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Weighted estimates for diffeomorphic extensions of homeomorphisms

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc domain and $\varphi : \mathbb{S}^1 \rightarrow \partial\Omega$ be a homeomorphism. Then there is a diffeomorphic extension $h : \mathbb{D} \rightarrow \Omega$ of φ . We study the relationship between weighted integrability of the derivatives of h and double integrals of φ and of φ^{-1} .

Keywords: Poisson extension, diffeomorphism, internal chord-arc domain.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Suppose that φ is a homeomorphism from the unit circle \mathbb{S}^1 onto $\partial\Omega$. Then, by Radó [13], Kneser [7], Choquet [3] and Lewy [10], the complex-valued Poisson extension h of φ is a diffeomorphism from \mathbb{D} onto Ω . We are interested in the integrability degrees of the derivatives of h . In 2007, G. C. Verchota [14] proved that the derivatives of h may fail to be square integrable but that they are necessarily p -integrable over \mathbb{D} for all $p < 2$. In 2009, T. Iwaniec, G. J. Martin and C. Sbordone improved on [5] by showing that the derivatives belong to weak- L^2 with sharp estimates. Actually

$$(1.1) \quad \int_{\mathbb{D}} |Dh(z)|^2 dz \approx \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|\varphi(\xi) - \varphi(\eta)|^2}{|\xi - \eta|^2} |d\xi| |d\eta|,$$

since harmonic functions minimize the L^2 -energy and the right-hand side of (1.1) is the trace norm of $\dot{W}^{1,2}(\mathbb{D})$. In [1], it was further shown that if additionally $\partial\Omega$ is a C^1 -regular Jordan curve then

$$(1.2) \quad \int_{\mathbb{D}} |Dh(z)|^2 dz < \infty \Leftrightarrow \int_{\partial\Omega} \int_{\partial\Omega} |\log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|| |d\xi| |d\eta| < \infty.$$

All the above results require the target domain to be convex.

If Ω is a bounded non-convex Jordan domain, then there exists a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \partial\Omega$ for which the harmonic extension fails to map \mathbb{D} homeomorphically onto Ω , see [3, 7]. Hence we cannot use the harmonic extension to produce a diffeomorphic extension. Nevertheless, (weighted) analogs of the results as (1.2) for diffeomorphic extensions in the case of an internal chord-arc Jordan domain exist, see [9]. For the definition of (internal) chord-arc domains, we refer to Definition 2.1. Notice that each bounded convex Jordan domain is a chord-arc domain. In this paper, we generalize the results in [9] to the weighted L^p -setting.

Let Ω be an internal chord-arc Jordan domain with the internal distance λ_Ω . Assume that $h : \mathbb{D} \rightarrow \Omega$ is a diffeomorphism and $\varphi : \mathbb{S}^1 \rightarrow \partial\Omega$ is a homeomorphism. Set $\delta(z) = 1 - |z|$. Given $p > 1, \alpha \in \mathbb{R}, \lambda \in \mathbb{R}$, we define

$$\begin{aligned} I_1(p, \alpha, \lambda, h) &= \int_{\mathbb{D}} |Dh(z)|^p \delta^\alpha(z) \log^\lambda(2\delta^{-1}(z)) dz, \\ I_2(p, \alpha, \lambda, h) &= \int_{\mathbb{D}} |Dh(z)|^p \log^\lambda(e + |Dh(z)|) \delta^\alpha(z) dz, \\ \mathcal{U}(p, \alpha, \lambda, \varphi) &= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{\lambda_\Omega^p(\varphi(\xi), \varphi(\eta))}{|\xi - \eta|^{p-\alpha}} \log^\lambda \left(e + \frac{\lambda_\Omega(\varphi(\xi), \varphi(\eta))}{|\xi - \eta|} \right) |d\eta| |d\xi|, \\ \mathcal{A}_{p,\alpha,\lambda}(t) &= \int_1^t -x^{1+\alpha-p} \log_2^\lambda(x^{-1}) dx \quad \forall t \geq 0, \\ \mathcal{V}(p, \alpha, \lambda, \varphi) &= \int_{\partial\Omega} \left(\int_{\partial\Omega} \mathcal{A}_{p,\alpha,\lambda}(|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|) |d\eta| \right)^{p-1} |d\xi|. \end{aligned}$$

Our main result is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc Jordan domain and $\varphi : \mathbb{S}^1 \rightarrow \partial\Omega$ be a homeomorphism. There is a diffeomorphic extension $h : \mathbb{D} \rightarrow \Omega$ of φ for which, for any $p > 1$, we have that*

(1) *if either $\alpha \in (p - 2, +\infty)$ and $\lambda \in \mathbb{R}$ or $\alpha = p - 2$ and $\lambda \in (-\infty, -1)$,*

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are finite.

(2) if either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$,

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are comparable to $\mathcal{U}(p, \alpha, \lambda, \varphi)$.

Moreover whenever $p \in (1, 2]$

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ dominate $\mathcal{V}(p, \alpha, \lambda, \varphi)$,

while

$\mathcal{V}(p, \alpha, \lambda, \varphi)$ controls both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$

for all $p \in [2, +\infty)$. Furthermore both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are in general comparable to $\mathcal{V}(p, \alpha, \lambda, \varphi)$ only for $p = 2$.

For any $p > 1$, there is no diffeomorphic extension $h : \mathbb{D} \rightarrow \Omega$ of φ for which $I_1(p, \alpha, \lambda, h) < +\infty$ for either $\alpha \in (-\infty, -1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1, +\infty)$; or for which $I_2(p, \alpha, \lambda, h) < +\infty$ for some $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$.

Motivated by (1.2), one could hope to use $\mathcal{V}(p, \alpha, \lambda, \varphi)$ to control both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$. Example 4.2 together with Example 4.3 shows that $\mathcal{V}(p, \alpha, \lambda, \varphi)$ is comparable to $I_1(p, \alpha, \lambda, h)$ or $I_2(p, \alpha, \lambda, h)$ only when $p = 2$. Theorem 1.1 does not cover the case where $p > 1$, $\alpha = -1$ and $\lambda \in (-\infty, -1)$. We will return to this case in a future paper.

The structure of this paper is the following. In the next section, we give some preliminaries. Section 3 is the proof of Theorem 1.1. The final section contains several examples related to Theorem 1.1 (2).

2 Preliminaries

By $s \gg 1$ and $t \ll 1$ we mean that s is sufficiently large and t is sufficiently small, respectively. By $f \lesssim g$ we mean that there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for every x . If $f \lesssim g$ and $g \lesssim f$ we may denote $f \approx g$. By \mathbb{N} and \mathbb{R} we denote the set of all positive integers and the set of all real numbers. Let \mathcal{L}^2 (respectively \mathcal{L}^1) be the 2-dimensional (1-dimensional) Lebesgue measure. For sets $E \in \mathbb{R}^2$ and $F \in \mathbb{R}^2$, let $\text{diam}(E)$ be the diameter of E , and $\text{dist}(E, F)$ be the Euclidean distance between E and F . Let $B(p, r)$ be the disk with center P and radius r .

Definition 2.1. A Jordan domain $\Omega \subset \mathbb{R}^2$ is an **internal chord-arc Jordan domain** if $\partial\Omega$ is rectifiable and there is a constant $C > 0$ such that for all $w_1, w_2 \in \partial\Omega$,

$$(2.1) \quad \ell(w_1, w_2) \leq C\lambda_\Omega(w_1, w_2),$$

where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial\Omega$ joining w_1 to w_2 , and $\lambda_\Omega(w_1, w_2)$ is the **internal distance** between w_1, w_2 , which is defined as

$$\lambda_\Omega(w_1, w_2) = \inf_{\alpha} \ell(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha \subset \Omega$ joining w_1 and w_2 ; if there is no rectifiable curve joining w_1 and w_2 , we set $\lambda_\Omega(w_1, w_2) = \infty$; cf. [12, Section 3.1] or [2, Section 2].

If (2.1) holds for the Euclidean distance instead of the internal distance, we call Ω be a **chord-arc domain**. Naturally, every chord-arc Jordan domain is an internal chord-arc domain, but there are internal chord-arc domains that fail to be chord-arc; e.g. the standard cardioid domain

$$\Delta = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 - 4x(x^2 + y^2) - 4y^2 < 0\}.$$

2.1 Dyadic decomposition

Given $j \in \mathbb{N}$ and $k = 1, \dots, 2^j$, let

$$(2.2) \quad I_{j,k} = [2\pi(k-1)2^{-j}, 2\pi k2^{-j}], \quad \Gamma_{j,k} = \{e^{i\theta} : \theta \in I_{j,k}\}.$$

Then $\{I_{j,k}\}$ is a dyadic decomposition of $[0, 2\pi]$ and $\{\Gamma_{j,k}\}$ is a dyadic decomposition of \mathbb{S}^1 . We call $\Gamma_{j,k}$ a j -level dyadic arc. Moreover we have that

$$(2.3) \quad \ell(\Gamma_{j,k}) \approx 2^{-j} \quad \forall j \in \mathbb{N} \text{ and } k = 1, \dots, 2^j.$$

Based on (2.2), there is a decomposition of the unit disk \mathbb{D} given by $\{Q_{j,k} : j \in \mathbb{N} \text{ and } k = 1, \dots, 2^j\}$, where

$$(2.4) \quad Q_{j,k} = \{re^{i\theta} : 1 - 2^{1-j} \leq r \leq 1 - 2^{-j} \text{ and } \theta \in I_{j,k}\}.$$

By (2.3) it follows that

$$(2.5) \quad \mathcal{L}^2(Q_{j,k}) \approx 2^{-2j} \approx \ell(\Gamma_{j,k})^2 \quad \forall j \in \mathbb{N} \text{ and } k = 1, \dots, 2^j.$$

Moreover there is a uniform constant $C > 0$ such that for any $Q_{j,k}$ there is a disk $B_{j,k}$ satisfying

$$(2.6) \quad B_{j,k} \subset Q_{j,k} \subset CB_{j,k}.$$

2.2 A_p weights

Definition 2.2. For a given $p \in (1, +\infty)$, a locally integrable function $w : \mathbb{R}^2 \rightarrow [0, +\infty)$ is an A_p weight if there is a constant $C > 0$ such that for any disk $B \subset \mathbb{R}^2$ we have that

$$\frac{1}{\mathcal{L}^2(B)} \int_B w(x) dx \leq C \left(\frac{1}{\mathcal{L}^2(B)} \int_B w(x)^{\frac{1}{1-p}} dx \right)^{1-p}.$$

Next, w is an A_1 weight if there is a constant $C > 0$ such that

$$\frac{1}{\mathcal{L}^2(B)} \int_B w(z) dz \leq Cw(x)$$

for each disk $B \subset \mathbb{R}^2$ and all $x \in B$.

For more information on A_p weights, we recommend [4, 6, 11]. Let $\delta(x) = \text{dist}(\mathbb{S}^1, x)$. Given $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$, we define

$$(2.7) \quad w_{\alpha, \lambda}(x) = \begin{cases} \delta(x)^\alpha \log^\lambda(2\delta^{-1}(x)) & 0 \leq |x| \leq 2, \\ \log^\lambda(2) & |x| \geq 2. \end{cases}$$

It is well known that $w_{\alpha, 0}$ belongs to A_p . We now generalize this to all $\lambda \in \mathbb{R}$.

Proposition 2.3. *Let $p \geq 1$ and $w_{\alpha, \lambda}$ be as in (2.7). Then $w_{\alpha, \lambda}$ is an A_p weight for all $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. The idea of proof is to use the Jones factorization of A_p weights (see [6]), i.e. we should prove $w_{\alpha, \lambda} = w_1 w_2^{1-p}$ for two A_1 weights w_1 and w_2 .

We first consider the case $\lambda \geq 0$. For a given $\alpha \in (-1, p-1)$, there uniquely exist $a_1 \in (0, 1)$ and $a_2 \in (0, 1)$ such that $\alpha = a_1(-1) + a_2(p-1)$. Set $\alpha_1 = -a_1$, $\alpha_2 = -a_2$, $\lambda_1 = p\lambda$ and $\lambda_2 = \lambda$. We define

$$(2.8) \quad w_1(x) = \begin{cases} \delta(x)^{\alpha_1} \log^{\lambda_1}(2\delta^{-1}(x)) & 0 \leq |x| \leq 2, \\ \log^{\lambda_1}(2) & |x| \geq 2, \end{cases}$$

and

$$(2.9) \quad w_2(x) = \begin{cases} \delta(x)^{\alpha_2} \log^{\lambda_2}(2\delta^{-1}(x)) & 0 \leq |x| \leq 2, \\ \log^{\lambda_2}(2) & |x| \geq 2. \end{cases}$$

We next prove that w_1 is an A_1 weight, i.e.

$$(2.10) \quad \int_B w_1(x) dx \lesssim \inf_{x \in B} w_1(x)$$

for every disk $B \subset \mathbb{R}^2$. Let $d_B = \text{dist}(B, \mathbb{S}^1)$.

