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Whitney forms and their extensions ☆

Jonni Lohi *, Lauri Kettunen

Faculty of Information Technology, University of Jyväskylä, PO Box 35, FI 40014, Finland

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ABSTRACT

Whitney forms are widely known as finite elements for differential forms. Whitney's original definition yields first order functions on simplicial complexes, and a lot of research has been devoted to extending the definition to nonsimplicial cells and higher order functions. As a result, the term Whitney forms has become somewhat ambiguous in the literature. Our aim here is to clarify the concept of Whitney forms and explicitly explain their key properties. We discuss Whitney's initial definition with more depth than usually, giving three equivalent ways to define Whitney forms. We give a comprehensive exposition of their main properties, including the proofs. Understanding of these properties is important as they can be taken as a guideline on how to extend Whitney forms to nonsimplicial cells or higher order functions. We discuss several generalisations of Whitney forms and check which of the properties can be preserved.

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* Corresponding author.
E-mail addresses: jonni.j.lohi@student.jyu.fi (J. Lohi), lauri.y.o.kettunen@jyu.fi (L. Kettunen).

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1. Introduction

Whitney forms first appeared in the book of Hassler Whitney [1], which did not originally have any relation to numerical mathematics or to finite element and finite difference kind of approaches. Instead, Whitney formulated a theory of p-dimensional integration in n-dimensional affine space with chains and cochains. In a proof relating the cohomology of flat cochains to simplicial cohomology, he introduced elementary flat cochains and the corresponding differential forms [1, VII, §11]. Jozef Dodziuk used these forms (or their generalisations onto a manifold, to be precise) to approximate continuous Hodge theory with combinatorial Hodge theory and introduced the name Whitney forms in his thesis [2]. Unlike Whitney's work, Dodziuk's ideas were closely related to finite difference approaches.

Whitney forms became popular within the computational electromagnetics community in late 1980s and early 1990s after the pioneering work of Alain Bossavit [3–8]. He revealed their immediate relation to mixed finite elements [9,10] and emphasised the benefits of presenting the field equations in terms of differential forms instead of scalar and vector fields. Thereafter cochains and Whitney forms were shown to yield a natural framework to explain the finite difference method and its relation to the finite element method [11–14]. Differential forms have since been accepted as an appropriate tool to present both of these methods [15–20], and Whitney forms are widely used to build finite-dimensional subspaces of differential forms; for more examples of the use of Whitney forms (or their proxy fields) in the literature, see e.g. [21–27].

Whitney's original definition yields first order functions on simplicial complexes. In practice, the assumption of simplices behind Whitney forms is restrictive. Hence, in the literature one can find extensions to other cell types [18,28,29]. Furthermore, there have also been attempts to generalise them to higher order functions [30–32]. While the literature recognises several extensions of Whitney forms, the usage of the term "Whitney forms" is not unambiguous. The term is used for different type of objects by different authors, and the other way around, some instances of Whitney forms are sometimes called with a completely different name.

In this paper we clarify the concept of Whitney forms and create a synthesis of papers published on them. Our aim is to explain explicitly the key properties of Whitney forms and provide foundations for extending Whitney forms beyond their original assumptions. For this, in Section 3, we discuss Whitney's initial definition in more depth than usually and give three equivalent definitions, each emphasising a certain aspect of Whitney forms. In Section 4 we give a comprehensive exposition of their main properties, including the proofs. To further clarify the concept of Whitney forms, in Section 5 we consider generalisations that are called Whitney forms in the literature and check which of the properties are preserved. This reveals the trade-offs involved in extending Whitney forms to non-simplicial complexes and higher order functions. That is, to bypass assumptions involved in Whitney's initial setting, one also has to give up on some properties.

Regarding our contribution to the scientific literature, there is no prior paper which systematically lists all the key properties of Whitney forms with proofs. Although the results included in this paper can be considered as known, there are new aspects and some technical details that have not been published before. Our definitions and results are given in the spirit of Whitney's book and do not require Lebesgue theory or Sobolev spaces. This includes Theorems 4.9 and 5.1, which bring the approximation property of finite element theory into Whitney's setting. The proof of Theorem 4.3 has also not appeared elsewhere. This theorem could also be shown using Proposition 4.4 and the known fact that constants are in the span of Whitney forms, but the authors are not aware of such a proof – or even the proof of Proposition 4.4 – in the literature.

2. Preliminaries and notation

In this section some of the prerequisite concepts are briefly recalled. We expect the reader is familiar with exterior algebra and differential forms (see e.g. [1, Chapters I–III]).

Standard Whitney forms are differential forms in a simplicial complex. *Simplicial complex* K is a finite set of simplices such that

- each face of every simplex in K is also in K.
- The intersection of two simplices in K is either a common face of theirs or the empty set.

Complexes consisting of more general cells can be defined similarly. As in the initial context of Whitney forms [1], we assume that the simplices are embedded in affine space and tile a domain Ω. For simplicity, we may take $\mathbb{R}^n$ as the affine space, keeping in mind that only the affine structure of $\mathbb{R}^n$ is required, so that Ω is a polyhedron in $\mathbb{R}^n$. The general case where Ω is a manifold is covered in Section 5, which discusses generalisations of Whitney forms. We denote simplices by labels σ and τ, and $\sigma = x_0 \dots x_p$ means that σ is the oriented p-simplex whose vertices are $x_0, \dots, x_p$ and whose orientation is implied by this order of vertices. S^p denotes the set of p-simplices and $\mathrm{vect}(\sigma)$ the vectorial volume of σ (i.e. the p-vector of σ, see [1, III, §1]).

Recall that to each 0-simplex x_i of K corresponds a *barycentric function* λ_i — it is the unique function which is affine in each simplex and whose value is one at x_i and zero at other 0-simplices. Barycentric functions are the main building block for Whitney forms. We remark that they are exclusive to simplicial complexes, but we will discuss the generalisation of barycentric coordinates for other cells than simplices when considering extensions of Whitney forms.

Differential p-form in a complex K [1, p. 226] is a set of smooth p-forms ω_σ in the cells σ of K satisfying the following *patch condition*: if τ is a face of σ, then the trace $\omega_\sigma|_\tau$ of ω_σ equals ω_τ in τ. In other words, $\langle \omega_\sigma(x), \alpha \rangle = \langle \omega_\tau(x), \alpha \rangle$ for all

$x \in \tau$ and all p-vectors α in the plane of τ. (Here and throughout the paper, we denote the action of a p-covector ω on a p-vector α by $\langle \omega, \alpha \rangle$.) This means that if τ is the cell for which $x \in \tau - \partial\tau$ and α is in the plane of τ, then $\langle \omega_\sigma(x), \alpha \rangle$ is the same for all σ containing x. Hence the set of p-forms ω_σ induces a single p-form ω such that $\langle \omega(x), \alpha \rangle$ is single-valued (i.e. well-defined) for such p-vectors α.

The patch condition ensures that differential p-forms in K can be integrated over p-cells in K. Denote by $F^p(K)$ the space of differential p-forms in K. Note that since the exterior derivative d commutes with trace, we have $\mathrm{d}\,\omega \in F^{p+1}(K)$ if $\omega \in F^p(K)$, but the Hodge star $\star\omega$ is not necessarily in $F^{n-p}(K)$.

When K is a simplicial complex, formal sums $\sum_{\sigma_i \in S^p} a_i\sigma_i$ of oriented p-simplices with real coefficients are called *p-chains of K* [1, App. II, §6]. These form a vector space $C_p(K)$ for which the p-simplices σ_i constitute a natural basis (here $\sigma_i = 1\sigma_i$, the sum in which $a_j = \delta_{ij}$, the Kronecker delta). The elements of the dual space $C_p^*(K)$ are *p-cochains of K*. Following [1], we use σ_i to denote also the cochain whose value is δ_{ij} at the chain σ_j. Then the p-simplices σ_i constitute the dual basis for $C_p^*(K)$, and also cochains can be written as formal sums of simplices. Negative coefficients indicate change of orientation so that $-\sigma$ is the simplex σ with opposite orientation. Chains and cochains for more general cell complexes are defined similarly.

Since p-forms can be integrated over p-cells, each p-form ω yields a p-cochain whose values on chains are determined by integration of ω. Namely, the *de Rham map* $\mathcal{C} : F^p(K) \to C_p^*(K)$ is a linear map defined by

$$\mathcal{C}\omega(\sum_i a_i\sigma_i) = \int_{\sum_i a_i\sigma_i} \omega = \sum_i a_i \int_{\sigma_i} \omega,$$

where the second equality is the definition of integration on p-chains. *Coboundary operator* $\mathrm{d} : C_p^*(K) \to C_{p+1}^*(K)$ is a linear map defined by $\mathrm{d}\,X(c) = X(\partial c)$. We use the same notation d as for the exterior derivative of forms. Stokes' theorem then implies that $\mathcal{C}\,\mathrm{d} = \mathrm{d}\,\mathcal{C}$.

3. Three equivalent definitions of Whitney forms

Whitney p-forms are a finite-dimensional subspace of differential p-forms in a simplicial complex K. To each p-simplex σ corresponds a Whitney p-form $\mathcal{W}\sigma$. Since σ also denotes a basis cochain of $C_p^*(K)$ (and linear maps are uniquely determined by their action on basis elements), this correspondence defines a linear map $\mathcal{W} : C_p^*(K) \to F^p(K)$. $\mathcal{W}$ is known as the Whitney map, and Whitney forms are its images. This is made precise in the following definition.

Definition 3.1. The Whitney 0-form corresponding to the 0-simplex x_i is the barycentric function $\mathcal{W}x_i = \lambda_i$. For $p > 0$, the Whitney p-form corresponding to the p-simplex $x_0 \dots x_p$ is [1, VII, 11.16]

$$\mathcal{W}(x_0 \dots x_p) = p! \sum_{i=0}^{p} (-1)^i \lambda_i \,\mathrm{d}\,\lambda_0 \wedge \cdots \wedge \widehat{\mathrm{d}\,\lambda_i} \wedge \cdots \wedge \mathrm{d}\,\lambda_p, \tag{3.1}$$

where $\widehat{}$ indicates a term omitted from the product.

For each p, the Whitney map $\mathcal{W} : C_p^*(K) \to F^p(K)$ is defined by setting

$$\mathcal{W}(\sum_{\sigma_i \in S^p} a_i\sigma_i) = \sum_{\sigma_i \in S^p} a_i \mathcal{W}(\sigma_i).$$

The image $\mathcal{W}(C_p^*(K)) = \mathrm{span}\{\mathcal{W}\sigma \mid \sigma \in S^p\} \subset F^p(K)$ is the space of Whitney p-forms and denoted by W^p.

Note that although the λ_i are not globally smooth, they are smooth in each simplex, so (3.1) defines a p-form in each simplex of K. The patch condition holds because barycentric functions in σ restrict to barycentric functions on the faces of σ (and trace commutes with $\wedge$ and d). Hence (3.1) yields a well-defined differential form in K. Note also that the right hand side of (3.1) changes sign when the orientation changes, so $\mathcal{W}(-\sigma) = -\mathcal{W}\sigma$ and the Whitney map is well-defined.

Since the definition of Whitney forms is the main issue here, we cover it in more detail than usually and from different viewpoints. First, we give an alternative but equivalent definition. Set $\mathcal{W}\sigma = 0$ in τ if σ is not a face of τ. If it is, say $\sigma = x_0 \dots x_p$ and τ has vertices $\{x_0, \dots, x_p, x_{p+1} \dots, x_q\}$, set [1, VII, 11.12]

$$\langle \mathcal{W}\sigma(x), \alpha \rangle = p! \frac{\alpha \wedge (x_{p+1} - x) \wedge (x_{p+2} - x_{p+1}) \wedge \cdots \wedge (x_q - x_{p+1})}{(x_1 - x_0) \wedge \cdots \wedge (x_q - x_0)} \quad \text{in } \tau; \tag{3.2}$$

that is, the value of the p-form $\mathcal{W}\sigma$ at point $x \in \tau$ is the p-covector whose value on a p-vector α is defined as the ratio of the two q-vectors in the plane of τ. This can be written equivalently as

$$\langle \mathcal{W}\sigma(x), \alpha \rangle = \frac{p!(q-p)!}{q!} \frac{\alpha \wedge \mathrm{vect}(x x_{p+1} \dots x_q)}{\mathrm{vect}(x_0 \dots x_p \dots x_q)},$$

from which we see that $\mathcal{W}\sigma$ in τ does not depend on the orientation of τ but changes sign when the orientation of σ changes. For $x \in y_0 \dots y_p \subset \tau$, (3.2) becomes

$$\langle \mathcal{W}\sigma(x), \mathrm{vect}(y_0 \dots y_p) \rangle = \frac{\mathrm{vect}(y_0 \dots y_p x_{p+1} \dots x_q)}{\mathrm{vect}(x_0 \dots x_q)}. \tag{3.3}$$

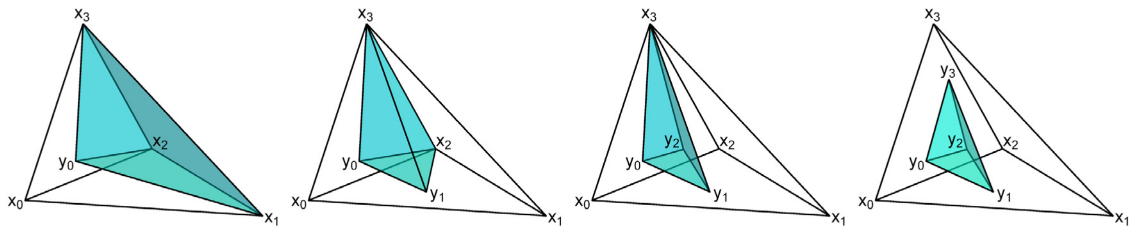

Fig. 1. Illustration of (3.3) in tetrahedron $\tau = x_0x_1x_2x_3$ for the cases $\sigma = x_0$, $\sigma = x_0x_1$, $\sigma = x_0x_1x_2$, and $\sigma = x_0x_1x_2x_3$. In each case, $\langle \mathcal{W}\sigma(x), \mathrm{vect}(y_0 \dots y_p)\rangle$ is the ratio of the highlighted volume and the volume of the whole tetrahedron. This holds for all $x \in y_0 \dots y_p$.

