Quark Mass Renormalization in Perturbative Quantum Chromodynamics in Light-Cone Gauge

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Abstract

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Perturbative Quantum Chromodynamics includes virtual particle perturbations which generate divergences. The divergences arise from the integration over the virtual momentum, and they are eliminated by a procedure called renormalization. In this thesis, the renormalization of a gluon loop corrected quark propagator, in the light-cone gauge, is studied. The light-cone gauge is known to be advantageous because it does not include Faddeev–Popov ghosts, and when deriving the DGLAP evolution equations. However, the calculations including light-cone gauge gluon propagator are challenging due to the unphysical pole in the gluon propagator. It appears that the light-cone gauge results for the pole mass of the one-loop corrected quark propagator and the complete self energy with the finite parts have not been explicitly listed in the literature.

In this thesis the unphysical pole in the gluon propagator is regulated with the Mandelstam–Leibbrandt prescription. The quark self energy is solved, including the finite terms, which is used to derive the effective quark propagator. The pole mass of the effective quark propagator is defined up to the order g_s^2, and it is found to be equal to the covariant gauge result. The quark field and mass renormalization counterterms in the $\overline{\text{MS}}$ scheme are determined and they are found to agree with the results in the literature.

Keywords: Quantum Chromodynamics, renormalization, ligh-cone gauge, pole mass

Tiivistelmä

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Kvarkin massarenormalisaatio kvanttiväridynamiikan häiriöteoriassa valokartiomitassa
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Kvanttiväridynamiikan häiriöteoriassa esiintyvät virtuaaliset häiriöt tuottavat äärettömyyksiä, jotka ilmenevät integroidessa virtuaalisen hiukkasen liikemäärän suhteen. Äärettömyydet poistetaan renormalisaatioksi kutsutulla menetelmällä. Tässä tutkielmassa renormalisoidaan valokartiomitassa kvarkkipropagaattoria, joka sisältää häiriön gluonisilmukan muodossa. Valokartiomitta on todettu hyödylliseksi käsiteltäessä DGLAP evoluutioyhtälöitä sekä sen vuoksi, että se ei sisällä Faddeev-Popov-aaveita. Kuitenkin laskut, joissa esiintyy valokartiomitan gluonipropagaattori, ovat haastavia gluonipropagaatorissa esiintyvän ylimääräisen navan vuoksi. Vaikuttaa siltä, että valokartiomitassa ratkaistuja kvarkin napamassaa sekä itseisenergiaa ei ole esitetty eksplisiittisesti kirjallisuudessa.

Tässä tutkielmassa $n \cdot q$ napaa käsitellään Mandelstam–Leibbrandt-menetelmällä. Kvarkin itseisenergia lasketaan äärelliset termit mukaanlukien, ja sen avulla muodostetaan efektiivinen kvarkkipropagaattori. Efektiivisen propagaattorin napamassa lasketaan kertaluokassa g_s^2. Tulokseksi saatu napamassa vastaa kovariantin mitan tulosta. Kvarkin kenttä- ja massarenormalisaatio suoritetaan myös $\overline{\text{MS}}$-skeemassa, ja saadut renormalisaatiotermit vastaavat kirjallisuudessa esiintyviä tuloksia.

Avainsanat: kvanttiväridynamiikka, renormalisaatio, valokartiomitta, napamassa

Contents

1 Introduction

Quantum Chromodynamics (QCD) is a nonabelian SU(3)-symmetric field theory describing the strong interaction of quarks and gluons, caused by the colour charge they carry. Hadrons and mesons are particles composed of the quarks, antiquarks, and gluons bound together by the colour confinement that prevents the existence of free quarks or gluons. To understand the internal structure of hadrons and mesons, one has to make experimental observations of high-energy scattering processes.

In perturbation theory physical observables have virtual and real corrections in high energies. When a real particle is emitted from a physical observable, one refers to a real correction. Whereas a virtual correction is understood as an emitted particle that couples back to the observable later in time, and hence cannot be detected. To have more precise predictions for experimental results, theorists have to consider more orders in the perturbation theory. In QCD, the gluon self-interactions generate complex perturbation structures which quickly make the analytical treatment quite difficult. Perturbations generate singularities in calculations, which are eliminated by renormalization.

In this thesis the virtual one-loop gluon correction to a quark propagator is discussed. The singularities of perturbative QCD are discussed in section 2. The gauge is chosen to be the light-cone gauge which general properties are considered in section 3.1. In the light-cone gauge, the unphysical pole in the gluon propagator produces difficulties in calculations. In this work the unphysical pole is regulated with a so-called Mandelstam–Leibbrandt prescription which is discussed in section 3.2. In section 4, the quark self energy is calculated. In section **??** the effective quark propagator is obtained and the pole mass is defined. The singularities from the effective propagator are renormalised in section 5.2.

The standard choice for the natural units is used: $\hbar = c = 1$. Calculations are done in both Minkowski and Euclidean space. Their connection, regarding the four-vectors and the Dirac matrices, is discussed in appendix B.

2 Perturbative QCD

2.1 Virtual corrections

The effective coupling constant in QCD decreases at high interacion scales i.e. small distances, and increases at low interacion scales i.e. large distances. At high interacion scales the so-called asymptotic freedom ensures the perturbation theory to work well. After some small scale point the quarks and gluons are bound by color confinement and the perturbation theory cannot be used. [1]

The corrections are divided into virtual and real corrections. The real correction stands for the emission of a real particle. A virtual correction is understood as an emitted particle that is later in time coupled back to the system, forming a loop in the Feynman diagram of the system, hence it is called a loop correction. Next-to-leading order (NLO) corrections are the first non-zero terms of the order higher than g_s, the next-to-next-to-leading order (NNLO) terms are the next non-zero terms in the higher order than (NLO), and so on. To get more precise results one has to count in higher order terms, and to sum together all the possible corrections of the wanted order.

The evaluation of the Feynman graphs containing virtual corrections will give divergences which arise from the integration over the momentum of the virtual particle. Those divergences are either infrared (IR) or ultraviolet (UV), corresponding to the square of the virtual momentum being zero or infinite respectively. The special case of the IR divergences are the so-called collinear divergences. The collinear divergences arise when dealing with massless quarks and four-momenta cancel each other in a denominator of propagator. The divergences can be regulated multiple ways, depending on the type of the divergence. Regulation does not eliminate the singularities but gives a way to handle them.

If the divergences end up in any physical quantity, such as the pole mass, the quantity has to be renormalised. The renormalization stands for redefining the bare quantities, appearing in the bare Lagrangian, by absorbing the divergences in them. Then one expresses the theory with the renormalised quantities and the counterterms

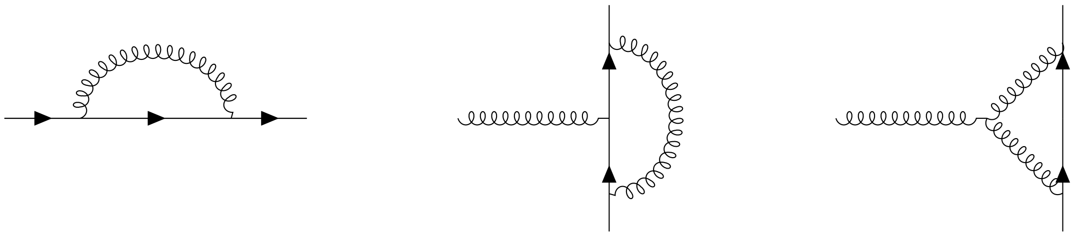

Figure 1. The one-loop corrections to the quark propagator and quark gluon vertex.

in which the divergences are located.

The LO correction to a propagator studied in this thesis is a virtual gluon loop as on the left side of Figure 1. If one would consider NLO corrections in an interaction, also the vertex corrections would have to be calculated. The NLO gluon loop vertices are seen on the right side of Figure 1.

2.2 Dimensional regularization

One method to handle the divergences arising from evaluation of Feynman diagrams is the so-called dimensional regularisation. Dimensional regularisation is based on understanding the number of spacetime dimensions D as a continuous, instead of a discrete variable and then increasing or decreasing the number of dimensions. The number of dimensions is increased when dealing with IR-divergences and decreased in the case of UV-divergences. The divergences are then identified as poles when dimension is analytically continuous near $D=4$.

In dimension D, the indices of Dirac matrices γ^μ and space-time vectors p run from 0 to $D-1$

$$p=(p^0,p^1,p^2,\ldots,p^{D-1}) \quad \text{and} \quad \gamma^0,\gamma^1,\ldots,\gamma^{D-1}. \tag{1}$$

The metric tensor in D dimensional Minkowski space is $g^{\mu\nu}=\mathrm{Diag}(1,-1,\ldots,-1)$, which yields

$$g^{\mu\nu}g_{\mu\nu}=D. \tag{2}$$

The Clifford algebra for gamma-matrices and the trace of the identity matrix remain intact in dimension D, but due to Eq. (2) some of the gamma-matrix identities are changed

$$
\begin{aligned}
\mathrm{Tr}(\mathbb{1}_D) &= 4 \\
\{\gamma^\mu,\gamma^\nu\} &= 2g^{\mu\nu}\mathbb{1}_D \\
\gamma^\mu\gamma^\nu\gamma_\mu &= -(D-2)\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu &= 4g^{\nu\lambda}\mathbb{1}_D + (D-4)\gamma^\nu\gamma^\lambda.
\end{aligned} \tag{3}
$$

The QCD action in D dimensions reads

$$
\begin{aligned}
S_{\mathrm{QCD}} = \int d^Dx\mathcal{L}_{\mathrm{QCD}} = \int d^Dx\Big[-\frac{1}{4}F^{\mu\nu,a}F^a_{\mu\nu} \\
+ \sum_q \left[i\left(\overline{\psi}_{q0}\right)_i \gamma^\mu \left((\partial_\mu)_{ij} + ig_sA^a_{0\mu}(t^a)_{ij}\right)\left(\psi_q\right)_j - m_{q0}\left(\overline{\psi}_{q0}\right)_i\left(\psi_q\right)_j \right]\Big],
\end{aligned} \tag{4}
$$

where $F^{\mu\nu,a}$ denotes the gluon field strength tensor defined as $F^a_{\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} - g_s f^{abc}A^b_{0\mu}A^c_{0\nu}$, the indices a,b,c denote color of gluons, i and j are color indices for quarks, and μ and ν are Lorentz indices, $A^a_{0\mu}$ is the bare gluon field, ψ_{q0} is the bare quark field, m_{q0} is a corresponding bare quark mass, f^{abc} is a SU(3) structure constant, g_s is the strong coupling constant, t^a is is a SU(3) generator, and the sum $\sum_q$ denotes a sum over quark flavors. The action is dimensionless i.e. $[S_{\mathrm{QCD}}] = [m]^0 = 1$. The spacetime integral has dimension $[L]^D$, where L denotes length. In natural units $[L] = [m]^{-1}$, which gives the dimension of the Lagrangian is $[m]^D$ and thus every term in QCD Lagrangian has dimension of $[m]^D$. From the quark mass term it can be seen the dimension of the wave function has to be $[\psi] = [m]^{(D-1)/2}$. And from the kinetic term of the gluon field one gets $[A^a_\mu] = [m]^{(D-2)/2}$. Examining the quark and gluon interaction term results in

$$
[g_s][A^a_\mu][\psi]^2 = [m]^D \quad \longrightarrow \quad [g_s] = [m]^{2-D/2}, \tag{5}
$$

which could also be derived from the gluon self-interaction terms. The dimension of the coupling constant can be expressed with an arbitrary mass parameter μ. Replacing the coupling constant in the Lagrangian with

$$
g_s \longrightarrow g_s\mu^{2-D/2}, \quad \text{where } [\mu] = [m] \text{ and } [g_s] = [m]^0, \tag{6}
$$

allows utilizing the same dimensionless coupling constant as in 4 dimensional theory.

The QCD-theory is renormalizable in four dimensions. In this thesis the dimension is chosen as

$$D \equiv 2\omega, \quad \text{where } \omega \equiv 2 - \epsilon. \tag{7}$$

Regulating UV-divergences requires $\epsilon > 0$ whereas regulating IR-divergences $\epsilon < 0$. Evaluating Feynman diagrams in 2ω dimensions will give ϵ^{-1} divergences, which can be eliminated with the renormalization counterterms. However, the dimensionally regulated IR and UV divergences are not always distinguishable in the final result. In QCD calculations the renormalization is used to remove UV-divergences, and IR-divergences are expected to cancel between different loop corrections of physical observables.

The renormalization counterterms can also contain finite terms. The choice of which finite terms are included defines the renormalization scheme. In the so-called minimal subtraction (MS) scheme only the divergent parts ϵ^{-1} are included to the counterterms. In the modified minimal subtraction ($\overline{\text{MS}}$) scheme counterterms include the terms

$$\frac{1}{\epsilon} - \gamma_E + \log(4\pi), \tag{8}$$

which arise from the dimensional regularization, and where γ_E denotes the Euler's constant. There are infinitely many ways to add finite terms to the counterterms, however the terms cannot have momentum dependence since they appear in the renormalised Lagrangian.

3 Light-Cone gauge

3.1 Properties of a light-cone gauge

In this thesis the one-loop correction of quark propagator is studied in the light-cone gauge. The light-cone gauge is an axial gauge where the gluon field satisfies the constraint

$$n \cdot A^a = 0, \tag{9}$$

where n is a fixed four-vector. The gauge fixing term restricting the degrees of freedom in the QCD-Lagrangian defined in Eq. (141) is

$$\mathcal{L}_{\text{fix}} = \frac{-1}{2\alpha}(n \cdot A^a)^2, \tag{10}$$

where α is an arbitrary parameter, usually taken to zero in the resulting Feynman rules. The light-cone gauge is characterised by the vector n being light-like i.e.

$$n^2 = 0. \tag{11}$$

The Euler-Lagrange equation for the non-interacting gluon field is

$$\frac{\partial \mathcal{L}_{\text{YM}}}{\partial A^{b\nu}} - \partial^\mu \frac{\partial \mathcal{L}_{\text{YM}}}{\partial(\partial_\mu A^{b\nu})} = \left(\frac{-1}{\alpha} n_\alpha n_\nu + \partial^2 g_{\alpha\nu} - \partial_\alpha \partial_\nu\right) \delta^{ab} A^{b\alpha} = 0, \tag{12}$$

where $\mathcal{L}_{\text{YM}}$ is the free Yang-Mills Lagrangian defined in appendix A. A gluon propagator is defined as a Green's function for the gluon field. The Green's function $G^{\alpha\beta}(q)$ for the gluon field satisfies

$$\begin{gathered}\left(\frac{-1}{\alpha} n_\alpha n_\nu + \partial^2 g_{\alpha\nu} - \partial_\alpha \partial_\nu\right) \delta^{ab} G^{\alpha\beta}(x-y) = i\delta^\beta_\nu \delta^{ab} \delta^4(x-y) \\ \int \frac{d^4q}{(2\pi)^4} G^{\alpha\beta}(q) \delta^{ab} \left(\frac{-1}{\alpha} n_\alpha n_\nu - q^2 g_{\alpha\nu} + q_\alpha q_\nu\right) e^{iq(x-y)} = \int \frac{d^4q}{(2\pi)^4} i\delta^\beta_\nu \delta^{ab} e^{iq(x-y)}.\end{gathered} \tag{13}$$

Noting the Green's function has to be symmetric with respect to the α and β

interchange, one can make an ansatz

$$G^{\alpha\beta}(q) = Ag^{\alpha\beta} + Bq^\alpha q^\beta + Cn^\alpha n^\beta + Dn^\alpha q^\beta + Eq^\alpha n^\beta. \tag{14}$$

Then matching the left and right sides of Eq. (13) one arrives at a solution

$$\delta^{ab} G^{\alpha\beta}(q) = \frac{-i\delta^{ab}}{q^2}\left(g^{\alpha\beta} - \frac{n^\alpha q^\beta + q^\alpha n^\beta}{n \cdot q} - \frac{\alpha q^2 q^\alpha q^\beta}{(n \cdot q)^2}\right). \tag{15}$$

Finally, taking a limit $\alpha \to 0$ and adding the Feynman $i\epsilon$-prescription, the Green's function (15) can be defined as the gluon propagator

$$\delta^{ab} D_{\mu\nu}(q) = \frac{-i\delta^{ab}}{q^2 + i\epsilon}\left(g_{\mu\nu} - \frac{n_\mu q_\nu + n_\nu q_\mu}{n \cdot q}\right). \tag{16}$$

It can be seen that for the gluon propagator applies

$$n^\mu D_{\mu\nu}(q) = n^\nu D_{\mu\nu}(q) = 0. \tag{17}$$

External gluons have only two physical polarization states since they are transversally polarized. However when taking a square of the invariant matrix element one essentially has gluon loops formed by the external gluons. In covariant gauges these ”gluon loops” are summed over also by the two unphysical polarization states and therefore the squared matrix element will have unphysical terms. To eliminate the unphysical terms one has to add so-called Faddeev-Popov ghosts to the Lagrangian. The ghosts are merely a mathematical tool to retain the wanted outcome for the squared matrix element.

