Neutrino density matrix formalism derived from Kadanoff-Baym equations

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Julkaisu on tekijänoikeussäännösten alainen. Teosta voi lukea ja tulostaa henkilökohtaista käyttöä varten. Käyttö kaupallisiin tarkoituksiin on kielletty.

Abstract

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When neutrinos propagate in matter they encounter two kinds of interactions with the medium: coherent and incoherent collisions. Due to the coherent collisions a background potential arises which modifies the energy eigenstates of neutrinos. Incoherent collisions, however, lead to quantum damping which affects the dynamical evolution of neutrinos.

In this thesis it is studied how a formalism, which describes mixing of relativistic neutrino fields, can be derived from the grounds of thermal quantum field theory. We begin by deriving the general Kadanoff-Baym (KB) equations in the Wigner space starting from the contour Schwinger-Keldysh euqation. Next the KB equations are solved using the coherent quasiparticle approximation (cQPA). From the cQPA scheme it follows that the phase space of the system contains completely novel shell solutions which can be recognized to carry information about non-local quantum coherence.

Thus, in this thesis we focus on discussing how a closed set of equations of motion, which take into account quantum coherence without any additional approximations, can be derived for propagating neutrinos. With these equations of motion it is possible to calculate scattering processes between coherent neutrino states to which no other existing model is capable of. In addition, they can be used to derive the neutrino density matrix formalism from more fundamental grounds than what has been done before.

Keywords: neutrino physics, thermal field theory, quantum field theory, kinetic transport theory, cQPA

Tiivistelmä

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Edetessään väliaineessa neutriinojen ja materian välillä tapahtuu kahdenlaisia törmäyksiä: koherentteja ja epäkoherentteja törmäyksiä. Koherentit törmäykset aiheuttavat taustapotentiaalin, joka muuttaa neutriinojen ominaisenergioita. Epäkoherentit törmäykset aiheuttavat puolestaan kvanttivaimennustekijöitä, jotka vaikuttavat neutriinotilojen dynaamiseen kehitykseen.

Tässä tutkielmassa tarkastellaan relativististen neutriinojen sekoittumista kuvaavan formalismin johtamista kvanttikenttäteoriaan pohjautuvista lähtökohdista. Tutkielmassa lähdetään liikkeelle Schwinger-Keldysh yhtälöstä, josta johdetaan yleiset Kadanoff-Baym (KB) yhtälöt Wigner-avaruudessa. KB-yhtälöt ratkaistaan käyttäen koherenttia kvasihiukkasapproksimaatiota (cQPA). cQPA-mallista seuraa, että systeemin faasiavaruudessa esiintyy täysin uudenlaisia koherenssikuoria, jotka voidaan tunnistaa sisältävän informaatiota ei-lokaalista kvanttikoherenssista.

Tutkielmassa siis tarkastellaan kuinka neutriinoille voidaan johtaa liikeyhtälöt, jotka huomioivat kvanttikoherenssin ilman lisäoletuksia. Johdettujen liikeyhtälöiden avulla voidaan laskea koherenttien neutriinotilojen välistä sirontaa, johon tämänhetkiset mallit eivät kykene. Lisäksi näiden liikeyhtälöiden avulla voidaan johtaa neutriinojen tiheysmatriisiformalismi yleisemmistä lähtökohdista kuin mitä on aikaisemmin tehty.

Avainsanat: neutriinofysiikka, äärellisen lämpötilan kenttäteoria, kvanttikenttäteoria, kineettinen kuljetusteoria, cQPA

Contents

1 Introduction

In modern physics quantum field theory (QFT) is understood as a collection of mathematical and conceptual tools used to describe elementary particles. There are lots of different formulations of QFT developed to describe various kinds of phenomena. For instance, in vacuum quantum field theory (in the zero temperature limit) one studies transition probabilities, while in thermal field theory it is more convenient to study expectation values of operators using imaginary or real time variables. The applications of quantum field theory are not restricted only to particle physics, but also includes e.g. effects studied in condensed matter physics. In a way, one could say that the quantum field theory gives us the most fundamental picture of the physics describing the surrounding world, as far as we know.

In the full generality, equations arising from the principles of QFT, which describe the dynamics of interacting quantum fields, are impossible to solve. Therefore, simplifying approximation schemes are needed. If one considers slowly varying background fields, weak interactions and assumes translational invariant correlators, the standard methods of quantum kinetic theory reduces the problem to the famous Boltzmann transport equations. These transport equations provide relatively good approximations for many situations. However, when considering out-of-equilibrium quantum systems there is no reason to assume translational invariance, i.e. thermal equilibrium, for the correlation functions. Due to the loss of the correlator's translational invariance, the Boltzmann transport equations can not be used and a more general approximation scheme is needed to study these situations.

In this thesis we consider the coherent quasiparticle approximation (cQPA) which holds also in many non-equilibrium systems. The key point of the cQPA is that one relinquishes the assumption of correlator's translational invariance. From this it follows that there exist completely new solutions in addition to the usual mass shell solutions. These novel shells are recognized to carry information about non-local quantum coherence which, in turn, means that using cQPA we can study phenomena where quantum coherence plays important role. Examples of situations where quantum coherence can not be neglected include inflation, preheating, electroweak

baryogenesis and neutrino flavor oscillations, just to mention a few.

The main goal of this thesis is to derive an equation in the cQPA limit in which incoherent collisions and neutrino flavor mixing are considered. From this equation it is possible to solve the equations of motion for neutrinos which take into account quantum coherence and matter effects from more fundamental grounds than what has been done before, and in principle they allow us to study situations beyond the capability of the ordinary density matrix formalism.

Since we are studying neutrino oscillations, we begin this thesis by reviewing the mixing of neutrino masses and the vacuum theory of neutrino oscillations in chapter (2). Matter can have a huge effect to neutrino propagation and in many situations these effects can not be neglected. In chapter (3) we therefore discuss the matter effects of neutrinos and derive the matter Hamiltonian for interacting neutrinos. One of the main conclusions made in chapter (3) is that due to incoherent neutrino scatterings the usual Hamiltonian formalism can not be used, and a different formalism is needed. However, we are interested in non-equilibrium systems where quantum coherence effects are significant, so simple Boltzmann transport equations are not sufficient. In chapter (4) we start to build a more general theory which is capable of describing many non-equilibrium systems while taking into account quantum coherence. We derive a superior form of the general Kadanoff-Baym equations in the viewpoint of gradient expansions starting from the contour Schwinger-Keldysh equation. These KB equations are, however, impossible to solve in full generality as such since they contain infinite order gradient terms. For this reason we need an approximation scheme which simplifies the KB equations, but takes the quantum coherence effects into account. In chapter (5) we introduce such approximation scheme called the coherent quasiparticle approximation (cQPA) and derive the cQPA equations for neutrinos starting from the KB equations. Especially, we show how a closed set of equations of motion, which include terms arising from quantum coherence, can be derived for neutrinos. Lastly, chapter (6) is devoted to summary, conclusions and discussion.

2 Neutrino oscillation

Neutrino oscillation is a quantum mechanical phenomenon, in which neutrinos can evolve to a state with different lepton number L without interacting with any other particle. It was proposed by Pontecorvo in the late 1950's [1, 2]. Neutrino oscillations are a consequence of nonzero neutrino masses, and mixing between the neutrino weak interaction (flavor) eigenstates and the (propagating) neutrino states of definite mass.

Since the 1960's neutrino oscillation phenomenon has been of a great theoretical and experimental importance, because it can shed light into the properties of neutrinos. For instance, in order to neutrino oscillation to be possible neutrinos must have nonzero masses. This in turn can help us to understand physics beyond the Standard Model and get us towards a more general theory of physics.

In this chapter we review the mixing of neutrino masses and the derivation of the standard neutrino oscillation probability in vacuum using the plane-wave approximation. These subjects have been discussed and reviewed in multiple papers, e.g in [3–10].

2.1 Neutrino mixing

Although for some leptons, like electron and muon, the flavor eigenstates have definite masses, there is no reason to assume that this would hold for neutrinos. In fact, it turns out that in theories beyond the SM where neutrinos are massive particles, the fields participating in the weak interaction processes do not in general diagonalize the mass matrix. For this reason it might not be clear what kind of mass terms there can be for neutrinos. Luckily, Hermicity and Lorentz-invariance give constraints for the possible mass terms, and we can get relations between the flavor and mass eigenstates. This section follows the outlines of ref. [11].

Assume that a right-handed neutrino field ν_R exist which is allowed by the symmetries of the SM. Then, we can write a Dirac mass term for N neutrino flavors

as

$$-\mathcal{L}_m^D = \bar{\nu}_R m_D \nu_L + \bar{\nu}_L m_D^\dagger \nu_R, \tag{2.1}$$

where we have introduced the chiral fields

$$\nu_L = \frac{1}{2}(1-\gamma_5)\nu = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ ... \end{pmatrix} \tag{2.2}$$

and

$$\nu_R = \frac{1}{2}(1+\gamma_5)\nu = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ ... \end{pmatrix}, \tag{2.3}$$

see Sec.(3.1) for more discussion about the chiral fields. Notice that we have not restricted the number of neutrino flavors, so this holds for arbitrary number of fields. From this it follows that in eq. (2.1) m_D is a complex $N \times N$ matrix and it can be diagonalized by a bi-unitary transformation:

$$M_D = U m_D V^\dagger, \tag{2.4}$$

i.e.

$$m_D = U^\dagger M_D V. \tag{2.5}$$

Here M_D is diagonal matrix. Using these we can write the Dirac mass term as

$$\mathcal{L}_M^D = -\bar{\nu}_{mR} M_D \nu_{mL} - \bar{\nu}_{mL} M_D^\dagger \nu_{mR}, \tag{2.6}$$

where

$$\nu_{mL} \equiv U^\dagger \nu_L \qquad \text{and} \qquad \nu_{mR} \equiv V^\dagger \nu_R \tag{2.7}$$

are the mass eigenfields. Now it easy to see that the $\nu_{R/L}$ fields correspond to neutrinos with definite mass since M_D is diagonal. The matrix U which relates the left-handed neutrino fields is the leptonic mixing matrix, or the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [12]. It is equivalent to the Cabibbo-Kobyashi-Maskawa matrix which is the mixing matrix for quarks [13, 14].

Consider next the left- and right-handed Majorana mass terms $\mathcal{M}_L$ and $\mathcal{M}_R$ in the case of N neutrino flavors. The corresponding part of the Lagrangian density

function reads

$$-\mathcal{L}_M^{\mathcal{M}} = \frac{1}{2}\overline{\nu_L^c}\mathcal{M}_{ML}\nu_L + \overline{\nu_R^c}\mathcal{M}_{MR}\nu_R + \text{h.c.}, \tag{2.8}$$

where we have used the charge conjugated fields, for example

$$\begin{aligned}\nu_L^c \equiv (\nu_L)^c &= C\overline{\nu_L}^T = C(\nu_L^\dagger\gamma^0)^T \\ &= C\gamma_0^T\frac{1+\gamma_5^T}{2}\nu^* = \frac{1-\gamma_5}{2}C\overline{\nu}^T = (\nu^c)_R \equiv \nu_R^c.\end{aligned} \tag{2.9}$$

Here C is the charge conjugation matrix and in chiral representation it can be written as $C = C^\dagger = C^T = C^{-1} = i\gamma^0\gamma^2$. Using properties of the Majorana neutrino fields one can show that the Majorana mass matrix is symmetric:

$$\overline{\nu_L^c}\mathcal{M}_{ML}\nu_L = \overline{\nu_L^c}\mathcal{M}_{ML}^T\nu_L, \tag{2.10}$$

and similarly for the right-handed field, see ref. [4] for details. Thus, the mass matrices $\mathcal{M}_L$ and $\mathcal{M}_R$ can be diagonalized by unitary transformations U and V, respectively:

$$U^T\mathcal{M}_{ML}U = M_{ML}, \tag{2.11}$$

where M_{ML} is the diagonal left-handed Majorana mass. Similar relation holds for the right-handed mass matrix. Now we can write the Majorana mass term as

$$\mathcal{L}_M^M = -\frac{1}{2}\overline{N_L^c}M_M N_L + \text{h.c.}, \tag{2.12}$$

where

$$N_L \equiv (\nu_{mL}, (\nu_{mR})^c) = (U^\dagger\nu_L, V^\dagger\nu_R^c) \tag{2.13}$$

and

$$M_M = \begin{pmatrix} M_{ML} & 0 \\ 0 & M_{MR} \end{pmatrix}. \tag{2.14}$$

As in the case of Dirac mass, we can identify that ν_{mL} are the left-handed components and ν_{mR} are the right-handed components of the massive Majorana fields. We also notice that the mixing matrix U relates again the left-handed neutrino fields and thus it is the PMNS-matrix.

An interesting property of Majorana mass terms, described by eq. (2.12), is that they are not invariant under constant phase shifts, for instance under the

transformation

$$\nu \to e^{i\phi}\nu \qquad \text{and} \qquad \nu^c \to e^{-i\phi}\nu^c. \tag{2.15}$$

This kind of mass terms are not allowed for charged leptons since the conservation of charge would be broken. However, neutrinos do not carry any charge and the Majorana masses in eq. (2.12) are possible. Another interesting property of eq. (2.15) is that it implies that Majorana masses break global symmetries, especially the lepton number conservation is lost. This symmetry break makes new processes like neutrinoless double-beta-decay possible to occur.

After we have derived the useful forms (2.6) and (2.12) for the Dirac and Majorana mass terms, we can immediately write down the most general neutrino mass term:

$$\mathcal{L}_M = -\frac{1}{2}\overline{N_L^c} M N_L + \text{h.c.}, \tag{2.16}$$

where N_L is as defined in eq. (2.13), but now the mass matrix M contains both the Dirac and Majorana mass terms:

$$M = \begin{pmatrix} M_{ML} & M_D^T \\ M_D & M_{MR} \end{pmatrix}. \tag{2.17}$$

Thus, we have shown that there exists mixing between the neutrino flavor and the mass eigenstates regardless of the form of the mass matrix M.

In this section the main point of discussion was the mixing of neutrinos. For a more detailed discussion about neutrino masses and their origins see refs. [4, 15].

Number of parameters in the leptonic matrix

Before moving to examine neutrino oscillations in vacuum, lets take a closer look to the mixing matrix U. The discussion follows closely ref. [16]. In general, a unitary $N \times N$ matrix can be parametrized by N^2 independent real parametrers. These parameters can be divided into

$$\frac{N(N-1)}{2} \qquad \text{mixing angles} \tag{2.18}$$

and

$$\frac{N(N+1)}{2} \qquad \text{phases.} \tag{2.19}$$

However, not all of these parameters are physical because some of them can be eliminated by rephasing the neutrino and charged lepton fields. This can be done since the Lagrangian, excluding the weak charged current (CC) part, does not change under transformations defined by eq. (2.15). In general, for Dirac neutrinos we can absorb N complex phases into the redefinitions of the charged lepton fields. This leaves us with $N(N-1)$ mixing parameters in which $N(N-1)/2$ are complex phases. One could expect that we can also eliminate N complex phases by redefining the neutrino fields. Nonetheless, this is not the case since one of these phases corresponds to an overall phase factor which leaves the neutrino CC part invariant. Using Noether's theorem this kind of invariance can be related to the conservation of lepton number (in the SM neutrino oscillations can not happen, since neutrinos are massless, and thus the lepton number is conserved). In other words, the overall phase factor corresponds to a physical observable and we cannot eliminate it. In the case of Majorana neutrinos there is a crucial difference with respect to the Dirac neutrinos: the Majorana mass term in eq. (2.8) is not invariant under global phase transformations defined by eq. (2.15). From this it follows that we can not eliminate any phases of the mixing matrix by redefining the neutrino fields.

Summarizing, for the mixing matrix in the case of Dirac neutrinos we have

$$\frac{N(N-1)}{2} \qquad \text{mixing angles} \tag{2.20}$$

and

$$\frac{(N-1)(N-2)}{2} \qquad \text{phases.} \tag{2.21}$$

For the Majorana neutrinos we have

$$\frac{N(N-1)}{2} \qquad \text{mixing angles} \tag{2.22}$$

$$\frac{N(N-1)}{2} \qquad \text{phases.} \tag{2.23}$$

The total number of mixing parameters rises very quickly since it is proportional to N^2. For this reason, the analysis of neutrino mixing becomes hard and technical when considering multiple neutrino flavors.

2.2 Derivation of the neutrino oscillation probability

We can assume neutrinos to be ultra-relativistic particles since masses of neutrinos are under 1 eV [17], while only neutrinos with energy of keV scale can be detected. In the standard plane-wave theory of neutrino oscillation it is assumed that neutrinos are detected or created as a flavor eigenstate given by [3]

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} |\nu_k\rangle \,, \tag{2.24}$$

where U is the leptonic mixing matrix (PMNS matrix), $|\nu_\alpha\rangle$ is a flavor eigenstate and $|\nu_k\rangle$ is a mass eigenstate. Greek indices ($\alpha, \beta, \gamma ... = e, \mu, \tau ...$) denote the flavor eigenstates and Latin indices ($i, j, k ... =$ 1,2,3...) refer to the mass eigenstates. Corresponding relation for antineutrinos reads

$$|\bar{\nu_\alpha}\rangle = \sum_k U_{\alpha k} |\bar{\nu_k}\rangle \,. \tag{2.25}$$

The only difference between mixing of neutrinos and antineutrinos is the complex conjugation of the leptonic mixing matrix.