Case 1: $d_B \geq \text{diam}(B)/2$. We have that

$$(2.11) \quad d_B \leq \delta(x) \leq 3d_B \quad \forall x \in B.$$

If $1 \leq d_B$, then $\delta(x) \geq 1$ for all $x \in B$. Therefore $w_1(x) = \log^{\lambda_1}(2)$ whenever $x \in B$. Of course (2.10) holds now. If $3d_B \leq 1$, then $w_1(x) = \delta(x)^{\alpha_1} \log^{\lambda_1}(2\delta^{-1}(x))$ for all $x \in B$. By (2.11) it hence follows that $w_1(x) \approx d_B^{\alpha_1} \log^{\lambda_1}(2d_B^{-1})$ whenever $x \in B$. Therefore (2.10) holds. If $d_B < 1 < 3d_B$, let $B_1 = \{x \in B : d_B < \delta(x) < 1\}$ and $B_2 = \{x \in B : 1 \leq \delta(x) < 3d_B\}$. Then $B = B_1 \cup B_2$ and

$$(2.12) \quad w_1(x) = \log^{\lambda_1}(2) \quad \text{whenever } x \in B_2.$$

Since

$$(2.13) \quad [t^{\alpha_1} \log^{\lambda_1}(2t^{-1})]' = t^{\alpha_1-1} \log^{\lambda_1}(2t^{-1}) \left(\alpha_1 - \frac{\lambda_1}{\log(2t^{-1})} \right) < 0,$$

for all $t \in (0, 1]$, we have that

$$(2.14) \quad w_1(x) \leq d_B^{\alpha_1} \log^{\lambda_1}(2d_B^{-1}) \leq \frac{\log^{\lambda_1}(6)}{3^{\alpha_1}} \quad \forall x \in B_1.$$

Combining (2.12) and (2.14) implies that

$$\begin{aligned} \int_B w_1(x) dx &= \frac{1}{\mathcal{L}^2(B)} \left(\int_{B_1} w_1 + \int_{B_2} w_1 \right) \\ &\leq \frac{1}{\mathcal{L}^2(B)} \left(\mathcal{L}^2(B_1) \frac{\log^{\lambda_1}(6)}{3^{\alpha_1}} + \mathcal{L}^2(B_2) \log^{\lambda_1}(2) \right) \\ &\lesssim \log^{\lambda_1}(2) = \inf_{x \in B} w_1(x). \end{aligned}$$

Case 2: $d_B < \text{diam}(B)/2$ and $\text{diam}(B) \leq 2/3$. Pick $x' \in \partial B$ and $x_0 \in \mathbb{S}^1$ such that $\text{dist}(B, \mathbb{S}^1) = |x' - x_0|$. Let $r_B = 3\text{diam}(B)/2$. Since

$$|x - x_0| \leq |x - x'| + |x' - x_0| \leq r_B$$

for all $x \in B$, we have $B \subset B(x_0, r_B)$. Let $E = \{x \in \mathbb{R}^2 : \text{dist}(x, \mathbb{S}^1) < r_B\}$. Then $B(x_0, r_B) \subset E$. Since $\mathcal{L}^2(B(x_0, r_B)) = \pi r_B^2$ and $\mathcal{L}^2(E) = 4\pi r_B$, the maximal number of pairwise disjoint open disks $B(x, r_B)$ with $x \in \mathbb{S}^1$ is less than $4r_B^{-1}$. We have that

$$(2.15) \quad \begin{aligned} \frac{1}{\mathcal{L}^2(B)} \int_B w_1(x) dx &\leq \frac{1}{\mathcal{L}^2(B)} \int_{B(x_0, r_B)} w_1(x) dx \\ &\lesssim \frac{r_B}{\mathcal{L}^2(B)} \int_E w_1(x) dx \approx \frac{1}{r_B} \int_0^{r_B} t^{\alpha_1} \log^{\lambda_1}(2t^{-1}) dt. \end{aligned}$$

Notice that

$$(2.16) \quad [t^{\alpha_1+1} \log^{\lambda_1}(2t^{-1})]' = t^{\alpha_1} \log^{\lambda_1}(2t^{-1}) \left(\alpha_1 + 1 - \frac{\lambda_1}{\log(2t^{-1})} \right) \quad t > 0.$$

Since $\lim_{t \rightarrow 0^+} \alpha_1 + 1 - \frac{\lambda_1}{\log(2t^{-1})} = \alpha_1 + 1$ and $\alpha_1 + 1 - \frac{\lambda_1}{\log(2t^{-1})}$ is decreasing with respect to $t > 0$, there exists $\epsilon \in (0, 1)$ determined by α_1 and λ_1 such that $\alpha_1 + 1 - \frac{\lambda_1}{\log(2\epsilon^{-1})} \geq (\alpha_1 + 1)/2$. We then obtain from (2.16) that

$$[t^{\alpha_1+1} \log^{\lambda_1}(2t^{-1})]' \geq \frac{\alpha_1 + 1}{2} t^{\alpha_1} \log^{\lambda_1}(2t^{-1})$$

for all $t \in [0, \epsilon r_B]$. Therefore

$$(2.17) \quad \int_0^{\epsilon r_B} t^{\alpha_1} \log^{\lambda_1}(2t^{-1}) dt = \frac{2(\epsilon r_B)^{\alpha_1+1}}{\alpha_1 + 1} \log^{\lambda_1}(2(\epsilon r_B)^{-1}) \lesssim r_B^{\alpha_1+1} \log^{\lambda_1}(2r_B^{-1})$$

Moreover by (2.13) we have that

$$(2.18) \quad \int_{\epsilon r_B}^{r_B} t^{\alpha_1} \log^{\lambda_1}(2t^{-1}) dt \leq (r_B - \epsilon r_B)(\epsilon r_B)^{\alpha_1} \log^{\lambda_1}(2(\epsilon r_B)^{-1}) \lesssim r_B^{\alpha_1+1} \log^{\lambda_1}(2r_B^{-1}).$$

Combining (2.15), (2.17) with (2.18) implies that

$$\frac{1}{|B|} \int_B w_1(x) dx \lesssim r_B^{\alpha_1} \log^{\lambda_1}(2r_B^{-1}).$$

Together with

$$r_B^{\alpha_1} \log^{\lambda_1}(2r_B^{-1}) = \inf_{t \in [0, r_B]} t^{\alpha_1} \log^{\lambda_1}(2t^{-1}) = \inf_{x \in E} w_1(x) \leq \inf_{x \in B} w_1(x),$$

we hence obtain (2.10).

Case 3: $d_B < \text{diam}(B)/2$ and $\text{diam}(B) > 2/3$. Let x' and x_0 be as in Case 2. Then $|x'| = 1 + \text{dist}(x', \mathbb{S}^1) \leq 1 + \text{diam}(B)2^{-1}$. Together with the fact that $|x - x'| \leq \text{diam}(B)$ for all $x \in B$, we have $B \subset B(0, 1 + r_B)$. Moreover by (2.17) and (2.18), we obtain that

$$\int_{B(0,2)} w_1(x) dx = \int_{B(0,1)} + \int_{B(0,2) \setminus B(0,1)} = 4\pi \int_0^1 t^{\alpha_1} \log^{\lambda_1}(2t^{-1}) dt \approx 1.$$

Therefore

$$(2.19) \quad \begin{aligned} \frac{1}{\mathcal{L}^2(B)} \int_B w_1(x) dx &\lesssim \frac{1}{r_B^2} \left(\int_{B(0,2)} + \int_{B(0,1+r_B) \setminus B(0,2)} \right) \\ &\lesssim \frac{1}{r_B^2} (\mathcal{L}^2(B(0,2)) + \log^{\lambda_1}(2) \mathcal{L}^2(B(0,1+r_B) \setminus B(0,2))) \\ &\lesssim 1. \end{aligned}$$

Moreover by the monotonicity of $t^{\alpha_1} \log^{\lambda_1}(2t^{-1})$ on $(0, +\infty)$, we have that

$$(2.20) \quad \log^{\lambda_1}(2) = \inf_{t \in [0, 1+r_B]} t^{\alpha_1} \log^{\lambda_1}(2t^{-1}) = \inf_{x \in B(0, 1+r_B)} w_1(x) \leq \inf_{x \in B} w_1(x).$$

By combining (2.19) with (2.20), we obtain (2.10).

By the analogous arguments as for (2.10), we obtain $w_2 \in A_1$. Therefore the Jones factorization theorem implies that $w_{\alpha, \lambda} \in A_p$ for all $\alpha \in (-1, p-1)$ and $\lambda \geq 0$.

When $\lambda < 0$, define w_1 and w_2 as in (2.8) and (2.9) with $\lambda_1 = -\lambda$, $\lambda_2 = 2\lambda(1-p)^{-1}$ and both α_1 and α_2 invariant. By the same arguments as for the case $\lambda \geq 0$, we obtain that $w_{\alpha, \lambda} \in A_p$ whenever $\alpha \in (-1, p-1)$ and $\lambda < 0$. \square

2.3 A class of functions

We define the Hardy-Littlewood maximal function for a Lebesgue measurable function f in \mathbb{R}^2 as

$$M_f(x) = \sup_{x \in B} \int_B |f(z)| dz = \sup_{x \in B} \frac{1}{|B|} \int_B |f(z)| dz$$

where the supremum is taken over all disks $B \subset \mathbb{R}^2$ containing x . Let $p \in (1, \infty)$ and w be a weight. It is well-known that

$$\int_{\mathbb{R}^2} |M_f(x)|^p w(x) dx \lesssim \int_{\mathbb{R}^2} |f(x)|^p w(x) dx$$

if and only if w is an A_p weight. We generalize this to weighted Orlicz spaces. We begin with some definitions.

Definition 2.4. Let \mathcal{F} be the collection of $\Phi : [0, \infty) \rightarrow [0, \infty)$, which is increasing and satisfies $\lim_{t \rightarrow 0} \Phi(t) = 0$ and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$. We say that $\Phi \in \mathcal{F}$ is the Young function, if Φ is convex on $[0, \infty)$ and $\lim_{t \rightarrow 0} \Phi(t)/t = \lim_{t \rightarrow \infty} t/\Phi(t) = 0$.

Definition 2.5. We say that a function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ satisfies the Δ_2 -condition if there is a constant $C > 0$ such that

$$\Phi(2t) \leq C\Phi(t)$$

for all $t \in [0, +\infty)$.

Let $\Phi \in \mathcal{F}$ satisfying the Δ_2 -condition. Put

$$h_\Phi(s) = \sup_{t > 0} \frac{\Phi(st)}{\Phi(t)} \quad s > 0.$$

We define the lower index of Φ by

$$i(\Phi) = \lim_{s \rightarrow 0} \frac{\log h_\Phi(s)}{\log s} = \sup_{0 < s < 1} \frac{\log h_\Phi(s)}{\log s}.$$

The quantity $i(\Phi)$ is well defined, see [8]. The following lemma is from [8, Theorem 2.1.1].

Lemma 2.6. *Let Φ be a Young function satisfying the Δ_2 -condition and w be a weight on \mathbb{R}^2 . Then the following conditions are equivalent:*

1. $\int_{\mathbb{R}^2} \Phi(M_f(x))w(x) dx \lesssim \int_{\mathbb{R}^2} \Phi(|f(x)|)w(x) dx$,
2. $w \in A_{i(\Phi)}$.

We next consider a special class of Young functions. Given $p > 1$ and $\lambda \in \mathbb{R}$, we set

$$(2.21) \quad \Phi_{p,\lambda}(t) = t^p \log^\lambda(e+t) \quad \text{for } t \in [0, +\infty).$$

Proposition 2.7. *Let $\Phi_{p,\lambda}$ be as in (2.21) with $p > 1$ and $\lambda \geq 0$. Then $\Phi_{p,\lambda}$ is a Young function and satisfies the Δ_2 -condition on $[0, \infty)$. Moreover $i(\Phi_{p,\lambda}) = p$.*

Proof. Simple calculations show that

$$(2.22) \quad \Phi'_{p,\lambda}(t) = \left(p \log(e+t) + \lambda \frac{t}{e+t} \right) t^{p-1} \log^{\lambda-1}(e+t)$$

and

$$(2.23) \quad \begin{aligned} \Phi''_{p,\lambda}(t) &= \left(p(p-1) \log^2(e+t) + \lambda e \frac{t}{(e+t)^2} \log(e+t) + R_{p,\lambda}(t) \right) \\ &\quad \times t^{p-2} \log^{\lambda-2}(e+t) \end{aligned}$$

where $R_{p,\lambda}(t) = \lambda(\lambda-1)(t(e+t)^{-1})^2 + \lambda(2p-1)t(e+t)^{-1} \log(e+t)$. Since $p \log(e+t) + \lambda t(e+t)^{-1} > 0$ for all $t \in (0, +\infty)$, it follows from (2.22) that $\Phi'_{p,\lambda}(t) > 0$ for all $t > 0$. Therefore $\Phi_{p,\lambda}$ is strictly increasing on $[0, \infty)$. If $\lambda \geq 1$, we have that

$$(2.24) \quad R_{p,\lambda}(t) \geq 0 \quad \text{for all } t \geq 0.$$

Whenever $0 \leq \lambda < 1$, since $t/(e+t) < 1$ and $\log(e+t) \geq 1$ for all $t \geq 0$ we have that

$$(2.25) \quad \begin{aligned} R_{p,\lambda}(t) &= \frac{t}{e+t} \left(\lambda(\lambda-1) \frac{t}{e+t} + \lambda(2p-1) \log(e+t) \right) \\ &\geq \frac{t}{e+t} (\lambda(\lambda-1) + \lambda(2p-1)) = \frac{t}{e+t} (\lambda^2 + 2\lambda(p-1)) \geq 0 \end{aligned}$$

for all $t \geq 0$. By (2.23), (2.24) and (2.25), we have that $\Phi_{p,\lambda}''(t) \geq 0$ for all $t \geq 0$. Therefore $\Phi_{p,\lambda}$ is convex on $[0, +\infty)$. Hence $\Phi_{p,\lambda}$ is a Young function whenever $p > 1$ and $\lambda \geq 0$.

Since both t^p and $\log^\lambda(e+t)$ satisfy the Δ_2 -condition on $[0, +\infty)$, $\Phi_{p,\lambda}$ satisfies Δ_2 -condition also. Since $h_{\Phi_{p,\lambda}}(s) = s^p$ whenever $s \in (0, 1)$, we have $i(\Phi_{p,\lambda}) = p$. \square

Remark 2.8. For $p > 1$, let $\Phi_{p,\lambda}$ be as in (2.21) with $\lambda \geq 0$. Assume that w is an A_p weight. Given a Lebesgue measurable function f , by Lemma 2.6 and Proposition 2.7 we have that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_f(x))w(x) dx \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(|f(x)|)w(x) dx.$$

Let $\Phi_{p,\lambda}$ be as in (2.21) with $p > 1$ and $\lambda < 0$. By (2.22) and (2.23), we have that both monotonicity and convexity of $\Phi_{p,\lambda}$ may fail whenever $t \ll 1$, but still hold for all $t \gg 1$. We modify $\Phi_{p,\lambda}$ in a neighborhood of the origin such that it satisfies all conclusions in Proposition 2.7.