In the light-cone gauge, the gluon propagator has a form that gives the relation in Eq. (17) which causes the ghosts to decouple from the gluons [2]. The crucial advantage of choosing a light-cone gauge is the feature of not having ghosts. Also the calculation of the so-called Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) evolution equations in the light-cone gauge is less complicated than in a covariant gauge. [1, 3, 4]

3.2 Mandelstam-Leibbrandt prescription

The Feynman integrals containing the gluon propagator defined in Eq. (16) include the pole $(n \cdot q)^{-1}$ which may yield additional divergences. These divergences can be regulated in multiple ways. In this thesis the regulation is carried out by using the so-called Mandelstam-Leibbrandt (ML) prescription [2]. The idea of the ML-prescription is to chance the denominator by adding a small imaginary shift which is later taken to zero after the Wick rotation.

Another light-cone feature $n^2 = 0$ yields ambiguity considering the values of the components of the vector n

$$(n^0)^2 = \mathbf{n}^2 \longrightarrow n^0 = \pm|\mathbf{n}|. \tag{18}$$

The constraint $n^2 = 0$ does not fix the n vector and therefore the value of the pole $(n \cdot q)^{-1}$ is not unique. The ML-prescription addresses this ambiguity by fixing the four-vector n as

$$n \equiv (|\mathbf{n}|, \mathbf{n}), \tag{19}$$

and then defining a new four-vector n^* as

$$n^* \equiv (|\mathbf{n}|, -\mathbf{n}). \tag{20}$$

For both these vectors the time component is positive which gives $n^* \cdot n > 0$.

The ML-prescription is constructed with the vectors n and n^* in Leibbrandt's way [2, 5] as

$$\frac{1}{n \cdot q} = \lim_{\theta \to 0} \frac{n^* \cdot q}{(n^* \cdot q)(n \cdot q) + i\theta} \quad \theta > 0. \tag{21}$$

This is equal to a form which was discovered by Mandelstam [6]

$$\frac{1}{n \cdot q} = \lim_{\theta \to 0} \frac{1}{n \cdot q + \frac{i\theta}{n^* \cdot q}} \quad \theta > 0, \tag{22}$$

from which it is easier to see the idea of ML-prescription being an imaginary shift in the denominator. In the literature the term $i\theta/(n^* \cdot q)$ is often written as $i\theta \text{sign}(n^* \cdot q)$, where the absolute value of the inner product is absorbed to θ. In this thesis the Leibbrandt's version (21) of ML-prescription is used.

The integral with the ML-prescription defined in Eq. (21) reads

$$\int d^D q \frac{n^* \cdot q}{(n^* \cdot q)(n \cdot q) + i\theta} = \int d^D q \frac{n^* \cdot q}{n_0^2 (q_0^2 - \frac{(\mathbf{n} \cdot \mathbf{q})^2}{n_0^2} + \frac{i\theta}{n_0^2})}, \tag{23}$$

from where one can see the poles are placed in the second and fourth quadrants of the complex plane i.e. $q_0 = \pm|\mathbf{n} \cdot \mathbf{q}|/|n_0| \mp i\theta/(2|n_0||\mathbf{n} \cdot \mathbf{q}|)$ as in Figure 2. Performing the Wick rotation as in appendix C.2, with the path Γ defined in Figure 2, one changes the time-like component of the gluon vector q as $q_0 = iq_4$ and then defines the Euclidean vector $q_E^2 = -q^2$. With transformations $n_0 = in_4$ and $n_0^{*0} = in_4^* = in_4$ it is possible to transfer all the vectors in Eq. (23) to Euclidean space. The identities regarding this transformation are derived in appendix B. After the Wick rotation and the transformation to Euclidean space the integral in Eq. (23) reads

$$i \int_E d^D q \frac{-(n^* \cdot q)_E}{-n_4^2 \left(-q_4^2 - \frac{(\mathbf{n} \cdot \mathbf{q})^2}{-n_4^2}\right)} = i \int_E d^D q \frac{-(n^* \cdot q)_E}{(n^* \cdot q)_E (n \cdot q)_E} = -i \int_E d^D q \left(\frac{1}{n \cdot q}\right)_E . \tag{24}$$

The advantage of transforming to Euclidean space is that one does not have to explicitly perform the Wick rotation when evaluating integrals with the pole $(n \cdot q)^{-1}$,

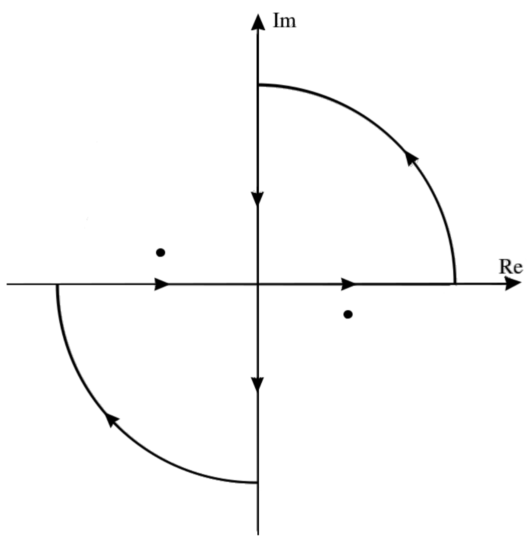

Figure 2. Path Γ in the complex plane with poles of the ML-prescription in the second and fourth quadrants.

but one can use the relation

$$\left(\frac{1}{n\cdot q}\right) = -\left(\frac{1}{n\cdot q}\right)_E, \tag{25}$$

where the Euclidean ML-prescription is defined as [5]

$$\left(\frac{1}{n\cdot q}\right)_E = \lim_{\theta^2\to 0}\frac{(n^*\cdot q)_E}{(n^*\cdot q)_E(n\cdot q)_E+\theta^2} \quad \theta^2 > 0. \tag{26}$$

If light-cone gauge gluon and quark propagators appear in the same Feynman diagram, the corresponding integral contains three types of poles $p^2+m_0^2+i\epsilon$, $q^2+i\epsilon$ and $n\cdot q$, where q and p are gluon and quark momenta respectively and m_0 is the quark mass. Since all the poles are located in the second and fourth quadrants on a complex plane, the Wick rotation is achievable. This is the crucial advantage of the ML-prescription compared to the conventional Principal Value (PV) [7] prescription which sets the poles of $(n\cdot q)^{-1}$ either first and fourth or second and third quadrants, and therefore prevents taking the Wick rotation.

4 One-loop quark self energy (Σ)

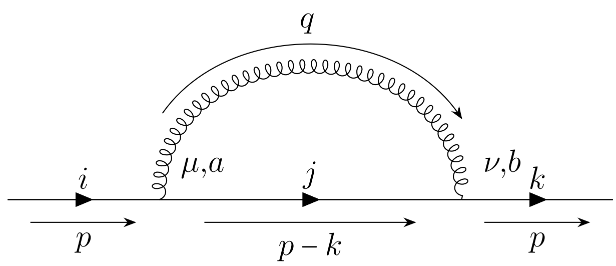

Figure 3. The leading order gluon loop correction to the quark propagator.

With the Feynman rules, from appendix A, the perturbed gluon propagator in Figure 3 reads

$$\begin{aligned}&\int \frac{d^4q}{(2\pi)^4}\frac{i}{\not{p}-m_0+i\epsilon}(ig\gamma^\mu t^a_{ji})\delta^{ab}D_{\mu\nu}(q)\frac{i}{\not{p}-\not{q}-m_0+i\epsilon}(ig\gamma^\nu t^b_{ik})\frac{i}{\not{p}-m_0+i\epsilon}\\ &=\frac{i}{\not{p}-m_0+i\epsilon}\Sigma\frac{i}{\not{p}-m_0+i\epsilon},\end{aligned} \tag{27}$$

where Σ is the self energy of the quark. For the SU(3) generator matrices one can use

$$(t^a)_{ji}(t^a)_{ik}=(t^at^a)_{ik}=C_F\delta_{ik}=\frac{4}{3}\delta_{ik}, \tag{28}$$

where C_F is the so-called Casimir operator. With Eq. (28) the self energy is

$$\begin{aligned}\Sigma&=-ig_s^2C_F\int\frac{d^4q}{(2\pi)^4}\gamma^\mu D_{\mu\nu}(q)\frac{\not{p}-\not{q}+m_0}{(p-q)^2-m_0^2+i\epsilon}\gamma^\nu\\ &=-g_s^2C_F\int\frac{d^4q}{(2\pi)^4}\gamma^\mu\frac{\not{p}-\not{q}+m_0}{(p-q)^2-m_0^2+i\epsilon}\gamma^\nu\frac{1}{q^2+i\epsilon}\left[g_{\mu\nu}-\frac{n_\mu q_\nu+q_\mu n_\nu}{n\cdot q}\right].\end{aligned} \tag{29}$$

The UV-divergences are regulated with the dimensional regularization by changing to 2ω dimensions defined in section 2.2. Using the identities in Eq. (3) the self energy reads

$$\Sigma = -\, g_s^2 \mu^{4-2\omega} C_F \int \frac{d^{2\omega} q}{(2\pi)^{2\omega}} \Bigg[\frac{-2(\omega-1)(\not{p} - \not{q}) + 2(\omega-1) m_0}{((p-q)^2 - m_0^2 + i\epsilon)(q^2 + i\theta)} - \frac{\not{n}\not{p}\not{q} + \not{q}\not{p}\not{n} - 2q^2 \not{n}}{((p-q)^2 - m_0^2 + i\epsilon)(q^2 + i\theta) n\cdot q} \Bigg], \tag{30}$$

where the pole $n \cdot q$ is regulated via the ML-prescription.

To get rid of the imaginary parts in the denominator of Eq. (30) one has to perform the Wick rotation with a pole $q_0 = \pm|\mathbf{q}| \mp i\epsilon$ from the gluon propagator and a pole $q_0 = \pm\sqrt{(\mathbf{p}-\mathbf{q})^2 + m_0^2} \mp i\epsilon$ from the quark propagator. The Wick rotation for the pole $n \cdot q$ goes as in section 3.2. With transforms $p^0 = ip^4, q^0 = iq^4$, $n^0 = in^4$ and $\gamma^0 = i\gamma^4$ one can transfer from Minkowski space to Euclidean space. The relations for four-vectors, gamma-matrices and trace identities in Euclidean space are derived in appendix B.

The self energy in Euclidean space is

$$\Sigma_E = -i g_s^2 \mu^{4-2\omega} C_F \int_E \frac{d^{2\omega} q_E}{(2\pi)^{2\omega}} \Bigg[\frac{2(\omega-1)(\not{p}_E - \not{q}_E) + 2(\omega-1) m_0}{((p_E - q_E)^2 + m_0^2) q_E^2} - \frac{\not{n}_E \not{p}_E \not{q}_E + \not{q}_E \not{p}_E \not{n}_E + 2 q_E^2 \not{n}_E}{((p_E - q_E)^2 + m_0^2) q_E^2 (n\cdot q)_E} \Bigg], \tag{31}$$

where $q_E = (q_4,\mathbf{q})$, $p_E = (p_4,\mathbf{p})$, $n_E = (n_4,\mathbf{n})$ are Euclidean vectors in 2ω dimensions and the Euclidean integral is over q_4,q_1,q_2 and q_3. The Euclidean ML-prescription for the pole $(n\cdot q)_E^{-1}$ is defined in Eq. (26). To simplify evaluation of Eq. (31) the vectors n and n^* can be fixed as

$$n = (n_0, \underline{0}, n_3) \quad \text{and} \quad \mathrm{n}^* = (\mathrm{n}_0, \underline{0}, -\mathrm{n}_3). \tag{32}$$

The relation $n_E^2 = 0$ gives

$$n_4^2 = -n_3^2. \tag{33}$$

The components, of the vectors q and p, perpendicular to n are denoted by $q_\perp$ and $p_\perp$.

4.1 Ansatz for the self energy

The straightforward integration of Eq. (31) is made challenging by the gamma-matrices in the numerator. This can be avoided by making an ansatz. When integrating with respect to q_E in Eq. (31) the result is proportional to the identity matrix $\mathbb{1}_{2\omega}$. When integrating with $\not{q}_E$ in Eq. (31) the result is going to be proportional to one of the matrices $\not{p}_E$, $\not{n}_E$ or $\not{n}^*{}_E$. If the integral, with $\not{q}_E$, is free of the variables n_E and n_E^*, the result is proportional to only $\not{p}_E$, otherwise it could be proportional to any of those matrices.

The challenging part in Eq. (31) is the integral

$$\int_E \frac{d^{2\omega} q_E}{(2\pi)^{2\omega}} \Bigg[\not{n}_E \not{p}_E \frac{\not{q}_E}{((p_E - q_E)^2 + m_0^2) q_E^2 (n \cdot q)_E} + \frac{\not{q}_E}{((p_E - q_E)^2 + m_0^2) q_E^2 (n \cdot q)_E} \not{p}_E \not{n}_E \Bigg] \tag{34}$$

which result can be proportional to the matrices $\not{p}, \not{n}$ and $\not{n}^*$. Examining what happens to the term $\not{n}_E \not{p}_E \not{q}_E + \not{q}_E \not{p}_E \not{n}_E$ when evaluating the integral (34) gives

$$\not{n}_E \not{p}_E \not{q}_E + \not{q}_E \not{p}_E \not{n}_E = \begin{cases} \not{n}_E \not{p}_E \not{p}_E + \not{p}_E \not{p}_E \not{n}_E = -2p_E^2 \not{n}_E, & \text{when } \int_E d^{2\omega} dq f(\not{q}_E) \propto \not{p}_E \\ \not{n}_E \not{p}_E \not{n}_E + \not{n}_E \not{p}_E \not{n}_E = -4(n \cdot p)_E \not{n}_E, & \text{when } \int_E d^{2\omega} dq f(\not{q}_E) \propto \not{n}_E \\ \not{n}_E \not{p}_E \not{n}^*{}_E + \not{n}^*{}_E \not{p}_E \not{n}_E = -4(n^* \cdot p)_E \not{n}_E + \not{n}_E \not{n}^*{}_E \not{p}_E + \not{p}_E \not{n}^*{}_E \not{n}_E, & \text{when } \int_E d^{2\omega} dq f(\not{q}_E) \propto \not{n}^*{}_E, \end{cases} \tag{35}$$

where it can be seen the integral value of Eq. (31) can be written in a way that depends on the sum $\not{n}_E \not{n}^*{}_E \not{p}_E + \not{p}_E \not{n}^*{}_E \not{n}_E$ and the matrices $\mathbb{1}_{2\omega}$, $\not{p}$, and $\not{n}$, but not on the matrix $\not{n}^*$.