To see this, note that $\mathrm{vect}(y_0 \dots y_p) \wedge (x_{p+1} - x) = \mathrm{vect}(y_0 \dots y_p) \wedge (x_{p+1} - y_p - (x - y_p)) = \mathrm{vect}(y_0 \dots y_p) \wedge (x_{p+1} - y_p)$ since $x - y_p$ is in the plane of $y_0 \dots y_p$.

Although volumes depend on the metric, their ratios do not, and the above formula is meaningful in affine space. This definition beautifully illustrates the geometric character of Whitney forms (see Fig. 1), while Definition 3.1 offers an explicit formula in terms of barycentric functions. Whitney showed that these two definitions are indeed equivalent.

Proposition 3.2. *The definition with the geometric formula (3.2) is equivalent to Definition 3.1.*

Proof. Let $\sigma = x_0 \dots x_p \in S^p$ and $\tau \in S^q$, and denote by $\mathcal{W}_1\sigma$ the Whitney form of σ given by (3.1) and by $\mathcal{W}_2\sigma$ that given by (3.2). To show that $\mathcal{W}_1\sigma = \mathcal{W}_2\sigma$ in τ, we first note that both $\mathcal{W}_1\sigma$ and $\mathcal{W}_2\sigma$ zero in τ if σ is not a face of τ. Moreover, both are affine in τ, are zero at those vertices of τ that are not in σ, and change sign when the orientation of σ changes. Hence it suffices to consider the case $\tau = x_0 \dots x_p x_{p+1} \dots x_q$ and show that $\mathcal{W}_1\sigma(x_0) = \mathcal{W}_2\sigma(x_0)$.

Since the edge vectors $x_i - x_0$ span the plane of τ, all p-vectors in τ can be written as linear combinations of their wedge products. Hence it suffices to show $\langle \mathcal{W}_1\sigma(x_0), \alpha\rangle = \langle \mathcal{W}_2\sigma(x_0), \alpha\rangle$ for p-vectors α of the form $\alpha = (x_{i_1} - x_0) \wedge \cdots \wedge (x_{i_p} - x_0)$ for $i_1 < \cdots < i_p$. Since $\lambda_i(x_0) = 0$ and $\langle \mathrm{d}\,\lambda_i(x_0), x_j - x_0\rangle = \delta_{ij}$ if $i \neq 0$, we have

$$\langle \mathcal{W}_1\sigma(x_0), (x_{i_1} - x_0) \wedge \cdots \wedge (x_{i_p} - x_0)\rangle = 0 \quad \text{if any of the indices } i_j \text{ are not in } \{1, \dots, p\}$$
$$\langle \mathcal{W}_1\sigma(x_0), (x_1 - x_0) \wedge \cdots \wedge (x_p - x_0)\rangle = p!$$

By (3.2) the same is true for $\mathcal{W}_2\sigma(x_0)$; hence $\mathcal{W}_1\sigma(x_0) = \mathcal{W}_2\sigma(x_0)$. □

At this point, it is instructive to briefly discuss Whitney's book [1] and the role of Whitney forms there. The book is about p-dimensional integration in n-dimensional space. What we call chains (and cochains) of K are called algebraic chains (and cochains) in [1] where p-chains have a more general meaning as p-dimensional domains of integration. Whitney starts from polyhedral p-chains – formal sums of polyhedral p-cells with real coefficients and invariance under subdivision – which form an infinite-dimensional vector space. This space can be equipped with a norm and then completed with respect to that norm; for example, the flat norm [1, V, §3] yields the space of flat p-chains $C_p^\flat$. Its (continuous) dual space $C_p^{\flat *}$ is the space of flat p-cochains and consists of bounded linear functionals $C_p^\flat \to \mathbb{R}$. Similarly, the sharp norm [1, V, §6] yields the spaces of sharp p-chains $C_p^\sharp$ and sharp p-cochains $C_p^{\sharp *}$.

We saw that Whitney forms correspond to (algebraic) cochains of a simplicial complex K, but they also correspond to certain flat cochains in K. This explains why Whitney forms are sometimes called flat forms. A correspondence between flat forms and flat cochains is made precise in Wolfe's theorem [1, IX, Theorem 7C]. Without going into details, p-form ω and p-cochain X correspond if $\int_\sigma \omega = X(\sigma)$ for all p-cells σ. In his work [1, VII, §11], Whitney defined a linear injection ϕ from the algebraic cochains of K to flat cochains in K, which he used to prove that the cohomology ring of flat cochains is isomorphic to that of algebraic cochains. The images of ϕ he called elementary flat cochains in K, and these are in correspondence with Whitney forms.

Whitney's theory of p-chains as p-dimensional domains of integration had some shortcomings. For instance, sharp chains do not have a continuous boundary operator, while the Hodge star of a flat form is not flat. The theory has since been extended by Jenny Harrison [33]. We need not go deeper into this. However, now that we have mentioned chains, we can briefly discuss another way to look at the definition of Whitney forms, as emphasised by Alain Bossavit [14,29,34]: approximating p-chains with algebraic p-chains.

To explain this, we extend the notation $\langle \omega, c\rangle := \int_c \omega$ for differential forms ω and chains c. This expression is bilinear and can be interpreted either as the evaluation of ω on c or (by duality) as the evaluation of c on ω. Similarly, denote $\langle X, c\rangle = X(c)$ for cochains X and chains c. Whitney forms have the property $\langle \mathcal{W}\sigma_j, \sigma_i\rangle = \delta_{ij}$ and hence enable one to approximate a p-form ω in W^p with $\tilde{\omega} = \sum_{\sigma_i \in S^p} \langle \omega, \sigma_i\rangle \mathcal{W}\sigma_i$. The approximation $\tilde{\omega}$ has the property that $\langle \tilde{\omega}, c\rangle = \langle \omega, c\rangle$ — not for all p-chains c, but for algebraic chains, namely those in $C_p(K)$. This has a dual viewpoint: one can approximate a p-chain c in $C_p(K)$ with $\tilde{c} = \sum_{\sigma_i \in S^p} \langle \mathcal{W}\sigma_i, c\rangle \sigma_i$, and the approximation $\tilde{c}$ has the property that $\langle \omega, c\rangle = \langle \omega, \tilde{c}\rangle$ — not for all p-forms ω, but for those in W^p. Letting $\mathcal{W}^t$ denote the map $c \mapsto \tilde{c}$, we have $\langle \mathcal{W}X, c\rangle = \langle X, \mathcal{W}^t c\rangle$ for all p-chains c and all $X \in C_p^*(K)$.

On the other hand, if we have such a map $\mathcal{W}^t$ to approximate p-chains in $C_p(K)$, this defines a map $\mathcal{W}$ from $C_p^*(K)$ to $F^p(K)$ by requiring that $\langle \mathcal{W}X, c\rangle = \langle X, \mathcal{W}^t c\rangle$ hold for all p-chains c and all $X \in C_p^*(K)$. This approach to the definition of Whitney forms is used e.g. in [14,29,31,34,35]. When suitable conditions are imposed for the map $\mathcal{W}^t$, this approach leads to the following, yet another equivalent definition of Whitney forms, which first appeared in [34]. Setting $\mathcal{W}x_i = \lambda_i$ for $p = 0$, the Whitney form corresponding to p-simplex σ for $p > 0$ is obtained recursively by

$$\mathcal{W}\sigma = \sum_{\tau \in S^{p-1}} \mathbf{d}_\tau^\sigma \lambda_{\sigma-\tau} \,\mathrm{d}\, \mathcal{W}\tau, \tag{3.4}$$

where $\mathbf{d}_\tau^\sigma$ is the incidence number relating τ and σ (which is 0 if τ is not a face of σ and ± 1 if it is, the sign depending on whether the orientations agree or not) and $\sigma - \tau$ denotes the vertex opposite to the $(p-1)$-face τ of σ.

It is easy to show that this definition is equivalent to Definition 3.1, after we first note that the exterior derivative of the Whitney p-form $\mathcal{W}(x_0 \dots x_p)$ for any p-simplex $x_0 \dots x_p \in S^p$ is

$$\mathrm{d}\,\mathcal{W}(x_0 \dots x_p) = p! \sum_{i=0}^{p} (-1)^i \,\mathrm{d}\,\lambda_i \wedge \mathrm{d}\,\lambda_0 \wedge \cdots \wedge \widehat{\mathrm{d}\,\lambda_i} \wedge \cdots \wedge \mathrm{d}\,\lambda_p = (p+1)!\, \mathrm{d}\,\lambda_0 \wedge \cdots \wedge \mathrm{d}\,\lambda_p. \tag{3.5}$$

Proposition 3.3. *The definition with the recursive formula (3.4) is equivalent to Definition 3.1.*

Proof. First note that writing $\sigma = x_0 \dots x_p$ we get

$$\sum_{\tau \in S^{p-1}} \mathbf{d}_\tau^\sigma \lambda_{\sigma-\tau} \,\mathrm{d}\, \mathcal{W}\tau = \sum_{i=0}^{p} (-1)^i \lambda_i \,\mathrm{d}\, \mathcal{W}(x_0 \dots \widehat{x_i} \dots x_p).$$

For $\sigma = x_0x_1$ this becomes $\lambda_0 \,\mathrm{d}\, \mathcal{W}x_1 - \lambda_1 \,\mathrm{d}\, \mathcal{W}x_0 = \lambda_0 \,\mathrm{d}\,\lambda_1 - \lambda_1 \,\mathrm{d}\,\lambda_0$, which is the same as $\mathcal{W}x_0x_1$ of Definition 3.1, proving the claim for 1-simplices. Suppose as induction hypothesis that it holds for $(p-1)$-simplices, and let $\sigma = x_0 \dots x_p$ be a p-simplex. By (3.5) we get

$$\begin{aligned}\sum_{\tau \in S^{p-1}} \mathbf{d}_\tau^\sigma \lambda_{\sigma-\tau} \,\mathrm{d}\, \mathcal{W}\tau &= \sum_{i=0}^{p} (-1)^i \lambda_i \,\mathrm{d}\, \mathcal{W}(x_0 \dots \widehat{x_i} \dots x_p) \\ &= \sum_{i=0}^{p} (-1)^i \lambda_i p!\, \mathrm{d}\,\lambda_0 \wedge \cdots \wedge \widehat{\mathrm{d}\,\lambda_i} \wedge \dots \mathrm{d}\,\lambda_p = \mathcal{W}(x_0 \dots x_p). \quad \square\end{aligned}$$

3.1. Proxy fields

The definition of Whitney forms does not require the notion of metric; only the affine structure of the ambient space is invoked. However, metric structure allows one to identify certain differential forms with scalar or vector fields, so-called proxy fields. Indeed, Whitney forms are often presented in terms of these proxy fields. To clarify such seemingly different definitions, let us look at the 3-dimensional case with Euclidean metric and standard orientation (so that right-hand rule is used for cross product).

0-forms are scalar functions, so there is no distinction between a 0-form and its proxy field. In each simplex of K, flat map $\flat$ from vector fields to 1-forms is defined by $\langle \flat u(x), v\rangle = u(x) \cdot v$; that is, the value of $\flat u$ at point x is the covector whose value on vector v is the dot product $u(x) \cdot v$. This is an isomorphism with inverse $\sharp$, and the proxy field of a 1-form ω is the vector field $\sharp\omega$. Similarly, if u is a vector field, the rule $v_1 \wedge v_2 \mapsto u(x) \cdot v_1 \times v_2$ defines a 2-form, and this yields a correspondence between vector fields and 2-forms. The proxy field of a 2-form ω can be written as $\sharp \star \omega$, where $\star$ is the Hodge star. Finally, a scalar field f defines a 3-form by the rule $v_1 \wedge v_2 \wedge v_3 \mapsto f(x)\det(v_1, v_2, v_3)$, and any 3-form is obtained this way from a scalar field f, its proxy field. When considered globally in K, the proxy fields of 1- and 2-forms in K have a well-defined tangential and normal component on inter-element boundaries (respectively).

In this case the proxy fields are perhaps more easily explained in terms of standard coordinates. The proxy field of the 1-form $\omega_1 \,\mathrm{d}\,x^1 + \omega_2 \,\mathrm{d}\,x^2 + \omega_3 \,\mathrm{d}\,x^3$ is the vector field $(\omega_1, \omega_2, \omega_3)$, the proxy field of the 2-form $\omega_{12} \,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 + \omega_{13} \,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 + \omega_{23} \,\mathrm{d}\,x^2 \wedge \mathrm{d}\,x^3$ is the vector field $(\omega_{23}, -\omega_{13}, \omega_{12})$, and the proxy field of the 3-form $\omega_{123} \,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 \wedge \mathrm{d}\,x^3$ is the scalar field ω_{123}. (This holds more generally when Ω is an oriented Riemannian manifold and $\{x^1, x^2, x^3\}$ is any positively oriented orthonormal frame.) When ω is a differential form, denote by $\omega^\sharp$ its proxy field.

