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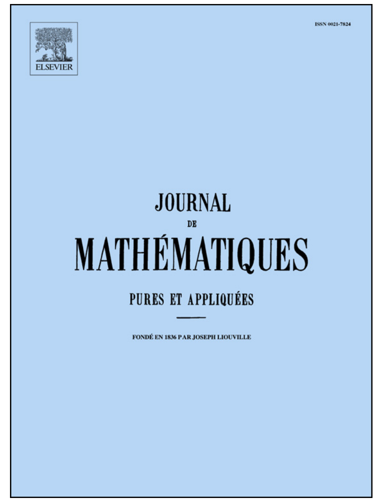

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Inverse problems for elliptic equations with power type nonlinearities

Dedicated to the memory of Yaroslav Kurylev

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Abstract

We introduce a method for solving Calderón type inverse problems for semilinear equations with power type nonlinearities. The method is based on higher order linearizations, and it allows one to solve inverse problems for certain nonlinear equations in cases where the solution for a corresponding linear equation is not known. Assuming the knowledge of a nonlinear Dirichlet-to-Neumann map, we determine both a potential and a conformal manifold simultaneously in dimension 2, and a potential on transversally anisotropic manifolds in dimensions $n \geq 3$. In the Euclidean case, we show that one can solve the Calderón problem for certain semilinear equations in a surprisingly simple way without using complex geometrical optics solutions.

Résumé

Dans cet article, on introduit une méthode pour résoudre les problèmes inverses de type Calderón pour les équations semi-linéaires avec des non-linéarités polynomiales. La méthode est basée sur des linéarisations d'ordre supérieur et elle permet de résoudre des problèmes inverses pour certaines équations non linéaires dans les cas où la solution d'une équation linéaire correspondante n'est pas connue. En supposant la connaissance de l'opérateur Dirichlet-Neumann non linéaire, nous déterminons simultanément un potentiel et une variété conforme en dimension 2, et un potentiel sur des variétés transversalement anisotropes de dimensions $n \geq 3$. Dans le cas euclidien, nous montrons que l'on peut résoudre le problème de Calderón pour certaines équations semi-linéaires d'une manière étonnamment simple sans utiliser de solutions optiques géométriques complexes.

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1. Introduction

In this paper we study inverse boundary value problems for nonlinear elliptic equations. A standard example of inverse problems for linear elliptic equations is the problem introduced by Calderón [1], where the objective is to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary. It is closely related to the problem of determining an unknown potential q in a Schrödinger operator $\Delta + q$ from boundary measurements, first solved in [2] in dimensions $n \geq 3$. There is an extensive theory concerning inverse boundary value problems for linear elliptic equations, and we refer to [3] for a survey.

It is also natural to consider analogous inverse problems under nonlinear settings. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary, and consider the reaction-diffusion equation

$$\partial_t w - \Delta w = a(x, w) \text{ in } \Omega \times \{t > 0\}.$$

Equations of this type arise in the modelling of chemical reactions, population dynamics and pattern formation [4]. Examples include the Fisher, Kolmogorov or logistic diffusion equations with quadratic nonlinearity (i.e. $a(x, w)$ is quadratic in w), the Newell-Whitehead-Segel equation with cubic nonlinearity, and equations in combustion involving polynomial or exponential nonlinearities.

A stationary solution $w(x, t) = u(x)$ satisfies the elliptic equation

$$\Delta u + a(x, u) = 0 \text{ in } \Omega.$$

The Dirichlet problem for this equation is related to maintaining a temperature (or concentration or population) f on the boundary. The boundary measurements for such an equation, provided that it is well-posed for some class of boundary values, may be encoded by a Dirichlet-to-Neumann map (DN map) Λ_a, which maps the boundary value f to the flux $\Lambda_a(f) = \partial_\nu u|_{\partial\Omega}$ of the corresponding equilibrium state across the boundary.

In fact, inverse problems for nonlinear elliptic equations have also been widely studied. A standard method, introduced in [5] in the parabolic case, is to show that the first linearization of the nonlinear DN map is actually the DN map of a linear equation, and to use the theory of inverse problems for linear equations. For the semilinear Schrödinger equation $\Delta u + a(x, u) = 0$, the problem of recovering the potential $a(x, u)$ was studied in [6, 7] in dimensions $n \geq 3$, and in [8, 7, 9] when $n = 2$. In addition, inverse problems have been studied for quasilinear elliptic equations [10, 11, 12, 13, 14], the degenerate elliptic p-Laplace equation [15, 16], and the fractional semilinear Schrödinger equation [17]. Certain Calderón type inverse problems for quasilinear equations on Riemannian manifolds were recently considered in [18]. We refer to the survey articles [19, 3] for further details on inverse problems for nonlinear elliptic equations.

Inverse problems have also been studied for hyperbolic equations with various nonlinearities. Many of the works mentioned above rely on a solution to a related inverse problem for a linear equation. This is in contrast to the study of inverse problems for nonlinear hyperbolic equations, where it has been realized that the nonlinearity can actually be beneficial in solving inverse problems.

By using the nonlinearity as a tool, some still unsolved inverse problems for hyperbolic linear equations have been solved for their nonlinear counterparts. For the scalar wave equation with a quadratic nonlinearity, Kurylev-Lassas-Uhlmann [20] proved that local measurements determine the global topology, differentiable structure and the conformal class of the metric g on a globally hyperbolic 4-dimensional Lorentzian manifold. The authors of [21] studied inverse problems for general semilinear wave equations on Lorentzian manifolds, and in [22] they studied analogous problem for the Einstein-Maxwell equations. For more inverse problems of nonlinear hyperbolic equations, we refer readers to [23, 24, 25, 26] and references there in.

In this work we introduce a method which uses nonlinearity as a tool that helps in solving inverse problems for certain nonlinear elliptic equations. The method is based on *higher order linearizations* of the DN map, and essentially amounts to using sources with several parameters and obtaining new linearized equations after differentiating with respect to these parameters. We demonstrate the scope of the method by solving Calderón type problems for three mathematical models.

The first model is the Calderón problem for a semilinear Schrödinger equation with quadratic nonlinearity,

$$\Delta u + qu^2 = 0 \text{ in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where $q \in C^\infty(\overline{\Omega})$ and $n \geq 2$. The solution to a related inverse problem with $a(x, u)$ in place of qu^2 is known under assumptions like $\partial_u a(x, u) \leq 0$ [6, 8, 7]. Theorem 1.1 proves uniqueness for the nonlinearity qu^2, which appears to be a new result. The method applies to more general models, but we begin with the equation (1.1) in order to introduce our approach in the simplest possible setting.

The second new result is Theorem 1.2, where we simultaneously determine the metric, the manifold and the potential up to gauge symmetry from the knowledge of the DN map of a semilinear Schrödinger equation on two-dimensional Riemannian surfaces. The analogous result for a linear Schrödinger equation is not known in this generality. Here we use nonlinearity to simultaneously determine the topology and the conformal structure of the Riemannian surface, as well as the potential, up to a natural gauge transformation.

The third result, Theorem 1.3, is the recovery of the potential q from the knowledge of the DN map of a Schrödinger operator with nonlinearity of the form qu^m, $m \geq 3$, on transversally anisotropic manifolds in dimensions $n \geq 3$. Transversally anisotropic manifolds are product type manifolds which appear in several works related to the anisotropic Calderón problem. Again, the solution to the analogous inverse problem for a linear equation is not known in this generality. Existing results will be discussed in more detail later in this introduction.

Let us introduce the mathematical setting for this article. We will denote by (M, g) a compact Riemannian manifold with C^∞ boundary ∂M, where

$\dim(M) = n$, $n \geq 2$. For example, one could have $M = \overline{\Omega}$ where Ω is a bounded C^∞ domain in $\mathbb{R}^n$, and g could be the Euclidean metric. Let $q \in C^\infty(M)$. We will consider semilinear elliptic equations of the form

$$\begin{cases} \Delta_g u + q u^m = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \tag{1.2}$$

where

$$m \in \mathbb{N} \text{ and } m \geq 2.$$

Here Δ_g is the Laplace-Beltrami operator, given in local coordinates by

$$\Delta_g u = \frac{1}{\det(g)^{1/2}} \sum_{a,b=1}^{n} \frac{\partial}{\partial x_a} \left(\det(g)^{1/2} g^{ab} \frac{\partial u}{\partial x_b} \right),$$

where $g = (g_{ab}(x))$ and $g^{-1} = (g^{ab}(x))$.

We will show that the Dirichlet problem (1.2) has a unique small solution u for sufficiently small boundary data $f \in C^s(\partial M)$, where $s > 2$ with $s \notin \mathbb{N}$. More precisely this means that there is $\delta > 0$ such that whenever $\|f\|_{C^s(\partial M)} \leq \delta$, there is a unique solution u_f to (1.2) with sufficiently small $C^s(M)$ norm (see Section 2 for more details on well-posedness). We will call u_f the unique small solution. Here C^s is the standard Hölder space for $s > 2$ with $s \notin \mathbb{N}$ (often written as $C^{k,\alpha}$ if $s = k + \alpha$ where $k \in \mathbb{Z}$ and $0 < \alpha < 1$), see e.g. [27, Section 13.8]. Hence, the DN map is defined by using the unique small solution in a following way:

$$\Lambda_{M,g,q} : C^s(\partial M) \to C^{s-1}(\partial M), \quad f \mapsto \partial_\nu u_f|_{\partial M},$$

where ∂_ν denotes the normal derivative on the boundary ∂M. In what follows, we use the notation $\Lambda_{M,g}$ to denote the DN map when $q = 0$. When $M = \Omega \subset \mathbb{R}^n$ and g is the identity matrix, we denote the DN map by Λ_q.

As a warm-up, we begin with a theorem that illustrates our method in a simple setting. This theorem is in $\mathbb{R}^n$ for $n \geq 2$, where Δ_g is the Euclidean Laplacian and $M = \overline{\Omega}$ with Ω a bounded smooth domain in $\mathbb{R}^n$.

Theorem 1.1 (Global uniqueness for a quadratic nonlinearity). *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Let $q_1, q_2 \in C^\infty(\overline{\Omega})$. Assume the DN maps Λ_{q_j} for the equations*

$$\begin{cases} \Delta u + q_j u^2 = 0 & \textit{in } \Omega, \\ u = f & \textit{on } \partial\Omega, \end{cases} \tag{1.3}$$

for $j = 1, 2$ satisfy

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$$

for all $f \in C^s(\partial\Omega)$ with $\|f\|_{C^s(\partial M)} < \delta$, where $\delta > 0$ is any sufficiently small number. Then $q_1 = q_2$ in Ω.

We will offer a detailed proof of Theorem 1.1 in Section 3, but let us briefly discuss the idea how to prove the theorem by using the method of *higher order linearization*. The second order linearization of the nonlinear DN map has already been used in the works [10, 11] related to nonlinear equations with matrix coefficients. First and second order linearizations were also used in [12] for a nonlinear conductivity equation (see also [28]). Under certain assumptions on the nonlinearity, by using the second order linearization, they can recover quadratic parts of the nonlinearity (see [12, Theorem 1.2 and Theorem 1.3]). In this work, we use similar ideas but obtain interesting new phenomena for related nonlinear inverse problems.

For the equation (1.3) with quadratic nonlinearity, the first linearization of the nonlinear DN map Λ_q, linearized at the zero boundary value, is just the DN map for the standard Laplace equation:

$$(D\Lambda_q)_0 : C^s(\partial\Omega) \to C^{s-1}(\partial\Omega), \quad f \mapsto \partial_\nu v_f|_{\partial\Omega},$$

where v_f is the unique solution of $\Delta v_f = 0$ in Ω with $v_f|_{\partial\Omega} = f$. Thus the first linearization does not carry any information about the unknown potential q. However, for a quadratic nonlinearity the *second linearization* $(D^2\Lambda_q)_0$, which is a symmetric bilinear map from $C^s(\partial\Omega) \times C^s(\partial\Omega)$ to $C^{s-1}(\partial\Omega)$, turns out to be very useful: it is characterized by the identity (see (2.9))

$$\int_{\partial\Omega} (D^2\Lambda_q)_0(f_1, f_2) f_3 \, dS = -2 \int_\Omega q v_{f_1} v_{f_2} v_{f_3} \, dx$$

where v_{f_j} is the harmonic function with boundary value f_j. See formula (1.4) bel . Thus we have the implications

$$\begin{aligned} &\Lambda_{q_1}(f) = \Lambda_{q_2}(f) \text{ for small } f \\ \Longrightarrow\ &(D^2\Lambda_{q_1})_0 = (D^2\Lambda_{q_2})_0 \\ \Longrightarrow\ &\int_\Omega (q_1 - q_2) v_1 v_2 v_3 \, dx = 0 \end{aligned}$$

for any functions $v_1, v_2, v_3 \in C^s(\overline{\Omega})$ that are harmonic in Ω.

The last statement is very close to the linearized Calderón problem for a *linear* Schrödinger equation (the difference is that here one has the product of three harmonic functions, instead of two). Choosing v_1 and v_2 to be harmonic exponentials as in the work of Calderón [1], and choosing $v_3 \equiv 1$, shows that the Fourier transform of $q_1 - q_2$ vanishes and hence $q_1 = q_2$. Thus, somewhat strikingly, we can solve a Calderón type inverse problem for the nonlinear equation $\Delta u + qu^2 = 0$ in a much simpler way than for the linear equation $\Delta u + qu = 0$ (the latter requires complex geometrical optics solutions as in [2]). The method also provides extremely simple reconstruction of the potential q, see Corollary 3.1.

We also mention that the second order linearization can be described as

$$(D^2\Lambda_q)_0(f_1, f_2) = \partial_{\epsilon_1}\partial_{\epsilon_2} u_{\epsilon_1 f_1 + \epsilon_2 f_2}|_{\epsilon_1=\epsilon_2=0} \quad \text{on } \partial\Omega. \tag{1.4}$$

That is, one considers boundary data

$$f = \epsilon_1 f_1 + \epsilon_2 f_2 \in C^s(\partial\Omega),$$

where ϵ_1, ϵ_2 are sufficiently small parameters, and takes the mixed derivative

$$\frac{\partial}{\partial\epsilon_1}\frac{\partial}{\partial\epsilon_2}\Big|_{\epsilon_1=\epsilon_2=0}$$

of the equation (1.3). This idea is similar to the recent works on inverse problems for nonlinear hyperbolic equations mentioned above, and it yields the equations

$$\Delta w_j = -2q_j v_{f_1} v_{f_2}, \tag{1.5}$$

for $j = 1, 2$, where $w_j = \frac{\partial}{\partial \epsilon_1}\frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_1=\epsilon_2=0} u_j$ and v_{f_j} are harmonic functions, i.e. solutions to the linearized equation $\Delta v = 0$. Taking the mixed derivative of the DN maps yields (see Section 2)

$$\partial_\nu w_1 = \partial_\nu w_2 \text{ on } \partial\Omega.$$

Subtracting the equations (1.5) for $j = 1, 2$ and integrating the resulting equation against the harmonic function v_{f_3} yields the desired formula

$$\int_\Omega (q_1 - q_2) v_{f_1} v_{f_2} v_{f_3}\, dx = 0$$

which was mentioned in the discussion above.

We move on to describe our next result. By using higher order linearizations we prove the following *simultaneous recovery* on a two-dimensional Riemannian surface.

Theorem 1.2 (Simultaneous recovery of metric and potential)**.** *Let (M_1, g_1) and (M_2, g_2) be two compact connected manifolds with mutual C^∞ smooth boundaries $\partial M_1 = \partial M_2 =: \partial M$ and $\dim(M_1) = \dim(M_2) = 2$. Let $m \geq 2$, and let Λ_{M_j, g_j, q_j} be the DN maps of*

$$\Delta_{g_j} u + q_j u^m = 0 \text{ in } M_j \tag{1.6}$$

for $j = 1, 2$. Let $s > 2$ with $s \notin \mathbb{N}$ and assume that

$$\Lambda_{M_1,g_1,q_1} f = \Lambda_{M_2,g_2,q_2} f \text{ on } \partial M,$$

for any $f \in C^s(\partial M)$ with $\|f\|_{C^s(\partial M)} \leq \delta$, where $\delta > 0$ is sufficiently small. Then:

(1) There exists a conformal diffeomorphism $J : M_1 \to M_2$ and a positive smooth function σ on M_1 such that for $x \in M_1$ we have

$$(\sigma J^* g_2)(x) = g_1(x),$$

with $J|_{\partial M} = \mathrm{Id}$ and $\sigma|_{\partial M} = 1$.

