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Trace Operators on Regular Trees

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Abstract: We consider different notions of boundary traces for functions in Sobolev spaces defined on regular trees and show that the almost everywhere existence of these traces is independent of the chosen definition of a trace.

Keywords: regular tree; trace operator; Newtonian space

MSC: 46E35, 31E05

1 Introduction

Let us begin with the classical setting. Consider the unit ball $B^n(0, 1)$ in the n-dimensional Euclidean space $\mathbb{R}^n$. If u belongs to the usual Sobolev space $W^{1,1}(B^n(0, 1))$ consisting of all integrable functions whose all first order distributional derivatives are also integrable over $B^n(0, 1)$, then u has a representative v for which the limit

$$\lim_{t\to 1} v(t\xi) \tag{1.1}$$

exists for almost every $\xi \in \partial B^n(0, 1)$. Here almost everywhere refers to the surface measure on $\partial B^n(0, 1)$. In this sense, u has a well defined trace almost everywhere on $\partial B^n(0, 1)$.

Towards a more constructive definition of a trace, let us extend u to a function $Eu \in W^{1,1}(\mathbb{R}^n)$. This is possible by classical extension theorems in [5, 24]. By the version of Lebesgue differentiation theorem for Sobolev functions [26, Section 5.14], the limit

$$\lim_{r\to 0} \frac{1}{m_n(B(x, r))} \int_{B(x,r)} Eu \, dm_n$$

exists for H^{n-1}-almost every x. Here m_n is the Lebesgue measure on $\mathbb{R}^n$ and H^{n-1} refers to the $(n-1)$-dimensional Hausdorff measure. It then follows from the (1,1)-Poincaré inequality that also

$$\lim_{r\to 0} \frac{1}{m_n(B(x, r)\cap B^n(0, 1))} \int_{B(x,r)\cap B^n(0,1)} u \, dm_n \tag{1.2}$$

exists for H^{n-1}-almost every x and also that, for almost every $\xi \in \partial B^n(0, 1)$ there is a value $Tu(\xi)$ for which

$$\lim_{r\to 0} \frac{1}{m_n(B(\xi, r)\cap B^n(0, 1))} \int_{B(\xi,r)\cap B^n(0,1)} |u(x) - Tu(\xi)| \, dm_n(x) = 0. \tag{1.3}$$

Thus we have three different possible traces, but it turns out that $Tu(\xi)$ coincides with the limits in (1.1) and (1.2) (for a suitable v) almost everywhere on $\partial B^n(0, 1)$. Moreover, by the (q, p)-Poincaré inequality (with

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$1 \le q < \infty$ when $p \ge n$ and $1 \le q \le \frac{pn}{n-p}$ when $1 \le p < n$), we may replace the term $|u(x) - Tu(\xi)|$ by $|u(x) - Tu(\xi)|^q$ in (1.3) if we assume that $u \in W^{1,p}(B^n(0,1))$. As usual, $W^{1,p}(B^n(0,1))$ requires p-integrability instead of integrability, both for the function and for all the first order distributional derivatives.

Let us next consider a weighted situation when $p > 1$. Suppose that $u \in W^{1,p}_{\mathrm{loc}}(B^n(0,1))$ and that

$$\int_{B^n(0,1)} |\nabla u(x)|^p \, w(x) \, dm_n(x) < \infty$$

for a positive weight function w. By again choosing a suitable representative v of u (with respect to m_n), one can check that v has a limit as in (1.1) for almost every ξ (with respect to the surface measure) provided that $w^{-1/(p-1)}$ is integrable over $B^n(0,1)$. This integrability condition is not necessary for the asserted existence of limits as seen by considering the weight w defined by setting $w(x) = |x|^{(p-1)n}$. If we replace the integrability assumption on $w^{-1/(p-1)}$ by the stronger requirement that w be a Muckenhoupt A_p-weight, then one can again use a Poincaré inequality to obtain analogs of (1.2) and (1.3) (with any power $1 \le q < p + \epsilon(w)$) and further that $Tu(\xi)$ can be chosen to be the limit from the analog of (1.1), see [4, Theorem 4.4].

There has been recent interest in establishing trace theorems for Sobolev-type functions in the setting of a metric measure space, see [16–18] (also the references therein). In this paper we consider the particular case of a weighted regular tree. Instead of giving the formal definition used in [2, 14, 15, 21, 22, 25], we give an equivalent definition in Section 2 below. Let us only give an intuitive description here. Our tree is a graph G that consists of a countable connected union of isometric copies of the unit interval $[0,1]$, distributed so that two given copies either intersect at a common vertex or do not intersect at all. We require that G comes with an integer $K \ge 1$ and a distinguished vertex, called the root 0, so that 0 is a vertex of K copies of $[0,1]$, and each other vertex is a vertex of $K+1$ such copies. We additionally ask that there are no loops in G : given two vertices, there is a unique collection of copies of the unit interval that connect these vertices. When $K = 1$, our regular tree is thus isometric to the interval $[0, \infty)$. The above being fixed, we call G a K-regular tree. Towards introducing weighted K-regular trees, we consider G as equipped with the natural path metric. Then any pair of points $x, y \in G$ are joined by a unique geodesic, denoted $[x, y]$. As usual, we define the boundary ∂G of G to consist of all the isometric embeddings of $[0, \infty)$ into G, with the requirement that the real number 0 maps to our root 0. Then our boundary points can be viewed as infinite geodesics starting from the root. We abuse notation and refer to the image of the embedding corresponding to $\xi \in \partial G$ by $[0, \xi)$. We equip ∂G with the natural probability measure ν as in Falconer [6] by distributing the unit mass uniformly on ∂G. Let $w, \lambda : [0, \infty) \to (0, \infty)$ be locally integrable functions. We define a measure μ and a metric d_λ on G by setting $\mu(A) = \int_A w(|z|) d_G(z)$, $d_\lambda(x, y) = \int_{[x,y]} \lambda(|z|) d_G(z)$, where $|z|$ is the path distance between 0 and z on G and $d_G(z)$ is the length element on G. See Section 2.1 for the precise definitions.

Given $1 \le p < \infty$, our space (G, d_λ, μ) is a metric measure space and hence one may define a Newtonian Sobolev space $N^{1,p}(G) := N^{1,p}(G, d_\lambda, \mu)$ based on upper gradients [9, 23]. As usual, we denote by $\dot{N}^{1,p}(G)$ the homogeneous version of $N^{1,p}(G)$.

Given $\xi \in \partial G$, we refer to points $x \in [0, \xi)$ by x_ξ. We begin with our analog of (1.1).

Definition 1.1. *Let G be a K-regular tree with metric d_λ and measure μ as above. Let f be a function defined on G. We define the arcwise trace of f at $\xi \in \partial G$ (along the corresponding geodesic), denoted by $T_R f(\xi)$, by setting*

$$T_R f(\xi) = \lim_{x_\xi \to \xi} f(x_\xi). \tag{1.4}$$

If the limit of (1.4) exists for ν-a.e $\xi \in \partial G$, then we say that the radial trace $T_R f$ exists.

