JYU DISSERTATIONS 272

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Traces for Function Spaces on Metric Measure Spaces

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Jyväskylä, May 2020

Zhuang Wang

List of included articles

This dissertation consists of an introductory part and the following four publications:

- **[A]** P. Koskela, T. Soto and Z. Wang, *Traces of weighted function spaces: dyadic norms and Whitney extensions,* Sci. China Math. 60 (2017) , no.11 1981-2010.
- **[B]** P. Koskela and Z. Wang, *Dyadic norm Besov-type spaces as trace spaces on regular trees*, Potential Anal., accepted. ArXiv:1908.06937.
- **[C]** P. Lahti, X. Li and Z. Wang, *Traces of Newton-Sobolev, Hajłasz-Sobolev, and BV functions on metric spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), accepted. ArXiv:1911.00533.
- **[D]** Z. Wang, *Characterization of trace spaces on regular trees via dyadic norms*, submitted. ArXiv:2004.03432.

The author of this dissertation has actively taken part in the research for the joint papers [A], [B] and [C].

INTRODUCTION

The characterization of the trace spaces (on the boundary of a domain) of Sobolev spaces and other function spaces has a long history. It can be traced back to [17] by Gagliardo in 1957, which gave a characterization of the trace space of the first order Sobolev space $W^{1,p}(\mathbb{R}^{n+1}_+)$, $1 < p < \infty$, $\mathbb{R}^{n+1}_+ := \{(x,r) : x \in \mathbb{R}^n, r > 0\}$, in terms of the convergence of a suitable double integral of the boundary values. More precisely, for any function $u \in W^{1,p}(\mathbb{R}^{n+1}_+)$, define the trace operator T by setting

$$Tu(x) = \lim_{r \to 0^+} u(x,r)$$

for those $x \in \mathbb{R}^n$, for which this limit exists. Then the trace operator $T : W^{1,p}(\mathbb{R}^{n+1}_+) \to B^{1-1/p}_{p,p}(\mathbb{R}^n)$ is linear and bounded for $1 < p < \infty$ and there exists a bounded linear extension operator that acts as a right inverse of T. Here the space $B^{1-1/p}_{p,p}(\mathbb{R}^n)$, consisting of all measurable functions f on $\mathbb{R}^n$ with

$$\|f\|^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+(1-1/p)p}}\,dx\,dy < \infty,$$

is nowadays commonly called a Besov space or a fractional Sobolev space. Actually, Gagliardo also proved in [17] that the trace operator $T : W^{1,1}(\mathbb{R}^{n+1}_+) \to L^1(\mathbb{R}^n)$ is a bounded linear surjective operator with a non-linear right inverse. Peetre showed in [41] that one can not find a bounded linear extension operator that acts as a right inverse of $T : W^{1,1}(\mathbb{R}^{n+1}_+) \to L^1(\mathbb{R}^n)$.

It is natural to seek for the trace spaces of Sobolev spaces associated with weights or the trace spaces for other function spaces, like Orlicz-Sobolev spaces. Early results considering the trace spaces of Sobolev spaces with weights of the form $x \to \operatorname{dist}(x,\mathbb{R}^n)^\alpha$ were given by Nikolskii, Lizorkin and Vašarin, see [31, 37, 56]. More recently, Tyulenev studied in [52, 53, 54, 55] the trace spaces of Sobolev spaces associated with more general Muckenhoupt A_p-weights. We also refer to [4, 27, 35, 41, 47, 49, 50] for more information on the traces of (weighted) Sobolev spaces. For the traces of Orlicz-Sobolev spaces (associated with weights), we refer to [12, 13, 16, 28, 39, 11, 29, 40].

Over the past two decades, analysis in general metric measure spaces has attracted a lot of attention as exhibited by [5, 7, 21, 22, 23, 24, 25]. The trace theory in the metric setting has been under development. Malý proved in [32] that the trace space of the Newtonian space $N^{1,p}(\Omega)$ is the Besov space $B^{1-\theta/p}_{p,p}(\partial\Omega)$ provided that Ω is a John domain for $p > 1$ (uniform domain for $p \ge 1$) that admits a p-Poincaré inequality and whose boundary $\partial\Omega$ is endowed with a codimensional-θ Ahlfors regular measure with $\theta < p$. We also refer to the paper [44] for studies on the traces of Hajłasz-Sobolev functions to porous Ahlfors regular closed subsets via a method based on hyperbolic fillings of a metric space, also see [9, 48]. It was shown in [30, 33] that the trace space of $\mathrm{BV}(\Omega)$ (functions of bounded variation) is $L^1(\partial X)$ whenever Ω is a bounded domain supporting 1-Poincaré inequality and the boundary $\partial\Omega$ is endowed with a codimensional-1 Ahlfors regular measure.

In this thesis, we study the traces of function spaces on metric measure spaces. In the paper [A], we revisit the Euclidean setting, viewing the upper half space $\mathbb{R}^{n+1}_+$ as a particularly nice metric space endowed with a weighted measure, and give characterizations of trace spaces of first order Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces via dyadic norms. We then study the trace problem in papers [B] and [D] on regular trees, dealing with first order Sobolev spaces and those Orlicz-Sobolev spaces whose Young function is of the form $t^p\log^\lambda(e+t)$. In the paper [C], we work on the relations of the traces among Newton-Sobolev, Hajłasz-Sobolev and bounded variation functions on metric measure spaces, and show that the trace spaces of those function classes coincide under suitable assumptions on the domain in question.

1. Revisiting the Euclidean setting

In this section, we deal with the upper half space $\mathbb{R}^{n+1}_+$ associated with the measure μ_α (where $\alpha > -1$) defined by

$$\mu_\alpha(E) = \int_E w_\alpha\, dm_{n+1},$$

where $w_\alpha\colon \mathbb{R}^{n+1}_+ \to (0,\infty)$ is the weight $(x_1, x_2, \cdots, x_{n+1}) \mapsto \min(1, |x_{n+1}|)^\alpha$ and m_{n+1} is the standard Lebesgue measure on $\mathbb{R}^{n+1}_+$. Then a straightforward calculation shows that

$$\mu_\alpha\big(B(x,r)\big) \approx r^{n+1+\alpha}$$

for all $x \in \mathbb{R}^n \times \{0\}$ and $0 < r \leq 1$.

First, we give the definitions of the relevant function spaces.

Definition 1.1. Suppose that μ is a Borel-regular measure on $\mathbb{R}^n$ such that every Euclidean ball has positive and finite μ-measure.

Let $p \in [1,\infty)$. Then $W^{1,p}(\mathbb{R}^n,\mu)$ is defined as the normed space of all the measurable functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that the first-order distributional derivatives of f coincide with functions in $L^1_{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{W^{1,p}(\mathbb{R}^n,\mu)} := \|f\|_{L^p(\mathbb{R}^n,\mu)} + \|\nabla f\|_{L^p(\mathbb{R}^n,\mu)} \tag{1.1}$$

is finite.

The space $W^{1,p}(\mathbb{R}^{n+1}_+,\mu_\alpha)$ is defined similarly, by replacing $\mathbb{R}^n$ and μ in (1.1) with $\mathbb{R}^{n+1}_+$ and μ_α, respectively.

In order to introduce the dyadic norms for the relevant fractional smoothness spaces, we recall the standard dyadic decompositions of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_+$. Denote by $\mathscr{Q}_n$ the collection of dyadic semi-open cubes in $\mathbb{R}^n$, of the form $Q := 2^{-k}\big((0,1]^n + m\big)$, where $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, and by $\mathscr{Q}_n^+$ for the cubes in $\mathscr{Q}_n$ which are contained in the upper half-space $\mathbb{R}^{d-1}\times(0,\infty)$. Write $\ell(Q)$ for the edge length of $Q \in \mathscr{Q}_n$, i.e. 2^{-k} in the preceding representation, and $\mathscr{Q}_{n,k}$ for the cubes $Q \in \mathscr{Q}_n$ such that $\ell(Q) = 2^{-k}$. If $x \in \mathbb{R}^n$ (resp. $x \in \mathbb{R}^{n+1}_+$) and $k \in \mathbb{Z}$, we write write Q^x_k for the unique cube in $\mathscr{Q}_n$ (resp. $\mathscr{Q}^+_{n+1}$) such that $x \in Q$ and $\ell(Q) = 2^{-k}$.

We say that Q and Q' in $\mathscr{Q}_n$ are neighbors and write $Q \sim Q'$ if $\frac{1}{2} \le \ell(Q)/\ell(Q') \le 2$ and $\overline{Q} \cap \overline{Q'} \ne \emptyset$. Note that every Q has a uniformly bounded number of neighbors.

Definition 1.2. Suppose that μ is a Borel-regular measure on $\mathbb{R}^n$ such that every Euclidean ball has positive and finite μ-measure.

