M, d) be a homogeneous metric space, H be a closed solvable subgroup of $\mathrm{Iso}(M, d)$ that acts transitively on M, and K be the stabiliser of a point in M; then K is a compact subgroup of H. From Remark 2.5, $Z(H) \cap K = \{e\}$.

In the case where K is connected and so is a torus, T say, then there is no loss of generality in supposing that $Z(H) \cap T = \{e\}$. Once we have done this, $\mathrm{nil}(H) \cap T$ is trivial, by Lemma 3.10.

4.1. **Derivations and automorphisms.** Here we prove some preliminary results and introduce a little more notation.

Remark 4.2. Suppose that L is a diagonalisable linear map on a Lie algebra $\mathfrak{g}$; then there is a direct sum eigenspace decomposition $\mathfrak{g} = \sum_\lambda \mathfrak{g}_\lambda$, where $LX = \lambda X$ for all $X \in \mathfrak{g}_\lambda$. It is well known that L is a derivation if and only if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all eigenvalues α and β. Indeed, if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, then

$$[LX, Y] + [X, LY] = (\alpha + \beta)[X, Y],$$

so if L is a derivation, then $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$. Conversely if $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$ for all $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ and all eigenvalues α and β, then by linearity $L[X, Y] = [LX, Y] + [X, LY]$ for all $X, Y \in \mathfrak{g}$ and L is a derivation.

Remark 4.3. If D is any derivation of a Lie algebra $\mathfrak{g}$, then $D\,\mathrm{rad}(\mathfrak{g}) \subseteq \mathrm{nil}(\mathfrak{g})$, by [41, Theorem 7, p. 74]. In particular, $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \subseteq \mathrm{nil}(\mathfrak{g})$. This implies that if $\mathfrak{v}$ is a subspace of $\mathfrak{g}$ and $\mathrm{nil}(\mathfrak{g}) \subseteq \mathfrak{v} \subseteq \mathrm{rad}(\mathfrak{g})$, then $\mathfrak{v}$ is an ideal in $\mathfrak{g}$.

The next lemma is certainly known, but we are not aware of a proof in the literature, so we provide one.

Lemma 4.4. *Suppose that $\mathfrak{g}$ is a real Lie algebra, and that $\mathfrak{d}$ is an abelian algebra of semisimple derivations of $\mathfrak{g}$. Then there are commuting abelian algebras $\mathfrak{d}_{\mathrm{r}}$ and $\mathfrak{d}_{\mathrm{i}}$ of semisimple derivations of $\mathfrak{g}$ such that every element of $\mathfrak{d}_{\mathrm{r}}$ has purely real eigenvalues, every element of $\mathfrak{d}_{\mathrm{i}}$ has purely imaginary eigenvalues, and every element D of $\mathfrak{d}$ may be written as a sum $D = D_{\mathrm{r}} + D_{\mathrm{i}}$, where $D_{\mathrm{r}} \in \mathfrak{d}_{\mathrm{r}}$ and $D_{\mathrm{i}} \in \mathfrak{d}_{\mathrm{i}}$.*

Proof. By considering the simultaneous eigenvalue decomposition of $\mathfrak{g}$ under the action of $\mathfrak{d}$, we may write the complexification $\mathfrak{g}_{\mathbb{C}}$ as a "sum of root spaces" $\sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where Φ is a finite subset of $\mathrm{Hom}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C})$ and $\mathfrak{g}_\alpha$ is the subspace of all $X \in \mathfrak{g}_{\mathbb{C}}$ such that $DX = \alpha(D)X$ for all $D \in \mathfrak{d}$. We write $\mathfrak{g}_\gamma = \{0\}$ if $\gamma \notin \Phi$.

Define $\bar{D} \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ by linearity and the condition that $\bar{D}X = \bar{\alpha}(D)X$ for all $X \in \mathfrak{g}_\alpha$ and all $\alpha \in \Phi$. Since $(\alpha(D) + \beta(D))^- = \bar{\alpha}(D) + \bar{\beta}(D)$, Remark 4.2 implies that $\bar{D}$ is a derivation. Further, $D + \bar{D}$ has real eigenvalues while $D - \bar{D}$ has purely imaginary eigenvalues. It remains to show that $\bar{D}$ restricts to a linear mapping of $\mathfrak{g}$.

By linear algebra, $\mathfrak{g}$ has a basis

$$\{X_j, Y_j, W_k : j \in \{1, \dots, J\}, k \in \{1, \dots, K\}\}$$

such that the subspaces $\mathrm{span}\{X_j, Y_j\}$ and $\mathrm{span}\{W_k\}$ are irreducible and invariant for $\mathfrak{d}$. In the complexification $\mathfrak{g}_{\mathbb{C}}$, each $D \in \mathfrak{d}$ is diagonalised in the basis

$$\{X_j + iY_j, X_j - iY_j, W_k : j \in \{1, \dots, J\}, k \in \{1, \dots, K\}\},$$

with eigenvalues λ_j and $\bar{\lambda}_j$ and μ_k, say; here the λ_j are strictly complex while the μ_k are real. By definition, $\bar{D}(X_j + iY_j) = \bar{\lambda}_j(X_j + iY_j)$ and $\bar{D}(X_j - iY_j) = \lambda_j(X_j - iY_j)$; it follows that

$$\bar{D}X_j = \mathrm{Re}\,\lambda_j X_j + \mathrm{Im}\,\lambda_j Y_j \qquad \text{and} \qquad \bar{D}Y_j = -\mathrm{Im}\,\lambda_j X_j + \mathrm{Re}\,\lambda_j Y_j.$$

Since also $\bar{D}W_k = \mu_k W_k$, it follows by $\mathbb{R}$-linearity that $\bar{D}$ preserves $\mathfrak{g}$, as required. □

Corollary 4.5 (After [49, Corollary 2.6])**.** *Suppose that $\mathfrak{g}$ is a Lie algebra and D is a derivation of $\mathfrak{g}$. Then we may write $D = D_{\mathrm{sr}} + D_{\mathrm{si}} + D_{\mathrm{n}}$, where each summand is a derivation of $\mathfrak{g}$, each summand commutes with the other summands, and D_{sr} is semisimple* *with real eigenvalues,* D_{si} *is semisimple with purely imaginary eigenvalues, and* D_{n} *is nilpotent. Moreover, the ranges* $\mathrm{Ran}(D_{\mathrm{sr}})$*,* $\mathrm{Ran}(D_{\mathrm{si}})$ *and* $\mathrm{Ran}(D_{\mathrm{n}})$ *are all subspaces of the range* $\mathrm{Ran}(D)$*.*

Proof. Bourbaki [13, Proposition 4, page 6] shows that we may write D as $D_{\mathrm{s}} + D_{\mathrm{n}}$, the commuting sum of a semisimple and a nilpotent derivation. Further, D_{s} decomposes as the commuting sum $D_{\mathrm{sr}} + D_{\mathrm{si}}$ of derivations, where the summands have real and purely imaginary eigenvalues, by Lemma 4.4. It remains to show that D_{sr} and D_{n} commute, and to examine the ranges.

We choose a Jordan basis for $\mathfrak{g}$ so that D is in real Jordan normal form; then in each block, the nilpotent part commutes with the real and imaginary parts of the diagonal part, and the ranges behave as claimed. □

4.2. **The lower central series.** We recall some standard facts (for more details, see, for example, [70, Section 3.5]). The lower central series of a Lie algebra $\mathfrak{g}$ is defined recursively:

$$\mathfrak{g}^{[0]} \coloneqq \mathfrak{g} \qquad \text{and} \qquad \mathfrak{g}^{[j+1]} = [\mathfrak{g}, \mathfrak{g}^{[j]}].$$

Then

$$\mathfrak{g}^{[0]} \supseteq \mathfrak{g}^{[1]} \supseteq \mathfrak{g}^{[2]} \supseteq \dots;$$

The subspaces in this series decrease strictly and then stabilise, that is, all later terms coincide. The series reaches $\{0\}$ if and only if $\mathfrak{g}$ is nilpotent; in this case, the *nilpotent length* ℓ *of* $\mathfrak{g}$ is the integer ℓ such that $\mathfrak{g}^{[\ell-1]} \neq \{0\}$ while $\mathfrak{g}^{[\ell]} = \{0\}$. Each $\mathfrak{g}^{[j]}$ is a characteristic ideal, that is, $D\mathfrak{g}^{[j]} \subseteq \mathfrak{g}^{[j]}$ for any derivation D of $\mathfrak{g}$. The lower central series of the complexification $\mathfrak{g}_{\mathbb{C}}$ is the complexification of the lower central series on $\mathfrak{g}$, that is, $(\mathfrak{g}_{\mathbb{C}})^{[j]} = (\mathfrak{g}^{[j]})_{\mathbb{C}}$.

4.3. **Modifications.** Many problems on Lie groups may be turned into linear problems on Lie algebras and solved. This is certainly the case for us. Corollary 2.25 shows that we are interested in examples of connected groups H with closed connected subgroups G_0, G_1 and K such that K is compact and $H = G_0 \rtimes K = G_1 \cdot K$. This implies that the corresponding Lie algebras satisfy $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{k}$ and $\mathfrak{g}_0$ is an ideal in $\mathfrak{h}$. In this situation, for all $X \in \mathfrak{g}_0$, there exists a unique $\sigma X \in \mathfrak{k}$ such that $X + \sigma X \in \mathfrak{g}_1$. Evidently, $\sigma : \mathfrak{g}_0 \to \mathfrak{k}$ is linear and

$$\mathfrak{g}_1 = \{X + \sigma X : X \in \mathfrak{g}_0\}.$$

The map σ and algebra $\mathfrak{g}_1$ are examples of what Gordon and Wilson [29] call a *modification map* and a *modification.* We shall be interested in modifications in the case where $\mathfrak{k}$ is the Lie algebra of a torus (so we write $\mathfrak{t}$).

The following technical lemma follows from Gordon and Wilson [29, Theorem 2.5]. We give a different proof.

Lemma 4.6. *Suppose that* $\mathfrak{h}$ *is a solvable Lie algebra of the form* $\mathfrak{n} \oplus \mathfrak{t}$*, where* $\mathfrak{n}$ *is a nilpotent ideal and* $\mathfrak{t}$ *is a toral subalgebra such that* $\mathfrak{t} \cap Z(\mathfrak{h}) = \{0\}$*. Suppose also that* $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{h}$ *such that* $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{t}$*. Then* $\mathfrak{g}$ *is an ideal in* $\mathfrak{h}$*.*

Proof. We are going to use induction on $\dim(\mathfrak{h})$. If $\dim(\mathfrak{h})$ is 0, 1 or 2, then $\mathfrak{g}$ is trivially an ideal. We assume for the rest of the proof that $\mathfrak{g}'$ is an ideal in $\mathfrak{h}'$ whenever $\mathfrak{h}'$, $\mathfrak{n}'$, $\mathfrak{t}'$ and $\mathfrak{g}'$ satisfy the hypotheses of the lemma and $\dim(\mathfrak{h}') < \dim(\mathfrak{h})$.

By "Weyl's unitarian trick", we may equip $\mathfrak{n}$ with an inner product such that the commuting family of linear maps $\mathrm{ad}(\mathfrak{t})$ is skew-symmetric. Since $\mathfrak{n}$ is $\mathrm{ad}(\mathfrak{t})$-invariant, so is each member $\mathfrak{n}^{[j]}$ of the lower central series (see Section 4.2), and there are (unique) $\mathrm{ad}(\mathfrak{t})$-invariant subspaces $\mathfrak{v}^{[j]}$ such that $\mathfrak{n}^{[j-1]} = \mathfrak{v}^{[j]} \oplus \mathfrak{n}^{[j]}$. Further, we may decompose the spaces $\mathfrak{v}^{[j]}$ into minimal $\mathrm{ad}(\mathfrak{t})$-invariant subspaces, of dimension 1 or 2, which we label $\mathfrak{w}_k$.

It is not hard to show inductively that $\mathfrak{n}^{[j]} = \mathrm{ad}(\mathfrak{v}^{[1]})^j\mathfrak{v}^{[1]} + \mathfrak{n}^{[j+1]}$, and hence $\mathfrak{v}^{[1]}$ generates $\mathfrak{n}$.

We consider $\mathfrak{g}$ as a modification of $\mathfrak{n}$, that is we choose $\sigma : \mathfrak{n} \to \mathfrak{t}$ such that

$$\mathfrak{g} = \{X + \sigma X : X \in \mathfrak{n}\}.$$

Since $\mathfrak{t}$ is abelian and $\mathfrak{g}$ is a subalgebra,

$$[X, Y] + [\sigma X, Y] - [\sigma Y, X] = [X + \sigma X, Y + \sigma Y] \in \mathfrak{g} \qquad \forall X, Y \in \mathfrak{n}.$$

All terms on the left-hand side lie in $\mathfrak{n} = \mathrm{Dom}(\sigma)$, and $\sigma(\mathfrak{g} \cap \mathfrak{n}) = \{0\}$, so

$$\sigma[X, Y] = \sigma[\sigma Y, X] - \sigma[\sigma X, Y] \qquad \forall X, Y \in \mathfrak{n}. \tag{4.1}$$

Step 1: a consequence of (4.1). Suppose that $\mathfrak{w}_j$ and $\mathfrak{w}_k$ are minimal $\mathrm{ad}(\mathfrak{t})$-invariant subspaces of $\mathfrak{n}$, and $\sigma[\mathfrak{w}_j, \mathfrak{w}_k] = \{0\}$. We claim, and shall now prove, that

$$\sigma[\sigma Y, X] = 0 \qquad \forall X \in \mathfrak{w}_j \quad \forall Y \in \mathfrak{w}_k, \tag{4.2}$$

or equivalently, $\sigma[\sigma X, Y] = 0$, since in light of our hypothesis,

$$\sigma[\sigma Y, X] = \sigma[\sigma X, Y] \qquad \forall X \in \mathfrak{w}_j \quad \forall Y \in \mathfrak{w}_k. \tag{4.3}$$

If $\dim(\mathfrak{w}_k) = 1$, then (4.2) holds, since $\mathrm{ad}(\sigma X)$ is skew-symmetric, so $[\sigma X, Y] = 0$; similarly (4.2) holds if $\dim(\mathfrak{w}_j) = 1$. If both $\mathfrak{w}_j$ and $\mathfrak{w}_k$ are 2-dimensional, then, as the dimension of the space of skew-symmetric maps of $\mathbb{R}^2$ is 1-dimensional, we may take an orthonormal basis $\{X_0, X_1\}$ of $\mathfrak{w}_j$ such that $\mathrm{ad}(\sigma X_0)|_{\mathfrak{w}_k} = 0$. This implies that

$$\sigma[\sigma Y, X_0] = \sigma[\sigma X_0, Y] = 0 \qquad \forall Y \in \mathfrak{w}_k.$$

Now there are two possibilities: either $[\sigma Y, X_0] = 0$ for all $Y \in \mathfrak{w}_k$, or there exists $Y \in \mathfrak{w}_k$ such that $[\sigma Y, X_0] \neq 0$. In the former case, the skew-symmetry of $\mathrm{ad}(\sigma Y)|_{\mathfrak{w}_j}$ implies that $\mathrm{ad}(\sigma Y)|_{\mathfrak{w}_j} = \{0\}$, for all $Y \in \mathfrak{w}_k$, and (4.2) holds. In the latter case, there exists $Y \in \mathfrak{w}_k$ such that $[\sigma Y, X_0] = X_1$ and hence $\sigma X_1 = 0$; coupled with the fact that $\mathrm{ad}(\sigma X_0)|_{\mathfrak{w}_k} = 0$, this shows that $\mathrm{ad}(\sigma X)|_{\mathfrak{w}_k} = 0$ for all $X \in \mathfrak{w}_j$ and (4.2) holds in this case too from (4.3).

Step 2: the case where $\mathfrak{n}$ *is abelian.* We recall that $\mathfrak{t} \cap Z(\mathfrak{h}) = \{0\}$.

Since

$$[\mathfrak{h}, \mathfrak{g}] = [\mathfrak{t}, \mathfrak{g}] + [\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{t}, \mathfrak{g}] + \mathfrak{g},$$

$\mathfrak{g}$ is an ideal if and only if $[\mathfrak{t}, \mathfrak{g}] \subseteq \mathfrak{g}$.

We consider the decomposition of $\mathfrak{n}$ into $\mathrm{ad}(\mathfrak{t})$-invariant subspaces $\mathfrak{w}_j$, as described in the second paragraph of this proof. Since $\mathfrak{n}$ is abelian, $[\mathfrak{w}_j, \mathfrak{w}_k] = \{0\}$ for all j and k. If $\sigma X = 0$ for all $X \in \mathfrak{w}_j$ and for all j, then $\mathfrak{g} = \mathfrak{n}$ and we are done. Otherwise, we fix a summand $\mathfrak{w}_j$ and $X \in \mathfrak{w}_j$ such that $\sigma X \neq 0$, and then our assumption that $\mathfrak{t} \cap Z(\mathfrak{h}) = \{0\}$ implies that there exists k such that $[\sigma X, \mathfrak{w}_k] \neq \{0\}$. Now $\sigma[\sigma X, Y] = 0$ for all $Y \in \mathfrak{w}_k$ and since $\mathrm{ad}(\sigma X)|_{\mathfrak{w}_k}$ is surjective, $\sigma Y = 0$ for all $Y \in \mathfrak{w}_k$. Then $\mathfrak{w}_k$ is a nontrivial ideal in $\mathfrak{h}$ that is contained in $\mathfrak{n}$ and in $\mathfrak{g}$. We may now write

$$\mathfrak{h}' = \mathfrak{n}' \oplus \mathfrak{t}' = \mathfrak{g}' \oplus \mathfrak{t}',$$

where

$$\mathfrak{h}' = \mathfrak{h}/\mathfrak{w}_k, \qquad \mathfrak{n}' = \mathfrak{n}/\mathfrak{w}_k, \qquad \mathfrak{g}' = \mathfrak{g}/\mathfrak{w}_k, \quad \text{and} \quad \mathfrak{t}' = (\mathfrak{t} + \mathfrak{w}_k)/\mathfrak{w}_k \simeq \mathfrak{t},$$

and it is easy to show that $\mathfrak{h}'$, $\mathfrak{n}'$, $\mathfrak{t}'$ and $\mathfrak{g}'$ satisfy the hypotheses of the lemma and $\dim(\mathfrak{h}') < \dim(\mathfrak{h})$, and so $\mathfrak{g}'$ is an ideal in $\mathfrak{h}'$ by the inductive assumption and hence $\mathfrak{g}$ is an ideal in $\mathfrak{h}$.

For the rest of the proof, we may and shall assume that $\mathfrak{n}$ is not abelian.

Step 3: the induction on dimension argument. Suppose that $\mathfrak{n}_0$ is a (nontrivial) ideal in $\mathfrak{h}$, that $\mathfrak{n}_0 \subseteq \mathfrak{n}^{[1]}$, and that $\sigma\mathfrak{n}_0 = \{0\}$, that is, $\mathfrak{n}_0 \subseteq \mathfrak{n}^{[1]} \cap \mathfrak{g}$. In this case, we may show that $\mathfrak{g}$ is an ideal by induction on dimension. Indeed, we may write

$$\mathfrak{h}' = \mathfrak{n}' \oplus \mathfrak{t}' = \mathfrak{g}' \oplus \mathfrak{t}',$$

where

$$\mathfrak{h}' = \mathfrak{h}/\mathfrak{n}_0, \qquad \mathfrak{n}' = \mathfrak{n}/\mathfrak{n}_0, \qquad \mathfrak{g}' = \mathfrak{g}/\mathfrak{n}_0, \quad \text{and} \quad \mathfrak{t}' = (\mathfrak{t} + \mathfrak{n}_0)/\mathfrak{n}_0 \simeq \mathfrak{t}.$$

By our inductive assumption, $\mathfrak{g}'$ is an ideal in $\mathfrak{h}'$, and hence $\mathfrak{g}$ is an ideal in $\mathfrak{h}$, as required.

Step 4: minimal $\mathrm{ad}(\mathfrak{t})$*-invariant subspaces.* Suppose that there exists a subspace $\mathfrak{w}_j$ such that $\sigma(\mathfrak{w}_j) = \{0\}$. Then for all $X \in \mathfrak{w}_j$, all $Y \in \mathfrak{n}$ and all $U \in \mathfrak{t}$,

$$\sigma[X, Y] = \sigma[\sigma X, Y] + \sigma[\sigma Y, X] = 0$$

since $\sigma X = 0$ by hypothesis and $[\sigma Y, X] \in \mathfrak{w}_j$, and

$$\sigma[U, [X, Y]] = \sigma[[U, X], Y] + \sigma[X, [U, Y]] = 0$$

similarly. Define

$$\mathfrak{n}_0 \coloneqq \mathfrak{w}_j + [\mathfrak{h}, \mathfrak{w}_j] + [\mathfrak{h}, [\mathfrak{h}, \mathfrak{w}_j]] + \dots;$$

then $\mathfrak{n}_0$ is an ideal in $\mathfrak{h}$ and $\sigma\mathfrak{n}_0 = \{0\}$, that is, $\mathfrak{n}_0 \subseteq \mathfrak{g}$.

There are now two possibilities: $\mathfrak{n}_0 \notin \mathfrak{v}^{[1]}$ or $\mathfrak{n}_0 \subseteq \mathfrak{v}^{[1]}$. In the first case, $\mathfrak{n}_1 \coloneqq \mathfrak{n}_0 \cap \mathfrak{n}^{[1]}$ is also an ideal which may be factored out much as in Step 3 to show that $\mathfrak{g}$ is an ideal by induction on dimension. Otherwise, $\mathfrak{n}_0$ is central in $\mathfrak{n}$ and an ideal in $\mathfrak{h}$, and may be factored out so that induction on dimension again shows that $\mathfrak{g}$ is an ideal.

Step 5: Denouement. Take $\mathfrak{w}_j \subseteq \mathfrak{n}^{[\ell-1]}$; then $\mathfrak{w}_j$ is an ideal in $\mathfrak{h}$, where ℓ is the nilpotent length of $\mathfrak{n}$. If $\dim(\mathfrak{w}_j) = 2$, then there exists $X \in \mathfrak{w}_k \subseteq \mathfrak{n}$ such that $\mathrm{ad}(\sigma X)|_{\mathfrak{w}_j} \neq 0$. Now $\sigma[\sigma X, Y] = 0$ for all $Y \in \mathfrak{w}_j$ by (4.2) and hence $\sigma\mathfrak{w}_j = \{0\}$. We may factor out $\mathfrak{w}_j$ and show that $\mathfrak{g}$ is an ideal by induction on dimension, as in Step 3. Otherwise, if $\dim(\mathfrak{w}_j) = 1$ and $\sigma\mathfrak{w}_j = \{0\}$, then $\mathfrak{w}_j$ is an ideal which we may factor out to apply the induction on dimension argument and show that $\mathfrak{g}$ is an ideal. Finally, if $\dim(\mathfrak{w}_j) = 1$ and $\sigma\mathfrak{w}_j \neq \{0\}$, there exists $\mathfrak{w}_k \subseteq \mathfrak{n}$ such that $[\sigma\mathfrak{w}_j, \mathfrak{w}_k] = \mathfrak{w}_k$, and now $\sigma\mathfrak{w}_k = \sigma[\sigma\mathfrak{w}_j, \mathfrak{w}_k] = \{0\}$ by (4.2), so again we may apply the result of Step 4 to conclude that $\mathfrak{g}$ is an ideal. □

4.4. **Split-solvability and the real-radical.** Recall that a solvable Lie algebra $\mathfrak{g}$ or corresponding Lie group G is said to be *split-solvable* (or *completely solvable*) if the eigenvalues of each $\mathrm{ad}(X)$, where $X \in \mathfrak{g}$, are real. If G is split-solvable and of polynomial growth, then the eigenvalues of each $\mathrm{ad}(X)$ are also purely imaginary, and so they are all zero, that is, G is nilpotent.

Theorem 4.7 (After Jablonski [40])**.** *Suppose that G is a connected Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ contains a unique maximal split-solvable ideal $\mathfrak{s}$. The analytic subgroup S of G corresponding to $\mathfrak{s}$ is closed, connected, and normal in G. If G is simply connected, then so is S.*

Proof. Jablonski [40, Proposition 2.1] showed that $\mathfrak{g}$ contains a unique maximal split-solvable ideal $\mathfrak{s}$.

We take S to be the Lie subgroup of G with Lie algebra $\mathfrak{s}$. The closure $\bar{S}$ is a connected normal solvable subgroup of G, contained in the radical R of G. Take $x \in \bar{S}$; we need to show that $x \in S$.

We recall that a subgroup S of G is closed if and only if $U \cap S$ is closed in U for a fixed open neighbourhood U of e. Hence we may suppose that x lies close enough to the identity that x and approximants x_n to x lie in $\exp(\mathfrak{s})$.

If $(x_n)_n$ is a sequence of elements of S that tend to x, then the characteristic polynomial p_n of each $\mathrm{Ad}(x_n)$ has only positive real roots. Since Ad is continuous, the characteristic polynomial p of $\mathrm{Ad}(x)$ is the limit of the p_n and has only roots in $[0, +\infty)$; as $\mathrm{Ad}(x)$ is invertible, 0 cannot be a root of p. It follows that the Lie algebra of $\bar{S}$ is a split-solvable ideal in $\mathfrak{g}$; by the maximality of $\mathfrak{s}$, $\bar{S} = S$, and S is closed.

By [15, end of Section II], all analytic subgroups of a connected simply connected solvable Lie group G are closed and simply connected. In particular, S is simply connected if G is simply connected. □

We call the Lie algebra $\mathfrak{s}$ and the group S of the theorem above the *real-radical* of $\mathfrak{g}$ and G, and denote them by $\mathrm{rrad}(\mathfrak{g})$ and $\mathrm{rrad}(G)$. The real-radical coincides with the nilradical in the special case where G is of polynomial growth.

The role of the real-radical is highlighted by the following simple result.

Lemma 4.8. *Suppose that H is a connected solvable Lie group with real-radical S, and T is a torus in H. Then $S \cap T \subseteq Z(H)$ and $\mathfrak{s} \cap \mathfrak{t} \subseteq Z(\mathfrak{h})$.*

Proof. If $x \in S \cap T$, then every eigenvalue of $\mathrm{Ad}(x)$ is of modulus 1 since $x \in T$ by "Weyl's unitarian trick", and is a positive real number since $x \in S$. Hence all eigenvalues are 1; since $\mathrm{Ad}(x)$ is semisimple because $x \in T$, $\mathrm{Ad}(x)$ is the identity operator, whence $x \in Z(H)$, and so $S \cap T \subseteq Z(H)$. *A fortiori* $\mathfrak{s} \cap \mathfrak{t} \subseteq Z(\mathfrak{h})$. □

If we are dealing with a metric space of the form H/T, where T is a torus in a connected solvable Lie group H such that $Z(H) \cap T = \{e\}$, then $S \cap T = \{e\}$. Such toral subgroups T are complemented.

Lemma 4.9. *Suppose that H is a connected solvable Lie group, and T is a toral subgroup of H such that $Z(H) \cap T = \{e\}$. Then there exists an ideal $\mathfrak{g}_0$ of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{t}$. Hence if H/T is simply connected, there exists a closed connected simply connected normal subgroup G_0 of H such that $H = G_0 \rtimes T$.*

Proof. Denote by $\mathfrak{h}$, $\mathfrak{n}$ and $\mathfrak{t}$ the Lie algebras of H, $\mathrm{nil}(H)$ and T; then $\mathfrak{n} \cap \mathfrak{t} = \{0\}$ by Lemma 4.8.

Take a subspace $\mathfrak{g}_0$ such that $\mathfrak{n} \subseteq \mathfrak{g}_0$ and $\mathfrak{g}_0 \oplus \mathfrak{t} = \mathfrak{h}$. Then $\mathfrak{g}_0$ is an ideal because every subspace of $\mathfrak{h}$ that includes $\mathfrak{n}$ is an ideal, because $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{n}$.

If H/T is simply connected, then the connected analytic subgroup G_0 of H with Lie algebra $\mathfrak{g}_0$ is closed and normal, and $H = G_0 \cdot T$ by Lemma 3.11, so $H = G_0 \rtimes T$. □

Split-solvable Lie subalgebras of solvable Lie algebras and hence connected split-solvable subgroups of solvable Lie groups have nice properties.

Theorem 4.10 (After Jablonski [40]). *Suppose that $\mathfrak{g}$ is a split-solvable subalgebra of a solvable Lie algebra $\mathfrak{h}$ and $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{t}$ and $Z(\mathfrak{h}) \cap \mathfrak{t} = \{0\}$. Then $\mathfrak{g}$ is the real-radical of $\mathfrak{h}$. If $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{t}$, then $\mathfrak{g}_1$ is also an ideal in $\mathfrak{h}$.*

Proof. We write $\mathfrak{n}$ and $\mathfrak{s}$ for the nilradical and real-radical of $\mathfrak{h}$.

First we are going to show that $\mathfrak{n} \subseteq \mathfrak{g}$. This implies immediately that $\mathfrak{g}$ is an ideal by Remark 4.3. Then, since $\mathfrak{g}$ is split-solvable by hypothesis, $\mathfrak{g}$ is contained in $\mathfrak{s}$. The hypotheses and Lemma 4.8 imply that

$$\dim(\mathfrak{h}) - \dim(\mathfrak{t}) = \dim(\mathfrak{g}) \leq \dim(\mathfrak{s}) \leq \dim(\mathfrak{h}) - \dim(\mathfrak{t}),$$

so $\mathfrak{g} = \mathfrak{s}$.

Since $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{t} \supseteq \mathfrak{n} \oplus \mathfrak{t}$, there is a unique linear mapping $\sigma : \mathfrak{n} \to \mathfrak{t}$ defined by the condition that

$$X + \sigma X \in \mathfrak{g} \qquad \forall X \in \mathfrak{n}.$$

Define

$$\tilde{\mathfrak{t}} \coloneqq \sigma(\mathfrak{n}), \qquad \tilde{\mathfrak{n}} \coloneqq \mathfrak{n}, \qquad \tilde{\mathfrak{h}} \coloneqq \mathfrak{n} \oplus \tilde{\mathfrak{t}}, \quad \text{and} \quad \tilde{\mathfrak{g}} \coloneqq \{X + \sigma X : X \in \mathfrak{n}\}.$$

Since $\mathfrak{t}$ is abelian, $\tilde{\mathfrak{t}}$ is a subalgebra of $\mathfrak{t}$; by Remark 4.3, $\tilde{\mathfrak{h}}$ is an ideal in $\mathfrak{h}$; and by linear algebra, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \cap \mathfrak{g}$; hence $\tilde{\mathfrak{g}}$ is a subalgebra of $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{t}}$.

Clearly, $\tilde{\mathfrak{n}}$ is a nilpotent ideal in $\tilde{\mathfrak{h}}$; if it were not the nilradical $\operatorname{nil}(\tilde{\mathfrak{h}})$ of $\tilde{\mathfrak{h}}$, then it would be a subalgebra thereof, and there would be some nonzero element U of $\tilde{\mathfrak{t}}$ in $\operatorname{nil}(\tilde{\mathfrak{h}})$. Consider $\operatorname{ad}(U)$ acting on $\tilde{\mathfrak{h}}$; this is semisimple since $\tilde{\mathfrak{t}}$ is toral, and nilpotent since $U \in \operatorname{nil}(\tilde{\mathfrak{h}})$, and hence $\operatorname{ad}(U)$ annihilates $\tilde{\mathfrak{h}}$. Since $U \in \tilde{\mathfrak{t}} \subseteq \mathfrak{t}$, $\operatorname{ad}(U)$ annihilates $\mathfrak{t}$ as $\mathfrak{t}$ is abelian and $\mathfrak{a}$ by the definition of $\mathfrak{a}$. Hence $\operatorname{ad}(U)$ annihilates $\mathfrak{h}$, that is, $U \in Z(\mathfrak{h}) \cap \mathfrak{t}$. We conclude that $U = 0$ and hence $\tilde{\mathfrak{n}}$ is also the nilradical of $\tilde{\mathfrak{h}}$.

We fix $X \in \tilde{\mathfrak{n}}$ and consider $\operatorname{ad}(X + \sigma X)$, acting on the complexified algebra $(\tilde{\mathfrak{g}})_{\mathbb{C}}$; suppose that

$$[X + \sigma X, Y + \sigma Y] = \lambda(Y + \sigma Y),$$

where $Y \in \tilde{\mathfrak{n}}_{\mathbb{C}} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. On the one hand, since $\mathfrak{g}$ is split-solvable, λ is real. On the other hand, $Y + \sigma Y \in \tilde{\mathfrak{n}}_{\mathbb{C}}$ since $\lambda \neq 0$, and so $\sigma Y = 0$. Now $Y \in \tilde{\mathfrak{n}}_{\mathbb{C}}^{[j]} \setminus \tilde{\mathfrak{n}}_{\mathbb{C}}^{[j+1]}$ (see Section 4.2 for the definition of the lower central series) for some j, whence

$$[\sigma X, Y] + \tilde{\mathfrak{n}}_{\mathbb{C}}^{[j+1]} = [X + \sigma X, Y] + \tilde{\mathfrak{n}}_{\mathbb{C}}^{[j+1]} = \lambda Y + \tilde{\mathfrak{n}}_{\mathbb{C}}^{[j+1]},$$

that is, λ is an eigenvalue of $\operatorname{ad}(\sigma X)$ acting on the complex quotient space $\tilde{\mathfrak{n}}_{\mathbb{C}}/\tilde{\mathfrak{n}}_{\mathbb{C}}^{[j+1]}$. Since $\operatorname{ad}(\sigma X)$ has purely imaginary eigenvalues, λ is purely imaginary.

These almost contradictory conclusions imply that all eigenvalues of $\operatorname{ad}(X + \sigma X)$, acting on $(\tilde{\mathfrak{g}})_{\mathbb{C}}$, are 0, and $\tilde{\mathfrak{g}}$ is nilpotent.

By Lemma 4.6, $\tilde{\mathfrak{g}}$ is an ideal in $\tilde{\mathfrak{h}}$; then $\tilde{\mathfrak{g}} \subseteq \tilde{\mathfrak{n}}$ as $\tilde{\mathfrak{n}}$ is the largest nilpotent ideal in $\tilde{\mathfrak{h}}$; for dimensional reasons, $\tilde{\mathfrak{n}} = \tilde{\mathfrak{g}}$. This completes the proof that $\mathfrak{n} \subseteq \mathfrak{g}$.

Now suppose that $\mathfrak{h}$ is a solvable Lie algebra with subalgebras $\mathfrak{g}$, $\mathfrak{g}_1$, and $\mathfrak{t}$ such that $\mathfrak{g}$ is a split-solvable ideal, $\mathfrak{t}$ is toral, and $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{t} = \mathfrak{g}_1 \oplus \mathfrak{t}$; we shall prove that $\mathfrak{g}_1$ is an ideal.

By the first part of this theorem, $\mathfrak{n} \subseteq \mathfrak{g}$ and $\mathfrak{g}$ is an ideal; now by "Weyl's unitarian trick", we may write $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$, where $[\mathfrak{t}, \mathfrak{a}] = \{0\}$. Much as before, there is a unique linear mapping $\sigma : \mathfrak{g} \to \mathfrak{t}$ such that

$$\mathfrak{g}_1 = \{X + \sigma X : X \in \mathfrak{g}\}.$$

As $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{h}$, $\mathfrak{g}_1$ is an ideal if and only if $[\mathfrak{t}, \mathfrak{g}_1] \subseteq \mathfrak{g}_1$. Now $[\mathfrak{t}, \sigma\mathfrak{g}] = \{0\}$, and $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$, where $[\mathfrak{t}, \mathfrak{a}] = \{0\}$, so

$$\begin{aligned} [\mathfrak{t}, \mathfrak{g}_1] &= \operatorname{span}\{[U, X + \sigma X] : U \in \mathfrak{t}, X \in \mathfrak{g}\} \\ &= \operatorname{span}\{[U, X] : U \in \mathfrak{t}, X \in \mathfrak{g}\} = [\mathfrak{t}, \mathfrak{g}] = [\mathfrak{t}, \mathfrak{n}], \end{aligned} \tag{4.4}$$

and hence $\mathfrak{g}_1$ is an ideal if and only if $[\mathfrak{t}, \mathfrak{n}] \subseteq \mathfrak{g}_1$.

Much as before, we define

$$\tilde{\mathfrak{t}} \coloneqq \mathfrak{t}, \qquad \tilde{\mathfrak{n}} \coloneqq \mathfrak{n}, \qquad \tilde{\mathfrak{h}} \coloneqq \mathfrak{n} \oplus \mathfrak{t}, \quad \text{and} \quad \tilde{\mathfrak{g}} \coloneqq \{X + \sigma X : X \in \mathfrak{n}\}.$$

Clearly $\tilde{\mathfrak{h}}$ is a subalgebra of $\mathfrak{h}$ and $\tilde{\mathfrak{h}} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{t}}$. Further, $[\mathfrak{t}, \mathfrak{a}] = \{0\}$ and $\mathfrak{h} = \mathfrak{a} \oplus \tilde{\mathfrak{h}}$, whence $Z(\tilde{\mathfrak{h}}) \cap \tilde{\mathfrak{t}} = Z(\mathfrak{h}) \cap \mathfrak{t} = \{0\}$; moreover, $[\tilde{\mathfrak{t}}, \tilde{\mathfrak{g}}] = [\tilde{\mathfrak{t}}, \tilde{\mathfrak{n}}]$ by the argument used to prove that $[\mathfrak{t}, \mathfrak{g}_1] = [\mathfrak{t}, \mathfrak{g}]$ in (4.4). From Lemma 4.6, $\tilde{\mathfrak{g}}$ is an ideal in $\tilde{\mathfrak{h}}$, and

$$[\mathfrak{t}, \mathfrak{n}] = [\tilde{\mathfrak{t}}, \tilde{\mathfrak{n}}] = [\tilde{\mathfrak{t}}, \tilde{\mathfrak{g}}] \subseteq \tilde{\mathfrak{g}} \subseteq \mathfrak{g}_1,$$

and hence $\mathfrak{g}_1$ is an ideal, as required. □

Corollary 4.11. *Suppose that G is a split-solvable subgroup of a connected solvable Lie group H and T is a toral subgroup of H such that $H = G \cdot T$ and $Z(H) \cap T = \{e\}$. Then G is normal in H and hence is the real-radical of H. If G_1 is a subgroup of H such that $H = G_1 \cdot T$, then G_1 is also normal in H.*

Proof. We reduce this proof to the previous result by considering the Lie algebras of the various groups and subgroups. The fact that the Lie algebra $\mathfrak{g}$ of G is an ideal and coincides with $\mathfrak{s}$ establishes that G is normal and the real-radical of H.