Using these results the ansatz can be constructed as

$$\Sigma_E = A + B\not{p}_E + C\left(\not{n}\not{n}^*\not{p} + \not{p}\not{n}^*\not{n}\right)_E + D\not{n}_E. \tag{36}$$

With the Euclidean trace identities, derived in appendix B, the constants in Eq. (36) can be solved

$$A = \frac{1}{4}\mathrm{Tr}(\Sigma_E), \tag{37}$$

$$B = -\frac{\mathrm{Tr}(\not{n}_E \Sigma_E)}{4(n \cdot p)_E}, \tag{38}$$

$$C = \frac{\mathrm{Tr}(\not{n}^*{}_E \Sigma_E) + 4(n^* \cdot p)_E B + 4(n^* \cdot n)_E D}{16(n^* \cdot n)_E (n^* \cdot p)_E}, \tag{39}$$

and

$$D = \frac{-4p_E^2 B + 8p_E^2 (n^* \cdot n)_E C - \mathrm{Tr}(\not{p}_E \Sigma_E)}{4(n \cdot p)_E}. \tag{40}$$

Plugging these in Eq. (36) the self energy results in

$$\begin{aligned}
\Sigma_E = {} & \frac{1}{4}\mathrm{Tr}(\Sigma_E) - \frac{\mathrm{Tr}(\not{n}_E \Sigma_E)}{4(n \cdot p)_E}\not{p}_E \\
& + \left[\frac{\mathrm{Tr}(\not{n}^*{}_E \Sigma_E) - \frac{(n^* \cdot p)_E}{(n \cdot p)_E}\mathrm{Tr}(\not{n}_E \Sigma_E)}{16(n^* \cdot n)_E (n^* \cdot p)_E} \right. \\
& \left. + \frac{p_E^2 \mathrm{Tr}(\not{n}^*{}_E \Sigma_E) + \frac{p_E^2 (n^* \cdot p)_E}{(n \cdot p)_E}\mathrm{Tr}(\not{n}_E \Sigma_E) - 2(n^* \cdot p)_E \mathrm{Tr}(\not{p}_E \Sigma_E)}{4(n^* \cdot p)_E \left(8(n^* \cdot p)_E (n \cdot p)_E - 4p_E^2 (n^* \cdot n)_E\right)} \right] \left(\not{n}\not{n}^*\not{p} + \not{p}\not{n}^*\not{n}\right)_E \\
& + \frac{p_E^2 \mathrm{Tr}(\not{n}^*{}_E \Sigma_E) + \frac{p_E^2 (n^* \cdot p)_E}{(n \cdot p)_E}\mathrm{Tr}(\not{n}_E \Sigma_E) - 2(n^* \cdot p)_E \mathrm{Tr}(\not{p}_E \Sigma_E)}{8(n^* \cdot p)_E (n \cdot p)_E - 4p_E^2 (n^* \cdot n)_E}\not{n}_E .
\end{aligned} \tag{41}$$

4.2 Trace of the self energy

To solve the constant A in the self energy ansatz (36)) one has to evaluate the trace of the self energy (31). With Feynman parametrization from appendix C.1 one has

$$\begin{aligned}
\mathrm{Tr}(\Sigma_E) &= -ig_s^2 \mu^{4-2\omega} C_F \int_E \frac{d^{2\omega}q}{(2\pi)^\omega} \left(\frac{8(\omega-1)m_0}{((p-q)^2 + m_0^2)q^2} \right)_E \\
&= -ig_s^2 \mu^{4-2\omega} C_F \int_E \frac{d^{2\omega}q}{(2\pi)^\omega} \int_0^1 dx \left(\frac{8(\omega-1)m_0}{[((p-q)^2 + m_0^2)x + (1-x)q^2]^2} \right)_E .
\end{aligned} \tag{42}$$

Utilizing the basic integral in Eq. (177) calculated in appendix C.3 and expanding in powers of ϵ, the trace becomes

$$\mathrm{Tr}(\Sigma_E) = -i\tilde{\alpha}_s C_F 8m + \frac{ig_s^2 C_F}{(4\pi)^2} 8m \left(1 + \int_0^1 dx \log \frac{x(1-x)p_E^2 + xm_0^2}{\mu^2} \right), \tag{43}$$

where

$$\tilde{\alpha}_s \equiv \frac{g_s^2 \Gamma(2-\omega)}{(4\pi)^\omega} = \frac{g_s^2}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \mathcal{O}(\epsilon) \right). \tag{44}$$

4.3 Trace of $\not{n}_E \Sigma_E$

The constants B,C and D in the self energy ansatz (36) depend on the trace

$$\begin{aligned}
\mathrm{Tr}(\not{n}_E \Sigma_E) &= i g_s^2 \mu^{4-2\omega} C_F \int_E \frac{d^{2\omega} q}{(2\pi)^\omega} \left(\frac{8(\omega-1)(p-q)\cdot n}{((p-q)^2+m_0^2)q^2} \right)_E \\
&= i g_s^2 \mu^{4-2\omega} C_F \int_0^1 dx \int_E \frac{d^{2\omega} q}{(2\pi)^\omega} \left(\frac{8(\omega-1)(p-q)\cdot n}{[((p-q)^2+m_0^2)x+(1-x)q^2]^2} \right)_E .
\end{aligned} \tag{45}$$

Using the basic integral in Eq. (177) calculated in appendix C.3 and expanding in powers of ϵ, the trace becomes

$$\begin{aligned}
\mathrm{Tr}(\not{n}_E \Sigma_E) &= i \tilde{\alpha}_s C_F 4(n\cdot p)_E \\
&\quad - \frac{i g_s^2 C_F}{(4\pi)^2} 4(n\cdot p)_E \left(1 + 2\int_0^1 dx (1-x) \log \frac{x(1-x)p_E^2 + x m_0^2}{\mu^2} \right).
\end{aligned} \tag{46}$$

4.4 Trace of $\not{p}_E \Sigma_E$

The constants B,C and D in the self energy ansatz (36) depend on the trace

$$\begin{aligned}
\mathrm{Tr}(\not{p}_E \Sigma_E) &= -i g_s^2 \mu^{4-2\omega} C_F \int_E \frac{d^{2\omega} q}{(2\pi)^\omega} \Bigg[\frac{8(\omega-1)p\cdot q + 8(2-\omega)p^2}{((p-q)^2+m_0^2)q^2} \\
&\quad + \frac{8 n\cdot p}{((p-q)^2+m_0^2) n\cdot q} - \frac{16(n\cdot p)(p\cdot q)}{((p-q)^2+m_0^2)q^2 n\cdot q} \Bigg]_E \\
&\equiv \mathrm{Tr}(\not{p}_E \Sigma_E)_A + \mathrm{Tr}(\not{p}_E \Sigma_E)_B + \mathrm{Tr}(\not{p}_E \Sigma_E)_C .
\end{aligned} \tag{47}$$

The two last integrals contain the pole $(n\cdot q)_E^{-1}$ which complicates the integration. With the Feynman parametrization, the first term in Eq. (47) is

$$\begin{aligned}
\mathrm{Tr}(\not{p}_E \Sigma_E)_A &= -i g_s^2 \mu^{4-2\omega} C_F \int_E \frac{d^{2\omega} q}{(2\pi)^\omega} \left(\frac{8(\omega-1)p\cdot q + 8(2-\omega)p^2}{((p-q)^2+m_0^2)q^2} \right)_E \\
&= -8 i g_s^2 \mu^{4-2\omega} C_F \int_0^1 dx \int_E \frac{d^{2\omega} q}{(2\pi)^\omega} \left(\frac{(\omega-1)p\cdot q + 8(2-\omega)p^2}{[x((p-q)^2+m_0^2)+(1-x)q^2]^2} \right)_E .
\end{aligned} \tag{48}$$

Using the basic integral in Eq. (177) and expanding in powers of ϵ one gets

$$\mathrm{Tr}(\not{p}_E \Sigma_E)_A = -i \tilde{\alpha}_s C_F 4 p_E^2 + \frac{i g_s^2 C_F}{(4\pi)^2} 4 p_E^2 \left(-1 + 2\int_0^1 dx x \log \frac{x(1-x)p_E^2 + x m_0^2}{\mu^2} \right). \tag{49}$$

The second term in Eq. (47), with the ML-prescription and Feynman parametrization, reads

$$
\begin{aligned}
&\mathrm{Tr}(\not{p}_E \Sigma_E)_B = \\
&- i g_s^2 \mu^{4-2\omega} C_F 8(n\cdot p)_E \int_E \frac{d^{2\omega}q}{(2\pi)^\omega} \left(\frac{n_4 q_4 - n_3 q_3}{((p-q)^2 + m_0^2)((n_4 q_4)^2 - (n_3 q_3)^2 + \theta^2)} \right)_E \\
&= -i g_s^2 \mu^{4-2\omega} C_F \frac{8(n\cdot p)_E}{n_4} \int_E \frac{d^{2\omega}q}{(2\pi)^\omega} \left(\frac{q_4 - a q_3}{((p-q)^2 + m_0^2)(q_4^2 + q_3^2 + \frac{\theta^2}{n_4^2})} \right)_E \\
&= -i g_s^2 \mu^{4-2\omega} C_F \frac{8(n\cdot p)_E}{n_4} \int_0^1 dx \int_E \frac{d^{2\omega}q}{(2\pi)^\omega} \\
&\quad \left(\frac{q_4 - a q_3}{[x((p-q)^2 + m_0^2) + (1-x)(q_4^2 + q_3^2 + \frac{\theta^2}{n_4^2})]^2} \right)_E , \qquad (50)
\end{aligned}
$$

where

$$
a \equiv \frac{n_3}{n_4}. \qquad (51)
$$

The denominator can be expressed as

$$
\begin{aligned}
&\underbrace{(q_4 - x p_4)}_{\equiv l}{}^2 + \underbrace{(q_3 - x p_3)}_{\equiv A}{}^2 + x\underbrace{(q_\perp - p_\perp)}_{\equiv B}{}^2 + \underbrace{x(1-x)(p_3^2 + p_4^2) + x m_0^2}_{\equiv C} + (1-x)\frac{\theta^2}{n_4^2} \\
&\equiv l^2 + A^2 + x B^2 + C + (1-x)\frac{\theta^2}{n_4^2},
\end{aligned} \qquad (52)
$$

where now θ^2 can be taken to zero. Changing the integration variable from q_4 to the l gives

$$
\begin{aligned}
&\mathrm{Tr}(\not{p}_E \Sigma_E)_B = \\
&- i g_s^2 \mu^{4-2\omega} C_F \frac{8(n\cdot p)_E}{n_4} \int_0^1 dx \int_E \frac{d^{2\omega-1}q}{(2\pi)^\omega} \int_{-\infty}^{\infty} dl \frac{l + x p_4 - a q_3}{[l^2 + A^2 + x B^2 + C]^2} \\
&= -i g_s^2 \mu^{4-2\omega} C_F \frac{8(n\cdot p)_E}{n_4} \int_0^1 dx \int_E \frac{d^{2\omega-1}q}{(2\pi)^\omega} \int_0^{\infty} dl 2 \frac{x p_4 - a q_3}{[l^2 + A^2 + x B^2 + C]^2}.
\end{aligned} \qquad (53)
$$

Using Eq. (163) from appendix C.1 to integrate over l gives

$$
\int_0^\infty dl 2 \frac{x p_4 - a q_3}{[t^2 + A^2 + x B^2 + C]^2} = \frac{\pi}{2} \frac{x p_4 - a q_3}{[A^2 + x B^2 + C]^{3/2}}. \qquad (54)
$$

With this, the trace in Eq. (53) becomes

$$\mathrm{Tr}(\not{p}_E\Sigma_E)_B = -ig_s^2\mu^{4-2\omega}C_F\frac{8(n\cdot p)_E}{n_4}\frac{\pi}{2}\int_0^1 dx\int_E\frac{d^{2\omega-1}q}{(2\pi)^\omega}\frac{xp_4-aq_3}{[A^2+xB^2+C]^{3/2}}. \tag{55}$$

Repeating the same steps when integrating with respect to q_3 gives

$$\mathrm{Tr}(\not{p}_E\Sigma_E)_B = -ig_s^2\mu^{4-2\omega}C_F\frac{8(n\cdot p)_E}{n_4}\pi\int_0^1 dx\int_E\frac{d^{2\omega-2}q_\perp}{(2\pi)^\omega}\frac{x(p_4-ap_3)}{xB^2+C}\quad. \tag{56}$$

With the basic integral in Eq. (177) from appendix C.3 and the indentity

$$\frac{p_4-ap_3}{2n_4}=\frac{n^*\cdot p}{n^*\cdot n}, \tag{57}$$

the trace is

$$\begin{aligned}\mathrm{Tr}(\not{p}_E\Sigma_E)_B &= -i\tilde{\alpha}_sC_F\frac{16(n\cdot p)_E(n^*\cdot p)_E}{(n^*\cdot n)_E}\\ &+\frac{ig_s^2C_F}{(4\pi)^2}\frac{16(n\cdot p)_E(n^*\cdot p)_E}{(n^*\cdot n)_E}\int_0^1 dx\log\left(\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right),\end{aligned} \tag{58}$$

where $\tilde{\alpha}_s$ is defined in Eq. (44).

The third trace in Eq. (47), with the ML-prescription and Feynman parametrization, is

$$\begin{aligned}\mathrm{Tr}(\not{p}_E\Sigma_E)_C &= ig_s^2\mu^{4-2\omega}C_F\int_E\frac{d^{2\omega}q}{(2\pi)^\omega}\left(\frac{16(n\cdot p)(p\cdot q)}{((p-q)^2+m_0^2)q^2n\cdot q}\right)_E\\ &= ig_s^2\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E}{n_4}\int_E\frac{d^{2\omega}q}{(2\pi)^\omega}\left(\frac{p\cdot q(q_4-aq_3)}{((p-q)^2+m_0^2)q^2(q_4^2+q_3^2+\frac{\theta^2}{n_4^2})}\right)_E\\ &= ig_s^2\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E}{n_4}\int_0^1 dx\int_0^{1-x}dy\int_E\frac{d^{2\omega}q}{(2\pi)^\omega}\\ &\left(\frac{2p\cdot q(q_4-aq_3)}{[x((p-q)^2+m_0^2)+yq^2+(1-x-y)(q_4^2+q_3^2+\frac{\theta^2}{n_4^2})]^3}\right)_E\end{aligned} \tag{59}$$

where $a=n_3/n_4$. One can use the relation

$$p_E^\mu\frac{\partial}{\partial p_E^\mu}\frac{1}{\left[x((p-q)_E^2+A\right]^2}=\frac{-4x(p_E^2-(p\cdot q)_E)}{\left[x((p-q)_E^2+A\right]^3} \tag{60}$$

and get

$$\begin{aligned}\mathrm{Tr}(\not{p}_E\Sigma_E)_C &= ig_s^2\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E}{n_4}\int_0^1 dx\int_0^{1-x}dy\int_E\frac{d^{2\omega}q}{(2\pi)^\omega}\\ &\left\{p^\mu\frac{\partial}{\partial p^\mu}\left[\frac{(q_4-aq_3)/(2x)}{[x((p-q)^2+m_0^2)+yq^2+(1-x-y)(q_4^2+q_3^2+\frac{\theta^2}{n_4^2})]^2}\right]\right.\\ &\left.+\frac{2p^2(q_4-aq_3)}{[x((p-q)^2+m_0^2)+yq^2+(1-x-y)(q_4^2+q_3^2+\frac{\theta^2}{n_4^2})]^3}\right\}_E.\end{aligned} \tag{61}$$

The denominators can be expressed as

$$\begin{aligned}&(\underbrace{q_4-xp_4}_{\equiv l})^2+(\underbrace{q_3-xp_3}_{\equiv A})^2+(x+y)\Bigg(\underbrace{q_\perp-\frac{xp_\perp}{x+y}}_{\equiv B}\Bigg)^2\\ &+\underbrace{x\left(1-\frac{x}{x+y}\right)p_\perp^2+x(1-x)(p_3^2+p_4^2)+xm_0^2}_{\equiv C}+(1-x-y)\frac{\theta^2}{n_4^2}\\ &\equiv l^2+A^2+(x+y)B^2+C+(1-x-y)\frac{\theta^2}{n_4^2}.\end{aligned} \tag{62}$$

Taking θ^2 to zero and changing the integration variable from q_4 to the l the trace becomes

$$\begin{aligned}\mathrm{Tr}(\not{p}_E\Sigma_E)_C &= ig_s^2\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E}{n_4}\int_0^1 dx\int_0^{1-x}dy\int_E\frac{d^{2\omega-1}q}{(2\pi)^\omega}\int_{-\infty}^{\infty}dl\\ &\left\{p^\mu\frac{\partial}{\partial p^\mu}\left[\frac{1/(2x)(l+xp_4-aq_3)}{[l^2+A^2+(x+y)B^2+C]^2}\right]+\frac{2p^2(l+xp_4-aq_3)}{[l^2+A^2+(x+y)B^2+C]^3}\right\}_E.\end{aligned} \tag{63}$$