Using the fact that the massive neutrino states $|\nu_k\rangle$ have definite masses m_k and energies E_k, we can write the time evolution of the massive neutrino states as

$$i\frac{d}{dt} |\nu_k(t)\rangle = \mathcal{H} |\nu_k(t)\rangle = E_k |\nu_k(t)\rangle \tag{2.26}$$

with energy eigenvalues

$$E_k = \sqrt{\mathbf{p}^2 + m_k^2}. \tag{2.27}$$

The Schrödinger equation (2.26) implies that the massive neutrino states evolve in time as plane waves:

$$|\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle \,. \tag{2.28}$$

It follows then that the time evolution of the neutrino flavor states is given by

$$|\nu_\alpha(t)\rangle = \sum_k U^*_{\alpha k} e^{-iE_k t} |\nu_k\rangle = \sum_{\beta=e,\mu,\tau} \left(\sum_k U^*_{\alpha k} e^{-iE_k t} U_{\beta k} \right) |\nu_\beta\rangle \,. \tag{2.29}$$

According to eq. (2.29) a state which is at time $t = 0$ a pure flavor state $|\nu_\alpha\rangle$ will evolve in time to a superposition of different flavor states (if the mixing matrix is

different from unity). The quantity in the brackets in eq. (2.29) is the amplitude ($A_{\nu_\alpha(t)\to\nu_\beta}$) for the $\nu_\alpha(t) \to \nu_\beta$ transition as a function of time. Therefore, the probability of the transition $\nu_\alpha \to \nu_\beta$ is

$$P_{\nu_\alpha\to\nu_\beta}(t) = \left|A_{\nu_\beta\to\nu_\alpha}(t)\right|^2 = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i(E_k - E_j)t}. \tag{2.30}$$

For ultra-relativistic neutrinos we can approximate the dispersion relation (2.27) as

$$E_k = \sqrt{\mathbf{p}^2 + m_k^2} \approx E + \frac{m_k^2}{2E}, \tag{2.31}$$

where $E = |\mathbf{p}|$ is the energy of the neutrinos when we neglect the mass. Thus, we get for the energy difference between two massive states relation

$$E_k - E_j \simeq \frac{m_k^2 - m_j^2}{2E} = \frac{\Delta m_{kj}^2}{2E}. \tag{2.32}$$

In the oscillation probability (2.30) there occurs the oscillation time t. In most of the oscillation experiments it is not even possible to measure the oscillation time, so we must convert it to a quantity that is known or can be measured. Often the most convenient choice is the source-detector distance L. For ultra-relativistic neutrinos a reasonable approximation is to assume $t \approx L$. In addition, it is convenient to define

$$\Delta_{kj} = \frac{\Delta m_{kj}^2 L}{2E} \tag{2.33}$$

and

$$W^{kj}_{\alpha\beta} = U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}. \tag{2.34}$$

Using these notations we can approximate the oscillation probability as

$$P_{\nu_\alpha\to\nu_\beta}(E, L) = \sum_{k,j} W^{jk}_{\alpha\beta} \exp\{-i\Delta_{kj}\}. \tag{2.35}$$

There are multiple different ways to express the neutrino oscillation probability (2.35), e.g.

$$P_{\nu_\alpha\to\nu_\beta}(E, L) = \delta_{\alpha\beta} - \sum_{\substack{k,j \\ k>j}} \left(4\sin^2(\Delta_{kj}/2)\,\mathrm{Re}\left\{W^{kj}_{\alpha\beta}\right\} - 2\sin^2(\Delta_{kj}/2)\,\mathrm{Im}\left\{W^{kj}_{\alpha\beta}\right\}\right), \tag{2.36}$$

which can easily be obtained by using definitions and properties of the trigonometric functions. An interested reader can check ref. [16] for more discussion about the subject.

It is worth noting that even if neutrino oscillations imply massive neutrinos, it is not possible to get any information about the absolute masses of neutrinos (except that m_k or m_j must be greater than $\left|\Delta^2_{kj}\right|$). One can only measure the squared-mass differences Δ_{kj} between the massive states.

Majorana phases and neutrino oscillations

As we discussed in Sec.(2.1) there may be extra complex phases in the leptonic mixing matrix if neutrinos are Majorana particles. However, we will show that these phases do not affect the oscillation probabilities (2.30).

In the case of Majorana neutrinos the leptonic mixing matrix can be written as a product of a unitary matrix U^D, which is similar to the neutrino mixing matrix for Dirac neutrinos, and a diagonal unitary matrix U^M whose elements are complex phases. We can therefore write the components of the neutrino mixing matrix for Majorana neutrinos as

$$U^M_{\alpha k} = U^D_{\alpha k} e^{i\phi_k} = U_{\alpha k} e^{i\phi_k}, \tag{2.37}$$

where the upstairs index M indicates that we are talking about Majorana neutrinos. Since the oscillation probability in eq. (2.30) depends on the mixing matrix only through $W^{kj}_{\alpha\beta}$, we can easily find out the effects of the Majorana phases.

$$W^{kj,M}_{\alpha\beta} = U^{*M}_{\alpha k} U^M_{\beta k} U^M_{\alpha j} U^{*M}_{\beta j} = U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} = W^{kj}_{\alpha\beta}. \tag{2.38}$$

From here it follows that the Majorana phases do not affect the oscillation probability at all. On the other hand, we can not get any information about the Majorana phases by studying neutrino oscillations. These statements hold generally for N neutrino flavors and also for oscillations in matter, which will be discussed in chapter (3).

Antineutrino case

Flavor neutrinos are produced in the weak interaction processes through the charged current. Antineutrinos $(\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau)$ are produced similarly in the CC weak interaction processes, but from antileptons ℓ^+_α in transitions $\ell^+_\alpha \to \bar{\nu}_\alpha$ or in pair creation processes

together with lepton ℓ_α^- (i.e. in creation of $\ell_\alpha^-\bar{\nu}_\alpha$ pair). Kinematics of massive antineutrinos are equivalent to neutrinos and the mixing of antineutrinos is described by eq. (2.25), so the derivation of the oscillation probability for antineutrinos proceeds identically as in the case of neutrinos. The only difference comes from the mixing relation, that is the complex conjugation of the mixing matrix. We can therefore immediately write down the oscillation probability for the antineutrinos:

$$P_{\bar{\nu}_\alpha\to\bar{\nu}_\beta}(E,L) = \sum_{k,j}(W_{\alpha\beta}^{jk})^* \exp\{-i\Delta_{kj}\}. \tag{2.39}$$

It is instructive to express the oscillation probability in a similar form as in eq. (2.36):

$$P_{\bar{\nu}_\alpha\to\bar{\nu}_\beta}(E,L) = \delta_{\alpha\beta} - \sum_{\substack{k,j\\k>j}} \Big(4\sin^2(\Delta_{kj}/2)\,\mathrm{Re}\Big\{W_{\alpha\beta}^{kj}\Big\} + 2\sin^2(\Delta_{kj}/2)\,\mathrm{Im}\Big\{W_{\alpha\beta}^{kj}\Big\}\Big). \tag{2.40}$$

We can now see that the oscillation probability for antineutrinos in eq. (2.40) differs from the neutrino oscillation probability (2.36) only by the sign of the imaginary part.

2.3 Two neutrino mixing and oscillations

Lets consider as an example the simplest possible neutrino oscillation scenario: assume that there exist only two flavor neutrinos ν_α and ν_β, where $\alpha \neq \beta$. Here the flavor states can be pure flavor states ($\alpha, \beta = e, \mu$ or τ) or a linear combination of them, for instance $\nu_\alpha = c_e\nu_e + c_\mu\nu_\mu$. When $N = 2$, we see from eqs. (2.21) and (2.20) that the mixing matrix can be parametrized by one mixing angle, say θ. Since the flavor states are linear superpositions of the two massive states ν_1 and ν_2, there exist just one squared mass difference:

$$\Delta m^2 \equiv \Delta m^2_{21} = m_2^2 - m_1^2. \tag{2.41}$$

Here we defined ν_2 to be the heavier one of the two mass states in order to have $\Delta m^2 > 0$.

We can choose the mixing matrix to be

$$U = \begin{pmatrix} U_{\alpha 1} & U_{\alpha 2} \\ U_{\beta 1} & U_{\beta 2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.42}$$

where $0 \leq \theta \leq \pi/2$. Now it is straightforward to calculate the oscillation probability from eq. ((2.36):

$$P_{\nu_\alpha \to \nu_\beta}(E, L) = \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2 L}{4E}\right). \tag{2.43}$$

One can define the oscillation length L_{kj}^{osc} to be

$$L_{kj}^{\text{osc}} = \frac{4\pi E}{\Delta m_{kj}^2}, \tag{2.44}$$

which is the distance at which the phase generated by Δm_{kj} becomes equal to 2π. Using this definition we can write the oscillation probability as

$$P_{\nu_\alpha \to \nu_\beta}(E, L) = \sin^2(2\theta) \sin^2\left(\frac{\pi L}{L^{\text{osc}}}\right). \tag{2.45}$$

From here it is obvious that $\sin^2(2\theta)$ is the amplitude of the oscillation and $\pi L/L^{\text{osc}}$ is the oscillation phase.

Once we know the transition probability it is easy to find out the survival probability $P_{\nu_\alpha \to \nu_\alpha}(E, L)$, i.e. the probability that neutrino does not change it's flavor during propagation from a source to a detector. One can use the unitarity of the transition probability and immediately get

$$P_{\nu_\alpha \to \nu_\alpha}(E, L) = 1 - P_{\nu_\alpha \to \nu_\beta}(E, L) = 1 - \sin^2(2\theta) \sin^2\left(\frac{\pi L}{L^{\text{osc}}}\right). \tag{2.46}$$

The above treatment of neutrino oscillations using two neutrino flavors is just an approximation. However, many detectors are not sensitive to three-neutrino mixing and therefore the data can be analyzed by using effective model with two-neutrino mixing, i.e using the results which were derived in this section. On the other hand, if the experiment is sensitive to three-neutrino mixing, one needs to take into account all three neutrino flavors. This makes the analysis much more complicated, since as we have shown in eqs. (2.20) and (2.21) the number of mixing parameters grows quickly when more neutrino flavors are added.

3 Neutrinos in medium

Until now, we have considered neutrinos in a vacuum, i.e. there are no particles which could interact with neutrinos and therefore affect their propagation. This is a good approximation when neutrinos propagate for example in air, since neutrinos interact with matter very weakly. However, when the medium in which neutrinos propagate is dense, like neutron stars, the situation is different and matter can affect significantly the propagation of neutrinos, especially to neutrino oscillations.

In 1978 L. Wolfstein discovered that neutrinos propagating in (constant density) matter are subject to a potential which is caused by coherent forward scattering of neutrinos from the medium [18]. This potential is equivalent to an index of refraction and it affects the neutrino mixing: the vacuum mixing angles are replaced by effective matter mixing angles [19, 20]. In the mid 1980's S.P. Mikheyev and A.Yu. Smirnov discovered that when neutrinos propagate in matter with varying density there exist resonance at which the effective mixing angle can have it's maximal value $\pi/4$, no matter whatever the vacuum mixing angle is. This effect is called the MSW-effect and it can not be ignored in situations where matter density varies widely. The solar neutrino problem is perhaps the most famous example which can be explained by the MSW-effect, for the details see refs. [21–23]. In addition to the coherent forward elastic scattering, neutrino propagation is affected by quantum damping which arises due to incoherent neutrino scatterings. The quantum damping can reduce the oscillation probability between neutrino flavors significantly.

This chapter consists of two parts. In the first part we study the refractive properties of neutrinos propagating in matter. We go through the solution of the relativistic Dirac equation very quickly and derive the matter Hamiltonian for mixing neutrinos. In addition, we discuss some of the most important matter effects of neutrino oscillation. The second part concerns about damping and how it affects neutrino oscillations. We review the density matrix formalism, which is capable of taking into account damping effects, for oscillating neutrinos in the early Universe.

3.1 Refractive properties

The following discussion summarizes ref. [24] and thus follows it closely. Let us begin by stating some notations. For a study of relativistic particles such as neutrinos, it is convenient to use the chiral representation in which the 4×4 Dirac matrices read

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{3.1}$$

with the 2×2 block matrices

$$\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma}) \qquad \text{and} \qquad \overline{\sigma}^\mu = (\mathbb{1}, -\boldsymbol{\sigma}). \tag{3.2}$$

Here σ_i's are the usual Pauli spin matrices. Using eqs. (3.1) and (3.2), we can write the chirality matrix,

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{3.3}$$

as

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{3.4}$$

Since $(\gamma^5)^2 = 1$, the eigenvalues of the chirality matrix are ± 1.

We denote the eigenfunctions of the chirality matrix with eigenvalues 1 and -1 by ψ_R and ψ_L, respectively:

$$\gamma^5\psi_R = \psi_R, \tag{3.5}$$

$$\gamma^5\psi_L = -\psi_L. \tag{3.6}$$

The chiral field ψ_R is called the right-handed field and ψ_L the left-handed field.

It is always possible to split a generic spinor ψ into it's right-handed and left-handed components:

$$\psi = \psi_R + \psi_L, \tag{3.7}$$

where

$$\psi_R = \frac{\mathbb{1} + \gamma^5}{2}\psi, \tag{3.8}$$

$$\psi_L = \frac{\mathbb{1} - \gamma^5}{2}\psi. \tag{3.9}$$

Often it is also useful to define the chirality projection operators

$$P_R \equiv \frac{\mathbb{1} + \gamma^5}{2}, \tag{3.10}$$

$$P_L \equiv \frac{\mathbb{1} - \gamma^5}{2}. \tag{3.11}$$

We are now ready to derive the Hamiltonian for oscillating neutrinos which propagate in medium. Consider neutrino scattering on electrons. Neutrinos interact with matter only through weak interaction (if we neglect gravity), that is through charged-current (CC) or neutral current (NC) reactions. The corresponding Dirac equation in momentum space reads [25]

$$((\not{p} - m_i)\delta_{ij} + \Sigma_{ij})\psi_j = 0, \tag{3.12}$$

where p is the neutrino four-momentum, i and j are flavor indices, and Σ_{ij} is the self-energy function. The lowest order thermal contributions to neutrino propagators can be obtained by calculating the one-loop self-energy diagrams shown in fig.(1). All active neutrinos are refracted in the thermal background due to the NC processes, fig.(1) parts b and c. However, in normal matter, which consist of quarks and electrons, there are no neutrinos present and we can neglect the Z-loop correction. What is more, the tadpole-correction is the same for all active neutrino flavors so it produces just an overall phase factor, and it does not affect neutrino mixing. We can conclude that the NC thermal corrections do not affect active neutrino mixing and therefore they do not affect the neutrino oscillation probabilities in normal matter. It is worth to notice that this is not the case in the early universe where neutrinos are part of the heat bath. Consider, for instance, a situation in which the asymmetries of ν_e and ν_μ are different. Then corrections arising from the Z-loops, fig.(1b), are different for ν_e and ν_μ which affect $\nu_e \leftrightarrow \nu_\mu$ mixing and this can further affect the oscillation probabilities, see ref. [26] for details.

In the case of the CC interactions it is a whole new story. Only electron neutrinos can interact with ordinary matter by coherent CC interactions. This follows simply from the conservation of the lepton number and from the fact that in normal matter there are no other charged leptons than electrons. Due to this and the above discussion about the NC interactions, we need to consider only the W-loop correction.

It is straightforward to calculate the W-loop self-energy correction starting from

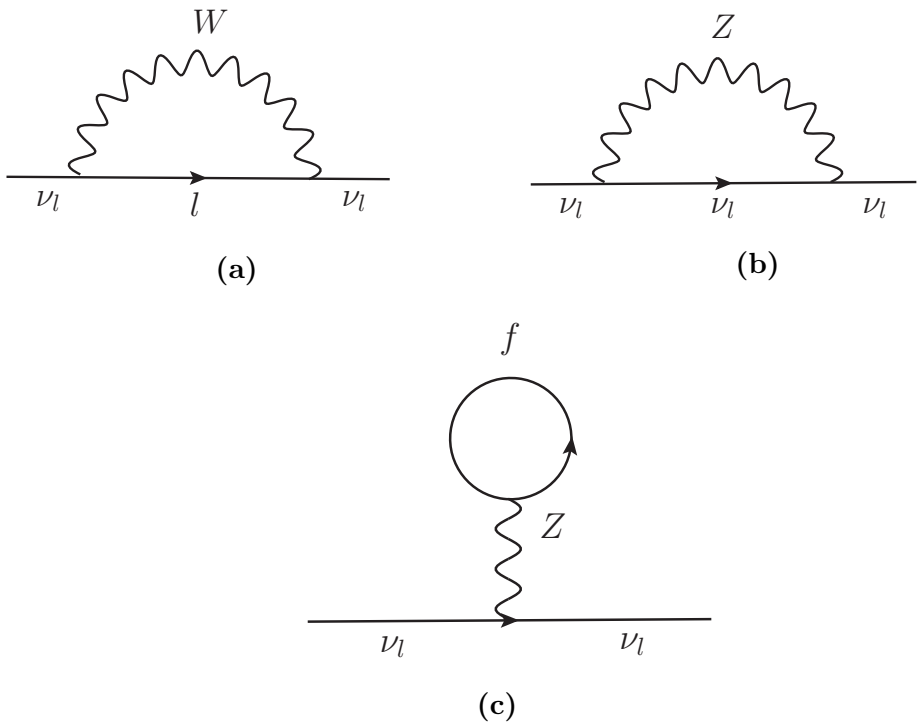

Figure (1) One-loop thermal neutrino self-energy corrections to neutrino propagators: a) W-loop, b) Z-loop, and c) Tadpole.

the low-energy CC weak interaction Lagrangian, but in this thesis we omit the exact derivation and give just the result:

$$\Sigma_{ij} = V_{eij}\frac{1}{2}\gamma^0(\mathbb{1} - \gamma^5), \tag{3.13}$$

with

$$V_{eij} = U_{ei}U^*_{ej}V_e = U_{ei}U^*_{ej}\sqrt{2}G_F n_e, \tag{3.14}$$

where n_e is the electron number density. Using the exact form of the self-energy function (3.13), the Dirac equation (3.12) can be written as

$$\Big[(\not{p} - m_i)\delta_{ij} + V_{eij}\frac{1}{2}\gamma^0(\mathbb{1} - \gamma^5)\Big]\psi_j = 0, \tag{3.15}$$

$$\Leftrightarrow \quad \Big[(\gamma^0 p_0 + \gamma^l p_l - m_i)\delta_{ij} + V_{eij}\frac{1}{2}\gamma^0(\mathbb{1} - \gamma^5)\Big]\begin{pmatrix}\psi_{Lk}\\ \psi_{Rk}\end{pmatrix} = 0, \tag{3.16}$$

We can write eq. (3.16) alternatively as a pair of equations:

$$\begin{cases} (p_0 \mathbb{1}_2 + \sigma \cdot \mathbf{p})\psi_{Lj} + V_{eij}\psi_{Lj} - m_j \mathbb{1}_2 \psi_{Rj} = 0 \\ (p_0 \mathbb{1}_2 - \sigma \cdot \mathbf{p})\psi_{Rj} - m_j \mathbb{1}_2 \psi_{Lj} = 0. \end{cases} \tag{3.17}$$

Let ϕ_λ be the helicity eigenstates satisfying the eigenvalue equation

$$\mathbf{p} \cdot \sigma \phi_\lambda = \lambda |\mathbf{p}| \phi_\lambda, \tag{3.18}$$

where λ labels the spin state of the spinor. One can always decompose a generic four-component spinor as a tensor product of two-component spinors:

$$\psi_\lambda = \begin{pmatrix} a_R^\lambda \\ a_L^\lambda \end{pmatrix} \otimes \phi_\lambda, \tag{3.19}$$

where $a_{R/L}^\lambda$ are complex numbers. Using the eigenvalue equation (3.18) and decomposition (3.19), we can write the Dirac equation (3.17) as

$$\begin{cases} (p_0 + \lambda |\mathbf{p}|) a_{Lj}^\lambda + V_{eij} a_{Lj}^\lambda = m_j a_{Rj}^\lambda \\ (p_0 - \lambda |\mathbf{p}|) a_{Rj}^\lambda = m_j a_{Lj}^\lambda, \end{cases} \tag{3.20}$$

and with little effort further as

$$\Big[(p_0^2 - |\mathbf{p}|^2 - m_i^2)\delta_{ij} + (p_0 - \lambda |\mathbf{p}|) V_{eij} \Big] a_{Lj}^\lambda = 0, \tag{3.21}$$

when we assume that the components of $\mathbf{a}$ are real.