Next, if G_1 satisfies the hypotheses of the theorem, then $\mathfrak{h}$ is a solvable Lie algebra with subalgebras $\mathfrak{g}_0$, $\tilde{\mathfrak{g}}$, and $\mathfrak{t}$ such that $\mathfrak{g}_0$ is a split-solvable ideal, $\mathfrak{t}$ is toral, $\mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_1$, and $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{t} = \mathfrak{g}_1 \oplus \mathfrak{t}$. By the preceding theorem $\mathfrak{g}_1$ is an ideal, and hence G_1 is normal in H. □

4.5. **Twisted versions of groups and isometry of solvable groups.** We begin by recalling some results from Section 2 and an observation that arises from the work of Alekseevskiĭ [2].

If a connected, simply connected Lie group G_0 acts simply transitively and isometrically on a metric manifold (M, d) and H is an isometry group of (M, d) containing G_0, then we may write $H = G_0 \cdot K$, where K is the stabiliser in H of a base point in M, and the condition $\bigcap_{h\in H}(hKh^{-1}) = \{e_H\}$ holds. We suppose that G_0 is normal in H. If G_1 is also contained in H and acts simply transitively and isometrically on M, then $H = G_1 \cdot K$. Hence there is a continuous bijection $\mathrm{T} : G_1 \to G_0$ and a continuous map $\Phi : G_1 \to K$ such that $g = \mathrm{T}(g)\Phi(g)$ for all $g \in G_1$. We check that

$$\mathrm{T}(gh)\Phi(gh) = gh = \mathrm{T}(g)\Phi(g)\mathrm{T}(h)\Phi(h) = \mathrm{T}(g)\mathrm{T}(h)^{\Phi(g)}\Phi(g)\Phi(h)$$

for all $g, h \in G_1$, where $\mathrm{T}(h)^{\Phi(g)} \coloneqq \Phi(g)\mathrm{T}(h)\Phi(g)^{-1}$; thus Φ is a continuous homomorphism and $\mathrm{T}(g) = g\Phi(g)^{-1}$, so both maps are smooth; further,

$$\mathrm{T}(gh) = \mathrm{T}(g)\mathrm{T}(h)^{\Phi(g)},$$

and T is a twisted homomorphism or cocycle. Further, $G_0 = \{g\Phi(g)^{-1} : g \in G_1\}$, as T is a bijection. We summarise this discussion in the following definition and lemma.

Definition 4.12. We write G_1 *is a twisted version of* G_0 to mean that there exists a Lie group H, containing G_0 and G_1 as closed subgroups, with G_0 normal, and a Lie group homomorphism $\Phi : G_1 \to K$, where K is a compact subgroup of H, such that $H = G_1 \cdot K$ and $G_0 = \{g\Phi(g)^{-1} : g \in G_1\}$. In this case, we also say that Φ is the *twisting homomorphism.*

Example 4.13. Let H denote the Lie group $(\mathbb{R}^2 \rtimes \mathrm{SO}(2)) \times \mathbb{R}$, and define closed subgroups $G_0 \coloneqq (\mathbb{R}^2 \rtimes \{0\}) \times \mathbb{R}$ and $G_1 \coloneqq \{(x, y, [\alpha], \alpha) : x, y, \alpha \in \mathbb{R}\}$, where $[\alpha]$ denotes the equivalence class of α in $\mathrm{SO}(2)$, which we may identify with $\mathbb{R}/2\pi\mathbb{Z}$. Now both G_0 and G_1 are normal subgroups of H. The compact subgroup $K = \{(0,0)\} \times \mathrm{SO}(2) \times \{0\}$ and the homomorphism $\Phi : G_1 \to K$ defined by $(x, y, [\alpha], \alpha) \mapsto \alpha$ satisfy $\{g\Phi(g)^{-1} : g \in G_1\} = G_0$, and hence G_1 is a twisted version of G_0. In this case, G_0 is also a twisted version of G_1, via the twisting homomorphism $\Phi' : G_0 \to K$, $(x, y, 0, \alpha) \mapsto -[\alpha]$. Thus the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ by rotations (embedded as G_1), and the direct product $\mathbb{R}^3$ (embedded as G_0), are twisted versions of each other.

Note that if G_1 is connected and solvable, then the closure $(\Phi(G_1))^-$ is connected, solvable and compact, and so is a torus; we often write T instead of K in this case. This remark leads to our next result.

Lemma 4.14. *Let (G_0, d) be a solvable metric Lie group, H be the connected component of the identity in* $\mathrm{Iso}(G_0, d)$*, K be the stabiliser in H of the point e in G, and T be a maximal torus of K. Suppose that G_0 is normal in H.*

Then, for a connected solvable Lie group G, the following are equivalent:

(i) *G may be made isometric to (G_0, d);*
(ii) *G may be embedded in $G_0 \rtimes T$ in such a way that $G \cdot T = G_0 \rtimes T$; and*
(iii) *G is a twisted version of G_0 with a twisting homomorphism $\Phi : G \to T$.*

Proof. We recall that maximal tori of compact Lie groups are conjugate; hence the group $G_0 \rtimes T$ does not depend on the choice of T, up to isomorphism.

Suppose that G may be made isometric to (G_0, d). From Corollary 2.25, there is an embedding of G into H such that $H = G \cdot K = G_0 \cdot K$, and $G_0 \cdot K = G_0 \rtimes K$ by assumption. The closure of the image of G in the quotient $(G_0 \rtimes K)/G_0$, which is isomorphic to K, is solvable, connected, and compact, hence a torus, and so contained in a maximal torus. This implies that $G \cdot T = G_0 \rtimes T$.

Conversely, if we may embed G into $G_0 \rtimes T$ in such a way that $G_0 \rtimes T = G \cdot T$, then we may embed G into $G_0 \rtimes K$, and it may be checked that $G_0 \rtimes K = G \cdot K$; again from Corollary 2.25, G may be made isometric to (G_0, d_0).

The equivalence of (ii) and (iii) follows from the discussion preceding Definition 4.12. □

In our situation, where we have solvable subgroups G_1 and G_2 of an isometry group H that we want to show are algebraically similar, it would seem to be desirable to have G_1 and G_2 normal in H, and a way to try to do this is to make H as small as possible. Our first two lemmas show that H may be taken to be solvable.

Proposition 4.15. *Suppose that H is a connected Lie group with a connected compact subgroup K such that H/K is simply connected and there exists a solvable subgroup G of H such that $H = G \cdot K$. Let H_0 be a maximal connected solvable subgroup of H that contains G. Then*

(i) *H_0 is unique up to conjugation in H;*
(ii) *$T := H_0 \cap K$ is a torus, and $H_0 = G \cdot T$; and*
(iii) *if G_1 is any connected solvable subgroup of H such that $H = G_1 \cdot K$, then there is a conjugate G_1^h of G_1 in H that is contained in H_0 and $H_0 = G_1^h \cdot T$.*

If moreover H acts effectively on H/K, then H_0 acts effectively on H_0/T.

Proof. As usual, denote by $\mathfrak{h}$, $\mathfrak{k}$, and so on the Lie algebras of H, K and so on.

By hypothesis, $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{k}$, and *a fortiori* $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{k}$. If H_1 is a maximal connected solvable subgroup of H that contains G, then $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{k}$, and by Lemma 3.19, $\mathfrak{h}_1$ is conjugate to $\mathfrak{h}_0$ under the action of the adjoint group of $\mathfrak{h}$, whence H_1 is conjugate to H_0 in H, and (i) holds.

Consider the action of H_0 on the quotient space H/K. Since G acts transitively, H_0 does so, and the stabiliser in H_0 of the point K in the quotient space H/K is $H_0 \cap K$. Now $H = G \cdot K$, so that $H_0 = G \cdot (H_0 \cap K) = G \cdot T$. Further, T is connected because H_0 is connected and $H_0 = G \cdot T$, solvable because H_0 solvable, and compact because it is a closed subgroup of K. Hence T is a torus.

If G_1 is a connected solvable subgroup of H such that $H = G_1 \cdot K$, then $\mathfrak{h} = \mathfrak{g}_1 + \mathfrak{k}$. If $\mathfrak{h}'$ is a maximal solvable subalgebra of $\mathfrak{h}$ that contains $\mathfrak{g}_1$, then $\mathfrak{h} = \mathfrak{h}' + \mathfrak{k}$, and there exists $h \in H$ such that $\mathfrak{h}_0 = \mathrm{Ad}(h)\mathfrak{h}'$. It follows that $\mathrm{Ad}(h)\mathfrak{g}_1 \subseteq \mathfrak{h}_0$, and it follows that $hG_1h^{-1} \subseteq H_0$ and $H_0 = hG_1h^{-1} \cdot T$.

Finally if H acts effectively on H/K, then so does the subgroup H_0, and H/K may be identified with H_0/T. □

Let G_1 and G_2 be connected simply connected solvable Lie groups, and suppose that H is a solvable Lie group with a toral subgroup T such that $H = G_1 \cdot T = G_2 \cdot T$ and $Z(H) \cap T = \{e\}$. Ideally, we would like to deduce that G_1 is a twisted version of G_2, or vice versa, but unfortunately this is not quite true; however, from Lemma 4.9, there is a connected, simply connected group G_0 such that $H = G_0 \rtimes T = G_1 \cdot T = G_2 \cdot T$, and hence both G_1 and G_2 are twisted versions of G_0. In the proof of Lemma 4.9, there were many possible choices for G_0, and it might be hoped that there is a choice with some additional properties that are useful and make it unique.

For instance, suppose that H is of polynomial growth. One might hope that G_0 is nilpotent, but this is not always so. However, one may define an abelian extension H^* of H, in which H is a normal subgroup, with a toral subgroup T^* containing T, such that $H^* = G_1 \cdot T^* = G_2 \cdot T^*$, whose nilradical N satisfies $H^* = N \rtimes T^*$. Then G_1 and G_2 are both twisted versions of N, which is known as the nilshadow of both G_1 and G_2. We shall describe a construction of the group H^* like that of Alexopoulos [3], Dungey, ter Elst and Robinson [21], and Breuillard [14], and show that one choice of G_0 is the real-radical of H^*.

4.6. **Hulls and real-shadows.** In this section, we sketch the proof of the following theorem, whose roots are in results of Cornulier [16, Section 2] and of Jablonski [40, Proposition 4.2], as well as earlier results of Gordon and Wilson [28, 29] and even earlier work of Auslander and Green [5].

Theorem 4.16. *Let G be a connected, simply connected solvable Lie group. Let T be a maximal torus in a maximal compact subgroup of the automorphism group of G, let H be the semidirect product $G \rtimes T$, and let G_0 be the real-radical of H. Then $H = G_0 \rtimes T$; further, there is a smallest subtorus J of T, unique up to isomorphism, such that*

$$G \rtimes J = G_0 \rtimes J. \tag{4.5}$$

Hence G is a twisted version of G_0, with twisting homomorphism into J, and vice versa.

Remark 4.17. Let $G^* = G \rtimes J = G_0 \rtimes J$. We call the group G^* the *hull* of G and the group G_0 the *real-shadow* of G; the corresponding Lie algebras are also called the *hull* and the *real-shadow* of $\mathfrak{g}$.

Proof. Maximal compact subgroups of $\mathrm{Aut}(G)$ are connected and conjugate in $\mathrm{Aut}(G)$, and maximal tori of a given maximal compact subgroup K are conjugate in K. Hence H is unique up to isomorphism, and so G_0 is too.

We now show that $H = G_0 \rtimes T$, using Lie algebra. We choose a maximal torus with some convenient properties. Let $\mathfrak{g}$ and $\mathfrak{n}$ be the Lie algebra of G and its nilradical. Take a Cartan subalgebra $\mathfrak{c}$ (see [13, pp. 13–15]) of $\mathfrak{g}$. The quotient $(\mathfrak{n} + \mathfrak{c})/\mathfrak{n}$ is a Cartan subalgebra of the abelian Lie algebra $\mathfrak{g}/\mathfrak{n}$, by [13, Corollary 2, page 14]; hence $\mathfrak{n} + \mathfrak{c} = \mathfrak{g}$. Hence we may take a subspace $\mathfrak{a}$ of $\mathfrak{c}$ such that

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}. \tag{4.6}$$

Denote by $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{n}}$ the corresponding projections of $\mathfrak{g}$ onto $\mathfrak{a}$ and $\mathfrak{n}$. Then

(a) $\mathrm{ad}_s(X)Y = 0$ and $[\mathrm{ad}_s(X), \mathrm{ad}_s(Y)] = 0$ for all $X, Y \in \mathfrak{c}$; and
(b) the map $X \mapsto \mathrm{ad}_s(\pi_{\mathfrak{a}} X)$ is a Lie algebra homomorphism from $\mathfrak{g}$ onto an abelian subalgebra of $\mathrm{der}(\mathfrak{g})$, the Lie algebra of derivations of $\mathfrak{g}$.

Item (a) is proved as part of the proof of Proposition III.1.1 of [21]; item (b) is Lemma 3.1 of [14]. (To be precise, these authors have a type (R) assumption, but, as they state, this is not needed.)

We define the homomorphism $\varphi : \mathfrak{g} \to \operatorname{der}(\mathfrak{g})$ by

$$\varphi(X) = \operatorname{ad}_{\mathrm{si}}(\pi_{\mathfrak{a}} X),$$

that is, the "imaginary part" (as in Lemma 4.4) of the semisimple derivation $\operatorname{ad}_{\mathrm{s}}(\pi_{\mathfrak{a}} X)$ constructed above. This homomorphism annihilates $\mathfrak{n}$, and also $\mathfrak{s}$, and its image is a toral subalgebra of $\operatorname{der}(\mathfrak{g})$. Consider the closure J in $\operatorname{Aut}(\mathfrak{g})$ of the analytic subgroup corresponding to $\varphi(\mathfrak{a})$. Hence J is a torus. (It is an abuse of notation to call this torus J, but we shall later check that it does satisfy (4.5), and so the abuse is justified.)

Let T be a maximal torus of $\operatorname{Aut}(\mathfrak{g})$ that contains J, and define the Lie algebra $\mathfrak{h}$ to be the semidirect sum algebra $\mathfrak{g} \oplus \mathfrak{t}$, with Lie product given by

$$[(X, D), (Y, E)] = ([X, Y] + D(Y) - E(X), 0) \tag{4.7}$$

for all $X, Y \in \mathfrak{g}$ and all $D, E \in \mathfrak{t}$. In this proof, we write elements of $\mathfrak{h}$ as ordered pairs rather than as sums as we feel that this helps understanding. The subspace $\mathfrak{g}_0$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}_0 = \{(X, -\varphi(X)) : X \in \mathfrak{g}\}.$$

(Again, we are abusing notation here, but proving the next claim justifies the abuse.) We claim that

(a) $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{t}$;

(b) the map $\tau : X \mapsto (X, -\varphi(X))$ is a bijection from $\mathfrak{g}$ to $\mathfrak{g}_0$, and further,

$$[\tau(X), \tau(Y)] = \tau([X, Y]_{\mathrm{rrad}}),$$

where

$$[X, Y]_{\mathrm{rrad}} = [X, Y] - \varphi(X)Y + \varphi(Y)X \qquad \forall X, Y \in \mathfrak{g}; \tag{4.8}$$

(c) $\mathfrak{g}_0$ is an ideal and is the real-radical of $\mathfrak{h}$.

Parts (a) and (b) follow immediately from the definitions.

Third, $\mathfrak{g}_0$ is an ideal since $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{n} \oplus \{0\} \subseteq \mathfrak{g}_0$, by (4.7) and Remark 4.3. To see that $\mathfrak{g}_0$ is split-solvable, we suppose that $X \in \mathfrak{g}$, $Y \in \mathfrak{g}_{\mathbb{C}} \setminus \{0\}$, and

$$([X, Y] - \varphi(X)Y + \varphi(Y)X, 0) = [(X, -\varphi(X)), (Y, -\varphi(Y))] = \lambda(Y, -\varphi(Y));$$

it will suffice to show that λ is real. If $\lambda \neq 0$, then $\varphi(Y) = 0$, so we may suppose that $Y \in \mathfrak{n}_{\mathbb{C}}$, whence, from (4.7), $\operatorname{ad}(X)Y - \varphi(X)Y = \lambda Y$, which implies that

$$\begin{aligned}(\operatorname{ad}_{\mathrm{sr}}(\pi_{\mathfrak{a}} X) + \operatorname{ad}_{\mathrm{n}}(\pi_{\mathfrak{a}} X) + \operatorname{ad}(\pi_{\mathfrak{n}} X))Y &= (\operatorname{ad}(\pi_{\mathfrak{a}} X) + \operatorname{ad}(\pi_{\mathfrak{n}} X) - \operatorname{ad}_{\mathrm{si}}(\pi_{\mathfrak{a}} X))Y \\ &= \lambda Y.\end{aligned}$$

Consider the complexified lower central series of $\mathfrak{n}$, that is, $\mathfrak{n}_{\mathbb{C}}^{[0]} = \mathfrak{n}_{\mathbb{C}}$, and $\mathfrak{n}_{\mathbb{C}}^{[j]} = [\mathfrak{n}_{\mathbb{C}}, \mathfrak{n}_{\mathbb{C}}^{[j-1]}]$ when $j > 0$. Since $Y \neq 0$, there exists $j \in \mathbb{N}$ such that $Y \in \mathfrak{n}^{[j]} \setminus \mathfrak{n}^{[j+1]}$. Now all the spaces $\mathfrak{n}^{[j]}$ are invariant under all derivations of $\mathfrak{n}_{\mathbb{C}}$, and in particular under $\operatorname{ad}_{\mathrm{sr}}(\pi_{\mathfrak{a}} X)$, $\operatorname{ad}_{\mathrm{n}}(\pi_{\mathfrak{a}} X)$ and $\operatorname{ad}(\pi_{\mathfrak{n}} X))$. Thus these operators have quotient actions on the quotient algebra $\mathfrak{n}^{[j]}/\mathfrak{n}^{[j+1]}$, which we write as $\operatorname{qad}_{\mathrm{sr}}(\pi_{\mathfrak{a}} X)$, $\operatorname{qad}_{\mathrm{n}}(\pi_{\mathfrak{a}} X)$ and $\operatorname{qad}(\pi_{\mathfrak{n}} X))$. Evidently, $\operatorname{qad}(\pi_{\mathfrak{n}} X)) = 0$, $\operatorname{qad}_{\mathrm{sr}}(\pi_{\mathfrak{a}} X)$ is semisimple with real eigenvalues, $\operatorname{qad}_{\mathrm{n}}(\pi_{\mathfrak{a}} X)$ is nilpotent, and the last two quotient operators commute. The eigenvalues of $\operatorname{qad}_{\mathrm{sr}}(\pi_{\mathfrak{a}} X)$ and of $\operatorname{qad}_{\mathrm{sr}}(\pi_{\mathfrak{a}} X) + \operatorname{qad}_{\mathrm{n}}(\pi_{\mathfrak{a}} X)$ coincide by [11, Theorem 1, p. A.VII.43]. So all the eigenvalues of $\operatorname{ad}(X, -\varphi(X))$ are real, and $\mathfrak{g}_0$ is indeed split-solvable.

From Theorem 4.10, $\mathfrak{g}_0$ is the real-radical of $\mathfrak{h}$; we shall now write it as $\mathfrak{g}_0$.

Next, we consider the groups that correspond to these Lie algebras. We have already seen that T is a torus. By Lemma 3.11, the connected analytic subgroup G_0 of H whose Lie algebra is $\mathfrak{g}_0$ is closed and $H = G_0 \cdot T$. Further, G_0 is normal in H as $\mathfrak{g}_0$ is an ideal in $\mathfrak{h}$. Thus $H = G \rtimes T = G_0 \rtimes T$.

We organised matters so that $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{t} = \mathfrak{g}_0 \oplus \mathfrak{t}$; however, by construction, $\mathfrak{g} \oplus \mathfrak{j} = \mathfrak{g}_0 \oplus \mathfrak{j}$, where $\mathfrak{j}$ is the Lie algebra of J, and there is no proper subtorus of J whose Lie algebra has this property. At group level, H/G may be identified with T in $\mathrm{Aut}(G)$ and J is the smallest subtorus of T that may be identified with the closure of G_0G/G therein. Thus

$$G \rtimes J = G_0 \rtimes J,$$

as required. We have seen that H and hence G_0 are unique up to isomorphism: it follows that J is too. □

We note that if G is split-solvable, then it is isomorphic to its real-shadow. If G is of polynomial growth, then its real-shadow coincides with its nilshadow, since in this case the real-radical and the nilradical of the hull are the same.

Remark 4.18. Suppose that G_0 is split-solvable, T_0 is a torus of automorphisms of G_0, and $H \coloneqq G_0 \rtimes T_0$. If G is a subgroup of H such that $H = G \cdot T_0$, then G_0 is the real-shadow of G.

This would be obvious if G were normal in H, but we have not assumed that this is true. However, if we take the smallest subtorus T_1 of T_0 such that $G \subseteq G_0 \rtimes T_1$, then G is normal in $G_0 \rtimes T_1$ by Corollary 4.11, and hence G_0 is the real-shadow of G.

4.7. **Applications to metric Lie groups.** Now we look at some of the consequences of the theory that we have developed, not only in Section 4, but also earlier.

We recall that a connected solvable Lie group is simply connected if and only if it is contractible. Thus a Lie group that may be made isometric to a connected simply connected solvable Lie group is contractible. By Remark 3.16, a contractible Lie group G may be written as $R \rtimes L$, where the radical R is simply connected and the Levi subgroup L is a direct product of finitely many (possibly zero) copies of the universal covering group of $\mathrm{SL}(2, \mathbb{R})$. Conversely, Theorem 3.24 shows that a Lie group G with the structure just described may be made isometric to a solvable Lie group. By contrast, if a Lie group G may be made isometric to a connected simply connected nilpotent Lie group, then G is contractible and of polynomial growth, and by Lemma 3.20, G is solvable.

These observations are the reason why we include a solvability hypothesis in many but not all of the upcoming results.

Corollary 4.19. *Let (G, d) be a connected simply connected solvable metric Lie group, H be a maximal connected solvable subgroup of $\mathrm{Iso}(G, d)$ containing G, and T be the stabiliser in H of the point e_G in G. Let G_0 be a normal subgroup of H such that $H = G_0 \rtimes T$, as in Lemma 4.9, and let G_1 be a connected solvable Lie group. Then the following are equivalent:*

(i) G_1 may be made isometric to (G, d);
(ii) G_1 may be embedded in H in such a way that $H = G_1 \cdot T$; and
(iii) G and G_1 are both twisted versions of G_0 with twisting homomorphisms into T.

Proof. If G_1 may be made isometric to (G, d), then G_1 is simply connected and there is an embedding of G_1 in $\mathrm{Iso}(G, d)$, by Theorem 2.21, hence an embedding of G_1 in H by Proposition 4.15, and so H contains closed disjoint subgroups G_1 and T, and $H = G_1 \cdot T$. Conversely, if G_1 may be embedded in H in such a way that $H = G_1 \cdot T$, then G_1 may be made isometric to (G, d) by Corollary 2.25.

The equivalence of (ii) and (iii) follows from Theorem 4.16. □

Corollary 4.20. *Let (G, d) be a connected simply connected solvable metric Lie group. Let G^* and G_0 be the hull and the real-shadow of G, and write $G^* = G \rtimes J$, as in Theorem 4.16. Then the following are equivalent:*

(i) G_0 *may be made isometric to* (G,d)*; and*
(ii) d *is invariant under conjugation by elements of* J*.*

Proof. If d is invariant under conjugation by elements of J, that is, if $d(jgj^{-1}, jhj^{-1}) = d(g,h)$ for all $g,h \in G$ and all $j \in J$, then we may view d as a G^*-invariant metric on G^*/J, hence as a G_0-invariant metric on G^*/J, and hence as a metric on G_0.

Conversely, if G_0 may be made isometric to (G,d), then we may embed G and G_0 into a maximal connected solvable subgroup H of $\mathrm{Iso}(G,d)$, by Proposition 4.15, and write $H = G_0 \cdot T = G \cdot T$ for a suitable torus T. By Corollary 4.11, $H = G_0 \rtimes T$. We may take a smaller subgroup H_0 of H of the form $G_0 \rtimes J$, where J is a subtorus of T, that is minimal subject to the requirement that $G \subseteq H_0$, and then, by Remark 4.18, G is normal in H_0, and H_0 and G_0 are the hull and real-shadow of G. We may identify G with H_0/J and the metric d is H_0-invariant, and *a fortiori* is J-invariant. □

We now restate (and expand slightly) Theorem C.

Theorem 4.21. *Let* G_0 *be a connected simply connected split-solvable Lie group,* T *be a maximal torus in* $\mathrm{Aut}(G_0)$*, and* d_0 *be a* T*-invariant metric on* G_0*. Let* G_1 *be a connected simply connected solvable Lie group. Then the following are equivalent:*
(i) G_1 *may be made isometric to* G_0*;*
(ii) G_1 *may be made isometric to* (G_0, d_0)*;*
(iii) G_0 *is the real-shadow of* G_1*;*
(iv) G_1 *may be embedded in* $H \coloneqq G_0 \rtimes T$ *in such a way that* $H = G_1 \cdot T$*; and*
(v) G_1 *is a twisted version of* G_0 *with twisting homomorphism into* T*.*

Proof. Before we start our proof, we note that the existence of a T-invariant metric d_0 on G_0 is shown in Corollary 2.18.

The equivalence of (ii), (iv) and (v) may be found in Lemma 4.14. The equivalence of (iii) and (iv) follows from Theorem 4.16 and Remark 4.18.

Evidently (ii) implies (i), so it will suffice to show that (i) implies (iii). If (i) holds, there is a metric d_0 on G_0 such that G_1 may be made isometric to (G_0, d_0); hence G_1 may be embedded in H, the connected component of the identity in $\mathrm{Iso}(G_0, d_0)$. By Proposition 4.15, there is a connected solvable subgroup H_0 of H and a torus T_0 in H_0 such that $H_0 = G_0 \cdot T_0$, and G_1 may also be embedded in H_0 in such a way that $H_0 = G_1 \cdot T_0$. By Corollary 4.11, $H_0 = G_0 \rtimes T_0$, and now G_0 is the real-shadow of G_1 by Remark 4.18. □

The following are corollaries of Theorem 4.21 and the theory that we have developed. This first follows immediately from the riemannian version of Corollary 2.18 (which is well known) and Theorem 4.21

Corollary 4.22. *Let* G_0 *be a connected simply connected split-solvable Lie group. Then there exists a riemannian metric* d_0 *on* G_0 *such that every connected simply connected solvable Lie group that may be made isometric to* G_0 *may be made isometric to* (G_0, d_0)*.*

Part of the next corollary also follows immediately from Theorem 4.21.

Corollary 4.23. *Let* G_1 *and* G_2 *be connected, simply connected solvable Lie groups. Then* G_1 *and* G_2 *may be made isometric if and only if their real-shadows are isomorphic.*

Proof. First, suppose that G_0 is the real-shadow of both G_1 and G_2, and take a metric d_0 on the real-shadow G_0 that is invariant under a maximal torus T of $\mathrm{Aut}(G_0)$. Then both G_1 and G_2 may be made isometric to (G_0, d_0).

Conversely, suppose that G_1 and G_2 are connected simply connected solvable Lie groups that admit admissible left-invariant metrics d_1 and d_2 such that (G_1, d_1) and

(G_2, d_2) are isometric. Let H be a maximal connected solvable subgroup of the Lie group $\mathrm{Iso}(G_1, d_1)$, and T be the stabiliser of the identity e of G_1 in H. By Corollary 4.19, there is a normal subgroup G of H such that

$$H = G \rtimes T = G_1 \cdot T = G_2 \cdot T.$$

Let T^* be a maximal torus of $\mathrm{Aut}(G)$ that contains T, and let G_0 be the real-radical of $H^* \coloneqq G \rtimes T^*$, so that $H^* = G_0 \rtimes T^*$ by Theorem 4.16. Now $G_1 \subseteq H_0 \subseteq H^*$ and $G_2 \subseteq H^*$ similarly. We may check that $H^* \coloneqq G_1 \cdot T^* = G_2 \cdot T^*$, using Lie algebra and Lemma 3.11. By Theorem 4.21, G_0 is the real-shadow of G_1 and of G_2. □

Of course, if G_1 and G_2 have the same real-shadow G_0, then not only may they be made isometric to G_0, but to (G_0, d_0), where d_0 is the metric of Corollary 4.22.

We have already observed that in the nilpotent case, stronger results are possible.

Corollary 4.24. *Let G_1 and G_2 be simply connected Lie groups and assume that G_1 is nilpotent. The following are equivalent:*

(i) G_2 and G_1 may be made isometric;
(ii) G_2 is solvable and of polynomial growth and G_1 is its nilshadow.

Proof. Since G_1 is simply connected and nilpotent, it is contractible and of polynomial growth. If G_1 and G_2 may be made isometric, then G_2 is contractible and of polynomial growth by Lemma 2.27, so is solvable by Lemma 3.20. Now (ii) follows from Theorem 4.21. It is immediate from Theorem 4.21 that (ii) implies (i). □

This leads to the following, which should be compared to a result of Kivioja and Le Donne [44].

Corollary 4.25. *If G_1 and G_2 are connected, simply connected nilpotent Lie groups, and both may be made isometric to the same connected Lie group G (not* a priori *solvable, and possibly with different metrics), then G is solvable and G_1 and G_2 are isomorphic.*

Proof. By the previous corollary, G is solvable and of polynomial growth, and both G_1 and G_2 are isomorphic to the nilshadow of G. □

In the preceding corollary, if "nilpotent" is replaced with "split-solvable", we cannot deduce that G must be solvable. However, if we replace "nilpotent" with "split-solvable" and we assume *a priori* that G is solvable, then the conclusion that G_1 and G_2 are isomorphic still holds, as they are both isomorphic to the real-shadow of G.

There are examples in the work of Gordon and Wilson [29, 28] and of Jablonski [40] where stronger results hold for split-solvable groups if an *a priori* assumption of unimodularity is included.

Our final corollaries are concerned with quasi-isometry rather than isometry. A general observation is that if two Lie groups may be made isometric using arbitrary admissible left-invariant metrics, then they may be made isometric for the derived semi-intrinsic metrics of (2.5), or for suitable riemannian metrics, as in Corollary 3.4, and hence they are quasi-isometric when equipped with admissible left-invariant proper quasigeodesic metrics, as all such metrics on a given group are quasi-isometric. We recall from Theorem 3.24 that a contractible homogeneous metric manifold (M, d) is homeomorphically roughly isometric to a connected, simply connected solvable metric Lie group. With an additional hypothesis of polynomial growth, more may be said.

Corollary 4.26. *Let (M, d) be a contractible homogeneous metric manifold. Suppose further that d is proper quasigeodesic and that M is of polynomial growth, as in* (2.14). *Then (M, d) is quasi-isometrically homeomorphic to a simply connected riemannian nilpotent Lie group.*

Proof. Theorem 3.24 shows that (M,d) is roughly isometrically homeomorphic to a simply connected solvable metric Lie group (H,d_H); by construction, (H,d_H) is proper quasigeodesic.

Let N be the nilshadow of H. By Theorem 4.21, there are metrics d'_H and d'_N on H and N such that (H,d'_H) and (N,d'_N) are isometric. We may assume that d'_H and d'_N are riemannian, by Corollary 4.22.

Finally, d is proper quasigeodesic and all admissible left-invariant proper quasigeodesic metrics on a Lie group are quasi-isometric, so the identity map on H is a quasi-isometry from d_H to d'_H. □

With a slightly weaker hypothesis, we obtain a slightly weaker conclusion.

Corollary 4.27. *Let (M,d) be a homogeneous metric space of polynomial growth, and suppose that the metric d is proper quasigeodesic. Then (M,d) is quasi-isometric to a connected simply connected nilpotent riemannian Lie group.*

Proof. Theorem 3.24 shows that (M,d) is roughly isometric to a simply connected solvable metric Lie group (H,d_H), which is a metric quotient of (N,d) with compact fibre, and hence also of polynomial growth.

We now repeat the argument of the previous corollary. □

If (M,d) is a homogeneous metric space of polynomial growth, then the argument above shows that there is an admissible metric d' on M, such that (M,d') is of polynomial growth and quasi-isometric to a connected simply connected nilpotent riemannian Lie group. For example, we may take d' to be a derived semi-intrinsic metric, as defined just before Lemma 2.3.

4.8. **Notes and remarks.**

4.3. Modifications. In the terminology of Gordon and Wilson [29], our Lemma 4.6 states that modifications of nilpotent Lie ideals are normal. Gordon and Wilson [29] proved the stronger result that modifications of nilpotent subalgebras are normal. However, our Theorem 4.10 shows that nilpotent subalgebras of solvable Lie algebras with a toral complement are ideals, and so our two results combined include their theorem.

4.4. Split-solvability and the real-radical. The real-radical, at the Lie algebra level, appears in the work of Jablonski [40]. In particular, the Lie algebra part of Theorem 4.7 and Theorem 4.10 are due to him. In the language of Gordon and Wilson [29], [28], the second part of Theorem 4.10 states that modifications of split-solvable groups are normal.

It was shown by Wolf [72] that a connected riemannian nilpotent group is normal in its isometry group. On the other hand, the examples of symmetric spaces of the noncompact type show that a riemannian split-solvable connected Lie group G need not be normal in its isometry group H; we may write $H = G \cdot K$, where K is the stabiliser of a base-point, but it is certainly false that $H = G \rtimes K$. So Theorem 4.10 and Corollary 4.11 are perhaps a little surprising.

One important way in which our approach differs from that of Gordon and Wilson is that we use Mostow's theorem [57] on maximal solvable subgroups to reduce questions of possible isometry of solvable groups to questions of possible isometry of solvable groups in a solvable supergroup. This enables us to avoid some of the complications that arise in dealing with general Lie groups and algebras.

4.5. Twisted versions of groups and isometry of solvable groups. Definition 4.12 is close to a proposal of Alekseevskiĭ [2], who used the expression *twisting* rather than twisted version (or rather his translator did). Actually, he considered the related question whether $\{g\Phi(g)^{-1} : g \in G_1\}$ is a subgroup if G_1 is normal and $\Phi : G_1 \to K$ is a homomorphism. His answer is not definitive, but the situation is now clearer due to the contributions of Gordon and Wilson [28, 29], who looked at the corresponding question at the Lie algebra level, namely, when $\{X + \varphi(X) : X \in \mathfrak{g}_1\}$ is a subalgebra.

4.6. Hulls and real-shadows. The idea of using a Cartan subalgebra of $\mathfrak{g}$ to find a good complement of $\mathrm{nil}(\mathfrak{g})$, as in Theorem 4.16, or to construct the nilshadow, seems to be due to Alexseevskiĭ. However, his class of solvable groups is restricted to those which arise in the study of riemannian homogeneous spaces of nonpositive curvature, and for these groups, the Cartan subalgebra $\mathfrak{a}$ is abelian; extra ideas are needed to deal with general solvable Lie groups. These are due to Alexopoulos (in the polynomial growth case).

The following example shows that not all the Cartan subalgebras that appear in the "shadow construction" are abelian. We take the Lie algebra $\mathfrak{g}$ with basis $\{U, V, X, Y, Z\}$ and commutation relations determined by linearity, antisymmetry and the nonzero basis commutation relations

$$[X, Y] = Z, \qquad [X, U] = U, \qquad [Y, V] = V.$$

This is a solvable extension of the abelian algebra $\mathrm{span}\{U, V\}$ by the nilpotent algebra $\mathrm{span}\{X, Y, Z\}$. The Cartan subalgebra $\mathrm{span}\{X, Y, Z\}$ is nilpotent and not abelian.

The nilshadow appears in work of Auslander and Green [5], where the group G^* is called the *hull* of G; it seems that the term nilshadow was first used in [6]. Interestingly, it seems that type (R) also appeared for the first time in [5]. Their construction of the nilshadow used ideas from the theory of algebraic groups. An alternative construction of the nilshadow, based on Lie algebras, appears in the work of Gordon and Wilson [28, 29], phrased in the language of modifications; their work was not restricted to groups of polynomial growth, and perhaps for this reason they did not make explicit the connection with the construction of Auslander and Green. The Lie algebraic construction of the nilshadow was found later by Alexopoulos [3], and developed by Dungey, ter Elst, and Robinson [21] and by Breuillard [14]. The nilshadow has been used quite extensively in the area of harmonic analysis on Lie groups, and in applications to nonriemannian metric geometry of Lie groups.

What we call the real-shadow is more recent. For groups that need not be of polynomial growth, the detailed investigation of Gordon and Wilson [28, 29] identified a special subgroup G_0, said to be in standard position, that is sometimes split-solvable. Cornulier [16] developed an object that he called the trigshadow, using techniques closer to those of Auslander and Green, and in particular working at group level rather than algebra level. In the recent work of Jablonski [40], which has roots in the work of Gordon and Wilson, the idea of a maximal split-solvable ideal appears and the real-shadow as viewed as a maximal split-solvable ideal of a larger Lie algebra.

We describe a construction of the hull G^* like that of Alexopoulos, Dungey, ter Elst and Robinson, and Breuillard.

Recall from Lemma 4.9 that if H is a solvable Lie group with a toral subgroup T such that $Z(H) \cap T = \{e\}$ and H/T is simply connected, then we may find a normal subgroup G_0 of H such that $H = G_0 \rtimes T$. Gordon and Wilson [28, 29] spend some effort on finding a choice of G_0 "in standard position". Essentially this is a group which is "as real as possible". From our point of view, the construction of G_0 proceeds, using Lie algebras, as follows: first, take a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{h}$ containing $\mathfrak{t}$ (this is possible), and then a subspace $\mathfrak{a}$ of $\mathfrak{c}$ such that $\mathfrak{h} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{t}$. Replace any $X \in \mathfrak{a}$ such that $\operatorname{ad}_{\mathrm{si}}(X)$, the imaginary part of the semisimple part of $\operatorname{ad}(X)$, as in Corollary 4.5, coincides with $\operatorname{ad}(U)$ for some U in $\mathfrak{t}$ by $X - U$. This produces a new subspace $\tilde{\mathfrak{a}}$ such that $\mathfrak{h} = \mathfrak{n} \oplus \tilde{\mathfrak{a}} \oplus \mathfrak{t}$. Let $\mathfrak{g}_0$ be $\mathfrak{n} \oplus \tilde{\mathfrak{a}}$.

A tool often used by Gordon and Wilson to construct nice subalgebras, such as the nilshadow of a solvable Lie algebra, is the Killing form, and orthogonal complements of compact subalgebras appear in their development, much as in Corollary 3.22.

4.7. Consequences and applications. Gordon and Wilson [29, Example 2.8] give examples of nonisomorphic connected simply connected solvable Lie groups G_1 and G_2 that are isometric, but they are not isometric to their real-shadow.