(2) Moreover, one can also recover the potential up to a natural gauge invariance in the sense that

$$\sigma q_1 = q_2 \circ J \text{ in } M_1.$$

We see that the conformal factor σ (and also the diffeomorphism J) couples to the potential. This is due to the *gauge symmetry* of the inverse problem:

$$\Lambda_{M_1, \sigma J^* g, \sigma^{-1} J^* q} = \Lambda_{J(M_1), g, q}$$

where J is a conformal diffeomorphism and σ is a positive smooth function satisfy the boundary conditions $J|_{\partial M} = \mathrm{Id}$ and $\sigma|_{\partial M} = 1$. For the linear equation

$\Delta_g u + qu = 0$, an analogous result has been proved when M is a domain in $\mathbb{R}^2$ with a Riemannian metric [29], when M is a manifold and the potentials are zero [30], and when the manifold M is a priori known [31]. The recovery of properties of both the manifold and potential is stated as an open question in [32], where further references to two-dimensional results are given. The proof of Theorem 1.2 uses the first linearization of the DN map to recover the metric and the manifold up to a conformal transformation. Then the second linearization is used to recover the potential on a single fixed manifold (up to the gauge symmetry).

The final new result in this article is to consider inverse problems for the semilinear Schrödinger equation on *transversally anisotropic* manifold. Let us recall the definition of a transversally anisotropic manifold.

Definition 1.1. *Let (M, g) be a compact oriented manifold with a C^∞ boundary and with $\dim M \geq 3$. (M, g) is called* transversally anisotropic *if $(M, g) \subset\subset (T, g)$, where $T = \mathbb{R} \times M_0$ and $g(x) = g(x_1, x') = e(x_1) \oplus g_0(x')$ for $x_1 \in \mathbb{R}$ and $x' \in M_0$. Here $(\mathbb{R}, e)$ is the Euclidean line and (M_0, g_0) is an $(n-1)$-dimensional compact manifold with a smooth boundary.*

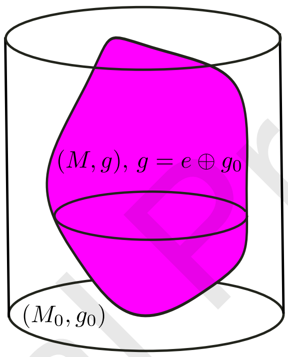

Figure 1: An example of a transversally anisotropic manifold (M, g).

An example of a transversally anisotropic manifold is visualized in Figure 1. Especially, if Ω is a domain in $\mathbb{R}^n$ and g_0 is any Riemannian metric on Ω, then $\mathbb{R} \times \Omega$ equipped with the metric

$$g(x_1, x') = \begin{bmatrix} 1 & 0 \\ 0 & g_0(x') \end{bmatrix}$$

is transversally anisotropic. For more details of inverse problems in transversally anisotropic geometries for linear equations, we refer readers to [33, 34].

We prove the following.

Theorem 1.3. *Let (M, g) be a transversally anisotropic manifold, let $q_j \in C^\infty(M)$, and let Λ_{q_j} be the DN maps for the equations*

$$\Delta_g u + q_j u^m = 0 \text{ in } M$$

for $j = 1, 2$, where we assume that

$$m \in \mathbb{N}, \quad m \geq 3.$$

If the DN maps satisfy

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$$

for all sufficiently small f, then $q_1 = q_2$ in M.

The higher order linearization method in this case reduces the proof of Theorem 1.3 to showing for any $m \geq 3$ that the identity

$$\int_M f v_1 \cdots v_{m+1} \, dV = 0 \tag{1.7}$$

holding for any $v_j \in C^\infty(M)$ with $\Delta_g v_j = 0$ in M, implies $f \equiv 0$. Thus we prove that the products of at least four harmonic functions on a transversally anisotropic manifold form a complete set. The main point is that the argument works for arbitrary transversally anisotropic manifolds without any restriction on the transversal geometry.

The solution to the analogous inverse problem for a linear equation $\Delta_g u + qu = 0$ on transversally anisotropic manifolds is only known under the additional assumption that the transversal manifold (M_0, g_0) has injective geodesic X-ray transform [33]. In the linearized version of that problem, the identity (1.7) only holds for $m = 1$ and one needs to prove that products of pairs of harmonic functions form a complete set. In [33] this is done by using complex geometrical optics solutions that concentrate near two-dimensional surfaces that are translates of geodesics on M_0. Using products of such solutions and their complex conjugates recovers certain integrals over geodesics in M_0, but does not yield pointwise information. In [34] products of solutions concentrating near two intersecting geodesics were used instead to recover microlocal information in the linearized inverse problem. The products are supported near finitely many points in M_0, but there is oscillation that prevents recovering more information. We also mention [35] that deals with the linearized problem on certain complex manifolds.

The idea behind the proof of Theorem 1.3 is that since one can use products of at least four harmonic functions, we can use solutions related to two intersecting geodesics on M_0 as well as their complex conjugates. The product of these four solutions is supported near finitely many points in M_0 and the product does not have high oscillations. This allows one to recover the potential completely.

We mention that the aim of this paper is not to work in the highest possible generality or to provide an extensive list of all possible applications of the higher order linearization method. For example, it is clear that the method applies to certain more general nonlinearities and less regular coefficients. These are left to forthcoming works. Here we have included applications that illustrate the power of the higher order linearization method.

Finally, we mention that before submitting this paper we became aware of an upcoming preprint of Ali Feizmohammadi and Lauri Oksanen, which simultaneously and independently proves a result similar to Theorem 1.3, and we agreed with them to publish the preprints of the results at the same time on the same preprint server. See [36].

The paper is organized as follows. In Section 2, we lay out the basic properties for semilinear elliptic equations that we use. This includes the well-posedness of the Dirichlet problem and higher order linearizations of the DN map. We use the higher order linearization approach to prove Theorem 1.1 in Section 3, Theorem 1.2 in Section 4, and Theorem 1.3 in Section 5, respectively.

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2. Preliminaries

In this section, we prove well-posedness of the Dirichlet problem for semilinear elliptic equations on Riemannian manifolds with small boundary data, and study higher order linearizations of the DN map. We assume that the Riemannian manifolds we consider are compact, C^∞ smooth and have C^∞ boundary.

We state the first result of this section for a general nonlinearity satisfying two conditions: Let Q be the semilinear elliptic operator

$$Q(u) := \Delta_g u + a(x,u), \tag{2.1}$$

where $a \in C^\infty(M \times \mathbb{R})$ satisfies the following two conditions:

$$a(x,0) = 0, \tag{2.2}$$

$$\text{The map } v \mapsto \Delta_g v + \partial_u a(\,\cdot\,,0)v \text{ is injective on } H_0^1(M). \tag{2.3}$$

The first condition ensures that $u \equiv 0$ is a solution, and the second states that the equation linearized at $u \equiv 0$ is well-posed.

The next result considers mappings between Banach spaces which are Fréchet differentiable. We refer the reader to [37, Section 10] and [38, Section 1.1] for basics about Fréchet differentiability.

Proposition 2.1 (Well-posedness)**.** *Let (M,g) be a compact Riemannian manifold with C^∞ boundary ∂M and let Q be the semilinear elliptic operator given by (2.1) satisfying (2.2) and (2.3). Let $s > 2$ with $s \notin \mathbb{Z}$. There exist $\delta, C > 0$ such that for any f in the set*

$$U_\delta := \{h \in C^s(\partial M)\,;\, \|h\|_{C^s(\partial M)} < \delta\},$$

there is a solution $u = u_f$ of

$$\begin{cases} \Delta_g u + a(x,u) = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \tag{2.4}$$

which satisfies

$$\|u\|_{C^s(M)} \le C\|f\|_{C^s(\partial M)}.$$

The solution u_f is unique within the class $\{w \in C^s(M)\,;\, \|w\|_{C^s(M)} \le C\delta\}$, and if $f \in C^\infty(\partial M)$, then $u_f \in C^\infty(M)$. Moreover, there are C^∞ Fréchet differentiable maps

$$\begin{aligned} S &: U_\delta \to C^s(M), & f &\mapsto u_f, \\ \Lambda &: U_\delta \to C^{s-1}(\partial M), & f &\mapsto \partial_\nu u_f|_{\partial M}. \end{aligned}$$

Proof. We prove the existence of solutions by using the implicit function theorem in Banach spaces [37, Theorem 10.6]. Let

$$X = C^s(\partial M), \quad Y = C^s(M), \quad Z = C^{s-2}(M) \times C^s(\partial M).$$

Consider the map

$$F : X \times Y \to Z, \quad F(f, u) = (Q(u), u|_{\partial M} - f).$$

We wish to show that F indeed maps to Z and is a C^∞ map. Note that since a is smooth, the map

$$u \mapsto a(x, u)$$

takes $C^s(M)$ to $C^s(M)$, and if $\|u\|_{C^s(M)} \leq K$ then $\|a(x, u)\|_{C^s(M)} \leq C(a, s, K)$ (these facts follow from a local coordinate computation). Thus F is well defined. If $u, v \in C^s(M)$ we use the Taylor formula

$$a(x, u + v) = \sum_{j=0}^{m} \frac{\partial_u^j a(x, u)}{j!} v^j + \int_0^1 \frac{\partial_u^{m+1} a(x, u + tv)}{m!} v^{m+1} (1 - t)^m \, dt.$$

Since $C^s(M)$ is an algebra, we have that when $\|v\|_{C^s(M)} \leq 1$ one has

$$\left\| a(x, u + v) - \sum_{j=0}^{m} \frac{\partial_u^j a(x, u)}{j!} v^j \right\|_{C^s(M)} \leq C_{m,a,u} \|v\|_{C^s(M)}^{m+1}.$$

This shows that $u \mapsto a(x, u)$ is a C^∞ map $C^s(M) \to C^s(M)$. Since the other parts of F are linear, F is a C^∞ map in the standard sense of [37, Definition 10.2].

Note that $F(0, 0) = 0$ by (2.2). The linearization of F at $(0, 0)$ in the u-variable is

$$D_u F|_{(0,0)} (v) = (\Delta_g v + \partial_u a(x, 0)v, v|_{\partial M}).$$

This is a homeomorphism $Y \to Z$ by (2.3). To see this, let $(w, \phi) \in Z = C^{s-2}(M) \times C^s(\partial M)$, and consider the Dirichlet problem

$$\begin{cases} (\Delta_g + \partial_u a(x, 0))v = w & \text{in } M, \\ v = \phi & \text{on } \partial M. \end{cases} \tag{2.5}$$

If a solution to (2.5) exists, it is unique by (2.3). Consequently, by using the Fredholm alternative (see e.g. [27, Proposition 1.9]), we may solve (2.5) in $H_0^1(M)$ for any source in $H^{-1}(M)$ and zero boundary value. Thus, we have solutions v_1 and v_2 in $H_0^1(M)$ to (2.5) with sources w and $-(\Delta_g + \partial_u a(x, 0))\Phi$ respectively, where $\Phi \in C^s(\overline{M})$ is a function with $\Phi|_{\partial M} = \phi \in C^s(\partial M)$. Then $v = v_1 + v_2 + \Phi$ is the unique solution in $H^1(M)$ to (2.5). We have the well-known Schauder estimate

$$\|v\|_{C^s(M)} \leq C \left(\|w\|_{C^{s-2}(M)} + \|\Phi\|_{C^s(\overline{M})} \right),$$

for some constant $C > 0$ independent of $w \in C^{s-2}(M)$ and $\Phi \in C^s(\overline{M})$, which shows that solutions to (2.5) depend continuously on w and ϕ. (We have included a proof of the Schauder estimate in the manifold setting in Appendix Appendix B.)

The implicit function theorem in Banach spaces [37, Theorem 10.6 and Remark 10.5] now yields that there is $\delta > 0$ and an open ball $U_\delta = B_X(0,\delta) \subset X$ and a C^∞ map $S : U_\delta \to Y$ such that whenever $\|f\|_{C^s(\partial M)} \le \delta$ we have

$$F(f, S(f)) = (0,0).$$

Since S is Lipschitz continuous and $S(0) = 0$, $u = S(f)$ satisfies

$$\|u\|_{C^s(M)} \le C\|f\|_{C^s(\partial M)}.$$

Moreover, by redefining δ if necessary $u = S(f)$ is the only solution to $F(f,u) = (0,0)$ whenever $\|f\|_{C^s(\partial M)} \le \delta$ and $\|u\|_{C^s(M)} \le C\delta$. We have proven the existence of unique small solutions of the Dirichlet problem (2.4) and the fact that the solution operator $S : U_\delta \to C^s(M)$ is a C^∞ map. Since the normal derivative is a linear map $C^s(M) \to C^{s-1}(\partial M)$, it follows that also Λ is a well defined C^∞ map $U_\delta \to C^{s-1}(\partial M)$. □

In the rest of the paper, we consider power type nonlinearities of the form $a(x,u) = q(x)u^m$, where $m \in \mathbb{N}$ and $m \ge 2$. For such nonlinearities, the higher order linearizations of the DN map will be particularly simple. We will consider complex solutions u to the boundary value problem (2.4). We remark that even though the Proposition 2.1 was proven for real valued solution the proposition remains valid for the nonlinearity $a(x,u) = q(x)u^m$ by analyticity in u. In the rest of the work we will consider complex valued solutions without separate notice.

The next proposition justifies the formal calculation that we may differentiate the equation

$$\Delta_g u_f + q(x)u_f^m = 0 \text{ in } M, \quad u_f|_{\partial M} = \epsilon_1 f_1 + \cdots + \epsilon_m f_m \tag{2.6}$$

in the ϵ_j variables to have equations corresponding to first and mth linearizations,

$$\Delta_g v_{f_k} = 0 \text{ and } \Delta_g w = -(m!)q v_{f_1} \cdots v_{f_m}.$$

The normal derivative of w is the mth linearization of the DN map of (2.6). Below, we write

$$(D^k f)_x(y_1, \ldots, y_k)$$

to denote the kth derivative at x of a mapping f between Banach spaces, considered as a symmetric k-linear form acting on $(y_1, \ldots, y_k)$. We refer to [38, Section 1.1], where the notation $f^{(k)}(x; y_1, \ldots, y_k)$ is used instead of $(D^k f)_x(y_1, \ldots, y_k)$.

For $f \in C^s(\partial M)$ with $s > 2, s \notin \mathbb{N}$, let us denote by v_f the unique solution of the Laplace equation

$$\Delta_g v_f = 0 \text{ in } M, \qquad v_f|_{\partial M} = f. \tag{2.7}$$

By using this notation, we have the following result.

Proposition 2.2. *Let $q \in C^\infty(M)$, and let Λ_q be the DN map for the semilinear elliptic equation*

$$\Delta_g u + q(x)u^m = 0 \text{ in } M, \tag{2.8}$$

where

$$m \in \mathbb{N} \text{ and } m \ge 2.$$

The first linearization $(D\Lambda_q)_0$ *of* Λ_q *at* $f = 0$ *is the DN map of the Laplace equation* (2.7) *such that*

$$(D\Lambda_q)_0 : C^s(\partial M) \to C^{s-1}(\partial M), \quad f \mapsto \partial_\nu v_f|_{\partial M}.$$

The higher order linearizations $(D^j\Lambda_q)_0$ *are identically zero for* $2 \leq j \leq m-1$.

The m*-th linearization* $(D^m\Lambda_q)_0$ *of* Λ_q *at* $f = 0$ *is characterized by the following identity: for any* $f_1, \ldots, f_{m+1} \in C^s(\partial M)$ *one has*

$$\int_{\partial M} (D^m\Lambda_q)_0(f_1, \ldots, f_m) f_{m+1}\, dS = -(m!) \int_M q v_{f_1} \cdots v_{f_{m+1}}\, dV \tag{2.9}$$

here each v_{f_k}, $k = 1, \ldots, m+1$, *is the solution to* (2.7) *with boundary value* $f = f_k$.

Proof. The nonlinearity $a(x, u) = q(x)u^m$ satisfies the conditions in Proposition 2.1, and thus the DN map $\Lambda_q = \partial_\nu S|_{\partial M}$ is well defined for small data. Here $S : f \mapsto u_f$ is the solution operator for the Dirichlet problem of the equation (2.8). To compute the derivatives of Λ_q at 0, it is enough to consider the derivatives of S. Let us write $f = f(\epsilon_1, \ldots, \epsilon_k) := \epsilon_1 f_1 + \cdots + \epsilon_k f_k$. The function $u_f = S(\epsilon_1 f_1 + \cdots + \epsilon_k f_k) \in C^s(M)$ depends smoothly on $\epsilon_1, \ldots, \epsilon_k$ since $S : U_\delta \to C^s(M)$ is C^∞ Fréchet differentiable by Proposition 2.1. Applying $\partial_{\epsilon_1} \cdots \partial_{\epsilon_k}|_{\epsilon_1 = \cdots = \epsilon_k = 0}$ to the Taylor's formula for C^∞ Fréchet differentiable mappings (see e.g. [38, Equation 1.1.7])

$$S(f) = \sum_{j=0}^{k} \frac{(D^j S)_0(f, \ldots, f)}{j!} + \int_0^1 \frac{(D^{k+1}S)_{tf}(f, \ldots, f)}{k!}(1-t)^k\, dt$$

implies that $(D^k S)_0$ may be computed using the formula

$$(D^k S)_0(f_1, \ldots, f_k) = \partial_{\epsilon_1} \cdots \partial_{\epsilon_k} u_f|_{\epsilon_1 = \cdots = \epsilon_k = 0}.$$

Moreover, since u_f is smooth in the ϵ_j variables and Δ_g is linear, we may differentiate the equation

$$\Delta_g u_f + q(x) u_f^m = 0, \qquad u_f|_{\partial M} = f \tag{2.10}$$

freely in the ϵ_j variables.