Theorem 3.4. *In a tetrahedron $x_0x_1x_2x_3$, the proxy fields of Whitney forms are*

$$\begin{aligned}(\mathcal{W}x_0x_1)^\sharp &= \lambda_0 \nabla\lambda_1 - \lambda_1 \nabla\lambda_0 \\ (\mathcal{W}x_0x_1x_2)^\sharp &= 2(\lambda_0 \nabla\lambda_1 \times \nabla\lambda_2 - \lambda_1 \nabla\lambda_0 \times \nabla\lambda_2 + \lambda_2 \nabla\lambda_0 \times \nabla\lambda_1)\end{aligned}$$

$$(\mathcal{W}x_0x_1x_2x_3)^\sharp = 6\Big(\lambda_0(\nabla\lambda_1 \times \nabla\lambda_2)\cdot\nabla\lambda_3 - \lambda_1(\nabla\lambda_0 \times \nabla\lambda_2)\cdot\nabla\lambda_3$$
$$+ \lambda_2(\nabla\lambda_0 \times \nabla\lambda_1)\cdot\nabla\lambda_3 - \lambda_3(\nabla\lambda_0 \times \nabla\lambda_1)\cdot\nabla\lambda_2\Big)$$

and their values at $x \in x_0x_1x_2x_3$ can be written as

$$(\mathcal{W}x_0x_1)^\sharp(x) = a \times x + b$$
$$(\mathcal{W}x_0x_1x_2)^\sharp(x) = cx + d$$
$$(\mathcal{W}x_0x_1x_2x_3)^\sharp(x) = \pm\frac{1}{|x_0x_1x_2x_3|},$$

where the vectors $a = \pm\frac{x_3-x_2}{6|x_0x_1x_2x_3|}$, $b = \mp\frac{x_3-x_2}{6|x_0x_1x_2x_3|} \times x_2$, and $d = \pm\frac{1}{3|x_0x_1x_2x_3|}x_3$ and the scalar $c = \mp\frac{1}{3|x_0x_1x_2x_3|}$ are constants and the signs depend on whether $\{x_1 - x_0, x_2 - x_0, x_3 - x_0\}$ is a right-handed frame or not.

Proof. Since the gradient ∇f of a function f is $(\mathrm{d}f)^\sharp$ and for 1-forms ω, η, and ξ we have

$$(\omega \wedge \eta)^\sharp = \omega^\sharp \times \eta^\sharp, \quad (\omega \wedge \eta \wedge \xi)^\sharp = (\omega^\sharp \times \eta^\sharp)\cdot\xi^\sharp,$$

the first part follows from Definition 3.1. Since the gradients of barycentric functions are constants, we omit the variable x from them and write

$$(\mathcal{W}x_0x_1)^\sharp(x) = \lambda_0(x)\nabla\lambda_1 - \lambda_1(x)\nabla\lambda_0 = \nabla\lambda_0\cdot(x - x_2)\nabla\lambda_1 - \nabla\lambda_1\cdot(x - x_2)\nabla\lambda_0$$
$$= (\nabla\lambda_0\cdot x)\nabla\lambda_1 - (\nabla\lambda_1\cdot x)\nabla\lambda_0 - (\nabla\lambda_0\cdot x_2)\nabla\lambda_1 + (\nabla\lambda_1\cdot x_2)\nabla\lambda_0$$
$$= (\nabla\lambda_0 \times \nabla\lambda_1) \times x - (\nabla\lambda_0 \times \nabla\lambda_1) \times x_2.$$

Here we used the identity

$$(a \times b) \times c = (a\cdot c)b - (b\cdot c)a. \tag{3.6}$$

Note that in place of x_2 in the vector b we could use any point of x_2x_3.

For any permutation $i_1i_2i_3i_4$ of 1234, the vector $(x_{i_2} - x_{i_3}) \times (x_{i_4} - x_{i_3})$ is orthogonal to $x_{i_2}x_{i_3}x_{i_4}$ and has length equal to $2|x_{i_2}x_{i_3}x_{i_4}|$. On the other hand, $\nabla\lambda_{i_1}$ is orthogonal to $x_{i_2}x_{i_3}x_{i_4}$ and has length equal to the reciprocal of the height of the tetrahedron with respect to the face $x_{i_2}x_{i_3}x_{i_4}$. Hence we have

$$\nabla\lambda_{i_1} = \pm\frac{(x_{i_2} - x_{i_3}) \times (x_{i_4} - x_{i_3})}{6|x_0x_1x_2x_3|}.$$

The sign is $+$ if $\{x_{i_2} - x_{i_3}, x_{i_4} - x_{i_3}, x_{i_1} - x_{i_3}\}$ is a right-handed frame and $-$ otherwise. Using (3.6) again we get

$$\nabla\lambda_{i_1} \times \nabla\lambda_{i_2} = \pm\frac{(x_{i_2} - x_{i_3}) \times (x_{i_4} - x_{i_3})}{6|x_0x_1x_2x_3|} \times \nabla\lambda_{i_2} = \pm\frac{x_{i_4} - x_{i_3}}{6|x_0x_1x_2x_3|},$$
$$(\nabla\lambda_{i_1} \times \nabla\lambda_{i_2})\cdot\nabla\lambda_{i_3} = \pm\frac{x_{i_4} - x_{i_3}}{6|x_0x_1x_2x_3|}\cdot\nabla\lambda_{i_3} = \frac{\mp 1}{6|x_0x_1x_2x_3|},$$

the signs depending as above. Using the handedness of $\{x_1-x_0, x_2-x_0, x_3-x_0\}$ to determine the signs for each permutation, these formulas yield

$$(\mathcal{W}x_0x_1)^\sharp(x) = (\nabla\lambda_0 \times \nabla\lambda_1) \times x - (\nabla\lambda_0 \times \nabla\lambda_1) \times x_2 = a \times x + b,$$
$$(\mathcal{W}x_0x_1x_2)^\sharp(x) = 2(\lambda_0(x)\nabla\lambda_1 \times \nabla\lambda_2 - \lambda_1(x)\nabla\lambda_0 \times \nabla\lambda_2 + \lambda_2(x)\nabla\lambda_0 \times \nabla\lambda_1)$$
$$= 2\left(\lambda_0(x)\left(\pm\frac{x_3 - x_0}{6|x_0x_1x_2x_3|}\right) - \lambda_1(x)\left(\pm\frac{x_1 - x_3}{6|x_0x_1x_2x_3|}\right) + \lambda_2(x)\left(\pm\frac{x_3 - x_2}{6|x_0x_1x_2x_3|}\right)\right)$$
$$= \pm\frac{1}{3|x_0x_1x_2x_3|}\Big(\big(\lambda_0(x) + \lambda_1(x) + \lambda_2(x)\big)x_3 - \lambda_0(x)x_0 - \lambda_1(x)x_1 - \lambda_2(x)x_2\Big)$$
$$= \pm\frac{1}{3|x_0x_1x_2x_3|}\Big(\big(1 - \lambda_3(x)\big)x_3 - \lambda_0(x)x_0 - \lambda_1(x)x_1 - \lambda_2(x)x_2\Big) = \pm\frac{x_3 - x}{3|x_0x_1x_2x_3|} = cx + d,$$
$$(\mathcal{W}x_0x_1x_2x_3)^\sharp(x) = 6\Big(\lambda_0(x)(\nabla\lambda_1 \times \nabla\lambda_2)\cdot\nabla\lambda_3 - \lambda_1(x)(\nabla\lambda_0 \times \nabla\lambda_2)\cdot\nabla\lambda_3$$
$$+ \lambda_2(x)(\nabla\lambda_0 \times \nabla\lambda_1)\cdot\nabla\lambda_3 - \lambda_3(x)(\nabla\lambda_0 \times \nabla\lambda_1)\cdot\nabla\lambda_2\Big) = 6\Big(\lambda_0(x)\frac{\pm 1}{6|x_0x_1x_2x_3|}$$
$$- \lambda_1(x)\frac{\mp 1}{6|x_0x_1x_2x_3|} + \lambda_2(x)\frac{\pm 1}{6|x_0x_1x_2x_3|} - \lambda_3(x)\frac{\mp 1}{6|x_0x_1x_2x_3|}\Big) = \pm\frac{1}{|x_0x_1x_2x_3|}. \quad \square$$

The proxy fields of Whitney forms first appeared in [10] and are sometimes called Whitney elements or Nedelec elements; 1-forms correspond to “edge elements”. Be aware that in some places the proxy fields are called just Whitney forms and are given as the definition of Whitney forms. We make the distinction that Whitney forms are always differential forms and Whitney elements mean their proxy fields.

4. Properties of Whitney forms

In this section we discuss the main properties of Whitney forms. Although these are mostly well-known, the kind of list that we have compiled is not easily found in the literature. In particular, we include proofs for all properties that are not evident from the discussion of Section 3. We also try to put emphasis on why these properties are relevant, to explain why one would like to preserve them for generalisations of Whitney forms.

Property 1: Whitney forms are differential forms in a complex

“Whitney forms are differential forms in a complex” concisely summarises their conformity properties on inter-element boundaries. Whitney p-form is an element of the space $F^p(K)$, so it is a set of p-forms ω_σ in the cells σ of K. Thanks to the patch condition in the definition of $F^p(K)$, we can consider this set of p-forms as a single p-form ω such that $\langle \omega(x), \alpha \rangle$ is well-defined for p-vectors α in the plane of the cell τ for which $x \in \tau - \partial\tau$. This reflects how finite element spaces for differential forms are built in FEEC theory [19,36] by first constructing them in each cell and then assembling the local constructions together.

This property ensures that Whitney forms can be used as conforming finite elements and p-forms can be integrated over p-cells in K. Perhaps most importantly, it prescribes what type of objects Whitney forms are in the first place. Hence we propose that all generalisations of Whitney forms should at the very least be differential forms in a complex to be called Whitney forms.

Property 2: W^p is isomorphic to $C_p^*(K)$

That Whitney forms correspond to the cells of K can already be seen from Definition 3.1: to the cochain σ corresponds the Whitney form $\mathcal{W}\sigma$, and Whitney p-forms are the images of the map $\mathcal{W} : C_p^*(K) \to F^p(K)$. The following proposition makes the correspondence more precise.

Proposition 4.1. *The map $\mathcal{W} : C_p^*(K) \to F^p(K)$ is an isomorphism onto its image W^p. Moreover, $\mathcal{CW}X = X$ for all $X \in C_p^*(K)$.*

Proof. For the first claim it suffices to show that $\mathcal{W}$ is injective, which follows from the second claim. To prove $\mathcal{CW}X = X$ for all $X \in C_p^*(K)$ it suffices to show that $\int_{\sigma_i} \mathcal{W}\sigma_j = \delta_{ij}$, whence the claim follows by linearity.

That $\int_{\sigma_i} \mathcal{W}\sigma_j = \delta_{ij}$ is perhaps most easily seen using (3.2) or (3.3) and Proposition 3.2. □

Because of this property, integrals on p-cells of K serve as unisolvent degrees of freedom for Whitney p-forms. This means that values of the integrals are in one-to-one correspondence with elements of W^p. Moreover, this correspondence is the simplest possible since $\int_{\sigma_i} \mathcal{W}\sigma_j = \delta_{ij}$. Note that $\sum_{\sigma_i \in S^p} a_i \mathcal{W}\sigma_i = 0$ implies $a_j = \int_{\sigma_j} \sum_{\sigma_i \in S^p} a_i \mathcal{W}\sigma_i = 0\ \forall j$, so the set $\{\mathcal{W}\sigma_i \mid \sigma_i \in S^p\}$ is linearly independent. Since it also spans W^p, it constitutes a basis for W^p.

There are two consequences. Firstly, we can interpolate the cochain $X \in C_p^*(K)$ with the p-form $\mathcal{W}X$, and the integrals of this interpolant match with the values of the cochain on p-simplices: $\mathcal{CW}X = X$. Secondly, we can approximate the p-form $\omega \in F^p(K)$ with the Whitney form $\mathcal{WC}\omega$, and the integrals of this approximation match with those of ω on p-simplices: $\mathcal{CWC}\omega = \mathcal{C}\omega$. Indeed, Whitney p-forms are commonly considered as a tool for either interpolating p-cochains or approximating differential p-forms.

Property 3: Whitney forms are first order polynomials in each cell

In each cell, barycentric functions are affine and hence their exterior derivatives are constant, so we see from Definition 3.1 that Whitney forms are affine. Hence they are at most first order polynomials in each cell. (That is, if $\omega \in W^p$, the function $x \mapsto \langle \omega(x), \alpha \rangle$ is a first order polynomial for each p-vector α.) This of course implies that they are also smooth in each cell.

Property 4: Whitney forms are affine invariant

In addition to being affine in each cell, Whitney forms are affine objects in the following two senses. First, their definition is meaningful in affine space without any choice of metric. Furthermore, they are invariant under affine transformations.