The universal covering group H of the group $\mathbb{R}^2 \rtimes \mathrm{SO}(2)$ of orientation-preserving rigid motions of $\mathbb{R}^2$ is a simply connected three-dimensional solvable Lie group that admits a left-invariant subriemannian metric d such that (H, d) is not bi-Lipschitz equivalent to any nilpotent group. Indeed, the two simply connected three-dimensional nilpotent Lie groups are the abelian group $\mathbb{R}^3$, which is the nilshadow of H, and the nonabelian Heisenberg group $\mathbb{H}$. However, if d is a suitable left-invariant subriemannian metric on H, then (H, d) is not even quasiconformally equivalent to either $\mathbb{R}^3$ or $\mathbb{H}$; see [23]. Nevertheless, (H, d) is locally bi-Lipschitz to $\mathbb{H}$ with the standard subriemannian metric.

Apropos of Theorem 4.10 and Corollary 4.11, in the riemannian case, the normality of a nilpotent Lie group N in its isometry group was proved by Wolf [72] and rediscovered by Wilson [71].

In the special case where (M, d) is of polynomial growth, so is every group G that acts simply transitively and isometrically on (M, d). If any such group G is nilpotent, then G is normal in $\operatorname{Iso}(M, d)$ by a theorem of Wolf [72], which is formulated for riemannian metrics but which extends to the case of general metrics by Corollary 3.4. This was extended to unimodular split-solvable groups by Gordon and Wilson [28, 29].

Corollary 4.25 was known for nilpotent G and arbitrary metrics, and for solvable G with riemannian metrics; see [2, 28, 29, 44, 71, 72]. Kivioja and Le Donne also showed that isometries of nilpotent metric Lie groups are affine, that is, are composed of translations and group automorphisms.

5. Characterisation of metrically self-similar Lie groups

In this section we prove Theorem D, which we renumber as Theorem 5.5.

One well known motivation of the study of metrically self-similar Lie groups is their appearance as stratified groups [24] or Carnot groups [59]. Another, perhaps less well known, is their appearance as the parabolic visual boundaries of negatively curved homogeneous riemannian manifolds. More precisely, Heintze [35] showed that every simply connected negatively curved homogeneous riemannian manifold is isometric to a riemannian Lie group (G, g) that is a semidirect product $N \rtimes_A \mathbb{R}$, where N is a simply connected nilpotent Lie group and at the Lie algebra level, $\mathbb{R}$ acts on $\mathfrak{n}$ by a derivation A whose eigenvalues have strictly positive real parts.

The parabolic visual boundary of (G, g) may be identified with the Lie group N, as we now explain. Let ξ be the geodesic ray whose support is $\{e_N\} \times \mathbb{R}^+$ in $N \rtimes_A \mathbb{R}$. The *parabolic visual boundary* is the set of infinite geodesics $\gamma \colon \mathbb{R} \to (G, g)$ that are asymptotic to ξ as $t \to +\infty$ and satisfy $\lim_{t\to\infty} d(\gamma(t), \xi(t)) = 0$. This set may be equipped with the Hamenstädt metric (see [36, p. 384])

$$d(\alpha, \beta) = \exp(-\tfrac{1}{2} \lim_{t\to+\infty} (2t - d_g(\alpha(-t), \beta(-t)))),$$

where d_g is the riemannian metric induced by g. Identification of the horosphere centered at ξ with the subset $N \times \{0\}$ gives a natural identification of the parabolic visual boundary and the Lie group N. Using this identification, one may show by direct computation that for every $t \in \mathbb{R}^+$ the automorphism of N with differential e^{tA} is a metric dilation of (N, d). Thus (N, d) is a metrically self-similar group.

We remark that the metric d need not be riemannian, or even geodesic. A simple example is when $N = \mathbb{R}^2$ and A is the diagonal matrix with diagonal entries 1 and 2. Here the group $\mathbb{R}^2 \rtimes_A \mathbb{R}$ may be given a negatively curved left-invariant riemannian metric that induces on its parabolic visual boundary, which is the topological space $\mathbb{R}^2$, a metric d that is bi-Lipschitz-equivalent to the product of $\mathbb{R}$ equipped with the usual metric and $\mathbb{R}$ equipped with the square root of the usual metric. Then $(\mathbb{R}^2, d)$ is a metrically self-similar group that is not geodesic. See [73] for more examples along these lines.

The structure of this section is the following. We show that a metric space satisfying the hypotheses of the theorem is doubling. Then we show that its isometry group G is a Lie group of polynomial growth, whence every Levi subgroup of G is compact. However, the metric space is contractible, so the stabiliser K of a point is a maximal compact subgroup, which contains the Levi subgroup. This allows us to find a subgroup S of G that is transverse to K, and this subgroup S induces the group structure on the metric space.

5.1. **Properties of metrically self-similar Lie groups.** We recall the definition of metrically self-similar Lie group and we present some examples and properties.

Definition 5.1. A *metrically self-similar Lie group* is a triple (G, d, δ), where G is a connected Lie group, d is an admissible left-invariant metric on G, and δ is an automorphism of G such that $d(\delta x, \delta y) = \lambda d(x, y)$ for some $\lambda \neq 1$.

The basic examples of metrically self-similar Lie groups are normed vector spaces of finite dimension where the dilation is scalar multiplication. Several other examples are already available when $G = \mathbb{R}^2$.

If $\alpha, \beta \geq 1$, then the automorphisms δ_λ corresponding to the matrix

$$\begin{pmatrix} \lambda^\alpha & 0 \\ 0 & \lambda^\beta \end{pmatrix}$$

are all dilations of factor λ for metrics including

$$d((x, y), (x', y')) = \max\{|x - x'|^{1/\alpha}, |y - y'|^{1/\beta}\}$$

or, when $\alpha = \beta$,

$$d(x, y) = \|x - y\|^{1/\alpha}$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^2$. In [48, Proposition 5.1], it is shown that there exists a homogeneous metric d whose spheres are fractals in $\mathbb{R}^2$ when $\alpha = 2$.

When $\alpha \geq 1$, the automorphisms δ_λ corresponding to the matrix

$$\lambda^\alpha \begin{pmatrix} \cos(\log \lambda) & -\sin(\log \lambda) \\ \sin(\log \lambda) & \cos(\log \lambda) \end{pmatrix}$$

are dilations of factor λ for the metric $d(x, y) = \|x - y\|^{1/\alpha}$, where $\|\cdot\|$ is the euclidean norm.

If $\alpha > 1$, then there is a left-invariant metric d on $\mathbb{R}^2$ for which the automorphisms δ_λ corresponding to the matrices

$$\begin{pmatrix} \lambda^\alpha & \lambda^\alpha \log(\lambda^\alpha) \\ 0 & \lambda^\alpha \end{pmatrix}$$

are dilations of factor λ. These dilations appear in [10] in the study of visual boundaries of Gromov hyperbolic spaces. See also [73] for further results and examples in $\mathbb{R}^n$.

Definition 5.2. A *(positive) grading* of a Lie algebra $\mathfrak{g}$ is a splitting $\mathfrak{g} = \bigoplus_{t\in\mathbb{R}^+} \mathfrak{v}_t$ such that $[\mathfrak{v}_s, \mathfrak{v}_t] \subseteq \mathfrak{v}_{s+t}$ for all $s,t \in \mathbb{R}^+$. A Lie group G is *gradable* if it is simply connected and its Lie algebra admits a grading.

Note that finitely many $\mathfrak{v}_t$ have positive dimension, because $\mathfrak{g}$ has finite dimension; further, a gradable group is nilpotent. When G is a gradable Lie group with Lie algebra grading $\mathfrak{g} = \bigoplus_{t\in\mathbb{R}^+} \mathfrak{v}_t$, we may define the *standard dilations* $\delta_\lambda : G \to G$ by requiring that $(\delta_\lambda)_* V = \lambda^t V$ for all $V \in \mathfrak{v}_t$. It is known that a metric d exists on G so that (G, d, δ_λ) is a metrically self-similar group if and only if $\mathfrak{v}_t = \{0\}$ for all $t \in (0,1)$, see [26]. For much more on gradable groups, see [50] and the references cited there.

Gradable groups are the only Lie groups that support a dilation, by the following theorem of Siebert [65].

Theorem 5.3. *Let G be a connected Lie group and suppose that there exists a Lie group automorphism $\delta : G \to G$ such that*

$$\lim_{n\to+\infty} \delta^n g = e_G \qquad \forall g \in G.$$

Then G is gradable, nilpotent and simply connected.

Corollary 5.4. *If (G, d, δ) is a metrically self-similar Lie group, then G is gradable, nilpotent and simply connected. Moreover, all metric dilations on (G, d) are Lie group automorphisms of G.*

Proof. Since a metrically self-similar Lie group admits a contractive automorphism, the first statement follows from Theorem 5.3. Recall that a metric dilation on a metric space (G, d) is a bijection $f : G \to G$ such that $d(f(x), f(y)) = \mu d(x, y)$ for all $x, y \in G$ and some $\mu \in (1, +\infty)$. Such a map is also an isometry from $(G, \mu d)$ to (G, d), and by [44, Proposition 2.4], isometries between connected nilpotent Lie groups are group isomorphisms composed with translations. ☐

5.2. **Proof of Theorem D.** We restate Theorem for the reader's convenience.

Theorem 5.5. *If a homogeneous metric space admits a metric dilation, then it is isometric to a metrically self-similar Lie group. Moreover, all metric dilations of a metrically self-similar Lie group are automorphisms.*

The last sentence in Theorem 5.5 was proved in Corollary 5.4. Throughout this section, we assume that (M, d) is a homogeneous metric space, $\lambda \in (1, +\infty)$, and δ is a bijection of M such that $d(\delta x, \delta y) = \lambda d(x, y)$ for all $x, y \in M$. Since M is locally compact and isometrically homogeneous, it is complete, and the Banach fixed point theorem shows that δ has a unique fixed point, o say. We prove a few preliminary results.

Lemma 5.6. *The metric space (M, d) is proper and doubling.*

Proof. The ball $B(o, r)$ is relatively compact for all sufficiently small r; using the dilation we see that this holds for all $r \in \mathbb{R}$, and (M, d) is proper.

We now show that (M, d) is a doubling metric space. Since the closed ball $\breve{B}(o, \lambda)$ is compact, there are points $x_1, \ldots, x_k \in \breve{B}(o, \lambda)$ such that

$$\breve{B}(o, \lambda) \subseteq \bigcup_{i=1}^{k} B(x_i, 1/2).$$

Take $R \in \mathbb{R}^+$, and define $n := \lfloor \log_\lambda R \rfloor$, so that $1 \le \lambda^{-n} R < \lambda$. Then

$$\delta^n B(x_i, 1/2) \subseteq \delta^n B(x_i, \lambda^{-n} R/2) = B(\delta^n x_i, R/2),$$

and so

$$B(o, R) = \delta^n(B(o, \lambda^{-n} R)) \subseteq \delta^n(B(o, \lambda)) \subseteq \bigcup_{i=1}^{k} B(\delta^n x_i, R/2).$$

Since (M, d) is isometrically homogeneous, (M, d) is doubling. □

Let H denote the connected component of the identity in $\mathrm{Iso}(M, d)$.

Lemma 5.7. *The space M is contractible, and H and M may be given analytic structures, compatible with their topologies, such that the Lie group H acts on M analytically and transitively. Moreover H is of polynomial growth.*

Proof. Define $\pi : H \to M$ by $\pi h := ho$ and $T : H \to H$ by $Th := \delta \circ h \circ \delta^{-1}$; then $\pi \circ T = \delta \circ \pi$. Let K be the maximal compact normal subgroup of H. Note that $T(K) = K$, since T is an automorphism of H. Then $\pi(K)$ is a compact subset of M: let $r := \max\{d(o, p) : p \in \pi(K)\}$. Then

$$\pi(K) = \pi T^{-1}(K) = \delta^{-1} \pi(K) \subseteq B(o, \lambda^{-1} r),$$

which implies that $r = 0$. Therefore $\pi(K) = \{o\}$, and K is contained in the stabiliser in H of the point o in M; by Remark 2.5, $K = \{e_H\}$. By Montgomery–Zippin structure theory (as in Theorem 3.1 and Corollary 3.3), H and M may be given analytic structures, compatible with their topologies, such that M is a manifold and the action of H on M is analytic.

Since M is a manifold and admits a metric dilation, it is compactly contractible, and hence contractible by Lemma 3.6. Since moreover M is doubling and proper by Lemma 5.6, it is of polynomial growth by Remark 2.30. By Lemma 2.27, H is a group of polynomial growth. □

Proof of Theorem 5.5. Let (M, d) be a homogeneous metric space. Let δ be a metric dilation of factor $\lambda \in (1, +\infty)$ and with fixed point o. Let H denote the connected component of the identity in $\mathrm{Iso}(M, d)$. By Lemma 5.7, H is a Lie group of polynomial growth (and hence is amenable) and M may be identified with H/K, where K is the stabiliser of o in H; further, M is contractible, so K is a maximal compact subgroup by Lemma 3.6.

We may now apply Lemma 3.22, and deduce that there exists a connected Lie subgroup G of H such that the restricted quotient map $h \mapsto h(o)$ from G to M is a homeomorphism. We use this homeomorphism to make G into a metrically self-similar Lie group isometric to (M, d).

Define the metric d_G on G by $d_G(h, h') = d(h(o), h'(o))$. It is clear that this is an admissible metric, and it is left-invariant because

$$d_G(hh', hh'') = d(h(h'(o)), h(h''(o))) = d(h'o, h''o) = d_G(h', h'')$$

for all $h, h', h'' \in G$. Further, define the map T on H by

$$Tg := \delta \circ g \circ \delta^{-1}.$$

Then T is a Lie group automorphism of H. Since $TK = K$, Lemma 3.22 implies that $TG = G$. Thus $T|_G$ is a Lie group automorphism of G.

We note that after the identification of G with M, the map $T|_G$ coincides with δ. Indeed,

$$(Th)(o) = (\delta h \delta^{-1})(o) = \delta(ho),$$

and the proof is complete. □

References

[1] A. Agrachev and D. Barilari, 'Sub-Riemannian structures on 3D Lie groups', *J. Dyn. Control Syst.* **18** (2012), 21–44.

[2] D. V. Alekseevskiĭ, 'Homogeneous Riemannian spaces of negative curvature', *Math. USSR–Sb.* **25** (1975), 87–109.

[3] G. Alexopoulos, 'Sub-Laplacians with drift on Lie groups of polynomial volume growth', *Mem. Amer. Math. Soc.* **155**, No. 739 (2002).

[4] S. A. Antonyan, 'Characterizing maximal compact subgroups', *Arch. Math. (Basel)* **98** (2012), 555–560.

[5] L. Auslander and L. W. Green, 'G-induced flows', *Amer. J. Math.* **88** (1966), 43–60.

[6] L. Auslander and R. Tolimieri, 'Splitting theorems and the structure of solvmanifolds', *Ann. of Math. (2)* **92** (1970), 164–173.

[7] R. W. Bagley and M. R. Peyrovian, 'A note on compact subgroups of topological groups', *Bull. Austral. Math. Soc.* **33** (1986), 273–278.

[8] V. N. Berestovskiĭ, 'Homogeneous manifolds with an intrinsic metric. II', *Sibirsk. Mat. Zh.* **30** (1989), 14–28, 225.

[9] V. Berestovskii and C. Plaut, 'Covering group theory for topological groups', *Topology Appl.* **114** (2001), 141–186.

[10] M. Bonk and B. Kleiner, 'Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary', *Geom. Topol.* **9** (2005), 219–246.

[11] N. Bourbaki, *Algebra II. Chapters 4–7.* Elements of Mathematics. Springer-Verlag, Berlin–Heidelberg–New York, 1990. Translated from the earlier French edition.

[12] N. Bourbaki, *Lie Groups and Lie Algebras. Chapters 1–3.* Elements of Mathematics. Springer-Verlag, Berlin, 1989. Translated from the earlier French edition.

[13] N. Bourbaki, *Lie Groups and Lie Algebras. Chapters 7–9.* Elements of Mathematics. Springer-Verlag, Berlin, 1989. Translated from the earlier French edition.

[14] E. Breuillard, 'Geometry of locally compact groups of polynomial growth and shape of large balls', *Groups Geom. Dyn.* **8** (2014), 669–732.

[15] C. Chevalley, 'On the topological structure of solvable groups', *Ann. of Math.* **42** (1941), 668–675.

[16] Y. de Cornulier, 'Dimension of asymptotic cones of Lie groups', *J. Topol.* **1** (2) (2008), 342–361.

[17] Y. Cornulier, 'Commability and focal locally compact groups', *Indiana Univ. Math. J.* **64** (1) (2015), 115–150.

[18] Y. Cornulier, 'On sublinear bilipschitz equivalence of groups', *Ann. Sci. Éc. Norm. Sup.* **5** (2019), 1201–1242.

[19] Y. Cornulier and P. de la Harpe, 'Metric geometry of locally compact groups'. Tracts in Mathematics 25. European Mathematical Society, Zürich, 2016.

[20] D. van Dantzig and B. L. van der Waerden, 'Über metrisch homogene Räume', *Abh. Math. Sem. Hamburgischen Univ.* **6** (1928), 367–376.

[21] N. Dungey, A. F. M. ter Elst, and D. W. Robinson, *Analysis on Lie Groups with Polynomial Growth.* Progress in Math., Vol. 214. Birkhäuser, Boston, 2003.

[22] K. Fässler and E. Le Donne, 'On the quasi-isometric and bi-Lipschitz classification of 3D Riemannian Lie groups', *Geom. Dedic.* (2020), `doi:10.1007/s10711-020-00532-8`.

[23] K. Fässler, P. Koskela, and E. Le Donne, 'Nonexistence of quasiconformal maps between certain metric measure spaces', *Intern. Math. Res. Not. IMRN* (16) (2015), 6968–6987.

[24] G. B. Folland, Gerald (1975), 'Subelliptic estimates and function spaces on nilpotent Lie groups', *Arkiv for Mat.* **13** (1975), 161–207.

[25] G. B. Folland and E. M. Stein, 'Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group', *Comm. Pure Appl. Math.* **27** (1974), 429–522.

[26] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups.* Mathematical Notes, Vol. 28. Princeton University Press, Princeton; University of Tokyo Press, Tokyo, 1982.

[27] H. Glöckner, Mathematics Review MR3365793.

[28] C. S. Gordon and E. N. Wilson, 'The fine structure of transitive Riemannian isometry groups. I', *Trans. Amer. Math. Soc.* **289** (1985), 367–380.

[29] C. S. Gordon and E. N. Wilson, 'Isometry groups of Riemannian solvmanifolds', *Trans. Amer. Math. Soc.* **307** (1988), 245–269.

[30] M. Gromov, 'Groups of polynomial growth and expanding maps', *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73.

[31] M. Gromov, 'Carnot–Carathéodory spaces seen from within', in: *Sub-Riemannian Geometry.* Progress in Math., Vol. 144, Birkhäuser, Basel, 1996, pp. 85–323.
[32] Y. Guivarc'h, 'Croissance polynomiale et périodes des fonctions harmoniques', *Bull. Soc. Math. France* **101** (1973), 333–379.
[33] W. Hebisch and A. Sikora, 'A smooth subadditive homogeneous norm on a homogeneous group', *Studia Math.* **96** (1990), 231–236.
[34] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces.* Graduate Studies in Math., Vol. 34. American Mathematical Society, Providence, 2001. Corrected reprint of the 1978 original.
[35] E. Heintze, 'On homogeneous manifolds of negative curvature', *Math. Ann.* **211** (1974), 23–34.
[36] S. Hersonsky and F. Paulin, 'On the rigidity of discrete isometry groups of negatively curved spaces', *Comment. Math. Helv.* **72** (1997), 349–388.
[37] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Volume 1.* Die Grundlehren der mathematischen Wissenschaften, Band 115. Springer-Verlag, New York, 1963.
[38] J.R. Hubbuck and R.M. Kane, 'The homotopy types of compact Lie groups', *Israel J. Math.* **51** (1985), 20–26.
[39] K. Iwasawa, 'On some types of topological groups', *Ann. of Math.* **50** (1949), 507–558.
[40] M. Jablonski, 'Maximal symmetry and unimodular solvmanifolds', *Pacific J. Math.* **298** (2019), 417–427.
[41] N. Jacobson, *Lie Algebras.* Dover, New York, 1979. Republication of the 1962 original.
[42] J. W. Jenkins, 'Growth of connected locally compact groups', *J. Funct. Anal.* **12** (1973), 113–127.
[43] E. Kehlet, 'Cross sections for quotient maps of locally compact groups', *Math. Scand.* **55** (1984), 152–160.
[44] V. Kivioja and E. Le Donne, 'Isometries of nilpotent metric groups', *J. École Polytechnique – Math.* **4** (2017), 473–482.
[45] A. W. Knapp, *Lie Groups Beyond an Introduction.* Progress in Math., Vol. 140. Birkhäuser, Boston, 2002. 2nd edn.
[46] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol. I.* Interscience Publishers, New York–London, 1963.
[47] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol. II.* Interscience Publishers, New York–London–Sydney, 1969.
[48] E. Le Donne and S. Nicolussi Golo, 'Regularity properties of spheres in homogeneous groups', *Trans. Amer. Math. Soc.* **370** (2018), 2057–2084.
[49] E. Le Donne and S. Nicolussi Golo, 'Metric Lie groups admitting dilations', *Arkiv för Mat.*, to appear.
[50] E. Le Donne and S. Rigot, 'Besicovitch Covering Property on graded groups and applications to measure differentiation', *J reine angew. Math.* **750** (2019), 241–297.
[51] E. Le Donne, 'A metric characterization of Carnot groups', *Proc. Amer. Math. Soc.* **143** (2015), 845–849.
[52] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, 3rd Edition. Studies in Mathematics 25. Walter de Gruyter, Berlin–Boston, 2013.
[53] J. Milnor, 'A note on curvature and fundamental group', *J. Differential Geom.* **2** (1968), 1–7.
[54] D. Montgomery and L. Zippin, *Topological Transformation Groups.* Krieger, Huntington, N.Y., 1974. Reprint of the 1955 original.
[55] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications.* Mathematical Surveys and Monographs, Vol. 91. American Mathematical Society, Providence, 2002.
[56] M. Moskowitz and R. Sacksteder, The exponential map and differential equations on real Lie groups, *J. Lie Theory* **13** (2003), 291–306.
[57] G. D. Mostow, 'On maximal subgroups of real Lie groups', *Ann. of Math.* **74** (1961), 503–517.
[58] G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces.* Annals of Mathematics Studies, 78. Princeton University Press, Princeton, 1973.
[59] P. Pansu, *Géometrie du groupe d'Heisenberg.* Thesis, Université Paris VII, Paris, 1982.
[60] P. Pansu, 'Croissance des boules et des géodésiques fermées dans les nilvariétés', *Ergodic Theory Dynam. Systems* **3** (1983), 415–445.
[61] P. Pansu, 'Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un', *Ann. of Math. (2)* **129** (1989), 1–60.
[62] K. Nomizu, 'Invariant affine connections on homogeneous spaces', *Amer. J. Math.* **76** (1954), 33–65.
[63] L. P. Rothschild and E. M. Stein, 'Hypoelliptic differential operators and nilpotent groups', *Acta Math.* **137** (1976), 247–320.

[64] H. Samelson, 'Topology of Lie groups', *Bull. Amer. Math. Soc.* **58** (1952), 2–37.
[65] E. Siebert, 'Contractive automorphisms on locally compact groups', *Math. Z.* **191** (1986), 73–90.
[66] M. Stroppel, *Locally Compact Groups*. EMS Textbooks in Math. European Mathematical Society, Zürich, 2006.
[67] A. S. Švarc, 'A volume invariant of coverings', *Doklady Akad. Nauk. SSR* **105** (1955), 32–34.
[68] J. Szenthe, 'On the topological characterization of transitive Lie group actions', *Acta Sci. Math. (Szeged)* **36** (1974), 323–344.
[69] T. Tao, *Hilbert's Fifth Problem and Related Topics*. Graduate Studies in Math., Vol. 153. American Mathematical Society, Providence, 2014.
[70] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*. Graduate Texts in Math., Vol. 102. Springer-Verlag, New York, 1984. Reprint of the 1974 edition.
[71] E. N. Wilson, 'Isometry groups on homogeneous nilmanifolds', *Geom. Dedicata* **12** (1982), 337–346.
[72] J. A. Wolf, 'Growth of finitely generated solvable groups and curvature of Riemanniann manifolds', *J. Differential Geom.* **2** (1968), 421–446.
[73] X. Xie, 'Large scale geometry of negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$' *Geom. Topol.* **18** (2015), 831–872.
[74] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*. Monographs in Mathematics, Birkhaüser, Basel, 1984.

School of Mathematics and Statistics, University of New South Wales, UNSW Sydney 2052, Australia

Department of Mathematics and Statistics, University of Jyväskylä, Jyväskylä FI-40014 Finland

Department of Mathematics and Statistics, University of Jyväskylä, Jyväskylä FI-40014 Finland
Current address: Department of Mathematics, University of Fribourg, Fribourg CH-1700 Switzerland

Department of Mathematics, University of Trento, via Calepina 14, I-38122 Trento, Italy
Current address: Department of Mathematics and Statistics, University of Jyväskylä, Jyväskylä FI-40014 Finland

School of Mathematics and Statistics, University of New South Wales, UNSW Sydney 2052, Australia

[C]

Metric equivalences of Heintze groups and applications to classifications in low dimension

Ville Kivioja, Enrico Le Donne and Sebastiano Nicolussi Golo

Preprint

METRIC EQUIVALENCES OF HEINTZE GROUPS AND APPLICATIONS TO CLASSIFICATIONS IN LOW DIMENSION

VILLE KIVIOJA, ENRICO LE DONNE, AND SEBASTIANO NICOLUSSI GOLO

Abstract. We approach the quasi-isometric classification questions on Lie groups by considering low dimensional cases and isometries alongside quasi-isometries. First, we present some new results related to quasi-isometries between Heintze groups. Then we will see how these results together with the existing tools related to isometries can be applied to groups of dimension 4 and 5 in particular. Thus we take steps towards determining all the equivalence classes of groups up to isometry and quasi-isometry. We completely solve the classification up to isometry for simply connected solvable groups in dimension 4, and for the subclass of groups of polynomial growth in dimension 5.

Contents

Date: April 1, 2021.
2010 *Mathematics Subject Classification.* 20F67, 53C23, 22E25, 17B70, 20F69, 30L10, 54E40.
Key words and phrases. isometries, quasi-isometries, homogeneous groups, Heintze groups.
V.K. and E.L.D. were partially supported by the European Research Council (ERC Starting Grant 713998 GeoMeG '*Geometry of Metric Groups*'). E.L.D. and S.N.G. were partially supported by the Academy of Finland (grant288501 '*Geometry of subRiemannian groups*' and by grant 322898 '*Sub-Riemannian Geometry via Metric-geometry and Lie- group Theory*'). V.K. was also supported by Emil Aaltonen Foundation.

0. Introduction

This paper is a contribution to various metric classifications of Lie groups. The study of quasi-isometries between solvable groups is an active area of research [Pan89, Sha04, Sau06, dC18, EFW12, EFW13, Dym10, Xie14, Xie15, CPS17, Pia17, Pal20]. Distinguished examples of solvable groups are Heintze groups, i.e., those solvable simply connected Lie groups that admit left-invariant Riemannian structures with negative sectional curvatures [Hei74]. Every Heintze group G is a semidirect product of $\mathbb{R}$ and a nilpotent graded Lie group N. The parabolic visual boundary of G has a structure of homogeneous group. Namely, the boundary may be identified with N equipped with a distance that has dilation properties. Moreover, a quasi-isometry between two Heintze groups induces a quasisymmetry between the associated nilpotent groups and vice versa [Pau96, BS00, dC18]. These quasisymmetries are, or induce, biLipschitz maps between the boundaries equipped with suitable homogeneous structures [LDX16, Pia17].

The main aim of this article is twofold: First, we introduce quasi-isometry invariants that finally distinguish some low dimensional Heintze groups. Second, we study a finer metric classification. We say that two Lie groups G and H *can be made isometric* if there are left-invariant Riemannian metrics ρ_G and ρ_H so that (G, ρ_G) is isometric to (H, ρ_H). This is an equivalence relation among simply connected solvable groups, and we find the equivalence classes in low dimension: we consider all simply connected solvable Lie groups in dimension 4 and those with polynomial growth in dimension 5. For each equivalence class, there is a Riemannian manifold for which each element of the class acts isometrically and simply transitively. In our construction, such a Riemannian manifold is a Lie group, which we call the "real-shadow". In particular, we make a contribution to the conjecture that claims that every two Heintze groups are either not quasi-isometric or they can be made isometric.

0.1. **Quasi-isometries of Heintze groups.** First we present our results related to distinguishing Heintze groups up to quasi-isometry equivalence. We work on the level of parabolic visual boundaries, thus our objects of interest are *homogeneous groups*, by which we mean pairs (N, α) where N is a simply connected nilpotent Lie group and α is a derivation of N, such that $N \rtimes_\alpha \mathbb{R}$ defines a Heintze group. We may assume that $N \rtimes_\alpha \mathbb{R}$ is purely real, i.e., that all the eigenvalues of α are real numbers. For a homogeneous group (N, α), we always consider the biLipschitz class of distances that are homogeneous under the one-parameter subgroups of automorphisms induced by the derivation α. This class may be empty in some cases, see Remark 1.3.

Below we use the following notation: If (N,α) is a homogeneous group and $\bigoplus_{\lambda>0} V^\alpha_\lambda$ is the decomposition of the Lie algebra of N by the generalised eigenspaces of the derivation α, then for every $s>0$ we denote by $(N,\alpha)^{(s)}$ the subgroup of N with the Lie algebra $\mathrm{LieSpan}(\bigoplus_{s\geq\lambda>0} V^\alpha_\lambda)$.

Theorem A. *Let (N_1,α) and (N_2,β) be purely real homogeneous groups that are biLipschitz equivalent via a map $F\colon N_1\to N_2$.*

(i) Then N_1 and N_2 are quasi-isometric as Riemannian Lie groups.
(ii) For every $p\in N_1$ and every $s\geq 1$ we have $F(p(N_1,\alpha)^{(s)})=F(p)(N_2,\beta)^{(s)}$ and the same holds for all the iterated normalisers of the subgroups $(N_1,\alpha)^{(s)}$ and $(N_2,\beta)^{(s)}$, respectively.

The proof of this result, which is inspired by the results of [CPS17], is presented in Section 2. We also present some examples to illustrate how this result helps to distinguish some particular pairs of low dimensional Heintze groups up to quasi-isometry. Notice that part (i) implies via [Pan89] that the Carnot groups associated to N_1 and N_2 as their asymptotic cones are isomorphic. In particular, the nilpotency steps of N_1 and N_2 agree.

0.2. **On the classification up to isometries.** To motivate and give some background, let us compare the state of the art of the classification up to isometry and quasi-isometry for two distinct subclasses of the class of solvable simply connected Lie groups: Heintze groups and solvable groups of polynomial growth (with nilpotent groups as main examples). These subclasses have some similarities when it comes to isometries and quasi-isometries. In both cases every group has "a representative with real roots" and those representatives are known to be distinguished by isometries, and are conjectured to be distinguished by quasi-isometries. More precisely, we have the following facts and folklore conjectures:

Proposition H1 (Alekseevskiĭ [Ale75])**.** *Every Heintze group can be made isometric to a purely real Heintze group.*

Proposition H2 (Gordon–Wilson [GW88])**.** *If two purely real Heintze groups can be made isometric, then they are isomorphic.*

Conjecture H3. *If two purely real Heintze groups are quasi-isometric then they are isomorphic.*

Proposition P1 (Breuillard [Bre14])**.** *Every simply connected solvable Lie group of polynomial growth can be made isometric to a nilpotent group.*

Proposition P2 (Wolf [Wol63])**.** *If two simply connected nilpotent Lie groups can be made isometric, then they are isomorphic.*

Conjecture P3. *If two simply connected nilpotent Lie groups are quasi-isometric then they are isomorphic.*

Recently, the articles [CKL$^+$21] and [Jab19] clarified quite a bit this field, when it comes to isometries. Now we know that to every simply connected solvable Lie group it is possible to canonically associate a completely solvable (a.k.a. split-solvable or real triangulable) Lie group, so called "real-shadow" of the group, which is unique up to isomorphism. In particular, this construction satisfies the following theorem.

Fact 0.1 (Corollary 4.23 in [CKL$^+$21])**.** *Let G and H be simply connected solvable Lie groups. Then G can be made isometric to H if and only if the real-shadows of G and H are isomorphic.*

This result, besides containing the information of propositions H1-2 and P1-2 above, implies that "can be made isometric" is an equivalence relation within the class of simply connected solvable Lie groups. Moreover, it implies that the isometric classification of such groups boils down to the algebraic problem of calculating their real-shadows. Remark that it is not known if Fact 0.1 holds when isometries are replaced by quasi-isometries. This is because the more general version, due to Y. Cornulier [dC18, Conjecture 19.113], of Conjecture H3 and Conjecture P3 is also open: whether two quasi-isometric completely solvable simply connected Lie groups are necessarily isomorphic or not.

Since Lie groups that can be made isometric are necessarily quasi-isometric, we are led to study the following problem: Which pairs of groups in the same quasi-isometry class can be made isometric? This problem is completely solved for groups of dimension 3 and it is surveyed in [FLD21]. One of the main contributions of the present article is to push towards a solution for simply connected groups of dimension 4. While we are not able to completely solve the quasi-isometry relations of 4-dimensional groups, we can solve the isometry relations: it is clear that Fact 0.1 is enough for that. However, we will also prove Theorem B below, which is a more explicit result and can be proved with elementary methods. In its statement, we denote by $\alpha_0 = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{nil}}$ the *real part* of the derivation α: we shall recall the relevant decomposition more precisely in Proposition 1.12.

Theorem B. *Let H be a simply connected Lie group and α a derivation of H. Then the Lie group $H \rtimes_\alpha \mathbb{R}$ can be made isometric to the Lie group $H \rtimes_{\alpha_0} \mathbb{R}$, where α_0 is the real part of α.*

In the category of solvable groups, the above result is a special case of Fact 0.1, but it may also provide information about isometry questions of non-solvable semidirect products. Notice that there is no assumptions on the eigenvalues of α.

Theorem B has practical value also within the family of solvable Lie groups: In Section 4 we find all the pairs of Lie groups that can be made isometric within the family of 4-dimensional simply connected solvable Lie groups. In Section 5 we do the same within the family of 5-dimensional simply connected solvable Lie groups of polynomial growth. The method is described as follows. Since the algebraic classification of Lie groups is known within these families, we first indicate all the completely solvable ones: these are the groups that are isomorphic to their real-shadows. Then for each solvable group G that is not completely solvable, we find a completely solvable group to which it is isometric by finding a suitable decomposition of G as a semi-direct product $H \rtimes_\alpha \mathbb{R}$ where H is completely solvable. This happens to be always possible within the families we investigate. Now we know from Theorem B that such a group G can be made isometric to the completely solvable group $H \rtimes_{\alpha_0} \mathbb{R}$, while Fact 0.1 then guarantees that $H \rtimes_{\alpha_0} \mathbb{R}$ is the real-shadow of G, and any other solvable group G' that can be made isometric to G must also have $H \rtimes_{\alpha_0} \mathbb{R}$ as the real-shadow.

The result we get in dimension 4 is summarised in the theorem below.

Theorem C. *Let G and H be simply connected solvable Lie groups of dimension 4. If G and H are both completely solvable, then they can be made isometric if and only if they are isomorphic. Instead, if at least one of them is not completely solvable, then they can be made isometric if and only if they belong to the same set of groups in the following list (the notation is w.r.t. the classification given by* [PSWZ76]*):*

(I) $\{\mathbb{R}^4,\ \mathbb{R} \times A_{3,6}\}$,
(II) $\{\mathbb{R} \times A_{3,1},\ A_{4,10}\}$,
(III_λ) $\{A_{4,5}^{\lambda,\lambda}\} \cup \{A_{4,6}^{a,b} : \lambda = \operatorname{sign}(ab)\min(|b/a|, |a/b|)\}$,
(IV) $\{A_{4,9}^{1}\} \cup \{A_{4,11}^{a} : a \in \,]0, \infty[\,\}$,
(V) $\{\mathbb{R} \times A_{3,3},\ A_{4,12}\} \cup \{\mathbb{R} \times A_{3,7}^{a} : a \in \,]0, \infty[\,\}$,
(VI) $\{\mathbb{R}^2 \times A_2\} \cup \{A_{4,6}^{a,0} : a \in \mathbb{R}\}$

Here (III_λ) *stands for distinct sets depending on parameter $\lambda \in \mathbb{R}\backslash\{0\}$. Hence the above list contains 5 sets (2 finite and 3 infinite) and one family of sets depending on a parameter.*

In Section 5 we find similar classification for simply connected solvable groups of polynomial growth in dimension 5. Table 3 in Section 5 summarises the results within this family.

Acknowledgements. We thank Pierre Pansu and Matias Carrasco Piaggio for constructive comments. In particular, Pansu helped us to strengthen an early version of Theorem A.(i).

1. Preliminaries

1.1. **Homogeneous groups.** We shall approach the quasi-isometric classification problems in Heintze groups by studying the biLipschitz maps on their boundary, and we now define precisely the terminology of this setting.

In this paper, we will always use the convention that if $N, H, G, \ldots$ are Lie groups, then the fraktur letters $\mathfrak{n}, \mathfrak{h}, \mathfrak{g}, \ldots$ denote their Lie algebras, and vice versa.

Definition 1.1. A pair (N, α) is a *homogeneous group* if N is a simply connected nilpotent Lie group and α is a derivation of N so that for each eigenvalue λ of α it holds $\mathrm{Re}(\lambda) > 0$. Further, we say that a homogeneous group (N, α) is

- *purely real*, if the eigenvalues of α are real numbers,
- *of Carnot type* if it is purely real, if α is diagonalisable over $\mathbb{R}$, and if the eigenspace corresponding to the smallest of the eigenvalues Lie-generates $\mathfrak{n}$.

Two homogeneous groups (N_1, α) and (N_2, β) are *isomorphic (as homogeneous groups)* if there is an isomorphism of Lie groups $F\colon N_1 \to N_2$ so that $\beta \circ F_* = F_* \circ \alpha$, where F_* is the Lie algebra isomorphism induced by F.

The data defining homogeneous groups exactly coincide with the data defining Heintze groups. The terms *purely real Heintze group* and *Heintze group of Carnot type* appear in the literature and correspond to the terms above, see for example [CPS17] and [dC18].

For purposes of classifications up to isometry or quasi-isometry, only the purely real homogeneous groups play a role due to the result of [Ale75] presented here as Proposition H1 in the introduction. Hence we will always assume that the derivation has real eigenvalues even in the cases when it would be not strictly necessary.

Next we discuss homogeneous distances on homogeneous groups.

