Journal Pre-proof

The price of multiobjective robustness: Analyzing solution sets to uncertain multiobjective problems

Anita Schöbel, Yue Zhou-Kangas

PII: S0377-2217(20)30849-3
DOI: https://doi.org/10.1016/j.ejor.2020.09.045
Reference: EOR 16791

To appear in: *European Journal of Operational Research*

Received date: 5 November 2018
Accepted date: 28 September 2020

Please cite this article as: Anita Schöbel, Yue Zhou-Kangas, The price of multiobjective robustness: Analyzing solution sets to uncertain multiobjective problems, *European Journal of Operational Research* (2020), doi: https://doi.org/10.1016/j.ejor.2020.09.045

- Mathematical comparison of minmax, lightly robust, and nominal efficient solutions.
- We prove that the lightly robust efficient solutions are good compromises.
- We propose a definition for the multiobjective price of robustness.
- We propose two strategies to support decision making.

The price of multiobjective robustness: Analyzing solution sets to uncertain multiobjective problems

Anita Schöbel[1] and Yue Zhou-Kangas*[2]

[1]Technische Universität Kaiserslautern, Faculty for Mathematics, and, Fraunhofer Institute for Industrial Mathematics ITWM, Germany, schoebel@mathematik.uni-kl.de
[2]University of Jyvaskyla, Faculty of Information Technology P.O.Box 35 (Agora), FI-40014, University of Jyvaskyla, Finland, yvonnezhou527@gmail.com

Abstract

Defining and finding robust efficient solutions to uncertain multiobjective optimization problems has been an issue of growing interest recently. Different concepts have been published defining what a "robust efficient" solution is. Each of these concepts leads to a different set of solutions, but it is difficult to visualize and understand the differences between these sets. In this paper we develop an approach for comparing such sets of robust efficient solutions, namely we analyze their outcomes under the nominal scenario and in the worst case using the upper set-less order from set-valued optimization. Analyzing the set of nominal efficient solutions, the set of minmax robust efficient solutions and different sets of lightly robust efficient solutions gives insight into robustness and nominal objective function values of these sets of solutions. Among others we can formally prove that lightly robust efficient solutions are good compromises between nominal efficient solutions and minmax robust efficient solutions. In addition, we also propose a measure to quantify the price of robustness of a single solution. Based on the measure, we propose two strategies which can be used to support a decision maker to find solutions to a multiobjective optimization problem under uncertainty. All our results are illustrated by examples.
Keywords: multiobjective robust optimization, decision making, uncertainty, price of robustness

1 Introduction

More and more complex optimization problems are being solved in the modern society. These problems are characterized by multiple conflicting objectives and they almost inevitably involve uncertainty due to imprecise data, uncertain future developments, uncertain consequences of decisions and so on. Multiobjective robust optimization is an evolving field specifically aiming at finding robust solutions that are sufficiently immune to uncertainty.

While the topic of robust multiobjective optimization is relatively young, robust optimization for single-objective optimization problems is well researched. For single-objective optimization problems, the standard book is (Ben-Tal, Ghaoui, & Nemirovski, 2009) with an extensive collection of results on the classical concept of minmax (or strict) robustness. In this concept, one looks for a solution which is feasible for all scenarios and has the best possible objective function value in the worst case. In other cases, a minmax robust solution does usually not have the best possible objective function value. This is known as *conservativeness* of robust optimization and has motivated researchers to quantify the resulting *price of robustness* (Bertsimas & Sim, 2004; Chassein & Goerigk, 2016) and to develop less conservative robustness concepts as in (Ben-Tal, Goryashko, Guslitzer, & Nemirovski, 2003; Goerigk, Knoth, Müller-Hannemann, Schmidt, & Schöbel, 2011, 2014; Liebchen, Lübbecke, Möhring, & Stiller, 2009; Schöbel, 2014), see also (Goerigk & Schöbel, 2016) for a survey.

In recent years, various robustness concepts have been developed to take uncertainty into account also for *multiobjective* optimization problems, see (Ide & Schöbel, 2016; Wiecek & Dranichak, 2016) for surveys of the many evolving robustness concepts. The easiest way to handle uncertainty in the input parameters is to identify the so-called *nominal scenario*, which is the most typical, the undisturbed, or the expected scenario, and to solve the problem only for this case. This means that the uncertainty is ignored and one receives a standard multiobjective optimization problem. The resulting (Pareto-) efficient solutions are called *nominal efficient*. However, nominal efficient solutions are not robust, i.e., they may be very bad in terms of their objective function values when the uncertainty realizes differently from the nominal scenario.

In order to take data uncertainty into account, different varieties of minmax robust efficiency, see e.g., (Ehrgott, Ide, & Schöbel, 2014; Fliege & Werner, 2014; Kuroiwa & Lee, 2012; Bokrantz & Fredriksson, 2017) have been proposed for multiobjective optimization problems under uncertainty. The idea of minmax robust efficiency is to optimize the objective functions in the worst case over all scenarios. The resulting solutions are called *minmax robust efficient*. The concept has raised attention in the literature: (Kuroiwa & Lee, 2012) take a set-valued point of view, (Fliege & Werner, 2014) study it in the context of portfolio selection problems, (Hassanzadeh, Nemati, & Sun, 2013) for Γ-uncertainty in the linear case and (Hassanzadeh, Nemati, & Sun, 2014) provide an interactive approach. (Kalantari, Dong, & Davies, 2016) consider the case in which only the constraints are uncertain, (Ehrgott et al., 2014; Bokrantz & Fredriksson, 2017; Schmidt, Schöbel, & Thom, 2019) provide scalarization approaches, (Raith, Schmidt, Schöbel, & Thom, 2018b) an approach for cardinality constrained uncertainty, and (Kuhn, Raith, Schmidt, & Schöbel, 2016; Raith, Schmidt, Schöbel, & Thom, 2018a) suggest approaches for shortest path problems. Recent results on feasible minmax robust solutions are given in (Wei, Chen, & Li, 2020a, 2020b) and approximation approaches are suggested in (Antczak, Pandey, Singh, & Mishra, 2020).

Since also in the multiobjective case minmax robust efficient solutions are rather conservative other concepts have been developed and analyzed, e.g., highly robust efficiency in (Georgiev, Luc, & Pardalos, 2013; M. A. Goberna, Jeyakumar, Li, & Vicente-Pérez, 2014; M. Goberna, Jeyakumar, Li, & Vicente-Pérez, 2015; Kuhn et al., 2016; Dranichak & Wiecek, 2019; M. A. Goberna, Jeyakumar, Li, & Vicente-Pérez, 2018; Rahimi & Soleimani-Damaneh, 2018, 2020), flimsily (or possibly) robust efficiency in (Bitran, 1980; Kuhn et al., 2016; Hladik, 2017), lightly robust efficiency in (Ide & Schöbel, 2016; Kuhn et al., 2016) and regret robustness in (Rivaz & Yaghoobi, 2018; Xidonas, Mavrotas, Hassapis, & Zopounidis, 2017; Rivaz, Yaghoobi, & Hladík, 2016). Scenario-based counterparts are investigated in (Botte & Schöbel, 2019; Engau & Sigler, 2020). In this paper, we focus on the concepts of minmax robust efficiency and of lightly robust efficiency. In the latter one looks for a robust solution still satisfying given tolerable nominal objective function values. For compactness of description, we refer to the nominal objective function values as the *nominal quality*.

While the relationships between different types of (robust) efficient solutions have been analyzed in (Ide & Schöbel, 2016), a comparison between their nominal quality, i.e., the objective function values in the nominal case, and their robustness has not been investigated so far. This is challenging since we are in a multiobjective setting in which we have to compare *sets* of robust efficient solutions (similar to Pareto fronts) with each other and not only single points.

The main contribution of this paper is to use set order relations to analyze the objective function values of a (robust efficient) solution in the nominal scenario and in the worst case. I.e., both extremes, the nominal (or most likely) and the most pessimistic view are taken into account. This approach is useful in two respects:

First, it is a theory-based approach to compare different robustness concepts in multiobjective optimization. Using minmax robustness and light robustness we demonstrate that this approach reflects the intuition one has about these concepts. In particular, we formally prove that minmax robustness is best in the worst case and that light robustness is a good compromise between nominal quality and robustness.

Second, our approach helps to understand and explain the meaning of robustness and hence may assist the decision maker. To this end, we extend the single-objective analysis between nominal quality and robustness of (Chassein & Goerigk, 2016; Schöbel, 2014) to multiobjective problems by defining the *price* and the *gain* of robustness for given solutions. The former specifies how much nominal quality is lost when moving a non-robust solution into the robust efficient set, the latter specifies how much robustness is increased in this case. Based on these definitions we sketch two approaches to find different types of solutions depending on the preferences of the decision maker and hence provide helpful information to find a balance between nominal quality and robustness.

The remainder of the paper is organized as follows: Section 2 formally introduces nominal efficiency, minmax robust efficiency, and lightly robust efficiency. Section 3 analyzes the relationships among the three different solution sets, both in the nominal scenario and in the worst case. Section 4 illustrates the results in some numerical examples, followed by Section 5 which introduces the price of robustness for a given solution and develops two strategies to assist decision making. Finally, Section 6 concludes the paper.

2 Nominal efficiency, minmax robust efficiency, and lightly robust efficiency

Let a feasible set $\mathfrak{X}$, an uncertainty set $\mathcal{U}$, and a function $f : \mathfrak{X} \times \mathcal{U} \to \mathbb{R}^k$ be given. We deal with the following *uncertain problem*

$$(P_{\mathcal{U}}) \qquad \left(\begin{array}{ll} \min & f(x,\xi) := \begin{pmatrix} f_1(x,\xi) \\ f_2(x,\xi) \\ \vdots \\ f_k(x,\xi) \end{pmatrix} \\ \text{s.t.} & x \in \mathfrak{X} \end{array} \right), \xi \in \mathcal{U}.$$

The elements of $\mathcal{U}$ are called *scenarios*. It is not known which scenario $\xi \in \mathcal{U}$ will occur making the above problem an uncertain multiobjective optimization problem. For any fixed choice of $\xi \in \mathcal{U}$ we have a deterministic multiobjective optimization problem

$$(P(\xi)) \qquad \left(\begin{array}{ll} \min & f(x,\xi) := \begin{pmatrix} f_1(x,\xi) \\ f_2(x,\xi) \\ \vdots \\ f_k(x,\xi) \end{pmatrix} \\ \text{s.t.} & x \in \mathfrak{X} \end{array} \right)$$

for which optimal solutions are defined in the sense of (Pareto-)efficiency:

Notation 1. • *Let $y, y' \in \mathbb{R}^k$. In what follows, notation $y' \leq y$ and $y' < y$ are both meant componentwise, i.e., $y'_i \leq y_i$ and $y'_i < y_i$, respectively, for all $i = 1, \ldots, k$. We say that y'* dominates *y if $y' \leq y$ and there exists some $i \in \{1, \ldots, l\}$ with $y'_i < y_i$.*

- *Let $x, x' \in \mathfrak{X}$. We say that x' dominates x if $f(x')$ dominates $f(x)$. A solution x is called efficient, if there does not exist a solution x' which dominates x.*

The *domination property* for the deterministic multiobjective optimization problem $(P(\xi))$ says: For every $x \in \mathfrak{X}$, either x is efficient or there exists an efficient solution $x' \in \mathfrak{X}$ which dominates x. It is known (e.g., in (Mordechai, 1986)) that the domination property holds for $(P(\xi))$ if $\mathfrak{X}$ is finite and if $\mathfrak{X}$ is compact and the objective functions $f_i(\cdot, \xi)$ are continuous in x for all $i = 1, \ldots, k$.