Using Eq. (163) from appendix C.1 to integrate over l one gets

$$\begin{aligned}\mathrm{Tr}(\not{p}_E\Sigma_E)_C &= \frac{i\pi}{4}g_s^2\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E}{n_4}\int_0^1 dx\int_0^{1-x}dy\int_E\frac{d^{2\omega-1}q}{(2\pi)^\omega}\\ &\left\{p^\mu\frac{\partial}{\partial p^\mu}\left[\frac{(xp_4-aq_3)/x}{[A^2+(x+y)B^2+C]^{3/2}}\right]+\frac{3p^2(xp_4-aq_3)}{[A^2+(x+y)B^2+C]^{5/2}}\right\}_E.\end{aligned} \tag{64}$$

Repeating the same steps when integrating with respect to q_3 gives

$$\mathrm{Tr}(\not{p}_E\Sigma_E)_C = i\pi g_s^2\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E}{2n_4}\int_0^1 dx\int_0^{1-x}dy\int_E\frac{d^{2\omega-2}q_\perp}{(2\pi)^\omega}$$
$$\left\{p^\mu\frac{\partial}{\partial p^\mu}\left[\frac{p_4-ap_3}{(x+y)B^2+C}\right]+\frac{2p^2(xp_4-xap_3)}{[(x+y)B^2+C]^2}\right\}_E. \tag{65}$$

Using the basic integral in Eq. (177) from appendix C.1 and expanding in ϵ one gets

$$\mathrm{Tr}(\not{p}_E\Sigma_E)_C =$$
$$i\tilde{\alpha}_s\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E(n^*\cdot p)_E}{(n^*\cdot n)_E}-\frac{ig_s^2C_F}{(4\pi)^2}\frac{16(n\cdot p)_E}{2n_4}\int_0^1 dx\int_0^{1-x}dy$$
$$\left\{p^\mu\frac{\partial}{\partial p^\mu}\left[\frac{p_4-ap_3}{x+y}\log\left(\frac{x}{x+y}\left(\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2\right)\right)\right]\right. \tag{66}$$
$$\left.-\frac{2p^2(p_4-ap_3)}{(1-\frac{x}{x+y})p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2}\right\}_E,$$

where $\tilde{\alpha}_s$ is defined in Eq. (44). By performing the derivative $p^\mu\frac{\partial}{\partial p^\mu}$, and using the relation

$$\frac{2(n\cdot p)(n^*\cdot p)}{n^*\cdot n} = p_0^2-p_3^2, \tag{67}$$

the trace results in

$$\mathrm{Tr}(\not{p}_E\Sigma_E)_C = i\tilde{\alpha}_s\mu^{4-2\omega}C_F\frac{16(n\cdot p)_E(n^*\cdot p)_E}{(n^*\cdot n)_E}-\frac{ig_s^2C_F}{(4\pi)^2}\frac{16(n\cdot p)_E(n^*\cdot p)_E}{(n^*\cdot n)_E}$$
$$\times\int_0^1 dx\int_0^{1-x}dy\frac{1}{(x+y)}\left\{2-\frac{2(p_E^2+m_0^2)}{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2}+\log\left(\frac{x}{x+y}\right)\right.$$
$$\left.+\log\left(\frac{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2}{\mu^2}\right)\right\}. \tag{68}$$

With the results in Eqs. (49), (58), and (68) the trace Eq. (47) is

$$\begin{aligned}
\mathrm{Tr}(\not{p}_E\Sigma_E) &= i\tilde{\alpha}_s C_F 4p_E^2 + \frac{ig_s^2 C_F}{(4\pi)^2}4p_E^2\left(-1+2\int_0^1 dx x\log\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right)\\
&+\frac{ig_s^2C_F}{(4\pi)^2}\frac{16(n\cdot p)_E(n^*\cdot p)_E}{(n^*\cdot n)_E}\left\{\int_0^1 dx\log\left(\frac{(1-x)(p_3^2+p_4^2)+m_0^2}{\mu^2}\right)\right.\\
&\int_0^1 dx\int_0^{1-x}dy\frac{1}{(x+y)}\left[2+\log\left(\frac{x}{x+y}\right)-\frac{2(p_E^2+m_0^2)}{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2}\right.\\
&\left.\left.+\log\left(\frac{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2}{\mu^2}\right)\right]\right\}\\
&\equiv -i\tilde{\alpha}_s C_F 4p_E^2+\frac{ig_s^2C_F}{(4\pi)^2}\left[4p_E^2\left(-1+2\int_0^1 dx x\log\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right)\right.\\
&\left.+\frac{16(n\cdot p)_E(n^*\cdot p)_E}{(n^*\cdot n)_E}\left(\mathcal{I}_1-\mathcal{I}_2\right)_E\right]
\end{aligned} \tag{69}$$

Where $\mathcal{I}_{1E}$ and $\mathcal{I}_{2E}$ expressed in Minkowski space read

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 dx\log\left(\frac{(1-x)(p_3^2-p_0^2)+m_0^2}{\mu^2}\right) = -1+\left(1+\frac{m_0^2}{p_3^2-p_0^2}\right)\log\left(\frac{p_3^2-p_0^2+m_0^2}{\mu^2}\right)\\
&-\frac{m_0^2}{p_3^2-p_0^2}\log\left(\frac{m_0^2}{\mu^2}\right)
\end{aligned} \tag{70}$$

and

$$\begin{aligned}
\mathcal{I}_2 = \int_0^1 dx\int_0^{1-x}\frac{dy}{(x+y)}&\left[2+\frac{2(p^2-m_0^2)}{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2-p_0^2)+m_0^2}\right.\\
&\left.+\log\left(\frac{x}{x+y}\right)+\log\left(\frac{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2-p_0^2)+m_0^2}{\mu^2}\right)\right].
\end{aligned} \tag{71}$$

The value of the integral $\mathcal{I}_2$ is calculated in appendix D resulting

$$\begin{aligned}\mathcal{I}_1 - \mathcal{I}_2 = \frac{p^2 - m_0^2}{p_0^2 - p_3^2}\Bigg[&\log(-p^2)\log\left(\frac{m_0^2 - p^2}{m_0^2}\right) + \log(m_0^2 - p^2)\log\left(\frac{m_0^2}{m_0^2 + p_3^2 - p_0^2}\right) \\ &+ \log(p_\perp^2)\log\left(\frac{m_0^2 + p_3^2 - p_0^2}{m_0^2 - p^2}\right) + \mathrm{Li}_2\left(\frac{p_3^2 - p_0^2}{m_0^2 - p^2}\right) + \mathrm{Li}_2\left(\frac{m_0^2}{-p_\perp^2}\right) \\ &- \mathrm{Li}_2\left(\frac{p_3^2 - p_0^2 + m_0^2}{-p_\perp^2}\right) - \mathrm{Li}_2\left(\frac{m_0^2(p_3^2 - p_0^2)}{p_\perp^2(p^2 - m_0^2)}\right) + \mathrm{Li}_2\left(\frac{p_3^2 - p_0^2}{-p_\perp^2}\right)\Bigg] \\ \equiv\ & \frac{(p^2 - m_0^2)(n^* \cdot n)}{2(n \cdot p)(n^* \cdot p)}\mathcal{I},\end{aligned} \tag{72}$$

where the relation

$$\frac{1}{p_3^2 - p_0^2} = \frac{n^* \cdot n}{2(n \cdot p)(n^* \cdot p)} \tag{73}$$

has been used and $\mathcal{I}$ denotes the terms inside of the square brackets.

4.5 Trace of $\not{p}^*_E \Sigma_E$

The constants C and D in the self energy ansatz (36) depend on the trace

$$\begin{aligned}\mathrm{Tr}(\not{n}^*{}_E\Sigma_E) = -ig_s^2\mu^{4-2\omega}C_F \int_E \frac{d^{2\omega}q}{(2\pi)^\omega}\Bigg[&\frac{8(\omega-1)n^* \cdot q + 8(2-\omega)n^* \cdot p}{((p-q)^2 + m_0^2)q^2} \\ &+ \frac{8n^* \cdot n}{((p-q)^2 + m_0^2)n \cdot q} - \frac{8(n^* \cdot n)(p \cdot q) + 8(n^* \cdot q)(n \cdot p)}{((p-q)^2 + m_0^2)q^2 n \cdot q}\Bigg]_E \\ \equiv\ & \mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_A + \mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_B + \mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_C + \mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D,\end{aligned} \tag{74}$$

where the first three integrals are analogous to $\mathrm{Tr}(\not{p}_E\Sigma_E)_A$, $\mathrm{Tr}(\not{p}_E\Sigma_E)_B$ and $\mathrm{Tr}(\not{p}_E\Sigma_E)_C$ calculated in section 4.4. Utilizing the earlier results one gets

$$\begin{aligned}\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_A = &-i\tilde{\alpha}_s C_F 4(n^* \cdot p)_E \\ &+ \frac{ig_s^2 C_F}{(4\pi)^2}4(n^* \cdot p)_E\left(-1 + 2\int_0^1 dx\, x\log\left(\frac{x(1-x)p_E^2 + xm_0^2}{\mu^2}\right)\right),\end{aligned} \tag{75}$$

$$\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_B = -i\tilde{\alpha}_s C_F 16(n^* \cdot p)_E + \frac{ig_s^2 C_F}{(4\pi)^2}16(n^* \cdot p)_E \mathcal{I}_{1E}, \tag{76}$$

and

$$\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_C = i\tilde{\alpha}_s C_F 8(n^* \cdot p)_E - \frac{ig_s^2 C_F}{(4\pi)^2}8(n^* \cdot p)_E \mathcal{I}_{2E}. \tag{77}$$

Now only $\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D$ needs to be calculated

$$\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D = ig_s^2\mu^{4-2\omega}C_F\int_E \frac{d^{2\omega}q}{(2\pi)^\omega}\left(\frac{8(n^*\cdot q)(n\cdot p)}{((p-q)^2+m_0^2)q^2 n\cdot q}\right)_E . \tag{78}$$

Applying the ML-prescription and Feynman parametrization gives

$$\begin{aligned}\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D &= ig_s^2\mu^{4-2\omega}C_F 8(n\cdot p)_E\int_0^1 dx\int_0^{1-x}dy\\ &\int_E \frac{d^{2\omega}q}{(2\pi)^\omega}\left(\frac{2(q_4-aq_3)^2}{\left[x((p-q)^2+m_0^2)+yq^2+(1-x-y)(q_4^2+q_3^2+\frac{\theta^2}{n_4^2})\right]^3}\right)_E ,\end{aligned} \tag{79}$$

where the denominator can be expressed as

$$\begin{aligned}&(\underbrace{q_4-xp_4}_{\equiv l})^2+(\underbrace{q_3-xp_3}_{A})^2+(x+y)\Bigg(\underbrace{q_\perp-\frac{xp_\perp}{x+y}}_{\equiv B}\Bigg)^2\\ &\underbrace{+x\left(1-\frac{x}{x+y}\right)p_\perp^2+x(1-x)(p_3^2+p_4^2)+xm_0^2}_{\equiv C}+(1-x-y)\frac{\theta^2}{n_4^2}\\ &\equiv l^2+A^2+(x+y)B^2+C+(1-x-y)\frac{\theta^2}{n_4^2}.\end{aligned} \tag{80}$$

Changing the integration variable from q_4 to l gives

$$\begin{aligned}&\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D =\\ &ig_s^2\mu^{4-2\omega}C_F 8(n\cdot p)_E\int_0^1 dx\int_0^{1-x}dy\int_E\frac{d^{2\omega}q}{(2\pi)^\omega}\left(\frac{2(l+xp_4-aq_3)^2}{\left[l^2+A^2+(x+y)B^2+C\right]^3}\right)_E\\ &=ig_s^2\mu^{4-2\omega}C_F 32(n\cdot p)_E\int_0^1 dx\int_0^{1-x}dy\int_E\frac{d^{2\omega-1}q}{(2\pi)^\omega}\\ &\int_0^\infty dl\left(\frac{l^2+(xp_4-aq_3)^2}{\left[l^2+A^2+(x+y)B^2+C\right]^3}\right)_E .\end{aligned} \tag{81}$$

Utilising the integral in Eq. (163) in appendix C.1 gives

$$\begin{aligned}\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D &= ig_s^2\mu^{4-2\omega}C_F 8(n\cdot p)_E \int_0^1 dx \int_0^{1-x} dy \\ &\int_E \frac{d^{2\omega-1}q}{(2\pi)^\omega}\frac{\pi}{4}\left(\frac{1}{\left[A^2+(x+y)B^2+C\right]^{3/2}} + 3\frac{(xp_4-aq_3)^2}{\left[A^2+(x+y)B^2+C\right]^{5/2}}\right)_E .\end{aligned} \tag{82}$$

Following the same steps when integrating with respect to q_3 one gets

$$\begin{aligned}\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D &= ig_s^2\mu^{4-2\omega}C_F 8(n\cdot p)_E \int_0^1 dx \int_0^{1-x} dy \\ &\int_E \frac{d^{2\omega-2}q}{(2\pi)^\omega}\pi\left(\frac{x^2(p_4-ap_3)^2}{\left[(x+y)B^2+C\right]^2}\right)_E .\end{aligned} \tag{83}$$

Using the basic integral in Eq. (177) from section C.3 and expanding in powers of ϵ, the trace becomes

$$\begin{aligned}\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D &= \frac{ig_s^2C_F}{(4\pi)^2}8(n\cdot p)_E(p_4-ap_3)^2\int_0^1 dx \\ &\int_0^{1-x} dy\left(\frac{x}{x+y}\frac{1}{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2}\right)_E .\end{aligned} \tag{84}$$

The relations Eq. (57) and Eq. (67) give

$$\begin{aligned}&\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)_D = \\ &\frac{ig_s^2C_F}{(4\pi)^2}8(n^*\cdot p)_E\int_0^1 dx\int_0^{1-x} dy\left(\frac{x}{x+y}\frac{p_4^2+p_3^2}{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2+p_4^2)+m_0^2}\right)_E \\ &\equiv \frac{ig_s^2C_F}{(4\pi)^2}8(n^*\cdot p)_E\mathcal{I}_{3E},\end{aligned} \tag{85}$$

where $\mathcal{I}_3$, expressed in Minkowski space, reads

$$\mathcal{I}_3 \equiv \int_0^1 dx\int_0^{1-x} dy\left(\frac{x}{x+y}\frac{p_3^2-p_0^2}{\left(1-\frac{x}{x+y}\right)p_\perp^2+(1-x)(p_3^2-p_0^2)+m_0^2}\right). \tag{86}$$

By combining the results in Eqs. (75), (76), (77), and (85), the final result for $\mathrm{Tr}(\not{n}^*{}_E\Sigma_E)$ is

$$\begin{aligned}\mathrm{Tr}(\not{n}^*{}_E\Sigma_E) &= -i\tilde{\alpha}_s C_F 12(n^*\cdot p)_E \\ &+ \frac{ig_s^2 C_F}{(4\pi)^2} 8(n^*\cdot p)_E\left(\int_0^1 dx x\log\left(\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right)+2\mathcal{I}_{1E}-\mathcal{I}_{2E}+\mathcal{I}_{3E}\right).\end{aligned} \tag{87}$$