Definition 1.2. Let (N, α) be a homogeneous group. A distance function ρ on the set N is said to be *homogeneous (for* (N, α)*)*, if ρ is left-invariant, induces the manifold topology of N, and for all $\lambda > 0$ we have $\rho(\delta_\lambda x, \delta_\lambda y) = \lambda \rho(x, y)$ for all $x, y \in N$, where δ_λ is the automorphism of N with the differential $(\delta_\lambda)_* = \mathrm{e}^{\log(\lambda)\alpha}$. The triple (N, α, ρ) is called *a homogeneous metric group* if the distance function ρ is homogeneous for (N, α).

Remark 1.3. In [LDNG19, Theorem B] it is characterised when a purely real homogeneous group (N, α) admits a distance ρ making it a homogeneous metric group: denoting by ν the smallest eigenvalue of α, a distance exists if and only if $\nu \geq 1$ and the restriction of α to its generalised eigenspace of eigenvalue 1 is diagonalisable over $\mathbb{R}$. Consequently, if (N, α) is a homogeneous group, then for every $\lambda > 1/\nu$, the homogeneous group $(N, \lambda\alpha)$ admits a distance ρ making it a homogeneous metric group, and this may or may not be true for $\lambda = 1/\nu$.

Remark 1.4. Given a homogeneous group (N, α), all the distance functions that are homogeneous for (N, α) are biLipschitz equivalent via the identity map. More generally, it is straightforward to prove the following statement. Let ρ and ρ' be two distances metrising the same topological space M. Suppose there is a transitive group of homeomorphisms acting by isometries for both of the distances. Suppose there is $o \in M$ and a bijection $\delta\colon M \to M$, fixing the point o, and $\lambda \in]0, 1[$ with

$$\rho(\delta(x), \delta(y)) = \lambda \rho(x, y) \qquad \text{and} \qquad \rho'(\delta(x), \delta(y)) = \lambda \rho'(x, y)\,, \qquad \forall x, y \in M\,.$$

Then ρ and ρ' are biLipschitz equivalent via the identity map of M.

Due to Remark 1.4, when considering biLipschitz maps between two homogeneous groups, it is not necessary to specify the homogeneous distance functions, provided they exist, for which we refer to Remark 1.3. Whenever we assume that two homogeneous groups are biLipschitz equivalent we mean that on both of them some homogeneous distances exist for which the metric spaces are biLipschitz equivalent.

The following result summarises the known correspondence between the quasi-isometries of Heintze groups and the biLipschitz maps on their boundaries. For a good exposition and list of references, see [CPS17, p.6].

Proposition 1.5. *Let (N_1,α) and (N_2,β) be homogeneous groups. Then the Heintze groups $N_1 \rtimes_\alpha \mathbb{R}$ and $N_2 \rtimes_\beta \mathbb{R}$ are quasi-isometric if and only if there exists $\lambda_1,\lambda_2>0$ so that $(N_1,\lambda_1\alpha)$ and $(N_2,\lambda_2\beta)$ are biLipschitz equivalent.*

Proof. The two Heintze groups $N_1 \rtimes_\alpha \mathbb{R}$ and $N_2 \rtimes_\beta \mathbb{R}$ are quasi-isometric if and only if there are $\lambda_1,\lambda_2>0$ so that $(N_1,\lambda_1\alpha)$ and $(N_2,\lambda_2\beta)$ are quasisymmetric [Pau96, BS00, dC18]. The constants are needed to ensure the existence of homogeneous distances, rather than quasidistances, see Remark 1.3.

If $(N_1,\lambda_1\alpha)$ and $(N_2,\lambda_2\beta)$ are biLipschitz equivalent, then they are quasisymmetric. Vice versa, suppose that $(N_1,\lambda_1\alpha)$ and $(N_2,\lambda_2\beta)$ are quasisymmetric. Without changing their biLipschitz class, we may assume that α and β have only real eigenvalues, see [LDNG19, Theorem C]. Up to changing the constants, we may assume that the smallest of the eigenvalues of $\lambda_1\alpha$ and $\lambda_2\beta$ agree.

If $(N_1,\lambda_1\alpha)$ is of Carnot type, then by [Pan89, Pia17, LDX16] we have that $(N_1,\lambda_1\alpha)$ and $(N_2,\lambda_2\beta)$ are isomorphic as homogeneous groups and thus biLipschitz equivalent. If $(N_1,\lambda_1\alpha)$ is not of Carnot type, then the quasisymmetry from $(N_1,\lambda_1\alpha)$ to $(N_2,\lambda_2\beta)$ is a biLipschitz map, by [Pia17, LDX16]. □

Next, we translate to our language Lemma 5.1 of [CPS17].

Proposition 1.6 ([CPS17])**.** *Let (N,α,ρ) be a purely real homogeneous metric group, and let λ_1 be the smallest of the eigenvalues of α. Then the Hausdorff dimension of any non-constant curve on N is at least λ_1, and the curve $t\mapsto \exp(tX)$ has Hausdorff dimension λ_1 if X is an eigenvector of α with eigenvalue λ_1.*

The next result is also a consequence of the work of [CPS17]. It tells us that whenever one is able to prove that some subgroups are preserved in the sense that all their left cosets are preserved, then the normalisers of these subgroups provide new invariants.

Proposition 1.7 ([CPS17])**.** *Let $F\colon (N_1,\alpha)\to(N_2,\beta)$ be a biLipschitz map between homogeneous groups, and suppose A_1 and A_2 are subgroups of N_1 and N_2, respectively. Let $\mathcal{N}(A_i)$ be the normaliser of A_i, for $i\in\{1,2\}$. If for all $p\in N_1$ we have $F(pA_1)=F(p)A_2$, then it holds $F(p\mathcal{N}(A_1))=F(p)(\mathcal{N}(A_2))$ for all $p\in N_1$.*

Proof. Fix $p,q\in N_1$. Then the following are equivalent statements

(i) $q\in p\mathcal{N}(A_1)$.
(ii) Hausdorff distance of qA_1 and pA_1 is finite.
(iii) Hausdorff distance of $F(qA_1)=F(q)A_2$ and $F(pA_1)=F(p)A_2$ is finite.
(iv) $F(q)\in F(p)\mathcal{N}(A_2)$.

Indeed, the equivalences (i)⇔(ii) and (iii)⇔(iv) are given by [CPS17, Lemma 3.2]. The equivalence (ii)⇔(iii) is a consequence of F being a biLipschitz map. □

We will need to understand quotients of homogeneous groups: for that the important lemma is the following straightforward consequence of the ideas of [LDR16] (see their results 2.8 and 2.10 in particular).

Lemma 1.8. *Suppose H is a normal subgroup of a homogeneous group (N,α). If $\mathfrak{h}$ is preserved under α, then the quotient N/H is a homogeneous group when equipped with the induced derivation $\hat{\alpha}$. Moreover, if ρ is a homogeneous distance on (N,α), then $\hat{\rho}$ given by*

$$\hat{\rho}(nH, mH) = \inf\{\rho(n, mh) : h \in H\}$$

is a homogeneous distance on $(N/H, \hat{\alpha})$ for which the projection $N \to N/H$ is a 1-Lipschitz map.

1.2. **Isometries of not necessarily solvable groups.** Above we said that two connected Lie groups G and H *can be made isometric* if there are left-invariant Riemannian metrics ρ_G and ρ_H so that (G,ρ_G) is isometric to (H,ρ_H). By [KLD17, Proposition 2.4] requiring the distances ρ_G and ρ_H to be Riemannian is not restrictive: We could suppose only that the distances are left-invariant and induce the respective manifold topologies. In any case, while quasi-isometries give a transitive relation between Lie groups, the relation by isometries is not transitive; we next wish to show an instructive example.

Proposition 1.9. *Both the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ can be made isometric to the group $\mathbb{S}^1 \times \mathrm{Aff}(\mathbb{R})^+$, but the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ cannot be made isometric (to each other).*

The argument for the fact that the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ cannot be made isometric is readily recorded in [FLD21, Proposition 2.11], but it goes back to Cornulier, and eventually to [Gor81, Theorem 2.2]. The first part of Proposition 1.9 may be deduced from [CKL+21, Theorem 3.24], but in this particular example the argument of [CKL+21] simplifies so much that we feel it is worth giving the following elementary proof.

Proof of the first part of Proposition 1.9. Let G denote either $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{R})$, and let d be a left-invariant admissible distance on G. In either case, G has the Iwasawa decomposition $G = ANK$, where the factor AN forms a subgroup isomorphic to $\mathrm{Aff}(\mathbb{R})^+$ and K is instead isomorphic to $\mathbb{S}^1$. Our aim is to construct from d a new metric d' in G and find a metric d'' on $AN \times K$ so that (G, d') is isometric to $(AN \times K, d'')$.

We define, taking the advantage of the compactness of K, a new distance function on G by the formula

$$d'(g,h) = \sup_{k\in K} d(gk, hk)\,.$$

It is trivial that d' satisfies the axioms of a distance function and that it is left-invariant. One may also see by a straightforward argument that any open d'-ball contains an open d-ball. Consequently, as $d'(g,h) \geq d(g,h)$ for all $g,h \in G$, then the distance d' induces the same topology as d.

We define d'' to be the pull-back distance on $AN \times K$ via the homeomorphism $\omega\colon (AN \times K) \to ANK$ given by $\omega(s,k) = sk^{-1}$. The resulting distance is left-invariant since for any fixed $k, k_1, k_2 \in K$ and $s, s_1, s_2 \in AN$ we have

$$\begin{aligned} d''((s,k)(s_1,k_1),(s,k)(s_2,k_2)) &= d'(ss_1(kk_1)^{-1}, ss_2, (kk_2)^{-1}) \\ &= \sup_{k'\in K} d(ss_1k_1^{-1}k^{-1}k', ss_2, k_2^{-1}k^{-1}k') \\ &= \sup_{k''\in K} d(ss_1k_1^{-1}k'', ss_2, k_2^{-1}k'') \\ &= \sup_{k''\in K} d(s_1k_1^{-1}k'', s_2, k_2^{-1}k'') \\ &= d'(s_1k_1^{-1}, s_2, k_2^{-1}) = d''((s_1,k_1),(s_2,k_2)) \, . \end{aligned}$$

In conclusion, the map ω is an isometry between (G, d') and $(AN \times K, d'')$. □

The proof of the following fact is just slightly more involved, and the details are recorded in [CKL+21, Theorem 3.24]. The main difference is that one does not have a compact factor K in the Iwasawa decomposition, but instead there is a non-compact central group involved.

Proposition 1.10. *The universal cover of the group* $\mathrm{SL}(2,\mathbb{R})$ *can be made isometric to the group* $\mathbb{R} \times \mathrm{Aff}(\mathbb{R})^+$.

Even if the transitivity of isometry-relation is shown by Proposition 1.9 to be false in general, we are not aware of counterexamples in the class of simply connected Lie groups. Moreover, Fact 0.1 implies the transitivity among simply connected solvable Lie groups. Despite Fact 0.1 some questions remain unanswered, like the following.

Question 1.11. Is there a non-solvable simply connected group G and two solvable groups S_1, S_2, so that both S_1 and S_2 can be made isometric to G (with different metrics) and S_1 and S_2 have different real-shadow, i.e., they cannot be made isometric?

1.3. **Algebraic tools for isometries.** The aim of this section is to recall the results related to the real-shadow of a simply connected solvable Lie group, so that after proving Theorem B in Section 3, we are able to link it to Fact 0.1 and real-shadows. We will make the link explicit in Corollary 3.1.

We start by recalling a decomposition result which is necessary both for the construction of the real-shadow and also for the statement of Theorem B. The ingredients of its proof are recorded in [LDNG19, Section 2] while it might be considered well known.

Proposition 1.12. *Let α be a derivation on a Lie algebra $\mathfrak{g}$. Then there are derivations α_{sr}, α_{si} and α_{nil} on $\mathfrak{g}$ satisfying the following properties:*

(i) The maps α, α_{sr}, α_{si} and α_{nil} all pairwise commute.
(ii) $\alpha = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{si}} + \alpha_{\mathrm{nil}}$.
(iii) The map α_{nil} is the nilpotent part of α.
(iv) The maps α_{sr} and α_{si} are semisimple.
(v) The spectrum of α_{sr} is real, and the spectrum of α_{si} is purely imaginary.

If $\alpha = \mathrm{ad}_X$ for a vector X of a Lie algebra, we denote $\mathrm{ad}_{\mathrm{s}}(X) = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{si}}$ and $\mathrm{ad}_{\mathrm{si}}(X) = \alpha_{\mathrm{si}}$; In the latter, "si" stands for semisimple and imaginary.

We recall some standard terminology: A Lie algebra $\mathfrak{g}$ is said to be *of type (R)* if all the eigenvalues of ad_X are purely imaginary for all $X \in \mathfrak{g}$. Instead, a Lie algebra is said to be *completely solvable*, if it is solvable and all these eigenvalues are real. The Lie algebra of a simply connected Lie group G is of type (R) if and only if G has polynomial growth [Jen73, Theorem 1.4], i.e., the Haar measure of the powers of neighbourhoods of identity grows bounded by a polynomial.

We recall here, using a slightly different viewpoint, the method of [CKL+21] to determine the real-shadow of a simply connected solvable Lie group. The arguments may be found inside the proof of Theorem 4.16 in [CKL+21].

Lemma 1.13. *Let $\mathfrak{g}$ be a solvable Lie algebra with nilradical $\mathfrak{n}$. Then there is a vector subspace $\mathfrak{a} \subseteq \mathfrak{g}$ so that*

(i) $\mathfrak{n} \oplus \mathfrak{a} = \mathfrak{g}$*,*
(ii) $\mathrm{ad}_{\mathrm{s}}(X)(Y) = 0$ *for all* $X, Y \in \mathfrak{a}$*, and*
(iii) $[\mathrm{ad}_{\mathrm{s}}(X), \mathrm{ad}_{\mathrm{s}}(Y)] = 0$ *for all* $X, Y \in \mathfrak{a}$.

Such a subspace $\mathfrak{a}$ is found by noticing that there is a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$; then $\mathfrak{a}$ may be chosen inside $\mathfrak{c}$ to complement $\mathfrak{n}$.

The following statement gives naturally a very constructive definition of the real-shadow in the level of Lie algebras.

Proposition 1.14. *Let $\mathfrak{g}$ be a solvable Lie algebra. Choose a vector subspace $\mathfrak{a} \subseteq \mathfrak{g}$ with the properties of Lemma 1.13 and let $\pi_{\mathfrak{a}}$ denote the projection to $\mathfrak{a}$ along $\mathfrak{n}$. Define a map*

$$\varphi_{\mathfrak{a}} \colon \mathfrak{g} \to \mathrm{der}(\mathfrak{g}) \qquad \varphi_{\mathfrak{a}}(X) = -\mathrm{ad}_{\mathrm{si}}(\pi_{\mathfrak{a}}(X))$$

Then

(i) $\varphi_{\mathfrak{a}}$ *is a homomorphism of Lie algebras, with Abelian image,*
(ii) the graph of $\varphi_{\mathfrak{a}}$*,* $\mathrm{Gr}(\varphi_{\mathfrak{a}}) = \{(X, \varphi_{\mathfrak{a}}(X)) \mid X \in \mathfrak{g}\}$*, is a completely solvable subalgebra of* $\mathfrak{g} \rtimes \mathrm{der}(\mathfrak{g})$*,*
(iii) if the vector space $\mathfrak{g}$ *is equipped with the operation defined by*

$$[X, Y]_{\mathbb{R}} = [X, Y] + \varphi_{\mathfrak{a}}(X)(Y) - \varphi_{\mathfrak{a}}(Y)(X)$$

then the map $X \mapsto (X, \varphi_{\mathfrak{a}}(X))$ *is a Lie algebra isomorphism from* $(\mathfrak{g}, [\cdot,\cdot]_{\mathbb{R}})$ *to* $\mathrm{Gr}(\varphi_{\mathfrak{a}})$.

Moreover, for every vector subspace $\mathfrak{a}' \subset \mathfrak{g}$ as in Lemma 1.13 we have that $\mathrm{Gr}(\varphi_{\mathfrak{a}})$ is isomorphic to $\mathrm{Gr}(\varphi_{\mathfrak{a}'})$.

Definition 1.15. Let $\mathfrak{g}$ be a solvable Lie algebra. Its *real-shadow* is the Lie algebra $\mathrm{Gr}(\varphi_{\mathfrak{a}})$ constructed as in Proposition 1.14.

The main result of [CKL+21] regarding this construction is that Fact 0.1 indeed holds for such a construction.

Remark 1.16. In many applications of low dimension, there is an Abelian subalgebra $\mathfrak{a}$ complementary to the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. Then such $\mathfrak{a}$ trivially satisfies Lemma 1.13 and can be used to construct the real-shadow. Another remark is that if $\mathfrak{g}$ is of type (R), then $\mathrm{ad}_{\mathrm{si}}(X) = \mathrm{ad}_{\mathrm{s}}(X)$ for any $X \in \mathfrak{g}$, and consequently, the real-shadow of a Lie algebra of type (R) is its nilshadow as defined in [DtER03].

1.4. **Algebraic tools for quasi-isometries.** When considering the class of simply connected solvable Lie groups, the algebraic tools relevant for our study of groups of dimension 4 and 5 up to quasi-isometry are the following invariants:

(Inv-1) Carnot groups are quasi-isometrically distinct among themselves by Pansu's Theorem [Pan89]. More generally, [Pan89] implies that if two simply connected nilpotent Lie groups are quasi-isometric, their associated Carnot groups are isomorphic.
(Inv-2) For nilpotent groups, the Betti numbers (by [Sha04]) and more generally the Lie algebra cohomology rings (by [Sau06]) are quasi-isometry invariants.
(Inv-3) For the groups of polynomial growth, their degree of growth is quasi-isometry invariant. It is because this degree is the Hausdorff dimension of the asymptotic cone.
(Inv-4) Two simply connected solvable groups are quasi-isometric if and only if their real-shadows are quasi-isometric. This is because these groups are quasi-isometric to their real-shadows by Fact 0.1.
(Inv-5) Topological dimension of the asymptotic cone, called *cone dimension*, is a quasi-isometry invariant.

As nilshadows were already treated above, we turn the attention here to the cone dimension. Cornulier proved in [dC08] that the cone dimension of a simply connected solvable Lie group agrees with the dimension of the exponential radical of the group. We turn this result into the following observation.

Proposition 1.17. *Let G be a simply connected completely solvable Lie group with Lie algebra $\mathfrak{g}$. Then the cone dimension of G equals the codimension of the subspace $\bigcap_{n\geq 1}\mathfrak{g}^n$ of $\mathfrak{g}$, where the subspace $\mathfrak{g}^n$ denotes the nth term in the lower central series of $\mathfrak{g}$.*

Proof. By [Osi02] (see also [dC08, Theorem 6.1]), the exponential radical R of G is a closed connected normal subgroup of G, the quotient group G/R has polynomial growth, and there is no closed connected normal subgroup R' so that the quotient G/R' would be of polynomial growth and have strictly larger dimension.

By [dC08, Theorem 1.1], the cone dimension of G, denoted by $\mathrm{conedim}(G)$, equals to the codimension of the exponential radical of G. By the above

$$\mathrm{conedim}(G) = \max\{\dim(\mathfrak{g}/\mathfrak{r}) : \mathfrak{r} \text{ ideal of } \mathfrak{g} \text{ and } \mathfrak{g}/\mathfrak{r} \text{ is of type (R)}\}$$

Moreover, since G is completely solvable, a quotient of $\mathfrak{g}$ is of type (R) if and only if it is nilpotent.

The terms of the lower central series are nested vector subspaces of $\mathfrak{g}$, and the condition $\mathfrak{g}^n = \mathfrak{g}^{n+1}$ for some n implies that $\mathfrak{g}^n = \mathfrak{g}^k$ for all $k \geq n$. Thus there is

$N \in \mathbb{N}$ so that $\bigcap_{n\geq 1} \mathfrak{g}^n = \mathfrak{g}^N$. The quotient $\mathfrak{g}/\mathfrak{g}^N$ is nilpotent, and we will show its dimension is maximal. Let $\mathfrak{q}$ be an ideal of $\mathfrak{g}$ so that $\mathfrak{g}/\mathfrak{q}$ is nilpotent of step s. It is enough to show $\mathfrak{g}^N \subset \mathfrak{q}$. Assuming the contrary, we have a non-zero vector $X \in \mathfrak{g}^N \backslash \mathfrak{q}$. Because $\mathfrak{g}^N = \mathfrak{g}^k$ for all $k \geq N$, we may express X as a bracket of arbitrary length. More precisely X may be expressed as a linear combination of a terms of the form $\mathrm{ad}_{X_1} \circ \cdots \circ \mathrm{ad}_{X_s}(X_{s+1})$ for some $X_i \in \mathfrak{g}$. It holds $X_i \notin \mathfrak{q}$ since $\mathfrak{q}$ is an ideal. Hence when X is considered as a non-zero element of the quotient $\mathfrak{g}/\mathfrak{q}$, it can be expressed as a bracket of length $s+1$, which contradicts the nilpotency step of the quotient. □

The above result implies that the cone dimension can be algorithmically calculated at the Lie algebra level.

2. On biLipschitz maps of homogeneous groups

It is conjectured that if two purely real Heintze groups are quasi-isometric, then they are isomorphic. Many quasi-isometry invariants are known, but still there are non-isomorphic pairs of purely real Heintze groups that are not distinguished by those invariants. In this section we present new quasi-isometry invariants for purely real Heintze groups: we prove Theorem A. Our analysis is based on the fact that two purely real Heintze groups are quasi-isometric if and only if their parabolic boundaries are biLipschitz equivalent, see Proposition 1.5.

In Section 2.1 we prove Theorem A.(i) stating that biLipschitz equivalent homogenous groups are quasi-isometric when equipped with Riemannian distances. One important consequence, see also Theorem 6.4 in appendix, is that the family of purely real Heintze groups with Abelian nilradical is closed under quasi-isometries among the family of purely real Heintze groups. In addition, the quasi-isometry relations within the family of purely real Heintze groups with Abelian nilradical are completely understood by the results of Xie [Xie14].

In Section 2.2, we prove that on the level of the boundary, the set of points reachable by curves of a given Hausdorff dimension can be algebraically computed and hence used as an invariant. This leads to Theorem A.(ii). Such a result will enable us to distinguish up to quasi-isometry some examples of low dimension that we discuss in Section 2.3.

We recall that in this paper, we will always use the convention that if $N, H, G, \ldots$ are Lie groups, then the fraktur letters $\mathfrak{n}, \mathfrak{h}, \mathfrak{g}, \ldots$ denote their Lie algebras, and vice versa.

2.1. Homogeneous biLipschitz implies Riemannian quasi-isometric.

In this section we prove Theorem A.(i). We follow a suggestion of Pansu for treating arbitrary homogeneous groups. However, in Section 6, we consider the case where one of the groups is Abelian. It is a less general setting, but the proof is direct and might be of independent interest.

Proof of Theorem A.(i). Given a metric space (M,d) and $\ell > 0$, we recall from [CKL+21] the definition of derived semi-intrinsic metric with parameter ℓ as

$$d_{[\ell]}(p,q) = \inf\Big\{\sum_{j=1}^{k} d(x_j,x_{j-1}) \mid x_0,\dots,x_k \in M,\ x_0 = p,\ x_k = q,\ d(x_j,x_{j-1}) \le \ell\Big\}\,.$$

It follows immediately from the definition, that if a map $f\colon (M,d) \to (M',d')$ is an L-Lipschitz-map with $L \ge 1$, then $d'_{[\ell]}(f(p),f(q)) \le L d_{[\ell/L]}(p,q)$. By [CKL+21, Lemma 2.3], for a homogeneous metric group (N,d) and $\ell > 0$, the function $d_{[\ell]}$ is a proper quasi-geodesic distance function inducing the topology of N. Thus, $d_{[\ell]}$ is quasi-isometric to any left-invariant Riemannian distance on N.

We conclude that if $F\colon N_1 \to N_2$ is an L-biLipschitz map between homogeneous metric groups (N_1,α,d^α) and (N_2,β,d^β), then the derived semi-intrinsic metrics satisfy the following inequalities:

$$\frac{1}{L} d^\alpha_{[L\ell]}(x,y) \le d^\beta_{[\ell]}(F(x),F(y)) \le L d^\alpha_{[\ell/L]}(x,y)\,.$$

Therefore, if D_1 and D_2 are left-invariant Riemannian distances, then the map $F\colon (N_1,D_1) \to (N_2,D_2)$ is a quasi-isometry. □

2.2. **Reachability sets.** Considering homogeneous groups up to biLipschitz equivalence, one obvious invariant is the set of those points that can be reached by curves starting from the identity element and having Hausdorff dimension at most s, for some fixed $s \ge 1$ (notice that curves have Hausdorff dimension at least 1). When (N,α) is a homogeneous group, we denote

$$R(s) = \{\gamma(1) \mid \gamma \in \mathcal{C}^0([0,1],N),\ \gamma(0) = 1_N,\ \mathcal{H}\text{-dim}(\gamma([0,1])) \le s\}$$

As one might expect and as we shall now prove, such a set may be computed as the subgroup $(N,\alpha)^{(s)} < N$ corresponding to the subalgebra $\mathrm{LieSpan}(\bigoplus_{0<\lambda\le s} V_\lambda)$. Here $\mathfrak{n} = \bigoplus_{\lambda>0} V_\lambda$ is the decomposition of the Lie algebra by the generalised eigenspaces of the derivation α. The fact that $R^\alpha(s) = (N,\alpha)^{(s)}$ makes this set into a practically usable invariant.

Theorem 2.1. *Let (N,α) be a purely real homogeneous group. Then $R(s) = (N,\alpha)^{(s)}$ for every $s \ge 1$.*

Proof. Fix $s \ge 1$. Using the Orbit Theorem, one may show (see [BL19, Proposition 2.26]) the following. Suppose W is subset of a Lie algebra $\mathfrak{g}$ so that W is invariant under scalar multiplication, i.e., $\mathbb{R}W = W$, and so that no proper subalgebra of $\mathfrak{g}$ contains W. Then $\bigcup_{k=1}^\infty (\exp(W))^k$ has non-empty interior in G, and since it is also a subgroup it holds $\bigcup_{k=1}^\infty (\exp(W))^k = G$. Applying this observation to $W = \bigcup_{0<\lambda\le s} V_\lambda$ and $G = (N,\alpha)^{(s)}$, we get that every element of $(N,\alpha)^{(s)}$ is a finite product of exponentials of vectors $X \in \bigcup_{0<\lambda\le s} V_\lambda$. Thus, to show that $R(s) \supset (N,\alpha)^{(s)}$, we only need to see that the flow lines $t \mapsto \exp(tX)$ have Hausdorff dimension at most s, for $X \in \bigcup_{0<\lambda\le s} V_\lambda$. By [CPS17, Lemma 5.1], we may assume that X is an eigenvector of α with eigenvalue $\lambda \le s$. Fix a homogeneous distance ρ, and set $L = \exp(\mathbb{R}X)$. Identifying L with $\mathbb{R}$, we get a distance to $\mathbb{R}$ that is homogeneous under the family of dilations induced by α. Hence by Remark 1.4, (L,ρ) is biLipschitz equivalent to $(\mathbb{R}, \|\cdot\|^{1/\lambda})$ and hence it has Hausdorff dimension λ.

To prove that $R(s) \subset (N,\alpha)^{(s)}$, denote $H_0 = (N,\alpha)^{(s)}$ and let then recursively H_k denote the normaliser of H_{k-1}. Consider the finite chain of subgroups $(N,\alpha)^{(s)} = H_0 < H_1 < \cdots < H_m = N$, where $m \geq 1$ is the first integer so that the repeated normaliser is the full space. Since nilpotent Lie algebras don't have non-trivial self-normalising subalgebras, such m exists. Fix a continuous curve $\gamma \colon [0,1] \to N$ with $\mathcal{H}$-dim$(\gamma([0,1])) \leq s$ and $\gamma(0) = 1_N$. We shall prove inductively that γ does not leave H_k for any $0 \leq k \leq m$.

The case $k = m$ of the induction is trivial. So we assume γ does not leave H_k for some $k \leq m$. Since H_{k-1} is normal in H_k, we may consider the quotient H_k/H_{k-1}. Observe that if a derivation α preserves a subalgebra $\mathfrak{q} < \mathfrak{n}$, then α necessarily preserves the normaliser of $\mathfrak{q}$. Therefore, since α preserves $(N,\alpha)^{(s)}$, then, by induction and Lemma 1.8, the quotient H_k/H_{k-1} is a homogeneous group. Moreover, the curve γ projects to the curve $\pi \circ \gamma$ of H_k/H_{k-1}, and Lemma 1.8 guarantees that the Hausdorff dimension of $\pi(\gamma([0,1]))$ is at most the Hausdorff dimension of $\gamma([0,1])$, so at most s.

Next, remark that all the generalised eigenspaces of α corresponding to eigenvalues less or equal to s are contained in the Lie algebra of $(N,\alpha)^{(s)}$, thus they are contained in H_{k-1}. This shows that all the eigenvalues of the derivation induced to H_k/H_{k-1} are strictly larger than s. Therefore, by Proposition 1.6, either $\pi \circ \gamma$ is constant or the Hausdorff dimension of $\pi(\gamma([0,1]))$ is strictly larger than s. Since the second case is ruled out, the curve $\pi \circ \gamma$ must be constant, i.e., $\gamma([0,1]) \subset H_{k-1}$ as the induction requires. We conclude that γ does not leave $H_0 = (N,\alpha)^{(s)}$ and hence $R(s) \subset (N,\alpha)^{(s)}$. ☐

Proof of Theorem A.(ii). As the set $R(s)$ is metrically defined, we get Theorem A.(ii) as immediate corollary of Theorem 2.1 when applying also Proposition 1.7. ☐

2.3. **Examples.** In this section we present some examples of pairs of Heintze groups trying to distinguish them up to quasi-isometry using the results that we proved.

- Ex 2.2 This is a pair of 7-dimensional Heintze groups with identical nilradical and derivations with identical diagonal form. Theorem A.(ii) distinguishes them.
- Ex 2.3 This is a pair of 5-dimensional Heintze groups with identical nilradical. This pair cannot be distinguished even with the new invariants we presented.
- Ex 2.4 This is a pair of 7-dimensional Heintze groups with different nilradical, but identical diagonal derivation. Theorem A.(i) distinguishes them.
- Ex 2.5 This is a pair of 10-dimensional Heintze groups with identical nilradical and derivations with identical diagonal form. Here the reachability sets don't distinguish the pair directly, but the normalisers can be used to distinguish them.
- Ex 2.6 This is a pair of 7-dimensional Heintze groups with different nilradical, but identical diagonal derivation. This pair cannot be distinguished even with the new invariants we presented.

In the next examples, we use the notation Heis for the standard Heisenberg group and Heis(5) for the 5-dimensional Heisenberg group. These are indexed by $A_{3,1}$ and $A_{5,4}$, respectively, in [PSWZ76], see also Section 4 and Section 5 later.

Example 2.2. Consider the Lie group $N = \mathrm{Heis} \times \mathbb{R}^3$ and two derivations on it

$$\alpha = \mathrm{diag}(1, 2, 3, 4, 5, 9) \qquad \text{and} \qquad \beta = \mathrm{diag}(4, 5, 9, 1, 2, 3)$$

Then the Heintze groups $N \rtimes_\alpha \mathbb{R}$ and $N \rtimes_\beta \mathbb{R}$ are not isomorphic: if they were, then α should be conjugate to β by an automorphism of $\mathfrak{n}$ (see for example [HKMT20, Proposition 4.7]). However, there is a unique linear endomorphism of $\mathfrak{n}$ that conjugates α to β and it is not an automorphism.

The invariant $R(2)$ distinguishes these homogeneous groups (N, α) and (N, β) by Theorem 2.1, as these sets have topological dimension 3 for (N, α) and 2 for (N, β).

Example 2.3. Consider the 4-dimensional Lie algebra $\mathrm{Heis} \times \mathbb{R}$ given by a basis $X_1, \dots, X_4$ with the only non-trivial bracket being $[X_1, X_2] = X_3$. Consider, for every parameter $a > 1$, the two linear maps given by matrices

$$\alpha = \begin{bmatrix} a-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} a-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

Both these maps are derivations of $\mathrm{Heis} \times \mathbb{R}$ with strictly positive eigenvalues, hence they define two 5-dimensional Heintze groups $(\mathrm{Heis} \times \mathbb{R}) \rtimes_\alpha \mathbb{R}$ and $(\mathrm{Heis} \times \mathbb{R}) \rtimes_\beta \mathbb{R}$. These Heintze groups are non-isomorphic, as they are the groups $A^a_{5,20}$ and $A^{a,a}_{5,19}$ in the classification [PSWZ76]. We are not aware of any method of distinguishing these Heintze groups up to quasi-isometry. We remark that while the Jordan-forms of the derivations are different, the Jordan form is not proven to be invariant in this generality. We also remark that these groups are sublinearly biLipschitz equivalent, by a result of Cornulier [dC11, Theorem 1.2], see also [Pal20, Theorem 3.2].

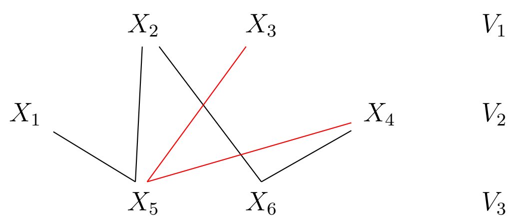

FIGURE 1. The graph representing schematically the bracket relations and positive gradings $V_1 \oplus V_2 \oplus V_3$ of Example 2.4.

Example 2.4. Consider the 6-dimensional vector space with a basis $X_1, \dots, X_6$ with two different structures of Lie algebra: Let $\mathfrak{n}_1$ be the Lie algebra given by the non-trivial bracket relations

$$[X_1, X_2] = X_5 \qquad [X_2, X_4] = X_6$$

This is denoted by $L_{6,8} = L_{5,8} \times \mathbb{R}$ in the classification [dG07]. Let $\mathfrak{n}_2$ be instead given by

$$[X_1, X_2] = X_5 \qquad [X_2, X_4] = X_6 \qquad [X_3, X_4] = X_5$$

This is denoted by $L_{6,22}(0)$ in the classification [dG07].

The linear map $\alpha = \mathrm{diag}(2, 1, 1, 2, 3, 3)$ in this basis is a derivation for both of these Lie algebra structures. For a schematic presentation, see Figure 1.

The homogeneous groups (N_1, α) and (N_2, α) cannot be biLipschitz-distinguished by Theorem A.(ii) but they can by Theorem A.(i) since the Lie algebras in question are stratifiable (even though homogeneous structures given are not of Carnot type).

Example 2.5. Consider the 10-dimensional Lie algebra $N = \mathrm{Heis}(5) \times \mathrm{Heis}(5)$ expressed as a vector space spanned by $X_1, \dots, X_{10}$ with the non-trivial bracket relations

$$[X_1, X_2] = X_5 \qquad [X_3, X_4] = X_5 \qquad [X_6, X_7] = X_{10} \qquad [X_8, X_9] = X_{10}$$

Consider the two derivations given by matrices

$$\alpha = \mathrm{diag}(1, 7, 3, 5, 8, 2, 6, 4, 4, 8) \qquad \text{and} \qquad \beta = \mathrm{diag}(1, 7, 4, 4, 8, 2, 6, 3, 5, 8)$$

The resulting Heintze groups $G_\alpha = N \rtimes_\alpha \mathbb{R}$ and $G_\beta = N \rtimes_\beta \mathbb{R}$ are not isomorphic: If they were, α and β should be conjugate by an automorphism of N (again, [HKMT20, Proposition 4.7]), but this is impossible. Indeed, the conjugating automorphism is forced to map $X_3 \mapsto X_8$ and $X_4 \mapsto X_9$ while in the same time keeping the basis-vectors X_1 and X_2 fixed, which is not conceivable.

Distinguishing these spaces is a bit more involved and demonstrates the combined power of Theorem 2.1 and Lemma 1.7. We cannot distinguish them directly via the reachability sets of prescribed Hausdorff dimension. However, the following works.

Suppose $F\colon (N, \alpha) \to (N, \beta)$ is a biLipschitz map. Then F must map $(N, \alpha)^{(6)}$ to $(N, \beta)^{(6)}$. By Theorem 2.1, $(N, \alpha)^{(6)}$ and $(N, \beta)^{(6)}$ both agree with the subgroup spanned by all the other basis vectors except X_2. This subgroup is again a homogeneous group, and it is Lie isomorphic to $\mathbb{R} \times \mathrm{Heis} \times \mathrm{Heis}(5)$. The original biLipschitz map induces a biLipschitz map of this subgroup equipped with the two different homogeneous structures (the derivations), call these groups (N_0, α_0) and (N_0, β_0). For these two homogeneous groups, consider now the subgroups

$$(N_0, \alpha_0)^{(4)} = \langle X_1, X_3, X_6, X_8, X_9, X_{10} \rangle \quad \text{and} \quad (N_0, \beta_0)^{(4)} = \langle X_1, X_3, X_4, X_5, X_6, X_8 \rangle$$

These are both isomorphic to the group $\mathbb{R}^3 \times \mathrm{Heis}$, so we did not yet distinguish the groups. However, the normalisers of these subgroups inside (N_0, α_0) and (N_0, β_0) are preserved by Lemma 1.7. These normalisers are

$$\mathcal{N}((N_0, \alpha_0)^{(4)}) = (N_0, \alpha_0)^{(4)} \oplus \langle X_5, X_7 \rangle \quad \text{and} \quad \mathcal{N}((N_0, \beta_0)^{(4)}) = (N_0, \beta_0)^{(4)} \oplus \langle X_{10} \rangle$$

which have different topological dimension and this prevents the existence of a biLipschitz map.

Example 2.6. Consider the 6-dimensional vector space with a basis $X_1, \dots, X_6$ with two different structures of Lie algebra: Let $\mathfrak{n}_1$ be the Lie algebra given by the non-trivial bracket relations

$$[X_1, X_2] = X_3 \qquad [X_1, X_3] = X_5 \qquad [X_1, X_4] = X_6 \qquad [X_2, X_4] = X_5\,.$$

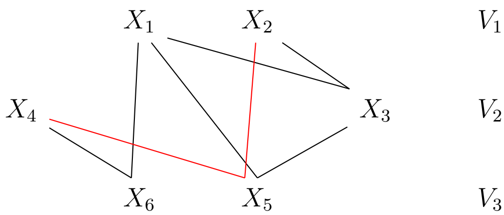

FIGURE 2. The graph representing schematically the bracket relations and positive gradings $V_1 \oplus V_2 \oplus V_3$ of Example 2.6.

This is denoted by $L_{6,23}$ in the classification [dG07]. Let $\mathfrak{n}_2$ be instead given by only the first three from above, i.e.,

$$[X_1, X_2] = X_3 \qquad [X_1, X_3] = X_5 \qquad [X_1, X_4] = X_6 .$$

This is denoted by $L_{6,25}$ in the classification [dG07].

The linear map $\alpha = \mathrm{diag}(1, 1, 2, 2, 3, 3)$ in this basis is a derivation for both of these Lie algebra structures. For a schematic presentation, see Figure 2.