For the results in this paper, let us assume that one of the following conditions is satisfied:

- $\mathfrak{X}$ and $\mathcal{U}$ are both finite sets,
- $\mathcal{U}$ is finite, $\mathfrak{X}$ is compact and $f(\cdot, \xi) : \mathfrak{X} \to \mathbb{R}^k$ is continuous in x for every fixed $\xi \in \mathcal{U}$,
- $\mathfrak{X}$ is finite, $\mathcal{U}$ is compact and $f(x, \cdot) : \mathcal{U} \to \mathbb{R}^k$ is continuous in ξ for every fixed $x \in \mathfrak{X}$,
- $\mathcal{U}$ and $\mathfrak{X}$ are both compact and $f : \mathfrak{X} \times \mathcal{U} \to \mathbb{R}^k$ is jointly continuous in (x, ξ).

Each of these assumptions guarantees that all minima and maxima exist, i.e., that

- $(P(\xi))$ has the domination property for all fixed $\xi \in \mathcal{U}$.
- $\max_{\xi \in \mathcal{U}} f(x, \xi)$ exists for every fixed $x \in \mathfrak{X}$,

The assumptions hold in many problems studied in the literature and in many applications.

For the uncertain problem, several concepts on how to define *robust efficiency* have been proposed. Here, we consider minmax robust efficiency and lightly robust efficiency. Our goal is to compare minmax robust efficient and lightly robust efficient solutions to the solutions we would obtain without considering robustness, i.e., the *nominal efficient solutions.*

2.1 Nominal efficiency

As usual in robust optimization (e.g., in (Bertsimas & Sim, 2004) and many other references) we assume that a *nominal scenario* $\hat{\xi} \in \mathcal{U}$ is known. This is the standard scenario one would usually take if robustness issues do not play a role. It might be the undisturbed or the most likely scenario, or it contains the parameters which have been measured without any deviation. We define $f^{\text{nom}}(x) := f(x, \hat{\xi})$ and the *nominal problem* as

$$(P^{\text{nom}}) \qquad \left(\begin{array}{ll} \min & f^{\text{nom}}(x) = \begin{pmatrix} f_1(x, \hat{\xi}) \\ f_2(x, \hat{\xi}) \\ \vdots \\ f_k(x, \hat{\xi}) \end{pmatrix} \\ \text{s.t.} & x \in \mathfrak{X} \end{array} \right).$$

Note that (P^{nom}) is a deterministic multiobjective optimization problem. It is the problem which is 'usually' solved, i.e., when no robustness is taken into account.

Definition 2. *We denote the set of efficient solutions to (P^{nom}) by X^{nom}. Solutions $x \in X^{\text{nom}}$ are called* nominal efficient*. For $x \in \mathfrak{X}$, we furthermore call $f^{\text{nom}}(x)$ its* nominal quality*.*

Since (P^{nom}) equals $(P(\hat{\xi}))$, it has the domination property.

Lemma 3. *For every $x \in \mathfrak{X}$ there exists $x' \in X^{\text{nom}}$ with $f^{\text{nom}}(x') \leq f^{\text{nom}}(x)$.*

2.2 Minmax robust efficiency

Minmax robustness is the most widely used concept in single-objective robust optimization (see a summary in (Ben-Tal et al., 2009)). Several generalizations to the multiobjective case have been proposed. Here we use the concept of *(point-based) minmax robustness* as proposed in (Fliege & Werner, 2014; Kuroiwa & Lee, 2012). In case of objective-wise uncertainty (called *owu* in (Ehrgott et al., 2014)), i.e., if the uncertainty in each of the objective functions is independent from the uncertainty in the other objective functions, point-based robustness coincides with set-based robustness (Ehrgott et al., 2014) and with hull-based robustness (Bokrantz & Fredriksson, 2017).

In order to find minmax robust efficient solutions, we define the worst case objective function f^{wc} as

$$f^{\text{wc}}(x) := \begin{pmatrix} \max_{\xi \in \mathcal{U}} f_1(x,\xi) \\ \max_{\xi \in \mathcal{U}} f_2(x,\xi) \\ \vdots \\ \max_{\xi \in \mathcal{U}} f_k(x,\xi) \end{pmatrix}.$$

The resulting optimization problem in the worst case is given as

$$(P^{\text{wc}}) \qquad \begin{pmatrix} \min & f^{\text{wc}}(x) = \begin{pmatrix} \max_{\xi \in \mathcal{U}} f_1(x,\xi) \\ \max_{\xi \in \mathcal{U}} f_2(x,\xi) \\ \vdots \\ \max_{\xi \in \mathcal{U}} f_k(x,\xi) \end{pmatrix} \\ \text{s.t.} & x \in \mathfrak{X} \end{pmatrix}.$$

The problem (P^{wc}) is again a deterministic multiobjective optimization problem.

Definition 4. *Let X^{wc} be the set of efficient solutions to (P^{wc}). Solutions $x \in X^{\text{wc}}$ are called* minmax robust efficient*. For $x \in \mathfrak{X}$, we furthermore call $f^{\text{wc}}(x)$ its* worst case objective value*.*

From our general assumptions we can conclude that (P^{wc}) has the domination property.

Lemma 5. *For every $x \in \mathfrak{X}$ there exists $x' \in X^{\text{wc}}$ with $f^{\text{wc}}(x') \leq f^{\text{wc}}(x)$.*

Proof. If $\mathfrak{X}$ is finite, it is trivial that the lemma holds. Otherwise, $\mathfrak{X}$ is compact and we have to distinguish two cases:

- Either $\mathcal{U}$ is finite and f is continuous in x for every fixed ξ. Then, f^{wc} is continuous as maximum of a finite set of continuous functions.
- Or both, $\mathfrak{X}$ and $\mathcal{U}$ are compact and f is jointly continuous in x and ξ. Then f^{wc} is continuous due to Berge's theorem, see, e.g., (Berge, 1963).

□

In case of objective-wise uncertainty (owu) there exists a worst case scenario $\overline{\xi} \in \mathcal{U}$ for which all objective functions simultaneously take their maxima. In this case, (P^{wc}) equals $(P(\overline{\xi}))$ and is hence a deterministic problem. This need not hold if the same uncertain parameter influences more than one of the objective functions.

2.3 Lightly robust efficiency

Finally, we introduce the concept of lightly robust efficient solutions. The idea comes from light robustness in single-objective optimization (see (Fischetti & Monaci, 2009; Schöbel, 2014)) and focuses on finding solutions which are not too bad in the nominal case. Light robustness was generalized in (Ide & Schöbel, 2016; Kuhn et al., 2016) to the multiobjective case as follows: one first determines the set of efficient solutions X^{nom} for the nominal scenario $\hat{\xi}$. We allow a lightly robust efficient solution to be a bit worse than an efficient solution in the nominal scenario. The deviation from the objective values in the nominal scenario should be bounded by some given $\varepsilon \in \mathbb{R}^k$, where ε_i bounds the deviation in objective function f_i^{nom}. In order to ensure this, we define for each $\hat{x} \in X^{\text{nom}}$

$$(P^{\text{light},\varepsilon}(\hat{x})) \qquad \left(\begin{array}{ll} \min & f^{\text{wc}}(x) = \begin{pmatrix} \max_{\xi\in\mathcal{U}} f_1(x,\xi) \\ \max_{\xi\in\mathcal{U}} f_2(x,\xi) \\ \vdots \\ \max_{\xi\in\mathcal{U}} f_k(x,\xi) \end{pmatrix} \\ \text{s.t.} & f^{\text{nom}}(x) \leq f^{\text{nom}}(\hat{x}) + \varepsilon \\ & x \in \mathfrak{X} \end{array} \right),$$

i.e., among all solutions which are only a bit worse than $\hat{x}$ in the nominal scenario we take the ones which are efficient in the worst case, i.e., which are minmax robust efficient within the set

$$F^{\text{light},\varepsilon}(\hat{x}) := \{x \in \mathfrak{X} : f^{\text{nom}}(x) \leq f^{\text{nom}}(\hat{x}) + \varepsilon\}$$

of feasible solutions to $(P^{\text{light},\varepsilon}(\hat{x}))$. Note that due to our assumptions, either $\mathfrak{X}$ is finite or f^{nom} is continuous. In both cases, $F^{\text{light},\varepsilon}(\hat{x})$ is closed.
In practice, the values of ε can be set by a decision maker. Based on the values of $f^{\text{nom}}(\hat{x})$, the decision maker can consider how much (s)he is willing to sacrifice. In Lemma 15 later in the paper, we will relate the choice of ε to the price of robustness.

Definition 6. *For $\hat{x} \in X^{\text{nom}}$, let $X^{\text{light},\varepsilon}(\hat{x})$ be the set of efficient solutions to $(P^{\text{light},\varepsilon}(\hat{x}))$. Solutions $x \in X^{\text{light},\varepsilon} := \bigcup_{\hat{x}\in X^{\text{nom}}} X^{\text{light},\varepsilon}(\hat{x})$ are called* lightly robust efficient.

Due to its closedness, the feasible set $F^{\text{light},\varepsilon}(\hat{x})$ is compact if $\mathfrak{X}$ is compact, and finite if $\mathfrak{X}$ is finite. With the same reasoning as for (P^{wc}) we hence conclude that the domination property holds for $(P^{\text{light},\varepsilon}(\hat{x}))$.

Lemma 7. *For every $x \in F^{\text{light},\varepsilon}(\hat{x})$ there exists $x' \in X^{\text{light},\varepsilon}(\hat{x})$ with $f^{\text{wc}}(x') \leq f^{\text{wc}}(x)$.*

3 Comparing sets of robust efficient solutions

In (non-robust) multiobjective optimization, the quality of a set of solutions $X \subseteq \mathfrak{X}$ is usually evaluated by looking at the image $f(X)$ of the solutions in the objective space. If X is the set of efficient solutions, their images $f(X)$ are called the *efficient front.* In order to compare the sets of nominal efficient solutions X^{nom}, of minmax robust efficient solutions X^{wc}, and of lightly robust efficient solutions $X^{\text{light},\varepsilon}$, we proceed similarly: we look at the images under the objective function f. However, the objective function values not only depend on $x \in \mathfrak{X}$ but also on the scenario which occurs; we hence get different objective function values for each scenario $\xi \in \mathcal{U}$ and an efficient point in the nominal scenario need not be an efficient point in other scenarios. To consider properties of a set $X \subseteq \mathfrak{X}$ of solutions (specifically for $X = X^{\text{nom}}, X = X^{\text{wc}}$, or $X = X^{\text{light},\varepsilon}$) we propose to evaluate X in the following two extreme cases:

- The first evaluation considers the nominal case, i.e.,

$$f^{\text{nom}}(X) = \{f(x,\hat{\xi}) : x \in X\}.$$

From a practical point of view such an evaluation makes sense since it shows what to expect in the most likely (or undisturbed) scenario. Clearly, $f^{\text{nom}}(X^{\text{nom}})$ shows the efficient front of the problem (P^{nom}).

- The second evaluation takes a robust perspective considering X under its component-wise worst case, i.e., we evaluate

$$f^{\text{wc}}(X) := \{f^{\text{wc}}(x) : x \in X\} = \left\{ \begin{pmatrix} \max_{\xi \in \mathcal{U}} f_1(x,\xi) \\ \vdots \\ \max_{\xi \in \mathcal{U}} f_k(x,\xi) \end{pmatrix} : x \in X \right\}.$$

Note that in contrast to $f^{\text{nom}}(X)$ which is always evaluated under the scenario $\hat{\xi} \in \mathcal{U}$, the scenarios which are relevant for evaluating $f^{\text{wc}}(x)$ depend on the objective function $f_i, i = 1, \ldots, k$ and on the point $x \in X$ itself.