Let first $k = 1$, so that $u = u_{\epsilon_1 f_1}$. Since $u_0 = 0$ and $m \geq 2$, the derivative of (2.10) in ϵ_1 evaluated at $\epsilon_1 = 0$ satisfies

$$\Delta_g(\partial_{\epsilon_1} u_f|_{\epsilon_1 = 0}) = 0, \qquad \partial_{\epsilon_1} u_f|_{\partial M} = f_1.$$

Thus the first linearization of the map S at $f = 0$ is

$$(DS)_0(f_1) = \partial_{\epsilon_1} u_{\epsilon_1 f_1}|_{\epsilon_1 = 0} = v_{f_1},$$

where v_{f_1} satisfies (2.7) with $f = f_1$.

For $2 \leq k \leq m-1$, applying $\partial_{\epsilon_1} \cdots \partial_{\epsilon_k}|_{\epsilon_1 = \cdots = \epsilon_k = 0}$ to (2.10) gives that

$$\Delta_g(\partial_{\epsilon_1} \cdots \partial_{\epsilon_k} u_f|_{\epsilon_1 = \cdots = \epsilon_k = 0}) = 0, \qquad \partial_{\epsilon_1} \cdots \partial_{\epsilon_k} u_f|_{\partial M} = 0,$$

since $\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}(q(x)u_f^m)$ is a sum of terms containing positive powers of u_f, which are equal to zero when $f=0$. Uniqueness of solutions for the Laplace equation implies that

$$(D^kS)_0(f_1,\dots,f_k)=0, \qquad 2\le k\le m-1.$$

When $k=m$, the only term in $\partial_{\epsilon_1}\cdots\partial_{\epsilon_m}(q(x)u_f^m)$ which does not contain second or higher order power of u_f is $q(x)(m!)(\partial_{\epsilon_1}u_f)\cdots(\partial_{\epsilon_m}u_f)$. This is the only nonzero term after setting $\epsilon_1=\ldots=\epsilon_m=0$, and thus the function

$$w:=(D^mS)_0(f_1,\dots,f_m)=\partial_{\epsilon_1}\cdots\partial_{\epsilon_m}u_f|_{\epsilon_1=\ldots=\epsilon_m=0}$$

solves

$$\Delta_g w=-q(x)(m!)v_{f_1}\cdots v_{f_m} \text{ in } M \tag{2.11}$$

with zero Dirichlet boundary values.

By linearity one has

$$(D^k\Lambda_q)_0=\partial_\nu(D^kS)_0|_{\partial M}.$$

The claims for $(D^k\Lambda_q)_0$ when $1\le k\le m-1$ follow immediately. For $k=m$ we observe that $(D^m\Lambda_q)_0(f_1,\dots,f_m)=\partial_\nu w|_{\partial M}$ satisfies

$$\int_{\partial M}(\partial_\nu w)f_{m+1}\,dS=\int_M(\langle dw,dv_{f_{m+1}}\rangle_g+(\Delta_g w)v_{f_{m+1}})\,dV,$$

where d denotes the exterior derivative on M. The integral of $\langle dw,dv_{f_{m+1}}\rangle_g$ vanishes since $w|_{\partial M}=0$ and $v_{f_{m+1}}$ is harmonic. The proposition follows by using (2.11). □

3. Proof of Theorem 1.1

In this section, we use the higher order linearization approach (in fact, the second order linearization of the DN map) to prove Theorem 1.1. We could use Proposition 2.2 to have the integral equation (3.6) below directly, even for the product of three harmonic functions instead of two (this is a stronger statement since one can always take the third harmonic function to be constant). The theorem would follow from this by using harmonic exponentials. However, we choose to give a direct hands-on approach that explains how to use the method.

Proof of Theorem 1.1. Let ϵ_1,ϵ_2 be sufficiently small numbers and let $f_1,f_2\in C^\infty(\partial M)$. Let the function $u_j:=u_j(x;\epsilon_1,\epsilon_2)\in C^s(M)$ be the unique small solution of

$$\begin{cases}\Delta u_j+q_ju_j^2=0 & \text{in } \Omega,\\ u_j=\epsilon_1f_1+\epsilon_2f_2 & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

for $j=1,2$ provided by Proposition 2.2. Let us differentiate (3.1) with respect to ϵ_ℓ so that

$$\begin{cases}\Delta\left(\frac{\partial}{\partial\epsilon_\ell}u_j\right)+2q_ju_j\left(\frac{\partial}{\partial\epsilon_\ell}u_j\right)=0 & \text{in } \Omega,\\ \frac{\partial}{\partial\epsilon_\ell}u_j=f_\ell & \text{on } \partial\Omega.\end{cases} \tag{3.2}$$

Inserting $\epsilon_1 = \epsilon_2 = 0$ into (3.2), shows that

$$\Delta v_j^{(\ell)} = 0 \text{ in } \Omega \text{ with } v_j^{(\ell)} = f_\ell \text{ on } \partial\Omega,$$

where

$$v_j^{(\ell)}(x) = \frac{\partial}{\partial \epsilon_\ell}\Big|_{\epsilon_1=\epsilon_2=0} u_j(x;\epsilon_1,\epsilon_2)\ .$$

Here we used $u_j(x;0,0) \equiv 0$. The functions v_j^ℓ are just harmonic functions defined in Ω with boundary data $f_\ell|_{\partial\Omega}$. By uniqueness of the Dirichlet problem for the Laplacian we have that

$$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega \text{ for } \ell = 1, 2. \tag{3.3}$$

Next, let us differentiate (3.2) with respect to ϵ_k for $k \neq \ell$. Then we have that

$$\begin{cases} \Delta\left(\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}u_j\right) + 2q_j u_j \left(\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}u_j\right) + 2q_j\left(\frac{\partial u_j}{\partial\epsilon_1}\right)\left(\frac{\partial u_j}{\partial\epsilon_2}\right) = 0 & \text{in } \Omega, \\ \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Again, evaluating at $\epsilon_1 = \epsilon_2 = 0$, the equation (3.4) becomes

$$\begin{cases} \Delta w_j + 2q_j v^{(1)} v^{(2)} = 0 & \text{in } \Omega, \\ w_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $w_j(x) = \left(\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}u_j\right)(x;0,0)$ and we used $u_j(x;0,0) \equiv 0$ for $j = 1, 2$ again. By using the fact that $\Lambda_{q_1}(\epsilon_1 f_1 + \epsilon_2 f_2) = \Lambda_{q_2}(\epsilon_1 f_1 + \epsilon_2 f_2)$ for small ϵ_1, ϵ_2, we have

$$\partial_\nu u_1|_{\partial\Omega} = \partial_\nu u_2|_{\partial\Omega},$$

and applying $\partial_{\epsilon_1}\partial_{\epsilon_2}|_{\epsilon_1=\epsilon_2=0}$ to this identity gives that

$$\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}.$$

Thus, by integrating the equation (3.5) over Ω (i.e. integrating against the harmonic function $v^{(3)} = 1$) and by using integration by parts we have

$$0 = \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, dS = \int_\Omega \Delta(w_1 - w_2)\, dx = 2\int_\Omega (q_2 - q_1) v^{(1)} v^{(2)}\, dx \tag{3.6}$$

where $v^{(1)}$ and $v^{(2)}$ are defined in (3.3). Therefore, by choosing f_1 and f_2 as the boundary values of the Calderón's exponential solutions [1],

$$v^{(1)}(x) := \exp((k + i\xi)\cdot x), \quad v^{(2)}(x) := \exp((-k + i\xi)\cdot x), \tag{3.7}$$

where k, $\xi \in \mathbb{R}^n$, $k \perp \xi$ and $|k| = |\xi|$, we obtain that the Fourier transformation of the difference $q_2 - q_1$ at -2ξ vanishes. As $\xi \in \mathbb{R}^n$ is arbitrary, we obtain $q_1 = q_2$. □

In the proof above we did not need to construct special solutions for an elliptic equation with unknown coefficients, such as complex geometrical optics solutions. The linearization technique allowed us to simply use known harmonic functions. This fact gives an extremely simple reconstruction in the setting of Theorem 1.1.

Corollary 3.1. *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Assume that $q \in C^\infty(\overline{\Omega})$, and let Λ_q be the DN map for the equation*

$$\Delta u + qu^2 = 0 \text{ in } \Omega. \tag{3.8}$$

Then

$$\widehat{q}(-2\xi) = -\frac{1}{2}\int_{\partial\Omega} \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\Big|_{\epsilon_1=\epsilon_2=0} \Lambda_q(\epsilon_1 f_1 + \epsilon_2 f_2)\, dS, \tag{3.9}$$

where f_1 and f_2 are the boundary values of the exponential solutions (3.7) and $\widehat{q}$ stands for the Fourier transform of q.

Proof. The proof of the reconstruction can be directly read from (3.6) in the proof of Theorem 1.1. □

We end of this section with a remark about the stability of the reconstruction formula in Corollary 3.1.

Remark 3.2. *Let us consider the stability of the solution of the inverse problem of Theorem 1.1, which regards determination of the potential q from the DN map of the equation $\Delta u + qu^2 = 0$. Let us assume as in Corollary 3.9 and adopt its notation with the difference that we consider the equation (3.8) with two different potentials q_1 and q_2 and the corresponding DN maps Λ_{q_1} and Λ_{q_2}. By the reconstruction formula (3.9) we have*

$$\widehat{q}_j(-2\xi) = -\frac{1}{2}\int_{\partial\Omega} (D^2\Lambda_{q_j})_0(f_1, f_2)\, dS, \text{ for } j = 1, 2.$$

By subtracting the above formula for $j = 1$ and $j = 2$ from each other, we obtain

$$(\widehat{q}_1 - \widehat{q}_2)(-2\xi) = -\frac{1}{2}\int_{\partial\Omega} ((D^2\Lambda_{q_1})_0 - (D^2\Lambda_{q_2})_0)(f_1, f_2)\, dS.$$

Now, we assume that

(1) $\|D^k(\Lambda_{q_1} - \Lambda_{q_2})_0\|_*$ is sufficiently small for $k = 0, 1, 2$, and

(2) $\|q_j\|_{H^1(\Omega)} \leq R$ for $j = 1, 2$,

where

$$\|T\|_* = \sup_{\|f_1\|_{C^s(\partial\Omega)} = \cdots = \|f_k\|_{C^s(\partial\Omega)} = 1} \|T(f_1, \ldots, f_k)\|_{C^{s-1}(\partial\Omega)}$$

for a bounded k-linear form $T : C^s(\partial\Omega) \times \cdots \times C^s(\partial\Omega) \to C^{s-1}(\partial\Omega)$. Next, by taking harmonic functions $v_{f_1} = v^{(1)}$, $v_{f_2} = v^{(2)}$ in Ω, where $v^{(1)}, v^{(2)}$ are the functions defined in (3.7), one can obtain that

$$\|q_1 - q_2\|_{L^2} \leq \omega\left(\|D^2(\Lambda_{q_1} - \Lambda_{q_2})_0\|_*\right), \tag{3.10}$$

where $\omega(t)$ is a modulus of continuity satisfying, for some $C = C(R)$,

$$\omega(t) \leq C|\log t|^{-\frac{2}{n+2}}, \quad 0 < t < \frac{1}{e}.$$

One can directly prove the logarithmic stability (3.10) by using standard arguments in stability for the Calderón problem, for example, see [39, Section 4].

4. Simultaneous recovery on two-dimensional Riemannian surfaces

We use the higher order linearization approach to simultaneously recover, from the DN map, the conformal class of a Riemannian surface and the potential of a semilinear Schrödinger operator up to the gauge symmetry. We use first order linearization to recover the conformal class of the manifold by using the result [30] (see also [18] for a recent alternative proof). Then by using the result [31] we recover the potential on the known conformal manifold (up to gauge).

Proof of Theorem 1.2. The proof is divided into two steps. We first recover the manifold and the conformal class of the Riemannian metric. After that we recover the potential on a known manifold up to the gauge symmetry.

Step 1. Recovering the conformal manifold.

Notice first by Proposition 2.2 that the equality

$$\Lambda_{M_1,g_1,q_1}(f) = \Lambda_{M_2,g_2,q_2}(f)$$

for all $f \in C^s(\partial M)$ with $\|f\|_{C^s(\partial M)} \leq \delta$, $\delta > 0$, implies that

$$(D\Lambda_{M_1,g_1,q_1})_0 = (D\Lambda_{M_2,g_2,q_2})_0.$$

By Proposition 2.2, the maps $(D\Lambda_{M_j,g_j,q_j})_0$, for $j = 1, 2$, are the DN maps of the linearizations of the equations $\Delta_{g_j} u + q_j u^m = 0$ in M_j at zero. The linearized equations are Laplace equations on (M_j, g_j). Since $D(\Lambda_{M_1,g_1,q_1})_0 = D(\Lambda_{M_2,g_2,q_2})_0$ we have that the DN maps of the Dirichlet problems

$$\begin{cases} \Delta_{g_j} v_j = 0 & \text{in } M_j, \\ v_j = f & \text{on } \partial M \end{cases}$$

agree. We are in the setting of the standard anisotropic Calderón problem on 2-dimensional Riemannian manifolds. We apply [18, Theorem 5.1] (with $\Gamma = \partial M$) to determine the manifold and the Riemannian metric up to a conformal transformation. That is, there exists a C^∞ smooth diffeomorphism $J : M_1 \to M_2$ such that

$$\sigma J^* g_2 = g_1$$

with $J|_{\partial M} = \mathrm{Id}$. Here $\sigma \in C^\infty(M_1)$ is a positive function with $\sigma|_{\partial M} = 1$. This completes the Step 1. of the proof.

Step 2. Recovering the potential.

We transform the equation (1.6) on the manifold (M_2, g_2) into an equation on the manifold (M_1, g_1) as follows. We denote

$$\widetilde{q}_2 = \sigma^{-1} q_2 \circ J \equiv \sigma^{-1} J^* q_2.$$

Let $f \in C^s(\partial M)$ with $\|f\|_{C^s(\partial M)} \leq \delta$ and let u_2 be the unique solution to

$$\Delta_{g_2} u_2 + q_2 u_2^m = 0 \text{ in } M_2 \text{ with } u_2 = f \text{ on } \partial M$$

given by Proposition 2.1. Let us denote

$$\widetilde{u}_2 := J^* u_2 \equiv u_2 \circ J.$$

Then $\widetilde{u}_2$ solves

$$\begin{aligned}
\Delta_{g_1} \widetilde{u}_2 + \widetilde{q}_2 (\widetilde{u}_2)^m &= \Delta_{\sigma J^* g_2} \widetilde{u}_2 + \widetilde{q}_2 (\widetilde{u}_2)^m \\
&= \sigma^{-1} \Delta_{J^* g_2} \widetilde{u}_2 + \sigma^{-1} (J^* q_2)(\widetilde{u}_2)^m \\
&= \sigma^{-1} J^* (\Delta_{g_2} u_2) + \sigma^{-1} (J^* q_2)(J^* u_2)^m \\
&= \sigma^{-1} J^* \left[\Delta_{g_2} u_2 + q_2 u_2^m \right].
\end{aligned}$$

Here we used the conformal invariance of the Laplace-Beltrami operator in dimension 2 in the second equality. In the third equality, the coordinate invariance of Laplace-Beltrami operator was used. Since u_2 solves $\Delta_{g_2} u_2 + q_2 u_2^m = 0$ in M_2, we consequently have that

$$\begin{cases} \Delta_{g_1} \widetilde{u}_2 + \widetilde{q}_2 (\widetilde{u}_2)^m = 0 & \text{in } M_1, \\ \widetilde{u}_2 = f & \text{on } \partial M. \end{cases} \tag{4.1}$$

Here we also used $J|_{\partial M} = \mathrm{Id}$.