Proposition 4.2. *Let $\sigma = x_0 \dots x_n$ and $\tau = y_0 \dots y_n$ be two n-simplices and $\varphi : \sigma \to \tau$ affine map such that $\varphi(x_i) = y_i$. Then*

$$\mathcal{W}(x_0 \dots x_p) = \varphi^*(\mathcal{W}(y_0 \dots y_p)) \quad \text{in } \sigma.$$

Proof. Let λ_i denote the barycentric coordinates in σ and μ_i those in τ. Since $\mu_i \circ \varphi$ is affine in σ and $(\mu_i \circ \varphi)(x_j) = \delta_{ij}$, it follows that $\varphi^*(\mu_i) = \mu_i \circ \varphi = \lambda_i$. Hence by the naturality of pullback with respect to wedge product and exterior derivative we have

$$\varphi^*(\mathcal{W}(y_0 \dots y_p)) = \varphi^*(p! \sum_{i=0}^{p} (-1)^i \mu_i \, \mathrm{d}\,\mu_0 \wedge \cdots \wedge \widehat{\mathrm{d}\,\mu_i} \wedge \cdots \wedge \mathrm{d}\,\mu_p)$$

$$= p! \sum_{i=0}^{p} (-1)^i \mu_i \, \mathrm{d}\,\varphi^*(\mu_0) \wedge \cdots \wedge \mathrm{d}\,\widehat{\varphi^*(\mu_i)} \wedge \cdots \wedge \mathrm{d}\,\varphi^*(\mu_p) = \mathcal{W}(x_0 \dots x_p). \quad \square$$

This property is useful because computations done in a reference simplex transfer to all simplices by affine transformations and hence need be done only once. For example, using (3.3),

$$\int_{\varphi(z_0 z_1)} \mathcal{W} y_0 y_1 = \int_{z_0 z_1} \varphi^*(\mathcal{W} y_0 y_1) = \int_{z_0 z_1} \mathcal{W} x_0 x_1 = \frac{\mathrm{vect}(z_0 z_1 x_2 \dots x_n)}{\mathrm{vect}(x_0 \dots x_n)} \quad \text{for } z_0 z_1 \subset \sigma.$$

This equality is also seen from (3.3), since volume ratios are preserved by affine transformations.

Property 5: locality

Whitney form $\mathcal{W}\sigma$ is nonzero only on those simplices that include σ as a face. Locality is needed to make system matrices sparse in numerical methods that utilise Whitney forms.

Property 6: Whitney forms constitute a partition of unity

Barycentric functions sum up to one, forming a partition of unity. The following theorem generalises this property for other Whitney forms.

Theorem 4.3. *In any q-simplex $\tau \in S^q$, for all points x and all p-vectors α in τ,*

$$\sum_{\sigma_i \in S^p} \langle \mathcal{W}\sigma_i(x), \alpha \rangle \, \mathrm{vect}(\sigma_i) = \alpha.$$

Proof. Suppose $\tau = x_0 \dots x_q \in S^q$ and $x \in \tau$. Since the edge vectors $x_i - x_0$ span the plane of τ, all p-vectors in τ can be written as linear combinations of their wedge products. Hence it suffices to consider the case $\alpha = (x_1 - x_0) \wedge \cdots \wedge (x_p - x_0)$, whereafter the claim follows by linearity.

At all points of τ

$$\langle \mathrm{d}\,\lambda_i, x_k - x_j \rangle = \begin{cases} 0 & \text{if } i \notin \{j, k\} \\ 1 & \text{if } i = k \\ -1 & \text{if } i = j \end{cases}$$

and hence for $i_1 < \cdots < i_p$ we have

$$\langle \mathrm{d}\,\lambda_{i_1} \wedge \cdots \wedge \mathrm{d}\,\lambda_{i_p}, (x_1 - x_0) \wedge \cdots \wedge (x_p - x_0) \rangle = \begin{cases} 0 & \text{if } \{i_1, \dots, i_p\} \not\subset \{0, \dots, p\} \\ (-1)^k & \text{if } \{i_1, \dots, i_p\} \subset \{0, \dots, \widehat{k}, \dots, p\} \end{cases}$$

Using this we see that

$$\langle \mathcal{W}(x_{i_0} \dots x_{i_p})(x), \alpha \rangle = 0 \quad \text{if at least two of the indices } i_j \text{ are not in } \{0, \dots, p\}$$

$$\langle \mathcal{W}(x_0 \dots x_{k-1} x_{i_k} x_{k+1} \dots x_p)(x), \alpha \rangle = p!(-1)^k \lambda_{i_k}(x)(-1)^k = p! \lambda_{i_k}(x) \quad \text{for } i_k \notin \{0, \dots, p\}$$

$$\langle \mathcal{W}(x_0 \dots x_p)(x), \alpha \rangle = p! \sum_{j=0}^{p} (-1)^j \lambda_j(x)(-1)^j = p! \sum_{j=0}^{p} \lambda_j(x)$$

Recalling that $\mathcal{W}\sigma_i = 0$ in τ if σ_i is not a face of τ, we can therefore write

$$\sum_{\sigma_i \in S^p} \langle \mathcal{W}\sigma_i(x), \alpha\rangle \operatorname{vect}(\sigma_i)$$
$$= p! \sum_{j=0}^{p} \lambda_j(x) \operatorname{vect}(x_0 \dots x_p) + \sum_{j=p+1}^{q} p!\lambda_j(x)\Big(\sum_{k=0}^{p} \operatorname{vect}(x_0 \dots x_{k-1}x_j x_{k+1} \dots x_p)\Big)$$

After rewriting the first term of the inner sum as

$$\operatorname{vect}(x_j x_1 \dots x_p) = \frac{1}{p!}(x_1 - x_j) \wedge \dots \wedge (x_p - x_j)$$
$$= \frac{1}{p!}\Big(x_1 - x_0 - (x_j - x_0)\Big) \wedge \dots \wedge \Big(x_p - x_0 - (x_j - x_0)\Big) = \frac{1}{p!}(x_1 - x_0) \wedge \dots \wedge (x_p - x_0)$$
$$- \frac{1}{p!}\sum_{l=1}^{p}(x_1 - x_0) \wedge \dots \wedge (x_{l-1} - x_0) \wedge (x_j - x_0) \wedge (x_{l+1} - x_0) \wedge \dots \wedge (x_p - x_0)$$
$$= \operatorname{vect}(x_0 \dots x_p) - \sum_{l=1}^{p} \operatorname{vect}(x_0 \dots x_{l-1}x_j x_{l+1} \dots x_p)$$

the other terms cancel, and we find out that

$$\sum_{\sigma_i \in S^p} \langle \mathcal{W}\sigma_i(x), \alpha\rangle \operatorname{vect}(\sigma_i) = p! \sum_{j=0}^{p} \lambda_j(x) \operatorname{vect}(x_0 \dots x_p) + \sum_{j=p+1}^{q} p!\lambda_j(x) \operatorname{vect}(x_0 \dots x_p)$$
$$= p! \operatorname{vect}(x_0 \dots x_p) = (x_1 - x_0) \wedge \dots \wedge (x_p - x_0) = \alpha. \quad \square$$

As we show next, this partition of unity property actually amounts to saying that W^p contains all constant forms.

Proposition 4.4. *Let $\widetilde{\mathcal{W}} : C_p^*(K) \to F^p(K)$ be any linear map such that $\mathcal{C}\widetilde{\mathcal{W}}X = X$ for all $X \in C_p^*(K)$, and denote by $\widetilde{W}^p$ its image in $F^p(K)$. Then*

i. *a p-form $\omega \in F^p(K)$ is in $\widetilde{W}^p$ if and only if $\widetilde{\mathcal{W}}\mathcal{C}\omega = \omega$.*
ii. *The partition of unity property of Theorem 4.3 holds for $\widetilde{\mathcal{W}}$ if and only if $\widetilde{W}^p$ contains all constant p-forms.*

Proof. i: If $\widetilde{\mathcal{W}}\mathcal{C}\omega = \omega$, then ω is in the image of $\widetilde{\mathcal{W}}$, while if $\omega \in \widetilde{W}^p$, then $\omega = \widetilde{W}X$ for some $X \in C_p^*(K)$, so $\widetilde{\mathcal{W}}\mathcal{C}\omega = \widetilde{\mathcal{W}}\mathcal{C}\widetilde{W}X = \widetilde{\mathcal{W}}X = \omega$.

ii: Suppose first that the partition of unity property holds, and let ω be a constant p-covector. For all points x and all p-vectors α

$$\langle \widetilde{\mathcal{W}}\mathcal{C}\omega(x), \alpha\rangle = \Big\langle \sum_{\sigma_i \in S^p}\Big(\int_{\sigma_i} \omega\Big)\widetilde{\mathcal{W}}\sigma_i(x), \alpha\Big\rangle = \sum_{\sigma_i \in S^p}\Big(\int_{\sigma_i} \omega\Big)\langle \widetilde{\mathcal{W}}\sigma_i(x), \alpha\rangle$$
$$= \sum_{\sigma_i \in S^p} \langle \omega, \operatorname{vect}(\sigma_i)\rangle \langle \widetilde{\mathcal{W}}\sigma_i(x), \alpha\rangle = \Big\langle \omega, \sum_{\sigma_i \in S^p} \langle \widetilde{\mathcal{W}}\sigma_i(x), \alpha\rangle \operatorname{vect}(\sigma_i)\Big\rangle = \langle \omega, \alpha\rangle.$$

Since this holds for all p-vectors α, the p-covectors $\widetilde{\mathcal{W}}\mathcal{C}\omega(x)$ and ω are the same, and since this holds for all x, we have $\widetilde{\mathcal{W}}\mathcal{C}\omega = \omega$. Hence $\omega \in \widetilde{\mathcal{W}}$.

Suppose then that $\widetilde{W}^p$ contains all constant p-forms, and take any point x and any p-vector α. Since constants are in $\widetilde{W}^p$, we have $\widetilde{\mathcal{W}}\mathcal{C}\omega = \omega$ for all p-covectors ω, and hence

$$\langle \omega, \alpha\rangle = \langle \widetilde{\mathcal{W}}\mathcal{C}\omega(x), \alpha\rangle = \Big\langle \sum_{\sigma_i \in S^p}\Big(\int_{\sigma_i} \omega\Big)\widetilde{\mathcal{W}}\sigma_i(x), \alpha\Big\rangle = \sum_{\sigma_i \in S^p}\Big(\int_{\sigma_i} \omega\Big)\langle \widetilde{\mathcal{W}}\sigma_i(x), \alpha\rangle$$
$$= \sum_{\sigma_i \in S^p} \langle \omega, \operatorname{vect}(\sigma_i)\rangle \langle \widetilde{\mathcal{W}}\sigma_i(x), \alpha\rangle = \Big\langle \omega, \sum_{\sigma_i \in S^p} \langle \widetilde{\mathcal{W}}\sigma_i(x), \alpha\rangle \operatorname{vect}(\sigma_i)\Big\rangle.$$

Since this holds for all p-covectors ω, we have $\alpha = \sum_{\sigma_i \in S^p} \langle \widetilde{\mathcal{W}}\sigma_i(x), \alpha\rangle \operatorname{vect}(\sigma_i)$, so the partition of unity property holds. $\square$

Corollary 4.5. *W^p contains all constant p-forms.*

This property ensures that approximating constants with Whitney forms yields exact approximations. It is useful in error analysis [29] and may be needed in convergence proofs [34].

Property 7: exactness

The exactness property or exact sequence property makes precise the good behaviour of Whitney forms with respect to the exterior derivative. We first state a closely related result. Recall that Stokes' theorem implies $\mathcal{C}\,\mathrm{d}\,\omega = \mathrm{d}\,\mathcal{C}\omega$ for all $\omega \in F^p(K)$. Similar property holds for the map $\mathcal{W}$.

Proposition 4.6. *$\mathcal{W}\,\mathrm{d}\,X = \mathrm{d}\,\mathcal{W}X$ for all p-cochains $X \in C_p^*(K)$.*

Proof. By linearity it is sufficient to consider the case $X = \sigma = x_0 \dots x_p$. Let $x_{i_1}, \dots, x_{i_m}$ be the vertices opposite to σ in those $(p+1)$-simplices that have σ as a face. Then the coboundary $\mathrm{d}\,\sigma$ can be written as $\mathrm{d}\,\sigma = \sum_{j=1}^m x_{i_j} x_0 \dots x_p$. By locality property $\mathcal{W}\,\mathrm{d}\,\sigma = 0 = \mathrm{d}\,\mathcal{W}\sigma$ in those simplices that do not have σ as a face, and in σ itself all $(p+1)$-forms are zero. Hence it suffices to show $\mathcal{W}\,\mathrm{d}\,\sigma = \mathrm{d}\,\mathcal{W}\sigma$ in any q-simplex $\tau \in S^q$ of the form $\tau = x_0 \dots x_p x_{p+1} \dots x_q$ for $q > p$.