The intuition is that under the nominal scenario, the set $f^{\text{nom}}(X^{\text{nom}})$ is better than the set of minmax robust efficient solutions $f^{\text{nom}}(X^{\text{wc}})$ while this result of comparison changes if we evaluate under the worst case objective function f^{wc}, i.e., $f^{\text{wc}}(X^{\text{wc}})$ is better than $f^{\text{wc}}(X^{\text{nom}})$. The set of lightly robust efficient solutions is expected to lie somewhere in between as they are claimed in (Ide & Schöbel, 2016; Kuhn et al., 2016) to be a good compromise between nominal quality and robustness.

In order to formulate these intuitions mathematically, we use a set-based order to compare two sets $Y_1, Y_2 \subseteq \mathbb{R}^k$:

Notation 8.

$$Y_1 \prec^{upp} Y_2 \text{ if for all } y \in Y_2 \text{ there exists } y' \in Y_1 \text{ with } y' \leq y$$

$$Y_1 \prec^{low} Y_2 \text{ if for all } y \in Y_1 \text{ there exists } y' \in Y_2 \text{ with } y \leq y'.$$

Denoting $\mathbb{R}^k_{\geq} = \{y \in \mathbb{R}^k : y_i \geq 0 \text{ for all } i = 1, \ldots, k\}$ as the nonnegative ordering cone, $Y_1 \prec^{upp} Y_2$ can equivalently be written as

$$Y_1 \prec^{upp} Y_2 \text{ if } Y_1 + \mathbb{R}^k_{\geq} \supseteq Y_2$$

which is known as the *upper set less order*, see (Khan, Tammer, & Zălinescu, 2015), and $Y_1 \prec^{low} Y_2$ can equivalently be written as

$$Y_1 \prec^{low} Y_2 \text{ if } Y_2 - \mathbb{R}^k_{\geq} \supseteq Y_1$$

which is known as the *lower set less order*, see again (Khan et al., 2015).

We first show that evaluating solutions in the worst case always gives a more pessimistic point of view than evaluating the same solutions in the nominal case, no matter what we choose as the set $X \subseteq \mathfrak{X}$.

Lemma 9. *For every set $X \subseteq \mathfrak{X}$ we have:*

(i) $f^{\text{nom}}(X) \prec^{upp} f^{\text{wc}}(X)$.

(ii) $f^{\text{nom}}(X) \prec^{low} f^{\text{wc}}(X)$.

Proof.

(i) : Let $y \in f^{\text{wc}}(X)$, i.e., $y = f^{\text{wc}}(x, \xi)$ for some $x \in X$. ~~and some $\xi \in \mathcal{U}$.~~ Define $y' := f^{\text{nom}}(x)$. Then $y' \in f^{\text{nom}}(X)$ and for each component $i = 1, \ldots, k$ we have

$$y'_i = f_i(x, \hat{\xi}) \leq \max_{\xi \in \mathcal{U}} f_i(x, \xi) = y_i,$$

hence $y' \leq y$.

(ii) : Let $y \in f^{\text{nom}}(X)$, i.e., $y = f^{\text{nom}}(x, \xi)$ for some $x \in X$ and some $\xi \in \mathcal{U}$. Define $y' := f^{\text{wc}}(x)$. Then $y' \in f^{\text{wc}}(X)$ and analogously to part (i) it follows $y \leq y'$.

□

It is more interesting to compare the different sets of robust efficient points with each other. We start by showing that under the upper set less order relation, X^{nom} is better than any other set $X \subseteq \mathfrak{X}$ in the nominal scenario, and X^{wc} is better than any other set $X \subseteq \mathfrak{X}$ under worst case evaluation.

Lemma 10. *For every set $X \subseteq \mathfrak{X}$ we have:*

(i) $f^{\text{nom}}(X^{\text{nom}}) \prec^{upp} f^{\text{nom}}(X)$.

(ii) $f^{\text{wc}}(X^{\text{wc}}) \prec^{upp} f^{\text{wc}}(X)$.

Proof.

(i) Let $y \in f^{\text{nom}}(X)$, i.e., $y = f^{\text{nom}}(x)$ for some $x \in X$. Due to the domination property for (P^{nom}) (Lemma 3), there exists $x' \in X^{\text{nom}}$ with $f^{\text{nom}}(x') \leq f^{\text{nom}}(x)$. Setting $y' := f^{\text{nom}}(x')$ shows the assertion.

(ii) Now let $y \in f^{\text{wc}}(X)$, i.e., $y = f^{\text{wc}}(x)$ for some $x \in X$. Due to the domination property for (P^{wc}) (Lemma 5) there exists $x' \in X^{\text{wc}}$ with $y' := f^{\text{wc}}(x') \leq f^{\text{wc}}(x) = y$, and the proof is complete.

□

Note that Lemma 10 does *not* hold under the lower set less order relation, not even if we only compare the sets X^{nom} and X^{wc} with each other in the nominal scenario. This is shown in the following small example. Note that the problem in this example only has $k = 2$ objective functions which are objective-wise independent (see (Ehrgott et al., 2014) for a formal definition); so the relation does not even hold under this rather special condition.

Example 1. *Let two scenarios $\mathcal{U} = \{\hat{\xi}, \bar{\xi}\}$ be given, consider a feasible set $\mathfrak{X}$ which contains only two elements $\mathfrak{X} = \{x_1, x_2\}$ and two objective functions. Let*

$$f(x_1, \hat{\xi}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad f(x_2, \hat{\xi}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
$$f(x_1, \bar{\xi}) = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \qquad f(x_2, \bar{\xi}) = \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

see Figure 1 for an illustration. In this example, we receive

$$X^{\text{nom}} = \{x_1, x_2\}$$

since their objective function values in the nominal scenario do not dominate each other. For f^{wc} we receive

$$f^{\text{wc}}(x_1) = \begin{pmatrix} \max\{2,5\} \\ \max\{1,5\} \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$
$$f^{\text{wc}}(x_2) = \begin{pmatrix} \max\{0,1\} \\ \max\{3,4\} \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

i.e.,

$$X^{\text{wc}} = \{x_2\}.$$

Hence,

$$f^{\text{nom}}(X^{\text{nom}}) = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\} \quad \textit{and} \quad f^{\text{nom}}(X^{\text{wc}}) = \left\{ \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}.$$

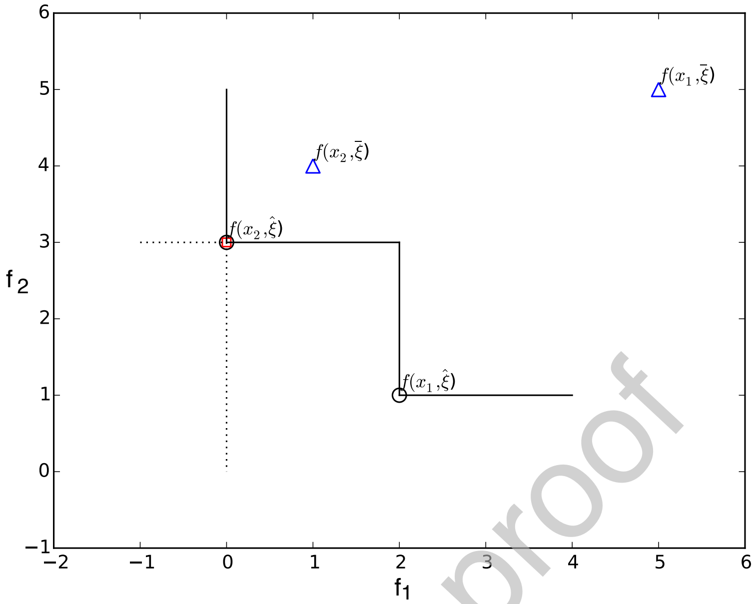

Figure 1: Illustration of Example 1. We compare $f^{\text{nom}}(X^{\text{nom}}) = \{f(x_1,\hat{\xi}), f(x_2,\hat{\xi})\}$ and $f^{\text{nom}}(X^{\text{wc}}) = \{f(x_2,\hat{\xi})\}$.

As Lemma 10 says, we have

$$f^{\text{nom}}(X^{\text{nom}}) \prec^{upp} f^{\text{nom}}(X^{\text{wc}}),$$

but for the lower set less order this does not hold. We even receive

$$f^{\text{nom}}(X^{\text{wc}}) \prec^{low} f^{\text{nom}}(X^{\text{nom}}).$$

We continue with analyzing the set of lightly robust efficient solutions $X^{\text{light},\varepsilon}$. We start with a simple observation which follows directly from Definition 6. For this we need to add the point ε to $f^{\text{nom}}(X^{\text{nom}})$ in the set-wise sense, i.e., for $A \subseteq \mathfrak{X}$, $\xi \in \mathbb{R}$ we have $f^{\text{nom}}(A) + \{\varepsilon\} = \{f^{\text{nom}}(x) + \varepsilon : x \in A\}$.

Lemma 11. *For every $\varepsilon \geq 0$ we have*

(i) $f^{\text{nom}}(X^{\text{light},\varepsilon}) \prec^{low} f^{\text{nom}}(X^{\text{nom}}) + \{\varepsilon\}$,

(ii) $f^{\text{nom}}(X^{\text{light},\varepsilon}) \prec^{upp} f^{\text{nom}}(X^{\text{nom}}) + \{\varepsilon\}$,

Proof.

(i) Let $y \in f^{\text{nom}}(X^{\text{light},\varepsilon})$, i.e., $y = f^{\text{nom}}(x)$ for $x \in X^{\text{light},\varepsilon}$. Then there exists a solution $\hat{x} \in X^{\text{nom}}$ such that x is an efficient solution to $(P^{\text{light},\varepsilon}(\hat{x}))$. In particular,

$$f^{\text{nom}}(x) \leq f^{\text{nom}}(\hat{x}) + \varepsilon.$$

We define $y' := f^{\text{nom}}(\hat{x}) + \varepsilon \in f^{\text{nom}}(X^{\text{nom}}) + \{\varepsilon\}$ and receive that $y = f^{\text{nom}}(x) \leq y'$.

(ii) Let $y \in f^{\text{nom}}(X^{\text{nom}}) + \{\varepsilon\}$, i.e., $y = f^{\text{nom}}(\hat{x}) + \varepsilon$ for some $\hat{x} \in X^{\text{nom}}$. Due to the domination property for $(P^{\text{light},\varepsilon}(\hat{x}))$ (Lemma 7) there exists $x' \in X^{\text{light},\varepsilon}(\hat{x})$. In particular,

$$f^{\text{nom}}(x') \leq f^{\text{nom}}(\hat{x}) + \varepsilon.$$

With $y' := f^{\text{nom}}(x')$ we hence receive $y' = f^{\text{nom}}(x') \leq f^{\text{nom}}(\hat{x}) + \varepsilon = y$.

□

Together with Lemma 10 we summarize

$$f^{\text{nom}}(X^{\text{nom}}) \prec^{upp} f^{\text{nom}}(X^{\text{light},\varepsilon}) \prec^{upp} f^{\text{nom}}(X^{\text{nom}}) + \{\varepsilon\},$$

i.e., for small $\varepsilon > 0$ the evaluation of X^{nom} and $X^{\text{light},\varepsilon}$ under the nominal scenario differs only slightly. The next lemma analyzes what might happen to lightly robust efficient solutions in the worst case.

Lemma 12.

(i) $f^{\text{wc}}(X^{\text{light},\varepsilon}) \prec^{upp} f^{\text{wc}}(X^{\text{nom}})$ *for all* $\varepsilon \geq 0$ *and*

(ii) $f^{\text{wc}}(X^{\text{light},\varepsilon_2}) \prec^{upp} f^{\text{wc}}(X^{\text{light},\varepsilon_1})$ *for all* $0 \leq \varepsilon_1 \leq \varepsilon_2$.

Proof.