Let $s \in (0,1)$, $p \in [1,\infty]$ and $q \in (0,\infty]$. Then the Besov space $\mathcal{B}^s_{p,q}(\mathbb{R}^n,\mu)$ is defined as the normed (or quasi-normed when $q<1$) space of all the functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^n,\mu)} := \|f\|_{L^p(\mathbb{R}^n,\mu)} + \Bigg(\sum_{k=0}^{\infty} 2^{ksq}\Big(\sum_{Q\in\mathscr{Q}_{n,k}} \mu(Q) \sum_{Q'\sim Q} \big|f_{Q,\mu} - f_{Q',\mu}\big|^p\Big)^{q/p}\Bigg)^{1/q} \tag{1.2}$$

(standard modification for $p=\infty$ and/or $q=\infty$) is finite.

Here and in the following, we use the notation

$$f_{Q,\mu} := \fint_Q f\,d\mu = \frac{1}{\mu(Q)}\int_Q f\,d\mu.$$

We omit μ from the notation and write f_Q when μ is the standard Lebesgue measure.

Definition 1.3. Suppose that μ is a Borel-regular measure on $\mathbb{R}^n$ such that every Euclidean ball has positive and finite μ-measure.

Let $s \in (0,1)$, $p \in [1,\infty)$ and $q \in (0,\infty]$. Then the Triebel-Lizorkin space $\mathcal{F}^s_{p,q}(\mathbb{R}^n,\mu)$ is defined as the normed (or quasi-normed when $q<1$) space of all the functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n,\mu)} := \|f\|_{L^p(\mathbb{R}^n,\mu)} + \Bigg(\int_{\mathbb{R}^n}\Big(\sum_{k=0}^{\infty} 2^{ksq} \sum_{Q'\sim Q^x_k} \big|f_{Q^x_k,\mu} - f_{Q',\mu}\big|^q\Big)^{p/q} d\mu(x)\Bigg)^{1/p} \tag{1.3}$$

(standard modification for $q=\infty$) is finite.

The spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^{n+1}_+,\mu_\alpha)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^{n+1}_+,\mu_\alpha)$ are defined similarly, by replacing $\mathbb{R}^n$ and μ with $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$ and μ_α in (1.2) and (1.3), respectively, and omitting the terms corresponding to the cubes $Q \in \mathscr{Q}_{n+1} \setminus \mathscr{Q}^+_{n+1}$ and $Q' \in \mathscr{Q}_{n+1} \setminus \mathscr{Q}^+_{n+1}$.

In case μ is the standard Lebesgue measure on $\mathbb{R}^n$, we omit μ from the notation of these three function spaces above and simply write $W^{1,p}(\mathbb{R}^n)$, $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$. The spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$ coincide with the standard Besov space $B^s_{p,q}(\mathbb{R}^n)$ and the standard Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$, see [A, Section 7.2]. We also refer to the seminal monographs [42] by Peetre and [49] by Triebel for spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$.

Next, we introduce the Whitney extension which plays a central role in the paper [A]. Given $Q \in \mathscr{Q}_{n,k}$, $k \in \mathbb{Z}$, let $\mathscr{W}(Q) := Q \times (2^{-k}, 2^{-k+1}] \in \mathscr{Q}^+_{n+1,k}$. Then it is easy to check that $\{\mathscr{W}(Q) : Q \in \mathscr{Q}_n\}$ is a Whitney decomposition of $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ with respect to the boundary $\mathbb{R}^n \times \{0\}$. Here we refer to [45, 25] for more information about Whitney decompositions. Further, let $\mathscr{Q}^0_n := \cup_{k\ge 0}\mathscr{Q}_{d,k}$.

For each $Q \in \mathscr{Q}_n^0$, pick a smooth function $\psi_Q \colon \mathbb{R}^{n+1}_+ \to [0,1]$ such that $\operatorname{Lip}\psi_Q \lesssim 1/\ell(Q)$, $\inf_{x\in\mathscr{W}(Q)} \psi_Q(x) > 0$ uniformly in $Q \in \mathscr{Q}_n^0$, $\operatorname{supp}\psi_Q$ is contained in an $\frac{\ell(Q)}{4}$-neighborhood of $\mathscr{W}(Q)$ and

$$\sum_{Q\in\mathscr{Q}_n^0} \psi_Q \equiv 1 \quad \text{in} \quad \bigcup_{Q\in\mathscr{Q}_n^0} \mathscr{W}(Q).$$

Let us point out that the sum above is locally finite – more precisely, it follows from the definition that

$$\operatorname{supp}\psi_Q \cap \operatorname{supp}\psi_{Q'} \neq \emptyset \quad \text{if and only if} \quad Q \sim Q'.$$

Definition 1.4. (i) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the Whitney extension $\mathcal{E}f \colon \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is defined by setting

$$\mathcal{E}f(x) = \sum_{Q\in\mathscr{Q}_n^0} \Big(\fint_Q f\, dm_n \Big) \psi_Q(x).$$

This definition gives rise in the obvious way to a linear operator $\mathcal{E}\colon L^1_{\mathrm{loc}}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^{d+1}_+)$.

(ii) Let $\mathbb{X} \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a quasinormed function space on $\mathbb{R}^n$, and let $\mathbb{Y}$ be a quasinormed function space on the weighted half-space $(\mathbb{R}^{n+1}_+, \mu)$. We say that $(\mathbb{X}, \mathbb{Y})$ is a *Whitney trace-extension pair* if $\mathcal{E}$ maps $\mathbb{X}$ continuously into $\mathbb{Y}$, if the trace function $\mathcal{R}f$ defined by

$$\mathcal{R}f(x) = \lim_{r\to 0} \fint_{B((x,0),r)\cap\mathbb{R}^{n+1}_+} f(y)\, d\mu(y),$$

is for all $f \in \mathbb{Y}$ well defined almost everywhere and belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $\mathcal{R}$ maps $\mathbb{Y}$ continuously into $\mathbb{X}$ and if additionally

$$\mathcal{R}(\mathcal{E}f) = f$$

pointwise almost everywhere for all $f \in \mathbb{X}$.

In the paper [A], we gave the following trace results for the Sobolev spaces $W^{1,p}(\mathbb{R}^{n+1}_+, \mu_\alpha)$, Besov spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^{n+1}_+, \mu_\alpha)$ and Triebel-Lizorkin spaces $\mathcal{F}^s_{p,q}(\mathbb{R}^{n+1}_+, \mu_\alpha)$.

Theorem 1.5. *(i) Let $1 \le p < \infty$ and $-1 < \alpha < p-1$. Then $\big(\mathcal{B}^{1-(\alpha+1)/p}_{p,p}(\mathbb{R}^n), W^{1,p}(\mathbb{R}^{n+1}_+, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

(ii) Let $0 < s < 1$, $1 \le p \le \infty$, $0 < q \le \infty$ and $-1 < \alpha < sp - 1$. Then $\big(\mathcal{B}^{s-(\alpha+1)/p}_{p,q}(\mathbb{R}^n), \mathcal{B}^s_{p,q}(\mathbb{R}^{n+1}_+, \mu_\alpha)\big)$ is a Whitney trace-extension pair.

(iii) Let $0 < s < 1$, $1 \le p < \infty$, $0 < q \le \infty$ and $-1 < \alpha < sp - 1$. Then $\big(\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^n), \mathcal{F}^s_{p,q}(\mathbb{R}^{n+1}_+, \mu_\alpha)\big)$ is a Whitney trace-extension pair.

The above result deals with the traces of functions defined on $\mathbb{R}^{n+1}_+$ to $\mathbb{R}^n$. It can be generalized to the case of $\mathbb{R}^{n+m}$ and $\mathbb{R}^n$, where $m \in \mathbb{N}$, $m \ge 1$. We refer to [A, Section 7.4] for more details.

2. Trace results on regular trees

In this section, we study the trace problem on regular trees. First, we introduce regular trees and their boundaries.

A *graph* G is a pair (V,E), where V is a set of vertices and E is a set of edges. Given vertices $x,y \in V$ are neighbors if x is connected to y by an edge. The number of the neighbors of a vertex x is referred to as the degree of x. A *tree* G is a connected graph without cycles.

We call a tree G a *rooted tree* if it has a distinguished vertex called the *root*, which we will denote by 0. The neighbors of a vertex $x \in G$ are of two types: the neighbors that are closer to the root are called *parents* of x and all other neighbors are called *children* of x. Each vertex has a unique parent, except for the root itself that has none. A K-regular tree is a rooted tree such that each vertex has exactly K children.

Let G be a K-regular tree with a set of vertices V and a set of edges E for some $K \geq 1$. For simplicity of notation, we let $X = V \cup E$ and call it a K-regular tree. We consider each edge as a geodesic of length one. For $x \in X$, let $|x|$ be the length of the geodesic from 0 to x, where we consider each edge to be an isometric copy of the unit interval. The geodesic connecting $x,y \in V$ is unique. We refer to it by $[x,y]$, and to its length by $|x-y|$. We write $x \leq y$ if $x \in [0,y]$. Then $|x-y| = |y| - |x|$. We say that a vertex $y \neq x$ is a descendant of the vertex x if $x \leq y$.