Since $2^{-1}(p+1)\log(e+t) \leq p\log(e+t) + \lambda t(e+t)^{-1}$ whenever $t \gg 1$, by (2.22) there is a constant $t_2 \gg 1$ such that

$$(2.26) \quad \frac{(p+1)\Phi_{p,\lambda}(t)}{2t} = \frac{p+1}{2}t^{p-1}\log^\lambda(e+t) \leq \Phi'_{p,\lambda}(t)$$

for all $t \geq t_2$. Without loss of generality, we assume that $\Phi_{p,\lambda}$ is strictly increasing and convex on $[t_2, \infty)$. Since $pt_2t^{p-1} + t^p \leq (p+1)t_2t^{p-1} \leq t_2^p\log^\lambda(e+t_2)$ for any $t \ll 1$, we have that

$$(2.27) \quad pt^{p-1}(t_2 - t) \leq \Phi_{p,\lambda}(t_2) - t^p$$

for all $t \ll 1$. Moreover when $t \leq t_2(p-1)/(p+1)$, we have that

$$(2.28) \quad \frac{\Phi_{p,\lambda}(t_2) - t^p}{t_2 - t} \leq \frac{\Phi_{p,\lambda}(t_2)}{t_2 - t} \leq \frac{(p+1)\Phi_{p,\lambda}(t_2)}{2t_2}.$$

Therefore by (2.26), (2.27) and (2.28), there exists a constant $t_1 \ll 1$ such that

$$pt_1^{p-1} \leq \frac{\Phi_{p,\lambda}(t_2) - t_1^p}{t_2 - t_1} \leq \frac{(p+1)\Phi_{p,\lambda}(t_2)}{2t_2} \leq \Phi'_{p,\lambda}(t_2).$$

Let $k = (\Phi_{p,\lambda}(t_2) - t_1^p)/(t_2 - t_1)$. Given $p > 1$ and $\lambda < 0$, we define

$$(2.29) \quad \Psi_{p,\lambda}(t) = \begin{cases} t^p & 0 \leq t < t_1, \\ k(t - t_1) + t_1^p & t_1 \leq t < t_2, \\ \Phi_{p,\lambda}(t) & t_2 \leq t. \end{cases}$$

Proposition 2.9. *The function $\Psi_{p,\lambda}$ is a Young function and satisfies the Δ_2 -condition on $[0, \infty)$. Moreover $i(\Psi_{p,\lambda}) = p$.*

Proof. It is easy to see that $\Psi_{p,\lambda}$ is strictly increasing, continuous and convex on $[0, +\infty)$. Hence $\Psi_{p,\lambda}$ is a Young function. To prove the Δ_2 -condition, it suffices to check that

$$(2.30) \quad \Psi_{p,\lambda}(2t) \leq C\Psi_{p,\lambda}(t)$$

for all $t \in [0, +\infty)$. In fact, (2.30) is trivial if either $t \geq t_2$ or $2t < t_1$. Whenever $t \in [t_1/2, t_2]$, by the monotonicity of $\Psi_{p,\lambda}$ we have that

$$\frac{\Psi_{p,\lambda}(2t)}{\Psi_{p,\lambda}(2t_2)} \leq 1 \leq \frac{\Psi_{p,\lambda}(t)}{\Psi_{p,\lambda}(t_1/2)}.$$

Let $s \ll 1$. Without loss of generality, we assume $s \leq t_1/t_2$. In order to prove $i(\Psi_{p,\lambda}) = p$, we first estimate $h_{\Psi_{p,\lambda}}(s)$. By (2.29), we have that

$$(2.31) \quad \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \begin{cases} s^p & \forall t \in (0, t_1), \\ \frac{(st)^p}{k(t-t_1)+t_1^p} \approx s^p & \forall t \in [t_1, t_2]. \end{cases}$$

Moreover we obtain that

$$(2.32) \quad \frac{s^p}{\log^\lambda(e+t_2)} \leq \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{(st)^p}{\Phi_{p,\lambda}(t)} \leq \frac{s^p}{\log^\lambda(e+t_1s^{-1})}$$

for all $t \in [t_2, t_1/s)$ and

$$(2.33) \quad \frac{t_1^p s^p}{t_2^p \log^\lambda(e+t_2s^{-1})} \leq \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{k(st-t_1)+t_1^p}{\Phi_{p,\lambda}(t)} \leq \frac{\Phi_{p,\lambda}(t_2)s^p}{t_1^p \log^\lambda(e+t_1s^{-1})}$$

for all $t \in [t_1/s, t_2/s)$. Assume $t \in [t_2/s, +\infty)$. It follows that

$$(2.34) \quad \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{\Phi_{p,\lambda}(st)}{\Phi_{p,\lambda}(t)} = s^p \left(\frac{\log(e+st)}{\log(e+t)} \right)^\lambda.$$

By the monotonicity of function $(\log(s) + \log(e+\cdot)) \log^{-1}(e+\cdot)$, we have that

$$\frac{\log(e+st)}{\log(e+t)} \geq \frac{\log(s) + \log(e+t)}{\log(e+t)} \geq \frac{\log(s) + \log(e+t_2s^{-1})}{\log(e+t_2s^{-1})} \geq \frac{\log(t_2)}{\log(e+t_2s^{-1})}$$

for all $t \geq t_2/s$. Hence we derive from (2.34) that

$$(2.35) \quad s^p \leq \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} \leq \log^\lambda(t_2) \frac{s^p}{\log^\lambda(e+t_2s^{-1})}.$$

Combining (2.31), (2.32), (2.33) with (2.35) implies that

$$(2.36) \quad s^p \lesssim h_{\Psi_{p,\lambda}}(s) \lesssim s^p \log^{-\lambda}(s^{-1})$$

whenever $s \ll 1$. By (2.36), we therefore have that $i(\Psi_{p,\lambda}) = p$. \square

Remark 2.10. For $p > 1$ and $\lambda < 0$, let $\Psi_{p,\lambda}$ be as in (2.29) and $\Phi_{p,\lambda}$ be as in (2.21). Analogously to Remark 2.8, we have that

$$(2.37) \quad \int_{\mathbb{R}^2} \Psi_{p,\lambda}(M_f(x))w(x) dx \lesssim \int_{\mathbb{R}^2} \Psi_{p,\lambda}(|f(x)|)w(x) dx.$$

Since $\lim_{t \rightarrow 0^+} \Psi_{p,\lambda}(t)/\Phi_{p,\lambda}(t) = 1$, it follows that

$$(2.38) \quad \Psi_{p,\lambda}(t) \approx \Phi_{p,\lambda}(t)$$

whenever $t \in [0, +\infty)$. Hence we derive from (2.37) that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_f(x))w(x) dx \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(|f(x)|)w(x) dx.$$

3 Proof of Theorem 1.1

We begin by proving the following special case of Theorem 1.1.

Theorem 3.1. *Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \rightarrow \mathbb{D}$ be the harmonic extension of φ . For any $p > 1$, we have that*

(1) *if either $\alpha \in (p - 2, +\infty)$ and $\lambda \in \mathbb{R}$, or $\alpha = p - 2$ and $\lambda \in (-\infty, -1)$,*

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are finite.

(2) *if either $\alpha \in (-1, p - 2)$ and $\lambda \in \mathbb{R}$, or $\alpha = p - 2$ and $\lambda \in [-1, +\infty)$, then*

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are comparable to $\mathcal{U}(p, \alpha, \lambda, \varphi)$.

Moreover whenever $p \in (1, 2]$

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ dominate $\mathcal{V}(p, \alpha, \lambda, \varphi)$,

while

$\mathcal{V}(p, \alpha, \lambda, \varphi)$ controls both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$

for all $p \in [2, +\infty)$. Furthermore both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are in general comparable to $\mathcal{V}(p, \alpha, \lambda, \varphi)$ only for $p = 2$.

(3) if either $\alpha \in (-\infty, -1)$ and $\lambda \in \mathbb{R}$, or $\alpha = -1$ and $\lambda \in [-1, +\infty)$, we have that $I_1(p, \alpha, \lambda, h) = \infty$. While $I_2(p, \alpha, \lambda, h) = \infty$ for all $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$.

Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism. Given $p > 1, \alpha \in \mathbb{R}, \lambda \in \mathbb{R}$, we define

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda$$

and

$$\mathcal{E}_2(p, \alpha, \lambda, \varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \Phi_{p,\lambda} \left(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \ell(\Gamma_{j,k})^{2+\alpha}$$

where $\Phi_{p,\lambda}(t)$ is from (2.21).

Lemma 3.2. *Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism. For any $p > 1, \alpha \in (-1, +\infty)$ and every $\lambda \in \mathbb{R}$, the dyadic energies $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ and $\mathcal{E}_2(p, \alpha, \lambda, \varphi)$ are equivalent.*

Proof. We first consider the case $\lambda \geq 0$. Let $\Phi_{p,\lambda}$ be as in (2.21). Since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ and $\ell(\Gamma_{j,k}) \approx 2^{-j}$ for all $j \in \mathbb{N}$ and $k \in \{1, \dots, 2^j\}$, by the monotonicity and Δ_2 -property of the standard logarithm we have that

$$\Phi_{p,\lambda} \left(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \lesssim \left(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right)^p \log^\lambda (e + 2\pi \cdot 2^j) \lesssim \left(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right)^p j^\lambda.$$

Hence

$$(3.1) \quad \mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi).$$

Given $p > 1$ and $\alpha \in (-1, +\infty)$, there is $\beta \in (0, 1)$ such that $\alpha > (1 - \beta)p - 1 > -1$. Define

$$\chi_{j,k} = \begin{cases} 1 & \text{if } \ell(\varphi(\Gamma_{j,k})) \geq 2^{-j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

We decompose $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ as

$$\begin{aligned} \mathcal{E}_1(p, \alpha, \lambda, \varphi) &= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda \chi_{j,k} \\ &\quad + \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda (1 - \chi_{j,k}) \\ (3.2) \quad &=: \mathcal{E}'_1(p, \alpha, \lambda, \varphi) + \mathcal{E}''_1(p, \alpha, \lambda, \varphi). \end{aligned}$$

Whenever $\ell(\varphi(\Gamma_{j,k})) \geq 2^{-j\beta}$, by (2.3) we have $j^\lambda \lesssim \log^\lambda(e + \ell(\varphi(\Gamma_{j,k}))\ell(\Gamma_{j,k})^{-1})$. Therefore

$$(3.3) \quad \begin{aligned} \mathcal{E}'_1(p, \alpha, \lambda, \varphi) &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^\lambda \left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \\ &= \mathcal{E}_2(p, \alpha, \lambda, \varphi). \end{aligned}$$

Moreover, by (2.3) we have that

$$(3.4) \quad \mathcal{E}''_1(p, \alpha, \lambda, \varphi) \leq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{-\beta jp} 2^{j(p-2-\alpha)} j^\lambda = \sum_{j=1}^{+\infty} 2^{-j((\beta-1)p+1+\alpha)} j^\lambda < \infty.$$

We conclude from (3.2), (3.3) and (3.4) that there is a constant $C > 0$ such that

$$(3.5) \quad \mathcal{E}_1(p, \alpha, \lambda, \varphi) \lesssim C + \mathcal{E}_2(p, \alpha, \lambda, \varphi).$$

From (3.1) and (3.5) it follows that

$$(3.6) \quad \mathcal{E}_1(p, \alpha, \lambda, \varphi) \text{ and } \mathcal{E}_2(p, \alpha, \lambda, \varphi) \text{ are comparable whenever } \lambda \geq 0.$$

Analogously to (3.6), we have that $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ and $\mathcal{E}_2(p, \alpha, \lambda, \varphi)$ are comparable whenever $\lambda < 0$. \square

Lemma 3.3. *Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \rightarrow \mathbb{D}$ be the Poisson homeomorphic extension of φ . For any $p > 1$, we have that $I_1(p, \alpha, \lambda, h) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi)$ whenever $\alpha \in (-1, +\infty)$ and $\lambda \in \mathbb{R}$, while $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ is controlled by $I_1(p, \alpha, \lambda, h)$ for all $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. We first prove that $I_1(p, \alpha, \lambda, h) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi)$ for all $\alpha > -1$ and all $\lambda \in \mathbb{R}$. Let $w_{\alpha, \lambda}$ be as in (2.7). For any $j \in \mathbb{N}$ and $1 \leq k \leq 2^j$, by (2.4) and (2.3) we have that

$$(3.7) \quad w_{\alpha, \lambda}(z) \approx 2^{-j\alpha} j^\lambda \approx \ell(\Gamma_{j,k})^\alpha j^\lambda$$

for all $z \in Q_{j,k}$. Hence

$$(3.8) \quad I_1(p, \alpha, \lambda, h) \approx \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{-j\alpha} j^\lambda \int_{Q_{j,k}} |Dh(z)|^p dz.$$

Let $\mathcal{P}(\Gamma_{j,k})$ be the technical decomposition of \mathbb{S}^1 based on $\Gamma_{j,k}$ in [9, Section 2.1]. As shown in [9, Proof (iv) \Rightarrow (i)], for any $j \in \mathbb{N}$ and $k = 1, \dots, 2^j$ we have that

$$(3.9) \quad |Dh(z)| \lesssim \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}.$$

for all $z \in Q_{j,k}$. Here $\Gamma_{n,m} \in \mathcal{P}(\Gamma_{j,k})$ and $\sharp i_n \leq 3$ for all $n \leq j$, see [9, Section 2.1]. Let $\alpha > -1$. There is $q_0 > 1$ such that $p/q_0 - 1 - \alpha < 0$. Denote by p_0 the exponent conjugate to q_0 . Via Hölder's inequality we derive from (3.9) that

$$(3.10) \quad \begin{aligned} |Dh(z)|^p &\lesssim \left(\sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n(\frac{1}{q_0} + \frac{1}{p_0})}} \right)^p \leq \left(\sum_{n \leq j} \sum_{m \in i_n} 2^{\frac{nq}{q_0}} \right)^{\frac{p}{q}} \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \\ &\approx 2^{\frac{jp}{q_0}} \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \end{aligned}$$

for all $z \in Q_{j,k}$. By (2.5), (3.8) and (3.10) we have that

$$(3.11) \quad I_1(p, \alpha, \lambda, h) \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{j(\frac{p}{q_0} - 2 - \alpha)} j^\lambda \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{q_0}}}.$$

Moreover given a dyadic arc $\Gamma_{n,m}$, for any $j \geq n$ it is shown in [9, Section 2.1] that

$$(3.12) \quad \sharp\{\Gamma : \Gamma \text{ is a } j\text{-level dyadic arc and } \Gamma_{n,m} \in \mathcal{P}(\Gamma)\} \leq 3 \cdot 2^{j-n}.$$