In τ we have

$$
\begin{aligned}
\mathcal{W}\,\mathrm{d}\,\sigma &= \mathcal{W}\Bigg(\sum_{i=p+1}^{q} x_i x_0 \dots x_p\Bigg) \\
&= \sum_{i=p+1}^{q} (p+1)!\Bigg(\lambda_i\,\mathrm{d}\,\lambda_0 \wedge \dots \wedge \mathrm{d}\,\lambda_p - \sum_{j=0}^{p} (-1)^j \lambda_j\,\mathrm{d}\,\lambda_i \wedge \mathrm{d}\,\lambda_0 \wedge \dots \wedge \widehat{\mathrm{d}\,\lambda_j} \wedge \dots \wedge \mathrm{d}\,\lambda_p\Bigg) \\
&= (p+1)!\Bigg(\sum_{i=p+1}^{q} \lambda_i\,\mathrm{d}\,\lambda_0 \wedge \dots \wedge \mathrm{d}\,\lambda_p - \sum_{j=0}^{p} (-1)^j \lambda_j\,\mathrm{d}\Bigg(\sum_{i=p+1}^{q} \lambda_i\Bigg) \wedge \mathrm{d}\,\lambda_0 \wedge \dots \wedge \widehat{\mathrm{d}\,\lambda_j} \wedge \dots \wedge \mathrm{d}\,\lambda_p\Bigg) \\
&= (p+1)!\Bigg(\sum_{i=p+1}^{q} \lambda_i\,\mathrm{d}\,\lambda_0 \wedge \dots \wedge \mathrm{d}\,\lambda_p + \sum_{j=0}^{p} (-1)^j \lambda_j\,\mathrm{d}\Bigg(\sum_{i=0}^{p} \lambda_i\Bigg) \wedge \mathrm{d}\,\lambda_0 \wedge \dots \wedge \widehat{\mathrm{d}\,\lambda_j} \wedge \dots \wedge \mathrm{d}\,\lambda_p\Bigg) \\
&= (p+1)!\Bigg(\sum_{i=p+1}^{q} \lambda_i\,\mathrm{d}\,\lambda_0 \wedge \dots \wedge \mathrm{d}\,\lambda_p + \sum_{j=0}^{p} \lambda_j\,\mathrm{d}\,\lambda_0 \wedge \dots \wedge \mathrm{d}\,\lambda_p\Bigg) = (p+1)!\,\mathrm{d}\,\lambda_0 \wedge \dots \wedge \mathrm{d}\,\lambda_p
\end{aligned}
$$

By (3.5), this is the same as $\mathrm{d}\,\mathcal{W}\sigma$. □

The exactness property follows from Proposition 4.6. The statement can be formulated as follows.

Proposition 4.7. *$\mathrm{d}\,W^p \subset W^{p+1}$, so we may consider the sequence*

$$0 \to \mathbb{R} \xrightarrow{\subset} W^0 \xrightarrow{\mathrm{d}} W^1 \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} W^n \xrightarrow{\mathrm{d}} 0.$$

In addition, if Ω has trivial homology, then this sequence is exact, so $\ker \mathrm{d}_p = \mathrm{im}\,\mathrm{d}_{p-1}$ for $p > 1$.

Proof. Any Whitney p-form is the image $\mathcal{W}X$ of some $X \in C_p^*(K)$, and $\mathrm{d}\,\mathcal{W}X = \mathcal{W}\,\mathrm{d}\,X$ then says that $\mathrm{d}\,\mathcal{W}X$ is the image of $\mathrm{d}\,X$ and hence a Whitney $(p+1)$-form. Thus $\mathrm{d}\,W^p \subset W^{p+1}$.

Trivial homology implies that also the cohomology groups are trivial, so every p-cochain $X \in C_p^*(K)$ for $p > 0$ such that $\mathrm{d}\,X = 0$ is a coboundary of some $(p-1)$-cochain Y. Suppose $\mathcal{W}X$ is a Whitney p-form such that $\mathrm{d}\,\mathcal{W}X = 0$. Then $\mathcal{W}\,\mathrm{d}\,X = \mathrm{d}\,\mathcal{W}X = 0$, and $\mathrm{d}\,X = 0$ by injectivity of $\mathcal{W}$. Hence $X = \mathrm{d}\,Y$ for some $Y \in C_{p-1}^*(K)$, and $\mathcal{W}X \in \mathrm{im}\,\mathrm{d}_{p-1}$ since $\mathrm{d}\,\mathcal{W}Y = \mathcal{W}\,\mathrm{d}\,Y = \mathcal{W}X$. Thus $\ker \mathrm{d}_p \subset \mathrm{im}\,\mathrm{d}_{p-1}$, and by $\mathrm{d}^2 = 0$ we get $\ker \mathrm{d}_p = \mathrm{im}\,\mathrm{d}_{p-1}$. □

This property is a standard requirement for finite element spaces in FEEC theory [19,36], and it may be decisive for the convergence of numerical methods. For example, in the case of computational electromagnetism it is useful in eliminating finite-dimensional solutions that do not correspond with solutions of Maxwell's equations in cavity resonators, as emphasised by Alain Bossavit [7,29,34].

Property 8: convergence

As discussed before, we can approximate a p-form ω with $\mathcal{W}\mathcal{C}\omega$, and the integrals of this approximation match with those of ω on all p-simplices of K. We also saw by Proposition 4.4 that this approximation is exact if and only if ω is in W^p. We have yet to show the desired property that $\mathcal{W}\mathcal{C}\omega$ converges to ω when the mesh is refined.

This is indeed true, as long as the simplices are not allowed to flatten limitlessly during the refinement process. To make this precise, we employ the metric of $\mathbb{R}^n$ to define the fullness $\Theta(\sigma)$ of the p-simplex σ as the ratio

$$\Theta(\sigma) = \frac{|\sigma|}{\mathrm{diam}(\sigma)^p}.$$

The simplices do not flatten limitlessly if there is a uniform lower bound for their fullness. Properties of fullness are discussed in [1]. We need only the following lemma.

Lemma 4.8. *Let $\sigma = x_0 \dots x_p$ be a p-simplex, and denote by h_i the distance from vertex x_i to the plane of the opposite $(p-1)$-face of σ. Let $x = \sum_{i=0}^{n} \lambda_i x_i$ be any point in σ. Then*

$$h_i \geq p!\Theta(\sigma)\operatorname{diam}(\sigma), \quad \operatorname{dist}(x, \partial\sigma) \geq p!\Theta(\sigma)\operatorname{diam}(\sigma) \min_{i\in\{0,\dots,p\}} \lambda_i.$$

Proof. Let τ_i be the $(p-1)$-face opposite to vertex x_i. Since $|\tau_i| \leq \frac{1}{(p-1)!}\operatorname{diam}(\tau_i)^{p-1}$ and $|\sigma| = \frac{1}{p}|\tau_i|h_i$,

$$h_i = \frac{p|\sigma|}{|\tau_i|} \geq \frac{p|\sigma|}{\frac{1}{(p-1)!}\operatorname{diam}(\tau_i)^{p-1}} \geq p!\Theta(\sigma)\operatorname{diam}(\sigma).$$

The distance from x to the plane of τ_i is $\lambda_i h_i$, so also the second claim follows. □

Now we are ready to prove the convergence property. A similar result has been proved by Jozef Dodziuk [2, Theorem 3.7], but our statement is slightly different and does not restrict to standard subdivisions. We are also in a position to give a much simpler proof using previous results. Below we use the Euclidean metric, as in the definition of fullness, but the choice of metric will only affect the result by up to a constant.

Theorem 4.9. *Let ω be a smooth p-form in Ω. There exists a constant C_ω such that*

$$|\mathcal{W}\mathcal{C}\omega(x) - \omega(x)| \leq \frac{C_\omega}{C_\Theta^p} h \quad \text{for all } x \in \tau \text{ in all } \tau \in S^n$$

whenever $h > 0$, $C_\Theta > 0$, and K is a simplicial complex in Ω such that $\operatorname{diam}(\sigma) \leq h$ *and* $\Theta(\sigma) \geq C_\Theta$ *for all simplices σ of K.*

Proof. It suffices to prove this for $\omega = \omega_I \, d x^{i_1} \wedge \dots \wedge d x^{i_p}$ where $1 \leq i_1 < \dots < i_p \leq n$. Since ω is smooth in the polyhedron Ω, ω_I admits a smooth extension to a neighbourhood of Ω. The partial derivatives of ω_I are hence bounded in Ω, and we can find a constant C_I such that $|\omega_I(x) - \omega_I(y)| \leq C_I|x-y|$ whenever $yx \subset \Omega$.

Fix $\tau \in S^n$ and $y \in \tau$. We can write

$$\begin{aligned} &\omega(x) = \omega_I(x)\, d x^{i_1} \wedge \dots \wedge d x^{i_p} = (\omega_I(y) + g(x))\, d x^{i_1} \wedge \dots \wedge d x^{i_p}, \quad \text{where} \\ &g(x) = \omega_I(x) - \omega_I(y), \quad |g(x)| \leq C_I|x-y| \leq C_I h \quad \text{if } x \in \tau. \end{aligned}$$

Using Proposition 4.4 and Corollary 4.5,

$$\begin{aligned} &\mathcal{W}\mathcal{C}\omega(x) = \omega_I(y)\, d x^{i_1} \wedge \dots \wedge d x^{i_p} + \mathcal{W}\mathcal{C}(g\, d x^{i_1} \wedge \dots \wedge d x^{i_p})(x), \\ &\mathcal{W}\mathcal{C}\omega(x) - \omega(x) = \mathcal{W}\mathcal{C}(g\, d x^{i_1} \wedge \dots \wedge d x^{i_p})(x) - g(x)\, d x^{i_1} \wedge \dots \wedge d x^{i_p}. \end{aligned}$$

When σ is a p-face of τ, we have $|\int_\sigma g\, d x^{i_1} \wedge \dots \wedge d x^{i_p}| \leq |\sigma|C_I h$, and hence in τ

$$|\mathcal{W}\mathcal{C}(g\, d x^{i_1} \wedge \dots \wedge d x^{i_p})(x)| = \left| \sum_{\sigma\subset\tau} \left(\int_\sigma g\, d x^{i_1} \wedge \dots \wedge d x^{i_p} \right) \mathcal{W}\sigma(x) \right| \leq \sum_{\sigma\subset\tau} |\sigma| C_I h |\mathcal{W}\sigma(x)|,$$

where the sum is over the p-faces σ of τ.

Now the affine invariance property proves useful since we can work in the standard n-simplex $\Delta^n = y_0 y_1 \dots y_n$, where $y_0 = (0, \dots, 0)$ and $y_i = (0, \dots, 0, 1, 0, \dots, 0)$ has 1 in the ith slot for $1 \leq i \leq n$. Consider one of the p-faces σ and label the vertices of $\tau = x_0 x_1 \dots x_n$ such that $\sigma = x_0 \dots x_p$. Let φ be the affine map from τ to Δ^n such that $\varphi(x_i) = y_i$. Proposition 4.2 and the pullback inequality $|f^*\omega(x)| \leq |Df(x)|^p \cdot |\omega(f(x))|$ of p-forms [1, II, 4.12] give

$$|\mathcal{W}\sigma(x)| = |\mathcal{W}(x_0 \dots x_p)(x)| = |\varphi^*(\mathcal{W}(y_0 \dots y_p))(x)| \leq |D\varphi(x)|^p |\mathcal{W}(y_0 \dots y_p)(\varphi(x))|.$$

Next we find a bound for $|D\varphi(x)|$. Denote by $z = \sum_{i=0}^{n} \frac{1}{n+1} x_i$ the barycentre of τ, and take v such that $|v| = 1$ and $|D\varphi(z)v| = \max_{|w|=1} |D\varphi(z)w|$. Let $t = \operatorname{dist}(z, \partial\tau)$; by Lemma 4.8 we have $t \geq \frac{n!}{n+1}\Theta(\tau)\operatorname{diam}(\tau)$. Now z and $z + tv$ are both in τ, so $\varphi(z)$ and $\varphi(z + tv)$ are in Δ^n, which has diameter $\sqrt{2}$. Since φ is affine,

$$|D\varphi(x)| = |D\varphi(z)| = |D\varphi(z)v| = \frac{|\varphi(z + tv) - \varphi(z)|}{t} \leq \frac{\sqrt{2}}{\frac{n!}{n+1}\Theta(\tau)\operatorname{diam}(\tau)}.$$

To compute $\mathcal{W}(y_0 \dots y_p)$ we note that the barycentric coordinates in Δ^n are $\lambda_i = x^i$ for $1 \le i \le n$ and $\lambda_0 = 1 - \sum_{i=1}^n x^i$. Hence

$$\begin{aligned}
&\mathcal{W}(y_0 \dots y_p) \\
&= p!\Big((1 - \sum_{i=1}^n x^i)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p + \sum_{j=1}^p (-1)^j x^j\,\mathrm{d}(1 - \sum_{i=1}^n x^i) \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^p\Big) \\
&= p!\Big((1 - \sum_{i=p+1}^n x^i)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p + \sum_{j=1}^p \sum_{i=p+1}^n (-1)^{p+j} x^j\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^p \wedge \mathrm{d}x^i\Big)
\end{aligned}$$

and

$$|\mathcal{W}(y_0 \dots y_p)| = p!\sqrt{\big(1 - \sum_{i=p+1}^n x^i\big)^2 + (n-p)\sum_{i=1}^p (x^i)^2} \le p!\sqrt{1+n-p} \quad \text{in } \Delta^n.$$

Using these estimates and the facts that $\operatorname{diam}(\tau) \ge \operatorname{diam}(\sigma)$ and $|\sigma| \le \frac{1}{p!}\operatorname{diam}(\sigma)^p$, we get

$$\begin{aligned}
&\sum_{\sigma \subset \tau} |\sigma| C_I h |\mathcal{W}\sigma(x)| \le \sum_{\sigma \subset \tau} |\sigma| C_I h \bigg(\frac{\sqrt{2}}{\frac{n!}{n+1}\Theta(\tau)\operatorname{diam}(\tau)}\bigg)^p p!\sqrt{1+n-p} \\
&\le \sum_{\sigma \subset \tau} C_I h \bigg(\frac{\sqrt{2}}{\frac{n!}{n+1}\Theta(\tau)}\bigg)^p \sqrt{1+n-p} = \frac{C_I \binom{n+1}{p+1}\big(\frac{\sqrt{2}(n+1)}{n!}\big)^p \sqrt{1+n-p}}{\Theta(\tau)^p} h.
\end{aligned}$$

This holds for all $\tau \in S^n$, so we may choose $C_\omega = C_I\binom{n+1}{p+1}\big(\frac{\sqrt{2}(n+1)}{n!}\big)^p\sqrt{1+n-p} + C_I$, and then

$$\begin{aligned}
&|\mathcal{WC}\omega(x) - \omega(x)| \le |\mathcal{WC}(g\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p})(x)| + |g(x)\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}| \\
&\le \sum_{\sigma \subset \tau} |\sigma| C_I h |\mathcal{W}\sigma(x)| + C_I h \le \frac{C_\omega}{C_\Theta^p} h \quad \text{for all } x \in \tau \text{ in all } \tau \in S^n,
\end{aligned}$$

which concludes the proof of the theorem. □

5. Generalisations of Whitney forms

To further clarify the concept of Whitney forms, we consider what other possibilities go by this name in the literature. In contrast to the three equivalent definitions given in Section 3, the Whitney forms considered in this section are generalisations of Whitney forms. By this we mean that they are not equivalent to the standard Whitney forms but are sufficiently related so that calling them by the same name is justified. As we shall see, they also preserve certain properties of standard Whitney forms.