The homogeneous groups (N_1, α) and (N_2, α) cannot be biLipschitz-distinguished by any method we know: The Lie algebra $L_{6,25}$ is the associated Carnot algebra of $L_{6,23}$ and the simply connected nilpotent Lie groups associated are not distinguished by the known quasi-isometric invariants, see [dC18, p. 339]. This rules out the usage of Theorem A.(i).

The only non-trivial reachability set is the reachability set for Hausdorff dimension 1, and it is the same subspace $\langle X_1, X_2, X_3, X_5 \rangle$ for both. Its normaliser contains in addition X_6 in both cases, and not X_4, the next repeated normaliser being the full space. This rules out the usage of Theorem A.(ii).

3. On isometries of semi-direct products

In this section we focus in proving Theorem B. We restate it for the reader's convenience in a longer form.

Theorem (Theorem B)**.** *Let H be a simply connected Lie group and α a derivation of H. Let $\alpha = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{si}} + \alpha_{\mathrm{nil}}$ be the decomposition to real, imaginary and nilpotent parts as in Proposition 1.12, and denote $\alpha_0 = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{nil}}$ Then the Lie group $H \rtimes_\alpha \mathbb{R}$ can be made isometric to the Lie group $H \rtimes_{\alpha_0} \mathbb{R}$.*

While Theorem B may be applied outside the family of solvable groups, also within the family of solvable groups sometimes it might be practical to directly apply Theorem B to find isometries between two solvable groups when neither of them is completely solvable. Indeed, we remark that $H \rtimes_{\alpha_0} \mathbb{R}$ does not need to be completely solvable when H is not completely solvable. This approach would avoid the work to find their common real-shadow as in Fact 0.1.

Proof of Theorem B. The groups $H \rtimes_\alpha \mathbb{R}$ and $H \rtimes_{\alpha_0} \mathbb{R}$ may be seen as acting by left-translations on the manifold $H \times \mathbb{R}$. Hence, the statement is proven by constructing a Riemannian metric on the manifold $H \times \mathbb{R}$ for which both these actions are by isometries. Denoting by 1 the element $(1_H, 0) \in H \times \mathbb{R}$, it is enough to construct a scalar product ρ on the tangent space $T_1(H \times \mathbb{R})$ with the following property (J)

(J) whenever two elements $g_1 \in H \rtimes_\alpha \mathbb{R}$ and $g_2 \in H \rtimes_{\alpha_0} \mathbb{R}$ satisfy $g_1(1) = g_2(1)$, then the differential of $g_1^{-1} \circ g_2$ is an isometry of the scalar product ρ

If ρ satisfies the property (J), then it can be transported to a Riemannian metric on $H \times \mathbb{R}$ with the desired properties.

For a derivation B on the Lie algebra of H, the map η_t^B will denote the automorphism with the differential e^{tB} for $t \in \mathbb{R}$. This automorphism is well defined and unique since H is assumed to be simply connected, and we have $\eta_t^B \circ \eta_s^B = \eta_{t+s}^B$. Remark that, since α_{si} is a semisimple map with purely imaginary eigenvalues, the subgroup $W = \{\eta_t^{\alpha_{\mathrm{si}}} \times \mathrm{Id} \colon t \in \mathbb{R}\} \subset \mathrm{Aut}(H) \times \{\mathrm{Id}\}$ is precompact. Thus we may choose a scalar product ρ on $T_1(H \times \mathbb{R})$ that is invariant under (the differentials of) the maps in the closure of W. We will next see that ρ has the property (J), thus finishing the proof.

A point $(h, t) \in H \times \mathbb{R}$ acts by left-translations with respect to the group law of $H \rtimes_\alpha \mathbb{R}$ on the manifold $H \times \mathbb{R}$ as

$$L^\alpha_{(h,t)}(m, s) = (h * \eta_t^\alpha(m), t + s) \tag{1}$$

and similarly for $H \rtimes_{\alpha_0} \mathbb{R}$ by replacing α with α_0. We deduce that if $L^\alpha_{(h,t)}(1) = L^{\alpha_0}_{(h',t')}(1)$, then $(h, t) = (h', t')$. Therefore, to establish the property (J), it is enough to show that the differential of the map

$$Q_{(h,t)} = (L^\alpha_{(h,t)})^{-1} \circ L^{\alpha_0}_{(h,t)}$$

is an isometry for the scalar product ρ for every (h, t). By a straightforward computation one now finds

$$Q_{(h,t)}(m, s) = (\eta^\alpha_{-t}(\eta_t^{\alpha_0}(m)), s) = (\eta^{\alpha_{\mathrm{si}}}_{-t}(m), s)\,.$$

This formula means that $Q_{(h,t)} \in W$, and since ρ was chosen to be invariant under W, we are done. □

We get the following corollary when combining Theorem B and Fact 0.1.

Corollary 3.1. *Let $\mathfrak{g}$ be a Lie algebra of the form $\mathfrak{g} = \mathfrak{h} \rtimes_\alpha \mathbb{R}$, where $\mathfrak{h}$ is completely solvable. Let $\alpha = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{si}} + \alpha_{\mathrm{nil}}$ be the decomposition of α as in Proposition 1.12. Then for $\alpha_0 = \alpha_{\mathrm{sr}} + \alpha_{\mathrm{nil}}$ it holds that the Lie algebra $\mathfrak{h} \rtimes_{\alpha_0} \mathbb{R}$ is the real-shadow of $\mathfrak{g}$.*

4. Dimension 4

The aim of this section is to prove Theorem C. Namely, we find all pairs of solvable simply connected 4-dimensional Lie groups that can be made isometric. We start from dimension 4 because dimensions 3 and below are already solved (for a survey, see [FLD21]).

4.1. **Solvable groups up to isometry.** The isomorphism classes of all simply connected solvable Lie groups are known in dimension 4. Thus determining within this family the pairs of non-isomorphic groups that can be made isometric reduces by Fact 0.1 to the determination of real-shadows of those solvable groups that are not completely solvable. Recall that by Fact 0.1 the relation “can be made isometric” is transitive, and hence *isometry (equivalence) classes* of groups are well defined objects.

The classification of simply connected Lie groups is equivalent to the classification of finite dimensional Lie algebras over $\mathbb{R}$. The list we shall use is given by Patera et al. [PSWZ76, Table I, p. 988], which in turn is based on the classification of Mubarakzjanov [Mub63a, Mub63b]. The list only contains the Lie algebras that are not direct products from lower dimension, and they are indexed from $A_{4,1}$ to $A_{4,12}$ with possible superscripts indicating one-parameter families. Table I in [PSWZ76] also contains the classification of 3D Lie algebras; in what follows we shall use those names from $A_{3,1}$ to $A_{3,9}$ together with $\mathbb{R}^n$ denoting the n-dimensional Abelian Lie algebra and A_2 denoting the unique non-Abelian 2D Lie algebra: the Lie group corresponding to A_2 was denoted by $\mathrm{Aff}^+(\mathbb{R})$ earlier.

In Table 1 we list all the simply connected completely solvable Lie groups of dimension 4. None of them can be made isometric to any other, and all the non-completely solvable groups (which in turn are listed in Table 2) can be made isometric to exactly one of these. In order to be able to divide the groups into families that seem to suit the purpose of classification up to isometry and quasi-isometry the best, in Table 1 we have relabelled the families in the left-most column, and we have written the unique completely solvable representative of the isometry class, in the notation of Patera et al., to the 2nd column. So the two left-most columns of Table 1 serve as a dictionary. We indicate the range of parameters immediately after the labels. Two concrete examples on how to read the table: our label (2) denotes the Lie algebra $\mathbb{R} \times A_{3,1}$ which is the direct product of the one-dimensional Abelian group and the Heisenberg group; instead, Lie algebra $(6, \frac{1}{2}, 1)$ denotes the Lie algebra $A_{4,5}^{1/2,1}$ of the classification of [PSWZ76].

The right-most column of Table 1 has a mark X if and only if the isometry class of this group consists of more than one isomorphism classes of simply connected solvable Lie groups. The third column is about quasi-isometric classification, and we come back to it in Section 4.3.

Table 2 lists all the remaining solvable Lie algebras of dimension 4. Namely, it lists those Lie algebras that are not completely solvable. Each of them has some completely solvable representative in its isometry class, namely the real-shadow. This real-shadow is indicated on the middle column, and our label for its isometry class is written in the right-most column. The computation of the real shadow is very simple after Corollary 3.1.

4.2. **Dropping the assumption of solvability.** We do not have many tools to treat non-solvable simply connected Lie groups. However, in dimension 4 there are no Levi decompositions other than the direct products (see [Mac99, p. 301]), and hence the only two non-solvable Lie groups are $\mathbb{R} \times \mathbb{S}^3$ and $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$, and we can

Our labelling	[PSWZ76]	**QI-type**	$*$
(1)	$\mathbb{R}^4$	poly growth	X
(2)	$\mathbb{R} \times A_{3,1}$	poly growth	X
(3)	$A_{4,1}$	poly growth	
$(4, a)$ $a \in \,]0, \infty[$	$A_{4,2}^a$	Heintze	
(5)	$A_{4,4}$	Heintze	
$(6, a, b)$ $a, b \in \,]0, 1],\ b > a$	$A_{4,5}^{a,b}$	Heintze	
$(7, a)$ $a \in \,]0, 1]$	$A_{4,5}^{a,a}$	Heintze	X
(8)	$A_{4,7}$	Heintze	
$(9, a)$ $a \in \,]0, 1[$	$A_{4,9}^a$	Heintze	
(10)	$A_{4,9}^1$	Heintze	X
$(11, a)$ $a \in \,]-\infty, 0[$	$A_{4,2}^a$	conedim 1	
$(12, a, b)$ $a, b \in \,]-1, 1[\setminus\{0\},\ b > a,\ a < 0$	$A_{4,5}^{a,b}$	conedim 1	
$(13, a)$ $a \in [-1, 0[$	$A_{4,5}^{a,a}$	conedim 1	X
$(14, a)$ $a \in \,]-1, 0[$	$A_{4,9}^a$	conedim 1	
(15)	$A_{4,8}$	conedim 1	
(16)	$A_{4,9}^0$	conedim 2	
(17)	$\mathbb{R} \times A_{3,2}$	conedim 2	
(18)	$\mathbb{R} \times A_{3,3}$	conedim 2	X
(19)	$A_2 \times A_2$	conedim 2	
$(20, a)$ $a \in \,]-1, 1[\setminus\{0\}$	$\mathbb{R} \times A_{3,5}^a$	conedim 2	
(21)	$\mathbb{R} \times A_{3,4}$	conedim 2	
(22)	$A_{4,3}$	conedim 3	
(23)	$\mathbb{R}^2 \times A_2$	conedim 3	X

TABLE 1. Completely solvable Lie algebras of dimension 4.

say something about these. We are thus interested if either of these two groups can be made isometric to some solvable groups, or if they can be made isometric to each other. For topological reasons, $\mathbb{R} \times \mathbb{S}^3$ cannot be made isometric to any other simply connected 4-dimensional group: it is the only group not homeomorphic to $\mathbb{R}^4$. The case of the group $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ however is more involved, and we will see next what we can say about it.

Lie algebra	**real-shadow**	**isometry class**
$\mathbb{R} \times A_{3,6}$	$\mathbb{R}^4$	(1)
$A_{4,10}$	$\mathbb{R} \times A_{3,1}$	(2)
$A_{4,6}^{a,b}$ $a, b \in]0, \infty[,\ a \leq b$	$A_{4,5}^{a/b,a/b}$	$(7, a/b)$
$A_{4,6}^{a,b}$ $a, b \in]0, \infty[,\ a > b$	$A_{4,5}^{b/a,b/a}$	$(7, b/a)$
$A_{4,11}^{a}$ $a \in]0, \infty[$	$A_{4,9}^{1}$	(10)
$A_{4,6}^{-a,b}$ $a, b \in]0, \infty[,\ a \leq b$	$A_{4,5}^{-a/b,-a/b}$	$(13, -a/b)$
$A_{4,6}^{a,-b}$ $a, b \in]0, \infty[,\ b < a$	$A_{4,5}^{-b/a,-b/a}$	$(13, -b/a)$
$\mathbb{R} \times A_{3,7}^{a}$ $a \in]0, \infty[$	$\mathbb{R} \times A_{3,3}$	(18)
$A_{4,12}$	$\mathbb{R} \times A_{3,3}$	(18)
$A_{4,6}^{a,0}$	$\mathbb{R}^2 \times A_2$	(23)

TABLE 2. Solvable but not completely solvable Lie algebras of dimension 4, and their real-shadows.

We know from Proposition 1.10 that $\widetilde{\mathrm{SL}}(2)$ can be made isometric to $\mathbb{R} \times A_2$, hence the groups $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ and $\mathbb{R}^2 \times A_2$ can be made isometric. Consequently, $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ must have cone dimension 3, which is the cone dimension of $\mathbb{R}^2 \times A_2$ as one may see from Proposition 1.17. Thus, checking the cone dimensions of solvable groups from Table 1, only the question remains whether or not $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ can be made isometric also to the group $A_{4,3}$ or to some groups in the family $A_{4,6}^{a,0}$ for $a \in \mathbb{R}$. Notice for example that the fact that $A_{4,3}$ and $\mathbb{R}^2 \times A_2$ cannot be made isometric does not rule out that $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ and $A_{4,3}$ can be made isometric, because $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ is not solvable. Similarly, if $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$ can be made isometric to $A_{4,6}^{a,0}$ for some $a \in \mathbb{R}$, it does not imply anything for $A_{4,6}^{a',0}$ with $a' \neq a$. One might wish to compare this phenomenon to [CKL+21, Theorem 4.21].

4.3. **Quasi-isometric classification of 4-dimensional groups.** We don't have a complete quasi-isometric classification of simply connected 4-dimensional Lie groups. In this section we show what is known about it. Recall that quasi-isometry equivalence classes are necessarily unions of the isometry classes, and these isometry classes we just established for simply connected solvable groups. Hence it is enough to consider the groups in Table 1 and the two non-solvable groups $\mathbb{R} \times \mathbb{S}^3$ and $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$.

Recall that the degree of polynomial growth is a quasi-isometric invariant. The degree of polynomial growth for $\mathbb{R} \times \mathbb{S}^3$ is 1, so it cannot be quasi-isometric either to any group in Table 1 or the group $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$. Consequently, the quasi-isometry class of $\mathbb{R} \times \mathbb{S}^3$ within the family of simply connected 4-dimensional groups, is a singleton.

About the group $\mathbb{R} \times \widetilde{\mathrm{SL}}(2)$, the only thing that we are able to say is that since it can be made isometric to $\mathbb{R}^2 \times A_2$, then it must have cone dimension 3.

For all completely solvable groups that are not Heintze groups and do not have polynomial growth, we have calculated, using Proposition 1.17, their cone dimensions and marked them to the third column titled "QI-type" of Table 1. The cone dimensions are quasi-isometry invariants by [dC11].

Recall that while Heintze groups have cone dimension 1, they are quasi-isometrically distinct from the non-Heintze groups of cone dimension 1 since in dimension 4 only the Heintze groups are Gromov hyperbolic by [dCT11] (see also [dC18, p. 277]).

The quasi-isometric classification of 4-dimensional purely real Heintze groups can be done by case-by-case study. However, a direct argument follows from Theorem A.(i), Proposition 1.5 and the results of Xie [Xie14], Carrasco Piaggio and Sequeira [CPS17, Theorem 1.3]: The purely real Heintze groups in Table 1 split into two categories

$$\begin{array}{llll} \text{nilradical } \mathbb{R}^3 & (4,a) & (5) & (6,a,b) \\ \text{nilradical Heis} & (8) & (9,a) & (10) \end{array}$$

Those with nilradical $\mathbb{R}^3$ are quasi-isometrically distinct from each other by [Xie14]. Those with nilradical Heis are quasi-isometrically distinct from each other by [CPS17, Theorem 1.3]. All the quasi-isometry relations between these two classes are excluded by Theorem A.(i). Thus, the quasi-isometry classes, isometry classes, and isomorphism classes all agree for purely real Heintze groups of dimension 4.

For the groups of polynomial growth, our classes (1), (2) and (3) are known to be quasi-isometry equivalence classes, because the completely solvable representatives (in this case, nilpotent representatives) are Carnot groups and quasi-isometric classification of Carnot groups is solved by Pansu [Pan89].

As a conclusion, we may present the following proposition.

Proposition 4.1. *Let $\mathcal{G}$ be the family of the isomorphism classes of 4-dimensional simply connected solvable groups that either have polynomial growth or are Heintze groups. Then two elements $G, H \in \mathcal{G}$ are quasi-isometric if and only if they can be made isometric. If the groups G and H are completely solvable, then they are quasi-isometric if and only if they are isomorphic.*

5. Dimension 5

The classification of real solvable Lie algebras is known in dimensions five also, see [PSWZ76]. However, due to the multitude of isomorphism classes, we rather restrict our attention to the groups of polynomial growth.

The first task is to determine a list of all simply connected solvable Lie groups of polynomial growth in dimension 5. We are not aware of a reference where this is done, so we have to do it by ourselves using the classification of real solvable Lie algebras presented in Patera et al. [PSWZ76, p. 989]. Notice that one can pretty quickly find all the candidates for groups of polynomial growth by excluding the Lie algebras with a bracket relation of the type $[e_i, e_j] = \lambda e_j$ for $\lambda \neq 0$: This is an obstruction of being polynomial growth, since all the eigenvalues of all the adjoint maps should be purely

Patera et al.	**de Graaf**	**nilshadow**	G_∞
$\mathbb{R}^5$			
$\mathbb{R}^2 \times A_{3,1}$			
$\mathbb{R} \times A_{4,1}$			
$A_{5,1}$	$L_{5,8}$		
$A_{5,2}$	$L_{5,7}$		
$A_{5,3}$	$L_{5,9}$		
$A_{5,4}$	$L_{5,4}$		
$A_{5,5}$	$L_{5,5}$		$\mathbb{R} \times A_{4,1}$
$A_{5,6}$	$L_{5,6}$		$A_{5,2}$
$\mathbb{R}^2 \times A_{3,6}$		$\mathbb{R}^5$	
$\mathbb{R} \times A_{4,10}$		$\mathbb{R}^2 \times A_{3,1}$	
$A_{5,17}^{s,0,0}$ $s \neq 0$		$\mathbb{R}^5$	
$A_{5,14}^0$		$\mathbb{R}^2 \times A_{3,1}$	
$A_{5,26}^{0,\varepsilon}$ $\varepsilon = \pm 1$		$A_{5,4}$	
$A_{5,18}^0$		$A_{5,1}$	

Table 3. Solvable Lie algebras of type (R) in dimension 5.

imaginary (see Section 1.3). The candidates so found are possible to check by hand if they have polynomial growth or not.

Taking into account the direct products, the full list of solvable simply connected Lie groups of polynomial growth is presented in Table 3. In the first 2 columns, we have recalled a dictionary between classifications presented in Patera et al. and that by de Graaf [dG07] for nilpotent Lie algebras. For nilpotent algebras that are not Carnot algebras, we have indicated their associated Carnot algebras in the 4th column. For non-nilpotent Lie algebras, we have indicated their nilshadow in the 3rd column.

From the algebraic classification given in Table 3 one may directly deduce the classification up to isometries and quasi-isometries (up to one open case we will mention soon) using the list of invariants we recorded in beginning of Section 1.4. Indeed, recalling invariant (Inv-4) and Remark 1.16, every group is isometric to its nilshadow, and in dimension 4 it happens that the nilshadows are always Carnot groups (Carnot groups are those with empty field both in “nilshadow” and in “G_∞”). Moreover, the nilshadows happen to be those Carnot groups that are not associated Carnot groups of some nilpotent non-Carnot groups. Hence the classification up to isometry and quasi-isometry is ready for the groups of polynomial growth and those Carnot groups that appear as their nilshadows. Only problem that remains after applying (Inv-1) is if $A_{5,5}$ or $A_{5,6}$ are quasi-isometric to their associated Carnot groups, recall that by [Wol63] isometries between non-isomorphic nilpotent groups cannot exist. The invariant (Inv-2) tells that $A_{5,5}$ is not quasi-isometric to its associated Carnot group $\mathbb{R} \times A_{4,1}$ (see [dC18, Section 19.7]), but the possible quasi-isometry relation between $A_{5,6}$ and $A_{5,2}$ remains unanswered by this analysis.

In conclusion, as was the case for the family of simply connected solvable Lie groups of dimension 4, we are unable to completely classify simply connected Lie groups of polynomial growth in dimension 5. However, here it is only one pair of groups whose possible quasi-isometry relation remains open: whether or not the Lie group $A_{5,6}$ is quasi-isometric to its associated Carnot group $A_{5,2}$. This question cannot be answered by the community for now.

6. Appendix: A direct proof in Abelian case

In this section we prove Theorem 6.4. It is a less general statement than Theorem A.(i), but the proof is completely different in spirit and might have independent interest and possibilities to generalise. The proof is highly inspired by the results of [CPS17].

The following definition appeared implicitly in [CPS17], but we prefer to have a name for it.

Definition 6.1. The *characteristic subalgebra* for a purely real homogeneous group (N, α) is the subalgebra $\mathfrak{h}_\alpha$ of $\mathfrak{n}$ constructed as follows. Consider a basis of $\mathfrak{n}$ where α is in Jordan form, and let λ_1 denote the smallest of the eigenvalues of α. Let V_{λ_1} be the subspace corresponding to the Jordan-blocks of α of eigenvalue λ_1 (i.e., the generalised eigenspace of eigenvalue λ_1). Let $\hat{V}_1 \subset V_{\lambda_1}$ be the sum of the subspaces corresponding to the Jordan blocks in V_{λ_1} of maximal size. Next, let $\mathcal{V}_1$ consist of eigenvectors of eigenvalue λ_1 inside $\hat{V}_1$, and finally define $\mathfrak{h}_\alpha = \mathrm{LieSpan}(\mathcal{V}_1)$. We further denote by H_α the subgroup of N with Lie algebra $\mathfrak{h}_\alpha$ and call it the *characteristic subgroup* of (N, α).

Remark 6.2. Equivalently, the characteristic subalgebra is defined as follows: Let $k \in \mathbb{N}$ be the unique integer such that $(\alpha|_{V_{\lambda_1}} - \lambda_1 \mathrm{Id})^k \neq 0$ and $(\alpha|_{V_{\lambda_1}} - \lambda_1 \mathrm{Id})^{k+1} = 0$. Then $\mathcal{V}_1 = \mathrm{Im}(\alpha|_{V_{\lambda_1}} - \lambda_1 \mathrm{Id})^k$ and $\mathfrak{h}_\alpha = \mathrm{LieSpan}(\mathcal{V}_1)$.

In the following, we list some facts related to characteristic subalgebras and subgroups.

Proposition 6.3. *(i) $\mathfrak{h}_\alpha = \mathfrak{n}$ if and only if (N, α) is of Carnot type.*

(ii) $\mathfrak{h}_\alpha$ is preserved under α.

(iii) Suppose $F\colon (N_1, \alpha) \to (N_2, \beta)$ is a biLipschitz map between two purely real homogeneous groups, and suppose $F(1_{N_1}) = 1_{N_2}$. Then $F(H_\alpha) = H_\beta$.

Proof. The part (i) follows by observing that both of the claims are equivalent to the condition $V_{\lambda_1} = \mathcal{V}_1$.

The part (ii) is proven by a straightforward induction on the length of a bracket in $\mathfrak{h}_\alpha$, using that $\mathcal{V}_1$ is preserved under α by construction.

The part (iii) is proven in [Pia17], see also [CPS17, p. 6] $\square$

Theorem 6.4. *Let (N_1, α) and (N_2, β) be purely real homogeneous groups that are biLipschitz equivalent. If N_1 is Abelian, so is N_2. Consequently (N_1, α) and (N_2, β) are isomorphic as homogeneous groups, by* [Xie14].

Proof. We prove the claim inductively on the topological dimension of the groups in question. The case $n = 1$ (and also $n = 2$) is true due to the lack of non-Abelian nilpotent groups. So assume the claim holds for groups of dimension k and less and

$$\dim(N_1) = \dim(N_2) = k + 1 \,. \tag{2}$$

Let $F\colon N_1 \to N_2$ be a biLipschitz map, which after post-composing with a left-translation we may assume to satisfy $F(1_{N_1}) = 1_{N_2}$. Thus, when H_α and H_β denote the respective characteristic subgroups, by Proposition 6.3.(iii) it holds

$$F(H_\alpha) = H_\beta \,. \tag{3}$$

If $H_\alpha = N_1$, then by Proposition 6.3.(i) the homogeneous group (N_1, α) is of Carnot type, and as a consequence of [Pia17, Theorem 1.9] the homogeneous group (N_2, β) is also of Carnot type. In this case, by Pansu's Theorem [Pan89], (N_1, α) and (N_2, β) are isomorphic as homogeneous groups. We are left to consider the case

$$H_\alpha \subsetneq N_1 \,. \tag{4}$$

From (2) and (4) we have $\dim(H_\alpha) \leq k$. Thus the induction assumption and (3) gives that H_β is Abelian because H_α is Abelian.

Moreover we claim that H_β is normal in N_2. Indeed, the normaliser of H_α is N_1 since N_1 is Abelian, hence by Proposition 1.7 the normaliser of H_β is N_2.

Next, we claim H_β is central in N_2. By the definition of the characteristic subalgebra we have $\mathfrak{h}_\beta = \mathrm{LieSpan}(\mathcal{V}_1)$, where $\mathcal{V}_1 \subset V_{\lambda_1}$, as in Definition 6.1. We know now that $\mathfrak{h}_\beta$ is an Abelian ideal, so $\mathcal{V}_1$ is Abelian and $\mathfrak{h}_\beta \subset V_{\lambda_1}$. Using the grading given by the generalised eigenspaces V_λ of β (see [Bou75, p. 16 Prop. 12]) and the fact that $\mathfrak{h}_\beta$ is an ideal of $\mathfrak{n}_2$ we get for all $H \in \mathfrak{h}_\beta$ and $X \in \mathfrak{n}_2$ that

$$[X, H] \in \mathfrak{h}_\beta \cap \bigoplus_{\lambda > \lambda_1} V_\lambda = \{0\} \,.$$

Hence $\mathfrak{h}_\beta$ is central in $\mathfrak{n}_2$.

Take W to be a complementary subspace to $\mathcal{V}_1$ inside V_{λ_1}. Define

$$\mathfrak{s}_\beta = W \oplus \bigoplus_{\lambda > \lambda_1} V_\lambda \,,$$

which is an ideal because it contains $[\mathfrak{n}_2, \mathfrak{n}_2]$. The subspaces $\mathfrak{s}_\beta$ and $\mathfrak{h}_\beta$ are in direct sum and they are both ideals, so the Lie algebra $\mathfrak{n}_2$ is the direct product of these two subalgebras: $\mathfrak{n}_2 = \mathfrak{s}_\beta \times \mathfrak{h}_\beta$. On the N_1 side, the same construction works but it is simpler because N_1 is Abelian. Anyway, we may decompose $\mathfrak{n}_1 = \mathfrak{h}_\alpha \times \mathfrak{s}_\alpha$, where $\mathfrak{s}_\alpha$ is an arbitrary complementary subspace to $\mathfrak{h}_\alpha$.

By Proposition 6.3.(ii) and the concrete formula for a homogeneous distance on the quotient given in Lemma 1.8, we have that the quotient groups N_1/H_α and N_2/H_β are biLipschitz equivalent purely real homogeneous groups. Their dimension is at most k, since the characteristic subgroups have at least dimension 1. Hence by induction, N_2/H_β is Abelian since N_1/H_α is Abelian. By the structure of direct products, N_2/H_β and S_β are isomorphic as Lie groups, hence S_β is Abelian. Since N_2 is a direct sum of two Abelian normal subgroups H_β and S_β, then N_2 is Abelian.

For the final statement, [Xie14, Theorem 1.1] tells that the Jordan forms of α and β are proportional. On the other hand, since the homogeneous groups are biLipschitz equivalent, the smallest of the eigenvalues of α and β must agree since by Proposition 1.6 we have that the common smallest eigenvalue λ_1 is the minimal Hausdorff dimension of curves. Therefore the Jordan forms of α and β agree and this is enough to give an isomorphism of homogeneous groups in the Abelian case. □

Remark 6.5. The proof above does not give a new proof of the main result of [Xie14], since it may happen that the complementary subspace W cannot be chosen to be preserved under the derivation β. Therefore, while N_2/H_β has a structure of a homogeneous group induced by β, the subgroup S^β is not preserved under β and does not inherit a structure of a homogeneous group.

Remark 6.6. Theorem 6.4 may also be proven from Theorem A.(ii) by an argument that we will next sketch, thereby giving a third proof for Theorem 6.4. A homogeneous group (N, α) is non-Abelian if and only if for some $s > 0$ the reachability set $(N, \alpha)^s$ is strictly larger than the subgroup corresponding to $\bigoplus_{0<\lambda\leq s} V_\lambda$. Suppose that homogeneous groups (N_1, α) and (N_2, β) are biLipschitz equivalent. On the one hand, the characteristic polynomials of α and β agree by [CPS17], and hence the dimensions of the generalised eigenspaces of the same eigenvalues agree. On the other hand, $(N_1, \alpha)^s$ and $(N_2, \beta)^s$ have the same dimension for every s by Theorem A.(ii). We conclude that, if N_1 is Abelian, then also N_2 is Abelian.

References

[Ale75] D. V. Alekseevskiĭ, *Homogeneous Riemannian spaces of negative curvature*, Mat. Sb. (N.S.) **96(138)** (1975), 93–117, 168. MR 0362145

[BL19] Costante Bellettini and Enrico Le Donne, *Sets with constant normal in carnot groups: properties and examples*, Preprint, arXiv:1910.12117 (2019).

[Bou75] N. Bourbaki, *Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées*, Actualités Scientifiques et Industrielles, No. 1364. Hermann, Paris, 1975. MR 0453824

[Bre14] Emmanuel Breuillard, *Geometry of locally compact groups of polynomial growth and shape of large balls*, Groups Geom. Dyn. **8** (2014), no. 3, 669–732. MR 3267520

[BS00] M. Bonk and O. Schramm, *Embeddings of gromov hyperbolic spaces*, Geometric and Functional Analysis GAFA **10** (2000), 266–306.

[CKL+21] Michael G. Cowling, Ville Kivioja, Enrico Le Donne, Sebastiano Nicolussi Golo, and Alessandro Ottazzi, *From homogeneous metric spaces to lie groups*, arXiv preprint arXiv:1705.09648v3 (2021).

[CPS17] Matias Carrasco Piaggio and Emiliano Sequeira, *On quasi-isometry invariants associated to a Heintze group*, Geom. Dedicata **189** (2017), 1–16. MR 3667336

[dC08] Yves de Cornulier, *Dimension of asymptotic cones of Lie groups*, J. Topol. **1** (2008), no. 2, 342–361. MR 2399134

[dC11] ———, *Asymptotic cones of Lie groups and cone equivalences*, Illinois J. Math. **55** (2011), no. 1, 237–259 (2012). MR 3006687

[dC18] ———, *On the quasi-isometric classification of locally compact groups*, New directions in locally compact groups, London Math. Soc. Lecture Note Ser., vol. 447, Cambridge Univ. Press, Cambridge, 2018, pp. 275–342. MR 3793294

[dCT11] Yves de Cornulier and Romain Tessera, *Contracting automorphisms and L^p-cohomology in degree one*, Ark. Mat. **49** (2011), no. 2, 295–324. MR 2826945

[dG07] Willem A. de Graaf, *Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2*, J. Algebra **309** (2007), no. 2, 640–653. MR 2303198

[DtER03] Nick Dungey, A. F. M. ter Elst, and Derek W. Robinson, *Analysis on Lie groups with polynomial growth*, Progress in Mathematics, vol. 214, Birkhäuser Boston, 2003. MR 2000440

[Dym10] Tullia Dymarz, *Large scale geometry of certain solvable groups*, Geom. Funct. Anal. **19** (2010), no. 6, 1650–1687. MR 2594617

[EFW12] Alex Eskin, David Fisher, and Kevin Whyte, *Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs*, Ann. of Math. (2) **176** (2012), no. 1, 221–260. MR 2925383

[EFW13] ———, *Coarse differentiation of quasi-isometries II: Rigidity for Sol and lamplighter groups*, Ann. of Math. (2) **177** (2013), no. 3, 869–910. MR 3034290

[FLD21] Katrin Fässler and Enrico Le Donne, *On the quasi-isometric and bilipschitz classification of 3d riemannian lie groups*, Geometriae Dedicata **210** (2021), 27–42.

[Gor81] C. Gordon, *Transitive riemannian isometry groups with nilpotent radicals*, Annales de l'Institut Fourier **31** (1981), no. 2, 193–204 (en). MR 82i:53040

[GW88] Carolyn S. Gordon and Edward N. Wilson, *Isometry groups of Riemannian solvmanifolds*, Trans. Amer. Math. Soc. **307** (1988), no. 1, 245–269. MR 936815

[Hei74] Ernst Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. **211** (1974), 23–34. MR 353210

[HKMT20] Eero Hakavuori, Ville Kivioja, Terhi Moisala, and Francesca Tripaldi, *Gradings for nilpotent lie algebras*, arXiv preprint arXiv:2011.06871 (2020).

[Jab19] Michael Jablonski, *Maximal symmetry and unimodular solvmanifolds*, Pacific J. Math. **298** (2019), no. 2, 417–427. MR 3936023

[Jen73] J. W. Jenkins, *Growth of connected locally compact groups*, J. Functional Analysis **12** (1973), 113–127. MR 0349895

[KLD17] Ville Kivioja and Enrico Le Donne, *Isometries of nilpotent metric groups*, J. Éc. polytech. Math. **4** (2017), 473–482. MR 3646026

[LDNG19] Enrico Le Donne and Sebastiano Nicolussi Golo, *Metric lie groups admitting dilations*, Preprint, arXiv:1901.02559, To appear in Arkiv för Matematik (2019).

[LDR16] Enrico Le Donne and Séverine Rigot, *Remarks about the Besicovitch covering property in Carnot groups of step 3 and higher*, Proc. Amer. Math. Soc. **144** (2016), no. 5, 2003–2013. MR 3460162

[LDX16] Enrico Le Donne and Xiangdong Xie, *Rigidity of fiber-preserving quasisymmetric maps*, Rev. Mat. Iberoam. **32** (2016), no. 4, 1407–1422. MR 3593530

[Mac99] M. A. H. MacCallum, *On the classification of the real four-dimensional lie algebras*, On Einstein's Path: Essays in Honor of Engelbert Schucking (Alex Harvey, ed.), Springer New York, New York, NY, 1999, pp. 299–317.

[Mub63a] G. M. Mubarakzjanov, *Classification of real structures of Lie algebras of fifth order*, Izv. Vysš. Učebn. Zaved. Matematika **1963** (1963), no. 3 (34), 99–106. MR 0155871

[Mub63b] ———, *Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element*, Izv. Vysš. Učebn. Zaved. Matematika **1963** (1963), no. 4 (35), 104–116. MR 0155872

[Osi02] D. V. Osin, *Exponential radicals of solvable Lie groups*, J. Algebra **248** (2002), no. 2, 790–805. MR 1882124

[Pal20] Gabriel Pallier, *Sublinear quasiconformality and the large-scale geometry of Heintze groups*, Conform. Geom. Dyn. **24** (2020), 131–163. MR 4127909

[Pan89] Pierre Pansu, *Métriques de carnot-carathéodory et quasiisométries des espaces symétriques de rang un*, Annals of Mathematics (1989), 1–60.

[Pau96] Frédéric Paulin, *Un groupe hyperbolique est déterminé par son bord*, Journal of the London Mathematical Society **54** (1996), no. 1, 50–74.

[Pia17] Matias Carrasco Piaggio, *Orlicz spaces and the large scale geometry of Heintze groups*, Math. Ann. **368** (2017), no. 1-2, 433–481. MR 3651580

[PSWZ76] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, *Invariants of real low dimension Lie algebras*, J. Mathematical Phys. **17** (1976), no. 6, 986–994. MR 404362

[Roe93] John Roe, *Coarse cohomology and index theory on complete riemannian manifolds*, Memoirs of the American Mathematical Society, 1993.

[Sau06] R. Sauer, *Homological invariants and quasi-isometry*, Geometric & Functional Analysis GAFA **16** (2006), no. 2, 476–515.

[Sha04] Yehuda Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), no. 2, 119–185.

[Wol63] Joseph A. Wolf, *On locally symmetric spaces of non-negative curvature and certain other locally homogeneous spaces*, Comment. Math. Helv. **37** (1962/1963), 266–295.

[Xie14] Xiangdong Xie, *Large scale geometry of negatively curved* $\mathbb{R}^n \rtimes \mathbb{R}$, Geom. Topol. **18** (2014), no. 2, 831–872. MR 3180486

[Xie15] ———, *Quasiisometries of negatively curved homogeneous manifolds associated with Heisenberg groups*, J. Topol. **8** (2015), no. 1, 247–266. MR 3335254

(Kivioja and Nicolussi Golo) Department of Mathematics and Statistics, University of Jyväskylä, 40014 Jyväskylä, Finland

(Le Donne) Department of Mathematics, University of Fribourg, Fribourg CH-1700, Switzerland

Email address: `kivioja.ville@gmail.com`

Email address: `enrico.ledonne@gmail.com`

Email address: `sebastiano2.72@gmail.com`

[D]

Gradings for nilpotent Lie algebras

Eero Hakavuori, Ville Kivioja, Terhi Moisala and Francesca Tripaldi

Preprint

GRADINGS FOR NILPOTENT LIE ALGEBRAS

EERO HAKAVUORI, VILLE KIVIOJA, TERHI MOISALA,
AND FRANCESCA TRIPALDI

ABSTRACT. We present a constructive approach to torsion-free gradings of Lie algebras. Our main result is the computation of a maximal grading. Given a Lie algebra, using its maximal grading we enumerate all of its torsion-free gradings as well as its positive gradings. As applications, we classify gradings in low dimension, we consider the enumeration of Heintze groups, and we give methods to find bounds for non-vanishing $\ell^{q,p}$ cohomology.

CONTENTS

Date: March 31, 2021.
2010 *Mathematics Subject Classification.* 17B70, 22E25, 17B40, 20F65, 20G20.
Key words and phrases. nilpotent Lie algebras, gradings, maximal gradings, positive gradings, stratifications, Carnot groups, classifications, large scale geometry, Heintze groups, lqp cohomology.

1. Introduction

1.1. **Overview.** A *grading* of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in S} V_\alpha \tag{1}$$

indexed by some set S in such a way that for each pair $\alpha, \beta \in S$ there exists $\gamma \in S$ such that

$$[V_\alpha, V_\beta] \subset V_\gamma.$$

In this paper, we will focus on Lie algebras defined over fields of characteristic zero and gradings indexed over torsion-free abelian groups, where the element γ is given by $\gamma = \alpha + \beta$.