Next, let u_1 be the unique solution to the nonlinear equation (1.6) on (M_1, g_1) with potential q_1 and boundary value f. We show that the following equation

$$\partial_{\nu_1} u_1 = \partial_{\nu_1} \widetilde{u}_2 \text{ on } \partial M, \tag{4.2}$$

holds by the assumption that $\Lambda_{M_1,g_1,q_1}(f) = \Lambda_{M_2,g_2,q_2}(f)$. Since $\Lambda_{M_1,g_1,q_1} = \Lambda_{M_2,g_2,q_2}$, it follows that if $u_1 = u_2 = f$ on ∂M, then by definition

$$\partial_{\nu_1} u_1 = \partial_{\nu_2} u_2 \text{ on } \partial M. \tag{4.3}$$

We calculate

$$\partial_{\nu_2} u_2 = \nu_2 \cdot du_2 = \nu_2 \cdot d(u_2 \circ J \circ J^{-1}) = (J_*^{-1} \nu_2) \cdot d\tilde{u}_2 = \nu_1 \cdot d\tilde{u}_2 = \partial_{\nu_1} \widetilde{u}_2. \tag{4.4}$$

Here $\cdot$ denotes the canonical pairing between vectors and covectors, and d is the exterior derivative of a function. For example $\nu_2 \cdot du_2 = g(\nu_2, \nabla u_2) = \sum_{k=1}^2 \nu_2^k \, \partial_k u_2$. In calculating (4.4), we used that $J : M_1 \to M_2$ is conformal diffeomorphism, $\sigma J^* g_2 = g_1$, with $J|_{\partial M} = \mathrm{Id}$ and $\sigma|_{\partial M} = 1$. By combining (4.3) and (4.4) we have (4.2). Since the solution $\widetilde{u}_2$ is unique, we have that

$$\Lambda_{M_1,g_1,q_1}(f) = \widetilde{\Lambda}_{M_1,g_1,\tilde{q}_2}(f), \tag{4.5}$$

for all $f \in C^s(\partial M)$ with $\|f\|_{C^s(\partial M)} \leq \delta$, where $\widetilde{\Lambda}_{M_1,g_1,\tilde{q}_2}$ stands for the DN map of the Dirichlet problem (4.1).

We apply Proposition 2.2 on the single Riemannian manifold (M_1, g_1) for the DN maps Λ_{M_1,g_1,q_1} and $\widetilde{\Lambda}_{M_1,g_1,\tilde{q}_2}$, which agree by (4.5). By Proposition 2.2 we have

$$(D^2 \Lambda_{M_1,g_1,q_1})_0 = (D^2 \widetilde{\Lambda}_{M_1,g_1,\tilde{q}_2})_0$$

and

$$\int_{M_1} (q_1 - \widetilde{q}_2) v_1 v_2 v_3 \, dV = 0,$$

where v_1, v_2, $v_3 \in C^s(M_1)$ are harmonic functions in (M_1, g_1). Choosing $v_3 = 1$, we get

$$\int_{M_1} (q_1 - \widetilde{q}_2) v_1 v_2 \, dV = 0$$

for any harmonic functions v_1 and v_2 in (M_1, g_1). We choose v_1 and v_2 to be complex geometrical optics solutions constructed in [31]. (See the construction in the proof of Proposition 5.1 in [31]. We note that the construction can in fact be significantly simplified in our case where v_1 and v_2 are actually harmonic. In this case, Carleman estimates are not needed and the construction in [35] would suffice.) As in [31, Proposition 5.1], this yields that

$$q_1 = \widetilde{q}_2 \text{ in } M_1.$$

This concludes the proof. □

5. Transversally anisotropic manifolds: simplified case

In this and the next section we prove Theorem 1.3, which will be a consequence of the following proposition. The proof of the proposition is based on the existence of special harmonic functions on transversally anisotropic manifolds. These harmonic functions were constructed in [33]. They have the property that if (M, g) is a transversally anisotropic manifold, i.e. $(M, g) \subset\subset \mathbb{R} \times M_0$, $g = e \oplus g_0$, then on the transversal manifold M_0 these harmonic functions concentrate near the geodesics of (M_0, g_0).

Proposition 5.1. *Let (M, g) be a transversally anisotropic manifold and assume that $m \geq 4$. If $f \in C^1(M)$ satisfies*

$$\int_M f u_1 \cdots u_m \, dV = 0 \tag{5.1}$$

for any $u_j \in C^\infty(M)$ with $\Delta_g u_j = 0$ in M, then $f \equiv 0$.

Theorem 1.3 follows immediately from Proposition 5.1:

Proof of Theorem 1.3. Let Λ_{q_j} be the DN map for the equation $\Delta_g u + q u^m = 0$ in M. If $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ for small f, then $(D^m \Lambda_{q_1})_0 = (D^m \Lambda_{q_2})_0$. Thus by Proposition 2.2, one has

$$\int_M (q_1 - q_2) v_1 \cdots v_{m+1} \, dV = 0$$

where $v_j \in C^s(M)$ are harmonic functions in M. Since $m \geq 3$, it follows from Proposition 5.1 that $q_1 = q_2$. □

The harmonic functions u_ℓ for $\ell = 1, 2, \dots, m$ on M used in the proof of Proposition 5.1 are of the form

$$e^{-s x_1} (\widetilde{v}_s(x') + r_s(x))$$

where x_1 is the coordinate along $\mathbb{R}$ and $x = (x_1, x')$. They may be considered as an analogue of complex geometrical optics solutions for transversally anisotropic manifolds. Here $\widetilde{v}_s$, $s = \tau + i\lambda$, is a so called *Gaussian beam quasimode* on M_0, i.e. an approximate eigenfunction concentrating near a geodesic on (M_0, g_0) with (slightly complex) large frequency s. The function r_s is a small correction term. Such harmonic functions were introduced in [40] and [33]. Since we need some additional properties of these harmonic functions, we include a few statements regarding these functions. We have placed their proofs in Appendix Appendix A for readers' convenience.

We say that a geodesic $\gamma : [0, T] \to M$ is *nontangential* if $\gamma(0)$ and $\gamma(T)$ are on ∂M, $\gamma(t) \in M^{\mathrm{int}}$ for $0 < t < T$, and $\dot\gamma(0)$ and $\dot\gamma(T)$ are not tangential to ∂M. We remark that we will apply the following proposition in the case where (M, g) is a transversal manifold (M_0, g_0).

Proposition 5.2 (Gaussian beams quasimodes)**.** *Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M, $\dim(M) = m$. Let $\gamma : [0, T] \to M$ be a nontangential geodesic, and let $\lambda \in \mathbb{C}$. For any $K \in \mathbb{N}$ and $k \in \mathbb{N}$, there is a family of functions $(\widetilde{v}_s) \subset C^\infty(M)$, where $s = \tau + i\lambda \in \mathbb{C}$ and $\tau \geq 1$, such that*

$$\begin{gathered}\|(-\Delta_g - s^2)\widetilde{v}_s\|_{H^k(M)} = O(\tau^{-K}),\\ \|\widetilde{v}_s\|_{L^4(M)} = O(1), \qquad \|\widetilde{v}_s\|_{L^4(\partial M)} = O(1)\end{gathered} \tag{5.2}$$

as $\tau \to \infty$. The functions $\widetilde{v}_s$ have the following properties: If $p \in \gamma([0, T])$, then there is $P \in \mathbb{N}$ such that on a neighborhood U of p the function $\widetilde{v}_s$ is a finite sum

$$\widetilde{v}_s = \widetilde{v}^{(1)} + \cdots + \widetilde{v}^{(P)} \tag{5.3}$$

where $t_1 < \ldots < t_P$ are the times in $[0, T]$ such that $\gamma(t_l) = p$. Each $\widetilde{v}^{(l)}$ has the form

$$\widetilde{v}^{(l)} = \tau^{-\frac{m-1}{8}} e^{is\Theta^{(l)}} a^{(l)}, \tag{5.4}$$

where each $\Theta = \Theta^{(l)}$ is a smooth complex function in U satisfying

$$\begin{aligned}&\Theta(\gamma(t)) = t, \quad \nabla\Theta(\gamma(t)) = \dot\gamma(t),\\ &\mathrm{Im}(\nabla^2\Theta(\gamma(t))) \geq 0, \quad \mathrm{Im}(\nabla^2\Theta)(\gamma(t))|_{\dot\gamma(t)^\perp} > 0,\end{aligned} \tag{5.5}$$

for t close to t_l. Here $a^{(l)}(\gamma(t)) = \tau^{\frac{m-1}{4}}(a_0^{(l)}(\gamma(t)) + O(\tau^{-1}))$ where $a_0^{(l)}(\gamma(t))$ is nonvanishing and independent of τ, and the support of $a^{(l)}$ can be taken to be in any small neighborhood of $\gamma([0, T])$ chosen beforehand.

We remark that if the geodesic γ in Proposition 5.2 has no self-intersections, then the formula for $\widetilde{v}_s$ simplifies to

$$\widetilde{v}_s(x) = \tau^{-\frac{m-1}{8}} e^{is\Theta(x)} a(x), \tag{5.6}$$

where $\Theta = \Theta(x)$ satisfies (5.5). We use this fact in Section 5 below.

The following proposition describes complex geometrical optics (CGO) solutions on transversally anisotropic manifolds.

Proposition 5.3 (CGO solutions)**.** *Let $(M, g) \subset\subset (\mathbb{R} \times M_0, g)$ be a transversally anisotropic manifold with $g = e \oplus g_0$. We write $x \in M$ as $x = (x_1, x') \in \mathbb{R} \times M_0$. Let R, $k \in \mathbb{N}$. There exists $\tau_0 \geq 1$ such that for any fixed real number λ and for* *any τ with $|\tau| \geq \tau_0$ there is a solution of the equation $-\Delta_g u = 0$ in M having the form*

$$u_s(x) = e^{-sx_1}(\widetilde{v}_s(x') + r_s(x)).$$

Here $s = \tau + i\lambda$, $\widetilde{v}_s$ is a family as in Proposition 5.2 in (M_0, g_0) (so that $m = n-1$) with $K = K(R, k)$ chosen large enough, and

$$\|r_s\|_{H^k(M)} = O(\tau^{-R}) \text{ as } |\tau| \to \infty.$$

Before we prove Proposition 5.1 in the general case, we show how the proposition is proved in a simplified setting. The main idea of the proof of Proposition 5.1 is more transparent in the simplified setting.

Proof of Proposition 5.1 in a simplified setting. Let $(M, g) \subset\subset (\mathbb{R} \times M_0, g)$ be a transversally anisotropic manifold, where (M_0, g_0) is a compact Riemannian manifold with smooth boundary and $g = e \oplus g_0$. We call M_0 the transversal manifold, and denote by x_1 the coordinate along $\mathbb{R}$. Let us make the following simplifying assumption:

Assumption A. For each point y_0 in the transversal manifold M_0, there exist two distinct nontangential geodesics γ and η that intersect at y_0 and which have no other intersection points, and also γ and η do not self-intersect.

This assumption is valid for instance when g_0 is a small perturbation of the Euclidean metric on a domain $M_0 \subset \mathbb{R}^{n-1}$, or more generally if (M_0, g_0) is a simple manifold (i.e. M_0 is diffeomorphic to a ball, ∂M_0 is strictly convex, and no geodesic in M_0 has conjugate points). We now prove Proposition 5.1 under Assumption A. The situation of the proof under Assumption A is depicted in Figure 5.

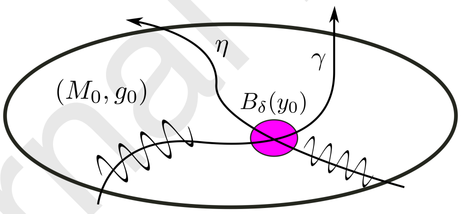

Figure 5: A product of once intersecting Gaussian beams localizes near their intersection point and oscillate.

Let $y_0 \in M_0$ and let γ and η be geodesics as described above. By applying Proposition 5.3, we have the following four harmonic functions in (M, g):

$$u_1(x) = e^{-(\tau+i\lambda)x_1}(\widetilde{v}_{\tau+i\lambda}(x') + r_1(x)), \qquad u_2(x) = \overline{e^{(\tau+i\lambda)x_1}(\widetilde{v}_{\tau+i\lambda}(x') + r_2(x))},$$
$$u_3(x) = e^{-\tau x_1}(\widetilde{w}_\tau(x') + r_3(x)), \qquad u_4(x) = \overline{e^{\tau x_1}(\widetilde{w}_\tau(x') + r_4(x))}.$$

Here $\widetilde{v}_{\tau+i\lambda}$ and $\widetilde{w}_\tau$ are the Gaussian beams introduced in Proposition 5.2 corresponding to the geodesics γ and η, respectively. Since by assumption γ and η do not have self-intersections, they are of the form

$$\widetilde{v}_{\tau+i\lambda}(x') = \tau^{-\frac{n-2}{8}} e^{i(\tau+i\lambda)\Phi(x')} a(x') \quad \text{and} \quad \widetilde{w}_\tau(x') = \tau^{-\frac{n-2}{8}} e^{i\tau\Psi(x')} b(x').$$

Here Φ and Ψ are phase functions that satisfy

$$\begin{aligned}
&\Phi(\gamma(t)) = t, \quad \nabla\Phi(\gamma(t)) = \dot\gamma(t), \quad \operatorname{Im}(\nabla^2\Phi(\gamma(t))) \geq 0, \quad \operatorname{Im}(\nabla^2\Phi)|_{\dot\gamma(t)^\perp} > 0,\\
&\Psi(\eta(t)) = t, \quad \nabla\Psi(\eta(t)) = \dot\eta(t), \quad \operatorname{Im}(\nabla^2\Psi(\eta(t))) \geq 0, \quad \operatorname{Im}(\nabla^2\Psi)|_{\dot\eta(t)^\perp} > 0,
\end{aligned}$$

and the functions a and b are amplitude functions. In particular, if z are normal coordinates on M_0 centered at y_0, one has

$$\operatorname{Im}(\Phi)(z) = \frac{1}{2}\operatorname{Hess}(\Phi)|_{y_0} z \cdot z + O(|z|^3) \tag{5.7}$$

where $\operatorname{Hess}(\Phi)|_{y_0}$ is the Hessian at y_0 in normal coordinates, and similarly for $\operatorname{Im}(\Psi)$. Without loss of generality, we may assume that the supports of a and b are contained in any fixed neighborhood of the graphs of γ and η. Furthermore, the functions r_ℓ, $1 \leq \ell \leq 4$, are small correction terms, which satisfy $\|r_\ell\|_{H^k(M)} = O(\tau^{-R})$ for any k, $R \in \mathbb{N}$ fixed beforehand. We choose some $k > n/4$, so that Sobolev embedding gives $\|r_\ell\|_{L^4(M)} = O(\tau^{-R})$.

Since $\widetilde v_{\tau+i\lambda}$ and $\widetilde w_\tau$ are supported near γ and η, respectively, and since γ and η intersect only at the single point y_0, the product $u_1u_2u_3u_4$ is supported near the point y_0 on the transversal manifold M_0. By applying the assumption (5.1) of Proposition 5.1 to the solutions u_ℓ, $1 \leq \ell \leq 4$, and extending f by zero to $\mathbb{R} \times M_0$, we have that

$$\begin{aligned}
0 &= \int_M f u_1u_2u_3u_4 \, dV\\
&= \int_{-\infty}^{\infty}\int_{M_0} f\, e^{-2i\lambda x_1} |\widetilde v_{\tau+i\lambda}|^2 |\widetilde w_\tau|^2 \, dV_{g_0}\, dx_1 + O\left(\tau^{-R}\right)\\
&= \int_{B_\delta(y_0)} \hat f(2\lambda, \cdot) e^{-2\tau \operatorname{Im}(\Phi+\Psi)} A \, dV_{g_0} + O\left(\tau^{-R}\right).
\end{aligned} \tag{5.8}$$

In the second equality, we used that $\|f\|_{L^\infty}$, $\|\widetilde v_{\tau+i\lambda}\|_{L^4(M_0)}$ and $\|\widetilde w_\tau\|_{L^4(M_0)}$ are $O(1)$ as $\tau \to \infty$ and that $\|r_\ell\|_{L^4(M)} = O(\tau^{-R})$. In the last equality, $B_\delta(y_0)$ is a geodesic ball of radius $\delta > 0$ in M_0 centered at $y_0 \in M_0$ that contains the support of

$$A = A(\tau) := \tau^{-\frac{n-2}{2}} e^{-2\lambda \operatorname{Re}(\Phi)} |a|^2 |b|^2, \tag{5.9}$$

and $\hat f$ denotes the partial Fourier transform of f with respect to the x_1-variable.