(i) Let $y = f^{\text{wc}}(x)$ for $x \in X^{\text{nom}}$. Then $x \in F^{\text{light},\varepsilon}(x)$, i.e., it is feasible for $(P^{\text{light},\varepsilon}(x))$. Due to the domination property for $(P^{\text{light},\varepsilon}(x))$ (Lemma 7) there exists $x' \in X^{\text{light},\varepsilon}(x)$ with $f^{\text{wc}}(x') \leq f^{\text{wc}}(x)$. We hence have $y' := f^{\text{wc}}(x') \leq f^{\text{wc}}(x) = y$.

(ii) Now let $y = f^{\text{wc}}(x)$ for $x \in X^{\text{light},\varepsilon_1}$. Then there exists $\hat{x} \in X^{\text{nom}}$ such that $x \in F^{\text{light},\varepsilon_1}(\hat{x})$. Since $\varepsilon_2 \geq \varepsilon_1$ we know that $x \in F^{\text{light},\varepsilon_2}(\hat{x})$. We again use the domination property (Lemma 7) for $(P^{\text{light},\varepsilon_2}(\hat{x}))$ and receive $x' \in X^{\text{light},\varepsilon_2}$ which satisfies $y' := f^{\text{wc}}(x') \leq f^{\text{wc}}(x) = y$.

□

Note that the statements of Lemma 12 do again not hold for the lower set less order. They also cannot be transferred to the nominal case, i.e., it is *not* true in general that

1. $f^{\text{nom}}(X^{\text{light},\varepsilon}) \prec f^{\text{nom}}(X^{\text{wc}})$ for $\varepsilon > 0$, and that
2. $f^{\text{nom}}(X^{\text{light},\varepsilon_1}) \prec f^{\text{nom}}(X^{\text{light},\varepsilon_2})$ for $0 \leq \varepsilon_1 < \varepsilon_2$,

neither for $\prec$ being the upper set less order nor for the lower set less order. This is illustrated next.

Example 2. *Let two scenarios* $\mathcal{U} = \{\hat{\xi}, \bar{\xi}\}$ *be given, consider a feasible set* $\mathfrak{X} = \{x_1, x_2, x_3\}$ *and two objective functions. Let*

$$f(x_1,\hat{\xi}) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad f(x_2,\hat{\xi}) = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix}, \quad f(x_3,\hat{\xi}) = \begin{pmatrix} 5 \\ 3.5 \end{pmatrix},$$

$$f(x_1,\bar{\xi}) = \begin{pmatrix} 10 \\ 10 \end{pmatrix}, \quad f(x_2,\bar{\xi}) = \begin{pmatrix} 8 \\ 8 \end{pmatrix}, \quad f(x_3,\bar{\xi}) = \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$

Then

$$X^{\text{nom}} = \{x_1\},\ X^{\text{light},1} = \{x_2\},\ X^{\text{light},2} = \{x_3\},\ X^{\text{wc}} = \{x_3\}$$

This example is illustrated in Figure 2.

The next lemma analyzes what happens in the nominal scenario when lightly robust efficient solutions and minmax efficient solutions are compared.

Lemma 13. *Let* $\varepsilon \geq 0$ *and* $x \in X^{\text{light},\varepsilon}$. *Then there does not exist* $x' \in X^{\text{wc}}$ *which is at least as good as* x *with respect to* f^{nom} *and dominates* x *with respect to* f^{wc}.

Proof. Let $x' \in X^{\text{wc}}$ and $x \in X^{\text{light},\varepsilon}$. Assume that $f^{\text{nom}}(x') \leq f^{\text{nom}}(x)$. We know that $f^{\text{nom}}(x) \leq f^{\text{nom}}(\hat{x}) + \varepsilon$ for some $\hat{x} \in X^{\text{nom}}$. So $f^{\text{nom}}(x') \leq f^{\text{nom}}(\hat{x}) + \varepsilon$ holds. Hence, $x' \in F^{\text{light},\varepsilon}(\hat{x})$ and due to $x' \in X^{wc}$ we conclude that $x' \in X^{\text{light},\varepsilon}(\hat{x})$, i.e., both x and x' are lightly robust efficient and consequently, do not dominate each other under f^{wc}. Thus, the lemma holds. □

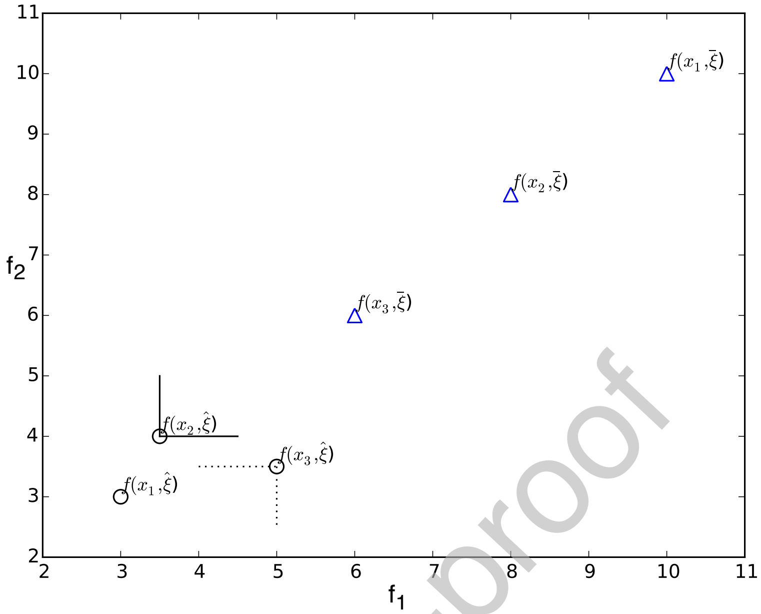

Figure 2: Illustration of Example 2. Lemma 12 neither holds for $\prec^{low}$ nor can it be mirrored to the nominal case for $\prec^{upp}$ and $\prec^{low}$.

What we have learned about lightly robust efficient solutions. We summarize our main findings with respect to lightly robust efficient solutions: First, the set of lightly robust efficient solutions lies (in the worst case) always between the set of nominal efficient and the set of worst case efficient solutions (see Lemma 10 and Lemma 12):

$$f^{\text{wc}}(X^{\text{wc}}) \prec^{upp} f^{\text{wc}}(X^{\text{light},\varepsilon_2}) \prec^{upp} f^{\text{wc}}(X^{\text{light},\varepsilon_1}) \prec^{upp} f^{\text{wc}}(X^{\text{nom}}) \tag{1}$$

for $\varepsilon_1 \leq \varepsilon_2$. Hence, choosing lightly robust efficient solutions might be a good compromise between nominal and minmax efficient solutions. Second, the larger ε is chosen, the more robustness we gain.

For the nominal scenario, due to Lemma 10 and Lemma 11 we furthermore know that

$$f^{\text{nom}}(X^{\text{nom}}) \prec^{upp} f^{\text{nom}}(X^{\text{light},\varepsilon}) \prec f^{\text{nom}}(X^{\text{nom}}) + \{\varepsilon\} \tag{2}$$

where the second relation holds for both, the upper set less order $\prec^{upp}$ and the lower set less order $\prec^{low}$, which means that in the nominal scenario, the set of lightly robust efficient solutions gets more similar to the set of nominal efficient solutions if ε is decreased.

4 Examples and Illustration

So far, we have analyzed the three different sets of nominal, lightly robust efficient and minmax robust efficient solutions in the nominal case and in the worst case. In this section, we illustrate our findings with some examples.

We first look at a simple problem where the uncertain parameter comes from an interval uncertainty set.

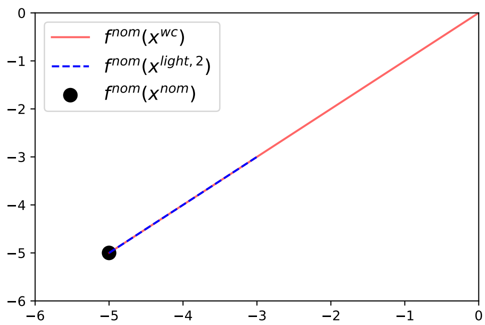

Figure 3: Evaluation in the nominal case of Example 3

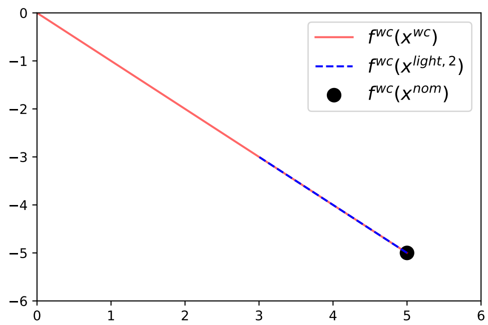

Figure 4: Evaluation in the worst case of Example 3

Example 3.

$$\begin{pmatrix} \min & f_1(x,\xi) = \xi x, \\ & f_2(x) = x \\ s.t. & -5 \leq x \leq 5 \end{pmatrix}_{\xi \in [-1,3]}. \tag{3}$$

The nominal scenario in this example is $\hat{\xi} = 1$. *The nominal efficient solution is* $X^{\text{nom}} = \{-5\}$. *The minmax robust efficient solutions are* $X^{\text{wc}} = \{x : x \in [-5, 0]\}$. *Given* $\varepsilon = 2$, *we have* $X^{\text{light},2} = \{x : x \in [-5, -3]\}$. *Evaluating* $X^{\text{nom}}, X^{\text{light}}$, *and* X^{wc} *in* f^{nom}, *the images are shown in Figure 3 and evaluating in* f^{wc}, *the images are shown in Figure 4.*

We observe that (1) *holds and see that it cannot be strengthened even for linear problems with interval uncertainty. However, regarding* (2) *in this simple example, we have that* $f^{\text{nom}}(X^{\text{wc}}) \prec^{upp} f^{\text{nom}}(X^{\text{nom}})$ *and* $f^{\text{nom}}(X^{\text{nom}}) \prec^{low} f^{\text{nom}}(X^{\text{wc}})$. *But also these observations are not true in general. They do not even hold for the easiest case of*

- *a linear problem with a single variable*
- *whose feasible region is an interval*
- *and with only a single uncertain parameter in only one objective function*
- *and an interval as uncertainty set.*

We illustrate this in the following counterexample with minor modification of (3)*:*

$$\begin{pmatrix} \min & f_1(x,\xi) = -\xi x, \\ & f_2(x) = x \\ s.t. & 0 \leq x \leq 5 \end{pmatrix}_{\xi \in [-1,3]}, \tag{4}$$

where the nominal case is $\hat{\xi} = 1$. *The nominal efficient solutions are* $X^{\text{nom}} = \{x : x \in [0, 5]\}$ *and the minmax robust efficient solutions are* $X^{\text{wc}} = \{0\}$. *Evaluating them in* f^{nom}, *the images are shown in Figure 5. The figure shows that*

$$f^{\text{nom}}(X^{\text{wc}}) \prec^{upp} f^{\text{nom}}(X^{\text{nom}}) \text{ and } f^{\text{nom}}(X^{\text{nom}}) \prec^{low} f^{\text{nom}}(X^{\text{wc}})$$

both do not hold.

Then we look at a bi-objective optimization problem given as follows.