Towards defining the metric of X, let $\epsilon > 0$, and set

$$d_X(x,y) = \int_{[x,y]} e^{-\epsilon|z|}\, d\,|z|.$$

Here $d\,|z|$ is the natural measure that gives each edge Lebesgue measure 1; recall that each edge is an isometric copy of the unit interval. Notice that $\operatorname{diam} X = 2/\epsilon$ if X is a K-ary tree with $K \geq 2$.

The boundary ∂X of a tree X is obtained by completing X with respect to the metric d_X. An element $\xi \in \partial X$ can be identified with an infinite geodesic starting at the root 0. Equivalently we employ the labeling $\xi = 0x_1x_2\cdots$, where x_i is a vertex in X with $|x_i| = i$, and x_{i+1} is a child of x_i. The extension of the metric to ∂X can be realized in the following manner. Given $\xi, \zeta \in \partial X$, pick an infinite geodesic $[\xi,\zeta]$ connecting ξ and ζ. Then $d_X(\xi,\zeta)$ is the length of the geodesic $[\xi,\zeta]$. Indeed, if $\xi = 0x_1x_2\cdots$ and $\zeta = 0y_1y_2\cdots$, let k be the integer with $x_k = y_k$ and $x_{k+1} \neq y_{k+1}$. Then

$$d_X(\xi,\zeta) = 2\int_k^{+\infty} e^{-\epsilon t}\, dt = \frac{2}{\epsilon} e^{-\epsilon k}.$$

For more details, see [6, 8, 10]. For clarity, we use ξ, ζ, ω to denote points in ∂X and x,y,z points in X.

We define a weighted measure μ_λ on the K-regular tree X by setting

$$d\mu_\lambda(x) = e^{-\beta|x|}(|x|+C)^\lambda\, d|x| \tag{2.1}$$

where $\beta > \log K$, $\lambda \in \mathbb{R}$ and $C \geq \max\{2|\lambda|/(\beta - \log K), 2(\log 4)/\epsilon\}$. We refer to [B, Section 2.2] for detailed discussions about the measure μ_λ. Then in [B] and [D], the trace spaces of the Newtonian space $N^{1,p}(X, \mu_\lambda)$, $1 \le p < \infty$ and the Orlicz-Sobolev space $N^{1,\Phi}(X, \mu_\lambda)$ with Young function $\Phi(t) = t^p \log^{\lambda'}(e+t)$, $1 \le p, \infty$, $\lambda' \in \mathbb{R}$ have been characterized. Before going into details, let us give some necessary definitions, including the definitions of $N^{1,p}(X, \mu_\lambda)$ and $N^{1,\Phi}(X, \mu_\lambda)$.

Let $u \in L^1_{\mathrm{loc}}(X, \mu_\lambda)$. We say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if

$$|u(z) - u(y)| \le \int_\gamma g \, ds_X \tag{2.2}$$

whenever $z, y \in X$ and γ is the geodesic from z to y, where ds_X denotes the arc length measure with respect to the metric d_X. In the setting of a tree any rectifiable curve with end points z and y contains the geodesic connecting z and y, and therefore the upper gradient defined above is equivalent to the definition which requires that inequality (2.2) holds for all rectifiable curves with end points z and y. We refer interested readers to [21, 24, 25, 46] for a more detailed discussion on upper gradients.

Definition 2.1. (i) The Newtonian space $N^{1,p}(X, \mu_\lambda)$, $1 \le p < \infty$, is defined as the collection of all the functions for which

$$\|u\|_{N^{1,p}(X,\mu_\lambda)} := \left(\int_X |u|^p \, d\mu_\lambda + \inf_g \int_X g^p \, d\mu_\lambda \right)^{1/p} < \infty,$$

where the infimum is taken over all upper gradients of u.

(ii) Let Φ be a Young function. Then the Orlicz space $L^\Phi(X)$ is defined by setting

$$L^\Phi(X, \mu_\lambda) = \left\{ u : X \to \mathbb{R} : u \text{ measurable}, \int_X \Phi(\alpha |u|) \, d\mu_\lambda < +\infty \text{ for some } \alpha > 0 \right\}.$$

The Orlicz space $L^\Phi(X, \mu_\lambda)$ is a Banach space equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(X,\mu_\lambda)} = \inf \left\{ k > 0 : \int_X \Phi(|u|/k) \, d\mu_\lambda \le 1 \right\}.$$

(iii) For any Young function Φ, the *Orlicz-Sobolev space* $N^{1,\Phi}(X, \mu_\lambda)$ is defined as the collection of all the functions u for which the norm of u defined as

$$\|u\|_{N^{1,\Phi}(X,\mu_\lambda)} = \|u\|_{L^\Phi(X,\mu_\lambda)} + \inf_g \|g\|_{L^\Phi(X,\mu_\lambda)}$$

is finite, where the infimum is taken over all upper gradients of u.

We refer to [51, section 2.2] and [38, 43] for more details about Young functions.

We equip ∂X with the natural probability measure ν by distributing the unit mass uniformly on ∂X. Then the boundary $(\partial X, \nu)$ is an Ahlfors Q-regular space with Hausdorff dimension $Q = \frac{\log K}{\epsilon}$. Hence

$$\nu(B_{\partial X}(\xi, r)) \approx r^Q = r^{\log K/\epsilon},$$

for any $\xi \in \partial X$ and $0 < r \le \operatorname{diam}(\partial X)$. For more details about the measure ν, we refer to [6, Lemma 5.2].

Inspired by the Euclidean setting, we try to characterize the trace spaces of $N^{1,p}(X,\mu_\lambda)$ and $N^{1,\Phi}(X,\mu_\lambda)$ by using dyadic-type norms. Towards this, we give a dyadic decomposition on the boundary ∂X of the K-ary tree X: Let $V_n = \{x_j^n : j = 1, 2, \cdots, K^n\}$ be the set of all n-level vertices of the tree X for any $n \in \mathbb{N}$, where a vertex x is of *n-level* if $|x| = n$. Then we have that

$$V = \bigcup_{n\in\mathbb{N}} V_n$$

is the set containing all the vertices of the tree X. For any vertex $x \in V$, denote by I_x the set

$$\{\xi \in \partial X : \text{the geodesic } [0,\xi) \text{ passes through } x\}.$$

Let $\mathscr{Q} = \{I_x : x \in V\}$ and $\mathscr{Q}_n = \{I_x : x \in V_n\}$ for any $n \in \mathbb{N}$. Then $\mathscr{Q}_0 = \{\partial X\}$ and we have

$$\mathscr{Q} = \bigcup_{n\in\mathbb{N}} \mathscr{Q}_n.$$

The set $\mathscr{Q}$ is called a dyadic decomposition of ∂X. Clearly, for any $n \in \mathbb{N}$ and $I \in \mathscr{Q}_n$, there is a unique element $\widehat{I}$ in $\mathscr{Q}_{n-1}$ such that I is a subset of it. It is easy to see that if $I = I_x$ for some $x \in V_n$, then $\widehat{I} = I_y$ with y the unique parent of x in the tree X. Hence the structure of the tree X gives the structure of our dyadic decomposition of ∂X.

In [B] and [D], we introduced the following Besov-type spaces.

Definition 2.2. (i) For $0 \le \theta < 1$, $p \ge 1$ and $\lambda \in \mathbb{R}$, the Besov-type space $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ consists of all the functions $f \in L^p(\partial X)$ for which the $\dot{\mathcal{B}}_p^{\theta,\lambda}$-dyadic energy of f defined as

$$\|f\|^p_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)} := \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda} \sum_{I\in\mathscr{Q}_n} \nu(I) \left|f_I - f_{\widehat{I}}\right|^p$$

is finite. The norm on $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ is

$$\|f\|_{\mathcal{B}_p^{\theta,\lambda}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)}.$$

(ii) Let Φ be the Young function $\Phi(t) = t^p \log^{\lambda_1}(e+t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \ge 0$. Then the Orlicz-Besov space $\mathcal{B}_\Phi^{\theta,\lambda_2}(\partial X)$ consists of all the functions $f \in L^\Phi(\partial X)$ whose norm defined as

$$\|f\|_{\mathcal{B}_\Phi^{\theta,\lambda_2}(\partial X)} := \|f\|_{L^\Phi(\partial X)} + \inf\left\{k > 0 : |f/k|_{\dot{\mathcal{B}}_\Phi^{\theta,\lambda_2}(\partial X)} \le 1\right\}$$

is finite, where for any $g \in L^1_{\text{loc}}(\partial X)$, the $\dot{\mathcal{B}}_\Phi^{\theta,\lambda_2}$-dyadic energy is defined as

$$|g|_{\dot{\mathcal{B}}_\Phi^{\theta,\lambda_2}(\partial X)} := \sum_{n=1}^{\infty} e^{\epsilon n(\theta-1)p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I)\Phi\left(\frac{|g_I - g_{\widehat{I}}|}{e^{-\epsilon n}}\right).$$

The study of traces on regular trees was initiated in [6], which dealt with the measure μ_λ in (2.1) when $\lambda = 0$. It was shown that the trace space of $N^{1,p}(X,\mu_0)$ is the Besov space $B^{\theta}_{p,p}(\partial X)$, where the smoothness exponent of the Besov space is

$$\theta = 1 - \frac{\beta/\epsilon - Q}{p}, \quad 0 < \theta < 1.$$

Here the space $B^{\theta}_{p,p}(\partial X)$ consists of all the functions $f \in L^p(\partial X)$ for which the seminorm $\|f\|_{\dot{B}^{\theta}_{p}(\partial X)}$ defined as

$$\|f\|^p_{\dot{B}^{\theta}_{p,p}} := \int_{\partial X}\int_{\partial X} \frac{|f(\zeta)| - f(\xi)|^p}{d_X(\zeta,\xi)^{\theta p}\nu(B(\zeta, d_X(\zeta,\xi)))} d\nu(\xi)\, d\nu(\zeta)$$

is finite.