The dispersion relation, i.e. the energy eigenstates of the propagating neutrinos, is given by the determinant of eq. (3.21):

$$\det\Big(p_0^2 - \omega_i^2 + (p_0 - \lambda |\mathbf{p}|) V_{eij} \Big) = 0 \tag{3.22}$$

with $\omega_i^2 = |\mathbf{p}|^2 + m_i^2$.

To avoid unnecessary mathematical complexity consider mixing between two neutrino flavors. We choose the mixing matrix to be

$$U = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{3.23}$$

where θ is the vacuum mixing angle. From the definition of V_{eij} (3.14) it follows that

$$\begin{aligned} V_{e11} &= \cos^2(\theta)V_e, \qquad V_{e22} = -\sin^2(\theta)V_e, \\ V_{e12} &= V_{e21} = \sin(\theta)\cos(\theta)V_e = \frac{1}{2}\sin(2\theta)V_e. \end{aligned} \tag{3.24}$$

Since neutrinos are ultra-relativistic particles we can approximate $p_0 \approx |\mathbf{p}|$, and because there are only left-handed neutrinos in the SM we can set $\lambda = -1$. Using these we obtain from eq. (3.22) that

$$\det\Big(p_0^2 - |\mathbf{p}|^2 - m_j^2 + (p_0 + |\mathbf{p}|)V_{eij}\Big) = 0, \tag{3.25}$$

$$\begin{aligned} \Leftrightarrow \quad (p_0 - |\mathbf{p}|)^2 - (p_0 - |\mathbf{p}|)\Big(\frac{-m_1^2 - m_2^2}{2|\mathbf{p}|} + V_{e11} + V_{e22}\Big) \\ + \Big(V_{e11} - \frac{m_1^2}{2|\mathbf{p}|}\Big)\Big(V_{e22} - \frac{m_2^2}{2|\mathbf{p}|}\Big) - V_{11}V_{22} = 0, \end{aligned} \tag{3.26}$$

of which solutions are

$$p_0 - |\mathbf{p}| = \frac{m_1^2 + m_2^2}{4|\mathbf{p}|} - \frac{V_{e11} + V_{e22}}{2} \pm \left[\Big(\frac{\delta m^2}{2|\mathbf{p}|} + V_{e22} - V_{e11}\Big)^2 + 4V_{e12}V_{e21}\right]^{1/2}. \tag{3.27}$$

Here we used a shorthand notation for the squared mass differences: $\delta m^2 = m_1^2 - m_2^2$. From eq. (3.27) we can immediately read off the energy eigenvalues in matter:

$$E_{1,2}^m = |\mathbf{p}| + \frac{m_1^2 + m_2^2}{4|\mathbf{p}|} - \frac{V_{e11} + V_{e22}}{2} \pm \left[\Big(\frac{\delta m^2}{2|\mathbf{p}|} + V_{e22} - V_{e11}\Big)^2 + 4V_{e12}V_{e21}\right]^{1/2}. \tag{3.28}$$

Especially, we notice that

$$\begin{aligned} E_2^m - E_1^m &= \left[\Big(\frac{\delta m^2}{2|\mathbf{p}|} + V_{e22} - V_{e11}\Big)^2 + 4V_{e12}V_{e21}\right]^{1/2} \\ &= \left[(V_e - \Delta\cos(2\theta))^2 + \Delta^2\sin^2(2\theta)\right]^{1/2} \equiv \Delta_m \end{aligned} \tag{3.29}$$

with $\Delta = \delta m^2/2|\mathbf{p}|$. Now we can express the matter eigenenergies in a compact form:

$$E_{1,2}^m = |\mathbf{p}| + \frac{m_1^2 + m_2^2}{4|\mathbf{p}|} - \frac{V_{e11} + V_{e22}}{2} \pm \Delta_m, \tag{3.30}$$

and what is more, we can write eq. (3.21) as

$$E_j^m a_{Lj}^\lambda \approx \Big[\Big(|\mathbf{p}| + \frac{m_i^2}{2|\mathbf{p}|}\Big)\delta_{ij} - V_{eij}\Big] a_{Lj}^\lambda, \tag{3.31}$$

where E_j^m is given by eq. (3.28). It is possible to write eq. (3.31) also in a matrix form:

$$\mathcal{H}^m \left|\nu_i\right\rangle = E^m \left|\nu_i\right\rangle, \tag{3.32}$$

where

$$E^m = \begin{pmatrix} E_1^m & 0 \\ 0 & E_2^m \end{pmatrix} \quad \text{and} \quad \mathcal{H}^m = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} - U \begin{pmatrix} V_e & 0 \\ 0 & 0 \end{pmatrix} U^\dagger \tag{3.33}$$

with U as the mixing matrix. From here it is easy to recognize that eq. (3.32) is just the usual Schrödinger equation and we can directly read off the Hamiltonian for neutrinos propagating in matter.

Even if we considered only two flavor mixing, the derivation of the Schrödinger equation in the case of N neutrino flavors proceeds exactly as above. The only difference is that the energy eigenstates have to be computed numerically. We also ignored terms originating from the NC processes, but we could have easily taken those terms into account. They would just add one more term to the Hamiltonian:

$$-U \begin{pmatrix} V_i^{NC} & 0 \\ 0 & V_j^{NC} \end{pmatrix} U^\dagger. \tag{3.34}$$

When neutrinos propagate in medium there exist important damping effects which we have omitted so far, for example the coherence damping. These damping effects cause inter alia that massive neutrino states, which have the same momentum, can have a continuous distribution of possible energy eigenvalues instead of just a few discrete eigenstates, as one could expect. However, in practice one has to assume that there exists only discrete energy eigenstates or the problem would become remarkably more complicated. States corresponding to the discrete eigenenergies are known as quasistates, and in the spirit the approximation scheme is know as the quasiparticle approximation. In sec.(5) we will discuss more about the quasiparticle approximation. Furthermore, the coherence damping, which arises due to incoherent neutrino collisions, can have significant impact on neutrino oscillation and it is

discussed in sec.(3.2).

Before moving to investigate quantum damping, lets let us analyze in more detail the dispersive matter effects on neutrino propagation. One of the main points of analyzing the time evolution of neutrinos is the choice of basis in which one works. Equation (3.32) is expressed in the flavor basis, but when investigating neutrinos in matter it is more convenient to go to a new basis called by the matter (eigenstate) basis $\left\{ |\tilde{\nu}_I\rangle \right\}$, i.e. the (effective) mass eigenbasis in matter. It is defined by demanding that it diagonalizes the full matter Hamiltonian in eq. (3.32). These matter eigenstates are related to the flavor states by a unitary matrix in a similar way as in vacuum:

$$|\nu_i\rangle = \sum_K \tilde{U}^*_{iK} |\tilde{\nu}_K\rangle \,, \tag{3.35}$$

where the capital letters denote the matter eigenstates, and the unitary matrix $\tilde{U}$ is called the effective leptonic mixing matrix and it can be parameterized in a similar manner as in vacuum. The only difference between U and $\tilde{U}$ is that in the effective leptonic mixing matrix the mixing parameters are replaced by effective parameters which are denoted by tildes, e.g. $\tilde{\theta}$. [1] The matter Hamiltonian in the new basis can be obtained by transformation

$$\tilde{\mathcal{H}}^m = \tilde{U}^\dagger \mathcal{H}^m \tilde{U}, \tag{3.36}$$

where

$$\tilde{U} = \begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix}, \tag{3.37}$$

since we are considering two neutrino mixing. From the diagonalization of the matter Hamiltonian, i.e. from eq. (3.36), one obtains relation for the effective mixing angle:

$$\begin{aligned} \sin^2(2\tilde{\theta}) =& \frac{\Delta^2 \sin^2(2\theta)}{(\Delta\cos(2\theta) - V_e)^2 + \Delta^2 \sin^2(2\theta)} \\ =& \left(\frac{\Delta}{\Delta_m}\right)^2 \sin^2(2\theta). \end{aligned} \tag{3.38}$$

[1] We switched our notation for the flavor eigenstates and the effective mass eigenstates. We are studying neutrino propagation in matter also in the following chapters, but we are using Greek letters to denote the Dirac indices. Therefore, we changed our notation already in this chapter in order to make it less confusing.

An interesting consequence of eq. (3.38) is that the effective mixing angle can have any value regardless of the vacuum mixing angle. Especially at resonance the effective mixing angle becomes $\pi/4$ leading to resonant condition:

$$\Delta \cos^2(2\theta) = V_e \quad \Longleftrightarrow \quad \frac{\delta m^2}{2E} \cos(2\theta) = \sqrt{2} G_F n_e. \tag{3.39}$$

This means that at resonance mixing between neutrino flavors is maximal and if the resonance region is wide enough, there can be total transitions between the two flavors considered here. This effect was discovered by Mikheev and Smirnov based on earlier work on matter effects by Wolfstein, and it is called the MSW-effect as discussed in the introduction of this section. Resonance can exist only if $\theta < \pi/4$ because V_e is positive in normal matter, we have chosen δm^2 to be positive and $\cos(2\theta) < 0$ if $\theta > \pi/4$.

On the contrary, for suitable matter densities (outside of the resonance region) there can be strong suppression of the oscillation probability caused by the matter potential. This strong suppression of the mixing is the analog of the "Turing paradox" in neutrino physics. According to the Turing paradox (or the "quantum zeno effect") quantum mechanical time evolution of a particle can be stopped by measuring the system frequently enough with respect to some chosen observable. This means that one can freeze out the system in it's initial state by measuring it frequently enough.

Similar phenomenon can happen when neutrinos propagate in medium. Every time when neutrino interacts with matter the time evolution of a neutrino state is disturbed. [2] Thus, if the average time between collisions is order of or shorter than the oscillation time (time that it takes on average from a neutrino to oscillate, e.g. change its flavor), the neutrino state can not evolve and the neutrino is frozen to some state. In other words, if coherence is slow process when compared to the interaction processes, neutrino states can not evolve but neutrinos may decay to other particles due to the interactions. It is not hard to see that this kind of process can have a huge effect on neutrino oscillation probabilities if the matter is dense enough. However, since neutrinos interact with matter very weakly, the existence of strong matter suppression requires enormous densities which can be reached only in extreme conditions, like in neutron stars or in the early universe.

[2] These interaction processes can be thought as measurements which stop the time evolution of the neutrino states.

3.2 Hard collisions and damping

When the matter Hamiltonian for oscillating neutrinos was derived in sec.(3.1), no damping effects were considered. We can not, however, neglect these phenomena in general, and they can have a huge impact on neutrino oscillations. In general damping effects can be divided into two categories: decoherence-like damping and decay-like damping. Here we will concentrate on the former one but the latter one can be added to the developed formalism. An example of the decay-like damping is the Landau damping.

In addition to the coherent forward scattering of neutrinos with the medium, there can be incoherent scatterings. In these collisions neutrinos interact with the medium in such a way that the coherent evolution of a neutrino state is interrupted, and the neutrino state collapses into some state ν_i with some specific probability. In a way one could describe these scattering events as measurements since they have similar effect on neutrino states as measurement has. If the time between these collisions is the order of or less than the oscillation time, coherent evolution of the neutrino states may be completely lost and the neutrino states are frozen to their initial values. This kind of damping is called the coherence damping.

Due to these damping effects the usual Hamiltonian formalism is not the optimal way to describe the dynamical evolution of neutrino states. Before moving to discuss the more general cQPA formalism, we consider the derivation of the density matrix formalism. It is simpler than the cQPA formalism but it is capable of taking into account finite temperature matter effects, including the damping terms. The density matrix formalism works also as an introduction to the cQPA formalism. The following discussion summarizes ref. [26] and all of the presented results are taken from there. We will also follow the notation of ref. [26] even if it differs from the notation used elsewhere in this thesis.

Density operator which describes the neutrino state (2.24) with a fixed momentum $\mathbf{k}$ is defined to be

$$\rho_{\mathbf{k}}(t) = \sum_{a,b} p_{a,b} \left|\mathbf{k}; a\right\rangle \left\langle \mathbf{k}; b\right|, \tag{3.40}$$

where a and b label the neutrino flavors and $p_{ab} = c_a c_b^*$. In general, density matrix is not diagonal, and for example the matrix elements of a fully coherent density matrix describing a pure state $|\psi\rangle = c_a |\nu_a\rangle + c_b |\nu_b\rangle$ is $\rho = |\psi\rangle \langle\psi|$ which has components $\rho_{aa} = |c_a|^2$ and $\rho_{ab} = c_a c_b^*$. According to the standard quantum mechanical interpretation

the diagonal elements give the probability for the neutrino system to be found from a specific pure state. On the other hand, the off-diagonal elements describe the degree of coherence in the system, that is they give information about the pureness of the neutrino state.

The time evolution of the density matrix can be written in terms of the time evolution operator and the density matrix at time $t = 0$:

$$\rho_{\mathbf{k}}(t) = e^{-iHt}\rho_{\mathbf{k}}(0)e^{iHt}. \tag{3.41}$$

Here H is the vacuum Hamiltonian, i.e. the first matrix in eq. (3.33) which consists of the energy eigenstates. From equation (3.41) it follows that if we know the density matrix at some instant of time, we can figure it out at any later time t.

Above the density matrix describes only neutrino states. However, we are considering situation in which neutrinos propagate in medium. This means that we have to define a more general density matrix which takes into account the thermal background:

$$\rho_{\mathrm{tot}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 E_{\mathbf{k}}} n_\nu(\mathbf{k}, T)\rho_{\mathbf{k}} \otimes \rho_{\mathrm{bg}}, \tag{3.42}$$

where $\rho_{\mathbf{k}}$ is given by eq. (3.40), $n_\nu(\mathbf{k}, T)$ is the thermal distribution function of neutrinos and ρ_{bg} is the density matrix which describes the rest of the system. In eq. (3.42) we assumed that the background is in thermal equilibrium, and there is neither coherence between different $\mathbf{k}$ states nor in the background nor between the neutrino states and the background. The background density matrix is defined as

$$\rho_{\mathrm{bg}} = \int \mathrm{d}\alpha \, |\alpha\rangle \langle\alpha| \,, \tag{3.43}$$

where the integral contains both the discrete quantum numbers, like spins, and the continuous quantum numbers, for example the momentums of the background particles. In other words, ρ_{bg} contains all the rest of the degrees of freedom which are not included in the neutrino density matrix $\rho_{\mathbf{k}}$. It is convenient to normalize the full density matrix as

$$\mathrm{Tr}[\rho_{\mathrm{tot}}] = N_\nu(T) \sum_i N_i(T), \tag{3.44}$$

where $N_\nu(T)$ is the number of the neutrino states and $N_i(T)$ is the particle number of species i.

Our goal is to figure out the evolution of the average neutrino flavor content which can be obtained from relation

$$\langle p_{ab} \rangle = \frac{1}{N_\nu(T)} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 E_\mathbf{k}} n_\nu(\mathbf{k}, T) p_{ab}(\mathbf{k}). \tag{3.45}$$

The first step towards the solution of eq. (3.45) is to define the full density matrix for a fixed momentum:

$$\rho_{\mathrm{tot}}(\mathbf{k}) = \rho_\mathbf{k} \otimes \rho_{\mathrm{bg}}. \tag{3.46}$$

Next we notice that since the neutrino flavor is conserved in elastic collisions, the full density matrix can be written after an elastic collision as

$$\rho'_{\mathrm{tot}}(\mathbf{k}) = \sum_{a,b} p_{ab}(\mathbf{k}) \int \mathrm{d}\alpha \, S_a \left| a; \mathbf{k}, \alpha \right\rangle \left\langle \alpha, \mathbf{k}; b \right| S_b^\dagger, \tag{3.47}$$

where $S_{a/b}$ is the usual scattering matrix acting on subspaces a/b with unitarity relation $S_a S_a^\dagger = 1$, and the neutrino states were combined with the background states. However, one can write these states separately whenever it is needed. The flavour oscillations are described by the reduced density matrix $\rho'(\mathbf{k})$ which is obtained by taking projections of eq. (3.47) with respect to the neutrino momentum and the background degrees of freedom. The resulting equation is

$$\rho'(\mathbf{k}) = \sum_{a,b} p_{ab}(\mathbf{k}) \Big[\int \mathrm{d}\alpha \left\langle \mathbf{k}, \alpha \right| S_b^\dagger S_a \left| \mathbf{k}, \alpha \right\rangle \Big] \left| a \right\rangle \left\langle b \right|. \tag{3.48}$$

Lets consider now a special case in the early Universe and assume that there exist just two neutrino flavors. One of these flavors is active and the other one is sterile. This specific situation is a really interesting one since active-sterile neutrino mixing could in principle generate lepton asymmetries, which would in turn affect, for instance, the primordial nucleosynthesis. We assume now that the active neutrino is electron neutrino and it is denoted by index e while sterile neutrinos are labelled with index x. In this case $S_e = 1 + iT_e$ and $S_x = 1$, where the T-matrix contains all the interactions. Noting that $\rho_{ab}(\mathbf{k}) = 2E_\mathbf{k} \sum_i N_i(T) p_{ab}(\mathbf{k})$, one obtains the background corrections to the equation of motion for p_{ex}:

$$\frac{\mathrm{d}p_{ex}(\mathbf{k})}{\mathrm{d}t}\Big|_{\mathrm{bg}} = -\Lambda_\mathbf{k} p_{ex}(\mathbf{k}), \tag{3.49}$$

where

$$\Lambda_{\mathbf{k}} \equiv \frac{1}{E_{\mathbf{k}}} \int \mathrm{d}\alpha \, \langle k, \alpha | \, iT_e \, | k, \alpha \rangle \, . \tag{3.50}$$

Since the early Universe can be considered as a dilute gas, we can approximate that the background consists of one particle states. This means that the T-matrix describes 2-2 forward elastic scattering processes. When one remembers that the integral in eq. (3.50) sums over the spins and momenta of the one particle states, it follows then from the optical theorem that

$$\Lambda_{\mathbf{k}} = \sum_i N_i(T) \Big(\frac{1}{2} \langle v_{\mathrm{rel}} \sigma^i(\mathbf{k}) \rangle + i \langle \mathrm{Re} \Big\{ T_e^i(\mathbf{k}) \Big\} \rangle \Big). \tag{3.51}$$

Here σ^i is the elastic cross section, v_{rel} is the relative speed between the neutrinos and the background particles, the sum is taken over all of the particle species present in the background, and the angle brackets denote thermal averaging.