We call $T_R f$ the radial trace since it is an analog of (1.1). The existence of a radial trace of a given function $f \in N^{1,p}(G)$ was studied in [2, 14, 15, 25]. In [14, Theorem 1.1-1.3], a characterization for the existence of $T_R f$ for all $f \in N^{1,p}(G)$ was given. In some special cases of metric d_λ and measure μ, $T_R f$ belongs to a Besov space, see [2, Theorem 6.1], [15, Theorem 1.1-1.4], [25, Theorem 1.1] for more details.

Let $x \in G$. Towards defining analogs of (1.2) and (1.3), we set

$$\Gamma_x := \{y \in G : x \in [0, y]\}.$$

Notice that Γ_x is also a K-regular tree if x is a vertex, obviously with root x.

Definition 1.2. *Let $1 \le q < \infty$ and G be a K-regular tree with metric d_λ and measure μ as above, with $\mu(G) < \infty$. Fix a function f defined on G. We say that the Lebesgue-point-type trace $T_L f$ of f on ∂G exists if*

$$T_L f(\xi) := \lim_{x_\xi \to \xi} \frac{1}{\mu(\Gamma_{x_\xi})} \int_{\Gamma_{x_\xi}} f(y)\, d\mu(y) \tag{1.5}$$

exists for ν-a.e $\xi \in \partial G$.
We say that the boundary trace of f of order q on ∂G exists if there is a function $T_q f : \partial G \to \mathbb{R}$ so that

$$\lim_{x_\xi \to \xi} \frac{1}{\mu(\Gamma_{x_\xi})} \int_{\Gamma_{x_\xi}} |f(y) - T_q f(\xi)|^q \, d\mu(y) = 0 \tag{1.6}$$

for ν-a.e $\xi \in \partial G$.

One can find versions of the two notions of traces in Definition 1.2 in literature under various names. We refer the readers to [7, Chapter 2], [19, Section 6.6], [20, Section 9.6],[26, Section 3.1] for discussions in the setting of Euclidean spaces, and [16–18] (also the references therein) for discussions in the setting of metric measure spaces. Notice that in the setting of a Muckenhoupt A_p-weight discussed above, the analogs of the traces $T_R f$, $T_L f$ and $T_q f$, $1 \le q \le p$, exist and actually coincide with each other almost everywhere on $\partial B^n(0, 1)$.

It is then natural to ask whether $T_R f$, $T_L f$, $T_q f$ exist (for suitable q) and coincide for a given function $f \in N^{1,p}(G, d_\lambda, \mu)$. Towards this, we recall a concept introduced in [14]. Let $1 \le p < \infty$. We set

$$R_1(\lambda, w) = \left\| \frac{\lambda(t)}{w(t) K^{j(t)}} \right\|_{L^\infty([0,\infty))} \tag{1.7}$$

and

$$R_p(\lambda, w) = \int_0^\infty \lambda(t)^{\frac{p}{p-1}} w(t)^{\frac{1}{1-p}} K^{\frac{j(t)}{1-p}}\, dt, \quad 1 < p < \infty \tag{1.8}$$

where $j(t)$ is the largest integer such that $j(t) \le |x| + 1$. Since we work with a fixed pair λ, w, we will usually refer to $R_p(\lambda, w)$ simply by R_p. One should view R_p as an analog of the isoperimetric profile of a Riemannian manifold in [11–13]. We assume in what follows that $\lambda^p w^{-1} \in L^{1/(p-1)}_{\text{loc}}([0, \infty))$ to make sure that the finiteness of R_p is a condition at infinity.

Our first result shows that the existence of any of $T_R f$, $T_L f$, $T_q f$, $1 \le q \le p$, for all $f \in N^{1,p}(G)$ is equivalent to the finiteness of R_p. Moreover, all these different traces of f coincide when $R_p < \infty$.

Theorem 1.3. *Let $1 \le p < \infty$ and G be a K-regular tree with metric d_λ and measure μ as above. Assume $\mu(G) < \infty$ and let $1 \le q \le p$. Then the following are equivalent:*

(i) $T_R f$ *exists for any* $f \in N^{1,p}(G)$.
(ii) $T_L f$ *exists for any* $f \in N^{1,p}(G)$.
(iii) $T_q f$ *exists for any* $f \in N^{1,p}(G)$.
(iv) $R_p < \infty$.

Moreover, if one of $T_R f$, $T_L f$, $T_q f$ exists for each $f \in N^{1,p}(G)$, then all of them exist and coincide ν-a.e on ∂G for a given f.

As a direct consequence of Theorem 1.3 we see that the existence of the trace operator T_q is independent of the value of $q \in [1, p]$. We do not know if one could even obtain this for all $q \in [1, p + \epsilon]$ for some $\epsilon > 0$ only depending on p, $R_p(\lambda, w)$, λ, w.

Based on the discussion in the beginning of our introduction, one should find Theorem 1.3 somewhat surprising since it does not seem possible to extend our functions to a larger underlying nice space and the finiteness of R_p should not, in general, imply the validity of Poincaré inequalities. In fact, the validity of

Poincaré inequalities under a doubling condition on (G, d_λ, μ) has very recently been characterized via a Muckenhoupt-type condition in [22]. The reason why we do not need a Poincaré inequality or a doubling measure and do not need to move to a representative when we consider T_R is basically that our space is locally one-dimensional.

Our second result deals with the coincidence of $N^{1,p}(G)$ and $\dot{N}^{1,p}(G)$. Here $\dot{N}^{1,p}(G)$ is the homogeneous version of $N^{1,p}(G)$.

Theorem 1.4. *Let $1 \le p < \infty$ and G be a K-regular tree with metric d_λ and measure μ with $\mu(G) < \infty$ as above. Suppose that $R_p < \infty$. Then $N^{1,p}(G) = \dot{N}^{1,p}(G)$.*

Consequently, Theorem 1.3 could alternatively be stated for $\dot{N}^{1,p}(G)$. In the case where $\mu(G) = \infty$, the homogeneous version of our Sobolev space is much larger than the non-homogeneous one. However, even under the assumption that $\mu(G) < \infty$, $R_p < \infty$ is not a necessary condition for $N^{1,p}(G) = \dot{N}^{1,p}(G)$. Example 3.8 in Section 3 shows that there exists a K-regular tree (G, d_λ, μ) so that $R_p = \infty$ and $\mu(G) < \infty$ but nevertheless $N^{1,p}(G) = \dot{N}^{1,p}(G)$.

The paper is organized as follows. In Section 2, we introduce K-regular trees and their boundaries, and Newtonian spaces. In Section 3, we give the proofs of Theorem 1.3 and Theorem 1.4.