Here and in the rest of this section, for given Banach spaces $\mathbb{X}(\partial X)$ and $\mathbb{Y}(X)$, we say that the space $\mathbb{X}(\partial X)$ is a trace space of $\mathbb{Y}(X)$ if and only if there is a bounded linear operator $T : \mathbb{Y}(X) \to \mathbb{X}(\partial X)$ and there exists a bounded linear extension operator $E : \mathbb{X}(\partial X) \to \mathbb{Y}(X)$ that acts as a right inverse of T, i.e., $T \circ E = \mathrm{Id}$ on the space $\mathbb{X}(\partial X)$.

It was observed in [B, Proposition 2.13] that $B^{\theta}_{p,p}(\partial X) = \mathcal{B}^{\theta,0}_{p}(\partial X)$. Our first result from [B] generalized the above trace result.

Theorem 2.3. *Let X be a K-ary tree with $K \geq 2$. Fix $\beta > \log K$, $\epsilon > 0$ and $\lambda \in \mathbb{R}$. Suppose that $p \geq 1$ and $p > (\beta - \log K)/\epsilon$. Then the Besov-type space $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$ is the trace space of $N^{1,p}(X, \mu_\lambda)$ whenever $\theta = 1 - (\beta - \log K)/\epsilon p$.*

Another motivation for the above theorem was to study the dyadic energy defined by [26], introduced for the regularity of space-filling curves. This dyadic energy turns out to be equivalent to a $\dot{\mathcal{B}}^{\theta,\lambda}_{p}(\partial X)$-energy. We refer the interested reader to the introduction of the paper [B] for more details.

In [D], we further generalized the above result to the Orlicz case where the Young function is of the form $\Phi(t) = t^p \log^{\lambda}(e + t)$.

Theorem 2.4. *Let X be a K-ary tree with $K \geq 2$ and let $\Phi(t) = t^p \log^{\lambda_1}(e + t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \geq 0$. Fix $\lambda_2 \in \mathbb{R}$ and assume that $p > (\beta - \log K)/\epsilon > 0$. Then the trace space of $N^{1,\Phi}(X, \mu_{\lambda_2})$ is the space $\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$ where $\theta = 1 - (\beta - \log K)/\epsilon p$.*

The next result from [D] identifies the Orlicz-Besov space $\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$ as the Besov space $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$.

Proposition 2.5. *Let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$. Let $\Phi(t) = t^p \log^{\lambda_1}(e + t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \geq 0$. Assume that $\lambda_1 + \lambda_2 = \lambda$. Then the Banach spaces $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$ and $\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$ coincide, i.e., $\mathcal{B}^{\theta,\lambda}_{p}(\partial X) = \mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$.*

By combining Theorem 2.4 and Proposition 2.5, we obtain the following result.

Corollary 2.6. *Let X be a K-ary tree with $K \geq 2$. Let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$. Assume that $p > (\beta - \log K)/\epsilon > 0$ and let $\theta = 1 - (\beta - \log K)/\epsilon p$. Let $\Phi(t) = t^p \log^{\lambda_1}(e + t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \geq 0$. Then the Besov-type space $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$ is the trace space of $N^{1,\Phi}(X, \mu_{\lambda_2})$ whenever $\lambda_1 + \lambda_2 = \lambda$.*

Let us go back to the trace result for the Newtonian space $N^{1,p}(X,\mu)$. Recall that we required in Theorem 2.3 (actually also in Theorem 2.4 and Corollary 2.6) that $p > (\beta - \log K)/\epsilon > 0$. The assumption that $\beta - \log K > 0$ is necessary in the sense that we need to make sure that the measure μ_λ on X is doubling; see [B, Section 2.2]. The requirement that $p > (\beta - \log K)/\epsilon$ will ensure that $\theta > 0$. So it is natural to consider the case $p = (\beta - \log K)/\epsilon \geq 1$. We obtained the following result in [B].

Theorem 2.7. *Let X be a K-ary tree with $K \geq 2$. Fix $\beta > \log K$, $\epsilon > 0$ and $\lambda \in \mathbb{R}$. Suppose that $p = (\beta - \log K)/\epsilon \geq 1$ and $\lambda > p-1$ if $p > 1$ or $\lambda \geq 0$ if $p = 1$. Then there is a bounded linear trace operator $T : N^{1,p}(X,\mu_\lambda) \to L^p(\partial X)$, defined via limits along geodesic rays. Here, $\lambda > p-1$ is sharp in the sense that for any $p > 1$, $\delta > 0$ and $\lambda = p - 1 - \delta$, there exists a function $u \in N^{1,p}(X,\mu_\lambda)$ so that $Tu(\xi) = \infty$ for every $\xi \in \partial X$.*

Moreover, for any $p = (\beta - \log K)/\epsilon \geq 1$, there exists a bounded nonlinear extension operator $E : L^p(\partial X) \to N^{1,p}(X)$ so that the trace operator $\widehat{T}$ defined via limits of $E(f)$ along geodesic rays for $f \in L^p(\partial X)$ satisfies $\widehat{T} \circ E = \mathrm{Id}$ on $L^p(\partial X)$.

A result similar to Theorem 2.7 for the weighted Newtonian space $N^{1,p}(\Omega, \omega\, d\mu)$ with a suitable weight ω was also established in [32] under the assumption that Ω is a bounded domain that admits a p-Poincaré inequality and whose boundary $\partial\Omega$ is endowed with a p-co-dimensional Ahlfors regular measure. In Theorem 2.7, for the case $p = (\beta - \log K)/\epsilon > 1$, we required that $\lambda > p-1$ to ensure the existence of limits along geodesic rays. In the case $p = (\beta - \log K)/\epsilon = 1$, these limits exist even for $\lambda = 0$, and there is a nonlinear extension operator that acts as a right inverse of the trace operator, similarly to the case of $W^{1,1}$ in Euclidean setting; see [17, 41].

Notice that $N^{1,p}(X,\mu_\lambda)$ is a strict subset of $N^{1,p}(X)$ when $\lambda > 0$. Hence except for the case $p = 1$ and $\lambda = 0$, Theorem 2.7 does not even tell whether the trace operator T is surjective or not. The following result shows that the trace operator T is actually not surjective when $p = (\beta - \log K)/\epsilon = 1$ and $\lambda > 0$, and gives a full characterization of the trace spaces of the Newtonian space $N^{1,1}(X,\mu_\lambda)$. Towards stating the result, we first define a Besov-type space.

Definition 2.8. For $\lambda > 0$, the Besov-type space $\mathcal{B}_1^\alpha(\partial X)$ consists of all the functions $f \in L^1(\partial X)$ for which the $\dot{\mathcal{B}}^{0,\lambda}$-dyadic energy of f defined as

$$\|f\|_{\dot{\mathcal{B}}_1^\alpha(\partial X)} = \sum_{n=1}^{\infty} \alpha(n)^\lambda \sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I)|f_I - f_{\widetilde{I}}|$$

is finite. Here $\alpha(n) = 2^n$ and for any $I = I_x \in \mathcal{Q}_{\alpha(n)}$ with $x \in V_{\alpha(n)}$ and $n \geq 1$, we denote $\widetilde{I} = I_y$ where $y \in V_{\alpha(n-1)}$ is the ancestor of x in X. The norm on $\mathcal{B}_1^\alpha(\partial X)$ is

$$\|f\|_{\mathcal{B}_1^\alpha(\partial X)} := \|f\|_{L^1(\partial X)} + \|f\|_{\dot{\mathcal{B}}_1^\alpha(\partial X)}.$$

We obtained the following characterization in [B].