5.1. Whitney forms on a manifold

In the initial context of Whitney forms, the simplicial complex K is embedded in affine space. In this subsection we consider the generalisation to the case where K is a smooth simplicial complex on a compact smooth manifold Ω. Now p-simplices are maps $\sigma : \Delta^p \to \Omega$ from the standard p-simplex Δ^p to Ω. The faces of σ are its restrictions $\sigma|_\tau$ to the faces τ of Δ^p. Since the q-faces of Δ^p can be identified with Δ^q, each q-face of σ yields a map from Δ^q to Ω. Hence the q-faces of σ are q-simplices.

In this subsection we assume K is a finite set of simplices σ such that

- The restriction of each $\sigma : \Delta^p \to \Omega$ to $\Delta^p - \partial\Delta^p$ is a diffeomorphism onto its image, and each point $x \in \Omega$ is contained in the image of exactly one such restriction
- Each face of every simplex in K is also in K
- The intersection of the images of two simplices in K is either the image of a common face of theirs or the empty set
- Each p-simplex has $p+1$ distinct vertices (0-faces), and no other p-simplex has this same set of vertices

Chains and cochains of K can be defined similarly as before. Now $\sigma = x_0 \dots x_p$ means that the p-simplex σ maps the vertices of Δ^p to $x_0, \dots, x_p$. Differential p-form in K is a set of smooth p-forms ω_σ in the images of the simplices σ of K satisfying the following patch condition: if τ is a q-face of σ, then the trace $\omega_\sigma|_{\tau(\Delta^q)}$ equals ω_τ in $\tau(\Delta^q)$. The exterior derivative and the de Rham map are well-defined.

Barycentric functions in Ω can be defined as follows. Let $x \in \Omega$ and let $\sigma = x_0 \dots x_p$ be the p-simplex (p depending on x) such that x is in the image of the restriction of σ to $\Delta^p - \partial\Delta^p$. Then the $\lambda_i(x)$ for $0 \le i \le p$ are the barycentric coordinates of $\sigma^{-1}(x)$ with respect to the corresponding vertices of Δ^p. For other vertices $\lambda_i(x) = 0$. The Whitney

p-form $\mathcal{W}\sigma$ corresponding to a p-simplex σ of K can now be defined with the same formula (3.1). Define the map $\mathcal{W} : C_p^*(K) \to F^p(K)$ by extending linearly and the space of Whitney p-forms W^p as its image. By the same arguments as before, $\mathcal{W}\sigma$ is in $F^p(K)$, so property 1 is fulfilled.

The definition given above amounts to taking pullback as follows. If $\sigma \in S^p$ is a face of $\tau \in S^q$ and σ' is the p-face of Δ^q such that $\sigma = \tau|_{\sigma'}$, then $\mathcal{W}\sigma$ is the pullback $\tau^{-1*}(\mathcal{W}\sigma')$ in $\tau(\Delta^q)$, where $\mathcal{W}\sigma'$ is the Whitney p-form corresponding to σ' in Δ^q. Hence

$$\int_{\sigma_i(\Delta^p)} \mathcal{W}\sigma_j = \int_{\tau(\sigma_i')} \tau^{-1*}(\mathcal{W}\sigma_j') = \int_{\sigma_i'} \mathcal{W}\sigma_j' = \delta_{ij}$$

if σ_i and σ_j are p-faces of τ, so our earlier discussion about property 2 applies here as well. If σ is not a face of τ, then $\mathcal{W}\sigma = 0$ in $\tau(\Delta^q)$, so property 5 holds too.

The same proof as before shows that property 7 holds. A convergence property similar to property 8 has been proved in [2] using standard subdivisions. For this Ω is assumed to be a Riemannian manifold so that the Riemannian metric induces a norm for p-covectors at each point of Ω.

However, not all of the properties are preserved. On a manifold we do not have the affine structure of affine space. We can no longer identify the tangent spaces of different points, so there are no such things as p-vector of σ or constant p-forms (for $p > 0$). Thus properties 3 and 6 do not make sense as such, and the partition of unity property only holds for 0-forms. Property 4 is also lost, although Proposition 4.2 works if σ is a diffeomorphism that preserves barycentric functions.

Although most of the properties of Whitney forms hold also when the complex K is on a manifold Ω, the affine character of Whitney forms – a central property in their initial context – is not visible on a manifold since there is no affine structure. This is the reason why we consider Whitney forms on a manifold to be generalisations of Whitney forms.

5.2. Higher order Whitney forms

Higher order finite elements are appreciated for better accuracy and convergence properties. There are also higher order Whitney forms, or at least this term has appeared in the literature several times [30–32,34,36,37]. In this subsection, we explain what these are and which properties of Whitney forms are preserved by their higher order generalisations. The discussion is limited to higher order differential forms on simplices. In the literature one can find higher order finite elements also on other cell types (see e.g. [38–43]). However, these are typically not called Whitney forms in the literature, and one would have to give up on even more of the properties, so we leave this kind of extensions out of scope of this paper.

Higher order Whitney forms are differential forms in a simplicial complex K. (Here the complex K is again embedded in affine space, and we assume Ω is a polyhedron in $\mathbb{R}^n$.) Property 1 is hence to be fulfilled by construction. We denote by W_k^p the space of Whitney p-forms of order k. We will next define W_k^p by giving a set of elements of $F^p(K)$ that span W_k^p.

Let $\mathcal{I}(n+1,k)$ denote the set of multi-indices with $n+1$ components that sum to k; that is, $\mathcal{I}(n+1,k)$ consists of arrays $\mathbf{k} = (k_0, k_1, \ldots, k_n)$ where the k_i are nonnegative integers such that $\sum_{i=0}^n k_i = k$. For a fixed n-simplex $\sigma = x_0 \ldots x_n$, denote by $\lambda_\sigma^{\mathbf{k}}$ the function $\prod_{i=0}^n (\lambda_i)^{k_i}$. This is a continuous function in Ω, and hence its product with any Whitney p-form is in $F^p(K)$. We may therefore define for $k \geq 1$

$$W_k^p = \operatorname{span}\{\lambda_\sigma^{\mathbf{k}} \mathcal{W}\tau \mid \sigma = x_0 \ldots x_n \in S^n,\ \mathbf{k} \in \mathcal{I}(n+1, k-1),\ \text{and } \tau \text{ is a } p\text{-face of } \sigma\}. \tag{5.1}$$

Note that $W_1^p = W^p$.

The spaces of higher order Whitney forms could also be defined using the Koszul operator [19,36]. In terms of their proxy fields, the 1-forms in 2D were first given in [9] and the 1- and 2-forms in 3D in [10]. They have subsequently been studied e.g. in [44–51]. It is shown in [52] that W_k^p by our definition is the same as the space $\mathcal{P}_k^- \Lambda^p$ in FEEC theory [19,36]. This space is constructed such that it includes all polynomials of order $\leq k-1$ and its elements are at most kth order polynomials in each simplex. Property 3 hence takes the obvious form for kth order Whitney forms.

Since W^p is already isomorphic to $C_p^*(K)$ and increasing the order increases the dimension of the space, one immediately sees that property 2 cannot hold. The de Rham map $\mathcal{C}$ from W_k^p to $C_p^*(K)$ is not injective, and we do not even have the map $\mathcal{W}$ from $C_p^*(K)$ to W_k^p. To approximate elements of $F^p(K)$ in W_k^p, one must first determine suitable degrees of freedom, as the integrals over p-cells no longer define a unique element of W_k^p. There are at least three ways to do this [37]. We consider the so-called small simplices of [32], for this yields us at least some kind of map from cochains to W_k^p and enables us to interpret generalisations of properties that involved the map $\mathcal{W}$.

Small simplices are homothetic images of the simplices of K. For a fixed n-simplex $\sigma = x_0 \ldots x_n$, each multi-index $\mathbf{k} \in \mathcal{I}(n+1, k-1)$ defines a map, which we denote by $\mathbf{k}_\sigma$, from σ to itself such that the point x whose barycentric coordinates are λ_i maps to the point whose barycentric coordinates are $\frac{\lambda_i + k_i}{k}$. In other words, $\mathbf{k}_\sigma$ is defined by

$$\mathbf{k}_\sigma : \sigma \to \sigma, \quad \lambda_0 x_0 + \cdots + \lambda_n x_n \mapsto \frac{\lambda_0 + k_0}{k} x_0 + \cdots + \frac{\lambda_n + k_n}{k} x_n.$$

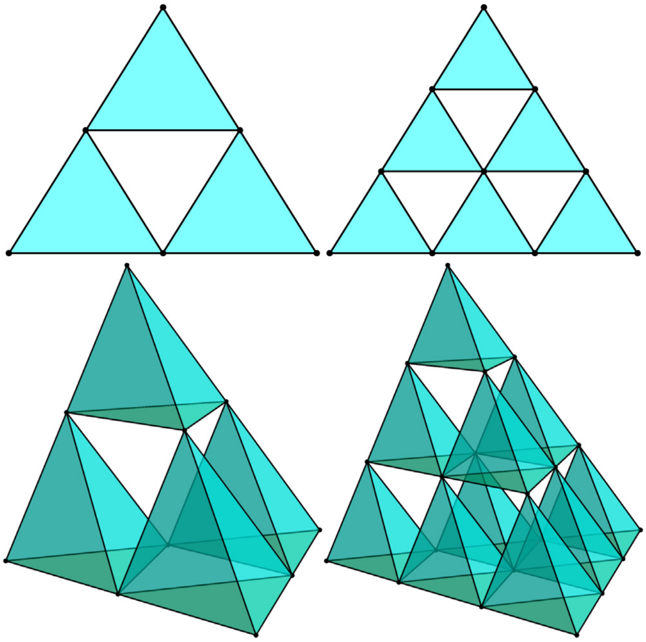

Fig. 2. Second and third order small simplices $\mathbf{k}_\sigma(\sigma)$ in the cases when σ is a triangle in two dimensions and a tetrahedron in three dimensions.

The set of kth order small p-simplices of K is

$$S_k^p = \{\mathbf{k}_\sigma(\tau) \mid \sigma = x_0 \dots x_n \in S^n,\ \mathbf{k} \in \mathcal{I}(n+1, k-1), \text{ and } \tau \text{ is a } p\text{-face of } \sigma\}. \tag{5.2}$$

Note from (5.1) and (5.2) that the small p-simplices of order k correspond exactly to the spanning p-forms of W_k^p. When required, we use label υ for elements of S_k^p and denote by $w(\upsilon)$ the corresponding p-form. See Fig. 2 for examples of small simplices.

Although the small n-simplices do not pave Ω, we can form a subdivision of K that contains the kth order small simplices of K as cells; denote this subdivision by K_k. Not all the cells of K_k are necessarily simplices, but it is a cell complex nevertheless, and we may hence consider p-chains $C_p(K_k)$, p-cochains $C_p^*(K_k)$, and the de Rham map of K_k.

Integrals over the small simplices S_k^p serve as degrees of freedom for W_k^p, but these are overdetermining, as the spanning p-forms in (5.1) are not linearly independent. To obtain unisolvent degrees of freedom, one can choose a subset of S_k^p such that the integrals over this subset uniquely determine an element of W_k^p by omitting redundant small simplices. This yields a linear map $\mathcal{V} : C_p^*(K_k) \to W_k^p$ such that the values of all cochains $X \in C_p^*(K_k)$ match with the integrals of $\mathcal{V}X$ on the chosen subset of S_k^p. Then we have $\mathcal{C}\mathcal{V}X = X$ for all $X \in \mathcal{C}(W_k^p)$ and $\mathcal{V}\mathcal{C}\omega = \omega$ for all $\omega \in W_k^p$ — this is closest to property 2 that one can get.

We immediately see that $w(\upsilon)$ is nonzero in n-simplex $\sigma \in S^n$ only if $\upsilon \subset \sigma$, so the spanning p-forms in (5.1) are local; this is the counterpart of property 5. Likewise, the affine invariance property continues to hold, and Proposition 4.2 now says $w(\mathbf{k}_\sigma(x_0 \dots x_p)) = \varphi^*(w(\mathbf{k}_\tau(y_0 \dots y_p)))$. It has been proved e.g. in [19] that also the exact sequence property holds.