An important example of a Lie algebra grading is the so called *maximal grading* (also known as *fine grading*), that is a grading that does not admit any proper refinement into smaller subspaces V_α. A classical example of such a maximal grading is the Cartan decomposition, which plays a fundamental role in representation theory and the classification of semisimple Lie algebras over $\mathbb{C}$, see for example [Hum78]. There has been a growing interest in the study of (maximal) gradings of semisimple Lie algebras since the paper [PZ89], see the survey [Koc09] or the monograph [EK13] for an overview. Moreover, a classification of maximal gradings of simple classical Lie algebras over algebraically closed fields of characteristic zero can be found in [Eld10].

Regarding nilpotent Lie algebras over algebraically closed fields of characteristic zero, an in depth study of maximal gradings over torsion-free abelian groups was carried out in [Fav73]. One of the main results in [Fav73] is that, considering the family of nilpotent Lie algebras $\mathfrak{g}$ of nilpotency step s and with abelianization $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of dimension r, there are only finitely many torsion-free maximal gradings, up to automorphisms of the free nilpotent Lie algebra of step s with r generators. This finiteness in the number of maximal gradings is in contrast with the existence of an uncountable number of non-isomorphic nilpotent Lie algebras in dimension 7 and higher.

There are two other special types of gradings of particular interest in the case of nilpotent Lie algebras: *positive gradings* and *stratifications* (also called *Carnot gradings*). A positive grading is a grading indexed over the reals such that in the direct sum decomposition (1) all the non-zero spaces V_α have positive indices $\alpha > 0$. A stratification is a positive grading for which V_1 generates $\mathfrak{g}$ as a Lie algebra.

Lie algebras with a stratification are the Lie algebras of Carnot groups. These groups have played a central role in the fields of geometric analysis, geometric measure theory, and large scale geometry, see [LD17] for a long list of references.

Positive gradings are important within the study of homogeneous spaces, as they appear directly in characterizations of such spaces.

First, any negatively curved homogeneous Riemannian manifold is a *Heintze group* $G \rtimes \mathbb{R}$ [Hei74], where G is a nilpotent Lie group and the action of $\mathbb{R}$ on G is given by a one-parameter family of automorphisms associated with a positive grading of G. Second, any connected locally compact group that admits a contracting automorphism is a positively gradable Lie group [Sie86]. In this latter result, the group structure and contracting automorphism may also be replaced by a metric structure and a dilation, see [CKLD$^+$17].

Another active area of research that contains several open problems related to positively gradable Lie groups is the quasi-isometric classification of locally compact groups. A survey on the topic can be found in [Cor18]. For instance, it is not known whether there exists a non-stratifiable positively gradable Lie group that is quasi-isometric to its asymptotic cone, nor whether all large-scale contractible groups are positively gradable, see [Cor19, Question 7.9]. The quasi-isometric classification is open also for Heintze groups, see [CPS17] for some known results.

1.2. **Main results.** In all of the following statements, let $\mathfrak{g}$ be a finite dimensional Lie algebra defined in terms of its structure coefficients and let F be the base field of $\mathfrak{g}$. That is, we assume we have a fixed basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ and a family of coefficients $\{c_{ij}^k \in F : i, j, k \in \{1, \ldots, n\}\}$ such that the Lie bracket is defined as

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

Our main result is the following.

Theorem 1.1. *Suppose the base field F is algebraically closed. Then there exists an algorithm that constructs a maximal grading of $\mathfrak{g}$.*

We also give explicit constructions for stratifications and positive gradings.

Theorem 1.2. *There exists an algorithm that constructs a stratification of $\mathfrak{g}$ or determines that one does not exist.*

Theorem 1.3. *Suppose the base field F is algebraically closed. Then there exists an algorithm that constructs a positive grading of $\mathfrak{g}$ or determines that one does not exist.*

Theorem 1.2 and Theorem 1.3 are constructive versions of the characterizations of stratifiability in [Cor16, Lemma 3.10] and existence of a positive grading in [Cor16, Proposition 3.22].

Using Theorem 1.1, we are able to enumerate all torsion-free gradings.

Theorem 1.4. *Suppose the base field F is algebraically closed. Then there exists an algorithm to compute a finite collection of gradings containing up to equivalence all the torsion-free gradings of $\mathfrak{g}$.*

A torsion-free grading is a grading that can be indexed over a torsion-free abelian group, and gradings are considered equivalent if there is an automorphism of the Lie algebra mapping layers of one grading to layers of the other. The precise definitions can be found in Section 2. The finite set we construct in Theorem 1.4 will in general contain redundant gradings, i.e., there may exist equivalent gradings in the collection. We eliminate this redundancy in the case of nilpotent Lie algebras of dimension up to 6 to find a complete classification up to equivalence of torsion-free gradings.

For applications related to positive gradings, we also give a method to extract from the complete list of Theorem 1.4 of all torsion-free gradings those that admit a positive realization, i.e., can be indexed over the positive reals:

Theorem 1.5. *Let $\mathfrak{g} = \bigoplus_{\alpha \in S} V_\alpha$ be a grading of $\mathfrak{g}$.*

- *(i) There exists an algorithm that constructs a positive realization of the grading or determines that one does not exist.*
- *(ii) If S is a finitely generated abelian group, then there exists an algorithm that constructs a positive realization such that the reindexing $S \to \mathbb{R}$ of layers is a homomorphism, or determines that one does not exist.*

We also give two applications of the enumeration of positive gradings obtained from the above results. First, we show that all non-equivalent positive gradings define non-isomorphic Heintze groups, see Proposition 4.7. In this way we are able to enumerate diagonal Heintze groups. Thus we give methods to tackle the problem of finding all Heintze groups with prescribed nilradical, which is a question already posed by Heintze in [Hei74].

Second, the enumeration of positive gradings gives a method to find better estimates for the non-vanishing of the $\ell^{q,p}$ cohomology of a nilpotent Lie group, which is a quasi-isometry invariant.

1.3. **Structure of the paper.** In Section 2 we recall various definitions and terminology related to gradings. The core concepts of realization, push-forward, and equivalence are defined in Subsection 2.1 and universal realizations are recalled in Subsection 2.2. Subsection 2.3 recalls how to study torsion-free gradings of a Lie algebra $\mathfrak{g}$ in terms of subtori of the derivation algebra $\mathrm{der}(\mathfrak{g})$. Maximal gradings and their universal property are covered in Subsection 2.4. Subsection 2.5 reduces Theorem 1.4 on enumeration of gradings to proving Theorem 1.1 on algorithmic construction of a maximal grading.

In Section 3 we give the remaining constructions for our main results. Subsection 3.1 covers Theorem 1.2 on stratifiability. Subsection 3.2 covers Theorem 1.3 and Theorem 1.5 on positive gradings. An alternate approach to deciding the existence of a positive realization is also described in Appendix A. Subsection 3.3 covers Theorem 1.1 on maximal gradings.

In Section 4 we give various applications of gradings to the study of Lie algebras and Lie groups. Subsection 4.1 shows how to use the maximal grading of a Lie algebra as a tool to detect decomposability of a Lie algebra, and how to reduce the dimensionality of the problem of deciding whether two Lie algebras are isomorphic. In Subsection 4.2 we classify up to equivalence the gradings of low dimensional nilpotent Lie algebras over $\mathbb{C}$. In Subsection 4.3 we cover the results on enumeration of Heintze groups. Finally, in Subsection 4.4 we present the method to find improved bounds for the non-vanishing of the $\ell^{q,p}$ cohomology.

2. Gradings

The contents of this section can, up to some modifications, be found in [Koc09, Section 3-4]. We nonetheless give here a self-contained presentation to better fit our constructive approach.

2.1. **Gradings and equivalences.** In this section we define some key notions related to gradings of Lie algebras, including equivalence, push-forwards and coarsenings. We also make a distinction between two different notions of grading, with the difference being whether the indexing plays a role or not.

Definition 2.1. A *grading* of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha\in S} V_\alpha$ such that for each $\alpha, \beta \in S$ either $[V_\alpha, V_\beta] = 0$ or there exists a unique $\gamma \in S$ such that $[V_\alpha, V_\beta] \subset V_\gamma$. When S is an abelian group A such that the unique element γ is given by $\gamma = a + b$, we say that the grading $\mathcal{V}$ is *over* A, or that $\mathcal{V}$ is an *A-grading*. In this case, A is the *grading group* of the grading $\mathcal{V}$.

The subspaces V_α are called the *layers* of the grading $\mathcal{V}$ and the elements $\alpha \in S$ such that $V_\alpha \neq 0$ are called the *weights* of $\mathcal{V}$. We will usually denote the set of weights by Ω. A basis of $\mathfrak{g}$ is said to be *adapted to* $\mathcal{V}$ if every element of the basis is contained in some layer of $\mathcal{V}$.

Definition 2.2. Suppose the indexing set S of a grading $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha\in S} V_\alpha$ can be embedded into an abelian group A such that $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$ for all $\alpha, \beta \in A$, where we define $V_\alpha = 0$ for $\alpha \notin S$. Then the resulting A-grading is called a *realization* of the grading $\mathcal{V}$.

Definition 2.3. A grading is called *torsion-free* if it admits a realization over a torsion-free (abelian) group.

In this paper, the notation $\langle X\rangle$ always refers to the span of X in the appropriate sense.

Example 2.4. Consider the 6-dimensional Lie algebra spanned by the vectors $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ with the non-trivial bracket relations

$$[X_1, Y_1] = Z_1 \qquad [X_2, Y_2] = Z_2.$$

The subspace decomposition

$$V_a = \langle X_1\rangle, \qquad V_b = \langle X_2\rangle, \qquad V_c = \langle Y_1, Z_2\rangle, \qquad V_d = \langle Z_1, Y_2\rangle$$

defines a grading. It can be realized over $\mathbb{Z}^2$ with the embedding

$$a \mapsto (1,0), \qquad b \mapsto (-1,0), \qquad c \mapsto (0,1), \qquad d \mapsto (1,1).$$

Definition 2.5. Let $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha\in A} V_\alpha$ be an A-grading for some abelian group A. Given an automorphism $\Phi \in \operatorname{Aut}(\mathfrak{g})$, an abelian group B and a homomorphism $f\colon A \to B$, we define the *push-forward grading* $f_*\Phi(\mathcal{V}) : \mathfrak{g} = \bigoplus_{\beta\in B} W_\beta$ over B, where

$$W_\beta = \bigoplus_{\alpha\in f^{-1}(\beta)} \Phi(V_\alpha).$$

When $\Phi = \mathrm{Id}$, we simply denote $f_* \mathrm{Id}(\mathcal{V}) = f_*\mathcal{V}$.

It is readily checked that the push-forward grading is indeed a B-grading in the sense of Definition 2.1.

Definition 2.6. Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha\in S_1} V_\alpha$ and $\mathcal{W} : \mathfrak{g} = \bigoplus_{\beta\in S_2} W_\beta$ be two gradings. If for every $\alpha \in S_1$ there exists $\beta \in S_2$ such that $V_\alpha \subset W_\beta$, then we say that $\mathcal{V}$ is a *refinement* of $\mathcal{W}$, and that $\mathcal{W}$ is a *coarsening* of $\mathcal{V}$.

Remark 2.7. If $\mathcal{W} = f_*\mathcal{V}$ for some homomorphism f, then $\mathcal{W}$ is a coarsening of $\mathcal{V}$. Such a map f is injective on the weights if and only if $\mathcal{V}$ and $\mathcal{W}$ are realizations of the same grading.

There are several different notions of equivalence of gradings in the literature. The two that we shall use are distinguished as *equivalence* and *group-equivalence* in [Koc09]. For brevity, we will refer to both notions as equivalence. Stated in terms of push-forwards, the group-equivalence notion of [Koc09] takes the following form:

Definition 2.8. An A-grading $\mathcal{V}$ and a B-grading $\mathcal{W}$ are said to be *equivalent* if there exist an automorphism $\Phi \in \operatorname{Aut}(\mathfrak{g})$ and a group isomorphism $f\colon A \to B$ such that $\mathcal{W} = f_*\Phi(\mathcal{V})$.

For gradings that admit realizations, the equivalence notion of [Koc09] can be rephrased through the previous notion as follows.

Definition 2.9. A grading $\mathfrak{g} = \bigoplus_{\alpha\in S_1} V_\alpha$ and a grading $\mathfrak{g} = \bigoplus_{\beta\in S_2} W_\beta$ over arbitrary indexing sets S_1, S_2 are said to be *equivalent* if they admit realizations as an A-grading and a B-grading that are equivalent in the sense of Definition 2.8.

Example 2.10. Consider two gradings $V_1 \oplus V_2$ and $V_1 \oplus V_3$ of $\mathbb{R}^2$ over $\mathbb{Z}$ with the same one-dimensional layers. The two gradings are equivalent in the sense of Definition 2.9, since the former is a realization of the second by the embedding $\{1,3\} \hookrightarrow \{1,2\} \subset \mathbb{Z}$, but they are not equivalent as $\mathbb{Z}$-gradings in the sense of Definition 2.8 as there does not exist an automorphism of $\mathbb{Z}$ mapping $\{1,3\} \to \{1,2\}$.

In the following lemma we demonstrate that, after possibly shrinking the grading groups, an A-grading and a B-grading are equivalent if and only if they are push-forwards of each other.

Lemma 2.11. *Let $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in A} V_\alpha$ be an A-grading and $\mathcal{W} : \mathfrak{g} = \bigoplus_{\beta \in B} W_\beta$ be a B-grading such that the weights of $\mathcal{V}$ and $\mathcal{W}$ generate the abelian groups A and B, respectively. If there exist homomorphisms $f \colon A \to B$ and $g \colon B \to A$ such that $\mathcal{W} = f_*\mathcal{V}$ and $\mathcal{V} = g_*\mathcal{W}$, then $\mathcal{V}$ and $\mathcal{W}$ are equivalent.*

Proof. Let us denote by Ω_A and Ω_B the sets of weights of $\mathcal{V}$ and $\mathcal{W}$. Notice first that by definition of the push-forward, $f(\Omega_A) = \Omega_B$ and $g(\Omega_B) = \Omega_A$, so both f and g are injective on weights. Moreover, we have for every $\alpha \in \Omega_A$ and $\beta \in \Omega_B$ the correspondence

$$V_\alpha = W_{f(\alpha)} = V_{g(f(\alpha))} \qquad \text{and} \qquad W_\beta = V_{g(\beta)} = W_{f(g(\beta))}.$$

Hence $f \colon \Omega_A \to \Omega_B$ is a bijection and $f^{-1} = g$ on Ω_B. Since Ω_A and Ω_B generate A and B as groups, we get that $f^{-1} = g$ on whole B. □

Notice that the assumption that the weights generate is indeed necessary: for instance, the gradings $\mathbb{R} = V_1$ over $\mathbb{Z}$ and $\mathbb{R} = V_{(1,0)}$ over $\mathbb{Z}^2$ are push-forward gradings of each other, but they are not equivalent.

2.2. **Universal gradings.** We do not in general require that the weights of an A-grading generate the grading group A in order to include e.g. gradings over $A = \mathbb{R}$ in the discussion. Moreover, weights of a grading may have additional relations coming from the ambient group structure, even when the corresponding layers are unrelated, see for instance Example 3.13. To build a satisfactory theory using homomorphisms between grading groups, we consider the notion of an (abelian) universal realization, see [Koc09, Section 3.3].

Definition 2.12. Let $\mathcal{V}$ be a grading of $\mathfrak{g}$. A *universal realization* of $\mathcal{V}$ is a realization $\widetilde{\mathcal{V}}$ as an A-grading such that for every realization of $\mathcal{V}$ as a B-grading with B abelian, there exists a unique homomorphism $f \colon A \to B$ such that the B-grading is the push-forward grading $f_*\widetilde{\mathcal{V}}$.

Observe that by Lemma 2.11, the universal realization of a grading is unique up to equivalence.

If a grading admits a realization, then it also admits a universal realization. The universal realization can be constructed by considering the free abelian group generated by the weights and quotienting out the grading relations, as described by the following algorithm.

Algorithm 2.13 (Universal realization)**.** *Input: a grading $\mathcal{V}$ that has a realization. Output: a universal realization $\widetilde{\mathcal{V}}$ of $\mathcal{V}$.*

(1) Let $\Omega = \{\alpha_1, \ldots, \alpha_n\}$ be the set of weights of $\mathcal{V}$ and let $B = \{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{Z}^n$. Set $R = \emptyset$.
(2) Repeat for all pairs $\alpha_i, \alpha_j \in \Omega$: If $0 \neq [V_{\alpha_i}, V_{\alpha_j}] \subseteq V_{\alpha_k}$ for some $\alpha_k \in \Omega$, extend R by $e_i + e_j - e_k$.
(3) For all $i = 1, \ldots, n$, set $\widetilde{V}_{\pi(e_i)} = V_{\alpha_i}$, where $\pi\colon \mathbb{Z}^n \to \mathbb{Z}^n/\langle R\rangle$ is the projection. Return the obtained $\mathbb{Z}^n/\langle R\rangle$-grading.

Proof of correctness. Consider a realization of $\mathcal{V}$ over an abelian group A and the homomorphism $\phi\colon \mathbb{Z}^n \to A$ defined by $\phi(e_i) = \alpha_i$ for all $1 \leq i \leq n$. Observe that by construction $R \subset \ker(\phi)$. Then the grading $\widetilde{\mathcal{V}}$ is well-defined: if $\pi(e_i) = \pi(e_j)$, then $e_i - e_j \in \langle R\rangle$ and we have $\alpha_i = \phi(e_i) = \phi(e_j) = \alpha_j$. Moreover, the obtained $\mathbb{Z}^n/\langle R\rangle$-grading is a universal realization of $\mathcal{V}$ by the universal property of quotients and arbitrariness of A. □

In the rest of the paper we will focus on gradings that admit torsion-free realizations. For such gradings, the universal realizations are gradings over some $\mathbb{Z}^k$, as demonstrated by the following lemma.

Lemma 2.14. *If $\mathcal{V}$ is a torsion-free grading, then the grading group of the universal realization of $\mathcal{V}$ is isomorphic to some $\mathbb{Z}^k$.*

Proof. Let $\widetilde{\mathcal{V}}$ be the universal realization of $\mathcal{V}$. By Algorithm 2.13, $\widetilde{\mathcal{V}}$ is a $\mathbb{Z}^n/\langle R\rangle$-grading for some subset $R \subset \mathbb{Z}^n$. The quotient $\mathbb{Z}^n/\langle R\rangle$ is isomorphic to a group $\mathbb{Z}^k \times G_t$, where G_t is some torsion group.

By assumption there exists a realization of $\mathcal{V}$ as an A-grading with A torsion-free. Since the image of G_t under a homomorphism must vanish in A, we conclude that there are no non-zero weights in G_t. Since a universal realization is generated by its weights, we conclude that $G_t = 0$, and $\widetilde{\mathcal{V}}$ is a $\mathbb{Z}^k$-grading. □

The following lemma is a part of [Koc09, Proposition 3.15], and we record it for later usage.

Lemma 2.15. *If a grading $\mathcal{V}$ is a coarsening of a grading $\mathcal{W}$, then every realization of $\mathcal{V}$ is a push-forward grading of the universal realization of $\mathcal{W}$.*

2.3. **Gradings induced by tori.** In this subsection we describe the correspondence between gradings of a Lie algebra $\mathfrak{g}$ and the split tori of its derivation algebra $\mathrm{der}(\mathfrak{g})$. In general, gradings of a Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with algebraic quasitori, see [Koc09, Section 4]. However, in this study we are only interested in cases when $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic zero and the gradings are over torsion-free abelian groups. In this setting, the characterization of gradings in terms of algebraic quasitori can be reduced to studying algebraic subtori of the derivation algebra $\mathrm{der}(\mathfrak{g})$.

For computational reasons, we will drop the algebraicity requirement for the subalgebras of $\mathrm{der}(\mathfrak{g})$. This means we lose the one-to-one correspondence described in [Koc09], but the less restrictive definition will be sufficient for our purposes. In particular, it will simplify the explicit construction of maximal gradings in terms of tori, see Subsection 3.3.

We start by defining split tori and gradings induced by them in the sense of [Fav73].

Definition 2.16. An abelian subalgebra $\mathfrak{t}$ of semisimple derivations of $\mathfrak{g}$ is called a *torus* of $\mathrm{der}(\mathfrak{g})$. If the torus $\mathfrak{t}$ is diagonalizable over the base field of $\mathfrak{g}$, it is called a *split torus*.

Lemma 2.17. *Let $\mathfrak{t}$ be a split torus of $\mathrm{der}(\mathfrak{g})$ and let $\mathfrak{t}^*$ be its dual as a vector space. For each $\alpha \in \mathfrak{t}^*$ define the subspace*

$$V_\alpha = \{X \in \mathfrak{g} : \delta(X) = \alpha(\delta)X \, \forall \delta \in \mathfrak{t}\}.$$

Then $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} V_\alpha$ is a $\mathfrak{t}^*$-grading.

Proof. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ that diagonalizes $\mathfrak{t}$. Since each vector X_i is an eigenvector of every derivation $\delta \in \mathfrak{t}$, there are well defined linear maps $\alpha_1, \dots, \alpha_n \in \mathfrak{t}^*$ determined by

$$\delta(X_i) = \alpha_i(\delta)X_i, \quad i = 1, \dots, n.$$

By construction $X_i \in V_{\alpha_i}$, so the direct sum $\bigoplus_{\alpha\in\mathfrak{t}^*} V_\alpha$ spans all of the Lie algebra $\mathfrak{g}$. The inclusion $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$ follows by linearity from the Leibniz rule $\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)]$ for all derivations $\delta \in \mathfrak{t}$ and vectors $X \in V_\alpha$ and $Y \in V_\beta$. $\square$

Definition 2.18. The $\mathfrak{t}^*$-grading of $\mathfrak{g}$ defined in Lemma 2.17 is called the *grading induced by the split torus* $\mathfrak{t}$.

See Example 3.13 for some gradings induced by tori in the Heisenberg Lie algebra.

For the purposes of Subsection 2.4, we need the following two lemmas. In Lemma 2.19 we link equivalences and push-forwards of gradings to relations between the inducing tori.

Lemma 2.19. *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two split tori of $\mathrm{der}(\mathfrak{g})$ with respective induced $\mathfrak{t}_1^*$-grading $\mathcal{V}$ and $\mathfrak{t}_2^*$-grading $\mathcal{W}$.*

(i) If there exists an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\Phi \circ \mathfrak{t}_1 \circ \Phi^{-1} = \mathfrak{t}_2$, then $\mathcal{V}$ and $\mathcal{W}$ are equivalent.

(ii) If $\mathfrak{t}_1 \subset \mathfrak{t}_2$, then there exists a homomorphism f so that $\mathcal{V} = f_*\mathcal{W}$.

Proof. To show (i), suppose that $\mathrm{Ad}_\Phi \mathfrak{t}_1 = \Phi \circ \mathfrak{t}_1 \circ \Phi^{-1} = \mathfrak{t}_2$ for some automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$. Let $f\colon \mathfrak{t}_1^* \to \mathfrak{t}_2^*$ be the linear isomorphism $f = \mathrm{Ad}^*_{\Phi^{-1}}$ given by $f(\alpha)(\delta) = \alpha(\Phi^{-1} \circ \delta \circ \Phi)$. Then

$$\begin{aligned}\Phi(V_\alpha) &= \{\Phi(X) : \delta(X) = \alpha(\delta)X \,\forall \delta \in \mathfrak{t}_1\}\\ &= \{Y : \Phi \circ \delta \circ \Phi^{-1}(Y) = \alpha(\delta)Y \,\forall \delta \in \mathfrak{t}_1\}\\ &= \{Y : \eta(Y) = f(\alpha)(\eta)Y \,\forall \eta \in \mathfrak{t}_2\} = W_{f(\alpha)}.\end{aligned}$$

Hence the gradings $\mathcal{V}$ and $\mathcal{W}$ are equivalent, as claimed.

Regarding (ii), suppose that $\mathfrak{t}_1 \subset \mathfrak{t}_2$. We claim that $\mathcal{V} = g_*\mathcal{W}$ through the restriction map $g\colon \mathfrak{t}_2^* \to \mathfrak{t}_1^*$, $g(\beta) = \beta|_{\mathfrak{t}_1}$. Indeed, fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ that diagonalizes the split torus $\mathfrak{t}_2$ (and hence also the subtorus $\mathfrak{t}_1$). Let $\beta_1, \ldots, \beta_n \in \mathfrak{t}_2^*$ be the maps defined by $\delta(X_i) = \beta_i(\delta)X_i$ and define $\alpha_i = \beta_i|_{\mathfrak{t}_1}$. By construction $X_i \in W_{\beta_i}$, $X_i \in V_{\alpha_i}$, and $g(\beta_i) = \alpha_i$, proving that $\mathcal{V} = g_*\mathcal{W}$. □

Finally, we observe that any torsion-free grading has a realization induced by a split torus.

Lemma 2.20. *Let $\mathcal{V}$ be a torsion-free grading. Then there exists a split torus $\mathfrak{t}$ whose induced $\mathfrak{t}^*$-grading is a realization of $\mathcal{V}$.*

Proof. Let $\mathcal{V}\colon \mathfrak{g} = \bigoplus_{\alpha \in A} V_\alpha$ be a realization of $\mathcal{V}$ over a torsion-free abelian group A and let A^* be the space of homomorphisms $A \to F$, where F is the base field of $\mathfrak{g}$. By reducing to the subgroup generated by the weights, we may assume A is isomorphic to $\mathbb{Z}^m$ for some $m \geq 1$. For each $\varphi \in A^*$ define the linear map

$$\delta_\varphi\colon \mathfrak{g} \to \mathfrak{g}, \quad \delta_\varphi(X) = \varphi(\alpha)X \quad \forall X \in V_\alpha.$$

We claim that $\mathfrak{t} = \{\delta_\varphi : \varphi \in A^*\}$ is a split torus that induces a realization for $\mathcal{V}$. Indeed, a direct computation shows that all the maps δ_φ are derivations. They are diagonalizable since by construction they are multiples of the identity on each layer V_α. Hence $\mathfrak{t}$ is a split torus.

Let then $\mathcal{W} : \mathfrak{g} = \bigoplus_{\beta \in \mathfrak{t}^*} W_\beta$ be the $\mathfrak{t}^*$-grading induced by $\mathfrak{t}$. Denote by Ω the set of weights of $\mathcal{V}$, and define a map $f\colon \Omega \to \mathfrak{t}^*$ by $f(\alpha)(\delta_\varphi) = \varphi(\alpha)$. Then f is well-defined: if $\varphi, \phi \in A^*$ are such that $\delta_\varphi = \delta_\phi$, then by the definition of $\mathfrak{t}$ we have $\varphi(\alpha) = \phi(\alpha)$ for all weights $\alpha \in \Omega$.

First, we show that $V_\alpha \subset W_{f(\alpha)}$ for every $\alpha \in A$. By the construction of the torus $\mathfrak{t}$, for each $X \in V_\alpha$ we have that

$$\delta_\varphi(X) = \varphi(\alpha)X = f(\alpha)(\delta_\varphi)X \quad \forall \delta_\varphi \in \mathfrak{t}.$$

By the definition of the grading $\mathcal{W}$, we then have $X \in W_{f(\alpha)}$ and so $V_\alpha \subset W_{f(\alpha)}$.

Next, we show that the map f is injective, which would prove that $V_\alpha = W_{f(\alpha)}$ for all $\alpha \in \Omega$ and so $\mathcal{W}$ would be a realization of $\mathcal{V}$, as claimed. Note that since A is isomorphic to $\mathbb{Z}^m$, for every non-zero $\alpha \in A$ there exists a homomorphism $\varphi \in A^*$ such that $\varphi(\alpha) \neq 0$. Therefore, if $\alpha, \alpha' \in \Omega$ are such that $f(\alpha) = f(\alpha')$, then by the construction of the map f we have

$$\varphi(\alpha - \alpha') = \varphi(\alpha) - \varphi(\alpha') = f(\alpha)(\delta_\varphi) - f(\alpha')(\delta_\varphi) = 0$$

for every homomorphism $\varphi\colon A \to F$. So $\alpha = \alpha'$ and f is injective, proving that $\mathcal{W}$ is a realization of $\mathcal{V}$. □

2.4. **Maximal gradings.** We now present the notion of maximal grading using maximal split tori and prove that a maximal grading has the universal property of push-forwards (see Proposition 2.23). The formulation through the derivation algebra will be convenient in the construction of maximal grading in Subsection 3.3. The universal property will be exploited in Subsection 2.5 where we give a method to construct all gradings over torsion-free abelian groups of a Lie algebra from a given maximal grading.

Definition 2.21. Let $\mathfrak{g}$ be a Lie algebra. A *maximal grading* of $\mathfrak{g}$ is the universal realization of the grading induced by a maximal split torus of $\mathrm{der}(\mathfrak{g})$.

Remark 2.22. The maximal grading of a Lie algebra is unique up to equivalence, since maximal split tori are all conjugate (see for instance, [Spr09, Theorem 15.2.6.]). Indeed, by Lemma 2.19(i) the conjugacy implies that any two maximal split tori induce equivalent gradings, so also their universal realizations are equivalent.

Proposition 2.23. *Let $\mathcal{W}$ be a maximal grading of $\mathfrak{g}$ and $\mathcal{V}$ a grading of $\mathfrak{g}$. Then every torsion-free realization of $\mathcal{V}$ is a push-forward of $\mathcal{W}$.*

Proof. Let $\mathcal{V}'$ be the realization of $\mathcal{V}$ as a $\mathfrak{t}^*$-grading induced by a split torus $\mathfrak{t}$ given by Lemma 2.20. Let also $\mathfrak{t}' \supset \mathfrak{t}$ be a maximal split torus in $\mathrm{der}(\mathfrak{g})$ with the induced grading $\mathcal{W}'$. By Lemma 2.19.(ii), the grading $\mathcal{V}'$ is a push-forward of $\mathcal{W}'$. In particular, $\mathcal{V}$ is a coarsening of $\mathcal{W}'$.

Since the maximal grading is unique up to equivalence by Remark 2.22, we may assume that $\mathcal{W}$ is the universal realization of $\mathcal{W}'$. Therefore, by Lemma 2.15 every realization of $\mathcal{V}$ is a push-forward grading of $\mathcal{W}$. □

Remark 2.24. It follows from Proposition 2.23 and the discussion in [Koc09, Section 3.6] that maximal gradings are universal realizations of fine gradings. In [Cor16, Definition 3.18], maximal gradings are defined as the gradings induced by maximal split tori in the automorphism group $\mathrm{Aut}(\mathfrak{g})$. [Cor16, Proposition 3.20] states that maximal gradings in the sense of [Cor16] have a universal property equivalent to Proposition 2.23, so by Lemma 2.11 any such grading is maximal also in the sense of Definition 2.21. The maximal gradings considered in [Fav73] are the gradings induced by maximal split tori.

2.5. **Enumeration of torsion-free gradings.** Following the method suggested in [Koc09, Section 3.7], we now give a simple way to enumerate a complete (and finite) set of universal realizations of gradings of a Lie algebra using the maximal grading. This reduces the proof of Theorem 1.4 to the construction of a maximal grading, which we cover in Subsection 3.3.

For the rest of this section, let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{W} : \mathfrak{g} = \bigoplus_{n\in\mathbb{Z}^k} W_n$ be a maximal grading of $\mathfrak{g}$ with weights Ω. Denote by $\Omega-\Omega$ the difference set $\Omega-\Omega = \{n-m \mid n, m \in \Omega\}$. For a subset $I \subset \Omega-\Omega$, let

$$\pi_I \colon \mathbb{Z}^k \to \mathbb{Z}^k/\langle I\rangle$$

be the canonical projection. We define the finite set

$$\Gamma = \{(\pi_I)_*\mathcal{W} \mid I \subset \Omega-\Omega,\ \mathbb{Z}^k/\langle I\rangle \text{ is torsion-free}\}.$$

Proposition 2.25. *The set Γ is, up to equivalence, a complete set of universal realizations of torsion-free gradings of $\mathfrak{g}$.*

Proof. Let $\mathcal{V}$ be the universal realization of some torsion-free grading. Due to Lemma 2.14, the grading group of $\mathcal{V}$ is some $\mathbb{Z}^m$. By Proposition 2.23, there exists a homomorphism $f\colon \mathbb{Z}^k \to \mathbb{Z}^m$ and an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathcal{V} = f_*\Phi(\mathcal{W})$. Let

$$I = \ker(f) \cap (\Omega - \Omega).$$

We are going to show that $\mathcal{V}' = (\pi_I)_*(\mathcal{W})$ is equivalent to $\mathcal{V}$. Then, a posteriori, $\mathbb{Z}^k/\langle I\rangle$ is torsion-free and we have $\mathcal{V}' \in \Gamma$, proving the claim.

First, since $\ker(\pi_I) = \langle I\rangle \subseteq \ker(f)$, by the universal property of quotients there exists a unique homomorphism $\phi\colon \mathbb{Z}^k/\langle I\rangle \to \mathbb{Z}^m$ such that $f = \phi \circ \pi_I$. In particular,

$$\mathcal{V} = f_*\Phi(\mathcal{W}) = \phi_*(\pi_I)_*\Phi(\mathcal{W}) = \phi_*\Phi(\mathcal{V}'),$$

so $\mathcal{V}$ is a push-forward grading of $\mathcal{V}'$.

Secondly, since also $\ker(f) \cap (\Omega-\Omega) = I \subseteq \ker(\pi_I) \cap (\Omega-\Omega)$, we deduce that $\mathcal{V}$ and $\Phi(\mathcal{V}')$ are realizations of the same grading. Since $\mathcal{V}$ is a universal realization, it follows that $\Phi(\mathcal{V}')$ is a push-forward grading of $\mathcal{V}$. Consequently, $\mathcal{V}'$ is a push-forward grading of $\mathcal{V}$. Since the grading group of a universal realization is generated by the weights, we get that the gradings $\mathcal{V}$ and $\mathcal{V}'$ are equivalent by Lemma 2.11, as wanted. □

Notice that some of the $\mathbb{Z}^k/\langle I\rangle$-gradings in Γ are typically equivalent to each other. From the classification point of view, a more challenging task is to determine the equivalence classes once the set Γ is obtained. In low dimensions, naive methods are enough to separate non-equivalent gradings, and for equivalent ones the connecting automorphism can be found rather easily.

In [HKMT20] we give a representative from each equivalence class of Γ for every 6-dimensional nilpotent Lie algebra over $\mathbb{C}$ and for an extensive class of 7-dimensional Lie algebras over $\mathbb{C}$. The results and the methods for distinguishing the equivalence classes of the obtained gradings are described in more detail in Subsection 4.2.

3. Constructions

3.1. Stratifications.

Definition 3.1. A *stratification* (a.k.a. *Carnot grading*) is a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} V_n$ such that V_1 generates $\mathfrak{g}$ as a Lie algebra. A Lie algebra $\mathfrak{g}$ is *stratifiable* if it admits a stratification.

In this section we show that constructing a stratification for a Lie algebra (or determining that one does not exist) is a linear problem and, consequently, prove Theorem 1.2. Our method is based on [Cor16, Lemma 3.10], which gives the following characterization of stratifiable Lie algebras:

Lemma 3.2. *A nilpotent Lie algebra $\mathfrak{g}$ is stratifiable if and only if there exists a derivation $\delta\colon \mathfrak{g} \to \mathfrak{g}$ such that the induced map $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is the identity map. Moreover, a stratification is given by the layers $V_i = \ker(\delta - i)$.*

The condition of Lemma 3.2 is straightforward to check in a basis adapted to the lower central series.

Definition 3.3. The lower central series of a Lie algebra $\mathfrak{g}$ is the decreasing sequence of subspaces

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \mathfrak{g}^{(3)} \supset \cdots,$$

where $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$. A basis $X_1, \dots, X_n$ of a Lie algebra $\mathfrak{g}$ is *adapted to the lower central series* if for every non-zero $\mathfrak{g}^{(i)}$ there exists an index $n_i \in \mathbb{N}$ such that $X_{n_i}, \dots, X_n$ is a basis of $\mathfrak{g}^{(i)}$. The *degree* of the basis element X_i is the integer $w_i = \max\{j \in \mathbb{N} : X_i \in \mathfrak{g}^{(j)}\}$.

Proposition 3.4. *Let $X_1, \dots, X_n$ be a basis adapted to the lower central series of a nilpotent Lie algebra $\mathfrak{g}$ defined over a field F. Let $w_1, \dots, w_n$ be the degrees of the basis elements and let $c_{ij}^k \in F$ be the structure coefficients in the basis. A linear map $\delta\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation that restricts to the identity on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ if and only if*

$$\delta(X_i) = w_i X_i + \sum_{w_j > w_i} a_{ij} X_j \tag{2}$$

such that, for each triple of indices i, j, k such that $w_k > w_i + w_j$, the coefficients $a_{ij} \in F$ satisfy the linear equation

$$c_{ij}^k(w_k - w_i - w_j) = \sum_{w_i < w_h \leq w_k - w_j} a_{ih} c_{hj}^k + \sum_{w_j < w_h \leq w_k - w_i} a_{jh} c_{ih}^k - \sum_{w_i + w_j \leq w_h < w_k} c_{ij}^h a_{hk}. \tag{3}$$

Proof. If $\delta \colon \mathfrak{g} \to \mathfrak{g}$ is a derivation that restricts to the identity on $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, then by Lemma 3.2 $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$$

such that $\delta|_{V_i} = i \cdot \mathrm{id}$. Since the terms of the lower central series are given in terms of the stratification as $\mathfrak{g}^{(i)} = V_i \oplus \cdots \oplus V_s$, it follows that $\delta(Y) \in i \cdot Y + \mathfrak{g}^{(i+1)}$ for any $Y \in \mathfrak{g}^{(i)}$. That is, a derivation δ restricting to the identity on $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is of the form (2) for some coefficients $a_{ij} \in F$.

It is then enough to show that (3) is equivalent to the Leibniz rule

$$\delta([X_i, X_j]) = [\delta(X_i), X_j] + [X_i, \delta(X_j)], \quad \forall i, j \in \{1, \ldots, n\}.$$

Indeed, this would prove that a linear map defined by (2) is a derivation if and only if the coefficients a_{ij} satisfy the linear system (3).