Theorem 2.9. *Let X be a K-ary tree with $K \geq 2$. Fix $\beta > \log K$, $\epsilon > 0$ and $\lambda > 0$. Suppose that $p = 1 = (\beta - \log K)/\epsilon$. Then the trace space of $N^{1,1}(X,\mu_\lambda)$ is the Besov-type space $\mathcal{B}_1^\alpha(\partial X)$.*

We stress that $\mathcal{B}_1^\alpha(\partial X)$ and $\mathcal{B}_1^{0,\lambda}(\partial X)$ are different spaces. More precisely, $\mathcal{B}_1^{0,\lambda}(\partial X)$ is a strict subspace of $\mathcal{B}_1^\alpha(\partial X)$, see Proposition 3.8 and Example 3.9 of [B]. Trace results similar to Theorem 2.9 in the Euclidean setting can be found in [18, 54].

3. Traces of $N^{1,1}$, $M^{1,1}$ and BV

Let us first recall some existing trace results. In [30], the authors studied the boundary traces, or traces for short, of BV functions in suitably regular domains. Typically, the boundary trace Tu of a function u in a domain Ω is defined by the condition

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - Tu(x)|\, d\mu = 0$$

for a.e. $x \in \partial\Omega$ with respect to the codimension-1 Hausdorff measure $\mathcal{H}$. In [33] the authors considered the corresponding extension problem, that is, the problem of finding a function whose trace is a prescribed L^1-function on the boundary. They showed that in sufficiently regular domains, the trace operator for BV functions is surjective, and that in fact the extension can always be taken to be a Newton-Sobolev function. This implies that the trace space of both $\mathrm{BV}(\Omega)$ and $N^{1,1}(\Omega)$ is $L^1(\partial X)$. In the Euclidean setting, it is know by [3] and [17] that the trace spaces of $W^{1,1}(\mathbb{R}^{n+1}_+)$ and $\mathrm{BV}(\mathbb{R}^{n+1}_+)$ coincide with each other, namely with the space $L^1(\mathbb{R}^n)$.

We would like to consider boundary traces from a different viewpoint. Unlike in the usual literature, we assume very little regularity of the domain, meaning that traces need not always exist. We are nonetheless able to show in various cases that for a given function, it is possible to find a more regular function that "achieves the same boundary values". In particular, if the original function has a boundary trace, then the more regular function has the same trace. This sheds further light on the extension problem. Not only considering BV- and $N^{1,1}$-functions, we include $M^{1,1}$-functions into discussion. We begin with necessary definitions.

In this section, we assume that (X, d, μ) is a complete metric space equipped with a doubling measure μ and supporting a $(1,1)$-Poincaré inquality. Here we call μ a doubling measure if there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d\mu(B(x,r)) < \infty$$

for every ball $B(x,r) := \{y \in X : d(y,x) < r\}$. By iterating the doubling condition, for every $0 < r \leq R$ and $y \in B(x,R)$, we have

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq 4^{-s}\left(\frac{r}{R}\right)^s, \tag{3.1}$$

for any $s \geq \log_2 C_d$. See [21, Lemma 4.7] or [5] for a proof of this. We fix such an $s > 1$ and call it the *homogeneous dimension*. We say that X supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_P r \fint_{B(x,\lambda r)} g\, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u\,d\mu := \frac{1}{\mu(B(x,r))}\int_{B(x,r)} u\,d\mu.$$

Let $\Omega \subset X$ be a nonempty open set. We can regard it as a metric space in its own right, equipped with the metric induced by X and the measure $\mu|_\Omega$ which is the restriction of μ to subsets of Ω. This restricted measure $\mu|_\Omega$ is a Radon measure, see [25, Lemma 3.3.11].

Definition 3.1. (i) We say that an open set Ω satisfies a *measure density condition* if there is a constant $c_m > 0$ such that

$$\mu(B(x,r)\cap\Omega) \geq c_m \mu(B(x,r)) \tag{3.2}$$

for every $x \in \overline{\Omega}$ and every $r \in (0, \operatorname{diam}(\Omega))$.

(ii) We say that Ω satisfies a *measure doubling condition* if the measure $\mu|_\Omega$ is a doubling measure, i.e., there is a constant $c_d > 0$ such that

$$0 < \mu(B(x,2r)\cap\Omega) \leq c_d \mu(B(x,r)\cap\Omega) < \infty \tag{3.3}$$

for every $x \in \overline{\Omega}$ and every $r > 0$.

Notice that if Ω satisfies the measure density condition, then it satisfies the measure doubling condition.

The Newtonian space $N^{1,1}(\Omega)$ is defined analogously as in Definition 2.1. So we only present the definitions of $\mathrm{BV}(\Omega)$ and $M^{1,1}(\Omega)$.

Given a function $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of u in Ω by

$$\|Du\|(\Omega) := \inf\left\{\liminf_{i\to\infty}\int_\Omega g_{u_i}\,d\mu :\, u_i \in N^{1,1}_{\mathrm{loc}}(\Omega), u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\},$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i in Ω. We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in \mathrm{BV}(\Omega)$, if $\|Du\|(\Omega) < \infty$. The BV norm is defined by

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

We refer to [1, 2, 14, 15, 19, 34, 57] for more information about bounded variation functions.

Towards the Hajłasz-Sobolev space, we define $M^{1,1}(\Omega)$ to be the set of all the functions $u \in L^1(\Omega)$ for which there exists $0 \leq g \in L^1(\Omega)$ and a set $K \subset \Omega$ of measure zero such that for all $x, y \in \Omega \setminus K$ we have the estimate

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)). \tag{3.4}$$

The corresponding norm is obtained by setting

$$\|u\|_{M^{1,1}(\Omega)} = \|u\|_{L^1(\Omega)} + \inf \|g\|_{L^1(\Omega)},$$

where the infimum is taken over all admissible functions g in (3.4). We refer to [20, 21] for more properties of the Hajłasz-Sobolev space $M^{1,1}(\Omega)$. The space $M^{1,1}_{c_H}(\Omega)$ is defined exactly in the same manner as the space $M^{1,1}(\Omega)$ except for one difference: in the definition of $M^{1,1}_{c_H}(\Omega)$, the condition (3.4) is assumed to hold only for points $x, y \in \Omega \setminus K$ that satisfy the condition

$$d(x,y) \leq c_H \cdot \min\{d(x, X\setminus\Omega), d(y, X\setminus\Omega)\}, \tag{3.5}$$

where $0 < c_H < 1$ is a constant.

We define the codimension-1 Hausdorff measure $\mathcal{H}$ of $A \subset X$ as

$$\mathcal{H}(A) := \lim_{R \to 0^+} \mathcal{H}_R(A),$$

where

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{j \in I} \frac{\mu(B(x_j, r_j))}{r_j} : A \subset \bigcup_{j \in I} B(x_j, r_j),\ r_j \le R,\ I \subset \mathbb{N} \right\}.$$

We give the following definitions for the boundary trace, or trace for short, of a function defined on an open set Ω.

Definition 3.2. Let $\Omega \subset X$ be an open set and let u be a μ-measurable function on Ω.

(i) A number $Tu(x)$ is the trace of u at $x \in \partial X$ if we have

$$\lim_{r \to 0^+} \fint_{B(x,r) \cap \Omega} |u - Tu(x)|\, d\mu = 0.$$

We say that u has a trace Tu in $\partial\Omega$ if $Tu(x)$ exists for $\mathcal{H}$-almost every $x \in \partial X$.

(ii) Let $\mathbb{X}(\Omega)$ be a function function space on Ω. A function space $\mathbb{Y}(\partial X, \mathcal{H})$ on ∂X is the trace space of $\mathbb{X}(\Omega)$ if the trace operator $u \mapsto Tu$ defined in (i) is a bounded linear surjective operator from $\mathbb{X}(\Omega)$ to $\mathbb{Y}(\partial X, \mathcal{H})$.

(iii) Let $\widetilde{\mathcal{H}}$ be a measure on ∂X. Let $\mathbb{X}(\Omega)$ be a Banach function space on Ω. A Banach space $\mathbb{Y}(\partial X, \widetilde{\mathcal{H}})$ on ∂X is the trace space of $\mathbb{X}(\Omega)$ with respect to $\widetilde{\mathcal{H}}$, if the trace operator $u \mapsto Tu$ defined in (i) by replacing $\mathcal{H}$ by $\widetilde{\mathcal{H}}$ is a bounded linear surjective operator from $\mathbb{X}(\Omega)$ to $\mathbb{Y}(\partial X, \widetilde{\mathcal{H}})$.

For BV functions we proved the following results in [C].

Theorem 3.3. *Let $u \in \mathrm{BV}(\Omega)$ and let s be the homogeneous dimension in (3.1).*

(i) There exists $v \in N^{1,1}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that

$$\fint_{B(x,r) \cap \Omega} |v - u|^{s/(s-1)}\, d\mu \to 0 \quad \text{as } r \to 0^+$$

uniformly for all $x \in \partial\Omega$.