4.6 Self energy result

Placing the trace results from sections 4.2, 4.3, 4.4, and 4.5 into the ansatz of the self energy, Eq. (41), gives the result for the self energy in Euclidean space

$$\begin{aligned}\Sigma_E &= -i\tilde{\alpha}_s C_F\left[2m_0+\not{p}_E+\left(\frac{\not{n}\not{n}^*\not{p}+\not{p}\not{n}^*\not{n}}{n^*\cdot n}\right)_E\right] \\ &+\frac{ig_s^2 C_F}{(4\pi)^2}\left[2m_0\left(1+\int_0^1 dx\log\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right)\right. \\ &+\left(1+2\int_0^1 dx(1-x)\log\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right)\not{p}_E+\left(2\int_0^1 dx\log\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right. \\ &+\left.\frac{2(n^*\cdot p)_E(n\cdot p)_E\left(\mathcal{I}_{2E}+\mathcal{I}_{3E}-\int_0^1 dx\log\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right)}{2(n^*\cdot p)_E(n\cdot p)_E-p_E^2(n^*\cdot n)_E}\right)\left(\frac{\not{n}\not{n}^*\not{p}+\not{p}\not{n}^*\not{n}}{2(n^*\cdot n)}\right)_E \\ &+\left.\left(2(\mathcal{I}_{2E}-\mathcal{I}_{1E})+\frac{p_E^2(n^*\cdot n)_E\left(\mathcal{I}_{2E}+\mathcal{I}_{3E}-\int_0^1 dx\log\frac{x(1-x)p_E^2+xm_0^2}{\mu^2}\right)}{2(n^*\cdot p)_E(n\cdot p)_E-p_E^2(n^*\cdot n)_E}\right)\frac{2(n^*\cdot p)_E}{(n^*\cdot n)_E}\not{n}_E\right],\end{aligned} \tag{88}$$

where $\tilde{\alpha}_s$ is defined in Eq. (44). Using the identities from appendix B one can transform back to Minkowski space

$$\Sigma = -i\tilde{\alpha}_s C_F\left[2m_0 - \not{p} + \frac{\not{n}\not{n}^*\not{p} + \not{p}\not{n}^*\not{n}}{n^*\cdot n}\right] + \frac{ig_s^2 C_F}{(4\pi)^2}\Bigg[2m_0\left(1 + \int_0^1 dx \log\frac{x(x-1)p^2 + xm_0^2}{\mu^2}\right)$$
$$-\left(1 + 2\int_0^1 dx(1-x)\log\frac{x(x-1)p^2 + xm_0^2}{\mu^2}\right)\not{p} + \Bigg(2\int_0^1 dx\log\frac{x(x-1)p^2 + xm_0^2}{\mu^2}$$
$$+\frac{2(n^*\cdot p)(n\cdot p)\left(\mathcal{I}_2 + \mathcal{I}_3 - \int_0^1 dx\log\frac{x(x-1)p^2+xm_0^2}{\mu^2}\right)}{2(n^*\cdot p)(n\cdot p) - p^2(n^*\cdot n)}\Bigg)\left(\frac{\not{n}\not{n}^*\not{p} + \not{p}\not{n}^*\not{n}}{2(n^*\cdot n)}\right)$$
$$-\left(2\mathcal{I}_2 - 2\mathcal{I}_1 + \frac{p^2(n^*\cdot n)\left(\mathcal{I}_2 + \mathcal{I}_3 - \int_0^1 dx\log\frac{x(x-1)p^2+xm_0^2}{\mu^2}\right)}{2(n^*\cdot p)(n\cdot p) - p^2(n^*\cdot n)}\right)\frac{2(n^*\cdot p)}{(n^*\cdot n)}\not{n}\Bigg]. \tag{89}$$

Since

$$\not{n}\not{n}^*\not{p} + \not{p}\not{n}^*\not{n} = 2(n^*\cdot p)\not{n} - 2(n\cdot p)\not{n}^* + 2(n^*\cdot n)\not{p}, \tag{90}$$

the self energy becomes

$$\begin{aligned}\Sigma = -i\tilde{\alpha}_s C_F\left[2m_0 + \not{p} + 2\frac{(n^*\cdot p)\not{n} - (n\cdot p)\not{n}^*}{n^*\cdot n}\right] + \frac{ig_s^2 C_F}{(4\pi)^2}\Bigg[2\left(1 + \int_0^1 dx\log\frac{\Delta}{\mu^2}\right)m_0 \\ +\left(-1 + 2\int_0^1 dx x\log\frac{\Delta}{\mu^2} + 2(n^*\cdot p)(n\cdot p)N\right)\not{p} \\ -\left(2\int_0^1 dx\log\frac{\Delta}{\mu^2} + 2(n^*\cdot p)(n\cdot p)N\right)\frac{(n\cdot p)\not{n}^*}{n^*\cdot n} \\ +\left(2\int_0^1 dx\log\frac{\Delta}{\mu^2} + 4\mathcal{I}_1 - 4\mathcal{I}_2 + 2\left((n^*\cdot p)(n\cdot p) - p^2(n^*\cdot n)\right)N\right)\frac{(n^*\cdot p)\not{n}}{n^*\cdot n}\Bigg],\end{aligned} \tag{91}$$

where

$$\tilde{\alpha}_s \equiv \frac{g_s^2\Gamma(2-\omega)}{(4\pi)^\omega} = \frac{g_s^2}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \mathcal{O}(\epsilon)\right), \tag{92}$$

$$\Delta \equiv x(x-1)p^2 + xm_0^2, \tag{93}$$

and

$$N \equiv \frac{\mathcal{I}_2 + \mathcal{I}_3 - \int_0^1 dx\log\frac{\Delta}{\mu^2}}{2(n^*\cdot p)(n\cdot p) - p^2(n^*\cdot n)}. \tag{94}$$

The integrals $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ are defined as

$$\mathcal{I}_1 = \int_0^1 dx \log\left(\frac{(1-x)(p_3^2 - p_0^2) + m_0^2}{\mu^2}\right),$$
$$\mathcal{I}_2 = \int_0^1 dx \int_0^{1-x} \frac{dy}{(x+y)}\left[2 + \frac{2(p^2 - m_0^2)}{\left(1 - \frac{x}{x+y}\right)p_\perp^2 + (1-x)(p_3^2 - p_0^2) + m_0^2}\right.$$
$$\left. + \log\left(\frac{x}{x+y}\right) + \log\left(\frac{\left(1 - \frac{x}{x+y}\right)p_\perp^2 + (1-x)(p_3^2 - p_0^2) + m_0^2}{\mu^2}\right)\right], \text{ and} \tag{95}$$
$$\mathcal{I}_3 = \int_0^1 dx \int_0^{1-x} dy \left(\frac{x}{x+y}\frac{p_3^2 - p_0^2}{\left(1 - \frac{x}{x+y}\right)p_\perp^2 + (1-x)(p_3^2 - p_0^2) + m_0^2}\right).$$

Using Eq. (67) shows that the denominator of Eq. (94) is $2n_0^2 p_\perp^2$, which suggests that N has a pole when $p_\perp^2 = 0$. However, it is shown in appendix D that also the numerator gives zero when $p_\perp^2 = 0$, thus Eq. (94) is finite in this limit. In appendix D it is also shown that the term $\mathcal{I}_1 - \mathcal{I}_2$ does not include poles.

The result for the one-loop corrected quark self energy, Eq. (91), contains UV-divergences in the form $\tilde{\alpha}_s$. The self energy is used to obtain results for physical quantities, such as the quark mass. This requires that the UV-divergences from the self energy have to be eliminated by the renormalization. Contrary to the free quark propagator, which has gamma-matrices only in the form $\not{p}$, the self energy in Eq. (91) includes also the matrices $\not{n}$ and $\not{n}^*$. This predicts that the renormalization in the light-cone gauge will be more complicated than in a covariant gauge, where the quark self energy has the same gamma-matrix structure as the free propagator. The renormalization of the quark mass and field are covered in the next chapter.

5 Renormalization

5.1 Pole mass of the quark in a light-cone gauge

The effective quark propagator is a sum of the unperturbed propagator and all the possible corrections. In this thesis only the one-loop correction is considered and the effective propagator is diagrammatically a geometric series of insertions of the one-loop self energy, as in Figure 4. The pole mass is the physical, measurable mass of the quark. Thus it is found out by finding the pole of the effective propagator.

The effective propagator, as in Figure 4, reads

$$\begin{aligned} iS_{\text{eff}} &= \frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0}\Sigma\frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0}\Sigma\frac{i}{\not{p} - m_0}\Sigma\frac{i}{\not{p} - m_0} + \ldots \\ &= \frac{i}{\not{p} - m_0}\left[1 + \Sigma\frac{i}{\not{p} - m_0} + \left(\Sigma\frac{i}{\not{p} - m_0}\right)^2 + \ldots\right], \end{aligned} \tag{96}$$

where the self energy Σ is defined in Eq. (91). Since Σ has a gamma-matrix structure, other than $\not{p}$, the multiplication order has to be unchanged. Summing as a geometric series Eq. (96) results in

$$iS_{\text{eff}} = \frac{i}{\not{p} - m_0}\frac{1}{1 - i\Sigma\frac{1}{\not{p} - m_0}} = i(\not{p} + m_0)\frac{1}{p^2 - m_0^2 - i\Sigma(\not{p} + m_0)}, \tag{97}$$

where one has used the relation

$$\frac{1}{\not{p} - m_0} = \frac{\not{p} + m_0}{p^2 - m_0^2}. \tag{98}$$

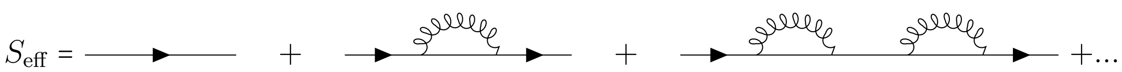

Figure 4. An effective quark propagator.

The denominator of Eq. (97) contains the term

$$
\begin{aligned}
&-i\Sigma(\not{p}+m_0) = \\
&-\left[\tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2}\left(-1+2\int_0^1 dx x \log\frac{\Delta}{\mu^2} + 2(n^*\cdot p)(n\cdot p)N\right)\right]p^2 \\
&-\left[2\tilde{\alpha}_s C_F - 2\frac{g_s^2 C_F}{(4\pi)^2}\left(1+\int_0^1 dx \log\frac{\Delta}{\mu^2}\right)\right]m_0^2 \\
&-\left[3\tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2}\left(1+2\int_0^1 dx (1+x) \log\frac{\Delta}{\mu^2} + 2(n^*\cdot p)(n\cdot p)N\right)\right]m_0\not{p} \\
&-\left[\tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2}\left(\int_0^1 dx \log\frac{\Delta}{\mu^2} + 2\mathcal{I}_1 - 2\mathcal{I}_2 + \big((n^*\cdot p)(n\cdot p) - p^2(n^*\cdot n)\big)N\right)\right] \\
&\times\frac{2(n^*\cdot p)\not{n}}{n^*\cdot n}(\not{p}+m_0) \\
&-\left[-\tilde{\alpha}_s C_F + \frac{g_s^2 C_F}{(4\pi)^2}\left(\int_0^1 dx \log\frac{\Delta}{\mu^2} + (n^*\cdot p)(n\cdot p)N\right)\right]\frac{2(n\cdot p)\not{n}^*}{n^*\cdot n}(\not{p}+m_0) \\
&\equiv -\mathcal{A}p^2 - \mathcal{B}m_0^2 - \mathcal{C}m_0\not{p} - \mathcal{D}\not{n}(\not{p}+m_0) - \mathcal{E}\not{n}^*(\not{p}+m_0).
\end{aligned}
\tag{99}
$$

In terms of the coefficients $\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$, and $\mathcal{E}$, the effective propagator in Eq. (97) reads

$$
iS_{\text{eff}} = \frac{i(\not{p}+m_0)}{\big(p^2(1-\mathcal{A}) - m_0^2(1+\mathcal{B})\big) - \mathcal{C}m_0\not{p} - \mathcal{D}\not{n}(\not{p}+m_0) - \mathcal{E}\not{n}^*(\not{p}+m_0)}, \tag{100}
$$

where the term $(\not{p}+m_0)$ in the numerator is understood to be multiplied from the left side. To determine the pole mass one must find the poles of this expression. When one multiplies the numerator and the denominator of the effective propagator in Eq. (100) from the right side with the term

$$
-\big(p^2(1-\mathcal{A}) - m_0^2(1+\mathcal{B})\big) - \mathcal{C}m_0\not{p} + (\not{p}-m_0)\mathcal{D}\not{n} + (\not{p}-m_0)\mathcal{E}\not{n}^*, \tag{101}
$$

the propagator becomes

$$
\begin{aligned}
&iS_{\text{eff}} = \\
&\frac{i(\not{p}+m_0)\left[-\big(p^2(1-\mathcal{A}) - m_0^2(1+\mathcal{B})\big) - \mathcal{C}m_0\not{p} + (\not{p}-m_0)\mathcal{D}\not{n} + (\not{p}-m_0)\mathcal{E}\not{n}^*\right]}{-\big(p^2(1-\mathcal{A}) - m_0^2(1+\mathcal{B})\big)^2 + \mathcal{C}^2 m_0^2 p^2 + 2\big((n\cdot p)\mathcal{D} + (n^*\cdot p)\mathcal{E}\big)\big(p^2(1-\mathcal{A}) - m_0^2(1+\mathcal{B}) - m_0^2\mathcal{C}\big) - (p^2-m_0^2)\mathcal{D}\mathcal{E}}.
\end{aligned}
\tag{102}
$$

The denominator of the propagator does not include gamma-matrix structures of any order in g_s^2. Thus, the propagator is now in a form from which it is possible to identify the physical mass of the quark. Since $\mathcal{A} + \mathcal{B} = \mathcal{C}$ the denominator is

$$(p^2 - m_0^2)\left[(p^2 - m_0^2)(-1 + 2\mathcal{A}) + m_0^2 2\mathcal{C} + 2\big((n \cdot p)\mathcal{D} + (n^* \cdot p)\mathcal{E}\big) + \mathcal{O}(g_s^4)\right]. \tag{103}$$

Opening the expression $(n \cdot p)\mathcal{D} + (n^* \cdot p)\mathcal{E}$ and using the identity (72) for $\mathcal{I}_1 - \mathcal{I}_2$ gives

$$\begin{aligned}
&(p^2 - m_0^2)\Bigg[(p^2 - m_0^2)(-1 + 2\mathcal{A}) + m_0^2 2\mathcal{C} \\
&+ \frac{g_s^2 C_F}{(4\pi)^2}\left(p^2 4(n^* \cdot p)(n \cdot p)N - (p^2 - m_0^2)4\mathcal{I}\right) + \mathcal{O}(g_s^4)\Bigg] \\
&= (p^2 - m_0^2)\left(-1 + 2\mathcal{A} + \frac{g_s^2 C_F}{(4\pi)^2}\left((n^* \cdot p)(n \cdot p)N - 4\mathcal{I}\right)\right) \\
&\times \left[p^2 - m_0^2\left(1 + 2\mathcal{C} + \frac{g_s^2 C_F}{(4\pi)^2}(n^* \cdot p)(n \cdot p)N\right) + \mathcal{O}(g_s^4)\right] \\
&= (p^2 - m_0^2)\left(-1 + 2\mathcal{A} + \frac{g_s^2 C_F}{(4\pi)^2}\left((n^* \cdot p)(n \cdot p)N - 4\mathcal{I}\right)\right) \\
&\times \left[p^2 - m_0^2\left(1 + \tilde{\alpha}_s C_F 6 - \frac{g_s^2 C_F}{(4\pi)^2}2\left(1 + 2\int_0^1 dx(1 + x)\log\frac{\Delta}{\mu^2}\right)\right) + \mathcal{O}(g_s^4)\right].
\end{aligned} \tag{104}$$

The numerator of Eq. (102) is

$$\begin{aligned}
&i(p^2 - m_0^2)\left[\not{p}\,(-1 + \mathcal{A}) - m_0\,(1 + \mathcal{B}) + \mathcal{D}\not{n} + \mathcal{E}\not{n}^*\right] \\
&= i(p^2 - m_0^2)\,(-1 + \mathcal{A})\left[\not{p} + m_0\,(1 + \mathcal{C}) - \mathcal{D}\not{n} - \mathcal{E}\not{n}^* + \mathcal{O}(g_s^4)\right].
\end{aligned} \tag{105}$$