Throughout this paper, the letter C (sometimes with a subscript) will denote positive constants that usually depend only on our space and may change at different occurrences; if C depends on $a, b, \ldots$ we write $C = C(a, b, \ldots)$. The notation $A \approx B$ means that there is a constant C such that $1/C \cdot A \le B \le C \cdot A$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a constant C such that $A \le C \cdot B$($A \ge C \cdot B$). For any function $f \in L^1_{\mathrm{loc}}(G)$ and any measurable subset $A \subset G$ of positive measure, we let $⨍_A f d\mu$ stand for $\frac{1}{\mu(A)} \int_A f d\mu$.

2 Preliminaries

2.1 Regular trees and their boundaries

A *graph* G is a pair (V, E), where V is a set of vertices and E is a set of edges. We call a pair of vertices $x, y \in V$ neighbors if x is connected to y by an edge. The degree of a vertex is the number of its neighbors. The graph structure gives rise to a natural connectivity structure. A *tree* G is a connected graph without cycles.

We call a tree G a *rooted tree* if it has a distinguished vertex called the *root*, which we will denote by 0. The neighbors of a vertex $x \in V$ are of two types: the neighbors that are closer to the root are called *parents* of x and all other neighbors are called *children* of x. Each vertex has a unique parent, except for the root itself that has none.

A *K-ary tree* G is a rooted tree such that each vertex has exactly K children. Then all vertices except the root of G have degree $K + 1$, and the root has degree K. We say that a tree G is *K-regular* if it is a K-ary tree for some $K \ge 1$.

Let G be a K-regular tree with a set of vertices V and a set of edges E for some $K \ge 1$. For simplicity of notation, we let $X = V \cup E$ and call it a K-regular tree. The geodesic connecting $x, y \in X$ is denoted by $[x, y]$. For any $x, y \in X$, let $|x - y|$ be the metric graph distance from x to y, that is, the metric graph length of the geodesic $[x, y]$ given by

$$|x - y| = l_G([x, y]) = \int_{[x,y]} d_G.$$

We denote by $|x|$ the metric graph distance from the root 0 to x. Then the metric graph distance between two vertices is the number of edges needed to connect them. Given a curve γ, we say that γ is an infinite geodesic in X if γ is a simple curve and $l_G(\gamma) = \infty$.

On our K-regular tree X, we define a measure μ and a metric d_λ by setting

$$d\mu(x) = w(|x|)\, d_G(x), \quad d_\lambda(x) = \lambda(|x|)\, d_G(x),$$

where $\lambda, w : [0, \infty) \to (0, \infty)$ are fixed with $\lambda, w \in L^1_{\mathrm{loc}}([0, \infty))$. For any two points $x, y \in X$, the distance between x and y, denoted $d_\lambda(x, y)$, is

$$d_\lambda(x, y) = \int_{[x,y]} d_\lambda = \int_{[x,y]} \lambda(|z|) d_G(z)$$

where $[x, y]$ is the unique geodesic between x, y. In particular, if $x \in [0, y]$ then the distance between $[x, y]$ is given by

$$d_\lambda(x, y) = \int_{|x|}^{|y|} \lambda(t) dt.$$

For any subset $A \subset X$, the measure of A, denoted $\mu(A)$, is

$$\mu(A) = \int_A d\mu = \int_A w(|x|) d_G(x).$$

The measure of our K-regular tree is

$$\mu(X) = \int_X d\mu = \int_0^\infty w(t) K^{j(t)} dt$$

where $j(t)$ is the largest integer such that $j(t) \le t + 1$.

We abuse notation and let $w(x)$ and $\lambda(x)$ denote $w(|x|)$ and $\lambda(|x|)$, respectively, for any $x \in X$, if there is no danger of confusion. We refer the interested readers to [14, 21, Section 2] for a discussion on this metric and this measure.

A tree is the quintessential Gromov hyperbolic space, and hence we can consider the visual boundary of the tree as in Bridson-Haefliger [3]. We define the boundary of our K-regular tree X, denoted ∂X, as the collection of all infinite geodesics in X starting at the root 0. Given two points $\xi, \zeta \in \partial X$, there is an infinite geodesic (ξ, ζ) in X connecting ξ and ζ.

To avoid confusion, points in X are denoted by Latin letters such as x, y and z, while for points in ∂X we use Greek letters such as ξ, ζ and η.

Given $z \in X$, we define the subtree with respect to the root z, denoted Γ_z, by setting

$$\Gamma_z := \{y \in X : z \in [0, y]\}.$$

Let $\partial\Gamma_z$ be the collection of $\xi \in \partial X$ with respect to all the infinite geodesics (in X) containing z and starting at the root 0. Then

$$\partial\Gamma_z := \{\xi \in \partial X : z \in [0, \xi)\}.$$

We equip ∂X with the natural probability measure ν as in Falconer [6] by distributing the unit mass uniformly on ∂X. Then for any subset $A \subset \partial X$, the boundary measure of A, denoted by $\nu(A)$, is

$$\nu(A) = \int_A d\nu.$$

For any $x \in X$ with $|x| = j$, if we denote by I_x (or $\partial\Gamma_x$) the set

$$\{\xi \in \partial X : \text{the geodesic } [0, \xi) \text{ passes through } x\},$$

then $\nu(I_x) = \nu(\partial\Gamma_x) = K^{-j}$. We refer to [2, Lemma 5.2] for more information on our boundary measure ν.

Let us assume that $\int_0^\infty \lambda(t) dt < \infty$ and let $\xi, \zeta \in \partial X$. We denote by (ξ, ζ) the infinite geodesic connecting ξ to ζ. Then (ξ, ζ) consists of the tails $[x, \xi)$ and $[x, \zeta)$ of the geodesics $[0, \xi)$ and $[0, \zeta)$ starting at the last common point x of $[0, \xi)$ and $[0, \zeta)$. We define the visual metric d_b on ∂X, see [3] for more details, by setting

$$d_b(\xi, \zeta) := \int_{(\xi,\zeta)} d_\lambda = 2 \int_{|x_{(\xi,\zeta)}|}^\infty \lambda(t) dt$$

for any $\xi, \zeta \in \partial X$, where $x_{(\xi,\zeta)}$ is the last common point of $[0, \xi)$ and $[0, \zeta)$.

Recall that a metric space $(\partial X, d_b)$ is an ultrametric space if for each triple of points $\xi, \zeta, \eta \in \partial X$ we have $d_b(\xi, \zeta) \le \max\{d_b(\xi, \eta), d_b(\eta, \zeta)\}$.