As for the partition of unity property, there are two interpretations. On one hand, Theorem 4.3 implies that in any n-simplex $\tau \in S^n$, for all p-vectors α and all points x in τ

$$\sum_{\substack{\tau \supset \sigma_i \in S^p \\ \mathbf{k} \in \mathcal{I}(n+1,k-1)}} \frac{(k-1)!}{k_0! k_1! \dots k_n!} \langle w(\mathbf{k}_\tau(\sigma_i))(x), \alpha \rangle \operatorname{vect}(\sigma_i) = \alpha;$$

this follows from the multinomial theorem. On the other hand, we have

$$\sum_{\upsilon_i \in S_k^p} \langle \mathcal{V}\upsilon_i(x), \alpha \rangle \operatorname{vect}(\upsilon_i) = \alpha. \tag{5.3}$$

To show (5.3), note that the requirement $\mathcal{C}\widetilde{\mathcal{W}}X = X$ for all $X \in C_p^*(K)$ in Proposition 4.4 can be replaced with $\mathcal{C}\widetilde{\mathcal{W}}X = X$ for all $X \in \mathcal{C}(\widetilde{W}^p)$ by requiring in addition that $\mathcal{C}$ be injective in $\widetilde{W}^p$. Using this for the map $\mathcal{V}$ yields (5.3), since W_k^p contains all constant p-forms.

Finally, for the convergence property one expects an improvement: higher order Whitney forms should enable higher order convergence. This is indeed true. The proof is similar as in the lowest order case, but we have included it below to bring also the higher order approximation property into Whitney's setting.

Theorem 5.1. *Let $\mathcal{V} : C^*_p(K_k) \to W^p_k$ be the linear map obtained with a choice of kth order small simplices as explained above, and let ω be a smooth p-form in Ω. There exists a constant $C_{\omega,k}$ such that*

$$|\mathcal{V}\mathcal{C}\omega(x) - \omega(x)| \le \frac{C_{\omega,k}}{C^p_\Theta} h^k \quad \text{for all } x \in \tau \text{ in all } \tau \in S^n$$

whenever $h > 0$, $C_\Theta > 0$, and K is a simplicial complex in Ω such that $\operatorname{diam}(\sigma) \le h$ *and* $\Theta(\sigma) \ge C_\Theta$ *for all simplices σ of K.*

Proof. It suffices to prove this for $\omega = \omega_I \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ where $1 \le i_1 < \cdots < i_p \le n$. Denote by T_y the $(k-1)$th order Taylor polynomial of ω_I at y. Since ω is smooth in the polyhedron Ω, we can find a constant C_I such that $|\omega_I(x) - T_y(x)| \le C_I |x-y|^k$ whenever $yx \subset \Omega$.

Fix $\tau \in S^n$ and $y \in \tau$. We can write

$$\begin{aligned} &\omega(x) = \omega_I(x) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = (T_y(x) + g(x)) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}, \quad \text{where} \\ &g(x) = \omega_I(x) - T_y(x), \quad |g(x)| \le C_I |x-y|^k \le C_I h^k \quad \text{if } x \in \tau. \end{aligned}$$

Since the constant $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ is in W^p (by Corollary 4.5) and T_y is in the span of the products $\lambda^{\mathbf{k}}$ with $\mathbf{k} \in \mathcal{I}(n+1, k-1)$ (it is a polynomial of order $k-1$), we see from (5.1) that $T_y \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ is in W^p_k. Hence $\mathcal{V}\mathcal{C}(T_y \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}) = T_y \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ and

$$\begin{aligned} &\mathcal{V}\mathcal{C}\omega(x) = T_y(x) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} + \mathcal{V}\mathcal{C}(g \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p})(x), \\ &\mathcal{V}\mathcal{C}\omega(x) - \omega(x) = \mathcal{V}\mathcal{C}(g \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p})(x) - g(x) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}. \end{aligned}$$

Denote by $\hat{S}^p_k$ the chosen subset of S^p_k and by $\hat{S}^p_k(\tau)$ its restriction to those small simplices that are in τ. The interpolant $\mathcal{V}\mathcal{C}(g \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p})$ is a linear combination $\sum_{\upsilon_i \in \hat{S}^p_k} \alpha_i w(\upsilon_i)$ of the spanning forms $w(\upsilon_i)$. Since $w(\upsilon) = 0$ in τ if $\upsilon \not\subset \tau$, it suffices to consider $\sum_{\upsilon_i \in \hat{S}^p_k(\tau)} \alpha_i w(\upsilon_i)$. Each coefficient α_i is a linear combination of the integrals $\int_{\upsilon_j} g \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$, $\upsilon_j \in \hat{S}^p_k(\tau)$. The coefficients of this latter linear combination are constant and affine-invariant quantities (determined by the inverse of the matrix A with components $A_{ij} = \int_{\upsilon_i} w(\upsilon_j)$). Hence there exists a constant C_α such that

$$|\alpha_i| \le C_\alpha \sum_{\upsilon_j \in \hat{S}^p_k(\tau)} \left| \int_{\upsilon_j} g \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \right| \le C_\alpha \sum_{\upsilon_j \in \hat{S}^p_k(\tau)} C_I h^k |\upsilon_j|$$

holds for all of the coefficients α_i. Using the facts that $\operatorname{diam}(\tau) \ge \operatorname{diam}(\upsilon_j)$ and $|\upsilon_j| \le \frac{1}{p!} \operatorname{diam}(\upsilon_j)^p$ and denoting by C_k the cardinality of $\hat{S}^p_k(\tau)$, we get

$$|\alpha_i| \le C_\alpha C_k C_I h^k \frac{1}{p!} \operatorname{diam}(\tau)^p.$$

To find a bound for the $|w(\upsilon_i)|$, suppose that υ_i is the image of the p-face $\sigma \subset \tau$. Then clearly $|w(\upsilon_i)(x)| \le |\mathcal{W}\sigma(x)| \; \forall x \in \tau$, and hence using the affine map to the standard n-simplex exactly in the same way as in the proof of Theorem 4.9 we find

$$|w(\upsilon_i)(x)| \le \left(\frac{\sqrt{2}}{\frac{n!}{n+1} \Theta(\tau) \operatorname{diam}(\tau)} \right)^p p! \sqrt{1+n-p} \quad \text{for all } x \in \tau.$$

Combining these estimates yields

$$\begin{aligned} &|\mathcal{V}\mathcal{C}(g \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p})(x)| = \left| \sum_{\upsilon_i \in \hat{S}^p_k(\tau)} \alpha_i w(\upsilon_i)(x) \right| \le \sum_{\upsilon_i \in \hat{S}^p_k(\tau)} |\alpha_i| |w(\upsilon_i)(x)| \\ &\le C_k^2 C_\alpha C_I \left(\frac{\sqrt{2}}{\frac{n!}{n+1} \Theta(\tau)} \right)^p \sqrt{1+n-p} \cdot h^k \quad \text{for all } x \in \tau. \end{aligned}$$

This holds for all $\tau \in S^n$, so we may choose $C_{\omega,k} = C_k^2 C_\alpha C_I \left(\frac{\sqrt{2}}{\frac{n!}{n+1}} \right)^p \sqrt{1+n-p} + C_I$, and then

$$\begin{aligned} &|\mathcal{V}\mathcal{C}\omega(x) - \omega(x)| \le |\mathcal{V}\mathcal{C}(g \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p})(x)| + |g(x) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}| \\ &\le C_k^2 C_\alpha C_I \left(\frac{\sqrt{2}}{\frac{n!}{n+1} \Theta(\tau)} \right)^p \sqrt{1+n-p} \cdot h^k + C_I h^k \le \frac{C_{\omega,k}}{C^p_\Theta} h^k \quad \text{for all } x \in \tau \text{ in all } \tau \in S^n. \quad \square \end{aligned}$$

5.3. Whitney forms on other cells than simplices

Standard Whitney forms are differential forms in a simplicial complex. For flexibility in modelling and mesh generation, also other kind of cells should be allowed, and there have been several approaches to generalising Whitney forms for nonsimplicial cells. In this subsection, we consider the case where K is a cell complex of convex polyhedral cells.

When moving to nonsimplicial cells, we would like to preserve at least properties 1 and 2 of Whitney forms, so we take these as a guideline. Firstly, as stated earlier, all Whitney forms should be differential forms in the complex K — elements of $F^p(K)$. Secondly, there should be a Whitney p-form $\mathcal{W}\sigma$ corresponding to each p-cell σ of K, so that we get a linear map $\mathcal{W} : C_p^*(K) \to F^p(K)$ whose image is the space of Whitney p-forms W^p. In addition, $\mathcal{W}$ should be an isomorphism onto its image, so that integrals over p-cells uniquely determine an element of W^p and serve as degrees of freedom. Without loss of generality, we may then also require that $\int_{\sigma_i} \mathcal{W}\sigma_j = \delta_{ij}$, which is probably the best-known property of Whitney forms. Property 5, locality, will be fulfilled by all constructions without further mention.

In general, properties 3 and 4 as such will be lost. This is inevitable: a first order polynomial would already be fixed by its values on $n + 1$ vertices, and there is no affine map like in Proposition 4.2 between more general cells. However, for some cell types there is a same kind of canonical map (maybe not affine) and Whitney forms on one cell move onto another through taking pullback. For example, any cube is obtained from the reference cube $[0, 1]^3$ with the obvious map after the image of one vertex is fixed, and to define Whitney forms on cubes it suffices to consider the reference cube. The same applies to for example triangular prisms and pyramids.

To define Whitney forms for a convex polyhedral cell, we may consider the cell and its faces as the cell complex K so that there is only one n-cell. In doing so, we must ensure that traces on faces depend only on the face itself, so that the same Whitney forms belong to $F^p(K)$ also in the case when K has many n-cells. The complex K may even contain different kind of cells, as long as traces on faces shared by two such cells are the same according to both constructions.

From the literature, we have chosen two constructions that we believe best preserve the properties of Whitney forms. These will be discussed below. More options can be found in the literature if one is willing to give up on more of the properties (see e.g. [53–56]). In particular we would like to mention [54], where the author shows a way to construct finite-dimensional spaces of differential forms on arbitrary polytopes in any dimension such that the basis p-forms correspond to the p-cells and the spaces fulfil the exact sequence property. It requires auxiliary spaces on a simplicial refinement of the complex, and as these one can use Whitney forms. However, the resulting forms are in $F^p(K')$ with respect to the refinement K' and not necessarily in $F^p(K)$ with respect to the initial complex K (discontinuities are allowed in the cells of K). Another downside is that explicit expressions for the basis forms are not given on general polytopes, so they might not be easily computable.

The rest of this subsection is divided into parts as follows. First we briefly discuss two relevant approaches to generalising Whitney forms. The first approach is based on the construction of [57] and generalised barycentric coordinates. The second approach [29] is based on geometric conation and extrusion operations and constructs Whitney forms for cells obtained with these operations recursively. Finally, we summarise the Whitney forms resulting from these approaches on cubes, triangular prisms, and pyramids in 3D.

5.3.1. Construction based on generalised barycentric functions

Whitney forms in a simplicial complex were built using barycentric functions. These are exclusive to simplicial complexes, but for nonsimplicial cells there are generalised barycentric coordinates, which are no longer unique. Suppose σ is a convex polyhedral p-cell in $\mathbb{R}^n$ with m vertices $x_1, \ldots, x_m$. Any set of m nonnegative functions $\lambda_i : \sigma \to \mathbb{R}$ are called generalised barycentric coordinates in σ if for all $x \in \sigma$

$$\sum_{i=1}^{m} \lambda_i(x) = 1, \quad \sum_{i=1}^{m} \lambda_i(x) x_i = x. \tag{5.4}$$

Note that generalised barycentric coordinates in σ restrict to generalised barycentric coordinates on its faces.

The functions λ_i are not uniquely determined by (5.4) for general cells, and there are different kind of generalised barycentric coordinates (see the references in [56] and [57]). On simplices, these all reduce to the standard barycentric coordinates. Generalised barycentric functions in K are defined after we choose barycentric coordinates in each cell such that their restrictions agree on inter-element faces. This is typically ensured by using the same kind of coordinates on incident cells [56].

In [56] and [57], Whitney forms are generalised for nonsimplicial cells by taking generalised barycentric functions as Whitney 0-forms and using the same formula (3.1) (without the multiplier $p!$) for 1- and 2-forms. This gives the 1-form $\lambda_i \,\mathrm{d}\lambda_j - \lambda_j \,\mathrm{d}\lambda_i$ for any two vertices x_i and x_j and the 2-form $\lambda_i \,\mathrm{d}\lambda_j \wedge \mathrm{d}\lambda_k - \lambda_j \,\mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_k + \lambda_k \,\mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j$ for any three vertices x_i, x_j, and x_k. (In [56] and [57], the forms are given in terms of their proxy fields.) These do not correspond to the cells of K, but they are used in [57] to construct finite elements in 2D and 3D that (although not called Whitney forms in [57]) actually better fulfil the properties of Whitney forms.

The construction of [57] uses Wachspress coordinates [58]. In both two and three dimensions, we get linear maps $\mathcal{W} : C_p^*(K) \to F^p(K)$ such that the spaces of Whitney p-forms $W^p = \mathcal{W}(C_p^*(K))$ constitute an exact sequence. Moreover, we have $\int_{\sigma_i} \mathcal{W}\sigma_j = \delta_{ij}$, and integrals over p-cells serve as degrees of freedom. In 2D any convex nondegenerate polygons are allowed, but in 3D the complex K is restricted by the additional requirement that the faces of the polyhedral cells be triangles or parallelograms.

As Whitney 0-forms we take the generalised barycentric functions resulting from Wachspress coordinates. In 2D, define the Whitney 1-forms corresponding to the edges of K such that their proxy fields are the $\mathbf{q}_i$ in Lemma 3.1 of [57] rotated 90 degrees counterclockwise and divided by the edge length. In 3D, define the Whitney 1- and 2-forms corresponding to the edges and the faces of K such that their proxy fields are the $\mathbf{p}_e$ and the $\mathbf{q}_f$ in Lemmas 4.7 and 4.6 of [57] divided by the edge length and the face area, respectively. Define the n-forms corresponding to the polygons/polyhedra of K such that their proxy fields equal the reciprocal of the area/volume in the corresponding polygon/polyhedron and zero elsewhere.