It can be shown that the imaginary part of Λ corresponds to the background correction of the self-energy function. This self-energy function can be absorbed complete into the Hamiltonian when considering the equation of motion for the density matrix. This is in complete analogy to a situation from which we have already seen an example of in this thesis: the neutrino vacuum Hamiltonian transforms into the matter Hamiltonian when we absorb the self-energy function into it. The real part of Λ, however, can not be absorbed into the Hamiltonian. This part corresponds to the inelastic scattering of neutrinos and, as discussed earlier in this section, it is responsible for the damping of the oscillations.

In the derivation of eq. (3.49) it was assumed that all of the neutrino collisions were elastic. It turns out that this approximation is not, after all, necessary: since the system is in thermal equilibrium, it follows that after each inelastic collision there must be another inelastic collision to maintain the equilibrium. The combination of these two collisions is then effectively equal to one elastic collision.

At this point we have derived most of the results which are needed to obtain the time evolution of the density matrix describing neutrinos propagating in medium. Our next task would be to specify the interactions and then find out the average flavor of a neutrino state by using eq. (3.45). However, let us consider a simpler situation by making the replacement

$$\langle p_{ab}(\mathbf{k}) \rangle \to p_{ab}(\langle \mathbf{k} \rangle), \tag{3.52}$$

i.e. we assume that the ensemble average evolves and is well represented by the evolution of the mean value. In the early Universe this turns out to be a sufficient approximation for our purposes. From this approximation it follows that in eq. (3.51) we have to also make the replacement

$$\langle v_{\rm rel}\sigma(\mathbf{k})\rangle \to v_{\rm rel}\sigma(\langle s\rangle = 2\langle\omega_{\mathbf{k}}\rangle^2) = \overline{\sigma}, \tag{3.53}$$

where $\langle\omega_{\mathbf{k}}\rangle$ is the neutrino energy eigenstate in matter.

It is more convenient to parametrize the density matrix in terms of the polarization vector $\mathbf{P}$. The matrix p_{ab} can be written in terms of $\mathbf{P}$ as

$$p_{ab}(t) = \frac{1}{2}(\mathbb{1} + \mathbf{P}(t)\cdot\boldsymbol{\sigma})_{ab}. \tag{3.54}$$

The polarization vector can be related to the flavor content of a neutrino state: $P_z = 1$ corresponds to a situation in which the state consists only of ν_e while $P_z = -1$ implies that the state is pure ν_x state. In other words, P_z gives the excess of electron neutrinos over sterile neutrinos in the sate. For this reason (in the flavor basis) the vector $\mathbf{P}$ is called the polarization vector. The dynamical evolution of the polarization vector is described by equation [3]

$$\frac{\mathrm{d}\mathbf{P}(\mathbf{t})}{\mathrm{d}t} = \mathbf{V}\times\mathbf{P} - D\mathbf{P}_T, \tag{3.55}$$

where the last term in the right hand side arises due to quantum damping. It is possible to get a relation between the matter Hamiltonian and the vector $\mathbf{V}$:

$$\mathcal{H} = \frac{1}{2}\mathbf{V}\cdot\boldsymbol{\sigma}. \tag{3.56}$$

Moreover, $\mathbf{V}$ can be divided into vacuum and medium parts:

$$\begin{aligned}\mathbf{V}_{\rm vac} &= \Delta_0\sin 2\theta_0\hat{\mathbf{e}}_x - \Delta_0\cos 2\theta_0\hat{\mathbf{e}}_z,\\ \mathbf{V}_{\rm med} &= V_e\hat{\mathbf{e}},\end{aligned} \tag{3.57}$$

where V_e is the effective energy of the electron neutrino in the early Universe, see eq. (2.23) in ref. [26]. Lastly, the damping factor can be straightforwardly read out from

[3] The damping term is proportional to the transverse component of the polarization vector since neutrino interactions with the background particles are in practice flavor diagonal.

eq. (3.51):

$$D = \sum_i \frac{1}{2} N_i(T)\overline{\sigma_{\mathbf{i}}}(T). \tag{3.58}$$

What is left to do is to determine the exact form of the damping factor D. Nowadays there exist machine-readable numerical data files where the damping factor has been calculated at a wide range of temperatures, see ref. [27] and references therein. Alternatively, one can reasonably approximate the scale of the damping factor in small temperature scales, for example in [26] it is approximated as

$$D \approx 0.25 G_F^2 T^5 \tag{3.59}$$

for $m_e \lesssim T \lesssim m_\mu$.

Now we have everything that is needed to figure out the density matrix describing oscillating neutrinos in medium. We do not perform the exact calculation here, but the process is simple: First, one solves the dynamical evolution of the polarization vector using eq. (3.55). Then, the evolution of the average neutrino flavor is figured out by using eqs. (3.45) and (3.52) - (3.54). Numerical results of this problem can be found for example from ref. [26].

In the last couple of pages we have introduced a general formalism which describes neutrino oscillations in matter while including quantum damping effects. In ref. [26][p.771-773] an instructive example is presented which explains in terms of **P** and **V** vectors how the neutrino states evolve, how the MSW-resonance arises, and the effects of damping. We do not go through this example here, but we present the main conclusions shortly. Recall that $P_z = 1$ corresponds to a situation in which the state is pure ν_e state and similarly $P_z = -1$ implies that the state is pure ν_x state. In this case the polarization vector **P** precesses around **V** as time passes and the MSW-resonance occurs when **V** is perpendicular to the z-axis.

The consequence of damping is that the transverse part of the polarization vector $\mathbf{P}_T$ shrinks. A situation in which the damping parameter D is large or the system is under it's influence long time, the coherent evolution of the system is lost and the oscillation is disturbed. This corresponds to a situation in which the polarization vector is parallel to the z-axis, i.e. $P_z = \pm 1$, and the fractions of the ν_e and ν_x states are frozen to some fixed values which are given by P_z. This is exactly the same situation about which we discussed in the beginning of this section, but now it is expressed in terms of the vectors **P** and **V**.

4 The Quantum transport theory approach

Quantum field theory (QFT) is a combination of quantum mechanics, classical field theory and special relativity. When one talks about QFT, one is often referring to the standard QFT in vacuum. If the energy scales of the particles under investigation are much larger than the temperature of the system and the density of the medium is low (that is $\mu \ll k$ and $T \ll k$, where μ is the chemical potential, T is the temperature of the heat bath and k is the momentum of the particles), interactions between the particles and their surroundings can be neglected and the system is treated in zero temperature limit. This is the heart of the vacuum QFT and makes it convenient to study transition amplitudes which describe the system completely. However, there are many situations in which thermal or finite density effects cannot be neglected, for instance in cosmology and in the theory of heavy ion collisions. This means that new kinds of methods are needed to handle these cases.

Finite temperature field theory or thermal field theory (FTFT) composes of methods which take into account effects resulting from finite temperature. In FTFT one is interested in thermal expectation values of observables rather than the transition probabilities. There are two formulations of FTFT which are widely used: the imaginary time and the real time formalism.

The imaginary time formalism, also known as the Matsubara formalism, is the oldest and the most used formulation of the field theory at finite temperature. It is based on the notion that a statistical ensemble in equilibrium at finite temperature can be described by a partition function, which is fully determined by the known density functional $\hat{\rho} = exp\{-\beta\hat{\mathcal{H}}\}$, and expressed in terms of path integrals. In the end real-time observables are obtained by analytic continuation. A more detailed discussion about the Matsubara formalism can be found e.g. from refs. [28, 29].

In this thesis we are mainly interested in systems out-of-equilibrium. However, the imaginary time formalism can not be used to study non-equilibrium situations. This follows from the fact that in general it is not possible to form a partition function at finite temperature describing out-of-equilibrium systems, because then the underlying density functional $\hat{\rho}$ is not known. On the contrary, the real time

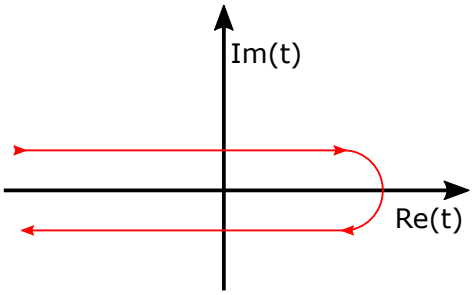

Figure (2) The Keldysh path in complex time.

formalism can be made to apply also to systems out-of-equilibrium. In addition, one benefit of the real time formalism is that there is no need to perform analytic continuations to obtain physical (real time) observables. There are a few different ways to formulate the real time theory, but here we will consider only the closed time path (CTP) formalism based on the Keldysh time path shown in fig.(2). An interested reader can check for instance refs. [28–30] for more thorough treatment of the subject.

In this chapter we firstly introduce the CTP formalism in terms of two-point functions and investigate some useful properties of these correlators. After this, we introduce the contour Schwinger-Dyson equation and derive the Kadanof-Baym (KB) equations from it. Lastly, we express the KB equations in Wigner space and rewrite them in an instructive form from the viewpoint of gradient expansion. This chapter will mainly follow the outlines and notations of refs. [31–33].

4.1 CTP formalism and Schwinger-Dyson equation

The CTP (or Schwinger-Keldysh) formalism was developed by Schwinger [34] and Keldysh [35]. The main point of the CTP formalism is that the real time variable is extended to a closed time path from some initial time t_{in} (often taken to be at $-\infty$) to final time t_f (often taken to be at ∞) and then back to t_{in}. This defines the so called Schwinger-Keldysh path in complex time which is shown in fig.(2), and makes it possible to study expectation values instead of the transition probabilities. In other words, in the CTP formalism we are studying "in-in" correlators instead of the standard QFT "in-out" correlators.[4]

In this thesis, and in QFT generally, the 2-point functions (Green's functions)

[4]By the "in-in" correlators we mean that the expectation values are $\langle \text{in}| A |\text{in}\rangle$ instead of the standard QFT "in-out" transition amplitudes $\langle \text{in}| A |\text{out}\rangle$.

are of special interest. We define a path ordered 2-point function (propagator) along the closed time path $\mathcal{C}$:

$$iS_{ij,\alpha\beta}(u,v) = \langle \mathcal{T}_C[\psi_{i,\alpha}(u)\bar{\psi}_{j,\beta}(v)]\rangle \equiv \mathrm{Tr}\Big\{\hat{\rho}\mathcal{T}_C[\psi_{i,\alpha}(u)\bar{\psi}_{j,\beta}(v)]\Big\}, \tag{4.1}$$

where ψ is the fermionic field, u and v are complex variables along the Schwinger-Keldys path, $\hat{\rho}$ is some unknown density operator which describes properties of the system and $\mathcal{T}_C$ defines time ordering along the contour $\mathcal{C}$. Time ordering in the contour $\mathcal{C}$ is such that the upper branch $\mathcal{C}_+$ in fig.(2) is earlier in time than the lower branch $\mathcal{C}_-$. When we express the 2-point correlator (4.1) in terms of real time variables, which run from t_{in} to t_{f}, it splits into four different parts (we will suppress the flavor indices (i, j) and the Dirac indices (α, β) when there is no risk of confusion):

$$\begin{aligned}
iS^{<}(u,v) &\equiv -iS^{+-}(u,v) \equiv \langle\bar{\psi}(v)\psi(u)\rangle,\\
iS^{>}(u,v) &\equiv -iS^{-+}(u,v) \equiv \langle\psi(u)\bar{\psi}(v)\rangle,\\
iS^{F}(u,v) &\equiv iS^{++}(u,v) \equiv \langle\mathcal{T}[\psi(u)\bar{\psi}(v)]\rangle,\\
iS^{\bar{F}}(u,v) &\equiv iS^{--}(u,v) \equiv \langle\bar{\mathcal{T}}[\psi(u)\bar{\psi}(v)]\rangle.
\end{aligned} \tag{4.2}$$

Here $\mathcal{T}$ ($\bar{\mathcal{T}}$) is the ordinary (reversed) time ordering operator, u_0 and v_0 are real time components, and plus and minus signs indicate how the time coordinates of u and v are oriented on the contour $\mathcal{C}$. Plus sign corresponds to the upper (positive) branch and minus sign to the lower (negative) branch, for example iS^{+-} corresponds to a situation in which u_0 is on the positive branch and v_0 is on the negative brach, i.e. u_0 is earlier in time than v_0. We can recognize that S^F and $S^{\bar{F}}$ are the Feynman (chronological) and the anti-Feynman (anti-chronological) propagators which can be written as

$$\begin{aligned}
S^{++}(u,v) &= \theta(u_0 - v_0)S^{>}(u,v) - \theta(v_0 - u_0)S^{<}(u,v),\\
S^{--}(u,v) &= \theta(v_0 - u_0)S^{>}(u,v) - \theta(u_0 - v_0)S^{<}(u,v).
\end{aligned} \tag{4.3}$$

In addition, $S^{<,>}$ are called the (2-point) Wightman functions and they are related to the self-correlation of ψ between the space time points u and v. These Wightman functions are in a key role in this thesis since we can figure out the dynamical evolution of the system by investigating them.

In the following sections it turns out to be convenient to define a few more propagators (Green's functions), which we shall introduce now and list some of their useful properties. First, we define the retarded and advanced propagators:

$$\begin{aligned} S^r(u,v) &\equiv \theta(u_0 - v_0)(S^> + S^<) = S^F + S^<, \\ S^a(u,v) &\equiv -\theta(v_0 - u_0)(S^> + S^<) = S^F - S^>. \end{aligned} \tag{4.4}$$

From eqs. (4.3) and (4.4) it follows then immediately that the propagators obey hermicity relations:

$$[iS^s(u,v)\gamma^0]^\dagger = iS^s(v,u)\gamma^0, \tag{4.5}$$

and

$$[iS^r(u,v)\gamma^0]^\dagger = -iS^a(v,u)\gamma^0, \tag{4.6}$$

where $s = <, >$. The Hermicity properties of the retarded and advanced propagators suggest us to divide the 2-point function into Hermitian and Antihermitian parts as

$$S^H \equiv \frac{1}{2}(S^a + S^r) \qquad \text{and} \qquad \mathcal{A} \equiv \frac{1}{2i}(S^a - S^r) = \frac{i}{2}(S^> + S^<), \tag{4.7}$$

where $\mathcal{A}$ is called the spectral function. Using the definition of the propagators (4.4), it easy to show that S^H and $\mathcal{A}$ obey the spectral relation:

$$S^H(u,v) = -i\mathrm{sgn}(u^0 - v^0)\mathcal{A}(u,v). \tag{4.8}$$

The path ordered 2-point Green's function $S_C(u,v)$ obeys the contour Schwinger-Dyson equation [30, 36]:

$$\int_C \mathrm{d}^4 z\, S_0^{-1}(u,z)S_C(z,v) = \delta_C^{(4)}(u-v) + \int_C \mathrm{d}^4 z\, \Sigma_C(u,z)S_C(z,v), \tag{4.9}$$

where S_0^{-1} is the inverse free fermion propagator, S is the full fermion propagator (4.1), Σ is the self-energy function and the contour time delta function is defined as

$$\delta_C^{(4)}(u-v) = \delta_C(u_C^0 - v_C^0)\delta^3(\mathbf{u} - \mathbf{v}). \tag{4.10}$$

The self-energy function is not specified, but in general it couples the 2-point functions to higher n-point functions. It can be derived perturbatively for example using the

2-particle irreducible effective action. The exact form of the self-energy is [37, 38]

$$\Sigma^{ab} \equiv -iab\frac{\delta\Gamma^2[S]}{\delta S^{ba}(v,u)}, \tag{4.11}$$

where $\Gamma^{(2)}$ is the 2PI effective action and indices $a, b = +$ or $-$ refer to the position of the arguments of u and v, respectively. However, the exact form of the self-energy function is not relevant for us right now and we will discuss about it later on. Currently it is enough to know that similar decomposition as (4.2) holds for Σ_C.