Proposition 2.1. *The metric space $(\partial X, d_b)$ is an ultrametric space under the assumption that $\int_0^\infty \lambda(t)\,dt < \infty$ and hence any two closed balls in ∂X are either disjoint or contain one another.*

Proof. For any $\xi_1, \xi_2, \xi_3 \in \partial X$, we let $x_{(\xi_i,\xi_j)}$ be the last common point of $[0, \xi_i)$ and $[0, \xi_j)$ for each $i, j \in \{1, 2, 3\}$. Let $k_{i,j} = |x_{(\xi_i,\xi_j)}|$ for each $i, j \in \{1, 2, 3\}$. Then $k_{12} \geq \min\{k_{13}, k_{23}\}$ and

$$d_b(\xi_i, \xi_j) = 2\int_{k_{ij}}^{\infty} \lambda(t)dt < \infty$$

for each $i, j \in \{1, 2, 3\}$. It follows that

$$d_b(\xi_1, \xi_2) \leq \max\{d_b(\xi_1, \xi_3), d_b(\xi_2, \xi_3)\}$$

for any triple of points $\xi_1, \xi_2, \xi_3 \in \partial X$. Thus $(\partial X, d_b)$ is an ultrametric space. The latter part of the proposition is a direct consequence of the ultrametric property of ∂X. The proof is complete. □

By Proposition 2.1, any two closed balls in ∂X are either disjoint or contain one another. Then (X, d_b, ν) is a Vitali metric measure space, i.e every subset A of ∂X and for every covering $\mathcal{B}$ of A by closed balls satisfying

$$\inf\{r : r > 0 \text{ and } \bar{B}(\xi, r) \in \mathcal{B}\} = 0$$

for each $\xi \in A$, where $\bar{B}(\xi, r) = \{\eta \in \partial X : d_b(\xi, \eta) \leq r\}$, there exists a pairwise disjoint subcollection $\mathcal{C} \subset \mathcal{B}$ such that

$$\nu\left(A \setminus \cup_{B\in\mathcal{C}} B\right) = 0.$$

By the Lebesgue differentiation theorem on a Vitali metric measure space in [10, Section 3.4], we obtain the following theorem.

Theorem 2.2. *Let $f \in L^1_{\mathrm{loc}}(\partial X, d_b, \nu)$. Assume that $\int_0^\infty \lambda(t)\,dt < \infty$. Then*

$$\lim_{r\to 0} \fint_{\bar{B}(\xi,r)} f(\eta)d\nu(\eta) = f(\xi)$$

for ν-a.e $\xi \in \partial X$, where $\bar{B}(\xi, r) = \{\eta \in \partial X : d_b(\xi, \eta) \leq r\}$.

2.2 Newtonian spaces

Let $1 \leq p < \infty$ and X be a K-regular tree with metric d_λ and measure μ as in Section 2.1. Let $f \in L^1_{\mathrm{loc}}(X, d_\lambda, \mu)$. We say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of f if

$$|f(y) - f(z)| \leq \int_\gamma g\, d_\lambda \tag{2.1}$$

whenever $y, z \in X$ and γ is the geodesic from y to z. In the setting of our tree, any rectifiable curve with end points z and y contains the geodesic connecting z and y, and therefore the upper gradient defined above is equivalent to the definition which requires that (2.1) holds for all rectifiable curves with end points z and y.

The notion of upper gradients was introduced in [9]. We refer the interested readers to [1, 8, 10, 23] for a more detailed discussion on upper gradients.

The *Newtonian space* $N^{1,p}(X) := N^{1,p}(X, d_\lambda, \mu)$, $1 \leq p < \infty$, is defined as the collection of all the functions f with finite $N^{1,p}$-norm

$$\|f\|_{N^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)}$$

where the infimum is taken over all upper gradients of u. If $f \in N^{1,p}(X)$, then it is continuous by (2.1); recall here our standing assumption that $\lambda^p w^{-1} \in L^{1/(p-1)}_{\mathrm{loc}}([0, \infty))$.

We define the *homogeneous Newtonian spaces* $\dot{N}^{1,p}(X)$, $1 \leq p < \infty$, as the collection of all the continuous functions f that have an upper gradient $0 \leq g \in L^p(X)$. The homogeneous $\dot{N}^{1,p}$-norm is given by

$$\|f\|_{\dot{N}^{1,p}(X)} := |f(0)| + \inf_g \|g\|_{L^p(X)}.$$

Here 0 is the root of our K-regular tree X and the infimum is taken over all upper gradients of f.

3 Proofs of Theorem 1.3-1.4

In this section, if we do not specifically mention, we always assume that $1 \le p < \infty$ and that X is a K-regular tree with metric d_λ and measure μ as in Section 2.1, with $\mu(X) < \infty$.

Let us first prove that $R_p(\lambda, w) < \infty$ together with $\mu(X) < \infty$ guarantee that our metric space is bounded.

Lemma 3.1. *Suppose that $\mu(X) < \infty$ and that $R_p(\lambda, w) < \infty$. Then $\int_0^\infty \lambda(t)\,dt < \infty$.*

Proof. For $p > 1$, the Hölder inequality gives

$$\int_0^\infty \lambda(t)dt \le \left(\int_0^\infty w(t)K^{j(t)}dt\right)^{1/p}\left(\int_0^\infty \lambda(t)^{\frac{p}{p-1}}w(t)^{\frac{1}{1-p}}(t)K^{\frac{j(t)}{1-p}}dt\right)^{(p-1)/p}.$$

Notice that $\int_0^\infty w(t)K^{j(t)}dt$ is precisely $\mu(X)$ and that the second term is $R_p^{\frac{p-1}{p}}$. Hence the claim follows for $p > 1$ since $\mu(X) < \infty$ and $R_p < \infty$. For $p = 1$, a similar idea gives $\int_0^\infty \lambda(t)dt \le \mu(X)R_1 < \infty$. The proof is complete. □

Let $\xi \in \partial X$. In what follows, the notation x_ξ means that $x_\xi \in [0, \xi)$. We set

$$\Gamma_x = \{y \in X : x \in [0, y]\} \text{ and } \partial\Gamma_x = \{\xi \in \partial X : x \in [0, \xi)\} \text{ for a given } x \in X.$$

Lemma 3.1 in [14], applied to the subtree Γ_z where $z \in X$, gives the following identity.

Lemma 3.2. *Let $u \in L^p(X)$. For any $z \in X$, we have that*

$$\int_{\partial\Gamma_z}\int_{[z,\xi)} |u(x)|^p K^{j(x)}\,d\mu(x)\,d\nu(\xi) = \int_{\Gamma_z} |u(x)|^p\,d\mu(x)$$

where $j(x)$ is the largest integer such that $j(x) \le |x| + 1$.

We also need the following formulation of Theorem 1.1 in [14].

Lemma 3.3. *Let $1 \le p < \infty$. Then $T_R f$ exists for each $f \in N^{1,p}(X)$ if and only if $R_p < \infty$.*

We begin by establishing the existence of two of the asserted limits.