From Fubini's theorem and (3.12) we obtain that

$$(3.13) \quad \begin{aligned} &\sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{j(\frac{p}{q_0} - 2 - \alpha)} j^\lambda \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \sum_{n \leq j} \sum_k 2^{j(\frac{p}{q_0} - 2 - \alpha)} j^\lambda \\ &\lesssim \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \sum_{n \leq j} 2^{j(\frac{p}{q_0} - 2 - \alpha)} j^\lambda 2^{j-n} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \ell(\varphi(\Gamma_{n,m}))^p 2^{n(\frac{p}{p_0} - 1)} \sum_{n \leq j} 2^{j(\frac{p}{q_0} - 1 - \alpha)} j^\lambda. \end{aligned}$$

Moreover when $p/q_0 - 1 - \alpha < 0$ we have that

$$(3.14) \quad \sum_{n \leq j} 2^{j(\frac{p}{q_0} - 1 - \alpha)} j^\lambda \approx 2^{n(\frac{p}{q_0} - 1 - \alpha)} n^\lambda.$$

By (3.11), (3.13), (3.14) and (2.3), we conclude that

$$I_1(p, \alpha, \lambda, h) \lesssim \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \ell(\varphi(\Gamma_{n,m}))^p 2^{n(p-2-\alpha)} n^\lambda \approx \mathcal{E}_1(p, \alpha, \lambda, \varphi).$$

We next prove that $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ is controlled by $I_1(p, \alpha, \lambda, h)$ for all $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. By [9, (3.17)], there is $j_0 > 1$ such that

$$(3.15) \quad \ell(\varphi(\Gamma_{j,k})) \lesssim \frac{1}{\ell(\Gamma_{j,k})} \int_{CQ_{j,k} \cap \mathbb{D}} |Dh(z)| dz$$

for all $j \geq j_0$ and $k \in \{1, \dots, 2^j\}$. Set $H(z) = |Dh(z)|\chi_{\mathbb{D}}(z)$. By (2.6) we have that

$$(3.16) \quad \int_{CQ_{j,k} \cap \mathbb{D}} |Dh(z)| dz \leq \int_{CC'B_{j,k}} H(z) dz \leq \int_{Q_{j,k}} M_H(w) dw,$$

where the last inequality comes from the fact that $\int_{CC'B_{j,k}} H(z) dz \leq M_H(w)$ for all $w \in Q_{j,k}$. Combining (3.15) with (3.16) implies that

$$(3.17) \quad \ell(\varphi(\Gamma_{j,k})) \lesssim \ell(\Gamma_{j,k}) \int_{Q_{j,k}} M_H(z) dz$$

for all $j \geq j_0$ and $k = 1, \dots, 2^j$. By Jensen's inequality and (2.5), we deduce from (3.17) that

$$(3.18) \quad \ell(\varphi(\Gamma_{j,k}))^p \lesssim \ell(\Gamma_{j,k})^{p-2} \int_{Q_{j,k}} M_H^p(z) dz.$$

By (3.7) and (3.18), there is then a constant $C > 0$ such that

$$(3.19) \quad \mathcal{E}_1(p, \alpha, \lambda, \varphi) \lesssim C + \sum_{j=j_0}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} M_H^p(z) w_{\alpha, \lambda}(z) dz \leq C + \int_{\mathbb{R}^2} M_H^p(z) w_{\alpha, \lambda}(z) dz.$$

Moreover, for any $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$, from Proposition 2.3 and Remark 2.8 it follows that

$$(3.20) \quad \int_{\mathbb{R}^2} M_H^p(z) w_{\alpha, \lambda}(z) dz \lesssim \int_{\mathbb{R}^2} H^p(z) w_{\alpha, \lambda}(z) dz = I_1(p, \alpha, \lambda, h).$$

By (3.19) and (3.20) we conclude that $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ is controlled by $I_1(p, \alpha, \lambda, h)$ for all $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. \square

Lemma 3.4. *Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \rightarrow \mathbb{D}$ be the Poisson homeomorphic extension of φ . For any $p > 1$, we have that $I_2(p, \alpha, \lambda, h) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi)$ whenever $\alpha \in (-1, +\infty)$ and $\lambda \in \mathbb{R}$, while $\mathcal{E}_2(p, \alpha, \lambda, \varphi)$ is controlled by $I_2(p, \alpha, \lambda, h)$ for all $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. We first consider that case $\lambda \geq 0$. Let $\Phi_{p,\lambda}$ be as in (2.21). Proposition 2.7 shows that $\Phi_{p,\lambda}(t)$ is increasing and satisfies Δ_2 -property on $[0, +\infty)$. From (3.7) and (3.9) we have that

$$(3.21) \quad \begin{aligned} I_2(p, \alpha, \lambda, h) &= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} \Phi_{p,\lambda}(|Dh(z)|) w_{\alpha,0}(z) dz \\ &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda} \left(\sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right). \end{aligned}$$

Moreover since $\ell(\varphi(\Gamma_{n,m})) \leq 2\pi$ for all $n \in \mathbb{N}$ and $m = 1, \dots, 2^n$, it follows that

$$\sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \lesssim \sum_{n \leq j} \frac{1}{2^{-n}} \lesssim 2^j.$$

for any $j \geq 1$. Therefore

$$(3.22) \quad \log^\lambda \left(e + \sum_{n \leq j} \sum_m \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right) \lesssim \log^\lambda(e + 2^j) \lesssim j^\lambda$$

for all $j \geq 1$. By (3.21) and (3.22) we obtain that

$$(3.23) \quad I_2(p, \alpha, \lambda, h) \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} j^\lambda \left(\sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right)^p.$$

The analogous arguments as for $I_1(p, \alpha, \lambda, h) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi)$ in Lemma 3.3 imply that

$$(3.24) \quad \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} j^\lambda \left(\sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right)^p \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi).$$

We conclude from (3.23) and (3.24) that $I_2(p, \alpha, \lambda, h) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi)$.

Applying $\Phi_{p,\lambda}$ to the both sides of (3.17), via Proposition 2.7 and Jensen's inequality we have that

$$(3.25) \quad \Phi_{p,\lambda} \left(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \lesssim \Phi_{p,\lambda} \left(\int_{Q_{j,k}} M_H(z) dz \right) \leq \int_{Q_{j,k}} \Phi_{p,\lambda}(M_H(z)) dz$$

for all $j \geq j_0$ and $k \in \{1, \dots, 2^j\}$. By (2.5), (3.7) and (3.25), we then obtain that

$$(3.26) \quad \begin{aligned} \mathcal{E}_2(p, \alpha, \lambda, \varphi) &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} \Phi_{p,\lambda}(M_H(z)) w_{\alpha,0}(z) dz \\ &\leq \int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_H(z)) w_{\alpha,0}(z) dz. \end{aligned}$$

Moreover, for any $\alpha \in (-1, p-1)$ it follows from Remark 2.8 that

$$(3.27) \quad \int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_H(z))w_{\alpha,0}(z) dz \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(H(z))w_{\alpha,0}(z) dz = I_2(p, \alpha, \lambda, h).$$

By (3.26) and (3.27) we conclude that $\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim I_2(p, \alpha, \lambda, h)$.

We next consider the case $\lambda < 0$. Let $\Psi_{p,\lambda}$ be as in (2.29). By the analogous arguments as for (3.21), we have that

$$(3.28) \quad \int_{\mathbb{D}} \Psi_{p,\lambda}(|Dh(z)|)w_{\alpha,0}(z) dz \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Psi_{p,\lambda} \left(\sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right).$$

Set $S_{j,k} = \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}$. It follows from (2.38) and (3.28) that

$$(3.29) \quad I_2(p, \alpha, \lambda, h) \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}(S_{j,k}).$$

Since $\alpha > -1$, there is $\beta > 0$ such that $\beta p \leq 1 + \alpha$. Define

$$\chi(j, k) = \begin{cases} 1 & \text{if } S_{j,k} < 2^{j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(3.30) \quad \begin{aligned} \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}(S_{j,k}) &= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}(\chi(j, k) S_{j,k}) \\ &+ \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}((1 - \chi(j, k)) S_{j,k}) =: \sum_1 + \sum_2. \end{aligned}$$

Since $\log^\lambda(e + S_{j,k}) \leq \log^\lambda(e) = 1$, we have that

$$(3.31) \quad \sum_1 \leq \sum_1 2^{-j(\alpha+2)} (S_{j,k})^p \leq \sum_{j=1}^{\infty} 2^{j(p\beta - \alpha - 1)} < \infty.$$

Whenever $S_{j,k} \geq 2^{j\beta}$, it follows that $\log^\lambda(e + S_{j,k}) \lesssim j^\lambda$. Via the analogous arguments as for $I_1(p, \alpha, \lambda, h) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi)$ in Lemma 3.3, it then follows that

$$(3.32) \quad \sum_2 \leq \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} j^\lambda S_{j,k}^p \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi).$$

From (3.29), (3.30), (3.31) and (3.32), we conclude that there is a constant $C > 0$ such that $I_2(p, \alpha, \lambda, h) \lesssim C + \mathcal{E}_1(p, \alpha, \lambda, \varphi)$.

By the analogous arguments as for $\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim I_2(p, \alpha, \lambda, h)$ whenever $\lambda \geq 0$, we have that

$$(3.33) \quad \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Psi_{p,\lambda} \left(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \lesssim \int_{\mathbb{R}^2} \Psi_{p,\lambda}(|Dh|(z)) w_{\alpha,0}(z) dz.$$

It follows from (2.38) that $\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim I_2(p, \alpha, \lambda, h)$. \square

Proof of Theorem 3.1 (1). From Lemma 3.3 and Lemma 3.4, we have that both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are dominated by $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ for all $p > 1$, $\alpha \in (-1, +\infty)$ and each $\lambda \in \mathbb{R}$. Moreover since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ for all $j \geq 1$ and $1 \leq k \leq 2^j$, we have that

$$\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \leq (2\pi)^{p-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) = (2\pi)^p.$$

Therefore both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are controlled by $\sum_{j=1}^{\infty} 2^{j(p-2-\alpha)} j^\lambda$ whenever $\alpha \in (-1, +\infty)$ and $\lambda \in \mathbb{R}$. Notice that $\sum_{j=1}^{\infty} 2^{j(p-2-\alpha)} j^\lambda < \infty$ whenever either $p-2 < \alpha$ and $\lambda \in \mathbb{R}$, or $p-2 = \alpha$ and $\lambda < -1$. We hence complete Theorem 3.1 (1). \square

By Example 4.4, there are homeomorphisms $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that, for their harmonic extensions $P[\varphi]$, both $I_1(p, \alpha, \lambda, P[\varphi])$ and $I_2(p, \alpha, \lambda, P[\varphi])$ may be finite or infinite for either some $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or some $\alpha = p-2$ and $\lambda \in [-1, +\infty)$. How can we characterize both $I_1(p, \alpha, \lambda, P[\varphi]) < \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) < \infty$? As shown in [9], double integrals of the inverse mapping over the boundary are potential choices.

Lemma 3.5. *Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism. For any $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, $\mathcal{V}(p, \alpha, \lambda, \varphi)$ is dominated by $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ whenever $p \in (1, 2]$; while $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ is controlled by $\mathcal{V}(p, \alpha, \lambda, \varphi)$ if $p \in [2, +\infty)$.*

Proof. We first consider the case $p \in (1, 2]$. Given $\xi \in \mathbb{S}^1$ and $t \geq 0$, set

$$E_t(\xi) = \{\eta \in \mathbb{S}^1 : |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)| < t\}.$$

By Fubini's theorem we have that

$$\begin{aligned}
\int_{\mathbb{S}^1} \mathcal{A}_{p,\alpha,\lambda}(|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|) |d\eta| &= \int_{\mathbb{S}^1} \int_{|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|}^1 -\mathcal{A}'_{p,\alpha,\lambda}(t) dt |d\eta| \\
&= \int_0^1 \int_{\mathbb{S}^1} -\mathcal{A}'_{p,\alpha,\lambda}(t) \chi_{E_t(\xi)} |d\eta| dt \\
(3.34) \qquad \qquad \qquad &= \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t) \mathcal{L}^1(E_t(\xi)) dt.
\end{aligned}$$

Moreover, from Jensen's inequality and Minkowski's inequality it follows that

$$\begin{aligned}
&\left(\int_{\mathbb{S}^1} \left(\int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t) \mathcal{L}^1(E_t(\xi)) dt \right)^{p-1} |d\xi| \right)^{\frac{1}{p-1}} \\
&\lesssim \left(\int_{\mathbb{S}^1} \left(\int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t) \mathcal{L}^1(E_t(\xi)) dt \right)^{\frac{1}{p-1}} |d\xi| \right)^{p-1} \\
&\leq \int_0^1 \left(\int_{\mathbb{S}^1} \left(-\mathcal{A}'_{p,\alpha,\lambda}(t) \mathcal{L}^1(E_t(\xi)) \right)^{\frac{1}{p-1}} |d\xi| \right)^{p-1} dt \\
(3.35) \qquad \qquad \qquad &= \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t) \left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_t(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1} dt.
\end{aligned}$$

Combining (3.34) with (3.35) implies that

$$\begin{aligned}
\mathcal{V}^{\frac{1}{p-1}}(p, \alpha, \lambda, \varphi) &\lesssim \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t) \left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_t(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1} dt \\
(3.36) \qquad \qquad \qquad &\leq \sum_{j=1}^{+\infty} \int_{2^{-j}}^{2^{1-j}} -\mathcal{A}'_{p,\alpha,\lambda}(t) dt \left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1}.
\end{aligned}$$

Since $E_{2^{1-j}}(\xi) \subset \cup_{i=k-1}^{k+1} \varphi(\Gamma_{j,i})$ for all $j \in \mathbb{N}$, $k = 1, \dots, 2^j$ and all $\xi \in \varphi(\Gamma_{j,k})$, we have that

$$\begin{aligned}
\left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1} &= \left(\sum_{k=1}^{2^j} \int_{\varphi(\Gamma_{j,k})} \mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1} \\
&\leq \left(\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) \left(\sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i})) \right)^{\frac{1}{p-1}} \right)^{p-1} \\
(3.37) \qquad \qquad \qquad &\leq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^{p-1} \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i})).
\end{aligned}$$

Moreover by Young's inequality we have that

$$(3.38) \quad \begin{aligned} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^{p-1} \ell(\varphi(\Gamma_{j,k-1})) &\leq \sum_{k=1}^{2^j} \frac{1}{p} \ell(\varphi(\Gamma_{j,k}))^p + \frac{p}{p-1} \ell(\varphi(\Gamma_{j,k-1}))^p \\ &\lesssim \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \end{aligned}$$

and

$$(3.39) \quad \begin{aligned} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^{p-1} \ell(\varphi(\Gamma_{j,k+1})) &\leq \sum_{k=1}^{2^j} \frac{1}{p} \ell(\varphi(\Gamma_{j,k}))^p + \frac{p}{p-1} \ell(\varphi(\Gamma_{j,k+1}))^p \\ &\lesssim \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p. \end{aligned}$$