Using Eqs. (104) and (105) the effective propagator (102) becomes

$$\begin{aligned}
iS_{\text{eff}} &= \frac{i\,(-1 + \mathcal{A})\left[\not{p} + m_0\,(1 + \mathcal{C}) - \mathcal{D}\not{n} - \mathcal{E}\not{n}^*\right]\left[-1 + 2\mathcal{A} + \frac{g_s^2 C_F}{(4\pi)^2}\left(4(n^* \cdot p)(n \cdot p)N - 4\mathcal{I}\right)\right]^{-1}}{p^2 - m_0^2\left(1 + \tilde{\alpha}_s C_F 6 - \frac{g_s^2 C_F}{(4\pi)^2}2\left(1 + 2\int_0^1 dx(1 + x)\log\frac{\Delta}{\mu^2}\right)\right) + \mathcal{O}(g_s^4)} \\
&= \frac{i\left[\not{p} + m_0\,(1 + \mathcal{C}) - \mathcal{D}\not{n} - \mathcal{E}\not{n}^*\right]\left[1 + \mathcal{A} + \frac{g_s^2 C_F}{(4\pi)^2}\left(4(n^* \cdot p)(n \cdot p)N - 4\mathcal{I}\right) + \mathcal{O}(g_s^4)\right]}{p^2 - m_0^2\left(1 + \tilde{\alpha}_s C_F 6 - \frac{g_s^2 C_F}{(4\pi)^2}2\left(1 + 2\int_0^1 dx(1 + x)\log\frac{\Delta}{\mu^2}\right)\right) + \mathcal{O}(g_s^4)} \\
&= \frac{i\left[\not{p} + m_0\,(1 + \mathcal{C}) - \mathcal{D}\not{n} - \mathcal{E}\not{n}^*\right]\left[1 + \tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2}\left(2\int_0^1 dx x\log\frac{\Delta}{\mu^2} - 1 + 4\mathcal{I} - 2(n^* \cdot p)(n \cdot p)N\right) + \mathcal{O}(g_s^4)\right]}{p^2 - m_0^2\left(1 + \tilde{\alpha}_s C_F 6 - \frac{g_s^2 C_F}{(4\pi)^2}2\left(1 + 2\int_0^1 dx(1 + x)\log\frac{\Delta}{\mu^2}\right)\right) + \mathcal{O}(g_s^4)}.
\end{aligned} \tag{106}$$

The effective propagator now has a single pole which is the squared pole mass of the quark

$$m_{\text{pole}}^2 \equiv m_0^2 \left(1 + 6\tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2} 2 \left(1 + 2 \int_0^1 dx (1+x) \log \frac{\Delta}{\mu^2} \right) \right). \tag{107}$$

Near the pole one can approximate $\log \frac{\Delta}{\mu^2} = \log \frac{x^2 m_0^2}{\mu^2}$, and the pole mass is found to be

$$m_{\text{pole}}^2 = m_0^2 \left(1 + 6\tilde{\alpha}_s C_F - 2 \frac{g_s^2 C_F}{(4\pi)^2} \left(3 \log \frac{m_0^2}{\mu^2} - 4 \right) \right). \tag{108}$$

The pole mass now contains a divergence when $\omega \to 2$, which is located in $\tilde{\alpha}_s$ defined in Eq. (92). It can be seen from Eq. (108) that the divergence is eliminated at order g_s^2 in the $\overline{\text{MS}}$-scheme if the bare mass term is set to be

$$m_0 = m - m3\tilde{\alpha}_s C_F, \tag{109}$$

where m is the renormalised mass in the $\overline{\text{MS}}$-scheme.

The effective propagator, Eq. (106), clearly does not satisfy the general propagator form since the numerator contains $\not{n}$ and $\not{n}^*$ matrices with finite and divergent factors. The terms $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ depend on the vectors n and n^*. The next step would be to attempt to absorb the problematic terms in the quark field and quark mass renormalization terms. However, the terms $n \cdot p$ and $n^* \cdot p$ cannot be included to the renormalization counterterms since they cannot depend on momenta. Since the Lorentz noninvariant inner products $n \cdot p$ and $n^* \cdot p$ have different dependence of the components of the four-vector p, one cannot take the mass-shell limit without fixing specific values for the components of the momentum p. The divergent parts are fortunately free from the Lorentz noninvariant terms, which makes it possible to eliminate divergences by the renormalization.

5.2 Renormalization of the quark wave function

The mass renormalization is already possible to do from Eq. (106) by redefining the bare mass term as in Eq. (109). However, in this section the mass and field renormalization are carried out proceeding as in Ref. [8].

The effective quark propagator in Eq. (106) can be expressed as the sum of the bare propagator and the one-loop correction terms as

$$
\begin{aligned}
iS_{\text{eff}} = iS_0 + iS_0 \Bigg[& \tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2} \left(2 \int_0^1 dx x \log \frac{\Delta}{\mu^2} - 1 + 4\mathcal{I} - 2(n^* \cdot p)(n \cdot p) N \right) \\
& + \frac{m_0^2}{p^2 - m_0^2} \left(6\tilde{\alpha}_s C_F - 4 \frac{g_s^2 C_F}{(4\pi)^2} \int_0^1 dx (1+x) \log \frac{\Delta}{\mu^2} \right) \Bigg] \\
& + i \frac{m_0 \mathcal{C}}{p^2 - m_0^2} - i \frac{\mathcal{D} \not{n} + \mathcal{E} \not{n}^*}{p^2 - m_0^2} + \mathcal{O}(g_s^4),
\end{aligned}
\tag{110}
$$

where $S_0 = (\not{p} - m_0)^{-1}$ is the bare propagator. This form could also be derived from Eq. (96). The renormalised effective propagator is obtained by adding a counterterm to the quark propagator in Eq. (110), as in Figure 5.

The renormalization counterterms are found by defining quark mass and wave function, with renormalised quantities m and ψ, as

$$
m_0 = m - \delta m \quad \psi_0 = \psi + M\psi, \tag{111}
$$

where δm is a number and M is a matrix, and both are proportional to g_s^2. It is shown in Ref. [8] that M contains gamma-matrices which is a light-cone gauge property. Therefore the renormalization cannot be done with the general conventions as in a covariant gauge where M would be a number.

When the redefined quantities in Eq. (111) are placed in the Lagrangian the Dirac part is written as

$$
\begin{aligned}
\overline{\psi}_0 (i\not{\partial} - m_0) \psi_0 &= \overline{(\psi + M\psi)} \left(i\not{\partial} - m + \delta m \right) (1 + M)\psi \\
&= \overline{\psi} (1 + \gamma^0 M^\dagger \gamma^0) \left(i\not{\partial} - m + \delta m \right) (1 + M)\psi,
\end{aligned}
\tag{112}
$$

which is a sum of the renormalised Dirac Lagrangian and the counterterm. Deriving

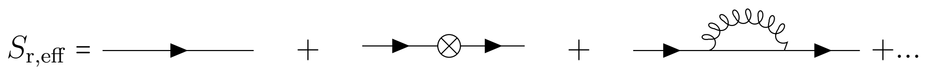

Figure 5. A renormalised effective quark propagator including counterterms.

the propagator form the right side of Eq. (112) gives

$$\begin{aligned}&\frac{i}{(1+\gamma^0 M^\dagger\gamma^0)\left(\not{p}-m+\delta m\right)(1+M)}=\frac{iS}{1+\delta m S+\gamma^0 M^\dagger\gamma^0+\left(\not{p}-m\right)MS+\mathcal{O}(g_s^4)}\\&=iS-iS^2\delta m-iS\gamma^0M^\dagger\gamma^0-iMS+\mathcal{O}(g_s^4),\end{aligned}\tag{113}$$

where $S=(\not{p}-m)^{-1}$. This can be identified as a sum of the renormalised free propagator and the counterterms, or as the renormalised effective propagator from where the one-loop corrections have been subtracted.

The one-loop correction part in Eq. (110) is renormalised by expressing it in terms of the renormalised mass, Eq. (111), and renormalised bare propagator $S=(\not{p}-m)^{-1}+\mathcal{O}(g_s^2)$. Then replacing the free propagator in Eq. (110) with the sum of the renormalised free propagator and the counterterms in Eq.(113) gives the result for the renormalised effective propagator

$$\begin{aligned}S_{\text{r,eff}}&=S-S^2\delta m-S\gamma^0M^\dagger\gamma^0-MS\\&+S\Bigg[\tilde{\alpha}_sC_F-\frac{g_s^2C_F}{(4\pi)^2}\left(2\int_0^1dx x\log\frac{\Delta}{\mu^2}-1+4\mathcal{I}-2(n^*\cdot p)(n\cdot p)N\right)\\&+\frac{m^2}{p^2-m^2}\left(6\tilde{\alpha}_sC_F-4\frac{g_s^2C_F}{(4\pi)^2}\int_0^1dx(1+x)\log\frac{\Delta}{\mu^2}\right)\Bigg]\\&+\frac{m\mathcal{C}}{p^2-m^2}-\frac{\mathcal{D}\not{n}+\mathcal{E}\not{n}^*}{p^2-m^2}+\mathcal{O}(g_s^4).\end{aligned}\tag{114}$$

Requiring that the divergent parts are cancelled by M and δm, the term

$$\begin{aligned}&-S^2\delta m-S\gamma^0M^\dagger\gamma^0-MS+S\left[\tilde{\alpha}_sC_F+\frac{m^2}{p^2-m^2}6\tilde{\alpha}_sC_F\right]+\frac{m}{p^2-m^2}3\tilde{\alpha}_sC_F\\&-\frac{\tilde{\alpha}_sC_F}{p^2-m^2}\frac{2(n^*\cdot p)\not{n}-2(n\cdot p)\not{n}^*}{(n^*\cdot n)}\end{aligned}\tag{115}$$

should be finite. After noting that $S2m(p^2-m^2)^{-1}+(p^2-m^2)^{-1}=S^2$, the coefficients of S^2 in Eq. (115) can be matched, which gives

$$\delta m=3\tilde{\alpha}_sC_Fm=\frac{g_s^2C_F}{(4\pi)^2}m\left(\frac{1}{\epsilon}-\gamma_E+\log(4\pi)\right)\tag{116}$$

in the $\overline{\text{MS}}$ scheme, as predicted from the pole of the effective propagator in Eq. (109).

Cancelling these within Eq. (115) and matching the coefficients of m gives

$$\gamma^0 M^\dagger \gamma^0 + M = \tilde{\alpha}_s C_F. \tag{117}$$

With the relation

$$\frac{2(n^*\cdot p)\not{n} - 2(n\cdot p)\not{n}^*}{(n^*\cdot n)} = \frac{\not{p}\not{n}^*\not{n} - \not{n}^*\not{n}\not{p}}{(n^*\cdot n)} \tag{118}$$

the coefficients of $\not{p}$ in Eq. (115) can be matched by taking

$$\not{p}\gamma^0 M^\dagger \gamma^0 + M\not{p} = \not{p}\tilde{\alpha}_s C_F - \tilde{\alpha}_s C_F \frac{\not{p}\not{n}^*\not{n} - \not{n}^*\not{n}\not{p}}{2n^*\cdot n}. \tag{119}$$

Using Eq. (117) in Eq. (119) gives

$$-\not{p}M + M\not{p} = -\tilde{\alpha}_s C_F \frac{\not{p}\not{n}^*\not{n} - \not{n}^*\not{n}\not{p}}{2n^*\cdot n}, \tag{120}$$

where it can be seen that M must be constructed as

$$M = \tilde{\alpha}_s C_F \left(\frac{\not{n}^*\not{n}}{n^*\cdot n} + x \right), \tag{121}$$

where x is a number, and the Hermitian conjugate multiplied with γ^0s is

$$\gamma^0 M^\dagger \gamma^0 = \tilde{\alpha}_s C_F \left(\frac{\not{n}\not{n}^*}{n^*\cdot n} + x \right) = \tilde{\alpha}_s C_F \left(-\frac{\not{n}^*\not{n}}{n^*\cdot n} + 2 + x \right). \tag{122}$$

Placing Eq. (122) in Eq. (117) one gets

$$\gamma^0 M^\dagger \gamma^0 + M = \tilde{\alpha}_s C_F \left(2 + 2x\right) = \tilde{\alpha}_s C_F, \tag{123}$$

which gives

$$x = \frac{-1}{2}. \tag{124}$$

Finally the wave function renormalization term is

$$M = \tilde{\alpha}_s C_F \left(\frac{\not{n}^*\not{n}}{n^*\cdot n} - \frac{1}{2} \right) = \frac{g_s^2 C_F}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) \right) \left(\frac{\not{n}^*\not{n}}{n^*\cdot n} - \frac{1}{2} \right) \tag{125}$$

Inserting Eqs. (116) and (125) in Eq. (114) the renormalised effective quark propagator reads

$$iS_{\mathrm{r,eff}} = \frac{i\left[\not{p} + m\left(1 + \mathcal{C}_{\mathrm{fin}}\right) - \mathcal{D}_{\mathrm{fin}}\not{n} - \mathcal{E}_{\mathrm{fin}}\not{n}^*\right]\left[1 - \frac{g_s^2 C_F}{(4\pi)^2}\left(2\int_0^1 dx x \log\frac{\Delta}{\mu^2} - 1 + 4\mathcal{I} - 2(n^*\cdot p)(n\cdot p)N\right) + \mathcal{O}(g_s^4)\right]}{p^2 - m^2\left(1 - \frac{g_s^2 C_F}{(4\pi)^2} 4\int_0^1 dx(1+x)\log\frac{\Delta}{\mu^2}\right) + \mathcal{O}(g_s^4)}, \quad (126)$$

where the subindex "fin" denotes the finite part of the constants defined as

$$\begin{aligned}
\mathcal{C}_{\mathrm{fin}} &\equiv \frac{-g_s^2 C_F}{(4\pi)^2}\left(1 + 2\int_0^1 dx(1+x)\log\frac{\Delta}{\mu^2} + 2(n^*\cdot p)(n\cdot p)N\right), \\
\mathcal{D}_{\mathrm{fin}} &\equiv \frac{-g_s^2 C_F}{(4\pi)^2}\left(\int_0^1 dx \log\frac{\Delta}{\mu^2} + 2\mathcal{I}_1 - 2\mathcal{I}_2 + \left((n^*\cdot p)(n\cdot p) - p^2(n^*\cdot n)\right)N\right)\frac{2(n^*\cdot p)}{n^*\cdot n}, \\
\mathcal{E}_{\mathrm{fin}} &\equiv \frac{g_s^2 C_F}{(4\pi)^2}\left(\int_0^1 dx \log\frac{\Delta}{\mu^2} + (n^*\cdot p)(n\cdot p)N\right)\Bigg]\frac{2(n\cdot p)}{n^*\cdot n},
\end{aligned} \quad (127)$$

where the parameters Δ, N, and $\mathcal{I}_1 - \mathcal{I}_2$ are defined in section 4.6, and the parameter $\mathcal{I}$ is denined in Eq. (72). The effective quark propagator is now free of the divergent parts but there are still the $n^* \cdot p$ and $n \cdot p$ dependent terms in $\mathcal{C}_{\mathrm{fin}}$,$\mathcal{D}_{\mathrm{fin}}$, and $\mathcal{E}_{\mathrm{fin}}$. The mass-shell limit is not unique but depends on how the vector p is constructed, since the terms $n^* \cdot p$ and $n \cdot p$ have different dependency on the different components of the momentum p. It seems that the renormalization of the quark propagator involving all the finite terms is not sufficient by itself to determine gauge invariant physical observables. This requires considering the one-loop correction to the quark self-energy with vertex corrections as in Figure 1.

The parameters δm and M, defined in Eqs. (116) and (125) respectively, agree with the results in Refs. [8, 9]. The gamma-matrix structure of M causes the renormalization to be different for the different spinor components of the quark field ψ, as stated in Ref. [8].