Lemma 3.4. *Let $1 \le q \le p$. If $\mu(X) < \infty$ and $R_p < \infty$, then $T_R f$ and $T_q f$ exist for any $f \in N^{1,p}(X)$. Moreover, $T_R f = T_q f$ ν-a.e. if $T_R f$ exists for each $f \in N^{1,p}(X)$.*

Proof. Suppose that $\mu(X) < \infty$ and $R_p < \infty$. Let $f \in N^{1,p}(X)$ and $g_f \in L^p(X)$ be an upper gradient of f. By Lemma 3.3, we obtain that $T_R f$ exists. To prove that $T_q f$ exists, it suffices to show that

$$\lim_{x_\xi \to \xi} \fint_{\Gamma_{x_\xi}} |f(y) - T_R f(\xi)|^q\,d\mu(y) = 0 \tag{3.1}$$

holds for ν-a.e $\xi \in \partial X$. By the Hölder inequality and the dominated convergence theorem, it follows from $1 \le q \le p$, (1.4), and (2.1) that for any $x_\xi \in [0, \xi)$,

$$\begin{aligned}\left(\fint_{\Gamma_{x_\xi}} |f(y) - T_R f(\xi)|^q d\mu(y)\right)^{1/q} &\le \lim_{z_\xi \to \xi}\left(\fint_{\Gamma_{x_\xi}} |f(y) - f(z_\xi)|^p d\mu(y)\right)^{1/p}\\ &\le \lim_{z_\xi \to \xi}\left(\fint_{\Gamma_{x_\xi}}\left(\int_{[y,z_\xi]} g_f\,d_\lambda\right)^p d\mu(y)\right)^{1/p}.\end{aligned}$$

Since $[y, z_\xi] \subset [y, x_\xi] \cup [x_\xi, \xi)$ for any $y, z_\xi \in \Gamma_{x_\xi}$, we have that

$$\left(\fint_{\Gamma_{x_\xi}}\left(\int_{[y,z_\xi]} g_f\,d_\lambda\right)^p d\mu(y)\right)^{1/p}$$

$$\leq \left(\fint_{\Gamma_{x_\xi}} \left(\int_{[y,x_\xi]} g_f d_\lambda \right)^p d\mu(y) \right)^{1/p} + \left(\fint_{\Gamma_{x_\xi}} \left(\int_{[x_\xi,\xi)} g_f d_\lambda \right)^p d\mu(y) \right)^{1/p}$$
$$= \left(\fint_{\Gamma_{x_\xi}} \left(\int_{[y,x_\xi]} g_f d_\lambda \right)^p d\mu(y) \right)^{1/p} + \int_{[x_\xi,\xi)} g_f d_\lambda =: H_1(x_\xi) + H_2(x_\xi).$$

To obtain (3.1), we only need to show that

$$\lim_{x_\xi \to \xi} H_1(x_\xi) = \lim_{x_\xi \to \xi} H_2(x_\xi) = 0 \text{ for } \nu\text{-a.e } \xi \in \partial X. \tag{3.2}$$

Suppose first that $p > 1$. By the Hölder inequality, a direct computation reveals that for any $[x, y]$ in X,

$$\begin{aligned}\left(\int_{[x,y]} g_f d_\lambda \right)^p &= \left(\int_{[x,y]} g_f(z) K^{j(z)/p} \frac{\lambda(z)}{w(z) K^{j(z)/p}} d\mu(z) \right)^p \\ &\leq \left(\int_{[x,y]} \left(\frac{\lambda(z)}{w(z) K^{j(z)/p}} \right)^{\frac{p}{p-1}} d\mu(z) \right)^{p-1} \int_{[x,y]} g_f^p(z) K^{j(z)} d\mu(z) \\ &\leq \left(2 \int_0^\infty \lambda(t)^{\frac{p}{p-1}} w(t)^{\frac{1}{1-p}} K^{\frac{j(t)}{1-p}} dt \right)^{p-1} \int_{[x,y]} g_f^p(z) K^{j(z)} d\mu(z) \\ &= 2^{p-1} R_p^{p-1} \int_{[x,y]} g_f^p(z) K^{j(z)} d\mu(z).\end{aligned} \tag{3.3}$$

Since $R_p < \infty$, substituting (3.3) into $H_1(\xi)$, $H_2(\xi)$ yields

$$H_1(x_\xi)^p \lesssim \fint_{\Gamma_{x_\xi}} \int_{[y,x_\xi]} g_f^p(z) K^{j(z)} d\mu(z) d\mu(y) \tag{3.4}$$

and

$$H_2(x_\xi)^p \lesssim \int_{[x_\xi,\xi)} g_f^p(z) K^{j(z)} d\mu(z). \tag{3.5}$$

For $p = 1$, by an argument similar to (3.3), without using the Hölder inequality, we also obtain that for any $[x, y]$ in X,

$$\int_{[x,y]} g_f \, d_\lambda \leq R_1 \int_{[x,y]} g_f(z) K^{j(z)} d\mu(z) \tag{3.6}$$

and hence that (3.4) and (3.5) also hold for $p = 1$.

Applying Lemma 3.2 for $\Gamma_z = X$ and $u = g_f$, it follows from $g_f \in L^p(X)$ that

$$\int_{[0,\xi)} g_f^p(z) K^{j(z)} d\mu(z) < \infty \tag{3.7}$$

for ν-a.e $\xi \in \partial X$. We conclude from (3.5) and (3.7) that

$$\lim_{x_\xi \to \xi} H_2(x_\xi) = 0 \tag{3.8}$$

for ν-a.e $\xi \in \partial X$. In order to get (3.2), we next estimate $H_1(x_\xi)$. By the Fubini theorem, (3.4) gives that

$$\begin{aligned}H_1(x_\xi)^p &\lesssim \fint_{\Gamma_{x_\xi}} \int_{\Gamma_{x_\xi}} g_f^p(z) K^{j(z)} \chi_{[y,x_\xi]}(z) d\mu(z) d\mu(y) \\ &= \frac{1}{\mu(\Gamma_{x_\xi})} \int_{\Gamma_{x_\xi}} g_f^p(z) K^{j(z)} \left(\int_{\Gamma_{x_\xi}} \chi_{[y,x_\xi]}(z) d\mu(y) \right) d\mu(z) \\ &= \frac{1}{\mu(\Gamma_{x_\xi})} \int_{\Gamma_{x_\xi}} g_f^p(z) K^{j(z)} \mu(\Gamma_z) d\mu(z).\end{aligned} \tag{3.9}$$