Combining (3.37), (3.38) with (3.39) implies that

$$(3.40) \quad \left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1} \lesssim \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p$$

for all $j \in \mathbb{N}$. Let

$$\Lambda_\lambda(t) = \begin{cases} t^{\lambda+1} & \lambda \neq -1, \\ \log t & \lambda = -1. \end{cases}$$

For any $j \in \mathbb{N}$, we have that

$$(3.41) \quad \int_{2^{-j}}^{2^{1-j}} t^{-1} \log_2^\lambda(t^{-1}) dt \approx - \int_{2^{-j}}^{2^{1-j}} d\Lambda_\lambda(\log_2(t^{-1})) = \Lambda_\lambda(j) - \Lambda_\lambda(j-1) \approx j^\lambda.$$

It follows (3.41) and (2.3) that

$$(3.42) \quad \int_{2^{-j}}^{2^{1-j}} -\mathcal{A}'_{p,\alpha,\lambda}(t) dt \approx 2^{j(p-2-\alpha)} \int_{2^{-j}}^{2^{1-j}} \frac{1}{t} \log_2^\lambda(t^{-1}) dt \approx \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda.$$

By combining (3.36), (3.40) with (3.42), we conclude that

$$\mathcal{V}^{\frac{1}{p-1}}(p, \alpha, \lambda, \varphi) \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda = \mathcal{E}_1(p, \alpha, \lambda, \varphi).$$

We next consider the case $p \in [2, +\infty)$. By the analogous arguments as for (3.36) we have that

$$(3.43) \quad \mathcal{V}^{\frac{1}{p-1}}(p, \alpha, \lambda, \varphi) \gtrsim \sum_{j=5}^{+\infty} \int_{\pi^{2^{1-j}}}^{\pi^{2^{2-j}}} -\mathcal{A}'_{p,\alpha,\lambda}(t) dt \left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{\pi^{2^{1-j}}}(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1}.$$

Since $\varphi(\Gamma_{j,k}) \subset E_{\pi^{2^{1-j}}}(\xi)$ for all $j \geq 5$, $k \in \{1, \dots, 2^j\}$ and all $\xi \in \varphi(\Gamma_{j,k})$, we have that

$$(3.44) \quad \begin{aligned} \left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{\pi^{2^{1-j}}}(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1} &= \left(\sum_{k=1}^{2^j} \int_{\varphi(\Gamma_{j,k})} \mathcal{L}^1(E_{\pi^{2^{1-j}}}(\xi))^{\frac{1}{p-1}} |d\xi| \right)^{p-1} \\ &\geq \left(\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) \ell(\varphi(\Gamma_{j,k}))^{\frac{1}{p-1}} \right)^{p-1} \\ &\geq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p. \end{aligned}$$

By (3.42), (3.43) and (3.44), there is a constant $C > 0$ such that

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) = \sum_{j=1}^4 \sum_{k=1}^{2^j} + \sum_{j=5}^{+\infty} \sum_{k=1}^{2^j} \lesssim C + \mathcal{V}^{\frac{1}{p-1}}(p, \alpha, \lambda, \varphi).$$

□

We next prove Theorem 3.1 (2).

Lemma 3.6. *Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism. For any $p \in (1, +\infty)$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$, we have that $\mathcal{U}(p, \alpha, \lambda, \varphi)$ and $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ are comparable.*

Proof. We first prove that $\mathcal{U}(p, \alpha, \lambda, \varphi)$ is controlled by $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$. Given $\xi \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$, let $\ell(\xi\eta)$ be the arc length of the shorter arc in \mathbb{S}^1 connecting ξ and η . Given $j \geq 1$ and $\xi \in \mathbb{S}^1$, set

$$A_j = \{(\xi, \eta) \in \mathbb{S}^1 \times \mathbb{S}^1 : \pi 2^{-j} < \ell(\xi\eta) \leq \pi 2^{1-j}\}$$

and $A_j(\xi) = \{\eta \in \mathbb{S}^1 : (\xi, \eta) \in A_j\}$. Notice that $\lambda_{\mathbb{D}}$ is the Euclidean distance. We have that

$$(3.45) \quad \begin{aligned} \mathcal{U}(p, \alpha, \lambda, \varphi) &= \sum_{j=1}^{+\infty} \int_{A_j} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \right) |\xi - \eta|^\alpha |d\eta| |d\xi| \\ &= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{\Gamma_{j,k}} \int_{A_j(\xi)} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \right) |\xi - \eta|^\alpha |d\eta| |d\xi|. \end{aligned}$$

Notice that

$$(3.46) \quad |\xi - \eta| \approx \ell(\Gamma_{j,k}) \text{ and } |\varphi(\xi) - \varphi(\eta)| \leq \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i})) \leq 2\pi$$

for all $j \in \mathbb{N}$, $k \in \{1, \dots, 2^j\}$, $\xi \in \Gamma_{j,k}$ and $\eta \in A_j(\xi)$. It then follows that

$$(3.47) \quad \begin{aligned} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \right) |\xi - \eta|^\alpha &\lesssim \left(\sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i})) \right)^p \ell(\Gamma_{j,k})^{\alpha-p} \log^\lambda(e + 2\pi \cdot 2^j) \\ &\lesssim \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))^p \ell(\Gamma_{j,k})^{\alpha-p} j^\lambda \end{aligned}$$

for all $\lambda \geq 0$, $\xi \in \Gamma_{j,k}$ and $\eta \in A_j(\xi)$. Since

$$(3.48) \quad \mathcal{L}^1(A_j(\xi)) \approx \ell(\Gamma_{j,k})$$

for all $j \in \mathbb{N}$, $k = 1, \dots, 2^j$ and $\xi \in \Gamma_{j,k}$, we derive from (3.45) and (3.47) that

$$\begin{aligned} \mathcal{U}(p, \alpha, \lambda, \varphi) &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))^p \ell(\Gamma_{j,k})^{\alpha-p} j^\lambda \int_{\Gamma_{j,k}} \int_{A_j(\xi)} |d\eta| |d\xi| \\ &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda = \mathcal{E}_1(p, \alpha, \lambda, \varphi) \end{aligned}$$

whenever $\lambda \geq 0$.

Since φ is homeomorphic, for any $j \in \mathbb{N}$ and $k \in \{1, \dots, 2^j\}$ there are $\xi'_{j,k} \in \Gamma_{j,k}$ and $\eta'_{j,k} \in A_j(\xi'_{j,k})$ such that

$$(3.49) \quad \begin{aligned} &\Phi_{p,\lambda} \left(\frac{|\varphi(\xi'_{j,k}) - \varphi(\eta'_{j,k})|}{|\xi'_{j,k} - \eta'_{j,k}|} \right) |\xi'_{j,k} - \eta'_{j,k}|^\alpha \\ &= \max \left\{ \Phi_{p,\lambda} \left(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \right) |\xi - \eta|^\alpha : \xi \in \Gamma_{j,k} \text{ and } \eta \in A_j(\xi) \right\}. \end{aligned}$$

Since $0 < \alpha + 1 < p$, there is $\beta \in (-1, 0)$ such that $0 < (1 + \beta)p < \alpha + 1$. Define

$$\chi(j, k) = \begin{cases} 1 & \text{if } |\varphi(\xi'_{j,k}) - \varphi(\eta'_{j,k})| \leq 2^{j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

From (3.45), (3.49), (3.48), (2.3) and (3.46), we obtain that

$$\begin{aligned}
\mathcal{U}(p, \alpha, \lambda, \varphi) &\leq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi'_{j,k}) - \varphi(\eta'_{j,k})|}{|\xi'_{j,k} - \eta'_{j,k}|} \right) \\
&= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi'_{j,k}) - \varphi(\eta'_{j,k})|}{|\xi'_{j,k} - \eta'_{j,k}|} \right) \chi(j, k) \\
(3.50) \quad &+ \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi'_{j,k}) - \varphi(\eta'_{j,k})|}{|\xi'_{j,k} - \eta'_{j,k}|} \right) (1 - \chi(j, k)) =: \sum_1 + \sum_2.
\end{aligned}$$

Since $\log^\lambda(e + |\varphi(\xi'_{j,k}) - \varphi(\eta'_{j,k})| |\xi'_{j,k} - \eta'_{j,k}|^{-1}) \leq 1$ for all $\lambda < 0$, $j \in \mathbb{N}$ and $1 \leq k \leq 2^j$, by (3.46) and (2.3) we have that

$$(3.51) \quad \sum_1 \lesssim \sum_{j=1}^{+\infty} 2^{-2j} \sum_{k=1}^{2^j} 2^{j((1+\beta)p-\alpha)} = \sum_{j=1}^{+\infty} 2^{j((1+\beta)p-\alpha-1)} < +\infty.$$

Moreover we derive from (3.46) that

$$\begin{aligned}
\sum_2 &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha-p} \left(\sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,k})) \right)^p \log^\lambda(2^{j(1+\beta)}) \\
(3.52) \quad &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda = \mathcal{E}_1(p, \alpha, \lambda, \varphi).
\end{aligned}$$

for all $\lambda < 0$. Combining (3.50), (3.51) with (3.52) implies that there is a constant $C > 0$ such that $\mathcal{U}(p, \alpha, \lambda, \varphi) \lesssim C + \mathcal{E}_1(p, \alpha, \lambda, \varphi)$ for all $\lambda < 0$.

We next prove that $\mathcal{U}(p, \alpha, \lambda, \varphi)$ dominates $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$. Given $j \geq 3$ and $\xi \in \mathbb{S}^1$, set

$$B_j = \{(\xi, \eta) \in \mathbb{S}^1 \times \mathbb{S}^1 : \pi 2^{2-j} < \ell(\xi\eta) \leq \pi 2^{3-j} \text{ with } \arg \eta > \arg \xi\}$$

and $B_j(\xi) = \{\eta \in \mathbb{S}^1 : (\xi, \eta) \in B_j\}$. We have that

$$(3.53) \quad \sum_{j=3}^{+\infty} \sum_{k=1}^{2^j} \int_{\Gamma_{j,k-1}} \int_{B_j(\xi)} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \right) |\xi - \eta|^\alpha |d\eta| |d\xi| = \mathcal{U}(p, \alpha, \lambda, \varphi).$$

Since φ is homeomorphic, for any $j \geq 3$ and $1 \leq k \leq 2^j$ there are $\xi''_{j,k} \in \Gamma_{j,k-1}$ and $\eta''_{j,k} \in B_j(\xi''_{j,k})$ such that

$$(3.54) \quad \begin{aligned} & \Phi_{p,\lambda} \left(\frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|} \right) |\xi''_{j,k} - \eta''_{j,k}|^\alpha \\ &= \min \left\{ \Phi_{p,\lambda} \left(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \right) |\xi - \eta|^\alpha : \xi \in \Gamma_{j,k-1} \text{ and } \eta \in B_j(\xi) \right\}. \end{aligned}$$

Notice that

$$(3.55) \quad |\xi''_{j,k} - \eta''_{j,k}| \approx \ell(\Gamma_{j,k}) \text{ and } 2 \geq |\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})| \gtrsim \ell(\varphi(\Gamma_{j,k}))$$

whenever $j \geq 3$ and $k \in \{1, \dots, 2^j\}$. Since $\mathcal{L}^1(B_j(\xi)) \approx \ell(\Gamma_{j,k})$ for all $j \geq 3$, $k = 1, \dots, 2^j$ and $\xi \in \mathbb{S}^1$, it follows from (3.53), (3.54) and (3.55) that

$$(3.56) \quad \sum_{j=3}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha} \Phi_{p,\lambda} \left(\frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|} \right) \lesssim \mathcal{U}(p, \alpha, \lambda, \varphi).$$

Moreover, for any $\lambda \leq 0$ we obtain from (3.55) that

$$(3.57) \quad j^\lambda \lesssim \log^\lambda(e + 2^{1+j}) \lesssim \log^\lambda \left(e + \frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|} \right)$$

for all $j \in \mathbb{N}$ and all $k = 1, \dots, 2^j$. From (3.55), (3.56) and (3.57), there is a constant $C > 0$ such that

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) = C + \sum_{j=3}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda \lesssim C + \mathcal{U}(p, \alpha, \lambda, \varphi)$$

for all $\lambda \leq 0$. For any $\lambda > 0$, by (3.55) and (3.56) there is a constant $C > 0$ such that

$$(3.58) \quad \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^\lambda(2^j \ell(\varphi(\Gamma_{j,k}))) \lesssim C + \mathcal{U}(p, \alpha, \lambda, \varphi).$$

Let β be same as in (3). Set

$$\chi_{j,k} = \begin{cases} 1 & \text{if } \ell(\varphi(\Gamma_{j,k})) \leq 2^{j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

We have that

$$\begin{aligned}
\mathcal{E}_1(p, \alpha, \lambda, \varphi) &= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda \chi_{j,k} \\
(3.59) \quad &+ \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda (1 - \chi_{j,k}) =: \sum^1 + \sum^2.
\end{aligned}$$

Moreover

$$(3.60) \quad \sum^1 \leq \sum_{j=1}^{+\infty} 2^{j((1+\beta)p-\alpha-1)} j^\lambda < \infty$$

and

$$(3.61) \quad \sum^2 \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^\lambda (2^j \ell(\varphi(\Gamma_{j,k}))).$$

From (3.59), (3.60), (3.61) and (3.58), we have that $\mathcal{U}(p, \alpha, \lambda, \varphi)$ controls $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ whenever $\lambda > 0$. \square

Proof of Theorem 3.1 (2). By Lemma 3.2, Lemma 3.3 and Lemma 3.4, for any $p > 1$ we have that both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are comparable to $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ whenever $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. By Lemma 3.6, we hence conclude comparability of both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ with $\mathcal{U}(p, \alpha, \lambda, \varphi)$ for all $p > 1, \alpha \in (-1, p-1)$ and every $\lambda \in \mathbb{R}$. By Lemma 3.5, we can dominate $\mathcal{V}(p, \alpha, \lambda, \varphi)$ by either $I_1(p, \alpha, \lambda, h)$ or $I_2(p, \alpha, \lambda, h)$ whenever $p \in (1, 2]$, while both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are controlled by $\mathcal{V}(p, \alpha, \lambda, \varphi)$ for all $p \in [2, +\infty)$. Moreover from Example 4.2 and Example 4.3, we have that $\mathcal{V}(p, \alpha, \lambda, \varphi)$ is comparable to either $I_1(p, \alpha, \lambda, h)$ or $I_2(p, \alpha, \lambda, h)$ only when $p = 2$. \square