The relation of the counterterm and the so-called renormalization constant Z in the Lehmann–Symanzik–Zimmermann (LSZ) reduction [10] is not as obvious as in the covariant gauge since the counterterm includes matrices. It is shown in Ref. [11] that the quark mass and field renormalization can be expressed as

$$m_0 = m - \delta m \quad \text{and} \quad \psi_0 = \sqrt{Z_2\tilde{Z}_2}\left(1 - (1 - \tilde{Z}_2{}^{-1})\frac{\not{n}^*\not{n}}{2(n^*\cdot n)}\right)\psi. \quad (128)$$

Using these the bare Dirac Lagrangian becomes

$$\overline{\psi}_0(i\not{\partial} - m_0)\psi_0 = \\ Z_2\tilde{Z}_2\overline{\psi}\left(1 - (1 - \tilde{Z}_2^{-1})\frac{\not{n}\not{n}^*}{2(n^*\cdot n)}\right)(i\not{\partial} - m + \delta m)\left(1 - (1 - \tilde{Z}_2^{-1})\frac{\not{n}^*\not{n}}{2(n^*\cdot n)}\right)\psi. \tag{129}$$

By keeping δm as Eq. (116) and defining

$$Z_2 \equiv 1 + \tilde{\alpha}_s C_F \quad \text{and} \quad \tilde{Z}_2 \equiv 1 - 2\tilde{\alpha}_s C_F, \tag{130}$$

and placing them in Eq. (129) the result is equivalent to Eq. (112). The definitions in Eq. (130) agree with the MS-scheme definitions in Ref. [9]. It is stated in Ref. [8] that terms containing the $\not{n}$ and $\not{n}^*$ matrices can be absorbed to the normalization of ψ to obtain covariant counterterms. One could speculate that by absorbing the factor $\left(1 - (1 - \tilde{Z}_2^{-1})\not{n}^*\not{n}(2n^*\cdot n)^{-1}\right)$ to the normalization of ψ, the renormalization constant in the LSZ reduction would be $Z = Z_2\tilde{Z}_2 = 1 - \tilde{\alpha}$, and Z would be a number as in a covariant gauge.

5.3 Pole mass of the quark propagator in a covariant gauge

It is interesting to see whether the quark pole mass in the light-cone gauge equals the pole mass in a covariant gauge. The covariant calculation follows the same lines as in sections 4 and 5, however it is much more straightforward. In this section the calculation of the covariant pole mass is outlined. The gluon propagator in a Lorenz gauge reads

$$D^{\mu\nu}(q) = \frac{-i\delta^{ab}}{q^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right). \tag{131}$$

Choosing a Feynman gauge i.e. fixing $\xi = 1$, and using Eq. (131) in Eq. (27) gives

$$\Sigma_{\text{cov}} = -g_s^2\mu^{4-2\omega}C_F\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\gamma^\mu\frac{g^{\mu\nu}}{q^2 + i\epsilon}\frac{\not{p} - \not{q} + m_0}{(p-q)^2 - m_0^2 + i\epsilon}\gamma^\nu. \tag{132}$$

With the same the Wick rotation as in section 4 the Euclidean self energy is

$$\Sigma_{cov,E} = -ig_s^2\mu^{4-2\omega}C_F\int_E\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{2\omega m_0 + 2(\omega-1)(\not{p}_E - \not{q}_E)}{q_E^2\left((p-q)_E^2 + m_0^2\right)}. \tag{133}$$

Utilizing the integrals calculated in sections 4.2 and 4.4, and then changing back to Minkowski space one gets

$$\begin{aligned}\Sigma_{\text{cov}} = -i\tilde{\alpha}_s C_F \left(4m_0 - \not{p}\right) + \frac{ig_s^2 C_F}{(4\pi)^2}\Bigg(2m_0\left(1 + 2\int_0^1 dx \log\frac{\Delta}{\mu^2}\right) \\ + \not{p}\left(-1 + 2\int_0^1 dx(x-1)\log\frac{\Delta}{\mu^2}\right)\Bigg).\end{aligned} \tag{134}$$

Inserting Eq. (134) into Eq. (97) the covariant effective propagator reads

$$iS_{\text{eff}}^{\text{cov}} \equiv \frac{i(\not{p} + m_0)}{p^2(1-\mathcal{A}_{\text{cov}}) - m_0^2(1+\mathcal{B}_{\text{cov}}) - m_0\not{p}(\mathcal{A}_{\text{cov}} + \mathcal{B}_{\text{cov}})}, \tag{135}$$

where

$$\begin{aligned}\mathcal{A}_{\text{cov}} &\equiv \tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2}\left(-1 + 2\int_0^1 dx(x-1)\log\frac{\Delta}{\mu^2}\right) \\ \text{and} \quad \mathcal{B}_{\text{cov}} &\equiv 2\tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2}2\left(1 + 2\int_0^1 dx \log\frac{\Delta}{\mu^2}\right).\end{aligned} \tag{136}$$

Expanding the effective propagator so that the denominator does not contain Dirac matrices gives the form

$$\begin{aligned}iS_{\text{eff}}^{\text{cov}} &= \frac{i(\not{p} - m_0)\left(p^2(1-\mathcal{A}_{\text{cov}}) - m_0^2(1+\mathcal{B}_{\text{cov}}) + m_0\not{p}(\mathcal{A}_{\text{cov}} + \mathcal{B}_{\text{cov}})\right)}{\left(p^2(1-\mathcal{A}_{\text{cov}}) - m_0^2(1+\mathcal{B}_{\text{cov}})\right)^2 - m_0^2 p^2(\mathcal{A}_{\text{cov}} + \mathcal{B}_{\text{cov}})^2} \\ &= i\frac{\not{p} + m_0(1 + \mathcal{A}_{\text{cov}} + \mathcal{B}_{\text{cov}})}{p^2 - m_0^2(1 + 2\mathcal{A}_{\text{cov}} + 2\mathcal{B}_{\text{cov}}) + \mathcal{O}(g_s^4)}(1 + \mathcal{A}_{\text{cov}}) + \mathcal{O}(g_s^4),\end{aligned} \tag{137}$$

It can be seen that the covariant gauge quark pole mass is

$$\begin{aligned}m_{\text{cov}}^2 &= m_0^2\left(1 + 2\mathcal{A}_{\text{cov}} + 2\mathcal{B}_{\text{cov}}\right) \\ &= m_0^2\left(1 + 6\tilde{\alpha}_s C_F - \frac{g_s^2 C_F}{(4\pi)^2}2\left(2\int_0^1 dx(1+x)\log\frac{\Delta}{\mu^2} + 1\right)\right),\end{aligned} \tag{138}$$

which is identical to the pole mass in light-cone gauge in Eq. (107). Clearly, also the mass renormalization in the $\overline{\text{MS}}$-scheme is identical.

The renormalised covariant effective propagator reads

$$iS_{\text{r,eff}}^{\text{cov}} = \frac{i(\not{p} + m)}{p^2 - m^2 + \mathcal{O}(g_s^4)}(1 + \mathcal{A}_{\text{cov}}) + \mathcal{O}(g_s^4), \tag{139}$$

where the term $1 + \mathcal{A}_{\text{cov}}$ is identified as the renormalization constant Z in the (LSZ) reduction [10]. In the covariant gauge the renormalization constant is related to the wave function and mass renormalization as

$$\psi_0^{\text{cov}} = \sqrt{Z}\psi^{\text{cov}} \quad \text{and} \quad Zm_0 = m - \delta m. \tag{140}$$

6 Conclusions

The aim of this thesis was to renormalize the quark propagator to one gluon loop, in the light-cone gauge, and to solve the pole mass of the propagator. The corresponding renormalization in the $\overline{\mathrm{MS}}$-scheme has been done before in Refs. [8, 9], of which the first one was used as reference in this thesis. In section 4 the one-loop quark self energy was derived using the ML-prescription to regulate the $n \cdot q$ pole. The finite parts of the quark self energy in Eq. (91) have been calculated before in Ref. [12] but, as it appears, have not been explicitly listed previously.

In section 5.1 the effective quark propagator was derived to the form in Eq. (106) from where the pole mass was defined. However, the numerator of Eq. (106) and the pole mass contain divergences which require cancellation by the quark field and mass renormalization, which were carried out in section 5.2. The field renormalization, Eq. (125), and mass renormalization, Eq. (109), are in the $\overline{\mathrm{MS}}$-scheme the same as obtained in Ref. [8], and in the MS-scheme they are equivalent to the results in Ref. [9]. The quark pole mass in the covariant Feynman gauge, derived in section 5.3, is equal to the light-cone gauge pole mass, Eq. (108). Also the mass renormalization in $\overline{\mathrm{MS}}$-scheme is the same in both gauges. These results agree with the expectation that the pole mass should be gauge invariant.

The final renormalised quark propagator, Eq. (126), includes finite gauge dependent and Lorentz noninvariant terms arising from the ML-prescription for the light-cone gauge. These terms make the mass-shell limit to depend on how one fixes the quark momentum p. Without taking the mass-shell limit they cannot be eliminated with the renormalization without having a momentum dependent counterterm. It remains as a question whether the finite leftover problematic terms cancel with the one-loop vertex corrections.

In this thesis it was seen that the method of defining the renormalization counterterms in light-cone gauge differs form the usual procedure. This is caused by the axial vector n which generates the $\not{n}$ and $\not{n}^*$ matrices to the renormalization counterpart $M\psi$ of the quark wave function. The the terms, including $\not{n}$ and $\not{n}^*$ matrices, can be absorbed to the normalization of the quark field so the counterterm in the renormalised Lagrangian would be free from noncovariant terms, as stated in Ref. [8]. The relation between the countertem and the renormalization constant in the LSZ reduction is more complicated than in the covariant gauge, due to the matrix feature of the counterterm.

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A QCD Lagrangian and Feynman rules

The QCD Lagrangian is

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{YM}} + \sum_q \left[i \left(\overline{\psi}_{q0}\right)_i \gamma^\mu \left((\partial_\mu)_{ij} + i g_s A^a_{0\mu} (t^a)_{ij} \right) \left(\psi_q\right)_j - m_{q0} \left(\overline{\psi}_{q0}\right)_i \left(\psi_q\right)_j \right] \\ + \mathcal{L}_{\text{fix}}, \tag{141}$$

where $\mathcal{L}_{\text{YM}}$ is the Yang-Mills Lagrangian, $A^a_{0\mu}$ is the bare gluon field, ψ_{q0} is the bare quark field and m_{q0} is a corresponding bare quark mass, g_s is the strong coupling constant, $\mathcal{L}_{\text{fix}}$ denotes the gauge fixing term, t^a is a SU(3) generator matrix, the sum $\sum_q$ denotes a sum over quark flavors, index a denotes a gluon color, i and j are color indices for quarks, and μ and ν are Lorentz indices. The Yang-Mills Lagrangian reads

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^{\mu\nu,a} F^a_{\mu\nu}, \tag{142}$$

where the gluon field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu, \tag{143}$$

where f^{abc} is a SU(3) structure constant and the indices a,b,c denote color of gluons.

The Feynman quark propagator is

$$\delta_{ij} S_F = i \delta_{ij} \frac{\not{p} + m_0}{p^2 - m_0^2}. \tag{144}$$

The quark-gluon vertex reads

$$-ig(t^a)_{ji}\gamma^\mu, \tag{145}$$

where the index j denotes color of the quark going out of the vertex.

B Rules for the Dirac matrices in Euclidean space

The Euclidean metric is

$$g_E^{\mu\nu} = \{1,1,1,1\}, \tag{146}$$

the inner product of two Euclidean four-vectors is

$$(a \cdot b)_E = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \tag{147}$$

and the slash notation is

$$\not{a}_E = a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 + a_4\gamma_4. \tag{148}$$

In Minkowski space the metric, the inner product and the slash notation are respectively

$$g^{\mu\nu} = \{1, -1, -1, -1\}, \tag{149}$$

$$(a \cdot b) = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \tag{150}$$

and

$$\not{a} = a_0\gamma_0 - a_1\gamma_1 - a_2\gamma_2 - a_3\gamma_3. \tag{151}$$

The transformation from Minkowski to Euclidean space can be performed in multiple ways. In this thesis it is carried out by redefining the time-like components of every four-vector and also the time-like Dirac matrix as

$$a_0 = i a_4 \quad \text{and} \quad \gamma_0 = i\gamma_4. \tag{152}$$

For the spatial parts the relation is

$$\begin{aligned} &(a_i)_M = (a_i)_E = (a^i)_E, \quad (a^i)_M = -(a_i)_E = -(a^i)_E, \\ &(\gamma_i)_M = (\gamma_i)_E = (\gamma^i)_E, \quad \text{and } (\gamma^i)_M = -(\gamma_i)_E = -(\gamma^i)_E. \end{aligned} \tag{153}$$

The transformation in Eq. (152) yields

$$a \cdot b = -a_4 b_4 - a_1 b_1 - a_2 b_2 - a_3 b_3 = -(a \cdot b)_E \tag{154}$$

for a dot product, and

$$\not{a} = -a_4 \gamma_4 - a_1 \gamma_1 - a_2 \gamma_2 - a_3 \gamma_3 = -\not{a}_E \tag{155}$$

for the slash notation. Identities for the γ^4 matrix are

$$\begin{aligned} \gamma^0 \gamma_0 &= i\gamma^4 i\gamma^4 = 1 \\ \longrightarrow \gamma^4 \gamma^4 &= -1 \end{aligned} \tag{156}$$

and

$$\begin{aligned} \gamma^0 \gamma^i &= i\gamma^4 \gamma^i = -\gamma^i i\gamma^4 \\ \longrightarrow \gamma^4 \gamma^i &= -\gamma^i \gamma^4. \end{aligned} \tag{157}$$

The Clifford algebra for Dirac matrices in Minkowski space is defined as

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{158}$$

which yields, together with Eq. (156) and Eq. (157), the Euclidean Clifford algebra to be

$$\{\gamma^\mu, \gamma^\nu\}_E = -2\delta^{\mu\nu}. \tag{159}$$

The most common trace identities for Minkowski space are

$$\begin{aligned} Tr\left(\not{a}\not{b}\right) &= 4(a \cdot b) \\ Tr\left(\not{a}\not{b}\not{c}\not{d}\right) &= 4\big((a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(c \cdot d)\big), \end{aligned} \tag{160}$$

which yields, together with Eq. (155) and Eq. (154), the Euclidean trace identities

$$\begin{aligned} Tr\left(\not{a}_E \not{b}_E\right) &= Tr\left(\not{a}\not{b}\right) = 4(a \cdot b) = -4(a \cdot b)_E \\ Tr\left(\not{a}_E \not{b}_E \not{c}_E \not{d}_E\right) &= Tr\left(\not{a}\not{b}\not{c}\not{d}\right) = 4\big((a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(c \cdot d)\big) \\ &= 4\big((a \cdot b)_E (c \cdot d)_E - (a \cdot c)_E (b \cdot d)_E + (a \cdot d)_E (c \cdot d)_E\big). \end{aligned} \tag{161}$$

C Integration methods

C.1 Feynman parametrization

The Feynman parametrization is a typical trick to evaluate loop integrals [10]

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^1 dx_1 ... \int_0^1 dx_n \frac{(n-1)!\delta(1 - \sum_{i=1}^n x_i)}{\left[x_1 A_1 + ... x_n A_n\right]^n}. \tag{162}$$

Another convenient integral to evaluate loop integrals is [13]

$$\int_0^\infty dz \frac{z^{\mu-1}}{(p + qz^\nu)^\alpha} = \frac{1}{\nu p^\alpha} \left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu)\Gamma(\alpha - \mu/\nu)}{\Gamma(\alpha)} \quad 0 < \frac{\mu}{\nu} < \alpha. \tag{163}$$

C.2 Wick rotation

Wick rotation is a complex analysis trick of removing imaginary parts of the denominators from integrals. An example integral in Minkowski space is of the form

$$\int d^D q \frac{f(q)}{\left[q^2 - \Delta + i\theta\right]^m}, \quad \Delta > 0, \quad \theta > 0. \tag{164}$$

The time-like integral is

$$\int_{-\infty}^{\infty} dq^0 \frac{f(q)}{\left[q_0^2 - \mathbf{q}^2 - \Delta + i\theta\right]^m}, \tag{165}$$

which has poles in the second and fourth quadrants in the complex plane

$$q^0 = \pm\sqrt{\mathbf{q}^2 + \Delta - i\theta} = \pm\sqrt{\mathbf{q}^2 + \Delta} \mp i\theta. \tag{166}$$

The Cauchy theorem states if the integrand is analytic within a simply connected region in a complex plane, then the integral over a piecewise smooth simple closed curve in this region gives zero. Regarding the integrand in Eq. (165) choosing an integration path that rules out the poles in the second and fourth quadrants as in

Figure 6, would yield the complex plane integration to be zero

$$\oint_\Gamma dq^0 \frac{f(q)}{\left[q_0^2 - \mathbf{q}^2 - \Delta + i\theta\right]^m} = 0. \tag{167}$$

Expressing Γ as a sum of its parts $\oint_\Gamma = \int_{-R}^{R} + \int_{Arc1} + \int_{iR}^{-iR} + \int_{Arc2}$ and then proving that the arc integrals vanish in the limit $R \to \infty$ [10], one gets a relation

$$\lim_{R\to\infty} \int_{-R}^{R} dq^0 \frac{f(q)}{\left[q_0^2 - \mathbf{q}^2 - \Delta + i\theta\right]^m} = -\int_{iR}^{-iR} dq^0 \frac{f(q)}{\left[q_0^2 - \mathbf{q}^2 - \Delta\right]^m}. \tag{168}$$

In Eq. (168) the $i\theta$ term is taken to zero in the latter integral since an infinitesimal imaginary shift does not matter in the imaginary axis. Transferring to Euclidean space by setting $q_0 = il_0$ i.e. $q^2 = -l_E^2$ in the latter integral in Eq. (168) one gets

$$\int_{-\infty}^{\infty} dq^0 \frac{f(q)}{\left[q_0^2 - \mathbf{q}^2 - \Delta + i\theta\right]^m} = i(-1)^m \int_{-\infty}^{\infty} dl^0 \frac{f(l)}{\left[l_E^2 + \Delta\right]^m}. \tag{169}$$

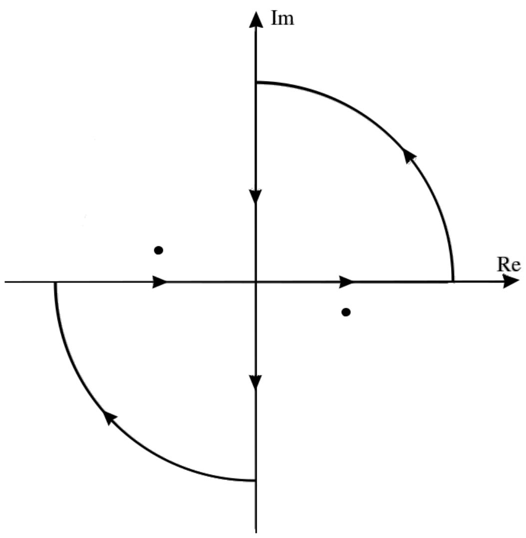

Figure 6. Path Γ in the complex plane with poles in the second and fourth quadrants.