(ii) Let $1 \le q < \infty$. Then there exists $v \in N^{1,1}(\Omega)$ such that

$$\fint_{B(x,r) \cap \Omega} |v - u|^{q}\, d\mu \to 0 \quad \text{as } r \to 0^+$$

uniformly for all $x \in \partial\Omega$.

Whenever there exists a BV extension of a given function defined on the boundary, it is possible to also find a Newtonian-Sobolev extension. We obtained in [C] the following corollary from above theorem.

Corollary 3.4. *The trace spaces of $\mathrm{BV}(\Omega)$ and $N^{1,1}(\Omega)$ are the same.*

Here and in the rest of this section, for two Banach function spaces $\mathbb{X}(\Omega)$ and $\mathbb{Y}(\Omega)$, that the trace spaces of $\mathbb{X}(\Omega)$ and $\mathbb{Y}(\Omega)$ are the same means that if the Banach function space $\mathbb{Z}(\partial X)$ is the trace space of $\mathbb{X}(\Omega)$, then it is also the trace space of $\mathbb{Y}(\Omega)$, and vice versa.

Corollary 3.4 is stronger than we expected; it says that we can obtain the existence of the trace and the trace space of $\mathrm{BV}(\Omega)$ by only knowing the existence of the trace and the trace space of $N^{1,1}(\Omega)$, which is nontrivial, since $N^{1,1}(\Omega)$ is a strict subset of $\mathrm{BV}(\Omega)$.

For the spaces $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$, we obtain the following result in in [C].

Theorem 3.5. *Suppose Ω satisfies the measure density condition (3.2). Then there exists $0 < c_H < 1$ such that for any $u \in N^{1,1}(\Omega)$, there is $v \in M^{1,1}_{c_H}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ satisfying $\|v\|_{M^{1,1}_{c_H}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}$ and*

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |v-u|\, d\mu = 0$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$, where $\mathcal{H}$ is the codimension-1 Hausdorff measure.

If additionally Ω is a uniform domain, then v can be chosen in $M^{1,1}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

Here a domain $\Omega \subset X$ is called *uniform* if there is a constant $c_U \in (0,1]$ such that every pair of distinct points $x, y \in \Omega$ can be connected by a curve $\gamma\colon [0,\ell_\gamma] \to \Omega$ parametrized by arc-length such that $\gamma(0) = x$, $\gamma(\ell_\gamma) = y$, $\ell_\gamma \le c_U^{-1} d(x,y)$, and

$$\operatorname{dist}(\gamma(t), X\setminus\Omega) \ge c_U \min\{t, \ell_\gamma - t\} \quad \text{for all} \quad t \in [0,\ell_\gamma].$$

More generally, instead of only studying the codimension-1 Hausdorff measure, we may study any arbitrary boundary measure $\widetilde{\mathcal{H}}$ on ∂X. Then we prove the following result.

Theorem 3.6. *Suppose Ω satisfies the measure doubling condition (3.3). Let $\widetilde{\mathcal{H}}$ be any Radon measure on $\partial\Omega$. Suppose that, for a given $u \in N^{1,1}(\Omega)$, there exists a function Tu such that*

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - Tu(x)|\, d\mu = 0$$

for $\widetilde{\mathcal{H}}$-a.e. $x \in \partial\Omega$. Then there exist $0 < c_H < 1$ and $v \in M^{1,1}_{c_H}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $\|v\|_{M^{1,1}_{c_H}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}$ and

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |v - Tu(x)|\, d\mu = 0$$

for $\widetilde{\mathcal{H}}$-a.e. $x \in \partial\Omega$.

If additionally Ω is a uniform domain, then v can be chosen in $M^{1,1}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

Similarly to Corollary 3.4, from Theorem 3.5 and Theorem 3.6 we obtain the following corollary.

Corollary 3.7. *Let $\Omega \subset X$ be a uniform domain and suppose that Ω satisfies the measure doubling condition (3.3). Then, for any given boundary measure $\widetilde{\mathcal{H}}$, the trace spaces of $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$ with respect to any boundary measure $\widetilde{\mathcal{H}}$ on ∂X are the same.*

A direct consequence of Corollary 3.4 and Corollary 3.7 is that under a proper setting, the trace spaces of BV, Newton-Sobolev space $N^{1,1}$, and Hajłasz-Sobolev space $M^{1,1}$ are the same. Hence we can obtain many trace results directly from the trace results for the Newton-Sobolev spaces in the literature.

Since $(\mathbb{R}^{n+1}_+, \mu_\alpha)$ and the weighted regular tree (X, μ_λ) are uniform and support an $(1,1)$-Poincaré inequality (see Example 5.7 and Example 5.10 of [C]) and the trace operator we used here is equivalent to the one in [B] on regular trees (see [36]), the trace results obtained for Sobolev spaces $W^{1,1}(\mathbb{R}^{n+1}, \mu_\alpha)$ in [A] and for Newtonian spaces $N^{1,1}(X, \mu_\lambda)$ in [B] can be applied directly to $\mathrm{BV}(\mathbb{R}^{n+1}, \mu_\alpha)$, $M^{1,1}(\mathbb{R}^{n+1}, \mu_\alpha)$ and $\mathrm{BV}(X, \mu_\lambda)$, $M^{1,1}(X, \mu_\lambda)$, respectively. We refer to [C, Section 5] for more applications of Corollary 3.4 and Corollary 3.7.

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Included articles

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Traces of weighted function spaces: dyadic norms and Whitney extensions

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TRACES OF WEIGHTED FUNCTION SPACES: DYADIC NORMS AND WHITNEY EXTENSIONS

PEKKA KOSKELA, TOMÁS SOTO, AND ZHUANG WANG

Abstract. The trace spaces of Sobolev spaces and related fractional smoothness spaces have been an active area of research since the work of Nikolskii, Aronszajn, Slobodetskii, Babich and Gagliardo among others in the 1950's. In this paper we review the literature concerning such results for a variety of weighted smoothness spaces. For this purpose, we present a characterization of the trace spaces (of fractional order of smoothness), based on integral averages on dyadic cubes, which is well adapted to extending functions using the Whitney extension operator.

1. Introduction

In 1957, Gagliardo [13] gave a characterization of the trace space of the first order Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, on a given Lipschitz domain $\Omega \subset \mathbb{R}^d$ in terms of the convergence of a suitable double integral of the boundary values. This work extended the earlier results by Aronszajn [1] and Slobodetskii and Babich [45] concerning the case $p = 2$. The trace space $B_p^{1-1/p}(\partial\Omega)$, consisting of all $(d-1)$-Hausdorff measurable functions u on $\partial\Omega$ with

$$\|u\|^p_{L^p(\partial\Omega,\mathcal{H}_{d-1})} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{(d-1)+(1-1/p)p}} d\mathcal{H}_{d-1}(x) d\mathcal{H}_{d-1}(y) < \infty, \tag{1}$$

is nowadays commonly called a fractional Sobolev space, a Slobodetskii space or a Besov space. Actually, Gagliardo also verified that the trace space of $W^{1,1}(\Omega)$ is $L^1(\partial\Omega)$ (see also [32] for a different proof of this fact). The norm estimates for the trace functions were obtained via Hardy inequalities, while the extension from the boundary was based on a Poisson-type convolution procedure. We refer to the seminal monographs by Peetre [40] and Triebel [49] for extensive treatments of the Besov spaces and related smoothness spaces.

A natural variant of this problem asks for the trace spaces associated to weights. Already in 1953, Nikolskii [38] had considered the trace problem for Sobolev spaces (for $p = 2$) with weights of the form $x \mapsto \operatorname{dist}(x, \partial\Omega)^\alpha$, where $-1 < \alpha < 1$. Other early related results were given by Lizorkin [29] and Vašarin [56]; see [37] and [33] for further references. More recently, Tyulenev [51, 52, 53, 54] has identified the traces of Sobolev and Besov spaces associated to more general Muckenhoupt A_p-weights. For related results concerning the traces of weighted Orlicz-Sobolev spaces, we refer to [11, 27, 39, 7, 8] and the references therein.

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On the other hand, a notable amount of recent research has focused on extending the theory of Sobolev spaces and related fractional smoothness spaces to the setting of metric measure spaces (including fractal subsets of Euclidean spaces); see e.g. [22] and the references therein as well as [18]. Works focusing on trace theorems for fractals and related subsets of a Euclidean space include [25, 41, 42, 50, 44, 23, 24, 5, 17] (we also refer to [55] for a recent result concerning traces on non-regular subsets of $\mathbb{R}^d$), while trace theorems in more general metric settings have been considered e.g. in [14, 43, 28, 31, 30]. In fact, the characterizations of fractional smoothness spaces as retracts of certain sequence spaces in [12], [18, Section 7] and [4, Proposition 6.3] can also be seen as abstract trace theorems.