Towards the proof of Theorem 3.1 (3), we have the following general result.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and $\varphi : \mathbb{S}^1 \rightarrow \partial\Omega$ be a homeomorphism. For any $p > 1$, there is no diffeomorphic extension $h : \mathbb{D} \rightarrow \Omega$ of φ for which $I_1(p, \alpha, \lambda, h) < +\infty$ for either $\alpha \in (-\infty, -1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1, +\infty)$; or for which $I_2(p, \alpha, \lambda, h) < +\infty$ for some $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$.*

Proof. Assume that there is a diffeomorphic extension $h : \mathbb{D} \rightarrow \Omega$ of φ for which $I_1(p, \alpha, \lambda, h) < +\infty$ for either $\alpha \in (-\infty, -1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1, +\infty)$. Then $h \in W^{1,p}(\mathbb{D}, \Omega)$. Let

$$\mathbb{S}_r = \{\xi \in \mathbb{R}^2 : |\xi| = r\} \text{ and } \text{osc}_{\mathbb{S}_r} h = \sup\{|h(\xi_1) - h(\xi_2)| : \xi_1, \xi_2 \in \mathbb{S}_r\}.$$

By the ACL-property of Sobolev mappings, we have that

$$(3.62) \quad \text{osc}_{\mathbb{S}_r} h \leq \int_{\mathbb{S}_r} |Dh(\xi)| |d\xi|$$

for \mathcal{L}^1 -a.e. $r \in [0, 1)$. By Jensen's inequality we derive from (3.62) that

$$(3.63) \quad \begin{aligned} (\text{osc}_{\mathbb{S}_r} h)^p &\leq (\text{osc}_{\mathbb{S}_r} h)^p r^{1-p} \lesssim \int_{\mathbb{S}_r} |Dh(\xi)|^p |d\xi| \\ &= w_{\alpha, \lambda}^{-1}(1-r) \int_{\mathbb{S}_r} |Dh(\xi)|^p w_{\alpha, \lambda}(1-r) |d\xi|. \end{aligned}$$

Let $\mathbb{D}_r = \{z \in \mathbb{R}^2 : |z| < r\}$. Since h is a homeomorphism, we have $\text{osc}_{\mathbb{D}_r} h = \text{osc}_{\mathbb{S}_r} h$. Hence

$$(3.64) \quad \text{osc}_{\mathbb{S}_r} h \text{ is increasing with respect to } r \in [0, 1).$$

Moreover $w_{\alpha, \lambda}(1-r) \approx 2^{-\alpha j} j^\lambda$ for all $j \geq 0$ and $r \in (1-2^{-j}, 1-2^{-j-1}]$. By (3.63), (3.64) and Fubini's theorem, we obtain that

$$(3.65) \quad \begin{aligned} \sum_{j=1}^{+\infty} (\text{osc}_{\mathbb{S}_{1-2^{-j}}} h)^p 2^{-(\alpha+1)j} j^\lambda &\leq \sum_{j=1}^{+\infty} \int_{1-2^{-j}}^{1-2^{-j-1}} (\text{osc}_{\mathbb{S}_r} h)^p w_{\alpha, \lambda}(1-r) dr \\ &\lesssim \sum_{j=1}^{+\infty} \int_{1-2^{-j}}^{1-2^{-j-1}} \int_{\mathbb{S}_r} |Dh(\xi)|^p w_{\alpha, \lambda}(1-r) |d\xi| dr \\ &= I_1(p, \alpha, \lambda, h). \end{aligned}$$

By the assumption at the beginning, we derive from (3.65) that

$$(3.66) \quad \sum_{j=1}^{+\infty} (\text{osc}_{\mathbb{S}_{1-2^{-j}}} h)^p 2^{-(\alpha+1)j} j^\lambda < +\infty$$

for either $\alpha < -1$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \geq -1$. Hence by (3.64) we have that $\text{osc}_{\mathbb{S}_{1-2^{-j}}} h = 0$ for all $j \geq 1$. Therefore there is a constant C such that $h(z) = C$ for all $z \in \mathbb{D}$. This contradicts the homeomorphicity of h . We conclude that the assumption at the beginning cannot hold.

We next assume that there is a diffeomorphic extension $h : \mathbb{D} \rightarrow \Omega$ of φ for which $I_2(p, \alpha, \lambda, h) < +\infty$ for some $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$. It is not difficult to see that $h \in W^{1,1}(\mathbb{D}, \Omega)$. We first let $\lambda \geq 0$. Proposition 2.7 shows that $\Phi_{p, \lambda}$ is convex. Analogously to (3.65), we have

$$(3.67) \quad \sum_{j=1}^{+\infty} \Phi_{p, \lambda} \left(\frac{\text{osc}_{\mathbb{S}_{1-2^{-j}}} \text{Re} h}{2\pi} \right) 2^{-(\alpha+1)j} \lesssim I_2(p, \alpha, \lambda, h).$$

Analogous arguments as below (3.66) imply that there is a contradiction under the above assumption. We next let $\lambda < 0$. Proposition 2.9 shows that $\Psi_{p,\lambda}$ is convex. Analogously to (3.67), we obtain from (2.38) that

$$\begin{aligned} \sum_{j=1}^{+\infty} \Psi_{p,\lambda} \left(\frac{\text{osc}_{\mathbb{S}^1-2^{-j}} \text{Re}h}{2\pi} \right) 2^{-(\alpha+1)j} &\lesssim \int_{\mathbb{D}} \Psi_{p,\lambda}(|Dh(z)|) w_{\alpha,0}(z) dz \\ &\approx I_2(p, \alpha, \lambda, h). \end{aligned}$$

Analogous arguments as below (3.66) imply that there is a contradiction under the above assumption. \square

Proof of Theorem 1.1. Let λ_Ω be the internal distance and $|\cdot|$ be the Euclidean distance. As the proof of [9, Theorem 1] shows that there exist a bi-Lipschitz mapping $g : (\mathbb{S}^1, |\cdot|) \rightarrow (\partial\Omega, \lambda_\Omega)$ and a diffeomorphic bi-Lipschitz extension $\tilde{g} : (\mathbb{D}, |\cdot|) \rightarrow (\Omega, \lambda_\Omega)$ of g . Let $h = \tilde{g} \circ P[g^{-1} \circ \varphi]$. Then $h : \mathbb{D} \rightarrow \Omega$ is a diffeomorphic extension of φ . Moreover

$$I_1(p, \alpha, \lambda, h) \approx I_1(p, \alpha, \lambda, P[g^{-1} \circ \varphi]), \quad I_2(p, \alpha, \lambda, h) \approx I_2(p, \alpha, \lambda, P[g^{-1} \circ \varphi]),$$

$$\mathcal{U}(p, \alpha, \lambda, \varphi) \approx \mathcal{U}(p, \alpha, \lambda, g^{-1} \circ \varphi), \quad \mathcal{V}(p, \alpha, \lambda, \varphi) \approx \mathcal{V}(p, \alpha, \lambda, g^{-1} \circ \varphi).$$

Hence Theorem 1.1 (1) and (2) follow from Theorem 3.1. By Lemma 3.7, we complete the proof of Theorem 1.1. \square

4 Examples

In this section, we give examples related to Theorem 3.1 (2). We first decompose $[0, 1]$. For a given $s \in (0, +\infty)$, let

$$(4.1) \quad j_n^s = \lfloor 2^{\frac{n}{s}} \rfloor$$

be the largest integer less than $2^{n/s}$. There is $n_0^s \geq 1$ such that

$$(4.2) \quad 2^{-2-j_n^s} \geq 2^{-j_{n+1}^s} \quad \text{and} \quad 2^{-j_n^s} \leq 4^{-n} \quad \forall n \geq n_0^s - 1.$$

Step 1. Let

$$I_1 = I_{1,1} = (a_{1,1}, a_{1,2}) \quad \text{where} \quad a_{1,1} = 4^{-1} \quad \text{and} \quad a_{1,2} = 1 - 4^{-1}.$$

Renumber the elements in $T_1 = \{0, 1\} \cup \partial I_1$ as $\{b_{1,i_1} : i_1 = 1, \dots, 4\}$ such that $b_{1,i_1'} < b_{1,i_1''}$ if $i_1' < i_1''$.

Step 2. Let

$$I_{2,1} = (b_{1,1} + 4^{-2}, b_{1,2} - 4^{-2}) \text{ and } I_{2,2} = (b_{1,3} + 4^{-2}, b_{1,4} - 4^{-2}).$$

Set $I_2 = \cup_{i=1}^2 I_{2,i}$, and renumber the elements in $T_2 = T_1 \cup \partial I_2$ as $\{b_{2,i_2} : i_2 = 1, \dots, 8\}$ such that $b_{2,i'_2} < b_{2,i''_2}$ if $i'_2 < i''_2$.

After Step (n-1), we have $\{I_{n-1,k_{n-1}} : k_{n-1} = 1, \dots, 2^{n-2}\}$, $I_{n-1} = \cup_{k_{n-1}=1}^{2^{n-2}} I_{n-1,k_{n-1}}$ and $T_{n-1} := T_{n-2} \cup \partial I_{n-1} = \{b_{n-1,i_{n-1}} : i_{n-1} = 1, \dots, 2^n\}$ where $b_{n-1,i'_{n-1}} < b_{n-1,i''_{n-1}}$ if $i'_{n-1} < i''_{n-1}$. In the following Step n, set

$$(4.3) \quad I_{n,k_n} := (b_{n-1,2k_n-1} + 4^{-n}, b_{n-1,2k_n} - 4^{-n}) \quad \text{for } k_n = 1, \dots, 2^{n-1}.$$

and $I_n = \cup_{k_n=1}^{2^{n-1}} I_{n,k_n}$. After renumbering the elements in $T_n = T_{n-1} \cup \partial I_n$ as above, we can proceed to Step (n+1). Moreover we must replace I_{n,k_n} in (4.3) by

$$(4.4) \quad I_{n,k_n} = (b_{n-1,2k_n-1} + 2^{-j_n^s}, b_{n-1,2k_n} - 2^{-j_n^s})$$

whenever $n \geq n_0^s$. Let $I = \cup_{n=1}^{\infty} I_n$ and $R = [0, 1] \setminus I$. Then $R \neq \emptyset$. We finally decompose $[0, 1]$ as

$$(4.5) \quad R \cup I.$$

We next give an estimate on the length of I_{n,k_n} . Since $\mathcal{L}^1(I_{n,k_n}) = 2^{-j_{n-1}} - 2^{1-j_n}$ for all $n \geq n_0 + 1$ and $k_n \in \{1, \dots, 2^{n-1}\}$, by the first inequality in (4.2) we have that

$$(4.6) \quad \mathcal{L}^1(I_{n,k_n}) \geq 2^{-1-j_{n-1}}.$$

for all $n \geq n_0 + 1$ and $k_n \in \{1, \dots, 2^{n-1}\}$. When $n = n_0$, from (4.4) and the second estimate in (4.2) we have that

$$(4.7) \quad \mathcal{L}^1(I_{n,k_n}) = 4^{1-n_0} - 2^{1-j_{n_0}} \geq 4^{-n_0+1/2} > 4^{-n_0}$$

for all $k_n = 1, \dots, 2^{n-1}$. Whenever $1 \leq n \leq n_0 - 1$ and $k_n \in \{1, \dots, 2^{n-1}\}$, we have $\mathcal{L}^1(I_{n,k_n}) = 4^{-n}$. Let $C_1(s) = \min\{2^{j_{n-1}-2n} : 1 \leq n \leq n_0\}$. Then

$$(4.8) \quad \mathcal{L}^1(I_{n,k_n}) \geq C_1(s)2^{-j_{n-1}}$$

for all $1 \leq n \leq n_0$ and $k_n \in \{1, \dots, 2^{n-1}\}$. By (4.6), (4.7) and (4.8), we obtain that there is a constant $C(s) > 0$ such that

$$(4.9) \quad \mathcal{L}^1(I_{n,k_n}) \geq C(s)2^{-j_{n-1}}$$

for all $n \in \mathbb{N}$ and $k_n \in \{1, \dots, 2^{n-1}\}$.

Define

$$(4.10) \quad f_{n,s}^1(x) = \sum_{k_n=1}^{2^{n-1}} \frac{2k_n - 1}{2^n} \chi_{I_{n,k_n}}(x) \text{ and } f_s^1(x) = \sum_{n=1}^{+\infty} f_{n,s}^1(x).$$

For any $x \in R$ and any $n \geq n_0^s$, there is $b_n \in \partial I_n$ such that $|b_n - x| = \inf_{b \in \partial I_n} |b - x|$. By (4.4) and (4.10), we have that

$$|b_n - x| \leq 2^{-j_n} \text{ and } |f_s^1(b_{n+1}) - f_s^1(b_n)| < 2^{-n-1}.$$

It follows that $\lim_{n \rightarrow +\infty} b_n = x$ and $\{f_s^1(b_n)\}$ is a Cauchy sequence. Therefore

$$(4.11) \quad f_s(x) = \begin{cases} f_s^1(x) & \text{if } x \in I, \\ \lim_{n \rightarrow +\infty} f_s^1(b_n) & \text{if } x \in R. \end{cases}$$

is a well-defined function on $[0, 1]$.

Proposition 4.1. *Let f_s be as in (4.11) with $s \in (0, +\infty)$. Then $f_s(0) = 0$, $f_s(1) = 1$ and f_s is increasing on $[0, 1]$. Moreover there is a constant $C(s) > 0$ such that*

$$(4.12) \quad |f(x) - f(y)| \log^s(|x - y|^{-1}) \leq C(s)$$

for all $x, y \in [0, 1]$ with $x \neq y$.

Proof. By (4.11), we have that $f_s(0) = \lim_{n \rightarrow \infty} f_s^1(2^{-j_n}) = \lim_{n \rightarrow \infty} 2^{-n} = 0$. Analogously $f_s(1) = 1$.