4.2 Kadanoff-Baym equations

We want to write the Schwinger-Dyson equation (4.9) in a different form to separate the dynamical and spectral properties of the system. This decomposition makes the analysis of the system more transparent and easier. To begin with, we list some useful identities for the contour integral and contour time delta function which follow when we express the complex Keldysh time in terms of real time variables:

$$\int_C \mathrm{d}^4 z \to \Sigma_a a \int_{t_{\mathrm{int}}}^{\infty} \mathrm{d}z^0 \int_V \mathrm{d}^3 z\,, \tag{4.12}$$

$$\delta_C(u_C^0 - v_C^0) \to a\delta_{ab}\delta(u^0 - v^0), \tag{4.13}$$

and

$$(\not{k} + m)S_0(u,v) \equiv \delta_C(u^0 - v^0) = \sigma_3\delta(u^0 - v^0). \tag{4.14}$$

Here σ_3 is the usual Pauli matrix and $a = \pm 1$ denotes on which branch the time argument lies. The additional a factors in the integral and in the delta function appear since the lower branch $\mathcal{C}_-$ runs backward in time. Using eqs. (4.10) and (4.12)-(4.14), we can immediately write the Schwinger-Dyson equation (4.9) in a matrix form:

$$\tilde{S}_0^{-1}(u,z) * S(z,v) = \sigma_3\delta^{(4)}(u-v) + \Sigma(u,z) * \sigma_3 S(z,v), \tag{4.15}$$

where

$$\tilde{S}_0^{-1} = \begin{pmatrix} S_0^{-1} & 0 \\ 0 & S_0^{-1} \end{pmatrix}, \quad S = \begin{pmatrix} S^F & -S^< \\ S^> & S^{\bar{F}} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \Sigma^F & -\Sigma^< \\ \Sigma^> & \Sigma^{\bar{F}} \end{pmatrix}. \tag{4.16}$$

We defined a shorthand notation $*$ for the convolution integral

$$f * g = (f * g)(u, v) = f(u, z) * g(z, v) \equiv \int_{t_{\text{int}}}^{\infty} \mathrm{d}z^0 \int_V \mathrm{d}^3 z \, f(\mathbf{u}, \mathbf{z}) g(\mathbf{z}, \mathbf{v}), \tag{4.17}$$

and we also suppressed the arguments of the propagators to simplify the notation. Writing down the matrices in eq. (4.15) explicitly, we get a set of equations

$$S_0^{-1} * S^F = \Sigma^F * S^F + \Sigma^< * S^> + \delta^{(4)}, \tag{4.18}$$
$$S_0^{-1} * S^< = \Sigma^F * S^< - \Sigma^< * S^{\bar{F}}, \tag{4.19}$$
$$S_0^{-1} * S^> = \Sigma^> * S^F - \Sigma^{\bar{F}} * S^>, \tag{4.20}$$
$$S_0^{-1} * S^{\bar{F}} = -\Sigma^> * S^< - \Sigma^{\bar{F}} * S^{\bar{F}} - \delta^{(4)}. \tag{4.21}$$

It is handy to notice that we can derive alternative forms for the retarded and advanced propagators using definitions (4.4):

$$S^r = -S^{\bar{F}} + S^> \qquad \text{and} \qquad S^a = -S^{\bar{F}} - S^<, \tag{4.22}$$

which also hold for Σ. Using these definitions, we can write eq. (4.19) as

$$S_0^{-1} * S^< = \Sigma^F * S^< - \Sigma^< * S^{\bar{F}}, \tag{4.23}$$
$$\Leftrightarrow \quad S_0^{-1} * S^< = [\Sigma^r - \Sigma^<] * S^< + \Sigma^< * [S^a + S^<], \tag{4.24}$$
$$\Leftrightarrow \quad [S_0^{-1} - \Sigma^r] * S^< = \Sigma^< * S^a. \tag{4.25}$$

For eq. (4.21) we get

$$S_0^{-1} * S^{\bar{F}} = -\Sigma^> * S^< - \Sigma^{\bar{F}} * S^{\bar{F}} - \delta^{(4)}, \tag{4.26}$$
$$\Leftrightarrow \quad -S_0^{-1} * [S^< + S^a] = -[\Sigma^a + \Sigma^<] * [S^a + S^<] - \Sigma^> * S^< - \delta^{(4)}, \tag{4.27}$$
$$\Leftrightarrow \quad [S_0^{-1} - \Sigma^a] * S^a = -S_0^{-1} * S^< + S_0^{-1} * S^< - [\Sigma^r - \Sigma^a] * S^< \tag{4.28}$$
$$+ [\Sigma^< + \Sigma^>] * S^< + \delta^{(4)}, \tag{4.29}$$
$$\Leftrightarrow \quad [S_0^{-1} - \Sigma^a] * S^a = \delta^{(4)}, \tag{4.30}$$

where we used eq. (4.24). Proceeding similarly for eqs. (4.18) and (4.20), we can

write the set of equations (4.18) - (4.21) compactly as

$$\Big([S_0^{-1} - \Sigma^p] * S^p\Big)(u,v) = \delta^{(4)}(u-v) \tag{4.31}$$

and

$$\Big([S_0^{-1} - \Sigma^s] * S^a\Big)(u,v) = \Big(\Sigma^s * S^a\Big)(u-v), \tag{4.32}$$

where $p = r, a$ and as earlier $s = <, >$. Equations (4.31) and (4.32) are called the pole and the KB equations, respectively. In general, the pole equations will determine the phase space properties of the system and the KB equations describe the dynamical evolution of the system. In the classical limit the KB equations will reduce to the familiar Boltzmann equation for the phase space number density [39].

4.3 KB equations in the Wigner space

It turns out to be useful to separate the internal (microscopic) and the external (macroscopic) degrees of freedom of the system from each other. This can be done by introducing the Wigner transformation of an arbitrary 2-point function [5]:

$$F(k,x) \equiv \int \mathrm{d}^4 r\, e^{ik\cdot r} F(x + r/2, x - r/2), \tag{4.33}$$

where $x \equiv (u+v)/2$ is the average coordinate and k is the internal momentum conjugate to the relative coordinate $r \equiv u - v$ which measures the non-locality of the coherence. In other words, r corresponds to microscopic scales and x to macroscopic scales.

At this point it is convenient to define explicitly the inverse free particle's Green function S_0^{-1}. We consider the following CP-violating Lagrangian:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \bar{\psi}_L m \psi_R + \bar{\psi}_R m^* \psi_L + \mathcal{L}_{\mathrm{int}}, \tag{4.34}$$

where $m(x) = m_R(x) + i m_I(x)$ is a complex, possibly spacetime dependent, mass matrix and $\mathcal{L}_{\mathrm{int}}$ is the interaction part of the Lagrangian (the exact form of $\mathcal{L}_{\mathrm{int}}$ is

[5] At this point we take $t_{\mathrm{in}} \to -\infty$. In addition, the Wigner transformation is actually just the usual Fourier transformation with respect to the average coordinate x.

not relevant yet and it will be discussed later on). From eq. (4.34) it follows that

$$S_0^{-1}(u,v) = \delta^{(4)}(u-v)i\not{\partial}, \tag{4.35}$$

since the mass is included in the retarded/advanced self-energy functions. What we mean by this can be seen by dividing the self-energy function into real and imaginary parts:

$$\Sigma^{r,a}(k,x) = \Sigma^H(k,x) \mp i\Sigma^{\mathcal{A}}(k,x), \tag{4.36}$$

where Σ^H is the Hermitian part and Σ^A is the Antihermitian part of the self-energy function. Next we separate the singular ($k-$independent) parts of the Hermitian self-energy function from the non-singular ($k-$dependent) parts:

$$\Sigma^H(k,x) = \Sigma^H_{\rm sg}(x) + \Sigma^H_{\rm nsg}(k,x). \tag{4.37}$$

As noted above, the singular part of the self-energy function,

$$\Sigma^H_{\rm sg}(x) = m(x) + \tilde{\Sigma}^H_{\rm sg}(x), \tag{4.38}$$

contains the mass term $m(x)$ appearing in the Lagrangian (4.34). $\tilde{\Sigma}^H_{sg}(x)$ denotes other, either exactly or approximately, local corrections. For example, at one loop level $\tilde{\Sigma}^H_{sg}$ consists only of the tadpole diagram (1c).

Now we can go back to discuss about the Wigner transformations. By using eq. (4.33), we can transform the pole equations (4.31) into the mixed (Wigner) representation:

$$\Big([S_0^{-1} - \Sigma^p] * S^p\Big)(u,v) = \delta^{(4)}(u-v), \tag{4.39}$$

$$\Leftrightarrow \quad \int \mathrm{d}^4r\, e^{ik\cdot r}\Big[S_0^{-1}(u,z) * S^p(z,v) - \Sigma^p(u,z) * S^p(z,v)\Big] = \int \mathrm{d}^4r\, e^{ik\cdot r}\delta^{(4)}(u-v), \tag{4.40}$$

$$\Leftrightarrow \quad e^{-i\lozenge}\Big\{S_0^{-1}(k,x)\}\{S^p(k,x)\} - e^{-i\lozenge}\{\Sigma^p(k,x)\}\{S^p(k,x)\} = 1. \tag{4.41}$$

The Moyal product, that is the $\lozenge$-operator, is a generalization of the ordinary Poisson brackets and it is defined as

$$\lozenge\{f\}\{g\} = \frac{1}{2}[\partial_x f \cdot \partial_k g - \partial_k f \cdot \partial_x g]. \tag{4.42}$$

In the derivation of eq. (4.41) we used a useful relation for the Wightman functions [37]:

$$\int \mathrm{d}^4(u-v)\, e^{ik\cdot(u-v)} \int \mathrm{d}^4 z\, f(u,z)g(z,v) = e^{-i\Diamond}\{f(k,x)\}\{g(k,x)\}. \tag{4.43}$$

However, Moyal products are not practical for obtaining gradient expansions and for this reason we need to find an another way to write eq. (4.42).

To begin with, we calculate the Wigner transformation of S_0^{-1}. It can be obtained easily by using the explicit form of the inverse free fermionic propagator (4.35):

$$\begin{aligned} S_0^{-1}(k,x) &= \int \mathrm{d}^4 r\, e^{ik\cdot r} S_0^{-1}\Big(x+\frac{r}{2}, x-\frac{r}{2}\Big) = \int \mathrm{d}^4 r\, e^{ik\cdot r} S_0^{-1}(u,v) \\ &= \int \mathrm{d}^4 r\, \delta^{(4)}(u-v) i\not{\partial}_v e^{ik\cdot r} = \int \mathrm{d}^4 r\, \delta^{(4)}(r) \not{k} e^{ik\cdot r} = \not{k}. \end{aligned} \tag{4.44}$$

Next we examine the Moyal products more closely.

$$\begin{aligned} e^{-i\Diamond}\{f(k,x)\}\{g(k,x)\} &= \sum_{n=0}^{\infty} \frac{-i}{2^n n!} \Big(\partial_x^f \cdot \partial_k^g - \partial_k^f \cdot \partial_x^g\Big)^n f \cdot g \\ &= \sum_{n=0}^{\infty} \frac{-i}{2^n n!} \sum_{l=0}^{n} \binom{n}{l} \Big(\partial_x^f \cdot \partial_k^g\Big)^l \Big(-\partial_k^f \cdot \partial_x^g\Big)^{n-l} f \cdot g \\ &= \sum_{n=0}^{\infty} \frac{-i}{2^n n!} \sum_{l=0}^{n} \frac{n!(-1)^{n-l}}{l!(n-l)!} \Big(\partial_x^f \cdot \partial_k^g\Big)^l \Big(\partial_k^f \cdot \partial_x^g\Big)^{n-l} f \cdot g \\ &= \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} \frac{(\frac{i}{2})^{n-l}(\frac{-i}{2})^l}{l!(n-l)!} \Big(\partial_x^f \cdot \partial_k^g\Big)^l \Big(\partial_k^f \cdot \partial_x^g\Big)^{n-l} f \cdot g \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\frac{-i}{2})^l}{l!} \Big(\partial_x^f \cdot \partial_k^g\Big)^l \frac{(\frac{i}{2})^m}{m!} \Big(\partial_k^f \cdot \partial_x^g\Big)^m f \cdot g \\ &= e^{\frac{i}{2}\partial_k^f \cdot \partial_x^g} e^{\frac{-i}{2}\partial_x^f \cdot \partial_k^g} f \cdot g. \end{aligned} \tag{4.45}$$

From eq. (4.45) it can be seen that the Moyal products can also be rearranged as

$$e^{-i\Diamond}\{f(k,x)\}\{g(k,x)\} = \Big[e^{\frac{i}{2}\partial_k^f \cdot \partial_x^g} + e^{\frac{-i}{2}\partial_x^f \cdot \partial_k^g} - 1 + \{\text{mixed terms}\}\Big] f \cdot g, \tag{4.46}$$

where e.g. $(\partial_k^f \cdot \partial_x^g)(\partial_x^f \cdot \partial_k^g) \in \{\text{mixed terms}\}$.

Now we know how to handle the Moyal products, so we can write the pole

equation in the mixed representation (4.41) as

$$\Big[e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}-1+\{\text{mixed terms}\}\Big]\{\not{k}\}\{S^p(k,x)\} - e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}[\Sigma^p(k,x)S^p(k,x)]=1. \tag{4.47}$$

From here we notice that the mixed terms do not give any contribution in eq. (4.47). This follows from the fact that the mixed terms differentiate the objects inside the curly brackets with respect to k and x, and the object inside the first curly brackets is just linear in k. Due to this linearity we can also drop all terms involving $(\partial_k\partial_x)^n$ whenever $n\geq 2$. Therefore, the pole equation (4.47) reads

$$\Big[\not{k}+\frac{i}{2}\not{\partial}_x\Big]S^p(k,x)-e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}[\Sigma^p(k,x)S^p(k,x)]=1. \tag{4.48}$$

Proceeding similarly as in the case of the pole equations, we can write the KB equations (4.32) in the mixed space as

$$e^{-i\Diamond}\Big\{S_0^{-1}(k,x)\}\{S^s(k,x)\}-e^{-i\Diamond}\{\Sigma^r(k,x)\}\{S^s(k,x)\}=e^{-i\Diamond}\{\Sigma^s(k,x)\}\{S^a(k,x)\}, \tag{4.49}$$

$$\Leftrightarrow\quad \Big[\not{k}+\frac{i}{2}\not{\partial}_x\Big]S^s(k,x)-e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}[\Sigma^r(k,x)S^s(k,x)] = e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}[\Sigma^s(k,x)S^a(k,x)]. \tag{4.50}$$

Combining the results, we have shown that equations (4.31) and (4.32) read in the Wigner space as

$$\not{\hat{K}}S^p(k,x)-e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}[\Sigma^p(k,x)S^p(k,x)]=1, \tag{4.51}$$

$$\not{\hat{K}}S^s(k,x)-e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}[\Sigma^r(k,x)S^s(k,x)] = e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^S}[\Sigma^s(k,x)S^a(k,x)], \tag{4.52}$$

where $\hat{K}=k+\frac{i}{2}\partial_x$.

It turns out to be useful to introduce yet another self-energy function:

$$\Sigma_{\text{out}}(k,x) \equiv \int \mathrm{d}^4 z\, e^{ik\cdot(x-z)}\Sigma(k,z) = e^{\frac{i}{2}\partial_x^\Sigma\cdot\partial_k^\Sigma}\Sigma(k,x), \tag{4.53}$$

where the last equality can be seen as follows: First we notice that

$$\begin{aligned}
&\int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot(x-z)}\Sigma_{\text{out}}(k,x) \\
&= \int \frac{\mathrm{d}^4 k\,\mathrm{d}^4 z'}{(2\pi)^4} e^{-ik\cdot(x-z)} e^{-ik\cdot(x-z')}\Sigma(x,z) \\
&= \int \mathrm{d}^4 z'\,\Sigma(x,z') \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot(z'-z)} \\
&= \Sigma(x,z).
\end{aligned} \tag{4.54}$$

Then by using the Wigner transformation defined in eq. (4.33), where now $r = x - z$, we get

$$\begin{aligned}
&\int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot r}\Sigma_{\text{Wig}}\Big(k, \frac{x+z'}{2}\Big) \\
&= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot r} \int \mathrm{d}^4 r'\, e^{ik\cdot r'}\Sigma\Big(\frac{x+z'}{2} + \frac{r'}{2}, \frac{x+z'}{2} - \frac{r'}{2}\Big) \\
&= \int \mathrm{d}^4 r'\,\Sigma(x,z') \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot(r-r')} = \Sigma(x,z),
\end{aligned} \tag{4.55}$$

where $\Sigma_{\text{Wig}}(k,x)$ denotes the Wigner transformed self-energy function. For now we use the downstairs index *Wig* for clarity even if it is easy to see from the arguments which function is expressed in the Wigner space and which is not. Expanding the self-energy function $\Sigma(x,z)$ as a Taylor series around point $(k,(x+z)/2) = (k, x-(x-z)/2)$

and integrating by parts we obtain

$$\begin{aligned}
\Sigma(x,z) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\cdot r}\Sigma_{\mathrm{Wig}}\Big(k, \frac{x+z}{2}\Big) \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\cdot(x-z)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Big(\partial_x^n \Sigma_{\mathrm{Wig}}(k,x)\Big)\Big(\frac{x-z}{2}\Big)^n \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Big(\partial_x^n \Sigma_{\mathrm{Wig}}(k,x)\Big)\Big(\frac{i}{2}\partial_k\Big)^n e^{-ik\cdot(x-z)} \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \Big(e^{\frac{i}{2}\partial_x\partial_k}\Sigma_{\mathrm{Wig}}(k,x)\Big) e^{-ik\cdot(x-z)} \\
&\equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\cdot(x-z)}\Sigma_{\mathrm{out}}(k,x)
\end{aligned} \tag{4.56}$$

from which eq. (4.53) follows directly.

We can now return to eq. (4.50) and express it in terms of $\Sigma_{\mathrm{out}}(k,x)$. To begin with, we notice that

$$\begin{aligned}
e^{-i\Diamond}\{\Sigma_{\mathrm{Wig}}(k,x)\}\{S_{\mathrm{Wig}}(k,x)\} &= e^{\frac{i}{2}\partial_x^\Sigma\cdot\partial_k^S} e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}[\Sigma(k,x)S(k,x)] \\
&= e^{\frac{i}{2}\partial_x^\Sigma\cdot\partial_k^S} e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}\Big(e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k^\Sigma}\Sigma_{\mathrm{out}}(k,x)\Big)S(k,x) \\
&= e^{\frac{-i}{2}\partial_x^\Sigma\cdot(\partial_k^\Sigma+\partial_k^S)} e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}\Sigma_{\mathrm{out}}(k,x)S(k,x) \\
&= e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k}\Sigma_{\mathrm{out}}(k+\frac{i}{2}\partial_x^s, x)S(k,x),
\end{aligned} \tag{4.57}$$

where we used that

$$e^{\frac{i}{2}\partial_k^\Sigma\cdot\partial_x^S}\Sigma_{\mathrm{out}}(k,x) = \Sigma_{\mathrm{out}}(k+\frac{i}{2}\partial_x^s, x). \tag{4.58}$$

In other words, we just shifted (or resummed) the self-energy function. It should also be noted that the derivatives are partial derivatives, for example ∂_x^Σ acts only on the x coordinate of Σ. In addition, ∂_k is "total" derivative, since it acts both on the self-energy function Σ and the propagator S.