Since the basis X_i is adapted to the lower central series, only the structure coefficients with large enough degrees are non-zero, i.e., we have

$$[X_i, X_j] = \sum_{w_k \geq w_i + w_j} c_{ij}^k X_k. \tag{4}$$

By direct computation using (2) and (4) we get the expressions

$$\begin{aligned}
[\delta(X_i), X_j] &= \sum_{w_k \geq w_i + w_j} c_{ij}^k w_i X_k + \sum_{w_h > w_i} \sum_{w_k \geq w_h + w_j} a_{ih} c_{hj}^k X_k \\
[X_i, \delta(X_j)] &= \sum_{w_k \geq w_i + w_j} c_{ij}^k w_j X_k + \sum_{w_h > w_j} \sum_{w_k \geq w_i + w_h} a_{jh} c_{ih}^k X_k \\
\delta([X_i, X_j]) &= \sum_{w_k \geq w_i + w_j} c_{ij}^k w_k X_k + \sum_{w_h \geq w_i + w_j} \sum_{w_k > w_h} c_{ij}^h a_{hk} X_k
\end{aligned}$$

Denoting $\sum_k B_{ij}^k X_k = \delta([X_i, X_j]) - [\delta(X_i), X_j] - [X_i, \delta(X_j)]$, we find that the equation $B_{ij}^k = 0$ is up to reorganizing terms equivalent to (3).

Finally, we observe that when $w_k \leq w_i + w_j$, the condition $B_{ij}^k = 0$ is automatically satisfied: for $w_k < w_i + w_j$ all of the sums are empty, and for $w_k = w_i + w_j$, the only remaining terms from the sums cancel out as

$$B_{ij}^k = c_{ij}^k w_k - c_{ij}^k w_i - c_{ij}^k w_j = 0. \qquad \square$$

The concrete criterion of Proposition 3.4 provides the algorithm of Theorem 1.2.

Algorithm 3.5 (Stratification)**.** *Input: A nilpotent Lie algebra* $\mathfrak{g}$. *Output: A stratification of* $\mathfrak{g}$ *or the non-existence of one.*

(1) *Construct a basis* $X_1, \ldots, X_n$ *adapted to the lower central series.*
(2) *Find a derivation* δ *as in (2) solving the linear system (3). If the system has no solutions, then* $\mathfrak{g}$ *is not stratifiable.*
(3) *Return the stratification with the layers* $V_i = \ker(\delta - i)$.

3.2. **Positive gradings.**

Definition 3.6. An $\mathbb{R}$-grading $\mathcal{V}\colon \mathfrak{g} = \bigoplus_{\alpha \in \mathbb{R}} V_\alpha$ is *positive* if $\alpha > 0$ for all the weights of $\mathcal{V}$. If such a grading exists for $\mathfrak{g}$, then $\mathfrak{g}$ is said to be *positively gradable.*

One of our main goals is to determine when a grading admits a positive realization, i.e., can be realized as a positive grading. A characterization is given in [Cor16, Proposition 3.22]. In the lemma and proposition below, we provide constructive proofs for this characterization.

Lemma 3.7. *Let* $m \geq 1$ *and let* $\mathcal{V}$ *be a* $\mathbb{Z}^m$*-grading of a Lie algebra* $\mathfrak{g}$. *Suppose the convex hull of the set of weights of* $\mathcal{V}$ *does not contain the origin. Then there exists a homomorphism* $f\colon \mathbb{Z}^m \to \mathbb{Z}$ *whose restriction on the weights is injective and positive.*

Proof. Let us consider the natural embedding of $\mathbb{Z}^m$ into $\mathbb{Q}^m$. Using the canonical inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{Q}^m$, we define for each vector $v \in \mathbb{Q}^m$, the corresponding open half-space

$$M_v = \{x \in \mathbb{Q}^m : \langle v, x \rangle > 0\}.$$

Denote by $\Omega = \{\alpha_1, \ldots, \alpha_N\}$ the set of weights of $\mathcal{V}$. Recall that the convex hull of a set is the intersection of all the affine half-spaces containing the set. Hence, because the convex hull of Ω does not contain the origin, it is contained in an open half-space $M_{v_0} \subset \mathbb{Q}^m$. Moreover, there exists a neighborhood B of v_0 such that the convex hull of Ω is contained in every half-space M_v with $v \in B$.

By construction all the inner products $\langle v, \alpha_i \rangle$ with $\alpha_i \in \Omega$ and $v \in B$ are strictly positive. Since B has non-empty interior, we may choose some $v \in B$ such that all the numbers $\langle v, \alpha_1 \rangle, \ldots, \langle v, \alpha_N \rangle$ are strictly positive and distinct. Rescaling v to eliminate denominators, we obtain a vector $\tilde{v} \in \mathbb{Z}^m$, and the map $f(\cdot) = \langle \tilde{v}, \cdot \rangle$ is the required homomorphism $\mathbb{Z}^m \to \mathbb{Z}$. Concretely, a valid vector $\tilde{v}$ can be found directly by just enumerating the points of $\mathbb{Z}^k$ with increasing distance from the origin and testing one by one if all the inner products with the weights are positive and distinct. ☐

Proposition 3.8. *Let* $\mathcal{W}$ *be a torsion-free grading. Then* $\mathcal{W}$ *admits a positive realization if and only if the convex hull of the set of weights of the universal realization of* $\mathcal{W}$ *does not contain the origin.*

Proof. We only need to prove the forward implication due to Lemma 3.7. Let $\mathcal{V}$ be a positive realization of $\mathcal{W}$ and let $\widetilde{\mathcal{W}}$ be the universal realization of $\mathcal{W}$, which by Lemma 2.14 is a $\mathbb{Z}^k$-grading. Then by the definition of a universal realization there is a homomorphism $f\colon \mathbb{Z}^k \to \mathbb{R}$ such that $\mathcal{V} = f_*\widetilde{\mathcal{W}}$. Consider the vector $v = (f(e_1), \dots, f(e_k)) \in \mathbb{R}^k$, where $e_1, \dots, e_k$ are the standard basis vectors of $\mathbb{Z}^k$, and express f as $f(\cdot) = \langle v, \cdot \rangle$. Since $\mathcal{V}$ is a positive grading, then for all weights α of $\widetilde{\mathcal{W}}$ we have $f(\alpha) > 0$, that is, $\langle v, \alpha \rangle > 0$. Hence all the weights belong to the open half-space determined by the vector v, and so the origin is not contained in their convex hull. $\square$

The above results give the following algorithm for Theorem 1.5.(i). We stress that we do not need to assume that the base field of $\mathfrak{g}$ is algebraically closed.

Algorithm 3.9 (Positive realization)**.** *Input: A torsion-free grading $\mathcal{V}$ for a Lie algebra $\mathfrak{g}$. Output: A positive realization of $\mathcal{V}$ or the nonexistence of one.*

(1) Compute the universal realization $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ using Algorithm 2.13. Let $\mathbb{Z}^k$ be the grading group of $\widetilde{\mathcal{V}}$.
(2) If the convex hull of the weights of $\widetilde{\mathcal{V}}$ contains the origin, then no positive realization exists.
(3) Otherwise, find a vector $v \in \mathbb{Z}^k$ so that the homomorphism $f\colon \mathbb{Z}^k \to \mathbb{Z}$, $f(\cdot) = \langle v, \cdot \rangle$ maps all weights of $\widetilde{\mathcal{V}}$ to distinct positive integers.
(4) Return the push-forward grading $f_*\widetilde{\mathcal{V}}$.

The algorithm for Theorem 1.5.(ii) is somewhat similar to Algorithm 3.9. Suppose we have an S-grading $\mathcal{V}$ for a finitely generated abelian group S, and we want to find a homomorphism $S \to \mathbb{R}$ turning it into a positive grading. If some element of S has torsion, then no such homomorphism can exist. Otherwise, S is isomorphic to $\mathbb{Z}^k$ for some $k \geq 1$ and we may follow steps 2-3 of the above algorithm to obtain a homomorphism $S \to \mathbb{R}$ giving a positive realization if one exists.

Remark 3.10. The argument of Lemma 3.7 can also be used to find a realization over $\mathbb{Z}$ for any torsion-free grading $\mathcal{V}$. Indeed, first consider the universal realization of $\mathcal{V}$ over some $\mathbb{Z}^k$. Then disregarding the discussion about half-spaces and positivity, find a push-forward to $\mathbb{Z}$ by constructing a vector v for which the projection is injective on the weights.

Remark 3.11. The results we have established can also be used to explicitly enumerate the positive gradings of a given Lie algebra $\mathfrak{g}$ over an algebraically closed field, in two different senses.

(i) Consider the maximal grading $\mathcal{V}$ of $\mathfrak{g}$ over $\mathbb{Z}^k$. Up to automorphism, positive gradings of $\mathfrak{g}$ are given by the projections from $\mathbb{Z}^k$ to $\mathbb{R}$ mapping the weights of $\mathcal{V}$ to strictly positive numbers. A parametrisation of these projections gives a parametrisation of positive gradings.

(ii) Construct all the gradings of $\mathfrak{g}$ as in Proposition 2.25 (using a maximal grading constructed in Algorithm 3.12). Then check one by one which of them admit positive realizations. This produces a finite list of positive gradings so that every positive grading of $\mathfrak{g}$ is equivalent in the sense of Definition 2.9 to one of the elements on the list.

The algorithm of Theorem 1.3 is given by applying Algorithm 3.9 to the maximal grading of a Lie algebra. Indeed, if some grading admits a positive realization, then by Proposition 2.23 the maximal grading admits a positive realization as well. For the maximal grading, a positive realization if one exists is given by Algorithm 3.9.

The existence of a positive realization of a grading can also be phrased as the existence of a solution to a linear system. This viewpoint gives rise to an alternate elementary algorithm to determine whether a positive realization exists, as we explain in Appendix A.

3.3. **Maximal gradings.** In this section we prove Theorem 1.1 by providing an algorithm to construct a maximal grading for a Lie algebra $\mathfrak{g}$ defined over an algebraically closed field of characteristic zero. In this setting, every torus is split. The method we use to compute maximal gradings is the following.

Algorithm 3.12 (Maximal grading)**.** *Input: A Lie algebra $\mathfrak{g}$ over an algebraically closed field F. Output: A maximal grading of $\mathfrak{g}$.*

(1) Compute a basis for the derivation algebra $\operatorname{der}(\mathfrak{g})$*. Set $B = \emptyset$.*
(2) Determine the $\mathfrak{t}^*$-grading $\mathcal{V} : \mathfrak{g} = \bigoplus_\lambda V_\lambda$ induced by the torus $\mathfrak{t} = \langle B \rangle$.
(3) Compute a basis $A_1, \ldots, A_n$ for the centralizer $C(\mathfrak{t}) \subset \operatorname{der}(\mathfrak{g})$.
(4) Compute the adjoint representation $\operatorname{ad}\colon C(\mathfrak{t}) \to \bigoplus_\lambda \mathfrak{gl}(\mathfrak{gl}(V_\lambda))$*.*
(5) Compute a basis $K_1, \ldots, K_m$ for $\ker(\operatorname{ad}) \subset C(\mathfrak{t})$*. If $K_i \notin \mathfrak{t}$ for some $i = 1, \ldots, m$, extend $\mathfrak{t}$ by K_i and go back to step 2.*
(6) Repeat for each $A = A_i$ and $A = A_i + A_j$, $i, j = 1, \ldots, n$: compute the Jordan decomposition $A = A_s + A_n$. If $A_s \notin \mathfrak{t}$, extend $\mathfrak{t}$ by A_s and go back to step 2.
(7) Compute and return the universal realization of the grading $\mathcal{V}$.

The rest of the section is devoted to proving the correctness of Algorithm 3.12 and to explaining the steps in more detail. Step 6 is the most involved part.

Step 1 is straightforward linear algebra. In step 2, the grading induced by the torus $\mathfrak{t}$ has a concrete description in terms of a fixed basis of $\mathfrak{t}$. Namely, a basis $\delta_1, \dots, \delta_k$ defines an isomorphism $\mathfrak{t}^* \to F^k$ and hence an equivalent push-forward grading over F^k. Expanding out the construction of Lemma 2.17 shows that the push-forward grading has the layers

$$V_\lambda = V_{(\lambda_1,\dots,\lambda_k)} = \bigcap_{i=1}^{k} E_{\delta_i}^{\lambda_i},$$

where $E_{\delta_i}^{\lambda_i}$ is the (possibly zero) eigenspace for the eigenvalue λ_i of the derivation δ_i.

Step 3 is another straightforward linear algebra computation. In step 4, the key observation is that any linear map $A \in C(\mathfrak{t})$ preserves the eigenspaces of all the derivations $\delta \in \mathfrak{t}$. Hence such a linear map A also preserves the layers V_λ of the F^k-grading. It follows that each map $\mathrm{ad}(A)$ restricts to a linear map $\mathrm{ad}(A)\colon \mathfrak{gl}(V_\lambda) \to \mathfrak{gl}(V_\lambda)$ for each weight λ. The direct sum of these representations gives the representation $\mathrm{ad}\colon C(\mathfrak{t}) \to \bigoplus_\lambda \mathfrak{gl}(\mathfrak{gl}(V_\lambda))$.

Step 5 captures the situation when the torus $\mathfrak{t}$ can be extended without refining the grading. Indeed, the elements of the kernel of ad are the elements $A \in C(\mathfrak{t})$ whose restrictions commute with all other maps in $\mathfrak{gl}(V_\lambda)$ for each weight λ. That is, they are the maps $A \in C(\mathfrak{t})$ such that each $A|_{V_\lambda}$ is a multiple of the identity. The eigenspaces of such maps are sums of the layers V_λ, so they do not further refine the grading induced by $\mathfrak{t}$, as seen in the following example.

Example 3.13. Let $\mathfrak{h}$ be the Heisenberg Lie algebra with the only bracket $[X, Y] = Z$. Consider the derivation

$$\delta\colon \mathfrak{h} \to \mathfrak{h}, \quad \delta(X) = X, \quad \delta(Y) = 2Y, \quad \delta(Z) = 3Z.$$

The grading induced by δ is $\mathfrak{h} = V_1 \oplus V_2 \oplus V_3 = \langle X\rangle \oplus \langle Y\rangle \oplus \langle Z\rangle$.

The centralizer of δ in $\mathrm{der}(\mathfrak{h})$ is the two-dimensional space $C(\delta) = \langle \delta_1, \delta_2\rangle$, where the two basis derivations are defined by

$$\begin{aligned} &\delta_1(X) = X, && \delta_1(Y) = 0, && \delta_1(Z) = Z,\\ &\delta_2(X) = 0, && \delta_2(Y) = Y, && \delta_1(Z) = Z. \end{aligned}$$

The one-dimensional Lie algebras $\mathfrak{gl}(V_i)$ are all abelian, so the adjoint representation $\mathrm{ad}\colon C(\delta) \to \mathfrak{gl}(\mathfrak{gl}(V_1)) \oplus \mathfrak{gl}(\mathfrak{gl}(V_2)) \oplus \mathfrak{gl}(\mathfrak{gl}(V_3))$ is just the zero map. Both $\{\delta, \delta_1\}$ and $\{\delta, \delta_2\}$ span strictly bigger tori than $\{\delta\}$, but neither torus further refines the original grading $\mathfrak{h} = V_1 \oplus V_2 \oplus V_3$: for instance, the grading induced by $\langle \delta, \delta_1\rangle$ is $V_{(1,1)} \oplus V_{(2,0)} \oplus V_{(3,1)} = \langle X\rangle \oplus \langle Y\rangle \oplus \langle Z\rangle$.

Step 6 is the most intricate part of Algorithm 3.12. To prove its correctness, we need to show that if $A_s \in \mathfrak{t}$ for all basis elements $A = A_i$ and all their sums $A = A_i + A_j$, then the torus $\mathfrak{t}$ is maximal. The proof is based on the efficient criterion of [dG17, Proposition 2.6.11]:

Lemma 3.14. *Let $\mathfrak{c}$ be a Lie algebra and let $X_1, \dots, X_n$ be a basis of $\mathfrak{c}$. If $\mathrm{ad}(X_i)$ is nilpotent for $1 \le i \le n$ and $\mathrm{ad}(X_i + X_j)$ is nilpotent for all $1 \le i < j \le n$, then $\mathrm{ad}(X)$ is nilpotent for all $X \in \mathfrak{c}$.*

To make use of the criterion Lemma 3.14, we also need the fact that the Jordan decomposition is preserved by the adjoint representation.

Lemma 3.15. *Let F be a field of characteristic zero. Let $A \in \mathfrak{gl}(n, F)$ be any linear map and $A = A_s + A_n$ its Jordan decomposition. Then $\mathrm{ad}(A) = \mathrm{ad}(A_s) + \mathrm{ad}(A_n)$ is the Jordan decomposition of the map $\mathrm{ad}(A)\colon \mathfrak{gl}(n, F) \to \mathfrak{gl}(n, F)$.*

Proof. By [dG17, Proposition 2.2.5], since the field F is perfect (as a field of characteristic zero), the adjoint map preserves both semisimplicity and nilpotency, so the map $\mathrm{ad}(A_s)$ is semisimple and the map $\mathrm{ad}(A_n)$ is nilpotent. Moreover, since the maps A_s and A_n commute, the Jacobi identity implies that the maps $\mathrm{ad}(A_s)$ and $\mathrm{ad}(A_n)$ also commute. The claim follows from the uniqueness of the Jordan decomposition. □

With the above results, we are able to conclude that if the semisimple parts of A_i and $A_i + A_j$ are contained in $\mathfrak{t}$ for all basis elements A_i, then the torus $\mathfrak{t}$ in step 6 of Algorithm 3.12 is maximal. First, by step 5 we have $\mathfrak{t} = \ker(\mathrm{ad})$ for the restricted adjoint representation $\mathrm{ad}\colon C(\mathfrak{t}) \to \bigoplus_\lambda \mathfrak{gl}(\mathfrak{gl}(V_\lambda))$ defined in step 4. Then for any $A \in C(\mathfrak{t})$ by Lemma 3.15 we find that $A_s \in \mathfrak{t}$ if and only if $\mathrm{ad}(A) = \mathrm{ad}(A_n)$, that is, if and only if $\mathrm{ad}(A)$ is nilpotent. By Lemma 3.14, if all $\mathrm{ad}(A_i)$ and $\mathrm{ad}(A_i + A_j)$ are nilpotent, then $\mathrm{ad}(A)$ is nilpotent for all $A \in C(\mathfrak{t})$. Hence $A_s \in \mathfrak{t}$ for all $A \in C(\mathfrak{t})$. In other words, no semisimple element $A_s \in C(\mathfrak{t}) \setminus \mathfrak{t}$ exists, so $\mathfrak{t}$ is maximal.

The final part of Algorithm 3.12 is step 7, where we replace the indexing by eigenvalues of the derivations of $\mathfrak{t}$ with indexing over some $\mathbb{Z}^k$ given by the universal realization. The precise method was described earlier in Algorithm 2.13. Since the construction of the first six steps of Algorithm 3.12 leads to a maximal torus of $\mathrm{der}(\mathfrak{g})$, by Definition 2.21 the output is a maximal grading of $\mathfrak{g}$.

Remark 3.16. The only part where we use the assumption that the base field is algebraically closed is in step 6. The significance of the assumption is that the Jordan decomposition and [dG17, Proposition 2.6.11] give us an efficient method to construct semisimple elements in $C(\mathfrak{t}) \setminus \mathfrak{t}$.

If the base field is not algebraically closed, we need to explicitly require that the constructed elements of $C(\mathfrak{t}) \setminus \mathfrak{t}$ are diagonalizable. The subset of diagonalizable elements of $C(\mathfrak{t}) \setminus \mathfrak{t}$ is a semialgebraic set, and constructions to extract points from such sets exist, see for instance [BPR06, Section 13] on the existential theory of the reals. The problem is that these methods are practical only in low dimensions, and the construction would be needed in dimension $\dim \mathfrak{gl}(\mathfrak{g}) = \dim(\mathfrak{g})^2$.

For Lie algebras defined over finite fields, more efficient randomized algorithms to find split tori exist, see [CM09] and [Roo13].

4. Applications

4.1. **Structure from maximal gradings.** In this subsection we show how maximal gradings may be used to find some structural information of Lie algebras. We start by studying how maximal gradings reveal the structure of a direct product. A similar result can be found in 1.6.5 of [Fav73].

Example 4.1. Consider the Lie algebra $L_{6,22}(1)$ in [CdGS12] with basis $\{X_1, \dots, X_6\}$, where the only non-zero bracket relations are

$$[X_1, X_2] = X_5, \ \ [X_1, X_3] = X_6, \ \ [X_2, X_4] = X_6, \ \ [X_3, X_4] = X_5.$$

In a basis $\{Y_1, \dots, Y_6\}$ adapted to the maximal grading, the bracket relations are

$$[Y_1, Y_2] = Y_3, \ \ [Y_4, Y_5] = Y_6.$$

From these bracket relations one sees more easily that the Lie algebra $L_{6,22}(1)$ is isomorphic to $L_{3,2} \times L_{3,2}$, where $L_{3,2}$ is the first Heisenberg Lie algebra.

We say that a split torus $\mathfrak{t} \subset \operatorname{der}(\mathfrak{g})$ is *non-degenerate* if the intersection of the kernels of the maps $D \in \mathfrak{t}$ is trivial. That is, a split torus is non-degenerate if and only if the $\mathfrak{t}^*$-grading it induces does not have zero as a weight.

We expect that the following result is known even without the non-degeneracy assumption, however we have been unable to locate a reference. We will therefore give a direct proof of the simpler claim.

Lemma 4.2. *Let* $\mathfrak{t}_1 \subset \operatorname{der}(\mathfrak{g}_1)$ *and* $\mathfrak{t}_2 \subset \operatorname{der}(\mathfrak{g}_2)$ *be non-degenerate maximal split tori. Then* $\mathfrak{t}_1 \times \mathfrak{t}_2$ *is a maximal split torus in* $\operatorname{der}(\mathfrak{g}_1 \times \mathfrak{g}_2)$.

Proof. Denoting $\mathfrak{t} = \mathfrak{t}_1 \times \mathfrak{t}_2$, let $D \in C(\mathfrak{t})$ be a diagonalizable derivation in the centralizer $C(\mathfrak{t})$. To show the maximality of $\mathfrak{t}$, it suffices to show that $D \in \mathfrak{t}$. In a basis adapted to the product we may represent

$$D = \begin{bmatrix} E_1 & F_1 \\ F_2 & E_2 \end{bmatrix},$$

where $E_1 \in \operatorname{der}(\mathfrak{g}_1)$, $E_2 \in \operatorname{der}(\mathfrak{g}_2)$, and $F_1 \colon \mathfrak{g}_2 \to \mathfrak{g}_1$ and $F_2 \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ are some linear maps. We are going to demonstrate that $E_1 \in \mathfrak{t}_1$, $E_2 \in \mathfrak{t}_2$ and $F_1 = F_2 = 0$, which would prove that $D = E_1 \times E_2 \in \mathfrak{t}$.

Let $D_1 \in \mathfrak{t}_1$. By assumption D commutes with $D_1 \times 0 \in \mathfrak{t}$, so a simple computation shows that E_1 commutes with D_1 and $D_1 F_1 = 0$. Since D_1 is arbitrary, we obtain $E_1 \in C(\mathfrak{t}_1)$. From the fact that $D_1 F_1 = 0$ for every $D_1 \in \mathfrak{t}_1$ we get

$$\operatorname{Im}(F_1) \subset \bigcap_{D_1 \in \mathfrak{t}_1} \ker(D_1) = \{0\},$$

where the last equality follows from the non-degeneracy of $\mathfrak{t}_1$. Consequently, $F_1 = 0$.

A similar argument shows that $E_2 \in C(\mathfrak{t}_2)$ and $F_2 = 0$. Since D is assumed diagonalizable, it follows that E_1 and E_2 are diagonalizable. Then by maximality of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ we have $E_1 \in \mathfrak{t}_1$ and $E_2 \in \mathfrak{t}_2$, which shows that $D = E_1 \times E_2 \in \mathfrak{t}$. $\square$

For gradings, the above lemma implies the following. Suppose $\mathcal{V} : \mathfrak{g}_1 = \bigoplus_{\alpha \in A} V_\alpha$ and $\mathcal{W} : \mathfrak{g}_2 = \bigoplus_{\beta \in B} W_\beta$ are maximal gradings of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and suppose zero is not a weight for either $\mathcal{V}$ or $\mathcal{W}$. Then

$$\mathcal{V} \times \mathcal{W} : \Big(\bigoplus_{(\alpha,0) \in A \times B} V_\alpha \times \{0\} \Big) \oplus \Big(\bigoplus_{(0,\beta) \in A \times B} \{0\} \times W_\beta \Big) \tag{5}$$

is a maximal grading of $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. Indeed, the gradings $\mathcal{V}$ and $\mathcal{W}$ are the universal realizations of gradings induced by the respective maximal split tori $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. By Lemma 4.2, the product torus $\mathfrak{t}_1 \times \mathfrak{t}_2$ is maximal. The universal realization of the grading induced by $\mathfrak{t}_1 \times \mathfrak{t}_2$ is equivalent to the product grading (5).

For a grading $\mathfrak{g} = \bigoplus_{\alpha \in \Omega} V_\alpha$, consider the graph with vertices Ω defined as follows: Whenever $0 \neq [V_\alpha, V_\beta] \subset V_\gamma$, we define edges between all the three vertices $\alpha, \beta, \gamma \in \Omega$. If the graph Ω admits a partition $\Omega = \Omega_1 \sqcup \Omega_2$ such that no edges exist between Ω_1 and Ω_2, then the Lie algebra $\mathfrak{g}$ is a direct product of the ideals $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Omega_1} V_\alpha$ and $\mathfrak{g}_2 = \bigoplus_{\beta \in \Omega_2} V_\beta$. In this situation we say the grading $\mathcal{V}$ *detects the product structure* $\mathfrak{g}_1 \times \mathfrak{g}_2$ of the Lie algebra $\mathfrak{g}$. We gather the observations made above into the following proposition.

Proposition 4.3. *If a Lie algebra $\mathfrak{g}$ is decomposable and the maximal gradings of the factor Lie algebras do not have zero as a weight, then the maximal grading of $\mathfrak{g}$ detects the product structure.*

We remark that while maximal gradings are able to detect product structures as indicated above, they are not able to detect some other algebraic properties. The Lie algebra $L_{6,24}(1)$ in [CdGS12] provides examples of two such phenomena. First, the layers of its maximal grading are not contained in the terms of its lower central series (this behavior can be also achieved by examples where the maximal grading is very coarse). Secondly, this Lie algebra has a "nice" basis (see [CR19] for the precise definition and its motivation), but it can be shown that no basis adapted to a maximal grading is nice.

Despite these negative results, maximal gradings have another structural application in simplifying the problem of deciding whether two Lie algebras are isomorphic or not.

Remark 4.4. If two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, then any isomorphism maps the maximal grading of $\mathfrak{g}_1$ to a maximal grading of $\mathfrak{g}_2$. Therefore, if the maximal gradings of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are given, then deciding if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic reduces to determining the existence of an isomorphism between the maximal gradings. In many cases this is significantly easier than naively solving the original isomorphism problem. For example, in low dimensions, the majority of the layers of the maximal grading are one-dimensional, in which case searching for possible isomorphisms becomes a combinatorial problem.

4.2. **Classification of gradings in low dimension.** Following the strategy outlined in [Koc09, Section 3.7], we classify torsion-free gradings, i.e., gradings that admit a torsion-free realization, in nilpotent Lie algebras of dimension up to 6 over $\mathbb{C}$. The main part of the classification is the construction of a maximal grading (Algorithm 3.12) and the enumeration of torsion-free gradings (Proposition 2.25). Here we will give a brief overview of the gradings of each Lie algebra.

We give a complete listing of the universal realizations for the 669 equivalence classes of gradings for the 46 complex Lie algebras of dimension up to 6 in [HKMT20]. We also include a similar listing for an extensive (but incomplete) family of 7 dimensional Lie algebras over $\mathbb{C}$. The listing in dimension 7 is incomplete because there are a few uncountable families of 7 dimensional complex Lie algebras that depend on a complex parameter λ. In these cases, following the study carried out in [Mag08], we focus on those singular values of λ for which either the Lie algebra cohomology or the adjoint cohomology have different dimensions compared to the rest of the Lie algebras in the same family. We will also include a few examples corresponding to non-singular values.

As a starting point we used the classifications of nilpotent Lie algebras given in [dG07] for dimensions less than 6, [CdGS12] for dimension 6, and [Gon98] for dimension 7. The classification up to dimension 6 has a pre-existing computer implementation in the GAP package [CdGSGT18]. However these Lie algebras are not always given in a basis adapted to any maximal grading, so we first compute the maximal grading using the methods described in Subsection 3.3 and switch to a basis adapted to the resulting grading.

The presentations we use for the nilpotent Lie algebras up to dimension 6 are listed in Table 1. The Lie brackets $[Y_a, Y_b] = Y_c$ are listed in the condensed form $ab = c$. Lie algebras $\mathfrak{g} \times \mathbb{R}^d$ with abelian factors have identical structure coefficients with the nonabelian factor $\mathfrak{g}$ and are omitted from the list. For example $L_{4,2} = L_{3,2} \times \mathbb{R}$ has the basis $Y_1, \dots, Y_4$ with the bracket relation $[Y_1, Y_2] = Y_3$ from $L_{3,2}$.

With all the maximal gradings computed, we enumerate all torsion-free gradings as in Proposition 2.25. For the classification up to equivalence, we first introduce some easy-to-check invariants for gradings. Recall that by Lemma 2.14, the grading groups of the obtained gradings are isomorphic to some groups $\mathbb{Z}^k$. The dimension k is called the

$L_{3,2}$	$12=3$					
$L_{4,3}$	$12=3$	$13=4$				
$L_{5,4}$	$41=5$	$23=5$				
$L_{5,5}$	$13=4$	$14=5$	$32=5$			
$L_{5,6}$	$12=3$	$13=4$	$14=5$	$23=5$		
$L_{5,7}$	$12=3$	$13=4$	$14=5$			
$L_{5,8}$	$12=3$	$14=5$				
$L_{5,9}$	$12=3$	$23=4$	$13=5$			
$L_{6,10}$	$23=4$	$51=6$	$24=6$			
$L_{6,11}$	$12=3$	$13=5$	$15=6$	$23=6$	$24=6$	
$L_{6,12}$	$23=4$	$24=5$	$31=6$	$25=6$		
$L_{6,13}$	$13=4$	$14=5$	$32=5$	$15=6$	$42=6$	
$L_{6,14}$	$12=3$	$13=4$	$14=5$	$23=5$	$25=6$	$43=6$
$L_{6,15}$	$12=3$	$13=4$	$14=5$	$23=5$	$15=6$	$24=6$
$L_{6,16}$	$12=3$	$13=4$	$14=5$	$25=6$	$43=6$	
$L_{6,17}$	$21=3$	$23=4$	$24=5$	$13=6$	$25=6$	
$L_{6,18}$	$12=3$	$13=4$	$14=5$	$15=6$		
$L_{6,19}(-1)$	$12=3$	$14=5$	$25=6$	$43=6$		
$L_{6,20}$	$12=3$	$14=5$	$15=6$	$23=6$		
$L_{6,21}(-1)$	$12=3$	$23=4$	$13=5$	$14=6$	$25=6$	
$L_{6,22}(0)$	$24=5$	$41=6$	$23=6$			
$L_{6,22}(1)$	$12=3$	$45=6$				
$L_{6,23}$	$12=3$	$14=5$	$15=6$	$42=6$		
$L_{6,24}(0)$	$13=4$	$34=5$	$14=6$	$32=6$		
$L_{6,24}(1)$	$12=3$	$23=5$	$24=5$	$13=6$		
$L_{6,25}$	$12=3$	$13=4$	$15=6$			
$L_{6,26}$	$12=3$	$24=5$	$14=6$			
$L_{6,27}$	$12=3$	$13=4$	$25=6$			
$L_{6,28}$	$12=3$	$23=4$	$13=5$	$15=6$		

TABLE 1. Lie algebras of dimension up to 6 over $\mathbb{C}$ in a basis adapted to a maximal grading.

rank of the grading. We recall also an invariant from [Koc09, Section 3.2]: The *type* of a grading is the tuple $(n_1, n_2, \dots, n_k)$, where k is the dimension of the largest layer, and each n_i is the number of i-dimensional layers.

From the full list of torsion-free gradings, we initially collect together gradings using the following criteria:

(1) The ranks of the gradings are equal.
(2) The types of the gradings are equal.
(3) There exists a homomorphism between the grading groups of the universal realizations mapping layers to layers of equal dimensions.

In this way we get for each Lie algebra families $I_1, I_2, \dots, I_k$ of gradings such that the gradings of I_i and I_j are not equivalent for $i \neq j$.

To compute the precise equivalence classes, we naively check if the gradings within each family I_i are equivalent. For each pair of an A-grading $\mathfrak{g} = \bigoplus_{\alpha \in A} V_\alpha$ and a B-grading $\mathfrak{g} = \bigoplus_{\beta \in B} W_\beta$, there are usually only a few homomorphisms $f\colon A \to B$ with $\dim V_\alpha = \dim W_{f(\beta)}$. For each such homomorphism f, we need to check whether there exists an automorphism $\Phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\Phi(V_\alpha) = W_{f(\beta)}$ for all weights α. These identities define a system of quadratic equations. Since we are working over an algebraically closed field, the system has no solution if and only if 1 is contained in the ideal defined by the polynomial equations. The dimensions of the layers are generally quite small in the cases we need to check, so Gröbner basis methods work well.

For nilpotent Lie algebras of dimension up to 6, an overview of our classification of gradings is compiled in Table 2. For each Lie algebra, we list its label in the classification of [CdGS12], the rank of its maximal grading (k), whether it is stratifiable or not (s?), the number of gradings (#), and the number of gradings with a positive realization ($\#\mathbb{Z}_+$).

Example 4.5. We present our method of classifying gradings explicitly in the simple case of the Lie algebra $L_{4,2} = L_{3,2} \times \mathbb{R}$ given in the basis $Y_1, \dots, Y_4$ with the only bracket $[Y_1, Y_2] = Y_3$. The maximal grading is over $\mathbb{Z}^3$ with the layers

$$V_{(1,0,0)} = \langle Y_1 \rangle, \quad V_{(0,1,0)} = \langle Y_2 \rangle, \quad V_{(1,1,0)} = \langle Y_3 \rangle, \quad V_{(0,0,1)} = \langle Y_4 \rangle.$$

Ignoring scalar multiples, the difference set $\Omega - \Omega$ of weights consists of the 6 elements e_1, e_2, $e_1 - e_2$, $e_1 - e_3$, $e_2 - e_3$, and $e_1 + e_2 - e_3$, where e_1, e_2, e_3 are the standard basis elements of $\mathbb{Z}^3$. Subsets of these points span the trivial subspace, 6 one-dimensional subspaces, 7 two-dimensional subspaces $\langle e_1, e_2 \rangle$, $\langle e_1, e_3 \rangle$, $\langle e_2, e_3 \rangle$, $\langle e_1 - e_3, e_2 \rangle$, $\langle e_1, e_2 - e_3 \rangle$, $\langle e_1 - e_3, e_2 - e_3 \rangle$, $\langle 2e_1 - e_3, 2e_2 - e_3 \rangle$, and the full space $\mathbb{Z}^3$.

In this case, each of these 15 subspaces S defines a torsion-free quotient $\mathbb{Z}^3/S$. For instance parametrizing the quotient $\pi\colon \mathbb{Z}^3 \to \mathbb{Z}^3/\langle e_1 - e_3, e_2 - e_3 \rangle$ as $\mathbb{Z}$ using the complementary line $\mathbb{Z}e_3$ gives the weights

$$\pi(e_1) = \pi(e_2) = \pi(e_3) = 1, \quad \pi(e_1 + e_2) = 2,$$

so a push-forward grading for the quotient $\mathbb{Z}^3/\langle e_1 - e_3, e_2 - e_3 \rangle$ is the $\mathbb{Z}$-grading

$$V_1 = \langle Y_1, Y_2, Y_4 \rangle, \quad V_2 = \langle Y_3 \rangle.$$

To determine the distinct equivalence classes out of the 15 gradings, we first consider the simple criteria listed earlier. The trivial grading and the maximal grading are distinguished by the rank. The six $\mathbb{Z}^2$-gradings all have 2 one-dimensional layers and 1 two-dimensional layer. There exists a homomorphism that preserves the dimensions of the layers for two pairs of the gradings: one between the quotients by $\langle e_1 \rangle$ and $\langle e_2 \rangle$, and one between the quotients by $\langle e_1 - e_3 \rangle$ and $\langle e_2 - e_3 \rangle$.

Name	k	s?	#	$\#\mathbb{Z}_+$	Name	k	s?	#	$\#\mathbb{Z}_+$
$L_{2,1}$	2	✓	2	2	$L_{6,9}$	3	✓	17	8
$L_{3,1}$	3	✓	3	3	$L_{6,10}$	3		23	8
$L_{3,2}$	2	✓	4	2	$L_{6,11}$	1		2	1
$L_{4,1}$	4	✓	5	5	$L_{6,12}$	2		9	4
$L_{4,2}$	3	✓	11	6	$L_{6,13}$	2		8	3
$L_{4,3}$	2	✓	6	2	$L_{6,14}$	1		2	1
$L_{5,1}$	5	✓	7	7	$L_{6,15}$	1		2	1
$L_{5,2}$	4	✓	26	15	$L_{6,16}$	2	✓	8	2
$L_{5,3}$	3	✓	22	9	$L_{6,17}$	1		2	1
$L_{5,4}$	3	✓	9	4	$L_{6,18}$	2	✓	8	2
$L_{5,5}$	2		7	3	$L_{6,19}(-1)$	3	✓	21	6
$L_{5,6}$	1		2	1	$L_{6,20}$	2	✓	8	3
$L_{5,7}$	2	✓	7	2	$L_{6,21}(-1)$	2	✓	6	2
$L_{5,8}$	3	✓	14	6	$L_{6,22}(0)$	3	✓	18	8
$L_{5,9}$	2	✓	5	2	$L_{6,22}(1)$	4	✓	32	15
$L_{6,1}$	6	✓	11	11	$L_{6,23}$	2		8	4
$L_{6,2}$	5	✓	52	31	$L_{6,24}(0)$	2		8	4
$L_{6,3}$	4	✓	60	27	$L_{6,24}(1)$	2		5	2
$L_{6,4}$	4	✓	29	13	$L_{6,25}$	3	✓	29	11
$L_{6,5}$	3		29	15	$L_{6,26}$	3	✓	10	5
$L_{6,6}$	2		8	6	$L_{6,27}$	3	✓	32	13
$L_{6,7}$	3	✓	31	11	$L_{6,28}$	2	✓	8	3
$L_{6,8}$	4	✓	52	25					

TABLE 2. Gradings of Lie algebras up to dimension 6 over $\mathbb{C}$

Out of the seven $\mathbb{Z}$-gradings, the four quotients by

$$\langle e_1, e_2\rangle, \langle e_1 - e_3, e_2\rangle, \langle e_1, e_2 - e_3\rangle, \langle e_1 - e_3, e_2 - e_3\rangle$$

define gradings with 1 one-dimensional layer and 1 three-dimensional layer, and the three quotients by

$$\langle e_1, e_3\rangle, \langle e_2, e_3\rangle, \langle 2e_1 - e_3, 2e_2 - e_3\rangle$$

define gradings with 2 two-dimensional layers. In both families there is exactly one pair of gradings admitting a homomorphism: the pair $\langle e_1 - e_3, e_2\rangle$ and $\langle e_1, e_2 - e_3\rangle$, and the pair $\langle e_1, e_3\rangle$ and $\langle e_2, e_3\rangle$.