Motivated by these works, we revisit the Euclidean setting, viewing the upper half-space $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty)$ as a particularly nice metric space endowed with a weighted measure. We shall introduce equivalent norms for the Besov spaces based integral averages on dyadic cubes. These norms are well adapted for studying the extension of functions defined on $\mathbb{R}^d$ to $\mathbb{R}^{d+1}_+$ via the natural *Whitney extension*. In contrast, the extension operator e.g. in [33] is based on the Poisson kernel.

Let us begin with a concrete example. We consider functions defined on the real line, but as we will later see, the discussion below generalizes to the setting of higher dimensions as well.

Given $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ and an interval $I \subset \mathbb{R}$, set

$$u(I) := \frac{1}{|I|}\int_I u(x)dx,$$

where $|I|$ is the length of the interval I. For each $k \in \mathbb{N}_0$, fix a dyadic decomposition of $\mathbb{R}$ into closed intervals $\{I_{k,i}\}_{i\in\mathbb{Z}}$ so that each $I_{k,i}$ has length 2^{-k} and $I_{k,i} \cap I_{k,j} \neq \emptyset$ exactly when $|i-j| \leq 1$. Consider the condition

$$\|u\|^2_{L^2(\mathbb{R})} + \sum_{k\in\mathbb{N}_0}\sum_{i\in\mathbb{Z}} |u(I_{k,i}) - u(I_{k,i+1})|^2 < \infty. \tag{2}$$

Now write $Q_{k,i} := I_{k,i} \times [2^{-k}, 2^{-k+1}]$ for all admissible k and i. Then these squares give us a Whitney decomposition of the upper half-plane $\mathbb{R}^2_+$. Pick a partition of unity in $\bigcup_{k,i} Q_{k,i}$ consisting of functions $\varphi_{k,i} \in C^\infty(\mathbb{R}^2_+)$ such that $|\nabla\varphi_{k,i}| \leq 5\cdot 2^k$ and the support of $\varphi_{k,i}$ is contained in a 2^{-k-2}-neighborhood of $Q_{k,i}$. For $u \in L^1_{\mathrm{loc}}(\mathbb{R})$, define

$$\mathcal{E}u := \sum_{k\in\mathbb{N}_0,\, i\in\mathbb{Z}} u(I_{k,i})\varphi_{k,i}. \tag{3}$$

Given $f \in W^{1,2}(\mathbb{R}^2_+)$, the trace function $u := \mathcal{R}f \colon \mathbb{R}^2 \to \mathbb{C}$, defined by

$$\mathcal{R}f(x) = \lim_{r\to 0}\frac{1}{m_2\big(B\big((x,0),r\big)\cap\mathbb{R}^2_+\big)}\int_{B((x,0),r)\cap\mathbb{R}^2_+} f(y)dm_2(y),$$

where m_2 stands for the 2-dimensional Lebesgue measure, is well-defined pointwise almost everywhere and satisfies the condition (2). Conversely, if $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfies (2), we have $\mathcal{E}u \in W^{1,2}(\mathbb{R}^2_+)$ with the expected norm bound and $\mathcal{R}(\mathcal{E}u) = u$ pointwise almost everywhere.

We conclude that $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ belongs to the trace space of $W^{1,2}(\mathbb{R}^2_+)$ if and only if (2) holds. Hence the condition (2) should characterize the space $B^{1/2}_2(\mathbb{R})$. This is indeed the case; a direct proof is given in Subsection 7.2 of the Appendix.

Let us next consider the following generalized form of the condition (2):

$$\|u\|^p_{L^p(\mathbb{R})} + \sum_{k\in\mathbb{N}_0} 2^{spk}2^{-k} \sum_{i\in\mathbb{Z}} |u(I_{k,i}) - u(I_{k,i+1})|^p < \infty, \tag{4}$$

where $1 < p < \infty$ and $0 < s < 1$. Above we saw that the choice $p = 2$ and $s = 1/2$ yields the trace space of $W^{1,2}(\mathbb{R}^2_+)$. Similarly, it turns out that the condition (4) characterizes the trace space of $W^{1,p}(\mathbb{R}^2_+)$ when $s = 1 - 1/p$. Where does this value of s come from? The so-called *differential dimension* of the space B^s_p over an n-dimensional Euclidean space is $s - n/p$, and the same holds for the space $W^{1,p}$ with 1 in place of s; see e.g. [49, Section 3.4.1]. Hence the order of smoothness s of the trace space should satisfy

$$s - \frac{1}{p} = 1 - \frac{2}{p},$$

which rewrites as $s = 1 - 1/p$.

Let us now try to extend a function $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ to a weighted Sobolev space, by requiring that

$$\int_{\mathbb{R}^2_+} |\mathcal{E}u(x)|^p \operatorname{dist}\big(x, \mathbb{R}\times\{0\}\big)^\alpha dx + \int_{\mathbb{R}^2_+} |\nabla(\mathcal{E}u)(x)|^p \operatorname{dist}\big(x, \mathbb{R}\times\{0\}\big)^\alpha dx < \infty, \tag{5}$$

where $\alpha > -1$ and $\mathcal{E}u$ is as defined as above. It turns out that when $\alpha \in (-1, p-1)$, the condition (5) is satisfied when u satisfies (4) with $s = 1 - (\alpha+1)/p$. On the other hand, since

$$\mu_\alpha\big(B\big((x,0), r\big) \cap \mathbb{R}^2_+\big) \approx r^{2+\alpha}$$

for all $x \in \mathbb{R}$ and $r > 0$, where μ_α is the measure associated to the weight $x \mapsto \operatorname{dist}(x, \mathbb{R})^\alpha$, we also see that $\alpha + 1 = (2+\alpha) - 1$ appears as a local codimension of $\mathbb{R}$ with respect to the metric measure space $(\mathbb{R}^2_+, \mu_\alpha)$. Hence the drop in the order of the derivative from one to the fractional order s is determined by p and this codimension.

Would the condition (2) allow us also to extend functions from $\mathbb{R}$ to a higher-dimensional weighted Euclidean space, e.g. $(\mathbb{R}^3, \mu_\alpha)$? If so, then the correct condition for the parameter α would be $\alpha > -2$ and the role of $2 + \alpha$ above should be taken by $3 + \alpha$. We recover $s = 1/2$ when $(\alpha + 2)/p = 1/2$, which for $p = 2$ gives $\alpha = -1$. This indeed works: (2) holds exactly when u is in the trace of $W^{1,2}(\mathbb{R}^3, \mu_{-1})$, where the measure μ_{-1} is associated to the weight $x \mapsto \operatorname{dist}(x, \mathbb{R}\times\{0\})^{-1}$ in $\mathbb{R}^3$, and in this case u can be extended as a function in $W^{1,2}(\mathbb{R}^3, \mu_{-1})$ with a suitable modification of the Whitney extension operator (3).

Can we find yet further function spaces whose traces are characterized by the condition (2) or the condition (4)? Towards this, let us mention that the space characterized by (4) coincides with the *diagonal Triebel-Lizorkin space* $\mathcal{F}^s_{p,p}(\mathbb{R})$. The scale of Triebel-Lizorkin spaces $\mathcal{F}^s_{p,q}(\mathbb{R}^d)$ on the d-dimensional Euclidean space, where $1 \le p < \infty$, $0 < q \le \infty$ and $0 < s < 1$, is another widely-studied family of fractional smoothness spaces that arise e.g. as the complex interpolation spaces between $L^p(\mathbb{R}^d)$ and $W^{1,p}(\mathbb{R}^d)$. The discussion above concerning the traces of weighted Sobolev spaces, with suitable modifications for the parameter ranges, turns out to hold for the traces of these function spaces as well. In particular, when $s \in (0,1)$ and $\alpha \in (-1, sp-1)$, the condition (4) with $s - (\alpha+1)/p$ in place of s characterizes the traces of the functions in $\mathcal{F}^s_{p,q}(\mathbb{R}^2_+, \mu_\alpha)$ for any admissible q. A similar trace theorem for the scale of Besov spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^2_+, \mu_\alpha)$

is formulated as Theorem 1.2 below. The precise definitions of these spaces are given in the next section.

Let us summarize the above discussion. The Whitney extension operator $\mathcal{E}$ extends a Besov space $B^s_p(\mathbb{R}) := \mathcal{B}^s_{p,p}(\mathbb{R})$ with given smoothness $s \in (0,1)$ linearly and continuously to a number of different (weighted) smoothness spaces defined on $\mathbb{R}^2_+$, the trace of all of whose equals $\mathcal{B}^s_{p,p}(\mathbb{R})$. Moreover, given $n \in \mathbb{N}$, a suitable variant of the Whitney extension operator $\mathcal{E}$ gives us a similar extension from $\mathcal{B}^s_{p,p}(\mathbb{R})$ to a variety of (weighted) function spaces defined on $\mathbb{R}^{1+n}$; this is discussed in detail in Subsection 7.4.