We next prove the monotonicity of f_s . Let $x_1 \in [0, 1]$, $x_2 \in [0, 1]$ with $x_1 \leq x_2$. If $x_1 \in I_{n,k'_n}$ and $x_2 \in I_{n,k''_n}$ with $k'_n \leq k''_n$, from (4.11) we have that

$$(4.13) \quad f_s(x_1) \leq f_s(x_2).$$

Assume $x_1 \in I_{n_1,k_{n_1}}$ and $x_2 \in I_{n_2,k_{n_2}}$ with $n_1 \neq n_2$. Let $q = |n_2 - n_1|$. If $n_1 < n_2$, from the construction of $\{I_{n,k_n}\}$ we have that $k_{n_2} \geq 2^q(k_{n_1} - 1) + 2^{q-1} + 1$. It then follows from (4.10) that

$$(4.14) \quad f_s(x_2) \geq \frac{2(2^q(k_{n_1} - 1) + 2^{q-1} + 1) - 1}{2^{n_1} 2^q} > f_s(x_1).$$

If $n_2 < n_1$, from the construction of $\{I_{n,k_n}\}$ we have that

$$k_{n_2} \geq \begin{cases} \left\lceil \frac{k_{n_1}}{2^q} \right\rceil + 1 & \text{if } 0 \leq \frac{k_{n_1}}{2^q} - \left\lceil \frac{k_{n_1}}{2^q} \right\rceil \leq 1/2, \\ \left\lceil \frac{k_{n_1}}{2^q} \right\rceil + 2 & \text{if } 1/2 < \frac{k_{n_1}}{2^q} - \left\lceil \frac{k_{n_1}}{2^q} \right\rceil < 1. \end{cases}$$

It follows that

$$(4.15) \quad 2k_{n_2} - 1 \geq 2\left(\frac{k_{n_1}}{2^q} + 1/2\right) - 1 = 2\frac{k_{n_1}}{2^q} \quad \text{if } 0 \leq \frac{k_{n_1}}{2^q} - \left\lfloor \frac{k_{n_1}}{2^q} \right\rfloor \leq 1/2$$

and

$$(4.16) \quad 2k_{n_2} - 1 \geq 2\left(\frac{k_{n_1}}{2^q} + 1\right) - 1 = 2\frac{k_{n_1}}{2^q} + 1 \quad \text{if } 1/2 < \frac{k_{n_1}}{2^q} - \left\lfloor \frac{k_{n_1}}{2^q} \right\rfloor < 1.$$

By combining (4.15) with (4.16), we deduce from (4.10) that

$$(4.17) \quad f_s(x_2) > f_s(x_1).$$

Assume $x_1 \in R$ and $x_2 \in I$. By (4.11), there is $\{b_n\} \subset \partial I$ such that $\lim_{n \rightarrow \infty} b_n = x_1$. Together with $x_1 < x_2$, it follows that $b_n < x_2$ whenever $n \gg 1$. Via the arguments for (4.13), (4.14) and (4.17), we have that

$$(4.18) \quad f_s^1(b_n) \leq f_s(x_2) \quad \forall n \gg 1.$$

By taking limit for (4.18), we have that

$$(4.19) \quad f_s(x_1) \leq f_s(x_2).$$

Assume either $x_1 \in I$ and $x_2 \in R$, or $x_1 \in R$ and $x_2 \in R$. Via analogous arguments as for (4.19), we can also prove $f_s(x_1) \leq f_s(x_2)$ at these two cases. By preceding arguments, we conclude that f_s is increasing on $[0, 1]$.

We next prove (4.12). Let $T_n = \{b_{n,i_n} : i_n = 1, \dots, 2^{n+1}\}$ with $n \in \mathbb{N}$ and $f_{i,s}^1$ be as in (4.10). For a given $n \in \mathbb{N}$, define

$$f_{n,s}^2(x) = \sum_{i=1}^{2^n} \left(\frac{2^{jn}}{2^n} (x - b_{n,2i-1}) + \frac{i-1}{2^n} \right) \chi_{[b_{n,2i-1}, b_{n,2i}]}(x),$$

$$(4.20) \quad f_{n,s}(x) = f_{n,s}^2(x) + \sum_{i=1}^n f_{i,s}^1(x).$$

Then $f_{n,s}$ is piecewise affine, increasing and continuous on $[0, 1]$. Furthermore we claim:

$$(i) \quad \lim_{n \rightarrow \infty} f_{n,s}(x_0) = f_s(x_0) \quad \text{for all } x_0 \in [0, 1],$$

(ii) there are constant $C(s) > 0$ and $N(s) > 0$ such that

$$\sup \{|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x - y|^{-1}) : x, y \in [0, 1] \text{ and } x \neq y\} \leq C(s)$$

for all $n \geq N(s)$.

If both (i) and (ii) hold, we can prove (4.12).

We first prove (i). Let $x_0 \in [0, 1]$. If $x_0 \in I$, without loss of generality we assume that $x_0 \in I_{n_0, k_{n_0}}$. From (4.11) and (4.20), we have that $f_n(x_0) = f(x_0)$ for all $n \geq n_0$. Therefore (i) holds. If $x_0 \in R$, from (4.11) there is $\{b_n\} \subset \partial I$ such that $\lim_{n \rightarrow \infty} b_n = x_0$ and $\lim_{n \rightarrow \infty} f_s^1(b_n) = f_s(x_0)$. Moreover by (4.20), we have that

$$|f_{n,s}(x_0) - f_s^1(b_n)| = |f_{n,s}(x_0) - f_{n,s}(b_n)| \leq 2^{-n}.$$

Together with $|f_{n,s}(x_0) - f_s(x_0)| \leq |f_{n,s}(x_0) - f_s^1(b_n)| + |f_s^1(b_n) - f_s(x_0)|$, we have that (i) also holds at this case.

We next prove (ii). Given $n \geq 1$, $x \in [0, 1]$ and $y \in [0, 1]$ with $x < y$, set

$$k_n(x, y) = \#\{I_{m, k_m} : I_{m, k_m} \subset [x, y] \text{ for } m = 1, \dots, n \text{ and } k_m = 1, \dots, 2^{m-1}\}.$$

Then $0 \leq k_n(x, y) \leq 2^n - 1$.

Assume $k_n(x, y) = 0$. If $x \in \cup_{m=1}^n I_m$, there are $m \in \{1, \dots, n\}$ and $k_m \in \{1, \dots, 2^{m-1}\}$ such that $x \in I_{m, k_m}$. For the location of y , possibly we have that

$$(4.21) \quad y \in I_{m, m_k}, \quad y \in I_{m, m_k+1}, \quad \text{or } y \in [0, 1] \setminus (\cup_{m=1}^n I_m).$$

If $y \in I_{m, m_k}$, by (4.20) we have that

$$f_{n,s}(x) = f_{n,s}(y) \quad \forall n \geq m.$$

If $y \in I_{m, m_k+1}$, then $|x - y| \geq 2^{-j_n}$. It follows from (4.20) that

$$(4.22) \quad |f_{n,s}(x) - f_{n,s}(y)| \log^s(|x - y|^{-1}) \leq 2^{-n} \log^s(2^{j_n}) < 1.$$

If $y \in [0, 1] \setminus (\cup_{m=1}^n I_m)$, there is $x_0 \in [x, y] \cap T_n$ such that

$$(4.23) \quad 0 < y - x_0 < 2^{-j_n} \text{ and } f_{n,s}(x) = f_{n,s}(x_0).$$

Since there is $n_1^s > 0$ such that $\log(2^{j_{n_1^s}}) - s > 0$, we have that

$$(4.24) \quad t \log^s(t^{-1}) \leq 2^{-j_n^s} \log^s(2^{j_n^s}) < 2^{-j_n}$$

for all $n \geq n_1^s$ and every $t \in (0, 2^{-j_n^s}]$. By (4.20), (4.23) and (4.24), we then have that

$$(4.25) \quad \begin{aligned} |f_{n,s}(x) - f_{n,s}(y)| \log^s(|x - y|^{-1}) &\leq \frac{|f_{n,s}(x_0) - f_{n,s}(y)|}{|x_0 - y|} |x_0 - y| \log^s(|x_0 - y|^{-1}) \\ &< \frac{2^{j_n}}{2^n} \frac{2^n}{2^{j_n}} = 1 \end{aligned}$$

whenever $n \geq N(s) := \max\{n_0^s, n_1^s\}$. If $x \in [0, 1] \setminus (\cup_{m=1}^n I_m)$, for the location of y we possibly have that

$$y \in [0, 1] \setminus (\cup_{m=1}^n I_m), \quad y \in \cup_{m=1}^n I_m.$$

If $y \in [0, 1] \setminus (\cup_{m=1}^n I_m)$, then $0 < y - x < 2^{-j_n}$. By (4.20) and (4.24) we have that

$$(4.26) \quad |f_{n,s}(x) - f_{n,s}(y)| \log^s(|x - y|^{-1}) = \frac{2^{j_n}}{2^n} |x - y| \log^s(|x - y|^{-1}) < 1$$

for all $n \geq N(s)$. If $y \in \cup_{m=1}^n I_m$, by analogous arguments as for (4.25) we have that

$$(4.27) \quad |f_{n,s}(y) - f_{n,s}(x)| \log^s(|x - y|^{-1}) < 1$$

for all $n \geq N(s)$. By (4), (4.22), (4.25), (4.26) and (4.27), we conclude that

$$(4.28) \quad |f_{n,s}(y) - f_{n,s}(x)| \log^s(|x - y|^{-1}) < 1$$

for all $n \geq N(s)$ and $k_n(x, y) = 0$.

Assume $k_n(x, y) \in \{1, \dots, 2^n - 1\}$. Define

$$x' = \inf\{e \in I_{m,k_m} : I_{m,k_m} \subset [x, y] \text{ for } m = 1, \dots, n \text{ and } k_m = 1, \dots, 2^{m-1}\}$$

and

$$y' = \sup\{e \in I_{m,k_m} : I_{m,k_m} \subset [x, y] \text{ for } m = 1, \dots, n \text{ and } k_m = 1, \dots, 2^{m-1}\}.$$

If $k_n(x, y) = 1$, by (4.20) we have that

$$(4.29) \quad f_{n,s}(x') = f_{n,s}(y').$$

If $2^m \leq k_n(x, y) \leq 2^{m+1} - 1$ for $m = 1, \dots, n - 1$, by (4.5), (4.9) and (4.20) we have that

$$|x - y| \geq \mathcal{L}^1(I_{n-m, k_{n-m}}) \geq C(s) 2^{-j_{n-m}^s}$$

and

$$|f_{n,s}(x') - f_{n,s}(y')| = \frac{2 + \dots + 2^m}{2^n} < 2^{m+1-n}.$$

Whenever $n \geq n_0^s + 1$, it follows from (4.1) that

$$(4.30) \quad \begin{aligned} |f_{n,s}(x') - f_{n,s}(y')| \log^s(|x - y|^{-1}) &\leq 2^{m+1-n} \log^s(C^{-1} 2^{j_{n-m-1}^s}) \\ &\leq C(s) 2^{m+1-n} j_{n-m-1}^s < C(s). \end{aligned}$$

Notice that there are two cases for the location of x

$$x \in (x' - 2^{-j_n}, x'], \quad x \in \cup_{m=1}^n I_m.$$

If $x \in (x' - 2^{-j_n}, x']$, by analogous arguments as for (4.26) we have that

$$(4.31) \quad |f_{n,s}(x) - f_{n,s}(x')| \log^s(|x - y|^{-1}) < 1 \quad \text{whenever } n \geq N(s).$$

If $x \in \cup_{m=1}^n I_m$, same arguments as (4.25) imply (4.31). Analogously, we have that

$$(4.32) \quad |f_{n,s}(y') - f_{n,s}(y)| \log^s(|x - y|^{-1}) < 1 \quad \text{whenever } n \geq N(s).$$

Since

$$\begin{aligned} &|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x - y|^{-1}) \\ &= (|f_{n,s}(x) - f_{n,s}(x')| + |f_{n,s}(x') - f_{n,s}(y')| + |f_{n,s}(y') - f_{n,s}(y)|) \log^s(|x - y|^{-1}), \end{aligned}$$

by (4.29), (4.30), (4.31) and (4.32) there is a constant $C(s) > 0$ such that

$$(4.33) \quad |f_{n,s}(x) - f_{n,s}(y)| \log^s(|x - y|^{-1}) \leq C(s)$$

whenever $n \geq N(s)$ and $k_n(x, y) \in \{1, \dots, 2^n - 1\}$. By (4.28) and (4.33), we finish the proof of (ii). \square

Let $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism. In the following we denote by $P[\varphi] : \mathbb{D} \rightarrow \mathbb{D}$ the harmonic extension of φ .

Example 4.2. For a given $p \in (1, 2)$, there is a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\mathcal{V}(p, p-2, 0, \varphi) < \infty$, $I_1(p, p-2, 0, P[\varphi]) = \infty$ and $I_2(p, p-2, 0, P[\varphi]) = \infty$.