The point here is as follows. The Hessians of $\operatorname{Im}(\Phi)$ and $\operatorname{Im}(\Psi)$ at y_0 are positive definite in directions orthogonal to $\dot\gamma$ and $\dot\eta$, respectively. Overall, the Hessians of $\operatorname{Im}(\Phi)$ and $\operatorname{Im}(\Psi)$ are also nonnegative definite. It follows that the sum of the Hessians of Φ and Ψ at y_0 is positive definite:

$$\operatorname{Im}(\operatorname{Hess}(\Phi) + \operatorname{Hess}(\Psi))|_{y_0} > 0. \tag{5.10}$$

By (5.10) and by choosing δ small enough, we have for some $C, c > 0$ that

$$|e^{-2\tau \operatorname{Im}(\Phi+\Psi)}| \leq C e^{-c\tau|z|^2} \text{ in } B_\delta(y_0), \tag{5.11}$$

where z are normal coordinates in $B_\delta(y_0)$ such that $z = 0 \in \mathbb{R}^{n-1}$ corresponds to $y_0 \in M_0$. By plugging (5.9) into $\int_{B_\delta(y_0)} \hat f(2\lambda, \cdot) e^{-2\tau \operatorname{Im}(\Phi+\Psi)} A \, dV_{g_0}$, we will make use of the change of variables $z \to \tau^{-1/2}z$ in the last integral in (5.8), which results in a Jacobian $\tau^{-\frac{n-1}{2}}$. Multiplying (5.8) by $\tau^{1/2}$, and using the new variables, we have as $\tau \to \infty$ that

$$\begin{aligned}
O(\tau^{-R+1/2}) &= \tau^{1/2}\int_M f u_1 u_2 u_3 u_4 \, dV_g \\
=&\tau^{1/2}\int_{B_\delta(y_0)} \hat{f}(2\lambda, z) e^{-2\tau \mathrm{Im}(\Phi(z)+\Psi(z))} A(z)\, dV_{g_0}(z) \\
=&\tau^{1/2}\tau^{-\frac{n-2}{2}}\int_{\mathbb{R}^{n-1}} \hat{f}(2\lambda, z) e^{-2\tau \mathrm{Im}(\Phi(z)+\Psi(z))} e^{-2\lambda \mathrm{Re}(\Phi(z))} |a(z)|^2 |b(z)|^2 \, dV_{g_0}(z) \\
=&\tau^{1/2}\tau^{-\frac{n-2}{2}}\tau^{-\frac{n-1}{2}} \\
&\times \int_{\mathbb{R}^{n-1}} \hat{f}\big(2\lambda, z/\tau^{1/2}\big) e^{-2\tau \mathrm{Im}(\Phi(z/\tau^{1/2})+\Psi(z/\tau^{1/2}))} e^{-2\lambda \mathrm{Re}(\Phi(z/\tau^{1/2}))} \\
&\qquad \times |a(z/\tau^{1/2})|^2 |b(z/\tau^{1/2})|^2 |g_0(z/\tau^{1/2})|^{1/2}\, dz \\
=&\tau^{-n+2}\int_{\mathbb{R}^{n-1}} \hat{f}\big(2\lambda, z/\tau^{1/2}\big) e^{-2\tau \mathrm{Im}(\Phi(z/\tau^{1/2})+\Psi(z/\tau^{1/2}))} e^{-2\lambda \mathrm{Re}(\Phi(z/\tau^{1/2}))} \\
&\qquad \times |\tau^{\frac{n-2}{4}}\big(a_0(z/\tau^{1/2}) + O(\tau^{-1})\big)|^2 |\tau^{\frac{n-2}{4}}\big(b_0(z/\tau^{1/2}) + O(\tau^{-1})\big)|^2 \\
&\qquad \times |g_0(z/\tau^{1/2})|^{1/2}\, dz \\
=:&\mathcal{K}(\tau).
\end{aligned}$$

Then we have

$$\lim_{\tau\to\infty} \mathcal{K}(\tau) = c\hat{f}(2\lambda, y_0).$$

Here the constant

$$c = |a_0(y_0)|^2 |b_0(y_0)|^2 \int_{\mathbb{R}^{n-1}} e^{-\left(\mathrm{Im}(\mathrm{Hess}(\Phi)+\mathrm{Hess}(\Psi))|_{y_0} z\right)\cdot z}\, dz \neq 0.$$

Here we have used (5.7) and the Lebesgue dominated convergence theorem, which was justified by the condition (5.11). We have also extended functions in the above integrals by zero outside $\mathrm{supp}(A) \subset B_\delta(y_0)$. Thus $\hat{f}(2\lambda, y_0) = 0$. Since $\lambda \in \mathbb{R}$ and $y_0 \in M_0$ were arbitrary, this proves Proposition 5.1 in this simplified case.

6. Transversally anisotropic manifolds: general case

To prove Proposition 5.1 on general transversally anisotropic manifolds, we need to consider the possibility where geodesics γ and η on M_0 may intersect at many different points and they may have self-intersections. The proof will be achieved by introducing additional parameters to the complex geometrical optics solutions of Proposition 5.3, and varying these parameters. These additional considerations make the proof of Proposition 5.1 more technical than in the simplified case considered in Section 5. We also use the following two lemmas, which will be proved in Appendix Appendix B.

Lemma 6.1 (Finitely many intersecting geodesics)**.** *Let (M_0, g_0) be a compact Riemannian manifold with strictly convex smooth boundary ∂M_0. There is a set E of measure zero in M_0 such that if $y_0 \in M_0 \setminus E$, there exist nontangential geodesics γ and η on M_0 that intersect at y_0, self-intersect only finitely many times and intersect each other only finitely many times.*

Lemma 6.2. *Let $f_1, \dots, f_N$ be compactly supported distributions in $\mathbb{R}$ such that for some distinct real numbers $a_1, \dots, a_N$ one has*

$$\sum_{j=1}^{N} \hat{f}_j(\lambda) e^{a_j \lambda} = 0, \qquad \lambda \in \mathbb{R}.$$

Then $f_1 = \dots = f_N = 0$.

We now prove Proposition 5.1 in the general case.

Proof of Proposition 5.1. We do the proof in several steps.

Step 1. Choice of the harmonic functions u_j.

By taking $u_j = 1$ for $j \geq 5$, it is sufficient to prove the result when $m = 4$. By the assumption we have that $M \subset\subset \mathbb{R} \times M_0$ with $g = e \oplus g_0$. The dimension of M is denoted by n, and so $\dim(M_0) = n - 1$. We may enlarge M_0 so that it has strictly convex boundary (first embed M_0 in some closed manifold M_1, remove a small geodesic ball from $M_1 \setminus M_0$ and glue a part with strictly convex boundary near the removed part). Let $E \subset M_0$ be as in Lemma 6.1 and let $y_0 \in M_0 \setminus E$. Let also γ and η be the geodesics on M_0 given by Lemma 6.1. That is, the nontangential geodesics γ and η intersect at y_0, self-intersect only finitely many times and intersect each other only finitely many times.

We denote the points x of $\mathbb{R} \times M_0$ by (x_1, x'). We apply Proposition 5.3 in the case where the parameter s in the proposition is set to $L(\tau + i\lambda)$, where $\tau \geq 1$ is sufficiently large, $L \geq 1$ is an additional large parameter that will be fixed later, and $\lambda \in \mathbb{R}$ is fixed. Thus we have harmonic functions of the form

$$\begin{aligned} u_1(x) &= e^{-L(\tau+i\lambda)x_1}(\widetilde{v}_{L(\tau+i\lambda)}(x') + r_1(x)), \\ u_2(x) &= \overline{e^{L(\tau+i\lambda)x_1}(\widetilde{v}_{L(\tau+i\lambda)}(x') + r_2(x))}, \end{aligned}$$

where $\Delta_g u_j = 0$ in M, $j = 1, 2$. Here

$$\widetilde{v}_{L(\tau+i\lambda)}(x')$$

is a Gaussian beam quasimode concentrating near the geodesic γ in M_0 of Proposition 5.2 and r_j, $j = 1, 2$, are remainder terms satisfying

$$\|r_j\|_{H^k(M)} = O(\tau^{-R})$$

as $\tau \to \infty$ where k, $R > 0$ can be chosen arbitrarily large. We have that

$$u_1 u_2 = e^{-2iL\lambda x_1} |\widetilde{v}_{L(\tau+i\lambda)}(x')|^2 + O_{L^2(M)}((L\tau)^{-R}). \tag{6.1}$$

By using Proposition 5.3 again, we choose solutions u_3 and u_4 now related to the geodesic η of the form

$$u_3 = e^{-(\tau+i\mu)x_1}(\widetilde{w}_{\tau+i\mu}(x') + r_3), \qquad u_4 = \overline{e^{(\tau+i\mu)x_1}(\widetilde{w}_{\tau+i\mu}(x') + r_4)},$$

where $\mu \in \mathbb{R}$ is fixed, $\|r_j\|_{H^k(M)} = O(\tau^{-R})$ as $\tau \to \infty$. Here $\widetilde{w}_{\tau+i\mu}$ is a Gaussian beam quasimode concentrating near η as in Proposition 5.2. Similarly as for u_1u_2 in (6.1), we have that

$$u_3u_4 = e^{-2i\mu x_1}|\widetilde{w}_{\tau+i\mu}(x')|^2 + O_{L^2(M)}(\tau^{-R}). \tag{6.2}$$

Step 2. The integral of f against $u_1u_2u_3u_4$.

By the assumption that f integrates to zero against products of four harmonic functions, we have

$$\int_M fu_1u_2u_3u_4\, dV = 0.$$

Using (6.1) and (6.2), this implies that

$$0 = \int_M f(x_1, x')e^{-2i(L\lambda+\mu)x_1}|\widetilde{v}_{L(\tau+i\lambda)}(x')|^2|\widetilde{w}_{\tau+i\mu}(x')|^2\, dV + O(\tau^{-R}).$$

If we extend f by zero to $\mathbb{R} \times M_0$ and denote the partial Fourier transform of f with respect to the x_1 variable by $\hat{f}(\lambda, x')$, then the previous identity becomes

$$0 = \int_{M_0} \hat{f}(2(L\lambda+\mu), \cdot\,)|\widetilde{v}_{L(\tau+i\lambda)}|^2|\widetilde{w}_{\tau+i\mu}|^2\, dV_{g_0} + O(\tau^{-R}). \tag{6.3}$$

Note that $\widetilde{v}_{L(\tau+i\lambda)}$ and $\widetilde{w}_{\tau+i\mu}$ can be chosen to be supported in arbitrarily small but fixed neighborhoods of γ and η, respectively. Thus if $p_1, \ldots, p_N$ are the distinct intersection points of geodesics γ and η in M_0, then the integral over M_0 in (6.3) is actually over $U_1 \cup \cdots \cup U_N$ where U_r is a small neighborhood of p_r in M_0, for $r = 1, \ldots, N$.

In the following, we denote

$$F(x') = F_{L\lambda+\mu}(x') := \hat{f}(2(L\lambda+\mu), x'). \tag{6.4}$$

Note for later purposes that

$$\|F\|_{C^1(M_0)} \lesssim \|f\|_{C^1(M)}.$$

(Here and in the rest of the paper the notation $a \lesssim b$ means, as usual, that there is a constant $c > 0$ such that $a \leq cb$.) Combining the above facts, we have that

$$\sum_{r=1}^{N} \tau^{\frac{1}{2}} \int_{U_r} F|\widetilde{v}_{L(\tau+i\lambda)}|^2|\widetilde{w}_{\tau+i\mu}|^2\, dV_{g_0} = o(1) \tag{6.5}$$

as $\tau \to \infty$. Here we also have multiplied (6.3) by a normalization factor $\tau^{\frac{1}{2}}$. It will be shown below that with the normalizing factor $\tau^{\frac{1}{2}}$, the left hand side has a nontrivial limit as $\tau \to \infty$.

Step 3. Analysis of the integrals in (6.5).

Fix now p to be one of the intersection points p_r, for some $r = 1, 2, \cdots, P$, and let us denote $U = U_r \subset M_0$ and $p = p_r \in M_0$. We consider the integral

$$\int_U F|\widetilde{v}_{L(\tau+i\lambda)}|^2|\widetilde{w}_{\tau+i\mu}|^2\, dV_{g_0}. \tag{6.6}$$

By Proposition 5.2, in U the quasimode $\widetilde{v}_{L(\tau+i\lambda)}$ is a finite sum

$$\widetilde{v}_{L(\tau+i\lambda)}|_U = \widetilde{v}^{(1)} + \cdots + \widetilde{v}^{(P)},$$

where $t_1 < \ldots < t_P$ are the times in $[0,T]$ such that $\gamma(t_j) = p$, each $\widetilde{v}^{(j)}$ has the form

$$\widetilde{v}^{(j)} = \tau^{-\frac{n-2}{8}} e^{iL(\tau+i\lambda)\Phi^{(j)}} a^{(j)},$$

where each $\Phi = \Phi^{(j)}$ is a smooth complex function in the neighborhood U of p satisfying

$$\Phi(\gamma(t)) = t, \quad \nabla\Phi(\gamma(t)) = \dot{\gamma}(t), \quad \mathrm{Im}(\nabla^2\Phi(\gamma(t))) \geq 0, \quad \mathrm{Im}(\nabla^2\Phi)|_{\dot{\gamma}(t)^\perp} > 0,$$

for t close to t_j, and each $a^{(j)}$ is supported in any fixed neighborhood of the graph of γ and $a^{(j)}(p) \neq 0$.

In a similar way, $\widetilde{w}_{\tau+i\mu}$ in U is a finite sum

$$\widetilde{w}_{\tau+i\mu}|_U = \widetilde{w}^{(1)} + \ldots + \widetilde{w}^{(Q)}$$

where $s_1 < \ldots < s_Q$ are the times in $[0,S]$ such that $\eta(s_k) = p$. The function $\widetilde{w}^{(k)}$ has the form

$$\widetilde{w}^{(k)} = \tau^{-\frac{n-2}{8}} e^{i(\tau+i\mu)\Psi^{(k)}} b^{(k)},$$

where each $\Psi = \Psi^{(k)}$ satisfies

$$\Psi(\eta(s)) = s, \quad \nabla\Psi(\eta(s)) = \dot{\eta}(s), \quad \mathrm{Im}(\nabla^2\Psi(\eta(s))) \geq 0, \quad \mathrm{Im}(\nabla^2\Psi)|_{\dot{\eta}(s)^\perp} > 0,$$

for s close to s_k, and each $b^{(j)}$ is supported in any fixed neighborhood of the graph of η and $b^{(j)}(p) \neq 0$.

Inserting the formulas for $\widetilde{v}_{L(\tau+i\lambda)}$ and $\widetilde{w}_{\tau+i\mu}$ in (6.6) yields that

$$\tau^{\frac{1}{2}} \int_U F|\widetilde{v}_{L(\tau+i\lambda)}|^2 |\widetilde{w}_{\tau+i\mu}|^2 \, dV_{g_0} = \sum_{j,k=1}^{P} \sum_{l,m=1}^{Q} I_{jklm} \tag{6.7}$$

where

$$\begin{aligned} I_{jklm} &= \tau^{\frac{1}{2}} \int_U F \widetilde{v}^{(j)} \overline{\widetilde{v}^{(k)}} \widetilde{w}^{(l)} \overline{\widetilde{w}^{(m)}} \, dV_{g_0} \\ &= \tau^{-\frac{n-1}{2}} \int_U e^{i\tau\Xi_{jklm}} A_{jklm} F \, dV_{g_0} \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \Xi_{jklm} &:= L\Phi^{(j)} - L\overline{\Phi}^{(k)} + \Psi^{(l)} - \overline{\Psi}^{(m)}, \\ A_{jklm} &:= e^{-L\lambda\Phi^{(j)} - L\lambda\overline{\Phi^{(k)}}} e^{-\mu\Psi^{(l)} - \mu\overline{\Psi^{(m)}}} a^{(j)} \overline{a}^{(k)} b^{(l)} \overline{b}^{(m)}. \end{aligned} \tag{6.9}$$

We will next analyze the integrals I_{jklm} in (6.8) and show that the only nontrivial contribution as $\tau \to \infty$ comes from the terms where $\nabla\Xi_{jklm}(p) = 0$. After this, we will fix the parameter L so that $\nabla\Xi_{jklm}(p) = 0$ will happen only when $j = k$ and $l = m$.

Step 4. Evaluation of I_{jklm} when $\nabla\Xi_{jklm}(p) = 0$.

Let j,k,l,m be the number such that $\Xi = \Xi_{jklm}$, which satisfies $\nabla\Xi(p) = 0$, and write

$$B := FA_{jklm},$$

where F is defined by (6.4). Writing z for a geodesic coordinate system in (M_0, g_0) with the origin at p, the phase function Ξ has the Taylor expansion

$$\Xi(z) = \Xi(0) + \frac{1}{2}Hz\cdot z + O(|z|^3).$$

Here $\Xi(0) = L(t_j - t_k) + s_l - s_m$ and

$$H = H_{jklm} = (\partial_{z_a z_b}\Xi)_{a,b}$$

the Hessian of Ξ in the z coordinates. Note that the imaginary parts of Hessians of $\Phi^{(j)}, \Phi^{(k)}, \Psi^{(l)}, \Psi^{(m)}$ at p are all positive semidefinite. Moreover, they are positive definite in the codimension one subspaces $\dot\gamma(t_j)^\perp$, $\dot\gamma(t_k)^\perp$, $\dot\eta(s_l)^\perp$, $\dot\eta(s_m)^\perp$, respectively. Thus it follows that

$$\mathrm{Im}(H) = \mathrm{Im}(\nabla^2(L(\Phi^{(j)} + \Phi^{(k)}) + \Psi^{(l)} + \Psi^{(m)}))|_p$$

is positive semidefinite, and moreover it is positive definite since the above codimension one subspaces span the whole tangent space at p. The last fact holds since $\dot\gamma(t_j) \neq \pm\dot\eta(s_l)$, which follows the geodesics γ and η are distinct and also not reparametrizations of each other.