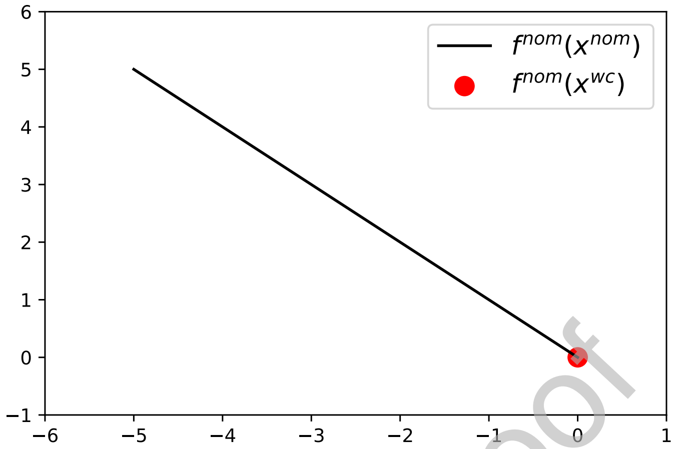

Figure 5: Evaluation in the nominal case of problem (4)

Example 4.

$$\begin{pmatrix} \min & f_1(x,\xi_1) = x_1^2 + \xi_1 x_2^2, \\ & f_2(x,\xi_2) = (\xi_2 x_1 - 5)^2 + (x_2 - 5)^2 \\ s.t. & 0 \le x_1 \le 4 \\ & 0 \le x_2 \le 3, \end{pmatrix}_{\xi_1 \in \mathcal{U}_1, \xi_2 \in \mathcal{U}_2}. \tag{5}$$

where $\mathcal{U}_1 = \{-6, -1, 0.5, 1\}$ *and* $\mathcal{U}_2 = \{-2, -1, 1, 2\}$ *and the nominal values for the uncertain parameters are* $\hat{\xi_1} = -1$ *and* $\hat{\xi_2} = 1$*. This problem is a variation of the Binh and Korn function (Binh & Korn, 1997), which minimizes two quadratic functions within the given ranges of decision variables. In this problem,* ξ_1 *and* ξ_2 *are independent from each other and there exists a single worst case scenario* $\xi_1 = 1$, $\xi_2 = -2$.

Figure 6 shows X^{wc}, $X^{\mathrm{light},15}$, $X^{\mathrm{light},10}$, *and* X^{nom} *evaluated in the nominal case. We can see the relationship*

$$f^{\mathrm{nom}}(X^{\mathrm{nom}}) \prec^{upp} f^{\mathrm{nom}}(X^{\mathrm{light},10}) \prec^{upp} f^{\mathrm{nom}}(X^{\mathrm{light},15}) \prec f^{\mathrm{nom}}(X^{\mathrm{nom}}) + 15.$$

Figure 7 illustrates X^{wc}, $X^{\mathrm{light},15}$, $X^{\mathrm{light},10}$, *and* X^{nom} *evaluated in the worst case. In the figure, we observe the results of Lemma 10 and Lemma 12, namely,*

$$f^{\mathrm{wc}}(X^{\mathrm{wc}}) \prec^{upp} f^{\mathrm{wc}}(X^{\mathrm{light},15}) \prec^{upp} f^{\mathrm{wc}}(X^{\mathrm{light},10}) \prec^{upp} f^{\mathrm{wc}}(X^{\mathrm{nom}}).$$

In this specific problem, we have

$$f^{\mathrm{nom}}(X^{\mathrm{light},10}) \prec^{upp} f^{\mathrm{nom}}(X^{\mathrm{light},15}) \prec^{upp} f^{\mathrm{nom}}(X^{wc})$$

even though it does not hold in general. The example also illustrates that $f^{\mathrm{nom}}(X^{\mathrm{nom}}) \prec^{low} f^{\mathrm{nom}}(X^{\mathrm{light},\varepsilon})$ *and* $f^{\mathrm{nom}}(X^{\mathrm{nom}}) \prec^{low} f^{\mathrm{nom}}(X^{\mathrm{wc}})$ *need not hold.*

We also observe that a solution in X^{nom} *can have very bad objective function values in the worst case compared to lightly and minmax robust efficient solutions. On the other hand, gaining minmax robust efficiency comes at a high price: there has to be a great sacrifice on the nominal quality of the solutions as the minmax robust efficient solutions are very far from the nominal solutions when evaluated in the nominal case. In this example, lightly robust solutions are a good compromise.*

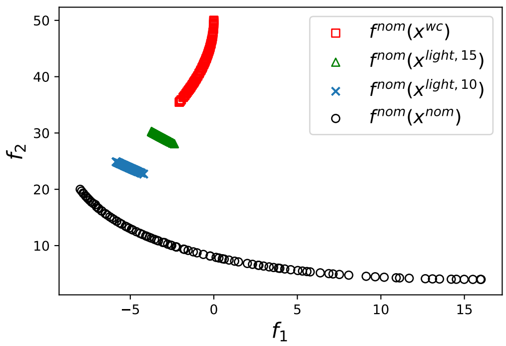

Figure 6: Evaluation in the nominal case of Example 4

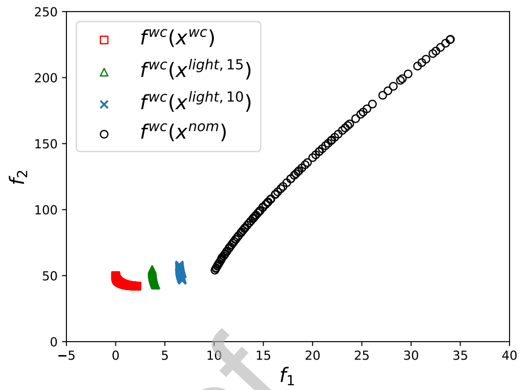

Figure 7: Evaluation in the worst case of Example 4

The example also illustrates a general property, namely, which of the depicted sets form "efficient fronts", i.e., in which cases the depicted points do not dominate each other. This holds per definition for $f^{\text{nom}}(X^{\text{nom}})$, $f^{\text{wc}}(X^{\text{wc}})$, and $f^{\text{wc}}(X^{\text{light},\varepsilon})$, i.e., we obtain a nominal efficient front for the nominal robust efficient points and a minmax robust efficient front for the minmax robust efficient and for the different lightly robust efficient points. In contrast to this, Figures 7 shows that the points in $f^{\text{wc}}(X^{\text{nom}})$ may dominate each other.

Next, we consider a more interesting example where the worst case depends on the solution x.

Example 5.

$$\begin{pmatrix} \min & f_1(x,\xi_1) = \xi_1 x_1 + x_2, \\ & f_2(x,\xi_2) = -x_1 - \xi_2 x_2 \\ s.t. & -2 \le x_1 \le 2 \\ & -2 \le x_2 \le 2, \end{pmatrix}_{\xi=(\xi_1,\xi_2)^T\in\mathcal{U}}. \tag{6}$$

The uncertainty set $\mathcal{U}$ is:

$$\mathcal{U} = \left\{ \begin{pmatrix} -3 \\ 1.5 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2.5 \end{pmatrix} \right\}.$$

The nominal values for the uncertain parameters are $\hat{\xi}_1 = -1$ and $\hat{\xi}_2 = 2$. Finding a worst case for some given and feasible x in this example can be written as the following two-objective optimization problem

$$\begin{pmatrix} \max & f_1(x,\xi_1) = \xi_1 x_1 + x_2, \\ & f_2(x,\xi_2) = -x_1 - \xi_2 x_2 \\ s.t. & \xi \in \mathcal{U} \end{pmatrix}. \tag{7}$$

We see that the worst case depends on the solution x: for $-2 \le x_1 \le 0$ and $0 \le x_2 \le 2$, the worst case is $\xi = (-3, 1.5)^T$. For $-2 \le x_1 \le 0$ and $-2 \le x_2 \le 0$, there does not exist a single worst case. We observe that no pair of the three scenarios in $\mathcal{U}$ dominates each other, hence the set of non-dominated solutions of the maximization problem (7) is $\mathcal{U}$ itself. Similarly, for $0 \le x_1 \le 2$,$-2 \le x_2 \le 0$, the worst case is $\xi = (1, 2.5)^T$ and for $0 \le x_1 \le 2$ and $0 \le x_2 \le 2$, the set of worst-case scenarios is again $\mathcal{U}$. Since we use point-based minmax robust efficiency, we need not worry about the existence of

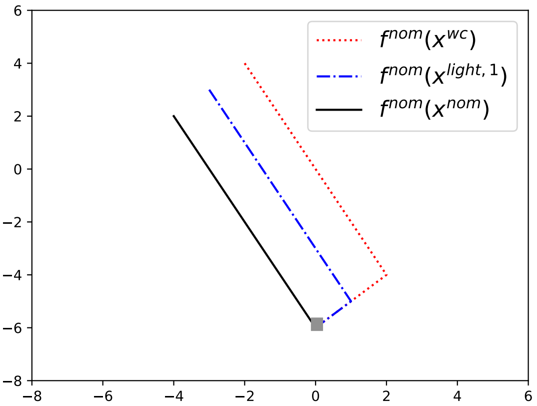

Figure 8: Evaluation in the nominal case of Example 5

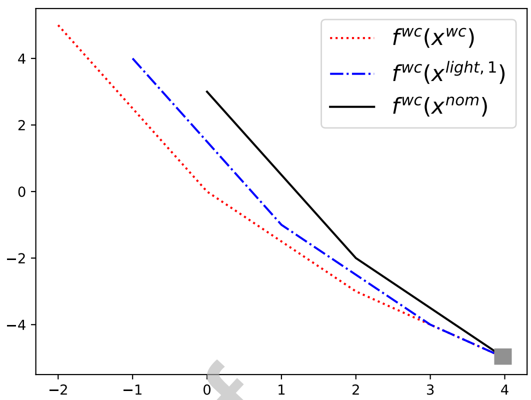

Figure 9: Evaluation in the worst case of Example 5

multiple worst-case scenarios, but compute the worst-case objective function f^{wc} *by taking the componentwise maximum. We hence receive*

$$\begin{aligned} X^{\text{nom}} &= \{(2, x_2) : -2 \leq x_2 \leq 2\}, \\ X^{\text{light},1} &= \{(1, x_2) : -2 \leq x_2 \leq 2\} \cup \{(x_1, 2) : 1 \leq x_1 \leq 2\}, \\ X^{\text{wc}} &= \{(0, x_2) : -2 \leq x_2 \leq 2\} \cup \{(x_1, 2) : 0 \leq x_1 \leq 2\}. \end{aligned}$$

Figure 8 shows $X^{\text{nom}}, X^{\text{light},1}$ *and* X^{wc} *in the nominal case and Figure 9 shows the three sets of solutions in the worst case. As shown in Figure 8, this example is in accordance with our results of Section 3:*

$$f^{\text{nom}}(X^{\text{nom}}) \prec^{upp} f^{\text{nom}}(X^{\text{light},1}) \prec^{upp} f^{\text{nom}}(X^{\text{nom}}) + 1.$$

In this example we also receive $f^{\text{nom}}(X^{\text{light},1}) \prec^{upp} f^{\text{nom}}(X^{wc})$ *although it does not hold for general problems. Figure 9 illustrates our results on the worst case evaluation, namely*

$$f^{\text{wc}}(X^{\text{wc}}) \prec^{upp} f^{\text{nom}}(X^{\text{light},1}) \prec^{upp} f^{wc}(X^{\text{nom}}).$$

The example has a particularity, namely the solution $x = (2, 2)^T$ *is a common element in all three sets of solutions and hence naturally a good choice as a final solution to gain best possible objective function values in both, the nominal and the worst case. The solution is indicated with a square in the two figures. We also observe that* $f^{\text{nom}}(X^{\text{nom}})$ *is not that far from the minmax robust efficient front in the worst case while the two sets differ significantly in the nominal case. Hence, in this example, much quality in the nominal case has to be sacrificed to gain minmax robust efficiency, i.e., the price to gain robustness is rather high.*

The observations on the examples motivate us to further analyze the trade-off between nominal quality and robustness of the solutions in the next section.

5 Utilizing the price of robustness in decision making

The price of robustness has been popular in single-objective robust optimization since its introduction in (Bertsimas & Sim, 2004). In this section we propose how to measure the price of robustness in a multiobjective setting, i.e., how much nominal quality has to be sacrificed in order to receive a minmax robust efficient solution. We first define the price of robustness, and then sketch ideas on how to utilize it to support a decision maker to find a desired solution which is satisfactory in both respects, nominal quality and robustness.