Note that

$$\frac{K^{j(z)}\mu(\Gamma_z)}{K^{j(x_\xi)}\mu(\Gamma_{x_\xi})} = \frac{\mu(X \setminus X^{|z|})}{\mu(X \setminus X^{|x_\xi|})} \le 1 \tag{3.10}$$

for any $z \in \Gamma_{x_\xi}$. Combining (3.9)-(3.10) with $\nu(\partial\Gamma_{x_\xi}) = K^{-j(x_\xi)}$, by Lemma 3.2, we obtain that for any $y_\xi \in [0, \xi)$ with $x_\xi \in [y_\xi, \xi)$,

$$\begin{aligned} H_1(x_\xi)^p &\lesssim \frac{1}{\nu(\partial\Gamma_{x_\xi})} \int_{\Gamma_{x_\xi}} g_f^p(z)d\mu(z) \\ &= \fint_{\partial\Gamma_{x_\xi}} \int_{[x_\xi,\eta)} g_f^p(z)K^{j(z)}d\mu(z)d\nu(\eta) \\ &\le \fint_{\partial\Gamma_{x_\xi}} \int_{[y_\xi,\eta)} g_f^p(z)K^{j(z)}d\mu(z)d\nu(\eta). \end{aligned} \tag{3.11}$$

Note that $G(\eta) := \int_{[y_\xi,\eta)} g_f^p(z)K^{j(z)}d\mu(z) \in L^1(\partial X)$ for any $y_\xi \in [0, \xi)$ by Lemma 3.2 and that $\int_0^\infty \lambda(t)dt < \infty$ by Lemma 3.1. Hence the Lebesgue differentiation theorem (see Theorem 2.2) gives that for each $y_\xi \in [0, \xi)$,

$$\lim_{x_\xi\to\xi} \fint_{\partial\Gamma_{x_\xi}} G(\eta)d\nu(\eta) = G(\xi) = \int_{[y_\xi,\xi)} g_f^p(z)K^{j(z)}d\mu(z)$$

for ν-a.e $\xi \in \partial X$. Hence (3.11) allows us to deduce that, for each $y_\xi \in [0, \xi)$,

$$\lim_{x_\xi\to\xi} H_1(x_\xi)^p \lesssim \int_{[y_\xi,\xi)} g_f^p(z)K^{j(z)}d\mu(z)$$

for ν-a.e $\xi \in \partial X$. Thanks to (3.7), letting $y_\xi \to \xi$, we obtain that

$$\lim_{x_\xi\to\xi} H_1(x_\xi) = 0 \tag{3.12}$$

for ν-a.e $\xi \in \partial X$. Combining (3.12) and (3.8), we obtain (3.2). Thus $T_R f$ and $T_q f$ exists for any $f \in N^{1,p}(X)$ if $R_p < \infty$.

Finally, if $T_R f$ exists for each $f \in N^{1,p}(X)$, then $R_p < \infty$ by Lemma 3.3, and the first part of our proof gives that $T_q f$ exists with $T_q f = T_R f$ ν–a.e for any $f \in N^{1,p}(X)$. The proof is complete. □

Lemma 3.5. *Let $1 \le q \le p$ and $f \in N^{1,p}(X)$. If $T_q f$ exists, then $T_L f$ also exists. Moreover, $T_q f = T_L f$ ν-a.e if $T_q f$ exists.*

Proof. The claim follows since

$$\left| T_q f(\xi) - \fint_{\Gamma_{x_\xi}} f(y)\, d\mu(y) \right| \le \fint_{\Gamma_{x_\xi}} |f(y) - T_q f(\xi)| d\mu(y) \le \left(\fint_{\Gamma_{x_\xi}} |f(y) - T_q f(\xi)|^q\, d\mu(y) \right)^{1/q} \to 0$$

when $x_\xi \to \xi$. □

Lemma 3.6. *If $R_p = \infty$, then there exists $f \in N^{1,p}(X)$ such that $T_L f$ does not exists.*

Proof. Let $\xi \in \partial X$. For each $n \in [0, \infty)$, we denote by $x_n(\xi)$ the point in $[0, \xi)$ with $|x_n(\xi)| = n$. It suffices to show that there exist a function $f \in N^{1,p}(X)$ and two sequences $\{n_i\}_{i=1}^\infty$, $\{m_i\}_{i=1}^\infty$ such that for any $\xi \in \partial X$,

$$\fint_{\Gamma_{x_{n_i}(\xi)}} f d\mu \ge \frac{2}{3} \text{ and } \fint_{\Gamma_{x_{m_i}(\xi)}} f d\mu \le \frac{1}{3} \tag{3.13}$$

for any $i \in \mathbb{N}$. Towards this, by Theorem 3.5 in [14], there exists a non-negative locally integrable function g on $[0, \infty)$ so that

$$\int_0^\infty g^p(t)w(t)K^{j(t)}dt < \infty \tag{3.14}$$

and

$$\int_0^\infty g(t)\lambda(t)dt = \infty. \tag{3.15}$$

Pick n_1 so that

$$\int_0^{n_1} g(t)\lambda(t)dt = 1. \tag{3.16}$$

As $\mu(X \setminus X^{n_1}) = \lim_{l_1\to\infty} \mu((X \setminus X^{n_1}) \cap X^{l_1})$, we find $l_1 \in \mathbb{N}$ with $n_1 \le l_1$ such that

$$\mu((X \setminus X^{n_1}) \cap X^{l_1}) \ge \frac{2}{3}\mu(X \setminus X^{n_1}).$$

Since

$$\mu(X \setminus X^n) = K^n \mu(\Gamma_{x_n(\xi)}) \text{ and } \mu((X \setminus X^n) \cap X^m) = K^n \mu(\Gamma_{x_n(\xi)} \cap X^m) \tag{3.17}$$

for any $\xi \in \partial X$ and for any $n, m \in \mathbb{N}$ with $n \le m$, the above estimates give

$$\frac{\mu(\Gamma_{x_{n_1}(\xi)} \cap X^{l_1})}{\mu(\Gamma_{x_{n_1}(\xi)})} \ge \frac{2}{3} \tag{3.18}$$

for any $\xi \in \partial X$. By (3.15) we find m_1 with $l_1 \le m_1$ such that

$$\int_{l_1}^{m_1} g(t)\lambda(t)dt = 1. \tag{3.19}$$

Since $\lim_{k_1\to\infty} \mu((X \setminus X^{m_1}) \cap X^{k_1}) = \mu(X \setminus X^{m_1})$, there exists k_1 with $m_1 \le k_1$ such that

$$\mu((X \setminus X^{m_1}) \cap X^{k_1}) \ge \frac{2}{3}\mu(X \setminus X^{m_1}).$$