Finally, we can write the pole equations and the KB equations (4.51) as

$$\begin{aligned}
&\not{K} S^p(k,x) - e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k}\Big[\Sigma^p_{\mathrm{out}}(\hat{K},x)S^p(k,x)\Big] = 1, \\
&\not{K} S^s(k,x) - e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k}\Big[\Sigma^r_{\mathrm{out}}(\hat{K},x)S^s(k,x)\Big] = e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k}\Big[\Sigma^s_{\mathrm{out}}(\hat{K},x)S^a(k,x)\Big].
\end{aligned} \tag{4.59}$$

This form of the KB equations is extremely useful for obtaining finite order gradient

expansions. The utility of eq. (4.59) arises from reorganization of the gradients into total derivatives which are fully controlled by the conjugate momentum, i.e. the external variation scale, of Σ. It is also worth to point out that all spacetime gradients acting on the correlation function S are included in the shifted (resummed) k argument of the self-energy function. In chapter 5 the troublesome derivation of eq. (4.59) turns out to be useful and will save us from huge amount of work. This form of the KB equations was first derived in ref. [31].

5 The cQPA equations and the spectral limit

We are finally at a point where we can start solving the KB equations (4.59) while taking into account quantum coherence effects. However, this still is not easy since eq. (4.59) contains infinite order gradients. Thus, in order to be able to solve the set of eqs. (4.59), we need to find out some approximation scheme to truncate the gradient expansion without losing information about the quantum coherence. Before moving to discuss the actual approximation scheme that we use in this thesis, let us briefly clarify some concepts.

Quasiparticle approximation is usually understood as a series of approximations leading to discrete set of energy eigenvalues, as discussed in sec.(3.1). Otherwise stated, in the QPA scheme the energy momentum relations are definite and the phase space of the propagators S^s consists of sharp shell structures. Necessary conditions for the QPA to be valid are weak interactions, slowly (adiabatically) varying background field and translational invariant correlators, i.e. that the correlators are close to thermal equilibrium. The QPA is one of the few known approximations which simplifies quantum mechanical many-body problems, so it is extremely useful e.g. in condensed matter physics.

The coherent quasiparticle approximation (cQPA) is an extension of the standard QPA scheme, since in the cQPA one relaxes the assumption of translational invariant correlators. From this it follows that in addition to the usual mass shell solutions new kind of singular shell solutions appear which are absent in the usual QPA. These new solutions are recognized to carry information about non-local quantum coherence, for instance between particles and antiparticles. Therefore, when out-of-equilibrium systems, where quantum coherence plays a role, are studied the cQPA can be very useful tool. Examples of this kind of situations include inflation, preheating, electroweak baryogenesis, leptogenesis, and neutrino flavor oscillations. The coherent quasiparticle approximation was introduced in ref. [40] and has been further developed in refs. [31, 33, 41–45].

In this chapter we firstly examine the necessary conditions for the coherent quasiparticle approximation to hold. Secondly, we derive another form of the KB

equations using the spectral function $\mathcal{A}$ and the Antihermitian part of the self-energy function Σ^A. Thirdly, we solve the pole equations, that is the spectral structure of the phase space, in the cQPA limit while assuming adiabatic background. Lastly, we use the spectral solution as an ansatz and substitute it to the full KB equations.

5.1 Weak interactions and the mean field limit

In the context of the quasiparticle approximation the limit of weak interactions means that the interaction width is negligible and we may take the limit $\Sigma^A \to 0$ when solving the phase space structure of the propagators. In this thesis we are interested in neutrinos which do interact with matter very weakly, so this approximation is well justified. On the contrary, when one studies the dynamical evolution of the correlators, it is necessary to include Σ^A. The reason for this is that Σ^A describes how the interactions affect the thermalization of an out-of-equilibrium system, and thus it can not be resummed into the propagators, but the corresponding non-equilibrium distribution has to be solved from the dynamical equations.

In general, for obtaining a spectral phase space structure for the 2-point correlators, it is not enough to neglect terms proportional to Σ^A [32]. It is also necessary to neglect all derivatives of the background fields, except those included in the resummation, to actually get singular shell solutions. This approximation of neglecting all but the lowest (zeroth) order derivatives of the background field and the derivatives included in the resummation is called the mean field (or the adiabatic) limit. The adiabatic limit together with the assumption of the weak interactions simplify the KB equations (4.59) extremely much and they will be the key approximations in sec.(5.2).

5.2 The KB equations in the cQPA limit

When we solve the spectral properties of the correlators $S^{<,>}$ in the cQPA limit, we write the KB equations in a slightly different form than we did in sec.(4.3). To begin with, we recall equations for S^H and $\mathcal{A}$ from eq. (4.7), and similar relations for the self-energy function Σ read explicitly

$$\Sigma^H \equiv \frac{1}{2}(\Sigma^a + \Sigma^r) \qquad \text{and} \qquad \Sigma^A \equiv \frac{1}{2i}(\Sigma^a - \Sigma^r) = \frac{i}{2}(\Sigma^> + \Sigma^<). \tag{5.1}$$

Using these definitions we obtain from eq. (4.31) that

$$\Big([S_0^{-1} - \Sigma^H + i\Sigma^A] * [S^H - i\mathcal{A}]\Big)(u,v) = \delta^{(4)}(u-v) \tag{5.2}$$

and

$$\Big([S_0^{-1} - \Sigma^H - i\Sigma^A] * [S^H + i\mathcal{A}]\Big)(u,v) = \delta^{(4)}(u-v). \tag{5.3}$$

Sum of eqs. (5.2) and (5.3) yields

$$\Big([S_0^{-1} - \Sigma^H] * S^H\Big)(u,v) + \Big(\Sigma^A * \mathcal{A}\Big)(u,v) = \delta^{(4)}(u-v), \tag{5.4}$$

while subtraction of these equations gives

$$\Big([S_0^{-1} - \Sigma^H] * \mathcal{A}\Big)(u,v) - \Big(\Sigma^A * S^H\Big)(u,v) = 0. \tag{5.5}$$

In addition, from eqs. (4.7), (4.32) and (5.1) we get that

$$\Big([S_0^{-1} - \Sigma^H + i\Sigma^A] * S^<\Big)(u,v) = \Big(\Sigma^s * [S^H + i\mathcal{A}]\Big)(u,v), \tag{5.6}$$

$$\Leftrightarrow \quad \Big([S_0^{-1} - \Sigma^H] * S^<\Big)(u,v) - \Big(\Sigma^< * S^H\Big)(u,v) = \Big(\Sigma^< * i\mathcal{A} - i\Sigma^A * S^<\Big)(u,v), \tag{5.7}$$

$$\Leftrightarrow \quad \Big([S_0^{-1} - \Sigma^H] * S^<\Big)(u,v) - \Big(\Sigma^< * S^H\Big)(u,v) = \frac{1}{2}\Big(\Sigma^> * S^< - \Sigma^< * S^>\Big)(u,v). \tag{5.8}$$

Thus, we have shown that the KB equations can we written alternatively as

$$\begin{aligned} &\Big([S_0^{-1} - \Sigma^H] * \mathcal{A}\Big)(u,v) - \Big(\Sigma^A * S^H\Big)(u,v) = 0, \\ &\Big([S_0^{-1} - \Sigma^H] * S^H\Big)(u,v) + \Big(\Sigma^A * \mathcal{A}\Big)(u,v) = \delta^{(4)}(u-v), \end{aligned} \tag{5.9}$$

and

$$\Big([S_0^{-1} - \Sigma^H] * S^<\Big)(u,v) - \Big(\Sigma^< * S^H\Big)(u,v) = \frac{1}{2}\Big(\Sigma^> * S^< - \Sigma^< * S^>\Big)(u,v). \tag{5.10}$$

Here equations (5.9) are the pole equations which fix the phase space structure of the Wightman functions, and (5.10) is one of the two KB equations which give the dynamical evolution of the propagators. We do not need to consider the KB equation for the other Wightman function $S^>$ since we get it from eq. (4.3) after we have

figured out the spectral function $\mathcal{A}$.

In the mixed representation, while assuming slowly varying background fields, the pole equations (5.9) and the KB equation (5.10) can be written as (proceeding similarly as in section (4.3))

$$\begin{aligned} \hat{\not{K}}\mathcal{A}(k,x) - [\Sigma^H(\hat{K},x)\mathcal{A}(k,x) + \Sigma^A(\hat{K},x)S^H(k,x)] &= 0, \\ \hat{\not{K}}S^H(k,x) - \big[\Sigma^H(\hat{K},x)S^H(k,x) + \Sigma^A(\hat{K},x)\mathcal{A}(k,x)\big] &= 1, \end{aligned} \tag{5.11}$$

and

$$\hat{\not{K}}S^<(k,x) - [\Sigma^H(\hat{K},x)S^<(k,x) + \Sigma^<(\hat{K},x)S^H(k,x)] = \mathcal{C}_{\rm coll}, \tag{5.12}$$

where

$$\begin{aligned} \mathcal{C}_{\rm coll} &\equiv e^{-i\Diamond}\{\Sigma^>(k,t)\}\{S^<(k,t)\} - e^{-i\Diamond}\{\Sigma^<(k,x)\}\{S^>(k,x)\} \\ &= e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k}[\Sigma^>_{\rm out}(\hat{K},x)S^<(k,x)] - e^{\frac{-i}{2}\partial_x^\Sigma\cdot\partial_k}[\Sigma^<_{\rm out}(\hat{K},x)S^>(k,x)] \\ &\approx \Sigma^>(\hat{K},x)S^<(k,x) - \Sigma^<(\hat{K},x)S^>(k,x). \end{aligned} \tag{5.13}$$

Here we dropped all gradient terms except those where ∂_x is acting on $S^<$, like $\hat{\not{K}}S^p(k,x)$ term, and those self-energy terms which are obtained when $\Sigma(\hat{K},x)$ is acting on the Wightman function.

Before moving on let us verify the approximation of dropping the gradient terms in eqs. (5.11) - (5.13). We recall the full KB equations (4.59), and the fact that we are assuming adiabatic background fields. When ∂_x is acting on the coherence solution, we can estimate to a leading order that $\partial_x S \sim kS$ with $k \sim 1/\lambda$. Here λ is the de Broglie wavelenght of the neutrinos and thus $k \sim 10^5\,1/\mathrm{m}$ at the very least. It is now obvious that $\partial_x S$ can lead to a large linear shift in the k argument of the self-energy function, and therefore we have to include these gradient terms at least to lowest order. On the contrary, to a leading order we can approximate $(\partial_x\Sigma)(\partial_k S) \sim 1/(kL)\Sigma S$, where L is the external variation scale of the self-energy function. Typically L is order of kilometers, e.g. in extreme dense neutron stars it is a few kilometers and in the Earth it is dozens or hundreds of kilometers. Thus, in most situations $kL \gg 1$ and consequently $(\partial_x\Sigma)(\partial_k S)$ terms are negligible and we can drop them out. Lastly, from the adiabatic background field it follows that $(\partial_x\Sigma)(\partial_k\Sigma) \ll \Sigma$ and we can also neglect these gradient terms.

The strategy for solving equations (5.11) and (5.12) involves two steps. First, we derive the collisionless solutions in the mean field limit. Secondly, we use this spectral quasiparticle form as an ansatz to the full KB equation. By solving the full KB equation we get the dynamical evolution of the shell functions. The second step will be discussed in sections (5.3) and (5.4).

5.2.1 The spectral solution

As discussed above, we start by studying the phase space properties of the correlators. Assume now weak interactions and the mean field limit. In general $\Sigma^s \sim \Sigma^A$ and since we are considering the limit $\Sigma^A \to 0$, we have to neglect terms proportional to Σ^A also in the collision term, i.e. in the RHS of eq. (5.12). In other words, we neglect all interaction terms and consider the collisionless situation. The pole equations (5.11) can be written in the collisionless limit as

$$\begin{aligned} \hat{\not{K}}\mathcal{A}(k,x) - \Sigma^H(k,x)\mathcal{A}(k,x) = 0, \\ \hat{\not{K}}S^H(k,x) - \Sigma^H(k,x)S^H(k,x) = 1, \end{aligned} \tag{5.14}$$

while the equation for the Wightman function, that is eq. (5.12), in the collisionless situation is

$$\hat{\not{K}}S^<(k,x) - \Sigma^H(\hat{K},x)S^<(k,x) - \Sigma^<(k,x)S^H(k,x) = 0. \tag{5.15}$$

The $\Sigma^< S^H$ term in eq. (5.15) only affects the phase space properties of the correlators, and it has to be dropped consistently with the collisionless pole equations [41]. The pole and the KB equations can hence be expressed as

$$\begin{aligned} [\hat{\not{K}} - \Sigma^H(k,x)]\mathcal{A}(k,x) = 0, \\ [\hat{\not{K}} - \Sigma^H(k,x)]S^H(k,x) = 1, \end{aligned} \tag{5.16}$$

and

$$[\hat{\not{K}} - \Sigma^H(\hat{K},x)]S^<(k,x) = 0. \tag{5.17}$$

Note that in general one has $\hat{K}$ in the Σ^H term in equation (5.17). We are anticipating the fact that the pole functions do not have rapidly oscillating pieces, so that in them $\hat{K} \to k$. However, in $S^<$ such solutions exist. In the end we avoid this complication due to the fact that we only need to consider singular Σ^H functions, which do not

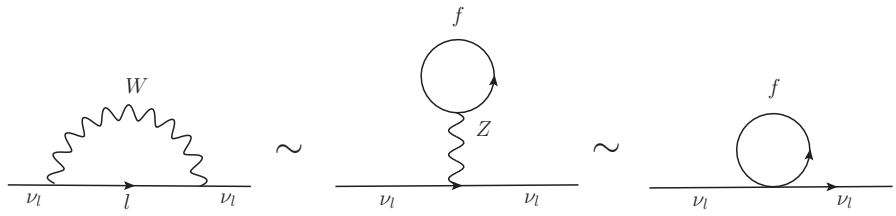

Figure (3) In a system where $T \ll M_{Z/W}$ the W-loop and the tadpole diagram become effectively equal to a tadpole-like diagram where the intermediate vector bosons are absent. Similar approximation holds for the Z-loop diagram.

have any k-dependence. Thus, the self-energy function and the propagator $S^<$ obey exactly the same kind of relations, and since equations (5.16) - (5.17) are decoupled it is then enough to solve either the self-energy function or the Wightman function and we can immediately write the solution for the other one too.

The reason why we need to consider only singular parts of the self-energy function is simple: In practice temperature of a system is usually much less than the mass of the Z and W bosons, that is $T \ll M_{Z/W}$ [6]. For this reason the non-singular (k-dependent) part of the self-energy function becomes effectively equal to the singular (k-independent) part of the self-energy function. Thus, we effectively have $\Sigma^H_{\mathrm{nsg},ij} \approx \Sigma^H_{\mathrm{sg},ij}$ where the self-energies are of the same form than in eq. (3.13). This is demonstrated at one-loop level in fig. (3).

Our goal is to find out the spectral shell solutions of the propagator and to figure out how to parametrize the Wightman function using these solutions. In practice this means that we have to solve eq. (5.17) using suitable basis matrices. The following subsections follow the outlines of ref. [46].

Multiplying eq. (5.17) from both sides by γ^0 and neglecting the non-singular parts of the self-energy function, we get

$$[k_0 + \frac{i}{2}\partial_t - \frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \boldsymbol{\alpha}\cdot\mathbf{k} - \gamma^0 m(x) - \gamma^0\tilde{\Sigma}^H_{\mathrm{sg}}(x) - \gamma^0\Sigma^H_{\mathrm{nsg}}(k,x)]\overline{S}^<(k,x) = 0, \tag{5.18}$$

$$\Leftrightarrow \quad [k_0 + \frac{i}{2}(\partial_t - \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}) - \mathcal{H}_{\mathbf{k}}(x)]\overline{S}^<(k,x) = 0, \tag{5.19}$$

[6] Example of a situation when this approximation might not be valid is the early universe, but e.g. in the sun relation $T \ll M_{Z/W}$ still holds.

where we used eq. (4.37) and defined the matter Hamiltonian: [7]

$$\mathcal{H}_{\mathbf{k},ij} = (\boldsymbol{\alpha}\cdot\mathbf{k})_i\delta_{ij} - \gamma^0 m_i\delta_{ij} - \gamma^0\tilde{\Sigma}^H_{\mathrm{sg,ij}} - \gamma^0\Sigma^H_{\mathrm{nsg},ij}. \tag{5.20}$$

Here i and j label fermion flavors, $\boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma}$ and $\overline{S}^< \equiv iS^<\gamma^0$. In addition, $\tilde{\Sigma}^H_{\mathrm{sg}}$ is the singular part and Σ^H_{nsg} is the non-singular part of the Hermitian self-energy function, as defined earlier in sec.(4.3). The ultimate goal of this thesis is to show how to derive the density matrix formalism for neutrinos propagating in matter. This means that we can not ignore any matter effects (at the level of classical limit) in order to get correct dispersion relations for the propagating neutrinos.

To carry the analysis further it is most convenient to go to a basis in which the matter Hamiltonian is diagonal, i.e. the mass eigenbasis in matter. Since the Hamiltonian is Hermitian, we can diagonalize it by a unitary transformation U. However, we are considering an adiabatic process, so the mixing matrix U depends on the spacetime points of the neutrinos. This means that we are actually working in the instantaneous mass eigenbasis in matter, or in other words in a basis which is rotating when time passes. Performing the diagonalization of the Hamiltonian leads to equation [8]

$$\begin{aligned}[k_0 + \frac{i}{2}\mathcal{D} - \mathcal{H}_{\mathbf{k}}(x)]\overline{S}^<(k,x) + \frac{i}{2}U^\dagger(x)[\mathcal{D}U(x)]\overline{S}^<(k,x) \\ + \frac{i}{2}\overline{S}^<(k,x)[\mathcal{D}U^\dagger(x)]U(x) = 0\end{aligned} \tag{5.21}$$

with

$$\mathcal{D} = \partial_t - \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}, \tag{5.22}$$

$$\mathcal{H}_{\mathbf{k},L}(x) \equiv \mathcal{H}_{\mathbf{k},L}(x)\delta_{LN} = U^\dagger_{Li}(x)\mathcal{H}_{\mathbf{k},\mathbf{ij}}(x)U_{jN}(x), \tag{5.23}$$

and

$$\overline{S}^<_{LN}(k,x) = U^\dagger_{Li}(x)\overline{S}^<_{ij}(k,x)U_{jN}(x). \tag{5.24}$$

The capital letters, e.g. L and N, denote the mass eigenstates in matter. We can

[7] Equation (5.19) is expressed in the flavor basis so due to the mass matrix the Hamiltonian is not diagonal.