Definition 14. *Let $x \in \mathfrak{X}$ be a feasible solution to $P_{\mathcal{U}}$. We define its* price of robustness *as the objective value of the minimization problem*

$$\text{price}(x) \quad = \quad \inf_{\bar{x} \in X^{\text{wc}}} \|f^{\text{nom}}(x) - f^{\text{nom}}(\bar{x})\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the infinity-metric.

price(x) quantifies the loss of nominal quality when moving x to a closest robust efficient solution $\bar{x}$ and hence measures the costs of making x robust.

Note that a minimum of the above optimization problem need not always exist, not even under the assumptions we stated in Section 2. However, a minimum exists for linear optimization problems and if the objective function f^{wc} is continuous and strictly quasiconcave (Benson & Sun, 1999). For $x \in X^{\text{nom}}$ being efficient in the nominal case, price(x) tells us how much nominal quality we have to sacrifice in one of the objective functions if we replace x by its (closest) robust efficient solution. Instead of using $\|\cdot\|_\infty$ we could also use another norm, e.g., $\|\cdot\|_1$ would give us the average nominal quality over all objective functions we lose when changing x to a minmax robust efficient solution. While the norm to be used may depend on the particular application, we suggest the $\|\cdot\|_\infty$ norm due to its inherent robustness (namely to consider the worst case) and due to its relation to the definition of light robustness which will be used below. Also, its meaning can be easily explained to a decision maker. In practice, if $\|\cdot\|_\infty$ is used, the objective function values should be normalized. Clearly, a point $x \in \mathfrak{X}$ is minmax robust efficient if and only if its price of robustness is zero, i.e.,

$$x \in X^{\text{wc}} \iff \text{price}(x) = 0.$$

Geometrically, for computing price(x), we have to project $f^{\text{nom}}(x)$ on the set $f^{\text{nom}}(X^{\text{wc}})$, i.e., the closest minmax robust efficient solution to x is chosen from X^{wc} with respect to $\|\cdot\|_\infty$. The situation is illustrated in Figure 10. In the figure, the nominal efficient solution x is marked by a bullet and the closest minmax robust efficient solution is marked by a filled square on the robust efficient front. The big square centered in the nominal efficient solution shows the unit ball of $\|\cdot\|_\infty$. While price(x) is easy to read off from such a figure in the biobjective case it is hard to compute in a general setting. The approach may be splitted into two steps: First, compute X^{wc} and find a suitable representation of $Y := f^{\text{nom}}(X^{\text{wc}})$, second project $y := f^{\text{nom}}(x)$ on Y. Note that in combinatorial optimization problems X^{wc} is finite and often of moderate size. In this case, price(x) can be computed by enumeration approaches. Doing this in one common step instead of splitting the computation into two steps may be possible by using methods to optimize over an efficient set. Optimization over an efficient set focuses on optimizing a function with the efficient set of a multiobjective optimization problem as feasible set. The methods are roughly developed from two directions: one is replacing the efficient set by optimality conditions and the other is to search for the solution in the objective space see e.g., (Benson, 1984; Yamamoto, 2002). Since price(x), $\|\cdot\|_\infty$ is nonlinear (but convex), approaches as in (Thoai, 2000; Yamada, Tanino, & Inuiguchi, 2000, 2001; Benson, 2012; Horst, Thoai, Yamamoto, & Zenke, 2007) can be used.

We finally describe a special case in which the price of robustness can be computed. Assume that $\mathfrak{X} = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polyhedron with a matrix $A \in \mathbb{R}^{m,n}$, and that $f_i(\cdot, \xi)$ for $i = 1, ..., k$ is linear in x for every fixed ξ and that $f_i(x, \cdot)$ for $i = 1, ..., k$ is either increasing or decreasing in ξ. Furthermore, let $\mathcal{U}$ be an interval uncertainty set

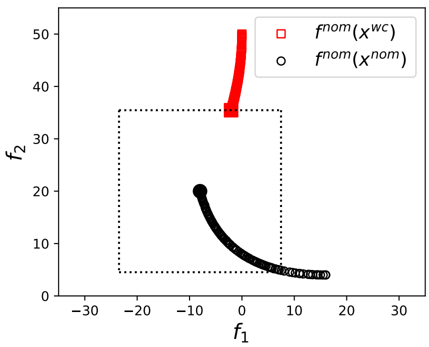

Figure 10: A nominal efficient solution and the closest minmax robust efficient solution

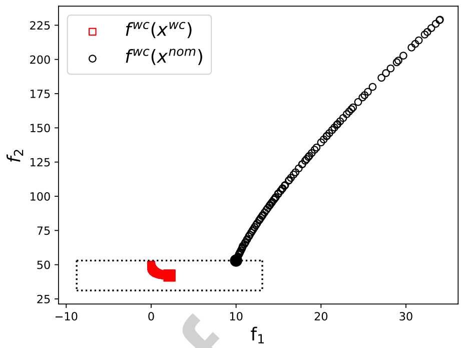

Figure 11: The gain of robustness of the two solutions in Figure 10

$([\underline{\xi}, \overline{\xi}])$. Then we can reformulate price(x) to an optimization problem with linear objective function and both linear and quadratic constraints:

Since $f_i(x, \cdot)$ for $i = 1, ..., k$ are either increasing or decreasing, we know that $f_i^{\text{wc}}(x)$ for $i = 1, ..., k$ is either $f_i(x, \underline{\xi})$ or $f_i(x, \overline{\xi})$ for $i = 1, ..., k$. Due to the linearity for fixed ξ we can rewrite (P^{wc}) as

$$(P^{\text{wc}}_{\text{linear}}) = \{\min Cx : Ax \leq b\},$$

where C is the matrix which contains the worst case coefficients of the objective functions.

Note that the price of robustness can be reformulated as:

$$\begin{array}{ll} \min & \alpha \\ \text{s.t.} & \bar{x} \in X^{\text{wc}} \\ & f_i^{\text{nom}}(x) - f_i^{\text{nom}}(\bar{x}) \leq \alpha \text{ for } i = 1, ..., k \\ & -f_i^{\text{nom}}(x) + f_i^{\text{nom}}(\bar{x}) \leq \alpha \text{ for } i = 1, ..., k \end{array} \tag{8}$$

Using the well-known optimality conditions for multiobjective linear optimization (see e.g., (Ehrgott, 2005)), $x \in \mathfrak{X}$ is efficient to $(P^{\text{wc}}_{\text{linear}})$ if and only if there exist $\lambda \in \mathbb{R}^k$ and $\mu \in \mathbb{R}^m$ such that $A^T\mu = \lambda^T C$ and $\lambda^T Cx = b^T\mu$. Thus, we can rewrite (8) as:

$$\begin{array}{ll} \min & \alpha \\ \text{s.t.} & A\bar{x} \leq b \\ & A^T\mu = \lambda^T C \\ & \lambda^T C\bar{x} = b^T\mu \\ & f_i^{\text{nom}}(x) - f_i^{\text{nom}}(\bar{x}) \leq \alpha \text{ for } i = 1, ..., k \\ & -f_i^{\text{nom}}(x) + f_i^{\text{nom}}(\bar{x}) \leq \alpha \text{ for } i = 1, ..., k \\ & \mu \geq 0 \\ & \lambda > 0. \end{array} \tag{9}$$

The relation to lightly robust efficient solutions is analyzed next. In Lemma 31 in (Ide & Schöbel, 2016) it was shown that there exists $\varepsilon \geq 0$ such that there exists a solution x to $(P^{\text{light},\varepsilon}(\hat{x}))$ which is minmax robust efficient, i.e., with price$(x) = 0$. We strengthen this result by specifying the size of ε and we extend it to the situation in which all solutions to $(P^{\text{light},\varepsilon}(\hat{x}))$ are minmax robust efficient.

Lemma 15. *Let $\hat{x} \in X^{\text{nom}}$ be given. Then the following hold.*

- $\varepsilon \geq \text{price}(\hat{x})$ *if and only if there is a solution* x *to* $(P^{\text{light},\varepsilon}(\hat{x}))$ *with* $\text{price}(x) = 0$.
- *Let* $\varepsilon \geq \sup_{\bar{x}\in X^{\text{wc}}} \|f^{\text{nom}}(\bar{x}) - f^{\text{nom}}(\hat{x})\|_\infty$. *Then all solutions* x *to* $(P^{\text{light},\varepsilon}(\hat{x}))$ *satisfy* $\text{price}(x) = 0$.

Proof.

- Let $\varepsilon \geq \text{price}(\hat{x})$, i.e., there exists $\bar{x} \in X^{\text{wc}}$ with $\|f^{\text{nom}}(\hat{x}) - f^{\text{nom}}(\bar{x})\|_\infty \leq \varepsilon$. Due to $\bar{x}$ being minmax robust efficient, it is an optimal solution to $(P^{\text{light},\varepsilon}(\hat{x}))$ and satisfies $\text{price}(\bar{x}) = 0$. On the other hand, let x be an optimal solution to $(P^{\text{light},\varepsilon}(\hat{x}))$ with $\text{price}(x) = 0$, i.e., $x \in X^{\text{wc}}$. Since $x \in F^{\text{light},\varepsilon}(\hat{x})$ we furthermore know $\|f_i^{\text{nom}}(x) - f_i^{\text{nom}}(\hat{x})\|_\infty \leq \varepsilon$ for all objectives $i = 1, \ldots, k$. We hence conclude

$$\varepsilon \geq \|f^{\text{nom}}(x) - f^{\text{nom}}(\hat{x})\|_\infty \geq \inf_{\bar{x}\in X^{\text{wc}}} \|f^{\text{nom}}(\bar{x}) - f^{\text{nom}}(\hat{x})\|_\infty = \text{price}(\hat{x}).$$

- Now let $\varepsilon \geq \sup_{\bar{x}\in X^{\text{wc}}} \|f^{\text{nom}}(\bar{x}) - f^{\text{nom}}(\hat{x})\|_\infty$, i.e., all $\bar{x} \in X^{\text{wc}}$ satisfy $\|f^{\text{nom}}(\hat{x}) - f^{\text{nom}}(\bar{x})\|_\infty \leq \varepsilon$, hence $X^{\text{wc}} \subseteq F^{\text{light},\varepsilon}(\hat{x})$. This implies $X^{\text{light},\varepsilon} = X^{\text{wc}}$. Consequently, every solution x to $(P^{\text{light},\varepsilon}(\hat{x}))$ is minmax robust efficient, and hence $\text{price}(x) = 0$.

□

The lemma identifies a relation between the price of robustness and the value of ε to be chosen in $(P^{\text{light},\varepsilon}(\hat{x}))$. This relation is used in the following two-stage strategy to assist a decision maker. There is also an interesting relation between the price of robustness and the maximum regret (as defined in (Rivaz et al., 2016)). Here, given a solution x, the regret of x is defined as the maximum over all objective functions, uncertain parameters and feasible solutions,

$$\text{regret}(x) = \max\{f_i(x,\xi) - f(y,\xi) : y \in \mathfrak{X}, \xi \in \mathcal{U}, i = 1, ..., k\}.$$

The following lemma shows that the regret of a minmax robust efficient solution $\bar{x}$ is an upper bound on the price of robustness of all solutions x that dominate $\bar{x}$ in the nominal case.

Lemma 16. *Let* $\bar{x} \in X^{\text{wc}}, x \in \mathfrak{X}$ *and* $f^{\text{nom}}(x) \leq f^{\text{nom}}(\bar{x})$. *Then* $\text{regret}(\bar{x}) \geq \text{price}(x)$.