Hence we have by (3.17) that

$$\frac{\mu(\Gamma_{x_{m_1}(\xi)} \cap X^{k_1})}{\mu(\Gamma_{x_{m_1}(\xi)})} \ge \frac{2}{3} \tag{3.20}$$

for any $\xi \in \partial X$. We continue by choosing n_2 with $k_1 \le n_2$ such that

$$\int_{k_1}^{n_2} g(t)\lambda(t)dt = 1. \tag{3.21}$$

By induction on n_1, l_1, m_1, k_1, n_2 with $n_1 \le l_1 \le m_1 \le k_1 \le n_2$, there exist four sequences $\{n_i\}_{i=1}^\infty, \{l_i\}_{i=1}^\infty, \{m_i\}_{i=1}^\infty, \{k_i\}_{i=1}^\infty$ such that $n_i \le l_i \le m_i \le k_i \le n_{i+1}$ and

$$\text{(3.18)-(3.21) hold for the corresponding pairs of indices } n_i, l_i, m_i, k_i, n_{i+1} \tag{3.22}$$

for any $i = 1, 2, \ldots$. Now we define a function f by setting $f(x) = 1$ if $x \in X^{n_1}$, and

$$f(x) = \begin{cases} 1 & \text{if } x \in X^{l_i} \setminus X^{n_i} \\ 1 - \int_{l_i}^{|x|} g(t)\lambda(t)dt & \text{if } x \in X^{m_i} \setminus X^{l_i} \\ 0 & \text{if } x \in X^{k_i} \setminus X^{m_i} \\ \int_{k_i}^{|x|} g(t)\lambda(t)dt & \text{if } x \in X^{n_{i+1}} \setminus X^{k_i} \end{cases} \tag{3.23}$$

for $i \ge 1$. Then by (3.16),(3.19),(3.21),(3.22),(3.23), we have that f is continuous, $0 \le f \le 1$, and g is an upper gradient of f. By (3.14) and the fact that $\mu(X) < \infty$, it follows that $f \in N^{1,p}(X)$. Combining (3.18),(3.20),(3.22),(3.23), we conclude that for any $\xi \in \partial X$, for any $i \in \mathbb{N}$,

$$\fint_{\Gamma_{x_{n_i}(\xi)}} f d\mu \ge \frac{1}{\mu(\Gamma_{x_{n_i}(\xi)})} \int_{\Gamma_{x_{n_i}(\xi)} \cap X^{l_i}} f\, d\mu = \frac{\mu(\Gamma_{x_{n_i}(\xi)} \cap X^{l_i})}{\mu(\Gamma_{x_{n_i}(\xi)})} \ge \frac{2}{3}$$

and

$$\fint_{\Gamma_{x_{m_i}(\xi)}} f\, d\mu = \frac{1}{\mu(\Gamma_{x_{m_i}(\xi)})} \int_{\Gamma_{x_{m_i}(\xi)}} f d\mu \le 1 - \frac{\mu(\Gamma_{x_{m_i}(\xi)} \cap X^{k_i})}{\mu(\Gamma_{x_{m_i}(\xi)})} \le \frac{1}{3}.$$

Thus (3.13) holds. The claim follows. □

Lemma 3.7. *Let $1 \le q \le p$. If one of $T_R f, T_L f, T_q f$ exists for each $f \in N^{1,p}(X)$, then all of them exist and coincide ν-a.e on ∂X for a given f.*

Proof. By Lemma 3.3-3.6, we have that $R_p < \infty$ if and only if one of $T_R f, T_L f, T_q f$ exists for each $f \in N^{1,p}(X)$. Then

$$T_R f,\, T_L f,\, T_q f \text{ exist if one of them exists} \tag{3.24}$$

for each $f \in N^{1,p}(X)$. By Lemma 3.4 and Lemma 3.5, we obtain that

$$T_R f = T_q f = T_L f \ \nu\text{-a.e if } T_R f, T_q f \text{ exist} \tag{3.25}$$

for each $f \in N^{1,p}(X)$. Combining (3.24)-(3.25), we conclude that $T_R f = T_q f = T_L f$ ν-a.e if one of $T_R f, T_q f, T_L f$ exists. The proof is complete. □

Proof of Theorem 1.3. $(i) \Leftrightarrow (iv)$ is given by Lemma 3.3.
$(iv) \Rightarrow (iii)$ is given by Lemma 3.4.
$(iii) \Rightarrow (ii)$ is given by Lemma 3.5.
$(ii) \Rightarrow (iv)$ is given by Lemma 3.6.
The latter part of the Theorem is given by Lemma 3.7. □

Proof of Theorem 1.4. Recalling that each $f \in N^{1,p}(X)$ is continuous, we have that $|f(0)| < \infty$ and hence $N^{1,p}(X) \subset \dot{N}^{1,p}(X)$. We are left to show that $\dot{N}^{1,p}(X) \subset N^{1,p}(X)$. It suffices to prove that

$$\|f\|_{L^p(X)} \lesssim \|f\|_{\dot{N}^{1,p}(X)}$$

for any $f \in \dot{N}^{1,p}(X)$. Let $f \in \dot{N}^{1,p}(X)$ and let g_f be an upper gradient of f. For any $x \in X$ we have

$$|f(x)| \le |f(0)| + \int_{[0,x]} g_f \, d_\lambda \tag{3.26}$$

where 0 is the root of X. By arguments (3.3), (3.6), it follows that for any $p \ge 1$,

$$\left(\int_{[0,x]} g_f \, d_\lambda \right)^p \le M \int_{[0,x]} g_f^p(y) K^{j(y)} \, d\mu(y). \tag{3.27}$$

where $M = \max\{2^{p-1} R_p^{p-1}, R_1\}$. By the Fubini theorem, we have from (3.26)-(3.27) that

$$\begin{aligned}
\|f\|_{L^p(X)} \le& \|f(0)\|_{L^p(X)} + \left\| \int_{[0,x]} g_f \, d_\lambda \right\|_{L^p(X)} \\
\le& \mu(X)^{1/p} |f(0)| + M^{\frac{1}{p}} \left(\int_X \int_{[0,x]} g_f^p(y) K^{j(y)} \, d\mu(y) d\mu(x) \right)^{1/p} \\
=& \mu(X)^{1/p} |f(0)| + M^{\frac{1}{p}} \left(\int_X g_f^p(y) K^{j(y)} \left(\int_X \chi_{[0,x]}(y) d\mu(x) \right) d\mu(y) \right)^{1/p} \\
=& \mu(X)^{1/p} |f(0)| + M^{\frac{1}{p}} \left(\int_X g_f^p(y) K^{j(y)} \mu(\Gamma_y) d\mu(y) \right)^{1/p}.
\end{aligned}$$

Since $K^{j(y)} \mu(\Gamma_y) = \mu(X \setminus X^{|y|}) \le \mu(X)$, the above estimate gives that

$$\|f\|_{L^p(X)} \le \mu(X)^{1/p} \|f(0)\| + \mu(X)^{1/p} M^{\frac{1}{p}} \|g_f\|_{L^p(X)}.$$

We conclude that for any $f \in \dot{N}^{1,p}(X)$,

$$\|f\|_{N^{1,p}(X)} = \|f\|_{L^p(X)} + \|g_f\|_{L^p(X)} \lesssim \|f\|_{\dot{N}^{1,p}(X)}.$$