[8] We do not specify the rotation matrix U explicitly, since it is a deep issue and beyond the scope of this thesis. Here it is enough to know that such matrix exists and it diagonalizes the Hamiltonian function. This topic will be discussed in ref. [47].

decompose eq. (5.21) in to two parts by using Hermicity, and the Hermitian part (H) is

$$2k^0\overline{S}^<(k,x) = \{\mathcal{H}_{\mathbf{k}}(x), \overline{S}^<(k,x)\}. \tag{5.25}$$

The (H)-equation (5.25) does not contain any derivatives, and thus it gives algebraic constraint equation for the propagator $S^<$.

In what follows we suppress the arguments k and x. It can be shown that an arbitrary complex 4×4 matrix in a spatially homogeneous and isotropic system can be expressed in terms of the set

$$\{\mathbb{1}, \gamma^0, \boldsymbol{\gamma}\cdot\hat{\mathbf{k}}, \boldsymbol{\alpha}\cdot\hat{\mathbf{k}}, \gamma^5, \gamma^0\gamma^5, \boldsymbol{\gamma}\cdot\hat{\mathbf{k}}\gamma^5, \boldsymbol{\alpha}\cdot\hat{\mathbf{k}}\gamma^5\}. \tag{5.26}$$

That is, the above set forms a basis which spans the homogeneous and isotropic subalgebra of the full Dirac algebra. Moreover, the above basis consists of elements which are helicity-diagonal, and the last basis matrix is actually the helicity operator, $\hat{h}_{\mathbf{k}} \equiv \boldsymbol{\alpha}\cdot\hat{\mathbf{k}}\gamma^5$. One could wonder why we are investigating homogeneous and isotropic system but it turns out to be extremely useful: Since we are assuming slowly varying background field, we can effectively treat the system at each point as if it were homogeneous and isotropic. The validity of this approximation is easy to see: In this thesis we are especially interested in the quantum effects, like quantum coherence, and the scale at which these effects show up is of the order of the neutrino's de Borglie wavelenght or less. In turn, importance of the matter effects can be estimated by the mean free path of the particles under investigation. The de Broglie wavelenght of neutrinos is order of 10^{-5} m while the neutrino mean free path is kilometers even in extreme dense neutron stars. Difference between these two scales is huge even if we considered extreme situation (neutron star). Thus, neutrinos propagate as free particles between infrequent collisions and we can assume the Hamiltonian to be locally helicity-diagonal. This means that the system is effectively locally homogeneous and isotropic.

It is convenient to introduce the energy and helicity projection operators:

$$P^e_{\mathbf{k},I} \equiv \frac{1}{2}\Big(\mathbb{1} + e\frac{\mathcal{H}_{\mathbf{k},I}}{\omega_{\mathbf{k},I}}\Big) \qquad \text{and} \qquad P^h_{\mathbf{k}} \equiv \frac{1}{2}\Big(\mathbb{1} + h\hat{h}_{\mathbf{k}}\Big). \tag{5.27}$$

Here $h = \pm 1$ is the helicity, $e = \pm 1$ is the energy sign index, and $\omega_{\mathbf{k},I}$ gives the energy eigenstates when the energy sign e ($+$ or $-$) in the projection matrix is defined.

It is not hard to show that the projection operators $P^e_{\mathbf{k},I}$ and $P^e_{\mathbf{k}}$ obey relations

$$\mathcal{H}_{\mathbf{k},I}P^e_{\mathbf{k},I} = P^e_{\mathbf{k},I}\mathcal{H}_{\mathbf{k},I} = e\omega_{\mathbf{k},I}P^e_{\mathbf{k},I}, \tag{5.28}$$

$$P^e_{\mathbf{k},I}P^e_{\mathbf{k},I} = P^e_{\mathbf{k},I}, \tag{5.29}$$

$$P^e_{\mathbf{k},I}P^{-e}_{\mathbf{k},I} = 0, \tag{5.30}$$

and

$$\hat{h}_{\mathbf{k}}P^h_{\mathbf{k}} = P^h_{\mathbf{k}}\hat{h}_{\mathbf{k}} = hP^h_{\mathbf{k}}, \tag{5.31}$$

$$P^h_{\mathbf{k}}P^{h'}_{\mathbf{k}} = \delta_{hh'}P^h_{\mathbf{k}}. \tag{5.32}$$

The reason behind defining the projection operators is that for mass indices I and J we can equally well parametrize the homogeneous and isotropic subalgebra as

$$P^h_{\mathbf{k}}P^e_{\mathbf{k},I}\gamma^0P^{e'}_{\mathbf{k},J}. \tag{5.33}$$

Necessity of the γ^0 matrix between the energy projection matrices is clear if one sets $I = J$: in this case the base matrices would not span the homogeneous and isotropic subspace completely if there were not the γ^0 matrix.

From the above discussion it follows that we can parametrize the correlator $\overline{S}^<$ without loss of generality as

$$\overline{S}^<_{IJ} = \sum_{h,e,e'} P^h_{\mathbf{k}}P^e_{\mathbf{k},I}\gamma^0P^{e'}_{\mathbf{k},J}D^{h,e,e'}_{\mathbf{k},IJ}, \tag{5.34}$$

where $D^{h,e,e'}_{\mathbf{k},IJ}$ are unknown spacetime and energy dependent coefficients [9]. Using this parametrization of the propagator together with the energy and helicity projection

[9] Even if we are considering homogeneous and isotropic system, the coefficients D depend on the spatial coordinates. This is a consequence of the fact that (globally) the background depends on the spatial coordinates since it changes adiabatically, and thus the coefficients D can have dependence of the spatial coordinates (e.g. momentum can be spatially dependent). However, the background changes so slowly that at each point (locally) the system can be treated as homogeneous and isotropic.

operators and their properties, the constraint equation (5.25) can be written as

$$\sum_{h,e,e'} 2k_0 P^h_{\mathbf{k}} P^e_{\mathbf{k},I} \gamma^0 P^{e'}_{\mathbf{k},J} D^{h,e,e'}_{\mathbf{k},IJ} = \sum_{h,e,e'} \Big[\mathcal{H}_{\mathbf{k},I} P^h_{\mathbf{k}} P^e_{\mathbf{k},I} \gamma^0 P^{e'}_{\mathbf{k},J} D^{h,e,e'}_{\mathbf{k},IJ} + P^h_{\mathbf{k}} P^e_{\mathbf{k},I} \gamma^0 P^{e'}_{\mathbf{k},J} D^{h,e,e'}_{\mathbf{k},IJ} \mathcal{H}_{\mathbf{k},J} \Big], \tag{5.35}$$

$$\Leftrightarrow \quad \begin{cases} \sum_{h,e} [k_0 - e\overline{\omega}_{\mathbf{k},IJ}] P^h_{\mathbf{k}} P^e_{\mathbf{k},I} \gamma^0 P^e_{\mathbf{k},J} D^{h,e,e}_{\mathbf{k},IJ} = 0 \\ \sum_{h,e} [k_0 - e\Delta\omega_{\mathbf{k},IJ}] P^h_{\mathbf{k}} P^e_{\mathbf{k},I} \gamma^0 P^e_{\mathbf{k},J} D^{h,e,-e}_{\mathbf{k},IJ} = 0 \end{cases} \tag{5.36}$$

with $\overline{\omega}_{\mathbf{k},IJ} \equiv (\omega_{\mathbf{k},I} + \omega_{\mathbf{k},J})/2$ and $\Delta\omega_{\mathbf{k},IJ} \equiv (\omega_{\mathbf{k},I} - \omega_{\mathbf{k},J})/2$. When we use the orthogonality of the projection operators, and take the helicity and energy projections of eq. (5.35) explicitly, we finally get

$$\begin{cases} [k_0 - e\overline{\omega}_{\mathbf{k},IJ}] D^{h,e,e}_{\mathbf{k},IJ} = 0 \\ [k_0 - e\Delta\omega_{\mathbf{k},IJ}] D^{h,e,-e}_{\mathbf{k},IJ} = 0 \end{cases} \tag{5.37}$$

for any h and e. These equations show that D's are generalized functions, and they have normalized spectral solutions:

$$\begin{aligned} D^{h,e,e}_{\mathbf{k},IJ} &= F^{m,h,e}_{\mathbf{k},IJ} \delta(k_0 - e\overline{\omega}_{\mathbf{k},IJ}), \\ D^{h,e,-e}_{\mathbf{k},IJ} &= F^{c,h,e}_{\mathbf{k},IJ} \delta(k_0 - e\Delta\omega_{\mathbf{k},IJ}), \end{aligned} \tag{5.38}$$

where $F^{m/c,h,e}_{\mathbf{k},IJ}$ are unknown spacetime-dependent complex functions. The function $F^{m,h,e}_{\mathbf{k},IJ}$ parametrizes the on-shell solutions while $F^{c,h,e}_{\mathbf{k},IJ}$ corresponds to the coherence shell solutions. Later on it turns out to be useful to define the particle distribution functions as

$$\begin{aligned} f^{m,h,e}_{\mathbf{k},IJ} &\equiv N^{m,h,e}_{\mathbf{k},IJ} F^{m,h,e}_{\mathbf{k},IJ}, \\ f^{c,h,e}_{\mathbf{k},IJ} &\equiv N^{c,h,e}_{\mathbf{k},IJ} F^{c,h,e}_{\mathbf{k},IJ}, \end{aligned} \tag{5.39}$$

where $N^{m/c,h,e}_{\mathbf{k},IJ}$ are normalization factors, which we choose to be

$$\begin{aligned} N^{m,h,e}_{\mathbf{k},IJ} &= \mathrm{Tr}\Big[P^h_{\mathbf{k}} P^e_{\mathbf{k},I} \gamma^0 P^e_{\mathbf{k},J} \gamma^0 \Big]^{-1/2}, \\ N^{c,h,e}_{\mathbf{k},IJ} &= \mathrm{Tr}\Big[P^h_{\mathbf{k}} P^e_{\mathbf{k},I} \gamma^0 P^{-e}_{\mathbf{k},J} \gamma^0 \Big]^{-1/2}. \end{aligned} \tag{5.40}$$

This choice of the normalization factors will show to be useful when the dynamical evolution of the distribution functions is solved.

It follows from eqs. (5.34), (5.38), (5.39) and (5.40) that the Wightman function can be parametrized as

$$\overline{S}^{<}_{IJ}(k,x) = \sum_{h,e} N^{m}_{\mathbf{k},IJ} P^{m,h,e}_{\mathbf{k},IJ} f^{m,h,e}_{\mathbf{k},IJ} \delta(k_0 - e\overline{\omega}_{\mathbf{k},IJ}) + N^{c,h,e}_{\mathbf{k},IJ} P^{c,h,e}_{\mathbf{k},IJ} f^{c,h,e}_{\mathbf{k},IJ} \delta(k_0 - e\Delta\omega_{\mathbf{k},IJ}), \tag{5.41}$$

where we defined a shorthand notation:

$$P^{m,h,e}_{\mathbf{k},IJ} = P^{h}_{\mathbf{k}} P^{e}_{\mathbf{k},I} \gamma^0 P^{e}_{\mathbf{k},J}, \tag{5.42}$$

and similarly for the other projection operators. Remarkable property of eq. (5.41) is that all k_0 dependece of the correlator $\overline{S}^{<}$ is in the delta functions. Later on when different momentum states are considered this fact will be of key importance. What is more, $k_0 = \overline{\omega}_{\mathbf{k},II}$ corresponds to the usual mass shell solution while $k_0 = \pm\Delta\omega_{\mathbf{k},II}$, $k_0 = \pm\overline{\omega}_{\mathbf{k},IJ}$ and $k_0 = \pm\Delta\omega_{\mathbf{k},IJ}$ $(I \neq J)$ correspond to completely novel solutions. The latter ones are identified as the coherence shell solutions which carry information about the flavor coherence in the particle or the antiparticle sectors separately ($\overline{\omega}_{\mathbf{k},IJ}$ term), or flavor coherence between the particle and the antiparticle sectors ($\Delta\omega_{\mathbf{k},IJ}$ terms). In other words, for instance the on-shell function $F^{m,h,e}_{\mathbf{k},IJ}(I \neq J)$ parametrizes flavor coherence between the mass eigenstates with energies $\pm\omega_I$ and $\pm\omega_J$. These new coherence solutions were first found in ref. [41] and have been further investigated in several papers, for example [31, 33, 40–45].

5.2.2 Spectral function

In the cQPA limit the free spectral function $\mathcal{A}$ satisfies similar equation as the propagator $S^{<}$, see eqs. (5.16) and (5.17). We can thus parametrize the spectral function exactly in the same way as the Wightman function (5.41):

$$\overline{\mathcal{A}}^{<}_{IJ}(k,x) = \sum_{h,e} P^{m,h,e}_{\mathbf{k},IJ} F^{m,h,e}_{\mathcal{A},\mathbf{k},IJ} \delta(k_0 - e\overline{\omega}_{\mathbf{k},IJ}) + P^{c,h,e}_{\mathbf{k},IJ} F^{c,h,e}_{\mathcal{A},\mathbf{k},IJ} \delta(k_0 - e\Delta\omega_{\mathbf{k},IJ}). \tag{5.43}$$

However, the spectral function must in addition obey the spectral sum rule

$$\int \mathrm{d}k_0 \, \overline{\mathcal{A}}_{IJ}(k,x) = \pi\delta_{IJ}. \tag{5.44}$$

Derivation of this sum rule is simple: we know that all physical field configurations must satisfy the canonical equal time anticommutation relation of the field operators:

$$\{\psi(t,u),\psi^\dagger(t,v)\} = -i\delta^{(3)}(\mathbf{u}-\mathbf{v}). \tag{5.45}$$

Using equations (4.2) and (4.7) it follows then directly from the anticommutation relation that

$$2\mathcal{A}(t,\mathbf{u};t,\mathbf{v})\gamma^0 = \delta^{(3)}(\mathbf{u}-\mathbf{v}). \tag{5.46}$$

When one Wigner transforms this relation, the spectral sum rule (5.44) is obtained.

The spectral sum rule is enough to completely fix the on-shell functions and the result is [46]:

$$F^{m,h,e}_{\mathcal{A},k,IJ} = e\pi\delta_{IJ}\omega_{\mathbf{k},I}/m_I \qquad \text{and} \qquad F^{c,h,e}_{\mathcal{A},k,IJ} = 0. \tag{5.47}$$

With these solutions the free spectral function becomes

$$\overline{\mathcal{A}}(k,x) = \pi\mathrm{sgn}(k_0)(\not{k}+m_I)\gamma^0\delta(k^2-m_I^2)\delta_{IJ}, \tag{5.48}$$

where m denotes the effective mass. This is the well-known form of the spectral function in the thermal quasiparticle limit, see [40] and references therein. From equation (5.48) it is evident that the spectral function does not have any dependence on the $k_0 = \Delta\omega_{\mathbf{k},IJ}$ coherence shell solutions. In fact, this is exactly as it should be: coherence is a dynamical phenomenon while the spectral function describes the properties of the one-particle phase space and hence it should not contain dynamics at all.

The point why we bothered to solve the spectral function from the pole equations is that now we do not need to consider the KB equation for the other Wightman function $S^>$. We can solve it easily using eq. (4.7):

$$\overline{S}^> = \overline{S}^< + 2i\overline{\mathcal{A}}. \tag{5.49}$$

5.3 General KB equations including thermal effects

Our goal is to find out the dynamical evolution of neutrinos propagating in medium. This means that we need to find out a closed set of equations of motion for the on-shell functions f. The next step towards these equations is to use the spectral solution (5.41) as an ansatz in the general KB equation (5.12). As discussed in sec. (5.1) when solving the dynamical evolution of the system the limit $\Sigma^A \to 0$ is not taken.

Before moving on to the actual calculation, let us discuss about a few important things. Firstly, the S^H-terms couple equations (5.11) and (5.12). This would make it remarkably more difficult to obtain a closed solution for the on-shell functions. However, it turns out that we can neglect terms proportional to S^H also when we are considering the general KB equation in the mean field limit. This term is associated with the width of the equilibrium part of the KB equation's solution and does not affect perturbations. In the limit $\Sigma^A \to 0$ it vanishes from the KB equation. This is non-trivial result which we shall take here for granted, see ref. [42] for some discussion. On the other hand, we neglected these terms during the derivation of the spectral solution and to be systematic in our calculations it is convenient to drop these terms also now.

Secondly, difficulties arise from the infinite gradient terms appearing in eq. (5.12). As discussed in sec.(5.2), we can not neglect terms in which $\Sigma(\hat{K}, x)$ is acting on the coherence solutions. An another way to see this is as follows: The coherence shell solutions are oscillating rapidly even if the background field is nearly constant. From this it follows that the higher order gradients, e.g. in the collision term, do not necessarily correspond to higher order derivatives of the background field and we can not drop them out. In sec. (4.3) we developed a formalism in order to write the general KB equations in such way that the infinite order derivatives are resummed. Thus, we already know how to handle these gradient terms conveniently.

Based on the above considerations, in the mean field limit the pole equations (5.11) and the KB equation (5.12) decouple, and by proceeding similarly as in

sec.(5.2), we obtain

$$[k_0 + \frac{i}{2}\mathcal{D} - \mathcal{H}_{\mathbf{k}}(x)]\overline{S}^{<}(k,x) + \frac{i}{2}U^{\dagger}(x)[\mathcal{D}U(x)]\overline{S}^{<}(k,x) + \frac{i}{2}\overline{S}^{<}(k,x)[\mathcal{D}U^{\dagger}(x)]U(x) = U^{\dagger}(x)\gamma^0 i\mathcal{C}_{\text{coll}}\gamma^0 U(x), \tag{5.50}$$

where the collision term is given by eq. (5.13). At this point it is useful to divide the collision term into real and imaginary part:

$$\mathcal{C}_{\text{coll}} = \mathcal{C}^{H}_{\text{coll}} + i\mathcal{C}^{A}_{\text{coll}}, \tag{5.51}$$

where $\mathcal{C}^{H}_{\text{coll}}$ is the Hermitian part and $\mathcal{C}^{A}_{\text{coll}}$ is the Antihermitian part of the self-energy. Now the Antihermitian part of eq.(5.50) reads

$$\begin{aligned} i\mathcal{D}\overline{S}^{<}(k,x) =& [\mathcal{H}_{\mathbf{k}}(x), \overline{S}^{<}(k,x)] + i[U^{\dagger}(x)(\mathcal{D}U(x)), \overline{S}^{<}(k,x)] \\ &+ 2iU^{\dagger}(x)\gamma^0\mathcal{C}^{H}_{\text{coll}}\gamma^0 U(x), \end{aligned} \tag{5.52}$$

where the Hamiltonian $\mathcal{H}_{\mathbf{k}}(x)$ is given by eq. (5.20). The (AH)-equation (5.52) contains time and spatial derivatives of the distribution functions and thus it gives the dynamical evolution of the neutrino states.