Then $\int_{\sigma_i} \mathcal{W}\sigma_j = \delta_{ij}$ by Lemmas 3.1, 4.7, and 4.6 of [57]. In 2D, it follows from Lemma 3.4 of [57] that constants are in W^p, and hence the partition of unity property holds by Proposition 4.4. In 3D this holds for certain types of cells by Lemma 4.14 of [57]. The counterpart to property 8 in 2D is Lemma 3.10 of [57], but we do not know if this has been proved in 3D yet. As discussed, properties 3 and 4 are lost. In general, Wachspress coordinates are rational functions. However, we remark that in the case of simplices everything reduces to normal Whitney forms. Thus, the construction of [57] truly generalises Whitney forms while preserving many of their properties.

5.3.2. Construction based on conation and extrusion

To present how Whitney forms for polytopal cells are obtained systematically, one approach is to first consider a systematic construction of the cells themselves. In [29] Whitney forms are defined recursively for cells that are obtained through conation and extrusion operations. Consider an n-dimensional cell σ with plane P in $\mathbb{R}^{n+1}$, a point $a \in \mathbb{R}^{n+1}$ outside P, and a vector v not parallel to P. Conation yields the $(n+1)$-dimensional cell

$$\operatorname{cone}(\sigma) = \{\lambda a + (1-\lambda)x \mid x \in \sigma,\ 0 \le \lambda \le 1\},$$

and extrusion yields the $(n+1)$-dimensional cell

$$\operatorname{extr}(\sigma) = \{x + \lambda v \mid x \in \sigma,\ 0 \le \lambda \le 1\}.$$

In [29], it is shown how Whitney forms lift up onto either of these $(n+1)$-dimensional cells, supposing we know them on σ.

The requirements (1)–(3) on page 1570 of [29] ensure $\int_{\sigma_i} \mathcal{W}\sigma_j = \delta_{ij}$, the exact sequence property, and the inclusion of constant p-forms in W^p (which by Proposition 4.4 implies the partition of unity property). Properties 3 and 4 are again understandably lost. In the case of simplices, this construction yields the usual Whitney forms (with repeated conation starting from a 0-cell). In 3D, other cell types that fit this approach are parallelepipeds (conation, extrusion, extrusion), pyramids (conation, extrusion, conation), and triangular prisms (conation, conation, extrusion).

Recently in [59], the authors combined these conation and extrusion techniques with their earlier construction [57] to define Whitney forms on polygon-based prisms and cones. The work [59] covers both theoretical analysis and implementation instructions. As mentioned in [59], any convex polyhedral cell can be divided into polygon-based cones by connecting the vertices with a chosen interior point. Hence one could define Whitney forms for cell complexes of arbitrary convex polyhedra by refining the complex this way — if one does not mind that the resulting forms are in $F^p(K')$ only with respect to the refined complex K'.

5.3.3. Formulas on cubes, triangular prisms, and pyramids

Finally, to show examples of Whitney forms on other cells than simplices, we give formulas of Whitney forms on cubes, triangular prisms, and pyramids. These three cell types are suitable for examples since the Whitney forms on them have sufficiently simple explicit formulas. In addition, both of the approaches we considered in this subsection yield these Whitney forms.

In all of the examples, we use Cartesian xyz-coordinates. The cell σ is defined by giving its vertices x_i in $\mathbb{R}^3$. Its edges are oriented so that $i < j$ for any edge x_ix_j, and its facets are oriented such that the normal vector (prescribed by the right hand rule) points outward.

Example 5.2 (*Cubes*)**.** Consider the cube σ with vertices

$$\begin{array}{llll} x_1 = (0,0,0) & x_2 = (1,0,0) & x_3 = (0,1,0) & x_4 = (1,1,0) \\ x_5 = (0,0,1) & x_6 = (1,0,1) & x_7 = (0,1,1) & x_8 = (1,1,1) \end{array}$$

The Whitney forms on σ are

$$\begin{array}{lll} \mathcal{W}x_1 = (1-x)(1-y)(1-z) & \mathcal{W}x_2 = x(1-y)(1-z) & \mathcal{W}x_3 = (1-x)y(1-z) \\ \mathcal{W}x_4 = xy(1-z) & \mathcal{W}x_5 = (1-x)(1-y)z & \mathcal{W}x_6 = x(1-y)z \\ \mathcal{W}x_7 = (1-x)yz & \mathcal{W}x_8 = xyz & \end{array}$$

$$\begin{array}{lll} \mathcal{W}x_1x_2 = (1-y)(1-z)\,\mathrm{d}x & \mathcal{W}x_3x_4 = y(1-z)\,\mathrm{d}x & \mathcal{W}x_5x_6 = (1-y)z\,\mathrm{d}x \\ \mathcal{W}x_7x_8 = yz\,\mathrm{d}x & \mathcal{W}x_1x_3 = (1-x)(1-z)\,\mathrm{d}y & \mathcal{W}x_2x_4 = x(1-z)\,\mathrm{d}y \\ \mathcal{W}x_5x_7 = (1-x)z\,\mathrm{d}y & \mathcal{W}x_6x_8 = xz\,\mathrm{d}y & \mathcal{W}x_1x_5 = (1-x)(1-y)\,\mathrm{d}z \\ \mathcal{W}x_2x_6 = x(1-y)\,\mathrm{d}z & \mathcal{W}x_3x_7 = (1-x)y\,\mathrm{d}z & \mathcal{W}x_4x_8 = xy\,\mathrm{d}z \end{array}$$

$$\begin{array}{lll} \mathcal{W}x_5x_6x_8x_7 = z\,\mathrm{d}x\wedge\mathrm{d}y & \mathcal{W}x_1x_3x_4x_2 = -(1-z)\,\mathrm{d}x\wedge\mathrm{d}y & \mathcal{W}x_1x_2x_6x_5 = (1-y)\,\mathrm{d}x\wedge\mathrm{d}z \\ \mathcal{W}x_3x_7x_8x_4 = -y\,\mathrm{d}x\wedge\mathrm{d}z & \mathcal{W}x_1x_5x_7x_3 = -(1-x)\,\mathrm{d}y\wedge\mathrm{d}z & \mathcal{W}x_2x_4x_8x_6 = x\,\mathrm{d}y\wedge\mathrm{d}z \\ \mathcal{W}\sigma = \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z & & \end{array}$$

The proxy fields of the 1- and 2-forms above first appeared in [10].

Example 5.3 (*Triangular Prisms*)**.** Consider the triangular prism σ with vertices

$$\begin{array}{lll} x_1 = (0, 0, 0) & x_2 = (1, 0, 0) & x_3 = (0, 1, 0) \\ x_4 = (0, 0, 1) & x_5 = (1, 0, 1) & x_6 = (0, 1, 1) \end{array}$$

The Whitney forms on σ are

$$\begin{array}{lll} \mathcal{W}x_1 = (1-y-x)(1-z) & \mathcal{W}x_2 = x(1-z) & \mathcal{W}x_3 = y(1-z) \\ \mathcal{W}x_4 = (1-y-x)z, & \mathcal{W}x_5 = xz, & \mathcal{W}x_6 = yz \\ \mathcal{W}x_1x_4 = (1-y-x)\,\mathrm{d}z & \mathcal{W}x_2x_5 = x\,\mathrm{d}z & \mathcal{W}x_3x_6 = y\,\mathrm{d}z \end{array}$$

$$\begin{array}{ll} \mathcal{W}x_1x_2 = (1-y)(1-z)\,\mathrm{d}x + x(1-z)\,\mathrm{d}y & \mathcal{W}x_2x_3 = -y(1-z)\,\mathrm{d}x + x(1-z)\,\mathrm{d}y \\ \mathcal{W}x_1x_3 = y(1-z)\,\mathrm{d}x + (1-x)(1-z)\,\mathrm{d}y & \mathcal{W}x_4x_5 = (1-y)z\,\mathrm{d}x + xz\,\mathrm{d}y \\ \mathcal{W}x_5x_6 = -yz\,\mathrm{d}x + xz\,\mathrm{d}y & \mathcal{W}x_4x_6 = yz\,\mathrm{d}x + (1-x)z\,\mathrm{d}y \\ \mathcal{W}x_1x_2x_5x_4 = (1-y)\,\mathrm{d}x\wedge\mathrm{d}z + x\,\mathrm{d}y\wedge\mathrm{d}z & \mathcal{W}x_2x_3x_6x_5 = -y\,\mathrm{d}x\wedge\mathrm{d}z + x\,\mathrm{d}y\wedge\mathrm{d}z \\ \mathcal{W}x_1x_4x_6x_3 = -y\,\mathrm{d}x\wedge\mathrm{d}z - (1-x)\,\mathrm{d}y\wedge\mathrm{d}z & \mathcal{W}x_1x_3x_2 = -2(1-z)\,\mathrm{d}x\wedge\mathrm{d}y \\ \mathcal{W}x_4x_5x_6 = 2z\,\mathrm{d}x\wedge\mathrm{d}y & \mathcal{W}\sigma = 2\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \end{array}$$

The proxy fields of the 1- and 2-forms above first appeared in [60].

Example 5.4 (*Pyramids*)**.** Consider the pyramid σ with vertices

$$x_1 = (0, 0, 0) \quad x_2 = (1, 0, 0) \quad x_3 = (0, 1, 0) \quad x_4 = (1, 1, 0) \quad x_5 = (0, 0, 1)$$

The Whitney forms on σ are

$$\begin{aligned}
&\mathcal{W}x_1 = \tfrac{(1-z-x)(1-z-y)}{1-z} \quad \mathcal{W}x_2 = \tfrac{x(1-z-y)}{1-z} \quad \mathcal{W}x_3 = \tfrac{(1-z-x)y}{1-z} \quad \mathcal{W}x_4 = \tfrac{xy}{1-z} \quad \mathcal{W}x_5 = z \\
&\mathcal{W}x_1x_2 = (1-z-y)\,\mathrm{d}x + \tfrac{x(1-z-y)}{1-z}\,\mathrm{d}z \quad \mathcal{W}x_2x_4 = x\,\mathrm{d}y + \tfrac{xy}{1-z}\,\mathrm{d}z \\
&\mathcal{W}x_3x_4 = y\,\mathrm{d}x + \tfrac{xy}{1-z}\,\mathrm{d}z \quad \mathcal{W}x_1x_3 = (1-z-x)\,\mathrm{d}y + \tfrac{(1-z-x)y}{1-z}\,\mathrm{d}z \\
&\mathcal{W}x_1x_5 = (z - \tfrac{yz}{1-z})\,\mathrm{d}x + (z - \tfrac{xz}{1-z})\,\mathrm{d}y + (1-x-y+\tfrac{xy}{1-z} - \tfrac{xyz}{(1-z)^2})\,\mathrm{d}z \\
&\mathcal{W}x_2x_5 = (-z + \tfrac{yz}{1-z})\,\mathrm{d}x + \tfrac{xz}{1-z}\,\mathrm{d}y + (x - \tfrac{xy}{1-z} + \tfrac{xyz}{(1-z)^2})\,\mathrm{d}z \\
&\mathcal{W}x_3x_5 = \tfrac{yz}{1-z}\,\mathrm{d}x + (-z + \tfrac{xz}{1-z})\,\mathrm{d}y + (y - \tfrac{xy}{1-z} + \tfrac{xyz}{(1-z)^2})\,\mathrm{d}z \\
&\mathcal{W}x_4x_5 = -\tfrac{yz}{1-z}\,\mathrm{d}x - \tfrac{xz}{1-z}\,\mathrm{d}y + (\tfrac{xy}{1-z} - \tfrac{xyz}{(1-z)^2})\,\mathrm{d}z \\
&\mathcal{W}x_1x_2x_5 = z\,\mathrm{d}x\wedge\mathrm{d}y + (2 - y - \tfrac{y}{1-z})\,\mathrm{d}x\wedge\mathrm{d}z - \tfrac{xz}{1-z}\,\mathrm{d}y\wedge\mathrm{d}z \\
&\mathcal{W}x_1x_5x_3 = z\,\mathrm{d}x\wedge\mathrm{d}y + \tfrac{yz}{1-z}\,\mathrm{d}x\wedge\mathrm{d}z + (-2 + x + \tfrac{x}{1-z})\,\mathrm{d}y\wedge\mathrm{d}z \\
&\mathcal{W}x_2x_4x_5 = z\,\mathrm{d}x\wedge\mathrm{d}y + \tfrac{yz}{1-z}\,\mathrm{d}x\wedge\mathrm{d}z + (x + \tfrac{x}{1-z})\,\mathrm{d}y\wedge\mathrm{d}z \\
&\mathcal{W}x_4x_3x_5 = z\,\mathrm{d}x\wedge\mathrm{d}y + (-y - \tfrac{y}{1-z})\,\mathrm{d}x\wedge\mathrm{d}z - \tfrac{xz}{1-z}\,\mathrm{d}y\wedge\mathrm{d}z \\
&\mathcal{W}x_1x_3x_4x_2 = -(1-z)\,\mathrm{d}x\wedge\mathrm{d}y - y\,\mathrm{d}x\wedge\mathrm{d}z + x\,\mathrm{d}y\wedge\mathrm{d}z \\
&\mathcal{W}\sigma = 3\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}$$

Whitney forms on pyramids first appeared in [28].

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