Proof. Let $\bar{x} \in X^{\text{wc}}, x \in \mathfrak{X}$ and $f^{\text{nom}}(\bar{x}) \geq f^{\text{nom}}(x)$. Then we have:

$$\begin{aligned}
\text{regret}(\bar{x}) &= \max\{f_i(\bar{x},\xi) - f_i(y,\xi) : y \in \mathfrak{X}, \xi \in \mathcal{U}, i = 1, ..., k\} \\
&\geq \max\{|f_i(\bar{x},\xi) - f_i(y,\xi)| : y \in \mathfrak{X}, \xi \in \mathcal{U}, f(\bar{x},\xi) \geq f(y,\xi), i = 1, ..., k\} \\
&= \max\{\|f(\bar{x},\xi) - f(y,\xi)\|_\infty : y \in \mathfrak{X}, \xi \in \mathcal{U}, f(\bar{x},\xi) \geq f(y,\xi)\} \\
&\geq \|f^{\text{nom}}(\bar{x}) - f^{\text{nom}}(x)\|_\infty \text{ since } \xi \in \mathcal{U}, x \in \mathfrak{X} \text{ and } f^{\text{nom}}(\bar{x}) \geq f^{\text{nom}}(x) \\
&\geq \inf_{\bar{y}\in X^{\text{wc}}} \|f^{\text{nom}}(\bar{y}) - f^{\text{nom}}(x)\|_\infty \\
&= \text{price}(x).
\end{aligned}$$

□

In practice it is preferable to choose a solution which is good in both respects, i.e., with respect to f^{nom} and with respect to f^{wc}. To find such a solution, we propose the following strategies that can be followed by a decision maker. In both strategies, we assume that the set of nominal efficient solutions X^{nom} and the set of minmax robust efficient solutions X^{wc} are already known.

A two-stage strategy. In the two-stage strategy, the decision maker may first concentrate on the nominal scenario and identify a most interesting nominal efficient solution $\hat{x}$ based on her/his preferences. This may be done with an interactive method, see e.g., (Branke, Deb, Miettinen, & Slowinski, 2008; Miettinen, 1999; Steuer, 1986). The interactive solution process also identifies what kind of values of the objective functions are desirable according to the preferences of the decision maker. In the second stage, the decision maker then takes robustness into account as follows: For the identified $\hat{x}$ we compute its price of robustness price$(\hat{x})$ together with its closest minmax robust efficient solution $\bar{x}$. If a closest solution does not exist we take $\bar{x}$ with $\|f^{\text{nom}}(\bar{x}) - f^{\text{nom}}(\hat{x})\|_\infty \approx \text{price}(\hat{x})$. Since $\bar{x}$ is the closest solution from X^{wc} to $\hat{x}$ it is likely that it is not too far from the preferences of the decision maker that have been already used in the nominal case. The price of robustness price$(\hat{x})$ is the nominal quality the decision maker has to sacrifice for changing $\hat{x}$ to this minmax robust efficient solution. This value should be compared with the *gain of robustness* which quantifies how much better the robust efficient solution is in the worst case than the nominal efficient solution.

Definition 17. *Let $\hat{x} \in X^{\text{nom}}$ be the solution of the decision maker's interest and $\bar{x} \in X^{\text{wc}}$ be a closest solution to $\hat{x}$ under f^{nom} (i.e., the minmax robust efficient solution found when computing* price$(\hat{x})$*). We define the* gain of robustness *of $\bar{x}$ as the distance between $f^{\text{wc}}(\hat{x})$ and $f^{\text{wc}}(\bar{x})$*

$$\text{gain}(\hat{x}, \bar{x}) = \|f^{\text{wc}}(\hat{x}) - f^{\text{wc}}(\bar{x})\|_\infty.$$

While the gain of robustness can be calculated for any pair of a feasible solution and a robust efficient solution, our aim here is to compare the nominal efficient solution of the decision maker's interest and the nearest minmax robust efficient solution. As an example to illustrate the gain of robustness, we use again the example depicted in Figure 10. Here we identified $\hat{x}$ and $\bar{x}$ such that $f^{\text{nom}}(\bar{x})$ is the closest point to $f^{\text{nom}}(\hat{x})$ on $f^{\text{nom}}(X^{\text{wc}})$. In Figure 11 both solutions, $\hat{x}$ and $\bar{x}$ are evaluated under f^{wc}. The dotted square illustrates the $\|\cdot\|_\infty$ norm used to determine the distance $\|f^{\text{wc}}(\hat{x}) - f^{\text{wc}}(\bar{x})\|_\infty$. Note that in the figure, the ranges of $f_1^{\text{wc}}(X^{\text{nom}})$ and $f_2^{\text{wc}}(X^{\text{nom}})$ are different which causes the dotted square looking like a dotted rectangle.

Being presented the values of price$(\hat{x})$ and of gain$(\hat{x}, \bar{x})$, the decision maker can then decide if it is worth to change the nominal solution from $\hat{x}$ to $\bar{x}$.

- If price$(\hat{x})$ is large or if gain$(\hat{x}, \bar{x})$ is small, the decision maker should keep the nominal efficient solution $\hat{x}$.
- It is preferable to change to $\bar{x}$ if the decision maker is very risk-averse, i.e., the decision maker wants to be prepared for the worst case, or if price$(\hat{x})$ is small, or if gain$(\hat{x}, \bar{x})$ is large compared to price$(\hat{x})$.
- If the decision maker does not want to sacrifice that much nominal quality but still wants to increase the robustness of the solution $\hat{x}$, (s)he can specify a maximum tolerable loss ε on the nominal quality by defining $\varepsilon_i \leq \text{price}(\hat{x})$, and solve $(P^{\text{light},\varepsilon})$ to find a lightly robust efficient solution x which
  - is still close to $\hat{x}$, i.e., it keeps the preferences of the decision maker in the nominal scenario,
  - has loss of nominal quality of at most ε,
  - and is the most robust solution among all solutions in $(P^{\text{light},\varepsilon})$, i.e. probably more reliable than $\hat{x}$.

Lexicographic strategies. If the decision maker has no specific preferences but is either mainly interested in the nominal quality or mainly interested in minimizing the risk, it might be appropriate to choose the nominal efficient solution $\hat{x} \in X^{\text{nom}}$ which has the smallest price of robustness (in the first case) or to compute the robust efficient solution $\bar{x} \in X^{\text{wc}}$ which is closest to the set of nominal efficient solutions X^{nom} (in the second case). In mathematical terms we solve

$$\min_{\hat{x} \in X^{\text{nom}}} \min_{\bar{x} \in X^{\text{wc}}} \|f^{\text{nom}}(\hat{x}) - f^{\text{nom}}(\bar{x})\|_\infty$$

and receive a pair of closest points $\hat{x}$ and $\bar{x}$. A risk-averse decision maker (without specific preferences otherwise) might then choose $\bar{x}$ while a decision maker mainly interested in nominal quality, again without specific preferences, can choose $\hat{x}$. The optimization problem can be geometrically solved when the sets X^{wc} and X^{nom} are known, but is otherwise hard to compute.

Illustration of the strategies

Example 6. *We continue Example 5 to illustrate the* ***two-stage strategy****.*

We selected three different nominal efficient solutions (which might reflect the individual preferences for three different decision makers): x^1 is the lexicographic solution with respect to f_1^{nom}, x^2 is some solution in which the decision maker wants to have a good value of f_2^{nom} but a not too bad value of f_1^{nom} and x^3 is the lexicographic solution with respect to f_2^{nom}. We computed the price of robustness $\text{price}(x^l)$, $l = 1, 2, 3$ *as illustrated in Figure 12. The figure shows the closest minmax robust efficient solutions for each of the three selected nominal efficient solutions. In this example, their* price*-values are:* $\text{price}(x^1) = 2$, $\text{price}(x^2) = 1.5$, *and* $\text{price}(x^3) = 0$ *and the corresponding gains are* $\text{gain}(x^1, \bar{x}^1) = 2$, $\text{gain}(x^2, \bar{x}^2) = 1$, *and* $\text{gain}(x^3, \bar{x}^3) = 0$. *Based on the values above, the decision maker can then make the choices described in the strategy. In our case, the decision maker with x^3 as her or his most preferred nominal efficient solution might be extremely satisfied because the nominal efficient solution is a minmax robust efficient solution. On the other hand, the decision makers having preferences for $\hat{x}^1$ and $\hat{x}^2$ observe that* $\text{gain}(\hat{x}^i, \bar{x}^i)$ *is rather low compared to what they would have to pay. If not over-conservative they probably keep the nominal efficient solutions found or they solve a light-robust problem with smaller values of ε.*

This could be the case for the decision maker who first identified $\hat{x}^1$ as the most interesting solution. He chooses $\varepsilon <$ $\text{price}(\hat{x}^1)$*, e.g., $\varepsilon = 1$. We solve $(P^{\text{light},1}(\hat{x}^1))$ to find the lightly robust efficient solutions. In this problem, we have only one lightly robust efficient solution as illustrated in Figure 13. In the figure, the chosen nominal efficient solution is marked with a black circle and the resulting lightly robust efficient solution is marked with a blue square. The two dotted black lines represent the region $f^{\text{nom}}(\hat{x}^1) + \{1\}$.*

We finally illustrate the ***lexicographic strategy****. The closest pair of points on the two fronts is depicted in Figure 12 as well. In this case, both sets have a common point, i.e., their distance is zero. We see that x^3 is the nominal efficient solution with the lowest price of robustness (a good choice for a decision maker who mainly cares for nominal quality) while it is also the minmax robust efficient solution which is closest to the set of nominal efficient solutions, i.e., a good choice for a risk-averse decision maker.*

6 Conclusion

In this paper, we formally analyzed nominal efficient solutions, minmax robust efficient solutions, and lightly robust efficient solutions to multiobjective optimization problems with uncertain parameters in the objective functions. We evaluated and compared the three different sets of solutions under the nominal scenario and in the worst case. We found that in the worst case, the set of minmax robust efficient solutions upper dominates the sets

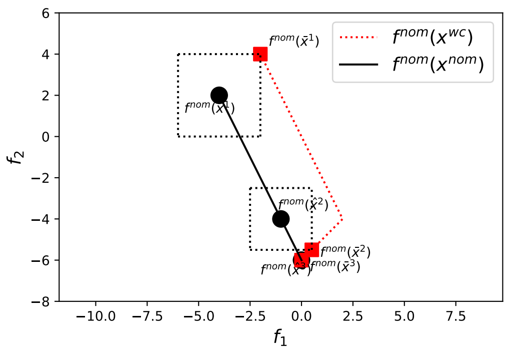

Figure 12: Nominal efficient solutions and their closest minmax robust efficient solutions.

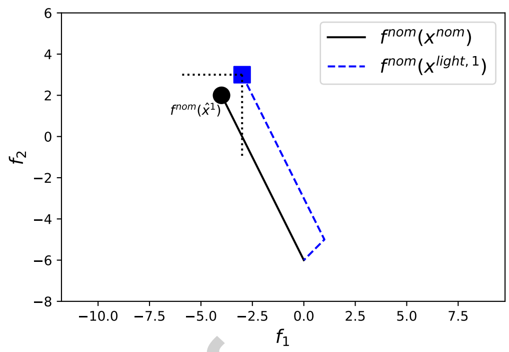

Figure 13: Chosen nominal efficient solution and the lightly robust efficient solution.

of nominal and lightly robust efficient solutions. We also found that in the nominal case, the set of nominal efficient solution upper dominates the set of lightly robust solutions and the set of lightly robust efficient solutions upper and lower dominates the shifted (with respect to ε) outcomes of the lightly robust efficient solutions. We illustrated their relationships with different examples.

In order to further analyze the trade-off between nominal quality and robustness, we proposed a measure for the *price of robustness*. We also analyzed its relationship to lightly robust efficient solutions. For supporting the decision maker to find a solution which is satisfactory in both nominal quality and robustness, we developed two strategies based on the *price of robustness*. We illustrated the utilization of the strategies in an example.