C.3 Basic integral

The dimensional regulation and the Wick rotation often gives an integral of the form

$$I = \int d^d l \frac{l^n}{[l^2 + \Delta]^m}, \tag{170}$$

where l is Euclidean. The integral can be transformed to spherical coordinates as

$$\int_E d^d l = \int d\Omega_d \int_0^\infty dl l^{d-1}, \tag{171}$$

where $d\Omega_d$ is differential solid angle of the d-dimensional unit sphere. It is shown in Ref. [10] that the angular integral gives

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{172}$$

Thus the basic integral is

$$I = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dl \frac{l^{d+n-1}}{[l^2 + \Delta]^m}. \tag{173}$$

First changing the integration variable to $dl = dl^2(2l)^{-1}$ and then defining a new variable $z = \Delta(l^2 + \Delta)^{-1}$ gives

$$\int_0^\infty dl^2 = \int_0^1 \frac{dz\Delta}{z^2}. \tag{174}$$

With this the basic integral becomes

$$\begin{aligned} I =& \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^1 dz \frac{\Delta}{z^2} \frac{z^m}{\Delta^m} \left(\frac{\Delta}{z} - \Delta \right)^{(d+n)/2-1} \\ =& \frac{\pi^{d/2}}{\Gamma(d/2)} \Delta^{(d+n)/2-m} \int_0^1 dz z^{m-(d+n)/2-1} (1-z)^{(d+n)/2-1}, \end{aligned} \tag{175}$$

from where one can recognize the Beta-function

$$B(\alpha, \beta) = \int_0^1 dz z^{\alpha-1} (1-z)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{176}$$

Using the Beta-function definition in Eq. (175) yields

$$I = \frac{\pi^{d/2}}{\Gamma(d/2)} \Delta^{(d+n)/2-m} \frac{\Gamma(m-(d+n)/2)\Gamma((d+n)/2)}{\Gamma(m)}. \tag{177}$$

D Calculations of $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$

The integral $\mathcal{I}_1$ can be evaluated straightforwardly

$$\begin{aligned}\mathcal{I}_1 = \int_0^1 dx \log\left(\frac{(1-x)(p_3^2 - p_0^2) + m_0^2}{\mu^2}\right) = -1 + \left(1 + \frac{m_0^2}{p_3^2 - p_0^2}\right)\log\left(\frac{p_3^2 - p_0^2 + m_0^2}{\mu^2}\right) \\ - \frac{m_0^2}{p_3^2 - p_0^2}\log\left(\frac{m_0^2}{\mu^2}\right).\end{aligned} \tag{178}$$

More effort has to put in solving $\mathcal{I}_2$ which is defined as

$$\begin{aligned}\mathcal{I}_2 &= \int_0^1 dx \int_0^{1-x} \frac{dy}{x+y}\Bigg[2 + \frac{2(p^2 - m_0^2)}{\left(1 - \frac{x}{x+y}\right)p_\perp^2 + (1-x)(p_3^2 - p_0^2) + m_0^2} + \log\left(\frac{x}{x+y}\right) \\ &+ \log\left(\frac{\left(1 - \frac{x}{x+y}\right)p_\perp^2 + (1-x)(p_3^2 - p_0^2) + m_0^2}{\mu^2}\right)\Bigg] \\ &= \int_0^1 dx \int_0^{1-x} dy\Bigg[\frac{2 + \log(x) - 2\log(x+y)}{x+y} \\ &+ \frac{2(p^2 - m_0^2)}{y\left(m_0^2 - p^2 - x(p_3^2 - p_0^2)\right) + x(1-x)(p_3^2 - p_0^2) + x m_0^2} \\ &+ \frac{1}{x+y}\log\left(\frac{y\left(m_0^2 - p^2 - x(p_3^2 - p_0^2)\right) + x(1-x)(p_3^2 - p_0^2) + x m_0^2}{\mu^2}\right)\Bigg],\end{aligned} \tag{179}$$

where

$$\int_0^1 dx \int_0^{1-x} dy \frac{2 + \log(x) - 2\log(x+y)}{x+y} = 2 \tag{180}$$

and

$$\begin{aligned}&\int_0^1 dx \int_0^{1-x} dy \frac{2(p^2 - m_0^2)}{y\left(m_0^2 - p^2 - x(p_3^2 - p_0^2)\right) + x(1-x)(p_3^2 - p_0^2) + x m_0^2} \\ &= \int_0^1 dx \frac{2(p^2 - m_0^2)}{m_0^2 - p^2 - x(p_3^2 - p_0^2)}\log\left(\frac{m_0^2 - (1-x)p^2}{x\left(m_0^2 + (1-x)(p_3^2 - p_0^2)\right)}\right).\end{aligned} \tag{181}$$

The last term in Eq. (179) gives dilogarithms

$$
\begin{aligned}
&\int_0^1 dx \int_0^{1-x} dy \frac{1}{x+y} \log\left(\frac{y\left(m_0^2 - p^2 - x(p_3^2 - p_0^2)\right) + x(1-x)(p_3^2 - p_0^2) + x m_0^2}{\mu^2} \right) \\
&= \int_0^1 dx \Bigg[-\log(x) \log\left(\frac{-p_\perp^2 x}{\mu^2} \right) + \mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2} \right) - \mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2 x} \right) \Bigg] \\
&= -2 + \log\left(\frac{-p_\perp^2}{\mu^2} \right) + \int_0^1 dx \Bigg[\mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2} \right) - \mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2 x} \right) \Bigg],
\end{aligned}
\tag{182}
$$

where partial integration for dilogarithms gives

$$
\begin{aligned}
&\int_0^1 dx \Bigg[\mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2} \right) - \mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2 x} \right) \Bigg] \\
&= x \left[\mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2} \right) - \mathrm{Li}_2\left(\frac{m_0^2 - p^2 - x(p_3^2 - p_0^2)}{p_\perp^2 x} \right) \right]\Bigg|_0^1 \\
&- \int_0^1 dx \Bigg[\log\left(-p_\perp^2\right) + \frac{x(p_3^2 - p_0^2)}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \log\left(m_0^2 + (1-x)(p_3^2 - p_0^2) \right) \\
&- \frac{m_0^2 - p^2}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \log\left(\frac{m_0^2 - (1-x)p^2}{x}) \right) \Bigg] \\
&= -\log\left(-p_\perp^2\right) + \int_0^1 dx \Bigg[- \frac{x(p_3^2 - p_0^2)}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \log\left(m_0^2 + (1-x)(p_3^2 - p_0^2) \right) \\
&+ \frac{m_0^2 - p^2}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \log\left(\frac{m_0^2 - (1-x)p^2}{x}) \right) \Bigg].
\end{aligned}
\tag{183}
$$

Combining the integrals Eqs. (180), (181), (182), and (183) gives the solution for $\mathcal{I}_2$

$$
\begin{aligned}
\mathcal{I}_2 &= \int_0^1 dx \log\left(\frac{m_0^2 + (1-x)(p_3^2 - p_0^2)}{\mu^2}\right) + \int_0^1 dx \frac{m_0^2 - p^2}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \Bigg[\log(x) \\
&+ \log\left(m_0^2 + (1-x)(p_3^2 - p_0^2)\right) - \log\left(m_0^2 - (1-x)p^2\right) \Bigg] \\
&= \int_0^1 dx \log\left(\frac{m_0^2 + (1-x)(p_3^2 - p_0^2)}{\mu^2}\right) - \frac{m_0^2 - p^2}{p_3^2 - p_0^2} \Bigg[\log(-p^2) \log\left(\frac{m_0^2 - p^2}{m_0^2}\right) \\
&+ \log(m_0^2 - p^2) \log\left(\frac{m_0^2}{m_0^2 + p_3^2 - p_0^2}\right) + \log(p_\perp^2) \log\left(\frac{m_0^2 + p_3^2 - p_0^2}{m_0^2 - p^2}\right) + \mathrm{Li}_2\left(\frac{p_3^2 - p_0^2}{m_0^2 - p^2}\right) \\
&+ \mathrm{Li}_2\left(\frac{m_0^2}{-p_\perp^2}\right) - \mathrm{Li}_2\left(\frac{p_3^2 - p_0^2 + m_0^2}{-p_\perp^2}\right) - \mathrm{Li}_2\left(\frac{m_0^2(p_3^2 - p_0^2)}{p_\perp^2(p^2 - m_0^2)}\right) + \mathrm{Li}_2\left(\frac{p_3^2 - p_0^2}{-p_\perp^2}\right) \Bigg],
\end{aligned} \tag{184}
$$

where the first term equals $\mathcal{I}_1$.

The integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ appear in the quark self energy, Eq. (91), together in a term $\mathcal{I}_1 - \mathcal{I}_2$. Its value when $p_\perp^2 = 0$ is

$$
\mathcal{I}_1 - \mathcal{I}_2 \overset{p_\perp^2=0}{=} \int_0^1 dx \frac{p^2 - m_0^2}{m_0^2 - (1-x)p^2} \log(x) = \frac{p^2 - m_0^2}{p^2} \mathrm{Li}_2\left(\frac{p^2}{p^2 - m_0^2}\right), \tag{185}
$$

which is finite when $p^2 = 0$ and when $p^2 = m_0^2$. The value of $\mathcal{I}_1 - \mathcal{I}_2$ is finite also, when $p^2 = 0$ without fixing the value of $p_\perp^2$.

The $\mathcal{I}_3$ is defined as

$$
\begin{aligned}
\mathcal{I}_3 &= \int_0^1 dx \int_0^{1-x} dy \left(\frac{x}{x+y} \frac{p_3^2 - p_0^2}{\left(1 - \frac{x}{x+y}\right) p_\perp^2 + (1-x)(p_3^2 - p_0^2) + m_0^2} \right) \\
&= \int_0^1 dx \int_0^{1-x} dy \left(\frac{x(p_3^2 - p_0^2)}{y\left(m_0^2 - p^2 - x(p_3^2 - p_0^2)\right) + x(1-x)(p_3^2 - p_0^2) + x m_0^2} \right) \\
&= \int_0^1 dx \frac{x(p_3^2 - p_0^2)}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \log\left(\frac{m_0^2 - (1-x)p^2}{x\left(m_0^2 + (1-x)(p_3^2 - p_0^2)\right)} \right).
\end{aligned} \tag{186}
$$

The integral $\mathcal{I}_3$ appears in the quark self energy, Eq. (91), only as a sum with $\mathcal{I}_2$. It is easier to handle those two integrals as a sum than individually, and the simplest way is to use the form where the integration is executed with respect to the Feynman parameter y only

$$\begin{aligned}
&\mathcal{I}_2 + \mathcal{I}_3 = \int_0^1 dx \log\left(\frac{m_0^2 + (1-x)(p_3^2 - p_0^2)}{\mu^2}\right) \\
&+ \int_0^1 dx \frac{m_0^2 - p^2}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \log\left(\frac{x(m_0^2 + (1-x)(p_3^2 - p_0^2))}{m_0^2 - (1-x)p^2}\right) \\
&+ \int_0^1 dx \frac{x(p_3^2 - p_0^2)}{m_0^2 - p^2 - x(p_3^2 - p_0^2)} \log\left(\frac{m_0^2 - (1-x)p^2}{x\left(m_0^2 + (1-x)(p_3^2 - p_0^2)\right)}\right) \\
&= \mathcal{I}_1 + \int_0^1 dx \left[\log(x) + \log\left(m_0^2 + (1-x)(p_3^2 - p_0^2)\right) - \log\left(m_0^2 - (1-x)p^2\right)\right] \\
&= 2\mathcal{I}_1 - 1 - \int_0^1 dx \log\left(\frac{m_0^2 - (1-x)p^2}{\mu^2}\right) \\
&= 2\mathcal{I}_1 - 2 - \int_0^1 dx \log\left(\frac{x(x-1)p^2 + xm_0^2}{\mu^2}\right).
\end{aligned} \tag{187}$$

The quark self energy, Eq. (91), contains a parameter N defined as

$$N = \frac{\mathcal{I}_2 + \mathcal{I}_3 - \int_0^1 dx \log\left(\frac{\Delta}{\mu^2}\right)}{2(n^* \cdot p)(n \cdot p) - p^2(n^* \cdot n)}, \tag{188}$$

where

$$\log\left(\frac{\Delta}{\mu^2}\right) = \log\left(\frac{x(x-1)p^2 + xm_0^2}{\mu^2}\right). \tag{189}$$

The fraction in Eq. (188) seemingly has a pole when $p^2(n^* \cdot n) = 2(n^* \cdot p)(n \cdot p)$ i.e. $p_\perp^2 = 0$, which however is not the case since also the numerator gives zero when $p_\perp^2 = 0$. The numerator reads

$$\begin{aligned}
&\mathcal{I}_2 + \mathcal{I}_3 - \int_0^1 dx \log\left(\frac{\Delta}{\mu^2}\right) \\
&= 2\int_0^1 dx \left[\log\left(\frac{m_0^2 + (1-x)(p_3^2 - p_0^2)}{\mu^2}\right) - \log\left(\frac{m_0^2 - (1-x)p^2}{\mu^2}\right)\right],
\end{aligned} \tag{190}$$

which equals zero when $p_\perp^2 = 0$ i.e. $p_0^2 - p_3^2 = p^2$. The value of N when $p_\perp^2 = 0$ is achieved using the L'Hôpital's rule

$$N \overset{p_\perp^2=0}{=} \frac{2}{(n^* \cdot n)p^2}\left(1 + \frac{m_0^2}{p^2}\log\left(\frac{m_0^2 - p^2}{m_0^2}\right)\right), \tag{191}$$

which has a pole in the $p^2 = 0$ limit. However, N is multiplied in the self energy, Eq. (4.6), always with $(n^* \cdot p)(n \cdot p)$ or with $p^2(n^* \cdot n)$, and thus the pole, when $p^2 = p_0^3 - p_3^2 = 0$, is canceled.