By Proposition 5.2, we may assume that the functions $\widetilde{v}^{(j)}$ and $\widetilde{w}^{(k)}$ are supported in arbitrary small neighborhoods of γ and η respectively. Thus, without loss of generality, we may assume that U is contained in any fixed open geodesic ball B_δ centered at p. Since $\mathrm{Im}(H)$ is positive definite, by choosing $\delta > 0$ small enough, we have

$$|e^{i\tau(\frac{1}{2}Hz\cdot z + O(|z|^3))}| \le e^{-c\tau|z|^2} \text{ in } U,$$

for some $c > 0$. This shows that one may indeed use Lebesgue dominated convergence theorem in the argument below.

One has

$$\begin{aligned} I_{jklm} &= \tau^{\frac{n-1}{2}} \int_U e^{i\tau\Xi} B\, dV_{g_0} \\ &= \tau^{\frac{n-1}{2}} e^{i\tau\Xi(0)} \int_{\mathbb{R}^{n-1}} e^{i\tau(\frac{1}{2}Hz\cdot z + O(|z|^3))} B(z)|g_0(z)|^{1/2}\, dz. \end{aligned} \tag{6.10}$$

We make use of the change variables $z \to \tau^{-1/2}z$ again, which brings a Jacobian factor $\tau^{-\frac{n-1}{2}}$ that cancels the power of τ in front. Note that, as $\tau \to \infty$, one has

$$\begin{aligned} &|g_0(z/\tau^{1/2})| \to 1, \\ &B(z/\tau^{1/2}) \to F(p)e^{-L\lambda t_j - L\lambda t_k} e^{-\mu s_l - \mu s_m} a^{(j)}(p)\overline{a^{(k)}(p)} b^{(l)}(p)\overline{b^{(m)}(p)}. \end{aligned}$$

Combining these facts and using the Lebesgue dominated convergence theorem yield that

$$I_{jklm} = e^{i\tau(L(t_j - t_k) + s_l - s_m)} c_{jklm} F(p) e^{-L\lambda t_j - L\lambda t_k} e^{-\mu s_l - \mu s_m} + o(1)$$

where

$$c_{jklm} = a^{(j)}(p)\overline{a^{(k)}(p)}b^{(l)}(p)\overline{b^{(m)}(p)} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{2}H_{jklm}z\cdot z}\,dz.$$

The last integral is finite since $\mathrm{Im}(H_{jklm})$ is positive definite. For later purposes we observe that H_{jjll} is purely imaginary, hence

$$c_{jjll} = |a_j(p)|^2|b_l(p)|^2 \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2}\mathrm{Im}(H_{jjll})z\cdot z}\,dz$$

where the last integral is positive. In particular, we have

$$I_{jjll} = c_{jjll}F(p)e^{-2L\lambda t_j}e^{-2\mu s_l} + o(1)$$

where $c_{jjll} > 0$.

Step 5. Evaluation of I_{jklm} when $\nabla\Xi_{jklm}(p) \neq 0$.

In Step 5, we showed that one always has $I_{jklm} = O(1)$ as $\tau \to \infty$. We will use a non-stationary phase argument to show that

$$I_{jklm} = O(\tau^{-1/2}) \text{ when } \nabla\Xi_{jklm}(p) \neq 0, \quad \text{as } \tau \to \infty.$$

The argument is similar to [33, end of proof of Proposition 3.1]. In order to accomplish the argument, we rewrite the integral I_{jklm} to bring out the oscillating part of $e^{i\tau\Xi_{jklm}}$. Let us denote $\varphi = \mathrm{Re}(\Xi_{jklm})$. Since $d\Phi^{(j)}(p), d\Psi^{(l)}(p)$ etc are real, we have $d\varphi(p) \neq 0$, and I_{jklm} may be written as

$$I_{jklm} = \tau^{\frac{1}{2}} \int_U e^{i\tau\varphi}F\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\,dV_{g_0}$$

where we define

$$\breve{v}^{(j)} = \tau^{-\frac{n-2}{8}}e^{-L\tau\mathrm{Im}(\Phi^{(j)})-L\lambda\Phi^{(j)}}a^{(j)}, \qquad \breve{w}^{(l)} = \tau^{-\frac{n-2}{8}}e^{-\tau\mathrm{Im}(\Psi^{(l)})-\mu\Psi^{(l)}}b^{(l)}.$$

Note that by Proposition 5.2, $\|\breve{v}^{(j)}\|_{L^4(M_0)} \lesssim 1$ and $\|\breve{v}^{(j)}\|_{L^4(U\cap\partial M)} \lesssim 1$ and the same bounds also hold for $\breve{v}^{(k)}$, $\breve{w}^{(l)}$ and $\breve{w}^{(m)}$. Write

$$e^{i\tau\varphi} = \frac{1}{i\tau}P(e^{i\tau\varphi}), \qquad Pw = \langle |d\varphi|^{-2}d\varphi, dw\rangle,$$

where we assume that U has been chosen so small that $d\varphi$ is nonvanishing in U. Since $F \in C^1$, we can integrate by parts to derive

$$\begin{aligned} I_{jklm} &= \tau^{\frac{1}{2}} \int_U \frac{1}{i\tau}P(e^{i\tau\varphi})F\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\,dV_{g_0} \\ &= \frac{1}{i\tau^{\frac{1}{2}}}\int_U e^{i\tau\varphi}P^t\left[F\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\right]dV_{g_0} \\ &\quad + \frac{1}{i\tau^{\frac{1}{2}}}\int_{U\cap\partial M}\frac{\partial_\nu\varphi}{|d\varphi|^2}e^{i\tau\varphi}F\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\,dS, \end{aligned} \tag{6.11}$$

where the boundary term only appears if $p \in \partial M$. Note that P^t is a first order differential operator.

The boundary term in (6.11) is $O(\tau^{-1/2})$ as $\tau \to \infty$ since $\|\breve{v}^{(j)}\|_{L^4(U\cap\partial M)} \lesssim 1$. Thus it remains to estimate the term

$$\frac{1}{i\tau^{\frac{1}{2}}}\int_U e^{i\tau\varphi}P^t\left[F\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\right]dV_{g_0}. \tag{6.12}$$

In (6.12), if the differential operator P^t acts F, one can estimate

$$\begin{aligned}&\left|\frac{1}{i\tau^{\frac{1}{2}}}\int_U e^{i\tau\varphi}(P^tF)\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\,dV_{g_0}\right|\\ \lesssim&\tau^{-1/2}\|F\|_{C^1(M)}\|\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\|_{L^1(M_0)}\end{aligned}$$

which is $O(\tau^{-1/2})$ as $\tau\to\infty$ since $\|\breve{v}^{(j)}\|_{L^4(M_0)}$, $\|\breve{v}^{(k)}\|_{L^4(M_0)}$, $\|\breve{w}^{(l)}\|_{L^4(M_0)}$ and $\|\breve{w}^{(m)}\|_{L^4(M_0)}$ are bounded by some positive constant. The worst behavior with respect to the parameter τ in (6.12) occurs when some derivative in P^t acts one of the factors $e^{-L\tau\mathrm{Im}(\Phi^{(j)})}$, $e^{-L\tau\mathrm{Im}(\Phi^{(k)})}$, $e^{-\tau\mathrm{Im}(\Psi^{(l)})}$ or $e^{-\tau\mathrm{Im}(\Psi^{(m)})}$. This brings a factor behaving like $\tau\nabla(\mathrm{Im}(\Phi^{(j)}))$ into the integrand, and since $\nabla(\mathrm{Im}(\Phi^{(j)}))$ vanishes on the geodesic γ, one can choose new coordinates $z=(z',z_{n-1})$ near 0 such that $|\nabla(\mathrm{Im}(\Phi^{(j)}))|\lesssim|z'|$. Thus the integral that one needs to estimate looks like

$$\frac{1}{i\tau^{\frac{1}{2}}}\int_U e^{i\tau\varphi}F\left[\tau\nabla(\mathrm{Im}(\Phi^{(j)}))\right]\breve{v}^{(j)}\overline{\breve{v}^{(k)}}\breve{w}^{(l)}\overline{\breve{w}^{(m)}}\,dV_{g_0}.$$

Unwinding the definitions of $\breve{v}^{(j)}$, $\breve{v}^{(k)}$, $\breve{w}^{(l)}$ and $\breve{w}^{(m)}$, we may rewrite this integral in the form given in (6.10), so that it is equal to

$$\frac{1}{i}\tau^{\frac{n-1}{2}}\int_U e^{i\tau\Xi}\nabla(\mathrm{Im}(\Phi^{(j)}))B\,dV_{g_0}. \tag{6.13}$$

Evaluating the integral (6.13) as in Step 4, and using the change of variables $z\to\tau^{-1/2}z$ together with $|\nabla(\mathrm{Im}(\Phi^{(j)}))|\lesssim|z'|$ brings an additional factor $\tau^{-1/2}$, this shows that this kind of integral is $O(\tau^{-1/2})$. Then we conclude the proof that

$$\nabla\Xi_{jklm}(p)\neq0 \implies \lim_{\tau\to\infty}I_{jklm}=O(\tau^{-1/2}).$$

Step 6. Evaluation of (6.5).

Recall from Step 3 that $p_1,\dots,p_N$ are the distinct intersection points of γ and η and that U_r were small neighborhoods of p_r. As in Step 4, for each r with $1\le r\le N$ let $t_1^{(r)}<\cdots<t_{P_r}^{(r)}$ be the times in $[0,T]$ such that $\gamma(t_j^{(r)})=p_r$, and let $s_1^{(r)}<\cdots<s_{Q_r}^{(r)}$ be the times in $[0,S]$ such that $\eta(s_j^{(r)})=p_r$. Thus on U_r we have that $\widetilde{v}_{L(\tau+i\lambda)}$ is of the form

$$\widetilde{v}_{L(\tau+i\lambda)}|_{U_r}=\widetilde{v}_r^{(1)}+\cdots+\widetilde{v}_r^{(P_r)},$$

where

$$\widetilde{v}_r^{(j)}=\tau^{-\frac{n-2}{8}}e^{iL(\tau+i\lambda)\Phi_r^{(j)}}a_r^{(j)},$$

where $\Phi_r^{(j)}$ and $a_r^{(j)}$ satisfy the properties of the phase and amplitude functions in Proposition 5.2. We write similarly for $\widetilde{w}_r^{(j)}$, $\Psi_r^{(j)}$ and $b_r^{(j)}$.

Going back to (6.5) and using (6.7), we have

$$\sum_{r=1}^{N}\sum_{j,k=1}^{P_r}\sum_{l,m=1}^{Q_r} I^{(r)}_{jklm} = o(1) \quad \text{as } \tau\to\infty,$$

where

$$I^{(r)}_{jklm} = \tau^{\frac{n-1}{2}} \int_{U_r} e^{i\tau\Xi^{(r)}_{jklm}} A^{(r)}_{jklm} F\, dV_{g_0}$$

where $\Xi^{(r)}_{jklm}$ and $A^{(r)}_{jklm}$ are defined in U_r as in (6.9) in Step 4.

The integrals $I^{(r)}_{jklm}$ were evaluated in Steps 5 and 6. If we define

$$c^{(r)}_{jklm} := \begin{cases} a^{(r)}_j(p_r)\overline{a^{(r)}_k(p_r)}b^{(r)}_l(p_r)\overline{b^{(r)}_m(p_r)} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{2}H^{(r)}_{jklm}z\cdot z}\, dz, & \nabla\Xi^{(r)}_{jklm}(p_r)=0,\\ 0, & \text{otherwise,}\end{cases}$$

then we get from Steps 5 and 6 that

$$\begin{aligned}\sum_{r=1}^{N}\sum_{j,k=1}^{P_r}\sum_{l,m=1}^{Q_r} & e^{i\tau\left[L(t^{(r)}_j - t^{(r)}_k)+s^{(r)}_l - s^{(r)}_m\right]} c^{(r)}_{jklm} F(p_r) \\ & \times e^{-L(\lambda t^{(r)}_j+\lambda t^{(r)}_k)} e^{-\mu s^{(r)}_l - \mu s^{(r)}_m} = o(1)\end{aligned} \tag{6.14}$$

as $\tau\to\infty$. In the above formula $F(p_r) = F_{L\lambda+\mu}(p_r)$ is defined in (6.4).

Step 6. Choosing L so that $\nabla\Xi^{(r)}_{jklm}(p_r)=0$ only when $j=k$ and $l=m$.

Next, we choose $L\in\mathbb{N}$ large enough, but fixed, so that

$$\nabla\Xi^{(r)}_{jklm}(p_r)\neq 0$$

for all $1\le r\le N$ unless $j=k$ and $l=m$. Once we have chosen such L, Step 5 shows that terms $I^{(r)}_{jklm}$ with $j\neq k$ or $l\neq m$ are negligible. We have

$$\begin{aligned}\nabla\Xi^{(r)}_{jklm}(p_r) &= L\nabla\Phi^{(j)}_r - L\nabla\overline{\Phi}^{(k)}_r + \nabla\Psi^{(l)}_r - \nabla\overline{\Psi}^{(m)}_r \\ &= L(\dot\gamma(t^{(r)}_j) - \dot\gamma(t^{(r)}_k)) + \dot\eta(s^{(r)}_l) - \dot\eta(s^{(r)}_m).\end{aligned} \tag{6.15}$$

Since the geodesic γ is transversal and thus it is not closed geodesic, we have

$$\dot\gamma(t^{(r)}_j) - \dot\gamma(t^{(r)}_k)\neq 0$$

for all $j\neq k$ for all r with $1\le r\le N$. Let us define two numbers α and β as

$$\begin{aligned}\alpha &:= \min\left\{|\dot\gamma(t^{(r)}_j) - \dot\gamma(t^{(r)}_k)| :\ 1\le r\le N,\ 1\le j,k\le P_r\right\} > 0,\\ \beta &:= \max\left\{|\dot\eta(s^{(r)}_l) - \dot\eta(s^{(r)}_m)| :\ 1\le r\le N,\ 1\le l,m\le Q_r\right\} > 0.\end{aligned}$$

We choose L that satisfies

$$L\ge \frac{\beta+1}{\alpha}.$$

Then we have that

$$|L(\dot\gamma(t_j^{(r)}) - \dot\gamma(t_k^{(r)})) + \dot\eta(s_l^{(r)}) - \dot\eta(s_m^{(r)})| \geq |L(\dot\gamma(t_j) - \dot\gamma(t_k))| - |\dot\eta(s_l) - \dot\eta(s_m)| \\ \geq L\alpha - \beta \geq 1 > 0.$$

Thus, by using (6.15), we have found that for any $L \geq \frac{\beta+1}{\alpha}$, and $j \neq k$, one has

$$\nabla \Xi_{jklm}^{(r)}(p_r) \neq 0.$$

For $1 \leq r \leq N$, assume then that $j = k$ and $l \neq m$. Then we have

$$\nabla \Xi_{jklm}^{(r)}(p_r) = L(\dot\gamma(t_j^{(r)}) - \dot\gamma(t_j^{(r)})) + \dot\eta(s_l^{(r)}) - \dot\eta(s_m^{(r)}) = \dot\eta(s_l^{(r)}) - \dot\eta(s_m^{(r)}) \neq 0,$$

since η is transversal and $l \neq m$.

In conclusion, the only case when $d\Xi_{jklm}^{(r)}(p_r) = 0$ is when $j = k$ and $l = m$.

Step 7. Conclusion of the proof.

Going back to the equation (6.14) and using the result in Step 6, and taking $\tau \to \infty$, we have

$$\sum_{r=1}^{N} \sum_{j=1}^{P_r} \sum_{l=1}^{Q_r} c_{jjll}^{(r)} F_{L\lambda+\mu}(p_r) e^{-2Lt_j^{(r)}\lambda} e^{-2s_l^{(r)}\mu} = 0, \tag{6.16}$$

where we have used that $e^{i\tau\left[L(t_j^{(r)} - t_k^{(r)}) + s_l^{(r)} - s_m^{(r)}\right]} = 1$, when $j = k$ and $l = m$.

Let $\lambda \in \mathbb{R}$ and choose μ so that

$$2L\lambda + 2\mu = 2\lambda,$$

which is equivalent to

$$\mu = (1 - L)\lambda \in \mathbb{R}.$$

Then (6.16) reads

$$\sum_{r=1}^{N} \sum_{j=1}^{P_r} \sum_{l=1}^{Q_r} c_{jjll}^{(r)} F_{\lambda}(p_r) e^{-2\lambda\left[L\left(t_j^{(r)} - s_l^{(r)}\right) + s_l^{(r)}\right]} = 0, \tag{6.17}$$

where $F_\lambda(p_r) = \hat f(2\lambda, p_r)$.