In all of these cases, the homomorphism between the quotients is induced by the isomorphism $f\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ swapping e_1 and e_2. All of the mentioned pairs of $\mathbb{Z}^2$- and $\mathbb{Z}$-gradings are in fact equivalent, since there is a corresponding Lie algebra automorphism swapping the basis elements Y_1 and Y_2 that preserves the subspaces $\langle Y_3\rangle$ and $\langle Y_4\rangle$. This reduces the list of 15 gradings down to 11 distinct equivalence classes. Universal realizations for each equivalence class of torsion-free gradings are listed in Table 3.

rank	type	layers
3	(4)	$V_{1,0,0} \oplus V_{0,1,0} \oplus V_{1,1,0} \oplus V_{0,0,1} = \langle Y_1 \rangle \oplus \langle Y_2 \rangle \oplus \langle Y_3 \rangle \oplus \langle Y_4 \rangle$
2	(2, 1)	$V_{0,0} \oplus V_{1,0} \oplus V_{0,1} = \langle Y_2 \rangle \oplus \langle Y_4 \rangle \oplus \langle Y_1, Y_3 \rangle$
2	(2, 1)	$V_{1,0} \oplus V_{0,1} \oplus V_{0,2} = \langle Y_4 \rangle \oplus \langle Y_1, Y_2 \rangle \oplus \langle Y_3 \rangle$
2	(2, 1)	$V_{1,0} \oplus V_{0,1} \oplus V_{1,1} = \langle Y_1, Y_4 \rangle \oplus \langle Y_2 \rangle \oplus \langle Y_3 \rangle$
2	(2, 1)	$V_{1,-1} \oplus V_{0,1} \oplus V_{1,0} = \langle Y_1 \rangle \oplus \langle Y_2 \rangle \oplus \langle Y_3, Y_4 \rangle$
1	(0, 2)	$V_0 \oplus V_1 = \langle Y_1, Y_4 \rangle \oplus \langle Y_2, Y_3 \rangle$
1	(0, 2)	$V_1 \oplus V_2 = \langle Y_1, Y_2 \rangle \oplus \langle Y_3, Y_4 \rangle$
1	(1, 0, 1)	$V_1 \oplus V_2 = \langle Y_1, Y_2, Y_4 \rangle \oplus \langle Y_3 \rangle$
1	(1, 0, 1)	$V_0 \oplus V_1 = \langle Y_1, Y_2, Y_3 \rangle \oplus \langle Y_4 \rangle$
1	(1, 0, 1)	$V_0 \oplus V_1 = \langle Y_1 \rangle \oplus \langle Y_2, Y_3, Y_4 \rangle$
0	(0, 0, 0, 1)	$V_0 = \langle Y_1, Y_2, Y_3, Y_4 \rangle$

TABLE 3. Gradings of the Lie algebra $L_{4,2}$

4.3. **Enumerating Heintze groups.** In this section, we present how our complete list of gradings for a given nilpotent Lie algebra $\mathfrak{g}$ can be used to determine a list of Heintze groups over $\mathfrak{g}$.

Definition 4.6. A *Heintze group* is a simply connected Lie group over $\mathbb{R}$ whose Lie algebra is a semidirect product of a nilpotent Lie algebra $\mathfrak{g}$ and $\mathbb{R}$ via a derivation $\alpha \in \mathrm{der}(\mathfrak{g})$ whose eigenvalues have strictly positive real parts.

Positive gradings for a given Lie algebra are naturally identified with diagonalizable derivations with strictly positive eigenvalues, see Subsection 2.3. Hence, to any positively graded Lie algebra $\mathfrak{g}$ we may associate a Heintze group over $\mathfrak{g}$. We shall call these groups *diagonal Heintze groups.*

The quasi-isometric classification of Heintze groups reduces to the study of so called *purely real Heintze groups*, for which the associated derivation has real eigenvalues. Purely real Heintze groups are equivalent to diagonal Heintze groups under a slightly weaker notion of equivalence (sublinear biLipschitz-equivalence, see [Cornulier Thm 1.2] and [Pal19, Thm 3.2]). Moreover, by [CPS17] if two diagonal Heintze groups are quasi-isometric, then their associated derivations are proportional. Hence, the quasi-isometric classification problem of diagonal Heintze groups can be approached by treating the algebraic problem of finding all the possible derivations defining non-isomorphic diagonal Heintze groups.

Proposition 4.7 is a tool for tackling the above mentioned algebraic problem using positive gradings. We will prove this result later in this section after discussing its role in the enumeration of Heintze groups.

Proposition 4.7. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\alpha, \beta \in \mathrm{der}(\mathfrak{g})$ diagonalizable derivations with strictly positive eigenvalues. If α and*

β define isomorphic Heintze groups, then they define equivalent $\mathbb{R}$-gradings.

The enumeration of positive gradings we have established immediately gives the corresponding enumeration of diagonal Heintze groups over $\mathfrak{g}$. The enumeration of positive gradings can be understood in two different ways, see Remark 3.11. The corresponding enumeration of Heintze groups has similar character: it is either a parametrization via the projections or a finite list that does not contain all the isomorphism classes of Heintze groups but a representative for each family in terms of the layers. If one is able to eliminate equivalent gradings from the enumeration of positive gradings, then by Proposition 4.7 the corresponding list of Heintze groups does not contain isomorphic Heintze groups.

Remark 4.8. The enumeration of Heintze groups has a few caveats:

(i) Already over $\mathfrak{g} = \mathbb{R}^2$ there are uncountably many isomorphism classes of Heintze groups given by the projections $(1,0) \mapsto 1$ and $(0,1) \mapsto a$ with $a > 0$.

(ii) Our methods are in general able to find maximal gradings only for Lie algebras over algebraically closed fields. On the contrary, the base field of Heintze groups is $\mathbb{R}$.

Before proving Proposition 4.7, we need the following lemmas.

Lemma 4.9. *Let $\mathfrak{g}$ be a Lie algebra. Let $\delta \in \operatorname{der}(\mathfrak{g})$ be a diagonalizable derivation and let $X \in \mathfrak{g}$ be an eigenvector of δ. Then*

$$\operatorname{Ad}_{\exp(X)} \circ \delta \circ \operatorname{Ad}_{\exp(-X)} = \delta - \operatorname{ad}_{\delta(X)} .$$

Proof. Let $Y_1, \dots, Y_n$ be a basis of $\mathfrak{g}$ that diagonalizes δ. Fix some $Y = Y_i$ and let w_Y and w_X be the eigenvalues of the eigenvectors Y and X. Since δ is a derivation and X and Y are eigenvectors, the vectors $\operatorname{ad}_X^k Y$ are also eigenvectors, and have the eigenvalues $kw_X + w_Y$. Using this fact, and recalling that $\operatorname{Ad}_{\exp(X)} = e^{\operatorname{ad}_X}$, see [Kna02, Proposition 1.91], we compute

$$\begin{aligned}
\delta \circ \operatorname{Ad}_{\exp(-X)}(Y) &= \delta\Big(\sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{-X}^k Y \Big) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} (kw_X + w_Y) \operatorname{ad}_{-X}^k Y \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \operatorname{ad}_{-X}^{k-1} [-w_X X, Y] + \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{-X}^k (w_Y Y) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{ad}_{-X}^k (-[w_X X, Y] + w_Y Y) \\
&= \operatorname{Ad}_{\exp(-X)}(-\operatorname{ad}_{\delta(X)} Y + \delta(Y)).
\end{aligned}$$

By cancellation of $\mathrm{Ad}_{\exp(X)}$ and $\mathrm{Ad}_{\exp(-X)}$, the claimed formula follows. □

Lemma 4.10. *Let $X, Y \in \mathfrak{g}$ be two vectors of a Lie algebra $\mathfrak{g}$. Then*

$$\mathrm{Ad}_{\exp(X)} \circ \mathrm{ad}_Y \circ \mathrm{Ad}_{\exp(-X)} = \mathrm{ad}_{\mathrm{Ad}_{\exp(X)} Y} .$$

Proof. Since the map $\mathrm{Ad}_{\exp(X)}$ is a Lie algebra homomorphism and is the inverse of $\mathrm{Ad}_{\exp(-X)}$, we have

$$\mathrm{Ad}_{\exp(X)}[Y, \mathrm{Ad}_{\exp(-X)} Z] = [\mathrm{Ad}_{\exp(X)} Y, Z]$$

for every $Z \in \mathfrak{g}$. □

Lemma 4.11. *Let $\delta \in \mathrm{der}(\mathfrak{g})$ be a diagonalizable derivation with all eigenvalues strictly positive. Then for every vector $Y \in \mathfrak{g}$ there exists a vector $X \in \mathfrak{g}$ such that $\mathrm{Ad}_{\exp(X)} \circ \delta \circ \mathrm{Ad}_{\exp(-X)} = \delta - \mathrm{ad}_Y$.*

Proof. For a vector $X \in \mathfrak{g}$, denote by $C_X \colon \mathrm{der}(\mathfrak{g}) \to \mathrm{der}(\mathfrak{g})$ the conjugation map

$$C_X(\eta) = \mathrm{Ad}_{\exp(X)} \circ \eta \circ \mathrm{Ad}_{\exp(-X)} .$$

Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ that diagonalizes δ. Consider the map

$$\Phi \colon \mathbb{R}^n \to \mathrm{der}(\mathfrak{g}), \quad \Phi(x_1, \dots, x_n) = C_{x_n X_n} \circ \dots \circ C_{x_1 X_1}(\delta).$$

By repeated application of Lemma 4.9 and Lemma 4.10, it follows that $\Phi(x) = \delta - \mathrm{ad}_{\phi(x)}$, where $\phi \colon \mathbb{R}^n \to \mathfrak{g}$ is the map

$$\begin{aligned}\phi(x_1, \dots, x_n) = \delta(x_n X_n) + \mathrm{Ad}_{\exp(x_n X_n)} \delta(x_{n-1} X_{n-1}) + \cdots \\ + \mathrm{Ad}_{\exp(x_n X_n)} \mathrm{Ad}_{\exp(x_{n-1} X_{n-1})} \cdots \mathrm{Ad}_{\exp(x_2 X_2)} \delta(x_1 X_1).\end{aligned} \tag{6}$$

Since the composition of conjugations is a conjugation, it suffices to prove that the map ϕ is surjective.

Let $w_1, \dots, w_n > 0$ be the eigenvalues of the vectors $X_1, \dots, X_n$ for the derivation δ. Since the maps $x_i \mapsto \mathrm{sign}(x_i) |x_i|^{w_i}$ are all invertible, the map $\phi \colon \mathbb{R}^n \to \mathfrak{g}$ is surjective if and only if the map $\tilde{\phi} \colon \mathbb{R}^n \to \mathfrak{g}$ defined by

$$\tilde{\phi}(x_1, \dots, x_n) = \phi(\mathrm{sign}(x_1) |x_1|^{w_1}, \dots, \mathrm{sign}(x_n) |x_n|^{w_n}) \tag{7}$$

is surjective.

Let $D_\lambda \in \mathrm{Aut}(\mathfrak{g})$, $\lambda > 0$, be the one-parameter family of dilations defined by the derivation δ, i.e., $D_\lambda = \exp(\delta \log \lambda)$. Then for each $i = 1, \dots, n$ the dilation is given by $D_\lambda(X_i) = \lambda^{w_i} X_i$ and we have the dilation equivariance

$$\mathrm{Ad}_{\exp(\lambda^{w_i} X_i)} \circ D_\lambda = D_\lambda \circ \mathrm{Ad}_{\exp(X_i)} .$$

Applying the above equivariance to the definition (7) we find that the map $\tilde{\phi}$ is D_λ-homogeneous, i.e., $\tilde{\phi}(\lambda x) = D_\lambda(\tilde{\phi}(x))$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. Since $\bigcup_{\lambda > 0} D_\lambda(U) = \mathfrak{g}$ for any neighborhood U of the identity it follows that the map $\tilde{\phi}$ is surjective if and only if it is open at zero. Since the change of parameters in (7) is a homeomorphism, the same is true also for the map ϕ.

By the definition (6), the map ϕ is smooth. The derivative of each summand $\mathrm{Ad}_{\exp(x_n X_n)} \cdots \mathrm{Ad}_{\exp(x_{i+1} X_{i+1})} \delta(x_i X_i)$ at zero is the map $x \mapsto \delta(x_i X_i)$, so the derivative $D_0\phi$ of the map ϕ at zero is

$$D_0\phi(x_1, \ldots, x_n) = \delta(x_1 X_1 + \cdots + x_n X_n).$$

By the strictly positive eigenvalue assumption, the map δ is invertible. Since $X_1, \ldots, X_n$ is a basis of $\mathfrak{g}$, it follows that the map ϕ is open at zero, concluding the proof. □

Proof of Proposition 4.7. Rescaling the derivations by a scalar, we may assume the smallest of the eigenvalues for both the derivations to be 1. Since the Heintze groups are assumed to be isomorphic, it is straightforward to see that there is a vector $X \in \mathfrak{g}$ so that the derivation α is conjugate by a Lie algebra automorphism of $\mathfrak{g}$ to the derivation $\beta + \mathrm{ad}_X$. We use Lemma 4.11 to find that actually α and β are conjugate. Applying Lemma 2.19(i) to the split tori spanned by α and β gives the desired result. □

4.4. **Bounds for non-vanishing $\ell^{q,p}$ cohomology.** Knowing all the possible positive gradings of a nilpotent Lie algebra $\mathfrak{g}$ has one further application in the realm of quasi-isometric classifications. Different positive gradings can be used to obtain better estimates in the computation of the $\ell^{q,p}$ cohomology of a nilpotent Lie group, which is a well-known quasi-isometry invariant.

By definition, the $\ell^{q,p}$ cohomology of a Riemannian manifold with bounded geometry is the $\ell^{q,p}$ cohomology of every bounded geometry simplicial complex quasi-isometric to it. A crucial result of [PR18] shows that in the case of contractible Lie groups, the $\ell^{q,p}$ cohomology of the manifold is isomorphic to its $L^{q,p}$ cohomology.

Definition 4.12. The $L^{q,p}$ *cohomology* of a nilpotent Lie group G is defined as

$$L^{q,p}H^\bullet(G) = \frac{\{\text{closed forms in } L^p\}}{d\big(\{\text{forms in } L^q\}\big) \cap L^p}.$$

In [PR18, Theorem 1.1] it is shown that the Rumin complex constructed on a Carnot group allows for sharper computations regarding $L^{q,p}H^\bullet(G)$ when compared to the usual de Rham complex. Defining and reviewing the properties of the Rumin complex $(E_0^\bullet, d_c)$ goes beyond the scope of this paper. For the following discussion, it is sufficient to know that the space of Rumin h-forms E_0^h is a subspace of the space of smooth differential h-forms of the underlying nilpotent Lie group G.

Definition 4.13. Let us consider a positive grading $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha \in \mathbb{R}} V_\alpha$. Then a left-invariant 1-form θ has *weight* α, that is $w(\theta) = \alpha$, if $\theta = X^*$

for $X \in V_\alpha$. In other words, θ is the dual of a vector field belonging to the subspace V_α at the identity of the group. In general, given a left-invariant h-form, we will say that it has *weight* p if it can be expressed as a linear combination of left-invariant h-forms $\theta_{i_1,\dots,i_h} = \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ such that $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = p$.

Given a positive grading $\mathcal{V} : \mathfrak{g} = \bigoplus_{\alpha\in\mathbb{R}} V_\alpha$, we call the quantity

$$Q = \sum_{\alpha\in\mathbb{R}_+} \alpha \dim V_\alpha$$

the *homogeneous dimension* of $\mathcal{V}$. We also define for each degree h the number

$$\delta N_{\min}(h) = \min_{\theta\in E_0^h} w(\theta) - \max_{\tilde\theta\in E_0^{h-1}} w(\tilde\theta)\,.$$

The following is [PR18, Theorem 1.1(ii)].

Theorem 4.14. *Let G be a Carnot group of homogeneous dimension Q. If*

$$1 \le p, q \le \infty \text{ and } \frac{1}{p} - \frac{1}{q} < \frac{\delta N_{\min}(h)}{Q}$$

then the $L^{q,p}$ cohomology of G in degree h does not vanish.

Moreover, in Theorem 9.2 of the same paper it is shown how the non-vanishing statement has a wider scope, as it can be applied to Carnot groups equipped with a homogeneous structure that comes from a positive grading. This result has been further extended in [Tri20] to arbitrary positively graded nilpotent Lie groups.

A natural question that stems from these considerations is whether it is possible to identify which choice of positive grading will yield the best interval for non-vanishing cohomology. This problem can be easily presented in terms of maximising the value of the fraction $\delta N_{\min}(h)/Q$ among all the possible positive gradings for a given Lie group G.

Let us describe the maximization procedure in more detail. Let $\mathcal{W} : \mathfrak{g} = \bigoplus_{n\in\mathbb{Z}^k} W_n$ be a maximal grading of $\mathfrak{g}$ and let Ω be the set of weights of $\mathcal{W}$. For $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$, let $\pi^{\mathbf{a}} \colon \mathbb{Z}^k \to \mathbb{R}$ be the projection given by $\pi^{\mathbf{a}}(e_i) = a_i$. Let

$$\mathbf{A}_+ = \{\mathbf{a} \in \mathbb{R}^k : \pi^{\mathbf{a}}(n) > 0 \,\forall n \in \Omega\}. \tag{8}$$

The push-forward $\pi^{\mathbf{a}}_*(\mathcal{W})$ is a positive grading if and only if $\mathbf{a} \in \mathbf{A}_+$.

In the sequel we shall identify any positive grading of $\mathfrak{g}$ with the corresponding vector $\mathbf{a} \in \mathbf{A}_+$. In particular, if θ is the dual of $X \in W_n$, then the weight of θ with respect to the grading $\pi^{\mathbf{a}}_*(\mathcal{W})$ is $w(\theta)_{\mathbf{a}} = \pi^{\mathbf{a}}(n)$. Then we want to find the value of the following expression for each degree h:

$$\max_{\mathbf{a}\in\mathbf{A}_+} \left\{ \frac{\min_{\theta\in E_0^h} w(\theta)_{\mathbf{a}} - \max_{\tilde\theta\in E_0^{h-1}} w(\tilde\theta)_{\mathbf{a}}}{Q_{\mathbf{a}}} \right\},$$

where $Q_{\mathbf{a}}$ is the homogeneous dimension of $\pi_*^{\mathbf{a}}(\mathcal{W})$.

A problem of this form can be converted into a linear optimization problem as follows:

1. replace $\min_{\theta\in E_0^h} w(\theta)_{\mathbf{a}}$ with a new variable x, and add the constraint $x \leq w(\theta)_{\mathbf{a}}$ for each $\theta \in E_0^h$;
2. replace $\max_{\tilde\theta\in E_0^{h-1}} w(\tilde\theta)$ with a new variable y, and add the constraint $y \geq w(\tilde\theta)_{\mathbf{a}}$ for each $\tilde\theta \in E_0^{h-1}$;
3. normalize the expression by imposing $Q_{\mathbf{a}} = 1$.

We are then left with the following expression for our original maximization problem

$$\begin{aligned} \text{Maximize} \quad & x - y \\ \text{subject to} \quad & x \leq w(\theta)_{\mathbf{a}} \quad \forall\, \theta \in E_0^h, \\ & y \geq w(\tilde\theta)_{\mathbf{a}} \quad \forall \tilde\theta \in E_0^{h-1}, \\ & Q_{\mathbf{a}} = 1, \quad \mathbf{a} \in \mathbf{A}_+ \end{aligned}$$

which can easily be solved by a computer, yielding the optimal bound for non-vanishing cohomology using the method of Theorem 4.14.

Example 4.15. Let us consider the non-stratifiable Lie group G of dimension 6, whose Lie algebra is denoted as $L_{6,10}$ in [dG07], with the non-trivial brackets

$$[X_1, X_2] = X_3\ ,\ [X_1, X_3] = [X_5, X_6] = X_4.$$

The space of Rumin forms in G is

$$\begin{aligned} E_0^1 &= \langle \theta_1, \theta_2, \theta_5, \theta_6 \rangle; \\ E_0^2 &= \langle \theta_{5,6} - \theta_{1,3}, \theta_{1,5}, \theta_{1,6}, \theta_{2,3}, \theta_{2,5}, \theta_{2,6} \rangle; \\ E_0^3 &= \langle \theta_{2,5,6} + \theta_{1,2,3}, \theta_{2,3,5}, \theta_{2,3,6}, \theta_{1,3,4} - \theta_{4,5,6}, \theta_{1,4,5}, \theta_{1,4,6} \rangle. \end{aligned}$$

For the Lie algebra $L_{6,10}$, the maximal grading is over $\mathbb{Z}^3$ with the layers

$$\begin{aligned} &V_{(0,1,0)} = \langle X_1 \rangle, && V_{(0,0,1)} = \langle X_2 \rangle, && V_{(0,1,1)} = \langle X_3 \rangle \\ &V_{(0,2,1)} = \langle X_4 \rangle, && V_{(1,0,0)} = \langle X_5 \rangle, && V_{(-1,2,1)} = \langle X_6 \rangle. \end{aligned}$$

The family of projections $\pi^{\mathbf{a}}\colon \mathbb{Z}^3 \to \mathbb{R}$ giving positive gradings is parametrized by $(a_1, a_2, a_3) = \mathbf{a} \in \mathbf{A}_+$ as in (8). The weights of left-invariant 1-forms are

$$\begin{aligned}
w(\theta_1)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,1,0) = a_2;\\
w(\theta_2)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,0,1) = a_3;\\
w(\theta_3)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,1,1) = a_2 + a_3;\\
w(\theta_4)_{\mathbf{a}} &= \pi^{\mathbf{a}}(0,2,1) = 2a_2 + a_3;\\
w(\theta_5)_{\mathbf{a}} &= \pi^{\mathbf{a}}(1,0,0) = a_1;\\
w(\theta_6)_{\mathbf{a}} &= \pi^{\mathbf{a}}(-1,2,1) = 2a_2 + a_3 - a_1.
\end{aligned}$$

From this computation we get the explicit expression

$$\mathbf{A}_+ = \{\mathbf{a} \in \mathbb{R}^3 : a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ -a_1 + 2a_2 + a_3 > 0\}$$

and the homogeneous dimension $Q_{\mathbf{a}} = 6a_2 + 4a_3$.

Let us first consider the bound for non-vanishing cohomology in degree 1. We express

$$\max_{\mathbf{a}\in\mathbf{A}_+} \left\{ \frac{\delta N_{\min}(1)}{Q_{\mathbf{a}}} \right\} = \max_{\mathbf{a}\in\mathbf{A}_+} \left\{ \frac{\min\{a_1, a_2, a_3, 2a_2 + a_3 - a_1\}}{6a_2 + 4a_3} \right\}.$$

as the linear optimization problem

$$\begin{aligned}
\text{Maximize} \quad & x\\
\text{subject to} \quad & x \le a_1,\ x \le a_2,\ x \le a_3,\\
& x \le 2a_2 + a_3 - a_1,\\
& 1 = 6a_2 + 4a_3,\\
& a_1, a_2, a_3 > 0,\ 2a_2 + a_3 - a_1 > 0.
\end{aligned}$$

A solver finds the solution $\frac{1}{10}$, which is obtained by choosing $a_1 = a_2 = a_3 = \frac{1}{10}$. Since the quantity $\frac{\delta N_{\min}(1)}{Q_{\mathbf{a}}}$ is scaling invariant, we find that the grading defined by $a_1 = a_2 = a_3 = 1$ gives $\ell^{q,p}H^1(G) \neq 0$ with the optimal bound $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$.

Similarly, once we re-express

$$\max_{\mathbf{a}\in\mathbf{A}_+} \left\{ \frac{\delta N_{\min}(2)}{Q_{\mathbf{a}}} \right\}.$$

as a linear optimization problem and feed it into a solver, we get the result $\frac{1}{10}$, obtained (up to rescaling) by taking $a_2 = a_3 = 2$ and $a_1 = 3$. Therefore $\ell^{q,p}H^2(G) \neq 0$ for $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$.

Likewise, we obtain the optimal bound $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$ for $\ell^{q,p}H^3(G) \neq 0$ by taking $a_1 = a_2 = a_3 = 1$.

Finally, by Hodge duality, see [Tri20, Theorem 7.3], we obtain the optimal bounds for $\ell^{q,p}$ cohomology in complementary degree, that is $\ell^{q,p}H^4(G) \neq 0$, $\ell^{q,p}H^5(G) \neq 0$, and $\ell^{q,p}H^6(G) \neq 0$, for $\frac{1}{p} - \frac{1}{q} < \frac{1}{10}$.

Remark 4.16. [PR18, Example 9.5] describes an explicit positive grading in the Engel group that gives an improved bound for the non-vanishing of the $L^{q,p}$ cohomology in degree 2. By a similar computation as the one shown in Example 4.15, one can verify that the value given in [PR18, Example 9.5] is indeed the optimal bound.

Appendix A. Existence of a positive realization

Example 2.4 motivates an alternate approach for deciding the existence of a positive realization. The grading in the example does not admit a positive realization: suppose by contradiction that there is an injection $\{a, b, c, d\} \to \mathbb{R}$ that gives a positive realization. The bracket relations of the Lie algebra imply the equations $a + c = d$ and $b + d = c$, which are impossible for strictly positive weights. Here the non-existence of a positive realization is found simply by considering the equations implied by the bracket relations of the layers.

In general, a grading can be realized over some abelian group A with the set of weights $\{\lambda_1, \dots, \lambda_k\}$ if and only if certain system of equations of the type $\lambda_i + \lambda_j = \lambda_h$ has a solution $(\lambda_1, \dots, \lambda_k)$ whose components are all distinct. This system consists of equations coming from the non-trivial bracket relations among the layers of the grading, see step 2 of Algorithm 2.13. A positive realization exists if and only if there is a solution in the group $A = \mathbb{R}$ with all weights strictly positive. Indeed, if there is a positive solution, then there is also a positive solution with distinct components as we shall see in the proof of Algorithm A.4.

Deciding if a solution exists with all components strictly positive is a classical problem in linear programming. By rescaling, we may replace the open conditions $\lambda_i > 0$ with the closed conditions $\lambda_i \geq 1$. By a change of variables $\mu_i = \lambda_i - 1$, we find that the linear problem for the existence of a positive realization is equivalent to an affine problem $\mu_i + \mu_j - \mu_h = -1$ with all the components μ_i non-negative.

Let A be the $N \times k$-matrix of coefficients of the affine problem and let $\mathbf{b} = (-1, \dots, -1) \in \mathbb{R}^N$. By getting rid of linearly dependent equations, we may assume that $\operatorname{rank}(A) = N \leq k$. Our goal is then an algorithm that either produces an element of the set

$$P = \{\mathbf{x} \in \mathbb{R}^k \mid A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0\} \tag{9}$$

or indicates that the set P is empty. Here we use the shorthand notation $\mathbf{x} \geq 0$ to mean that all the components of the vector $\mathbf{x}$ are non-negative. Notice that the set P of solutions is closed and convex.

There is a vast literature on how to solve linear programming problems, and we refer to the book [FP93]. This approach and in particular Lemma A.3 below are essentially from section 2.4 of that book. We state them here for completeness.

Definition A.1. Let $K \subset \mathbb{R}^n$ be a convex set. We say that a point $x \in K$ is an *extremal point* if it cannot be expressed as a non-trivial convex combination of the points of K.

Lemma A.2. *Let $K \subset \mathbb{R}^n$ be a closed convex non-empty set for which $\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$. Then K contains at least one extremal point.*

Proof. Consider the lexicographic order $\prec$ on $\mathbb{R}^n$, where $\mathbf{x} \prec \mathbf{y}$ if there exists some index $i \in \{1, \ldots, n\}$ such that $\mathbf{x}_j = \mathbf{y}_j$ for all $j < i$ and $\mathbf{x}_i < \mathbf{y}_i$. Observe that if $\mathbf{x} \prec \mathbf{y}$, then

$$\mathbf{x} \prec t\mathbf{x} + (1-t)\mathbf{y} \tag{10}$$

for every $0 < t < 1$. Since $\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$, there exists a lexicographic minimum $\mathbf{x}_{\min} \in K$. It follows from (10) that the point $\mathbf{x}_{\min}$ cannot be expressed as a non-trivial convex combination. □

Lemma A.3. *Let P be the set of non-negative solutions of a system $A\mathbf{x} = \mathbf{b}$ as in (9). Let $\mathbf{x} \in \mathbb{R}^k$ be an extremal point of P. Then there exists an invertible $N \times N$ matrix B whose columns are chosen from the matrix A, such that up to a permutation of components, $\mathbf{x} = (B^{-1}(\mathbf{b}), 0, \ldots, 0) \in \mathbb{R}^k$.*

Proof. By permuting the basis, we can express $\mathbf{x} = (\mathbf{x}_+, 0, \ldots, 0)$, where $\mathbf{x}_+ \in \mathbb{R}^p$ for some $p \leq k$ and $\mathbf{x}_+ > 0$. Let A_+ be the matrix consisting of the first p columns of A. First we show that the matrix A_+ has rank p. Suppose towards a contradiction that there is a non-zero vector $\mathbf{w} \in \mathbb{R}^p$ such that $A_+\mathbf{w} = 0$. Let $\delta > 0$ be so small that the vectors

$$\mathbf{z}_1 = \mathbf{x}_+ + \delta\mathbf{w} \qquad \mathbf{z}_2 = \mathbf{x}_+ - \delta\mathbf{w}$$

both satisfy $\mathbf{z}_1, \mathbf{z}_2 \geq 0$. For both $i \in \{1, 2\}$, let $\mathbf{u}_i = (\mathbf{z}_i, 0, \ldots, 0) \in \mathbb{R}^k$. Then

$$A\mathbf{u}_i = A_+\mathbf{z}_i = A_+\mathbf{x}_+ = A\mathbf{x} = \mathbf{b},$$

so $\mathbf{u}_i \in P$ are both solutions. Now the solution $\mathbf{x}$ can be represented as a non-trivial convex combination $\mathbf{x} = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$ of solutions, which contradicts the assumption that $\mathbf{x}$ is an extremal point. We conclude that the rank of A_+ must be p. Since $\operatorname{rank}(A) = N$, we deduce also $p \leq N$.

If $p < N$, then since $\operatorname{rank}(A) = N$ it is possible to form an invertible matrix B by adding some further columns of A to the matrix A_+. If instead $p = N$, then continue with $B = A_+$. Let $\bar{\mathbf{x}} = (\mathbf{x}_+, 0, \ldots, 0) \in \mathbb{R}^N$. Then

$$B\bar{\mathbf{x}} = A_+\mathbf{x}_+ = A\mathbf{x} = \mathbf{b}$$

so $\bar{\mathbf{x}} = B^{-1}(\mathbf{b})$ and the claim follows. □

Algorithm A.4 (Existence of a positive realization)**.** *Input: A grading $\mathcal{V}$ of a Lie algebra $\mathfrak{g}$. Output: Decision if $\mathcal{V}$ admits a positive realization.*

(1) Form the $N \times k$ matrix A associated with the problem and set $\mathbf{b} = (-1, \ldots, -1) \in \mathbb{R}^N$.
(2) For each invertible $N \times N$ matrix B formed from the columns of the matrix A do the following: Compute $\mathbf{x} = B^{-1}(\mathbf{b})$. If $\mathbf{x} \geq 0$, then the grading admits a positive realization.
(3) Otherwise, the grading has no positive realization.

Proof of correctness. Let P be as in (9). Lemma A.3 implies that step 2 constructs all the extremal points of P. By Lemma A.2, if no extremal points are found, the set P is empty and no positive realization exists. We still need to argue that if P is non-empty, then a positive realization exists.

A priori, even if some $\mathbf{x} = B^{-1}(\mathbf{b}) \geq 0$, the corresponding weights $x_i + 1, \ldots, x_k + 1$ do not necessarily define a realization of the original grading, since in general these weights are non-distinct and hence define a coarser grading. By Lemma 2.15 this coarser grading is a push-forward grading of the universal realization of the original grading, via a homomorphism from some $\mathbb{Z}^m$ to $\mathbb{R}$ mapping the weights to $x_1 + 1, \ldots, x_k + 1$. The homomorphism is realized as a projection to some line of $\mathbb{R}^m$, as in the proof of Lemma 3.7. By perturbing this line, it is always possible to find another homomorphism that is injective on the weights and maps all the weights to strictly positive reals. Hence there is also a positive realization of the original grading. $\square$

Acknowledgements. All of the authors were supported by the Academy of Finland (grant 288501 Geometry of subRiemannian groups and by grant 322898 Sub-Riemannian Geometry via Metric-geometry and Lie-group Theory) and by the European Research Council (ERC Starting Grant 713998 GeoMeG Geometry of Metric Groups). E.H. was also supported by the Vilho, Yrjö and Kalle Väisälä Foundation, and by the SISSA project DIP_ECC_MATE_CoordAreaMate_0459 - Dipartimenti di Eccellenza 2018 - 2022 (CUP: G91|18000050006). V.K. was also supported by the Emil Aaltonen foundation. F.T. was also supported by the University of Bologna, funds for selected research topics, and by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 777822 GHAIA Geometric and Harmonic Analysis with Interdisciplinary Applications, and the Swiss National Foundation grant 200020_191978.

References

[BPR06] Saugata Basu, Richard Pollack, and Marie-Françoise Roy, *Algorithms in real algebraic geometry*, second ed., Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, Berlin, 2006. MR 2248869

[CdGS12] Serena Cicalò, Willem A. de Graaf, and Csaba Schneider, *Six-dimensional nilpotent Lie algebras*, Linear Algebra Appl. **436** (2012), no. 1, 163–189. MR 2859920

[CdGSGT18] S. Cicalò, W. de Graaf, C. Schneider, and T. GAP Team, *LieAlgDB, a database of lie algebras, Version 2.2*, `https://gap-packages.github.io/liealgdb/`, Apr 2018, Refereed GAP package.

[CKLD+17] Michael G. Cowling, Ville Kivioja, Enrico Le Donne, Sebastiano Nicolussi Golo, and Alessandro Ottazzi, *From homogeneous metric spaces to Lie groups*, arXiv e-prints (2017), arXiv:1705.09648.

[CM09] Arjeh M. Cohen and Scott H. Murray, *An algorithm for Lang's Theorem*, J. Algebra **322** (2009), no. 3, 675–702. MR 2531217

[Cor16] Yves Cornulier, *Gradings on Lie algebras, systolic growth, and cohopfian properties of nilpotent groups*, Bull. Soc. Math. France **144** (2016), no. 4, 693–744. MR 3562610

[Cor18] Yves de Cornulier, *On the quasi-isometric classification of locally compact groups*, New directions in locally compact groups, London Math. Soc. Lecture Note Ser., vol. 447, Cambridge Univ. Press, Cambridge, 2018, pp. 275–342. MR 3793294

[Cor19] Yves Cornulier, *On sublinear bilipschitz equivalence of groups*, Ann. ENS **52** (2019), no. 5, 1201–1242.

[CPS17] Matias Carrasco Piaggio and Emiliano Sequeira, *On quasi-isometry invariants associated to a Heintze group*, Geom. Dedicata **189** (2017), 1–16. MR 3667336

[CR19] Diego Conti and Federico A. Rossi, *Construction of nice nilpotent Lie groups*, J. Algebra **525** (2019), 311–340. MR 3911646

[dG07] Willem A. de Graaf, *Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2*, J. Algebra **309** (2007), no. 2, 640–653. MR 2303198

[dG17] Willem Adriaan de Graaf, *Computation with linear algebraic groups*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2017. MR 3675415

[EK13] Alberto Elduque and Mikhail Kochetov, *Gradings on simple Lie algebras*, Mathematical Surveys and Monographs, vol. 189, American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013. MR 3087174

[Eld10] Alberto Elduque, *Fine gradings on simple classical Lie algebras*, J. Algebra **324** (2010), no. 12, 3532–3571. MR 2735398

[Fav73] Gabriel Favre, *Système de poids sur une algèbre de Lie nilpotente*, Manuscripta Math. **9** (1973), 53–90. MR 349780

[FP93] Shu-Cherng Fang and Sarat Puthenpura, *Linear optimization and extensions: Theory and algorithms*, Prentice Hall, 1993.

[Gon98] Ming-Peng Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and R)*, ProQuest LLC, Ann Arbor, MI, 1998, Thesis (Ph.D.)–University of Waterloo (Canada). MR 2698220

[Hei74] Ernst Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. **211** (1974), 23–34. MR 353210

[HKMT20] Eero Hakavuori, Ville Kivioja, Terhi Moisala, and Francesca Tripaldi, *ehaka/lie-algebra-gradings: v1.0*, November 2020, https://doi.org/10.5281/zenodo.4267884.

[Hum78] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978, Second printing, revised. MR 499562

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1920389

[Koc09] Mikhail Kochetov, *Gradings on finite-dimensional simple Lie algebras*, Acta Appl. Math. **108** (2009), no. 1, 101–127. MR 2540960

[LD17] Enrico Le Donne, *A primer on Carnot groups: homogenous groups, Carnot-Carathéodory spaces, and regularity of their isometries*, Anal. Geom. Metr. Spaces **5** (2017), no. 1, 116–137. MR 3742567

[Mag08] L. Magnin, *Adjoint and trivial cohomologies of nilpotent complex Lie algebras of dimension* ≤ 7, Int. J. Math. Math. Sci. (2008), Art. ID 805305, 12. MR 2461422

[PR18] Pierre Pansu and Michel Rumin, *On the* $\ell^{q,p}$ *cohomology of Carnot groups*, Ann. H. Lebesgue **1** (2018), 267–295. MR 3963292

[PZ89] J. Patera and H. Zassenhaus, *On Lie gradings. I*, Linear Algebra Appl. **112** (1989), 87–159. MR 976333

[Roo13] Dan Roozemond, *Computing split maximal toral subalgebras of Lie algebras over fields of small characteristic*, J. Symbolic Comput. **50** (2013), 335–349. MR 2996884

[Sie86] Eberhard Siebert, *Contractive automorphisms on locally compact groups*, Math. Z. **191** (1986), no. 1, 73–90. MR 812604

[Spr09] T. A. Springer, *Linear algebraic groups*, second ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009. MR 2458469

[Tri20] Francesca Tripaldi, *The Rumin complex on nilpotent Lie groups*, arXiv e-prints (2020), arXiv:2009.10154.

(Hakavuori) SISSA
Email address: `eero.hakavuori@sissa.it`

(Kivioja) University of Jyväskylä
Email address: `kivioja.ville@gmail.com`

(Moisala) University of Jyväskylä
Email address: `moisala.terhi@gmail.com`

(Tripaldi) University of Bern
Email address: `francesca.tripaldi@math.unibe.ch`