The two strategies rely on the measure *price of robustness* which can be computed if the set of minmax robust efficient solutions is known and small. Research on the computation of X^{wc} is ongoing and has been sketched in the introduction. Future research directions are iterative methods and approximations of X^{wc}. Computing the price of robustness opens other lines of research. Both, two-step procedures using projection techniques as well as optimizing over the efficient set are possible approaches. Efficient algorithms for finding the pair of a nominal efficient solution and a minmax robust efficient solution of smallest distance is also a topic for future research.

Our current measure *price of robustness* depends on a fixed nominal efficient solution. Another interesting future research direction is to quantify the *price of robustness* of the whole set of nominal efficient solutions. It is also interesting to investigate a relative version of the price of robustness, i.e., how much percent of the nominal objective function value has to be sacrificed. Finally, note that light robustness and the *price of robustness* is only one way of dealing with the trade-off between robustness and nominal quality. Other possibilities include regret robustness (Rivaz et al., 2016) or to use multi-attribute utility functions, see e.g., (Keeney & Raiffa, 1993) to decide on the robustness level of the solution to be chosen. The relation to the trade-off specified in this paper seems to be an interesting topic, maybe starting with the single-objective case.

Acknowledgments

We thank the mobility grant of the Science Council at the University of Jyvaskyla. We thank Prof. Kaisa Miettinen for useful comments during the preparation of the

manuscript.

References

Antczak, T., Pandey, Y., Singh, V., & Mishra, S. K. (2020). On approximate efficiency for nonsmooth robust vector optimization problems. *Acta Mathematica Scientia*, *40*(3), 887–902.

Benson, H. P. (1984). Optimization over the efficient set. *Journal of Mathematical Analysis and Applications*, *98*, 562–580.

Benson, H. P. (2012). An outcome space algorithm for optimization over the weakly efficient set of a multiple objective nonlinear programming problem. *Journal of Global Optimization*, *52*(3), 553 - 574.

Benson, H. P., & Sun, E. (1999). New closedness results for efficient sets in multiple objective mathematical programming. *Journal of Mathematical Analysis and Applications*, *238*(1), 277-296.

Ben-Tal, A., Ghaoui, L. E., & Nemirovski, A. (2009). *Robust optimization.* Princeton and Oxford: Princeton University Press.

Ben-Tal, A., Goryashko, A., Guslitzer, E., & Nemirovski, A. (2003). Adjustable robust solutions of uncertain linear programs. *Mathematical Programming A*, *99*, 351-376.

Berge, C. (1963). *Topological spaces.* Oliver and Boyd.

Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations Research*, *52*(1), 35-53.

Binh, T. T., & Korn, U. (1997). Mobes: A multiobjective evolution strategy for constrained optimization problems. In *proceeding of the third international conference on genetic algorithms* (p. 176-182).

Bitran, G. R. (1980). Linear multiple objective problems with interval coefficients. *Management Science*, *26*, 694-706.

Bokrantz, R., & Fredriksson, A. (2017). Necessary and sufficient conditions for pareto efficiency in robust multiobjective optimization. *European Journal of Operational Research*, *262*(2), 682–692.

Botte, M., & Schöbel, A. (2019). Dominance for multi-objective robust optimization concepts. *European Journal of Operational Research*, *273*, 430-440.

Branke, J., Deb , K., Miettinen, K., & Slowinski, R. (Eds.). (2008). *Multiobjective optimization, interactive and evolutionary approaches.* Springer.

Chassein, A., & Goerigk, M. (2016). A bicriteria approach to robust optimization. *Computers and Operations Research*, *66*, 181-189.

Dranichak, G. M., & Wiecek, M. M. (2019). On highly robust efficient solutions to uncertain multiobjective linear programs. *European Journal of Operational Research*, *273*(1), 20 - 30.

Ehrgott, M. (2005). *Multicriteria optimization.* Springer, Berlin, Heidelberg.

Ehrgott, M., Ide, J., & Schöbel, A. (2014). Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, *239*, 17-31.

Engau, A., & Sigler, D. (2020). Pareto solutions in multicriteria optimization under uncertainty. *European Journal of Operational Research*, *281*(2), 357-368.

Fischetti, M., & Monaci, M. (2009). Light robustness. In R. K. Ahuja, R. Möhring, & C. Zaroliagis (Eds.), *Robust and online large-scale optimization* (p. 61-84). Springer.

Fliege, J., & Werner, R. (2014). Robust multiobjective optimization & applications in portfolio optimization. *European Journal of Operational Research*, *234*(2), 422-433.

Georgiev, P., Luc, D., & Pardalos, P. (2013). Robust aspects of solutions in deterministic multiple objective linear programming. *European Journal of Operational Research*.

Goberna, M., Jeyakumar, V., Li, G., & Vicente-Pérez, J. (2015). Robust solutions to multi-objective linear programs with uncertain data. *European Journal of Operational Research*, *242*, 730-743.

Goberna, M. A., Jeyakumar, V., Li, G., & Vicente-Pérez, J. (2014). Robust solutions of multiobjective linear semi-infinite programs under constraint data uncertainty. *SIAM Journal on Optimization*, *24*(3).

Goberna, M. A., Jeyakumar, V., Li, G., & Vicente-Pérez, J. (2018). Guaranteeing highly robust weakly efficient solutions for uncertain multi-objective convex programs. *European Journal of Operational Research*, *270*(1), 40–50.

Goerigk, M., Knoth, M., Müller-Hannemann, M., Schmidt, M., & Schöbel, A. (2011). The Price of Robustness in Timetable Information. In A. Caprara & S. Kontogiannis (Eds.), *11th workshop on algorithmic approaches for transportation modelling, optimization, and systems* (Vol. 20, p. 76-87). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

Goerigk, M., Knoth, M., Müller-Hannemann, M., Schmidt, M., & Schöbel, A. (2014). The Price of Strict and Light Robustness in Timetable Information. *Transportation Science*, *48*, 225-242.

Goerigk, M., & Schöbel, A. (2016). Algorithm engineering in robust optimization. In L. Kliemann & P. Sanders (Eds.), *Algorithm engineering: Selected results and surveys* (p. 245-279).

Hassanzadeh, F., Nemati, H., & Sun, M. (2013). Robust optimization for multiobjective programming problems with imprecise information. *Procedia Computer Science*, *17*, 357 - 364.

Hassanzadeh, F., Nemati, H., & Sun, M. (2014). Robust optimization for interactive multiobjective programming with imprecise information applied to R&D project portfolio selection. *European Journal of Operational Research*, *238*(1), 41-53.

Hladik, M. (2017). On the relation of possibly efficiency and robust counterparts in interval multiobjective programming. In A. Sforza & C. Sterle (Eds.), *Optimization and decision science: Methodologoies and applications* (p. 335-344). Springer.

Horst, R., Thoai, N. V., Yamamoto, Y., & Zenke, D. (2007). On optimization over the efficient set in linear multicriteria programming. *Journal of Optimization Theory and Applications*, *134*(3), 433 - 443.

Ide, J., & Schöbel, A. (2016). Robustness for uncertain multi-objective optimization: A survey and analysis of different concepts. *OR Spectrum*, *38*(1), 235-271.

Kalantari, M., Dong, C., & Davies, I. (2016). Multi-objective robust optimisation of unidirectional carbon/glass fibre reinforced hybrid composites under flexural loading. *Composite Structures*, *138*, 264–275.

Keeney, R., & Raiffa, H. (1993). *Decisions with multiple objectives: Preferences and value trade-offs.* Cambridge University Press.

Khan, A., Tammer, C., & Zălinescu, C. (2015). *Set-valued optimization.* Berlin Heidelberg: Springer.

Kuhn, K., Raith, A., Schmidt, M., & Schöbel, A. (2016). Bicriteria robust optimization. *European Journal of Operational Research*, *252*, 418-431.

Kuroiwa, D., & Lee, G. M. (2012). On robust multiobjective optimization. *Vietnam Journal of Mathematics*, *40*(2&3), 305–317.

Liebchen, C., Lübbecke, M., Möhring, R. H., & Stiller, S. (2009). The concept of recoverable robustness, linear programming recovery, and railway applications. In R. K. Ahuja, R. Möhring, & C. Zaroliagis (Eds.), *Robust and online large-scale optimization* (p. 1-27). Springer.

Miettinen, K. (1999). *Nonlinear multiobjective optimization.* Kluwer Academic Publishers.

Mordechai, I. H. (1986). The domination property in multicriteria optimization. *Journal of Mathematical Analysis and Applications*, *114*(1), 7-16.

Rahimi, M., & Soleimani-Damaneh, M. (2018). Robustness in deterministic vector optimization. *Journal of Optimization Theory and Applications*, *179*(1), 137–162.

Rahimi, M., & Soleimani-Damaneh, M. (2020). Characterization of the norm-based robust solutions in vector optimization. to appear. *Journal of Optimization Theory and Applications*.

Raith, A., Schmidt, M., Schöbel, A., & Thom, L. (2018a). Extensions of labeling algorithms for multi-objective uncertain shortest path problems. *Networks*, *72*(1), 84-127.

Raith, A., Schmidt, M., Schöbel, A., & Thom, L. (2018b). Multi-objective minmax robust combinatorial optimization with cardinality-constrained uncertainty. *European Journal of Operational Research*, *267*, 628-642.

Rivaz, S., & Yaghoobi, M. (2018). Weighted sum of maximum regrets in an interval MOLP problem. *International Transactions in Operational Research*, *25*(5), 1659-1676.

Rivaz, S., Yaghoobi, M. A., & Hladík, M. (2016). Using modified maximum regret for finding a necessarily efficient solution in an interval MOLP problem. *Fuzzy Optimization and Decision Making*, *15*(3), 237-253.

Schmidt, M., Schöbel, A., & Thom, L. (2019). Min-ordering and max-ordering scalarization methods for multi-objective robust optimization. *European Journal of Operational Research*, *275*, 446-459.

Schöbel, A. (2014). Generalized light robustness and the trade-off between robustness and nominal quality. *Mathematical Methods of Operations Research*, *80*(2), 161-191.

Steuer, R. E. (1986). *Multiple criteria optimization: Theory, computation, and applications.* John Wiley & Sons, Inc.

Thoai, N. V. (2000). Conical algorithm in global optimization for optimizing over efficient sets. *Journal of Global Optimization*, *18*(4), 321–336.

Wei, H.-Z., Chen, C.-R., & Li, S.-J. (2020a). Characterizations of multiobjective robustness on vectorization counterparts. *Optimization*, *69*(3), 493–518.

Wei, H.-Z., Chen, C.-R., & Li, S.-J. (2020b). A unified approach through image space analysis to robustness in uncertain optimization problems. *Journal of Optimization Theory and Applications*, *184*(2), 466–493.

Wiecek, M. M., & Dranichak, G. M. (2016). Robust multiobjective optimization for decision making under uncertainty and conflict. In *Optimization challenges in complex, networked and risky systems* (p. 84-114). INFORMS.

Xidonas, P., Mavrotas, G., Hassapis, C., & Zopounidis, C. (2017). Robust multiobjective portfolio optimization: A minimax regret approach. *European Journal of Operational Research*, *262*(1), 299 - 305.

Yamada, S., Tanino, T., & Inuiguchi, M. (2000). Conical algorithm in global optimization for optimizing over efficient sets. *Journal of Global Optimization*, *16*(3), 197–217.

Yamada, S., Tanino, T., & Inuiguchi, M. (2001). An inner approximation method incorporating a branch and bound procedure for optimization over the weakly efficient set. *European Journal of Operational Research*, *133*(2), 267 - 286. (Multiobjective Programming and Goal Programming)

Yamamoto, Y. (2002). Optimization over the efficient set: overview. *Journal of Global Optimization*, *22*(1), 285-317.