We will conclude the proof by using Lemma 6.2. In order to use Lemma 6.2, we want to show that if

$$(r_1, j_1, l_1) \neq (r_2, j_2, l_2) \tag{6.18}$$

then

$$L(t_{j_1}^{(r_1)} - s_{l_1}^{(r_1)}) + s_{l_1}^{(r_1)} \neq L(t_{j_2}^{(r_2)} - s_{l_2}^{(r_2)}) + s_{l_2}^{(r_2)}. \tag{6.19}$$

We will redefine L so that the above is true. We define two sets of real numbers as follows:

$$\mathcal{Q}_1 = \cup_{r,r'=1}^{N} \cup_{j,k=1}^{P_r} \left\{ t_j^{(r)} - t_k^{(r')} \right\}, \qquad \mathcal{Q}_2 = \cup_{r,r'=1}^{N} \cup_{l,m=1}^{Q_r} \left\{ s_l^{(r)} - s_m^{(r')} \right\},$$

and

$$\widetilde{\alpha} = \min_{d_1\in\mathcal{Q}_1, d_2\in\mathcal{Q}_2, d_1\neq d_2} |d_1 - d_2|, \qquad \widetilde{\beta} = \max_{d\in\mathcal{Q}_2} |d|.$$

Finally, we redefine the number L in Step 6 as

$$L = \max\left\{\frac{\widetilde{\beta}}{\widetilde{\alpha}} + 1, \frac{\beta+1}{\alpha}\right\},$$

where α, β are the numbers given in Step 6.

Let $(r_1, j_1, l_1) \neq (r_2, j_2, l_2)$, then we want to show that

$$L(t_{j_1}^{(r_1)} - s_{l_1}^{(r_1)}) + s_{l_1}^{(r_1)} \neq L(t_{j_2}^{(r_2)} - s_{l_2}^{(r_2)}) + s_{l_2}^{(r_2)}. \tag{6.20}$$

Let us set

$$d_1 = t_{j_1}^{(r_1)} - t_{j_2}^{(r_2)} \text{ and } d_2 = s_{l_1}^{(r_1)} - s_{l_2}^{(r_2)}.$$

We have following cases:

(a) Assume that $d_1 = d_2$ and suppose that

$$L(t_{j_1}^{(r_1)} - s_{l_1}^{(r_1)}) + s_{l_1}^{(r_1)} = L(t_{j_2}^{(r_2)} - s_{l_2}^{(r_2)}) + s_{l_2}^{(r_2)}, \tag{6.21}$$

then we have $L(d_1 - d_2) + d_2 = 0$. It follows that $s_{l_1}^{(r_1)} = s_{l_2}^{(r_2)}$. Thus $l_1 = l_2$ and $r_1 = r_2$. Since $d_1 = d_2$, we also have $t_{j_1}^{(r_1)} = t_{j_2}^{(r_2)}$. Thus, $j_1 = j_2$ holds, which leads to a contradiction to $(r_1, j_1, l_1) \neq (r_2, j_2, l_2)$. Thus we must have (6.20).

(b) Assume then that $d_1 \neq d_2$ and that (6.21) holds. Then we have

$$L = \frac{d_2}{d_1 - d_2}.$$

However, this cannot be true since

$$L \geq \frac{\widetilde{\beta}}{\widetilde{\alpha}} + 1 > \frac{d_2}{d_1 - d_2}.$$

Thus again we have (6.20).

We have shown that $(r_1, j_1, l_1) \neq (r_2, j_2, l_2)$ implies (6.20). We now go back to (6.17) and use Lemma 6.2 together with the condition (6.20) and the fact that $F_\lambda(p_r) = \hat{f}(2\lambda, p_r)$. This yields that

$$c_{jjll}^{(r)} \hat{f}(2\lambda, p_r) = 0$$

for all (r, j, l). In Step 4 we proved that $c_{jjll}^{(r)} > 0$ for all (r, j, l), showing that $\hat{f}(2\lambda, p_r) = 0$ for all $\lambda \in \mathbb{R}$ and all r. Since the point y_0 in Step 1 was one of the points p_r, it follows that $f(x_1, y_0) = 0$ for all $x_1 \in \mathbb{R}$. By Step 1 this is true for almost every y_0 in (M_0, g_0), and by the continuity of f one gets that $f \equiv 0$ as required. $\square$

Appendix A. Complex geometrical optics solutions

In this appendix we provide several results which were used in the previous sections, related to Gaussian beam quasimodes and CGO type solutions on transversally anisotropic manifolds. The results are based on limiting Carleman weights, which were introduced in [41] and applied to inverse problems on conformally transversally anisotropic manifolds in [40, 33].

We first recall the construction of Gaussian beam quasimodes, i.e. approximate eigenfunctions concentrating near a geodesic, from [33]. Let (M, g) be a compact Riemannian manifold with smooth boundary.

Proposition Appendix A.1 (Gaussian beams quasimodes)**.** *Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M, $\dim(M) = m$. Let $\gamma : [0, T] \to M$ be a nontangential geodesic, and let $\lambda \in \mathbb{C}$. For any $K \in \mathbb{N}$ and $k \in \mathbb{N}$, there is a family of functions $(\widetilde{v}_s) \subset C^\infty(M)$, where $s = \tau + i\lambda \in \mathbb{C}$ and $\tau \geq 1$, such that*

$$\begin{aligned} \|(-\Delta_g - s^2)\widetilde{v}_s\|_{H^k(M)} = O(\tau^{-K}), \\ \|\widetilde{v}_s\|_{L^4(M)} = O(1), \qquad \|\widetilde{v}_s\|_{L^4(\partial M)} = O(1) \end{aligned} \tag{A.1}$$

as $\tau \to \infty$. The functions $\widetilde{v}_s$ have the following properties: If $p \in \gamma([0, T])$, then there is $P \in \mathbb{N}$ such that on a neighborhood U of p the function $\widetilde{v}_s$ is a finite sum

$$\widetilde{v}_s = \widetilde{v}^{(1)} + \cdots + \widetilde{v}^{(P)} \tag{A.2}$$

where $t_1 < \cdots < t_P$ are the times in $[0, T]$ such that $\gamma(t_l) = p$. Each $\widetilde{v}^{(l)}$ has the form

$$\widetilde{v}^{(l)} = \tau^{-\frac{m-1}{8}} e^{is\Theta^{(l)}} a^{(l)} \tag{A.3}$$

where each $\Theta = \Theta^{(l)}$ is a smooth complex function in U satisfying

$$\begin{aligned} &\Theta(\gamma(t)) = t, \quad \nabla\Theta(\gamma(t)) = \dot{\gamma}(t), \\ &\mathrm{Im}(\nabla^2\Theta(\gamma(t))) \geq 0, \quad \mathrm{Im}(\nabla^2\Theta)(\gamma(t))|_{\dot{\gamma}(t)^\perp} > 0, \end{aligned} \tag{A.4}$$

for t close to t_l. Here $a^{(l)}(\gamma(t)) = \tau^{\frac{m-1}{4}}(a_0^{(l)}(\gamma(t)) + O(\tau^{-1}))$ where $a_0^{(l)}(\gamma(t))$ is nonvanishing and independent of τ, and the support of $a^{(l)}$ can be taken to be in any small neighborhood of $\gamma([0, T])$ chosen beforehand.

Proof. We choose

$$\widetilde{v}_s = \tau^{-\frac{m-1}{8}} v_s, \qquad s = \tau + i\lambda, \tag{A.5}$$

where v_s are the Gaussian beam quasimodes constructed in [33, Proposition 3.1]. Recall from the displayed formula after [33, equation (3.5)] that

$$v_s = \sum_{j=0}^{r} \tilde{\chi}_j v_s^{(j)} \tag{A.6}$$

where $\tilde{\chi}_j$ are cutoff functions independent of s, and $v_s^{(j)}$ are quasimodes of the form $v_s^{(j)} = e^{is\Theta^{(j)}} a^{(j)}$ near the geodesic segments $\gamma(I^{(j)})$, where $\Theta^{(j)}$ is a complex phase function, $a^{(j)}$ is an amplitude and $I^{(j)}$ is a closed interval in $[0, T]$. Then it follows that (A.2), (A.3) and (A.4) are satisfied by (A.5) and by the construction in [33, Proposition 3.1]. We only need to verify the conditions in (A.1) about the Sobolev H^k decay estimate and the L^4 normalization condition for $\widetilde{v}_s$. By (A.6) it is enough to do this for a single function $\widetilde{v}_s^{(j)} = \tau^{-\frac{m-1}{8}} e^{is\Theta^{(j)}} a^{(j)}$. For simplicity, we will drop the index j from the notation.

Let us write $\Gamma = \gamma(I)$ and let $p \in \Gamma$. Recall from [33, Proposition 3.1] that a is a smooth function of the form

$$a(t, y) = \tau^{\frac{m-1}{4}} \left(a_0 + s^{-1} a_{-1} + \cdots + s^{-N} a_{-N}\right) \chi(y/\delta'). \tag{A.7}$$

Here (t, y) are Fermi coordinates (see e.g. [33, Lemma 3.5]) near $\gamma(I)$, $a_0(t, 0)$ is a nonvanishing function independent of s, χ is a smooth cutoff function supported in the unit ball in $\mathbb{R}^{m-1}$, and $\delta' > 0$ is a fixed number that can be taken to be very small. From the latter it follows that the support of a can be taken to be in any small neighborhood of Γ chosen beforehand. The constant $N \in \mathbb{N}$ will be chosen sufficiently large depending on K and k. We have chosen $\tau^{-\frac{m-1}{8}}$ as the normalization factor in (A.5), which will lead to the L^4 normalization condition in (A.1).

As shown in [33, Proposition 3.1], the functions v_s are approximate eigenfunctions for $-\Delta_g$ in the sense that the function

$$f := (-\Delta_g - s^2) v_s, \tag{A.8}$$

which describes the error of v_s from being a true eigenfunction, is of order $O(\tau^{-K})$ in $L^2(M)$ if N is chosen large enough. We next show that the function f is of the order $O(\tau^{-K})$ in $H^k(M)$ when N is sufficiently large.

The function f in (A.8) was calculated in [33, Proposition 3.1] to have the form

$$\begin{aligned} f = e^{is\Theta} \tau^{\frac{m-1}{4}} &\left(s^2 h_2 a + s h_1 + \cdots + s^{-(N-1)} h_{-(N-1)} - s^{-N} \Delta_g a_{-N}\right) \chi(y/\delta') \\ &+ e^{is\Theta} \tau^{\frac{m-1}{4}} s b\, \tilde{\chi}(y/\delta'), \end{aligned} \tag{A.9}$$

where for each j the function h_j vanishes to order N on Γ, the function $b = b(t, y)$ vanishes near Γ, and $\tilde{\chi}$ is a smooth function with $\tilde{\chi} = 0$ for $|y| \geq 1/2$. We have that

$$|e^{is\Theta}| \leq C_1 e^{-\tau c|y|^2}, \tag{A.10}$$

for $t \in I$ and $|y|$ small enough by the latter two properties in (A.4). We take δ' to be so small that (A.10) holds on the support of f. Thus we have that

$$|f| \leq C_2 \tau^{\frac{m-1}{4}} e^{-\tau c|y|^2} \left(\tau^2 |y|^{N+1} + \tau^{-N} + \tau\, O(|y|^\infty)\right),$$

where the term $\tau^2 |y|^{N+1}$ corresponds to the terms in (A.9) with h_j as a factor, the term τ^{-N} to the term $s^{-N} \Delta_g a_{-N}$, and the term $\tau\, O(|y|^\infty)$ to the term with $b\, \tilde{\chi}(y/\delta')$ (which vanishes near $y = 0$). Moreover, taking k derivatives of f brings at most k powers of $s \in \mathbb{C}$ to the front of the expression, or reduces the degree of vanishing of h_j on Γ by at most k. This gives

$$|\nabla^k f| \leq C_3 \tau^{\frac{m-1}{4}} e^{-\tau c|y|^2} \sum_{l=0}^{k} \tau^{k-l} (\tau^2 |y|^{N+1-l} + \tau^{-N} + \tau O(|y|^\infty)).$$

Thus, by taking $N = N(K, k)$ to be large enough, and by using polar coordinates and the standard formula $\int_0^\infty r^l e^{-\tau c r^2} dr \sim \tau^{-\frac{l+1}{2}}$ for $l \geq 0$ we obtain that

$$\|(-\Delta_{g_0} - s^2) v_s\|_{H^k(M)} = O(\tau^{-K}).$$

Since $\widetilde{v}_s = \tau^{-\frac{m-1}{8}} v_s$, the same is true for $\widetilde{v}_s$.

It remains to show that $\|\widetilde{v}_s\|_{L^4(M)} = O(1)$ and $\|\widetilde{v}_s\|_{L^4(\partial M)} = O(1)$. Again it is enough to consider a single function $v_s = e^{is\Theta}a$. By (A.10) and (A.7), one has

$$|v_s(t,y)| \le Ce^{-\tau c|y|^2}\tau^{\frac{m-1}{4}}\chi(y/\delta').$$

Computing the L^4 norm gives

$$\|v_s\|_{L^4}^4 = O(\tau^{\frac{m-1}{2}}).$$

Due to the normalization factor $\tau^{-\frac{m-1}{8}}$, we have $\|\widetilde{v}_s\|_{L^4(M)}^4 = O(1)$. To calculate $\|\widetilde{v}_s\|_{L^4(\partial M)}$ (see [33, Proposition 3.1] for a similar computation for the $L^2(\partial M)$ norm), we note that since γ is nontangential we may locally write ∂M in the Fermi coordinates (t,y) as the the set $\{(t(y),y) : |y| < \epsilon\}$ for some smooth function $t = t(y)$ and for some $\epsilon > 0$. Since the geodesic $\gamma(t)$ intersects ∂M at two points, it follows that $\|\widetilde{v}_s\|_{L^4(\partial M)}^4$ is a sum of two integrals of the form

$$\int_{\{|y|<\epsilon\}} |\widetilde{v}_s(t(y),y)|^4\, dS(y) \le C_3 \int_{\mathbb{R}^{m-1}} \tau^{\frac{m-1}{2}} e^{-\tau c|y|^2} dy = O(1).$$

Thus we also have $\|\widetilde{v}_s\|_{L^4(\partial M)} = O(1)$, which concludes the proof. $\square$

We record next a Carleman estimate from [40, Lemma 4.3]. The statement involves the following technical assumptions. We assumed that (M,g) is a compact Riemannian manifold with smooth boundary. Without loss of generality, we may assume that (M,g) is embedded in a compact manifold (N,g) without boundary. The function φ is assumed to be a limiting Carleman weight in (U,g), where U is open in N and M is compactly contained in U (see [40, Definition 1.1]).