Thus $\dot{N}^{1,p}(X) \subset N^{1,p}(X)$ which finishes the proof. □

Example 3.8. *Let $w(t) = e^{-\beta j(t)}$ and $\lambda(t) = e^{-\varepsilon j(t)}$ with $\varepsilon, \beta > 0$ and $\beta > \log K + \varepsilon p$. Then (X, d_λ, μ) is a metric measure space as in Section 2.1 with $\mu(X) < \infty$, $R_p = \infty$ for any $1 \le p < \infty$ but nevertheless $N^{1,p}(X) = \dot{N}^{1,p}(X)$.*

It is obvious that $\mu(X) < \infty$ and $R_p = \infty$ for any $1 \le p < \infty$. Indeed, since $(\beta - \log K) > \varepsilon p > 0$ we have that

$$\mu(X) = \int_0^\infty w(t)K^{j(t)}dt = \int_0^\infty e^{-(\beta - \log K)j(t)}\,dt < \infty.$$

For any $1 \le p < \infty$, as $(\beta - K - \varepsilon p) > 0$ we obtain that

$$R_p = \int_0^\infty \lambda(t)^{\frac{p}{p-1}} w(t)^{\frac{1}{1-p}} K^{\frac{j(t)}{1-p}}\,dt = \int_0^\infty e^{\frac{(\beta - \log K - \varepsilon p)j(t)}{p-1}}\,dt = \infty \text{ for } p > 1,$$

and

$$R_1 = \left\| \frac{\lambda(t)}{w(t)K^{j(t)}} \right\|_{L^\infty([0,\infty))} = \left\| e^{(\beta - \log K - \varepsilon)j(t)} \right\|_{L^\infty([0,\infty))} = \infty.$$

As in the proof of Theorem 1.4 we have that $N^{1,p}(X) \subset \dot{N}^{1,p}(X)$. Hence we only need to prove that $\dot{N}^{1,p}(X) \subset N^{1,p}(X)$. It suffices to show that for any $f \in \dot{N}^{1,p}(X)$,

$$\|f\|_{L^p(X)} \lesssim \|f\|_{\dot{N}^{1,p}(X)}.$$

Let g_f be an upper gradient of f. For $p > 1$, we have by the Hölder inequality that

$$\begin{aligned} |f(x)| \le |f(0)| + \int_{[0,x]} g_f d_\lambda =& |f(0)| + \int_{[0,x]} g_f(z) e^{-\varepsilon j(z)} d_G(z) \\ \le& |f(0)| + \left(\int_{[0,x]} g_f^p(z) d_G(z) \right)^{1/p} \left(\int_{[0,x]} e^{\frac{p\varepsilon j(z)}{1-p}} d_G(z) \right)^{\frac{p-1}{p}} \\ \le& |f(0)| + C_1^{\frac{p-1}{p}} \left(\int_{[0,x]} g_f^p(z) d_G(z) \right)^{1/p} \end{aligned}$$

for any $x \in X$, where

$$C_1 = \int_0^\infty e^{\frac{p\varepsilon j(t)}{1-p}}\,dt = \frac{p-1}{p\varepsilon}.$$

For $p = 1$, since $d_\lambda(z) = e^{-\varepsilon j(z)} d_G(z) \le d_G(z)$ we have that

$$|f(x)| \le |f(0)| + \int_{[0,x]} g_f d_\lambda \le |f(0)| + \int_{[0,x]} g_f(z) d_G(z).$$

Let $C = \max\{C_1^{\frac{p-1}{p}}, 1\}$. By the Fubini theorem, it follows that for any $p \ge 1$,

$$\begin{aligned} \|f\|_{L^p(X)} \le& \|f(0)\|_{L^p(X)} + \left\| C \left(\int_{[0,x]} g_f^p(z) d_G(z) \right)^{1/p} \right\|_{L^p(X)} \\ =& \|f(0)\|_{L^p(X)} + C \left(\int_X \left(\int_X g_f^p(z) \chi_{[0,x]}(z) d_G(z) \right) e^{-\beta j(x)} d_G(x) \right)^{1/p} \\ =& \mu(X)^{1/p} |f(0)| + C \left(\int_X g_f^p(z) \left(\int_X \chi_{[0,x]}(z) e^{-\beta j(x)} d_G(x) \right) d_G(z) \right)^{1/p}. \end{aligned} \tag{3.28}$$

For any $z \in X$, we have that

$$\begin{aligned} e^{\beta j(z)} \int_X \chi_{[0,x]}(z) e^{-\beta j(x)} d_G(x) =& e^{\beta j(z)} \int_{\Gamma_z} e^{-\beta j(x)} d_G(x) \\ =& e^{\beta j(z)} \int_{j(z)}^\infty e^{-\beta j(t)} K^{j(t)-j(z)} dt \\ =& e^{\beta j(z)} K^{-j(z)} \left. \frac{e^{-\beta j(t)} K^{j(t)}}{-\beta + \log K} \right|_{j(z)}^\infty = \frac{1}{\beta - \log K}. \end{aligned}$$

Since $\mu(X) < \infty$, $d\mu(z) = e^{-\beta j(z)} d_G(z)$, $C < \infty$, and $\beta - \log K > \varepsilon p > 0$, inserting this into (3.28) yields

$$\|f\|_{L^p(X)} \le \mu(X)^{1/p}|f(0)| + \frac{C}{(\beta - \log K)^{1/p}}\|g_f\|_{L^p(X)} \lesssim \|f\|_{\dot N^{1,p}(X)}$$

as desired.

Remark 3.9. *By Lemma 3.1 we know that $\int_0^\infty \lambda(t)\,dt < \infty$ under the assumptions that $\mu(X) < \infty$ and $R_p < \infty$. In this case, the diameter of X with respect to d_λ is finite and we could consider balls in X that have their centers on ∂X. Towards this, recall that (η, ζ) refers to the geodesic between $\eta, \zeta \in \partial X$. Given $\xi \in \partial X$ and $x_\xi \in [0, \xi)$, we let*

$$B_{x_\xi} = \left\{(\eta, \zeta) \in X : \eta, \zeta \in B_{\partial X}\left(\xi, 2\int_{|x_\xi|}^{\infty} \lambda(t)dt\right)\right\}$$

where $B_{\partial X}(\xi, r)$ is the ball with radius r and center at ξ in $(\partial X, d_b)$ as in Section 2.1. Then B_{x_ξ} is an analog of the intersection of a domain and a ball with center ξ at boundary in the classical setting, and

$$\Gamma_{x_\xi} = B_{x_\xi} \text{ for each } x_\xi \in [0, \xi)$$

for any $\xi \in \partial X$ in our setting. This gives us a justification to consider the traces T_L, T_q in Definition 1.2 to be analogs of (1.2)-(1.3). *We do not know if we could replace B_{x_ξ} by $B_X(\xi, r)$ in general in the definitions of our traces. It is easy to check that one can do so if μ is assumed to be doubling.*

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