5.4 The final task: Density matrix equations for $f^{m,h,e}_{IJ}$

When the phase space of the correlators was studied in sec.(5.2.1), we observed that there exist coherence shell solutions which carry information about various different coherence effects. In addition, we derived parametrization (5.41) for the Wightman function $S^{<}$ in collisionless situation using the mass and the coherence shell solutions. Since this spectral solution encodes information about several different coherence effects and our strategy is to use the spectral solution as an ansatz in the (AH)-equation (5.52), also the (AH)-equation contains the information about these coherence effects. However, we are only interested in flavor coherence in the particle or the antiparticle sector and thus we can neglect terms which describe any other coherence effect.

The dynamical evolution of the system should be described by the (AH)-equation, but the singular shell structure of the Wightman function $S^{<}$ makes things more

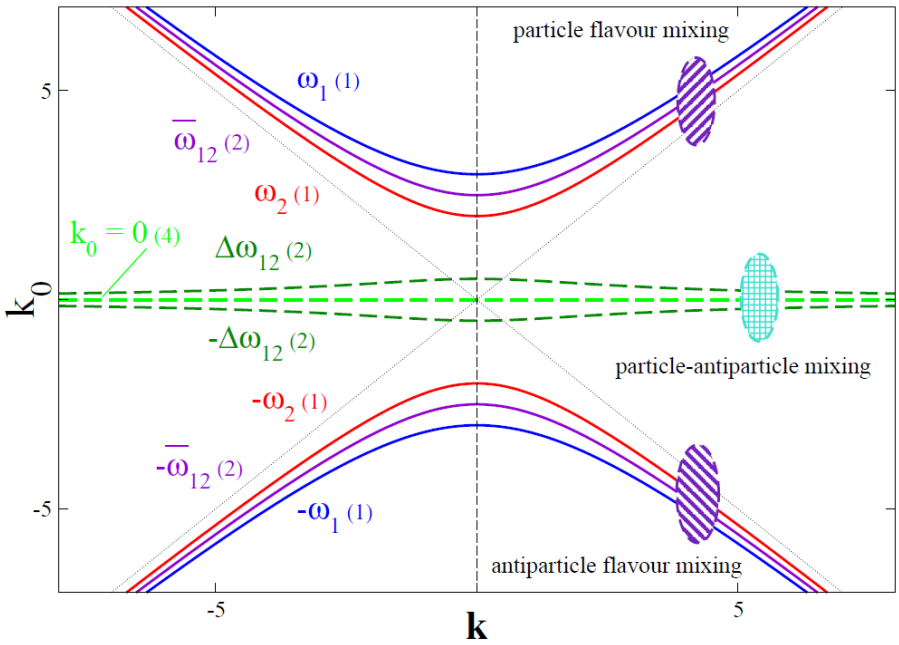

Figure (4) The cQPA shell structure in the case of two neutrino mixing is shown. The blue and the red line denotes the heavy ($m_1 = 5$ in arbitrary units) and the light ($m_2 = 3$) neutrino state, respectively. The curves are labelled by their eigenenergies and all particle-antiparticle flavor coherence solutions are shown with dashed green lines. The number in the paranthesis next to each eigenvalue gives the degeneracy of the corresponding solution. The figure is taken from ref. [45] with the Author's permission.

complicated. This singular structure suggests us to integrate the (AH)-equation over the momentum, and actually it is essential since distributions are well defined only inside an integral. To quantify the reasoning behind the integration let us consider an example. Assume that we are studying solar neutrinos. This means that we know that there are neutrinos instead of antineutrinos, the direction from which the neutrinos hit the detector, and we have information about the energy and momentum of the solar neutrinos (at some level). In figure (4) the cQPA shell structure in the case of two neutrino mixing is shown. The purple blobs correspond to flavor coherence in the particle (the upper blob) and the antiparticle (the lower blob) sectors, while the green blob in the middle corresponds to flavor coherence between the particle and the antiparticle sectors. The mass shells are denoted by the blue and red lines, and the coherence shells are denoted by the purple line and the green dashed lines. Due to the information about the solar neutrinos that we have, we know that there can be only flavor coherence in the particle sector which corresponds to the upper half of the fig. (4). However, the inaccuracy of the neutrino momentum

measurements is significantly larger than the energy differences between the different shell solutions. Therefore, we can not determine on which shell the neutrino is and we have to integrate over a part of the momentum phase space determined by the measurement accuracy. Nevertheless, the range of the momentum phase space over which we integrate is huge when compared to the differences between the shell solutions, and it makes no difference to integrate over the whole phase space. On the other hand, if we had perfect measurements with no inaccuracy (that is all the possible information about the system), there would be no need for the integration since we could tell on which shell the neutrino is. Motivated by this example, it is convenient to define a weight function that encodes the amount of information avaible, and which can be used to define the physical density matrix which takes the above discussion into account. We do not treat this subject more closely here, but see ref. [41] for some discussion.

It turns out that for fermions it is enough to integrate over the zeroth component of the momentum to find out a closed set of equations of motion for the on-shell functions [32]. According to this and the above discussion, we substitute the spectral solution (5.41) into eq. (5.52), integrate over the zeroth component of the momentum, and in addition take the helicity and energy projections of the resulting equation [10]:

$$
\begin{aligned}
&[i\mathcal{D}N^{m,h,e}_{\mathbf{k},IJ}]f^{m,h,e}_{\mathbf{k},IJ}P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IJ}P^{m,h,e}_{\mathbf{k},JI} + N^{m,h,e}_{\mathbf{k},IJ}f^{m,h,e}_{\mathbf{k},IJ}P^{m,h,e}_{\mathbf{k},JI}[i\mathcal{D}P^{m,h,e}_{\mathbf{k},IJ}]P^{m,h,e}_{\mathbf{k},JI} \\
&\quad + N^{m,h,e}_{\mathbf{k},IJ}[i\mathcal{D}f^{m,h,e}_{\mathbf{k},IJ}]P^{m,h,e}_{\mathbf{k},IJ}P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},JI} \\
=&e(\omega_I-\omega_J)N^{m,h,e}_{\mathbf{k},IJ}f^{m,h,e}_{\mathbf{k},IJ}P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IJ}P^{m,h,e}_{\mathbf{k},JI} \\
&\quad + \sum_{L,N}\Big(N^{m,h,e}_{\mathbf{k},NJ}f^{m,h,e}_{\mathbf{k},NJ}P^{m,h,e}_{\mathbf{k},JI}U^{\dagger}_{IL}[i\mathcal{D}U_{LN}]P^{m,h,e}_{\mathbf{k},NJ}P^{m,h,e}_{\mathbf{k},JI} \\
&\quad - N^{m,h,e}_{\mathbf{k},IL}f^{m,h,e}_{\mathbf{k},IL}P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IL}U^{\dagger}_{LN}[i\mathcal{D}U_{NJ}]P^{m,h,e}_{\mathbf{k},JI} \\
&\quad + 2iP^{m,h,e}_{\mathbf{k},JI}U^{\dagger}_{IL}\gamma^0\mathcal{C}^{H}_{\mathrm{coll},LN}\gamma^0U_{NJ}P^{m,h,e}_{\mathbf{k},JI}\Big).
\end{aligned}
\tag{5.53}
$$

This equation contains also sums over the Dirac indices even if these are not explicitly denoted. The simplest way to proceed and to get rid of the Dirac structure is to take trace over them. When one uses the properties of the energy and helicity projections operators (5.28)-(5.30) and (5.31) - (5.32), and basic properties of traces, eq. (5.53)

[10] Due to the singular structure of the Wightman function, the integration over the zeroth component of the momentum effectively projects neutrinos to the mass shells. In other words, even if the coherence shells exist we do not perform the calculations on them.

can be written as

$$
\begin{aligned}
&\Big([i\mathcal{D}N^{m,h,e}_{\mathbf{k},IJ}]f^{m,h,e}_{\mathbf{k},IJ} + N^{m,h,e}_{\mathbf{k},IJ}[i\mathcal{D}f^{m,h,e}_{\mathbf{k},IJ}]\Big)\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IJ}\Big] \\
&+ N^{m,h,e}_{\mathbf{k},IJ}f^{m,h,e}_{\mathbf{k},IJ}\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},JI}(i\mathcal{D}P^{m,h,e}_{\mathbf{k},IJ})\Big] \\
=&2e\Delta\omega_{IJ}N^{m,h,e}_{\mathbf{k},IJ}f^{m,h,e}_{\mathbf{k},IJ}\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IJ}\Big] + \sum_{L,N}2i\Big(\,\mathrm{Tr}\Big[U^{\dagger}_{IL}\gamma^0\mathcal{C}^{H}_{\mathrm{coll},LN}\gamma^0U_{NJ}P^{m,h,e}_{\mathbf{k},JI}\Big] \\
&+ N^{m,h,e}_{\mathbf{k},NJ}f^{m,h,e}_{\mathbf{k},NJ}\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},NJ}P^{m,h,e}_{\mathbf{k},JI}U^{\dagger}_{IL}(i\mathcal{D}U_{LN})\Big] \\
&- N^{m,h,e}_{\mathbf{k},IL}f^{m,h,e}_{\mathbf{k},IL}\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IL}U^{\dagger}_{LN}(i\mathcal{D}U_{NJ})\Big]\Big).
\end{aligned}
\tag{5.54}
$$

Next we note that

$$
\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IJ}\Big] = \frac{m_Im_J}{\omega_I\omega_J}(N^{m,h,e}_{\mathbf{k},IJ})^{-1},
\tag{5.55}
$$

and by using eq. (5.55) we can write the (AH)-equation (5.54) as

$$
\begin{aligned}
\partial_t f^{m,h,e}_{\mathbf{k},IJ} - \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}f^{m,h,e}_{\mathbf{k},IJ} =& -2ie\Delta\omega_{IJ}f^{m,h,e}_{\mathbf{k},IJ} \\
&+ \frac{2\omega_I\omega_J}{m_Im_J}N^{m,h,e}_{\mathbf{k},IJ}\sum_{L,N}\Big(\,\mathrm{Tr}\Big[U^{\dagger}_{IL}\gamma^0\mathcal{C}^{H}_{\mathrm{coll},LN}\gamma^0U_{NJ}P^{m,h,e}_{\mathbf{k},JI}\Big]\Big) \\
&- (N^{m,h,e}_{\mathbf{k},IJ})^{-1}(\mathcal{D}N^{m,h,e}_{\mathbf{k},IJ})f^{m,h,e}_{\mathbf{k},IJ} \\
&- \frac{\omega_I\omega_J}{m_Im_J}N^{m,h,e}_{\mathbf{k},IJ}\sum_{L,N}\Big(N^{m,h,e}_{\mathbf{k},IJ}f^{m,h,e}_{\mathbf{k},IJ}\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},JI}(i\mathcal{D}P^{m,h,e}_{\mathbf{k},IJ})\Big] \\
&- N^{m,h,e}_{\mathbf{k},NJ}f^{m,h,e}_{\mathbf{k},NJ}\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},NJ}P^{m,h,e}_{\mathbf{k},JI}U^{\dagger}_{IL}(\mathcal{D}U_{LN})\Big] \\
&+ N^{m,h,e}_{\mathbf{k},IL}f^{m,h,e}_{\mathbf{k},IL}\,\mathrm{Tr}\Big[P^{m,h,e}_{\mathbf{k},JI}P^{m,h,e}_{\mathbf{k},IL}U^{\dagger}_{LN}(\mathcal{D}U_{NJ})\Big]\Big).
\end{aligned}
\tag{5.56}
$$

Now the (AH)-equation (5.56) has a typical structure for a density matrix equation: the derivatives of the on-shell functions $f^{m,h,e}_{\mathbf{k},IJ}$ are separated from collision terms, source terms and Liouville terms. The right hand side of the first line of eq. (5.56), i.e. terms which come from the commutator of the Hamiltonian and the Wightman function, corresponds to the $\mathbf{V}\times\mathbf{P}$ term in eq. (3.55). The collision term in the second line of the (AH)-equation corresponds to the $D\mathbf{P}_T$ term in the density matrix equation (3.55) and also collisions between the other particles in the system. The third and the fourth lines of eq. (5.56) are the source terms and they contain

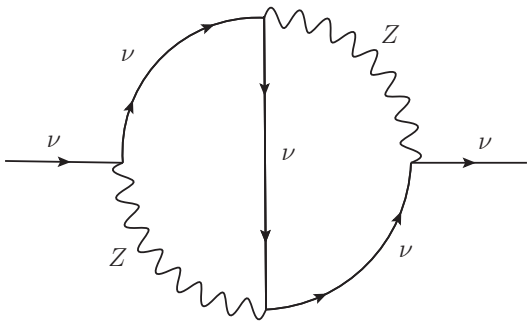

Figure (5) Shown is the two-loop Z-diagram contributing to the self-energy function Σ.

derivatives of the projection operators. The Liouville terms, which cause rotation of the basis, are on the fifth and the sixth line of the (AH)-equation.

All traces appearing in the (AH)-equation (5.56) depend on the self-energy function. At this point it is therefore necessary to specify the interactions and calculate the self-energies. Detailed analysis of this topic is beyond the scope of this thesis. However, we discuss shortly how the self-energies are calculated. Complete derivation of a closed set of equations of motions for propagating neutrinos will be presented in ref. [47].

As discussed in sec.(5.1), Σ^A is relevant for solving the dynamical evolution of the distribution functions. In the case of neutrinos it turns out that Σ^A actually needs to be expanded up to second order, i.e. at the fourth order in the electroweak coupling constant, in order to include the 2-2 scattering processes. This means that two-loop graphs must be calculated of which an example is given in figure (5). The precise determination of what diagrams to include is nontrivial: one has to deal with the issue related to double counting graphs, and to handle this problem conveniently it requires theoretical tool that is not developed in this thesis. Also, reducing the Liouville terms in the evolution equation (5.56) still requires some detailed calculations which we need to postpone to further work.

Even if there are a few topics which need further investigation, we can still see that when the source and the Liouville terms are neglected in the (AH)-equation (5.56), we get the usual density matrix equation (3.55). After all, this was our goal all the time, i.e. to show that the density matrix formalism can be derived from more fundamedal grounds than what has been done before.

6 Conclusions

In this thesis the main goal was to study how a closed set of equations of motion, which take into account quantum coherence and neutrino mixing, can be derived for neutrinos. We started out by reviewing neutrino physics and investigating neutrino propagation in medium when only elastic forward scatterings were considered. In this way, we were able to derive the matter Hamiltonian which defines the energy eigenvalues of neutrinos in medium. However, when neutrinos propagate in matter there exist also incoherent scatterings which affect the dynamical evolution of the system. Due to these incoherent scatterings the usual Hamiltonian formalism can not be used to describe propagation of neutrinos.

In sec.(4) we examined the quantum transport theory and how more general formalism, which is capable of describing neutrinos in medium, can be constructed. We discussed closely how a general Kadanoff-Baym (KB) equations can be obtained from the contour Schwinger-Dyson equation. Especially, we derived a superior form (4.59) of the KB equations in the viewpoint of gradient expansion: The KB equations (4.59) contain infinite order derivatives and are impossible to solve as such. We need therefore an approximation scheme which tells us how to handle the infinite order gradients. When applying such approximation scheme to the KB equations, advantages of eq. (4.59) are revealed.

The approximation scheme used in this thesis to simplify the general KB equations is called the coherent quasiparticle approximation (cQPA). In section (5) we studied the basic assumptions and properties of the cQPA. The most important feature of the cQPA is that it relinquishes approximation of translational invariance (this is assumed e.g. in the usual quasiparticle approximation). From this it follows that in the phase space there exist completely novel coherence shell solutions which are recognized to carry information about non-local quantum coherence. The cQPA scheme then gives us a way to solve the dynamical evolution of non-equilibrium systems while taking into account quantum coherence effects.

After the approximation scheme was defined in sec.(5), we expressed the KB equations in the cPQA limit. From these equations we solved the spectral properties

of the phase space and wrote down an equation from which the equations of motion for neutrinos can be solved. The process of solving the dynamical equation is outside the scope of this thesis, but we discussed shortly what is left to do and how the actual solution can be obtained. The complete derivation of the equations of motion for neutrinos will be presented in ref. [47].

These equations of motion are remarkable in two ways. Firstly, they take into account neutrino flavor mixing, quantum coherence and matter effects completely. This means that one can use them to describe scattering processes between coherent neutrino states which no other existing model is capable of doing. One application target for the derived equations are supernovae: When a supernova is born neutrinos are trapped for a few seconds in the supernova. Regardless of multiple research on the subject, it remains unclear what happens to the coherent evolution of the neutrino states in this extreme process. The equations of motion discussed in this thesis may help to solve this problem. There are also several other phenomena concerning about supernovae, for instance how neutrinos and neutrino oscillation affect energetics of supernova explosions, which can be studied using these equations. Secondly, we have shown that the density matrix formalism can be derived from more fundamental grounds than what has been done before.

In this thesis we used the so called spectral limit when solving the phase space properties of the system. There exists, however, a more general approximation (the mean field limit) which is consistent with the cQPA scheme. In this limit the reasoning behind some phenomena, like the necessity of considering 2-loop diagrams, becomes evident and can be understood properly. There are also other interesting topics that need to be studied more carefully. Firstly, the explicit form of the rotation matrix, i.e. matrix that diagonalizes the Hamiltonian function in the mass eigenbasis, is a deep issue since it can actually be energy and spacetime dependent, and contain chirality structures. In this thesis we diagonalized the Hamiltonian without specifying the exact form of the rotation matrix and just assumed that there exists such a matrix, but in general this is a nontrivial problem. Secondly, the Liouville terms depend on the derivatives of these rotation matrices. Thus, it is not obvious whether or not the Liouville terms give significant corrections to the equations of motion. Lastly, since the coherence solutions are oscillating rapidly even if adiabatic background fields are assumed, it needs further research to determine how significant the spatial derivatives of the distribution functions are, and how these

terms depend on the circumstances. In ref. [47] the above topics will be discussed, the derivation of the equations of motion will be finished using the local limit and numerical examples will be given.

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