Below, the space $H^s_{\mathrm{scl}}(N)$ stands for the *semiclassical Sobolev space* with a small parameter $h > 0$, see e.g. [40]. We define $H^s_{\mathrm{scl}}(M)$ by restriction, i.e. $H^s_{\mathrm{scl}}(M) = \{u|_M \,;\, u \in H^s_{\mathrm{scl}}(N)\}$. If $s \ge 0$ is an integer, then H^s_{scl} has the equivalent norm

$$\|u\|_{H^s_{\mathrm{scl}}} \sim \left(\sum_{l=0}^{s} h^l \|\nabla^l u\|_{L^2}\right)^{1/2}.$$

Lemma Appendix A.2 (Carleman estimate [40])**.** *Let (M,g), U, N, and the limiting Carleman weight φ be as described above. Let $s \in \mathbb{R}$. There exist two constants $C_s > 0$ and $0 < h_s \le 1$ such that for all functions $u \in C_c^\infty(M^{\mathrm{int}})$ and all $0 < h < h_s$ one has the inequality*

$$\|e^{\frac{\varphi}{h}}u\|_{H^{s+1}_{\mathrm{scl}}(N)} \le C_s h \|e^{\frac{\varphi}{h}}\Delta_g u\|_{H^s_{\mathrm{scl}}(N)}.$$

Proposition Appendix A.3 (H^s solvability)**.** *Let $s \ge 0$. Under the conditions in Lemma Appendix A.2, there exist constants $C_s, h_s > 0$ such that for $0 < h < h_s$ and for any function $f \in H^s_{\mathrm{scl}}(M)$ there is a solution $u \in H^{s+1}_{\mathrm{scl}}(M)$ to the equation*

$$e^{\frac{\varphi}{h}}\Delta_g e^{-\frac{\varphi}{h}}u = f$$

satisfying

$$\|u\|_{H^{s+1}_{\mathrm{scl}}(M)} \le C_s h \|f\|_{H^s_{\mathrm{scl}}(M)}.$$

Proof. The proof is for the most parts the same as that of [40, Proposition 4.4]. We consider the conjugated operator

$$P = e^{-\frac{\varphi}{h}} h^2 \Delta_g \, e^{\frac{\varphi}{h}}.$$

Let $f \in H^s_{\mathrm{scl}}(M)$, so that by definition there is $\tilde{f} \in H^s_{\mathrm{scl}}(N)$ with $\tilde{f}|_M = f$ and $\|\tilde{f}\|_{H^s_{\mathrm{scl}}(N)} \le C\|f\|_{H^s_{\mathrm{scl}}(M)}$. Consider the subspace

$$E = P^*(C^\infty(N))$$

of $H^{-s-1}_{\mathrm{scl}}(N)$ and the linear form L defined on E by

$$L(P^*v) = \langle f, v\rangle_M = \langle \tilde{f}, v\rangle_N, \quad v \in C^\infty_c(M^{\mathrm{int}}).$$

By Lemma Appendix A.2 the linear form L is well defined: if $P^*v_1 = P^*v_2$, then

$$|\langle f, v_1 - v_2\rangle_{L^2(M)}| \le \|\tilde{f}\|_{L^2(N)} \|v_1 - v_2\|_{L^2(N)} \le Ch \|e^{\frac{\varphi}{h}} \Delta_g \, e^{-\frac{\varphi}{h}} (v_1 - v_2)\|_{L^2(N)} = 0.$$

We also have that

$$\begin{aligned} |L(P^*v)| &\le \|\tilde{f}\|_{H^s_{\mathrm{scl}}(N)} \|v\|_{H^{-s}_{\mathrm{scl}}(N)} \le C_s \|f\|_{H^s_{\mathrm{scl}}(M)} h \|e^{\frac{\varphi}{h}} \Delta_g \, e^{-\frac{\varphi}{h}} v\|_{H^{-s-1}_{\mathrm{scl}}(N)} \\ &= C_s \|f\|_{H^s_{\mathrm{scl}}(M)} h^{-1} \|P^*v\|_{H^{-s-1}_{\mathrm{scl}}(N)}. \end{aligned}$$

By the Hahn-Banach theorem, there is an extension $\hat{L}$ of L which is a bounded functional on $H^{-s-1}_{\mathrm{scl}}(N)$ with norm $\|\hat{L}\| \le C_s h^{-1} \|f\|_{H^s(M)}$. Since the dual of $H^{-s-1}_{\mathrm{scl}}(N)$ is $H^{s+1}_{\mathrm{scl}}(N)$, there exists a function $\tilde{u} \in H^{s+1}_{\mathrm{scl}}(N)$ such that $\hat{L}(v) = \langle \tilde{u}, v\rangle_{L^2(N)}$ and $\|\tilde{u}\|_{H^{s+1}_{\mathrm{scl}}(N)} \le C_s h^{-1} \|f\|_{H^s_{\mathrm{scl}}(M)}$. Then $u = \tilde{u}|_M$ is the desired solution, since for all $v \in C^\infty_c(M^{\mathrm{int}})$ we have that

$$\langle Pu, v\rangle = \langle u, P^*v\rangle = \hat{L}(P^*v) = L(P^*v) = \langle f, v\rangle.$$

This completes the proof. □

Recall that by definition a transversally anisotropic manifold is a Riemannian manifold (M, g) compactly contained in $\mathbb{R} \times M_0$ with the metric $g = e \oplus g_0$. The coordinate x_1 along $\mathbb{R}$ is then a limiting Carleman weight [40, Lemma 2.9]. The following proposition constructs complex geometrical optics (CGO) solutions in this setting, based on the Gaussian beam quasimodes given in Proposition 5.2.

Proposition Appendix A.4 (CGO solutions)**.** *Let (M, g) be a transversally anisotropic manifold compactly contained in $I \times M_0$ with $g = e \oplus g_0$. Let also $R, k \in \mathbb{N}$. There exists $\tau_0 \ge 1$ such that for any fixed real number λ and for any τ with $|\tau| \ge \tau_0$ there is a solution of the equation $-\Delta_g u = 0$ in M having the form*

$$u_s = e^{-sx_1}(\widetilde{v}_s + r_s), \tag{A.11}$$

where $s = \tau + i\lambda$, x_1 is the coordinate along $\mathbb{R}$, $\widetilde{v}_s$ is a family as in Proposition 5.2 in (M_0, g_0) (so that $m = n - 1$) with $K = K(R, k)$ chosen large enough, and

$$\|r_s\|_{H^k(M)} = O(\tau^{-R}) \text{ as } |\tau| \to \infty.$$

Proof. A straightforward calculation, done in [33] after Proposition 2.1 there, shows that a function u_s of the form (A.11) is a solution to $\Delta_g u_s = 0$ provided that

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}(e^{-i\lambda x_1} r_s) = f, \tag{A.12}$$

where

$$f = -e^{i\lambda x_1}(-\Delta_{g_0} - s^2)\widetilde{v}_s.$$

Since $\widetilde{v}_s$ is independent of x_1, we have by (A.1) that

$$\|f\|_{H^k_{\mathrm{scl}}(M)} = O(\tau^{-K}).$$

Proposition Appendix A.3 with $h = \tau^{-1}$ shows that there is a solution $r_s \in H^k(M)$ to (A.12) satisfying

$$\|r_s\|_{H^{k+1}_{\mathrm{scl}}(M)} \leq \frac{C}{\tau}\|f\|_{H^k_{\mathrm{scl}}(M)} = O(\tau^{-K-1}).$$

Thus

$$\|r_s\|_{H^k(M)} \leq \tau^k \|r_s\|_{H^k_{\mathrm{scl}}(M)} = O(\tau^{k-K-1}),$$

and the required decay $\|r_s\|_{H^k(M)} = O(\tau^{-R})$ follows by taking $K > R + k - 1$. □

Appendix B. Some lemmas

We prove some lemmas which we have used in the previous sections. We begin with a well-known Schauder estimate (see e.g. [42] for domains in $\mathbb{R}^n$ and [43, Proposition 8.10] for manifolds with boundary). We include a proof since we could not find a direct reference for the statement that we need. The spaces $C^s(M)$, $s \in \mathbb{R}$ and $s \geq 0$, of functions on a smooth manifold are equipped with a norm, which is given with respect to a partition of unity $\{\varphi^\alpha\}$ subordinate to an atlas $\{(G_\alpha, U_\alpha)\}$ of the manifold as

$$\|v\|_{C^s(M)} = \sum_\alpha \|(G_\alpha^{-1})^*(\varphi^\alpha v)\|_{C^s(\mathbb{R}^n)}.$$

Using a different partition of unity gives an equivalent norm on a compact manifold. We refer to [44, Theorem 2.23] for properties of a partition of unity on a manifold with boundary.

Lemma Appendix B.1 (Schauder estimate)**.** *Let (M, g) be a compact C^∞ Riemannian manifold with C^∞ boundary ∂M. Let $F \in C^{s-2}(M)$ and $f \in C^s(\partial M)$, for some $s > 2$ and $s \notin \mathbb{N}$. Assume that the map*

$$v \mapsto (\Delta_g + c(x))\, v \tag{B.1}$$

is injective on $H^1_0(M)$.

Let $v \in H^1(M)$ be the unique solution of

$$\begin{cases} \Delta_g v + cv = F & \text{in } M, \\ v = f & \text{on } \partial M. \end{cases} \tag{B.2}$$

Then there exists a positive constant $C > 0$ independent of v, f and F such that

$$\|v\|_{C^s(M)} \leq C\left(\|F\|_{C^{s-2}(M)} + \|f\|_{C^s(\partial M)}\right). \tag{B.3}$$

Proof. Consider the linear map

$$S : C^{s-2}(M) \times C^s(\partial M) \to H^1(M), \quad S(F, f) = v.$$

This map is well defined since $C^{s-2}(M) \subset H^{-1}(M)$ and $C^s(\partial M) \subset H^{1/2}(\partial M)$, and since the assumption that $v \mapsto (\Delta_g + c(x))\, v$ is injective on $H_0^1(M)$ ensures that there is a unique weak solution $v \in H^1(M)$. The map S is also bounded since

$$\|v\|_{H^1(M)} \le C(\|F\|_{H^{-1}(M)} + \|f\|_{H^{1/2}(\partial M)}) \le C(\|F\|_{C^{s-2}(M)} + \|f\|_{C^s(\partial M)}).$$

We claim that the range of S is in $C^s(M)$. If this is the case, then S will be continuous $C^{s-2}(M) \times C^s(\partial M) \to C^s(M)$ by the closed graph theorem (if $(F_j, f_j) \to (F, f)$ in $C^{s-2}(M) \times C^s(\partial M)$ and $S(F_j, f_j) \to w$ in $C^s(M)$, then $S(F_j, f_j) \to S(F, f)$ in $H^1(M)$ showing that $w = S(F, f)$). This implies (B.3) for some $C > 0$ independent of v, F and f and proves the theorem.

The fact that the range of S is in $C^s(M)$ follows directly from the corresponding statement on subsets of $\mathbb{R}^n$ after passing to local coordinates. Let $\{(\Omega_\alpha, G_\alpha)\}_{\alpha=1}^K$ be an atlas of M, where

$$G_\alpha : \Omega_\alpha \to \Omega'_\alpha \subset \mathbb{R}^n.$$

Let us introduce notations for the coordinate representations of the relevant functions and the operator P. We define a family elliptic operators $\{P_\alpha\}_{\alpha=1}^K$ as

$$P_\alpha h := (G_\alpha^{-1})^* \left[(\Delta_g + c(x))\, G_\alpha^* h\right], \qquad h \in C_c^\infty(\Omega'_\alpha).$$

These are second order elliptic operators with C^∞-smooth coefficients on $\Omega'_\alpha \subset \mathbb{R}^n$. Let us also denote

$$v_\alpha = (G_\alpha^{-1})^* v, \quad F_\alpha = (G_\alpha^{-1})^* F \text{ and } f_\alpha := (G_\alpha^{-1})^* f.$$

We have that v_α solves

$$\begin{cases} P_\alpha v_\alpha = F_\alpha & \text{on } \Omega'_\alpha \subset \mathbb{R}^n \\ v_\alpha = f_\alpha & \text{on } \partial\Omega'_\alpha \end{cases} \tag{B.4}$$

by the coordinate invariance of the operator $\Delta_g + c$. By Schauder estimates for domains in $\mathbb{R}^n$, see [42, Section 6.4], we have that $v_\alpha \in C^s(\Omega'_\alpha)$. This proves that $v \in C^s(M)$ as required. □

Lemma Appendix B.2. *Let $f_1, \ldots, f_N$ be compactly supported distributions in $\mathbb{R}$ such that for some distinct real numbers $a_1, \ldots, a_N$ one has*

$$\sum_{j=1}^N \hat{f}_j(\lambda) e^{a_j \lambda} = 0, \qquad \lambda \in \mathbb{R}.$$

Then $f_1 = \ldots = f_N = 0$.

Proof. Suppose without loss of generality that $a_1 > a_2 > \ldots > a_N$. Then

$$\hat{f}_1(\lambda) = -\sum_{j=2}^N e^{-(a_1 - a_j)\lambda} \hat{f}_j(\lambda).$$

By the Paley-Wiener-Schwartz theorem there are $C, M > 0$ so that

$$|\hat{f}_j(\lambda)| \le C(1+|\lambda|)^M, \qquad \lambda \in \mathbb{R}.$$

Write $\delta = a_1 - a_2 > 0$. Since $a_1 - a_j \ge \delta$ for $j \ge 2$, it follows that

$$|\hat{f}_1(\lambda)| \le \begin{cases} C(1+|\lambda|)^M, & \lambda \le 0, \\ C(1+\lambda)^M e^{-\delta\lambda}, & \lambda \ge 0. \end{cases}$$

However, no nontrivial compactly supported distribution f_1 can have the above decay for its Fourier transform. To see this, note that

$$e^{\epsilon\lambda}\hat{f}_1(\lambda) \in \mathscr{S}'(\mathbb{R}), \qquad 0 \le \epsilon < \delta.$$

Thus using [38, Theorem 7.4.2] there exists an analytic function U in $\{0 < \mathrm{Im}(t) < \delta\}$ so that the Fourier transform of $e^{\epsilon\lambda}\hat{f}_1(\lambda)$ is $U(\,\cdot\, + i\epsilon)$. By [38, Remark after Theorem 7.4.3] the limit of $U(\,\cdot\, + i\epsilon)$ in $\mathscr{S}'(\mathbb{R})$ as $\epsilon \to 0$ is the Fourier transform of $\hat{f}_1(\lambda)$, i.e. $2\pi f_1(-\,\cdot\,)$. Fix some interval $I \subset \mathbb{R}$ that is outside the support of $f_1(-\,\cdot\,)$, and consider the rectangle $Z = I \times (0, \delta)$. For ϵ close to 0, one has

$$|U(t+i\epsilon)| \le \|e^{\epsilon\lambda}\hat{f}_1(\lambda)\|_{L^1} \lesssim \|(1+|\lambda|)^M e^{\epsilon\lambda}\|_{L^1(\mathbb{R}_-)} + 1 \lesssim \epsilon^{-M}.$$

Since the limit of $U(\,\cdot\, + i\epsilon)$ in $\mathscr{D}'(I)$ is $2\pi f_1(-\,\cdot\,)|_I = 0$, by [38, Theorem 3.1.15] one has $U = 0$ in Z. Now U is analytic, so $U = 0$ in $\{0 < \mathrm{Im}(t) < \delta\}$ and $f_1 = 0$. Repeating this argument gives that $f_2 = \ldots = f_N = 0$. $\square$

Lemma Appendix B.3 (Intersecting geodesics)**.** *Let (M_0, g_0) be a compact Riemannian manifold with strictly convex smooth boundary. There is a set E of zero measure in (M_0, g_0) such that if $y_0 \in M_0 \setminus E$, there exist nontangential geodesics γ and η on M_0 that intersect at y_0, self-intersect only finitely many times and intersect each other only finitely many times.*

Proof. By [45, Lemma 3.1], there is a set E of zero measure in (M_0, g_0) so that all points in $M_0 \setminus E$ lie on some nontangential geodesic between boundary points. Fix a point $y_0 \in M_0 \setminus E$ and a direction $v_0 \in S_{y_0}M_0$ so that the geodesic $\gamma : [0, T] \to M_0$ through (y_0, v_0) is a nontangential geodesic between boundary points. Without loss of generality, we may assume that γ is a unit speed geodesic (i.e. $|\dot{\gamma}| = 1$). The property of a geodesic being nontangential is not changed under small perturbations. Therefore, we may find $w_0 \in S_{y_0}M_0$ close to v_0 so that $w_0 \ne v_0$ and the unit speed geodesic $\eta : [0, S] \to M_0$ through (y_0, w_0) is also a nontangential geodesic between boundary points. We may arrange so that the geodesics γ and η are such that their graphs do not coincide (in fact, γ can only self-intersect at y_0 finitely many times [46, Lemma 7.2], and it is enough choose w_0 near v_0 that is different from the corresponding finitely many tangent vectors of γ and their negatives).

We next show that two distinct geodesics γ and η whose graphs do not coincide can intersect only finitely many times. Assume the opposite, that there are infinitely many intersection points $\{p_k\}_{k\in\mathbb{N}}$ and intersection times $\{t_k\}_{k\in\mathbb{N}}$, $\{s_k\}_{k\in\mathbb{N}}$ satisfying

$$\gamma(t_k) = p_k = \eta(s_k), \text{ for all } k \in \mathbb{N}.$$

Since M is compact, $t_k \in [0,T]$ and $s_k \in [0,S]$, by passing to subsequences and using continuity of unit speed geodesics γ, η, we may assume that $\gamma(t_k) \to \gamma(t_0) = p_0$ and $\eta(s_k) \to \eta(s_0) = p_0$ as $k \to \infty$, for some $t_0 \in [0,T]$, $s_0 \in [0,S]$ and $p_0 \in M$.

In addition, we denote the tangent vectors $V_\gamma := \dot\gamma(t_0)$ and $V_\eta := \dot\eta(s_0)$. By using the continuity of $\dot\gamma(t)$, $\dot\eta(s)$ and the compactness of the unit sphere, we have (by passing to subsequences again) that

$$\lim_{k\to\infty} \dot\gamma(t_k) = V_\gamma \text{ and } \lim_{k\to\infty} \dot\eta(s_k) = V_\eta.$$

Now, it is clear that $V_\gamma \neq \pm V_\eta$, by using the fact that the graphs of γ and η do not coincide. The injectivity radius at p_0 is positive. However, since γ and η intersect in all geodesic balls $B_\epsilon(p_0)$ for any $\epsilon > 0$, this is a contradiction. This shows that two different nontangential geodesics can only intersect finitely many times. □

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