Proof. We first introduce a class of self-homeomorphisms on \mathbb{S}^1 and their properties. Let f_s be as in (4.11) with $s \in (0, +\infty)$. Define

$$(4.34) \quad g_s(x) = \frac{f_s(x) + x}{2} \quad x \in [0, 1].$$

Then $g_s : [0, 1] \rightarrow [0, 1]$ is strictly increasing and continuous, i.e. g_s is homeomorphic. Moreover by (4.1), there is a constant $C(s) > 0$ such that

$$(4.35) \quad |g_s(x) - g_s(y)| \leq C(s) \log^{-s}(|x - y|^{-1})$$

for all $x, y \in [0, 1]$ with $x \neq y$. Let $\arg z \in (-\pi, \pi]$ be the principal value of the argument z . Define

$$(4.36) \quad \varphi_s(z) = \exp \left(i2\pi \left[g_s \left(\frac{\arg z}{2\pi} \right) - g_s \left(\frac{1}{2} \right) + \frac{1}{2} \right] \right) \quad z \in \mathbb{S}^1.$$

Then $\varphi_s : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is homeomorphic and $\varphi(e^{i\pi}) = e^{i\pi}$. Next we prove that

$$(4.37) \quad |\varphi_s(z_1) - \varphi_s(z_2)| \lesssim \log^{-s}(|z_1 - z_2|^{-1})$$

for all $z_1, z_2 \in \mathbb{S}^1$ with $z_1 \neq z_2$. Let $\Gamma(z_1, z_2)$ be the arc in \mathbb{S}^1 joining z_1 to z_2 with smaller length. Denote by $\ell(\Gamma(z_1, z_2))$ the length of $\Gamma(z_1, z_2)$. In order to prove (4.37), it is enough to consider the case $\ell(\Gamma(z_1, z_2)) \ll 1$. If $e^{i\pi} \notin \Gamma(z_1, z_2)$, we have that

$$|\arg z_1 - \arg z_2| \approx |z_1 - z_2| \text{ and } \left| g_s \left(\frac{\arg z_1}{2\pi} \right) - g_s \left(\frac{\arg z_2}{2\pi} \right) \right| \approx |\varphi_s(z_1) - \varphi_s(z_2)|$$

whenever $\ell(\Gamma(z_1, z_2)) \ll 1$. Together with (4.35), we then have that

$$(4.38) \quad \begin{aligned} |\varphi_s(z_1) - \varphi_s(z_2)| &\approx \left| g_s \left(\frac{\arg z_1}{2\pi} \right) - g_s \left(\frac{\arg z_2}{2\pi} \right) \right| \\ &\lesssim \log^{-s}(|\arg z_1 - \arg z_2|^{-1}) \approx \log^{-s}(|z_1 - z_2|^{-1}). \end{aligned}$$

If $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi(z_1), \varphi(e^{i\pi}))) > \ell(\Gamma(\varphi(e^{i\pi}), \varphi(z_2)))$, there is $z_0 \in \Gamma(z_1, e^{i\pi})$ such that

$$(4.39) \quad |\varphi_s(z_1) - \varphi_s(z_2)| \lesssim |\varphi_s(z_1) - \varphi_s(z_0)|.$$

Same arguments as for (4.38) imply that

$$(4.40) \quad |\varphi_s(z_1) - \varphi_s(z_0)| \lesssim \log^{-s}(|z_1 - z_0|^{-1}) \lesssim \log^{-s}(|z_1 - z_2|^{-1}).$$

Combining (4.39) with (4.40) therefore implies that (4.37) holds when $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi_s(z_1), \varphi_s(e^{i\pi}))) > \ell(\Gamma(\varphi(e^{i\pi}), \varphi_s(z_2)))$. Analogously, we can prove that (4.37) holds when $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi_s(z_1), \varphi_s(e^{i\pi}))) \leq \ell(\Gamma(\varphi_s(e^{i\pi}), \varphi_s(z_2)))$.

Let $p \in (1, 2)$. There is $s \in (1, +\infty)$ such that $p - 1 < 1/s < 1$. Based on this s , we obtain a homeomorphism $\varphi = \varphi_s : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, where φ_s is from (4.36). By

Jensen's inequality and (4.37), we have that

$$\begin{aligned}\mathcal{V}(p, p-2, 0, \varphi) &= \int_{\mathbb{S}^1} \left(\int_{\mathbb{S}^1} \log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|^{-1} |d\eta| \right)^{p-1} |d\xi| \\ &\leq \left(\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|^{-1} |d\eta| |d\xi| \right)^{p-1} \\ &\lesssim \left(\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} |\xi - \eta|^{-\frac{1}{s}} |d\eta| |d\xi| \right)^{p-1} < +\infty.\end{aligned}$$

Let n_0^s be as in (4.2) with s chosen above. For any $n \geq n_0^s$ and any $j_n < j \leq j_{n+1}$, by (4.34) and (4.11) we have that

$$\begin{aligned}(4.41) \quad \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p &= 2\pi \sum_{k=1}^{2^j} \mathcal{L}^1(g_s([(k-1)2^{-j}, k2^{-j}]))^p \\ &\gtrsim \sum_{k=1}^{2^j} (f_s(k2^{-j}) - f_s((k-1)2^{-j}))^p = 2^{n(n+1)(1-p)}.\end{aligned}$$

Notice that $j_{n+1} - j_n \approx 2^{n/s}$ whenever $n \geq n_0$. We then derive from (4.41) that

$$\mathcal{E}_1(p, p-2, 0, \varphi) \geq \sum_{n=n_0}^{+\infty} \sum_{j_n < j \leq j_{n+1}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \gtrsim \sum_{n=n_0}^{+\infty} 2^{n(n+1)(1-p)} = +\infty.$$

By Lemma 3.2, Lemma 3.3 and Lemma 3.4, it follows that $I_1(p, p-2, 0, P[\varphi]) = \infty$ and $I_2(p, p-2, 0, P[\varphi]) = \infty$. \square

Example 4.3. For a given $p \in (2, +\infty)$, there is a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\mathcal{V}(p, p-2, 0, \varphi) = \infty$, $I_1(p, p-2, 0, P[\varphi]) < \infty$ and $I_2(p, p-2, 0, P[\varphi]) < \infty$.

Proof. Since $p \in (2, +\infty)$, there is $s \in (0, 1)$ such that $p-1 > 1/s > 1$. Based on this chosen s , we obtain a homeomorphism $\varphi = \varphi_s : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, where φ_s is from (4.36). In order to prove $\mathcal{V}(p, p-2, 0, \varphi) = \infty$, by Jensen's inequality it suffices to prove that

$$(4.42) \quad \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|^{-1} |d\eta| |d\xi| = +\infty.$$

For any $\sigma \in \mathbb{S}^1$ and $\tau \in \mathbb{S}^1$, let $\ell(\sigma, \tau)$ be the arc length of the shorter arc in \mathbb{S}^1 joining σ and τ . Let n_0^s be from (4.2) with s chosen above. For any $n \geq n_0^s$, set

$$\Gamma_n = \{(\xi, \eta) \in \mathbb{S}^1 \times \mathbb{S}^1 : \pi 2^{1-j_{n+1}} < \ell(\varphi^{-1}(\xi), \varphi^{-1}(\eta)) \leq \pi 2^{1-j_n}\}.$$

We have that

$$\begin{aligned}
\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|^{-1} |d\eta| |d\xi| &\geq \sum_{n=n_0}^{+\infty} \int_{\Gamma_n} \log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|^{-1} |d\eta| |d\xi| \\
(4.43) \qquad \qquad \qquad &\gtrsim \sum_{n=n_0}^{+\infty} j_n \int_{\Gamma_n} |d\eta| |d\xi|.
\end{aligned}$$

Given $n \geq n_0^s$ and $k = 1, \dots, 2^n$, let

$$\Gamma'_{n,k} = \exp(i2\pi[b_{n,2k-1}, 2^{-j_{n+1}} + b_{n,2k-1}]),$$

$$\Gamma''_{n,k} = \exp(i2\pi[2^{-j_n} - 2^{-j_{n+1}} + b_{n,2k-1}, 2^{-j_n} + b_{n,2k-1}]).$$

For any $\xi \in \varphi(\Gamma'_{n,k})$ and $\eta \in \varphi(\Gamma''_{n,k})$, we have that

$$(4.44) \qquad 2\pi(2^{-j_n} - 2^{1-j_{n+1}}) \leq \ell(\varphi^{-1}(\xi), \varphi^{-1}(\eta)) \leq \pi \cdot 2^{1-j_n}.$$

Notice that by (4.2) we have that $2^{-j_{n+1}} < 2^{-j_n} - 2^{1-j_{n+1}}$ whenever $n \geq n_0^s$. It then follows from (4.44) that

$$(4.45) \qquad \varphi(\Gamma'_{n,k}) \times \varphi(\Gamma''_{n,k}) \subset \Gamma_n$$

for all $n \geq n_0^s$ and all $k = 1, \dots, 2^n$. Moreover from (4.36), (4.34) and (4.11), it follows that

$$\begin{aligned}
\ell(\varphi(\Gamma'_{n,k})) &= 2\pi \mathcal{L}^1(g([b_{n,2k-1}, 2^{-j_{n+1}} + b_{n,2k-1}])) \\
(4.46) \qquad \qquad &\geq \pi(f_s(2^{-j_{n+1}}) - f_s(0)) = \pi 2^{-n-1}.
\end{aligned}$$

for all $n \geq n_0^s$ and all $k = 1, \dots, 2^n$. Similarly

$$(4.47) \qquad \ell(\varphi(\Gamma''_{n,k})) \geq \pi 2^{-n-1}.$$

Since $(\varphi(\Gamma'_{n,k}) \times \varphi(\Gamma''_{n,k})) \cap (\varphi(\Gamma'_{n,j}) \times \varphi(\Gamma''_{n,j})) = \emptyset$ for all $n \geq n_0^s$ and $k, j \in \{1, \dots, 2^n\}$ with $k \neq j$, it follows (4.45), (4.46) and (4.47) that

$$(4.48) \qquad \int_{\Gamma_n} |d\eta| |d\xi| \geq \sum_{k=1}^{2^n} \int_{\varphi(\Gamma'_{n,k}) \times \varphi(\Gamma''_{n,k})} |d\xi| |d\eta| \geq \pi^2 2^{-n-2}$$

for all $n \geq n_0^s$. Combining (4.43) with (4.48) hence implies that

$$\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|^{-1} |d\eta| |d\xi| \gtrsim \sum_{n=n_0}^{+\infty} \frac{j_n}{2^n} \approx \sum_{n=n_0}^{+\infty} \frac{2^{\frac{n}{s}}}{2^n} = +\infty.$$

Therefore (4.42) is complete.

For any $n \geq n_0$ and $j_n < j \leq j_{n+1}$, by (4.36), (4.34), (4.11) and Jensen's inequality we have that

$$\begin{aligned}
\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p &= 2\pi \sum_{k=1}^{2^j} \mathcal{L}^1(g_s([(k-1)2^{-j}, k2^{-j}]))^p \\
&\lesssim \sum_{k=1}^{2^j} (f_s(k2^{-j}) - f_s((k-1)2^{-j}))^p + \sum_{k=1}^{2^j} 2^{-pj} \\
(4.49) \qquad &= 2^{(1-p)(n+1)} + 2^{(1-p)j}
\end{aligned}$$

Notice $j_{n+1} - j_n \approx 2^{n/s}$ whenever $n \geq n_0^s$. We then derive from (4.49) that

$$\begin{aligned}
\sum_{j=j_{n_0+1}}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p &\lesssim \sum_{n=n_0}^{+\infty} \sum_{j_n < j \leq j_{n+1}} 2^{(1-p)(n+1)} + \sum_{j=j_{n_0+1}}^{+\infty} 2^{(1-p)j} \\
(4.50) \qquad &\approx \sum_{n=n_0}^{+\infty} 2^{n(1-p+\frac{1}{s})} + \sum_{j=j_{n_0+1}}^{+\infty} 2^{(1-p)j} < +\infty.
\end{aligned}$$

Since $\sum_{j=1}^{j_{n_0}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p$ is finite, it follows from (4.50) that $\mathcal{E}_1(p, p-2, 0, \varphi) < +\infty$. Moreover by Lemma 3.3 and Lemma 3.4 we have that $I_1(p, p-2, 0, P[\varphi]) < +\infty$ and $I_2(p, p-2, 0, P[\varphi]) < +\infty$. \square

Example 4.4. *There is a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that both $I_1(p, \alpha, \lambda, P[\varphi]) < +\infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) < +\infty$ hold for all $p > 1$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. Moreover for any $p > 1$, there is a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $I_1(p, \alpha, \lambda, P[\varphi]) = \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) = \infty$ whenever either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$.*

Proof. Take $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ as the identity mapping. We have that

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) \approx \sum_{j=1}^{+\infty} 2^{j(p-2-\alpha)} j^\lambda 2^j (2^{1-j}\pi)^p \approx \sum_{j=1}^{+\infty} 2^{-j(1+\alpha)} j^\lambda < +\infty$$

whenever $p > 1$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. Therefore by Lemma 3.3 and Lemma 3.4 both $I_1(p, \alpha, \lambda, P[\varphi])$ and $I_2(p, \alpha, \lambda, P[\varphi])$ are finite now.

For a given $p > 1$, set j_n in (4.1) as $\lceil e^{2^{n(p-1)}} \rceil$. There is $n_0 \geq 1$ such that (4.2) holds for all $n \geq n_0 - 1$. By following the arguments for (4.5), we have f as in (4.11). Moreover by same arguments as in the proof of Proposition 4.1, there is a constant $C > 0$ depending only on p such that

$$|f(x) - f(y)| \log^{\frac{1}{p-1}} \log(|x-y|^{-1}) \leq C$$

for all $x, y \in [0, 1]$ with $x \neq y$. As in (4.36), we finally obtain a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. For any $n \geq n_0$ and $j_n < j \leq j_{n+1}$, by analogous arguments for (4.41) we have that

$$(4.51) \quad \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \gtrsim 2^{n(1-p)}.$$

Notice that $\sum_{j_n < j \leq j_{n+1}} j^{-1} \approx \log j_{n+1} - \log j_n \gtrsim 2^{n(p-1)}$ for all $n \geq n_0$. For any $\lambda \in [-1, +\infty)$ it then follows from (4.51) that

$$(4.52) \quad \begin{aligned} \mathcal{E}_1(p, p-2, \lambda, \varphi) &\geq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p j^{-1} \geq \sum_{n=n_0}^{+\infty} \sum_{j_n < j \leq j_{n+1}} j^{-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \\ &\gtrsim \sum_{n=n_0}^{+\infty} 2^{n(p-1)} \cdot 2^{n(1-p)} = +\infty. \end{aligned}$$

For any $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$, we have that $2^{j(p-2-\alpha)} j^\lambda \gtrsim j^{-1}$ whenever $j \gg 1$. Without loss of generality, we assume that $2^{j(p-2-\alpha)} j^\lambda \gtrsim j^{-1}$ for all $n \geq n_0$ and $j_n < j \leq j_{n+1}$. Hence from (4.52) we have that

$$(4.53) \quad \begin{aligned} \mathcal{E}_1(p, \alpha, \lambda, \varphi) &\geq \sum_{n=n_0}^{+\infty} \sum_{j_n < j \leq j_{n+1}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p 2^{j(p-2-\alpha)} j^\lambda \\ &\gtrsim \sum_{n=n_0}^{+\infty} \sum_{j_n < j \leq j_{n+1}} \frac{1}{j} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p = +\infty \end{aligned}$$

for all $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$. By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we conclude from (4.52) and (4.53) that for any $p > 1$ there is a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $I_1(p, \alpha, \lambda, P[\varphi]) = \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) = \infty$ whenever either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$. \square

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