To finish discussion, let us state our main results more precisely. Given a pair of function spaces (X, Y), we say that they are a *Whitney trace-extension pair* if X is the trace space of Y in the usual sense and the extension from X to Y is obtained using the natural Whitney extension – this notion is also defined more precisely in **Definition 2.6** below. The measure μ_α (where $\alpha > -1$) below stands for the measure on $\mathbb{R}^{d+1}_+$ defined by

$$\mu_\alpha(E) = \int_E w_\alpha \, dm_{d+1}, \tag{6}$$

where $w_\alpha\colon \mathbb{R}^{d+1}_+ \to (0,\infty)$ is the weight $(x_1, x_2, \cdots, x_{d+1}) \mapsto \min(1, |x_{d+1}|)^\alpha$ and m_{d+1} is the standard Lebesgue measure on $\mathbb{R}^{d+1}_+$. Finally, the definitions of the relevant function spaces are given in **Section 2** below.

First off, we have the following trace theorem for the first-order Sobolev spaces.

Theorem 1.1. *Let $1 \le p < \infty$ and $-1 < \alpha < p-1$. Then $\big(\mathcal{B}^{1-(\alpha+1)/p}_{p,p}(\mathbb{R}^d), W^{1,p}(\mathbb{R}^{d+1}_+, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

The analogous trace theorem for the Besov scale reads as follows.

Theorem 1.2. *Let $0 < s < 1$, $1 \le p \le \infty$, $0 < q \le \infty$ and $-1 < \alpha < sp - 1$. Then $\big(\mathcal{B}^{s-(\alpha+1)/p}_{p,q}(\mathbb{R}^d), \mathcal{B}^s_{p,q}(\mathbb{R}^{d+1}_+, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

Finally, the trace theorem for the Triebel-Lizorkin spaces reads as follows.

Theorem 1.3. *Let $0 < s < 1$, $1 \le p < \infty$, $0 < q \le \infty$ and $-1 < \alpha < sp - 1$. Then $\big(\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d), \mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

We present a refinement of the case $p = 1$ of Theorem 1.1, where the Sobolev space $W^{1,1}(\mathbb{R}^{d+1}_+, \mu_\alpha)$ is replaced by a *Hardy-Sobolev space* $h^{1,1}(\mathbb{R}^{d+1}_+, \mu_\alpha)$, in **Section 6**. The variants of the results above with higher Euclidean codimension are given in **Subsection 7.4** of the Appendix.

The paper is organized as follows. In Section 2 we give the definitions relevant to our main results and recall some basic properties of the spaces and measures in question. Sections 3 through 6 contain the proofs of the aforementioned trace theorems. The Appendix (Section 7) deals with various technicalities that we saw fit to postpone from the other sections.

2. Definitions and preliminaries

In this section we present the definitions of the relevant function spaces and the Whithey extension operator. Before this, let us introduce some notation that will be used throughout the paper.

Notation. (i) The majority of this paper will deal with extensions of functions defined on the Euclidean space $\mathbb{R}^d$ to the half-space $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty)$. The dimension $d \in \mathbb{N} := \{1,2,3,\cdots\}$ will be fixed throughout the paper. The d-dimensional Lebesgue measure will be denoted by m_d. When talking about measures μ defined on $\mathbb{R}^{d+1}_+$, we may abuse notation by writing $\mu(B(x,r))$ for $\mu(B(x,r)\cap\mathbb{R}^{d+1}_+)$ when e.g. $x \in \mathbb{R}^d \times \{0\}$.

(ii) If (X,μ) is a measure space and A is a μ-measurable subset of X with $0 < \mu(A) < \infty$, we shall write

$$f_{A,\mu} := \fint_A f\,d\mu := \frac{1}{\mu(A)}\int_A f\,d\mu$$

whenever the latter quantity is well-defined, i.e. when $f \in L^1(A,\mu)$ or $f(x) \geq 0$ for μ-almost every $x \in A$. We may omit μ from the notation and simply write f_A when μ is the Lebesgue measure on an Euclidean space and there is no risk of confusion.

(iii) While $L^1_{\text{loc}}(\mathbb{R}^d)$ stands for the space of (complex-valued) locally integrable functions on $\mathbb{R}^d$ in the usual sense, we use the notation $L^1_{\text{loc}}(\mathbb{R}^{d+1}_+)$ with a slightly different meaning: it refers to the space functions that are integrable on bounded subsets of $\mathbb{R}^{d+1}_+$.

(iv) If f and g are two non-negative functions on the same domain, we may use the notation $f \lesssim g$ with the meaning that $f \leq Cg$ in the domain, where the constant $C > 0$ is usually independent of some parameters obvious from the context. The notation $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$.

Definition 2.1. *Suppose that μ is a Borel regular measure on $\mathbb{R}^d$ such that every Euclidean ball has positive and finite μ-measure.*

Let $p \in [1,\infty)$. Then $W^{1,p}(\mathbb{R}^d,\mu)$ is defined as the normed space of measurable functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that the first-order distributional derivatives of f coincide with functions in $L^1_{\text{loc}}(\mathbb{R}^d)$ and

$$\|f\|_{W^{1,p}(\mathbb{R}^d,\mu)} := \|f\|_{L^p(\mathbb{R}^d,\mu)} + \|\nabla f\|_{L^p(\mathbb{R}^d,\mu)} \tag{7}$$

is finite.

The space $W^{1,p}(\mathbb{R}^{d+1},\mu)$ is defined similarly, by replacing $\mathbb{R}^d$ with $\mathbb{R}^{d+1}_+$ in (7).

In order to formulate the dyadic norms of the relevant fractional smoothness spaces, we recall the standard dyadic decompositions of $\mathbb{R}^d$ and $\mathbb{R}^{d+1}_+$. Denote by $\mathscr{Q}_d$ the collection of dyadic semi-open cubes in $\mathbb{R}^d$, i.e. the cubes of the form $Q := 2^{-k}\big((0,1]^d + m\big)$, where $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^d$, and $\mathscr{Q}^+_d$ for the cubes in $\mathscr{Q}_d$ which are contained in the upper half-space $\mathbb{R}^{d-1} \times (0,\infty)$. Write $\ell(Q)$ for the edge length of $Q \in \mathscr{Q}_d$, i.e. 2^{-k} in the preceding representation, and $\mathscr{Q}_{d,k}$ for the cubes $Q \in \mathscr{Q}_d$ such that $\ell(Q) = 2^{-k}$. If $x \in \mathbb{R}^d$ (resp. $x \in \mathbb{R}^{d+1}_+$) and $k \in \mathbb{Z}$, we may write write Q^x_k for the unique cube in $\mathscr{Q}_d$ (resp. $\mathscr{Q}^+_{d+1}$) such that $x \in Q$ and $\ell(Q) = 2^{-k}$.

We say that Q and Q' in $\mathscr{Q}_d$ are neighbors and write $Q \sim Q'$ if $\frac{1}{2} \leq \ell(Q)/\ell(Q') \leq 2$ and $\overline{Q} \cap \overline{Q'} \neq \emptyset$. Note that every Q has a uniformly finite number of neighbors.

Definition 2.2. *Suppose that μ is a Borel regular measure on $\mathbb{R}^d$ such that every Euclidean ball has positive and finite μ-measure.*

Let $s \in (0,1)$, $p \in [1,\infty]$ and $q \in (0,\infty]$. Then the Besov space $\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)$ is defined as the normed (or quasi-normed when $q<1$) space of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)} := \|f\|_{L^p(\mathbb{R}^d,\mu)} + \Big(\sum_{k=0}^{\infty} 2^{ksq}\Big(\sum_{Q\in\mathscr{Q}_{d,k}} \mu(Q) \sum_{Q'\sim Q} \big|f_{Q,\mu} - f_{Q',\mu}\big|^p\Big)^{q/p}\Big)^{1/q} \tag{8}$$

(standard modification for $p=\infty$ and/or $q=\infty$) is finite.

Definition 2.3. *Suppose that μ is a Borel regular measure on $\mathbb{R}^d$ such that every Euclidean ball has positive and finite μ-measure.*

Let $s \in (0,1)$, $p \in [1,\infty)$ and $q \in (0,\infty]$. Then the Triebel-Lizorkin space $\mathcal{F}^s_{p,q}(\mathbb{R}^d,\mu)$ is defined as the normed (or quasi-normed when $q<1$) space of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^d,\mu)} := \|f\|_{L^p(\mathbb{R}^d,\mu)} + \Big(\int_{\mathbb{R}^d}\Big(\sum_{k=0}^{\infty} 2^{ksq} \sum_{Q'\sim Q^x_k} \big|f_{Q^x_k,\mu} - f_{Q',\mu}\big|^q\Big)^{p/q} d\mu(x)\Big)^{1/p} \tag{9}$$

(standard modification for $q=\infty$) is finite.

The spaces $\mathcal{B}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu)$ are defined similarly, by replacing $\mathbb{R}^d$ with $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty)$ in (8) and (9) respectively, and omitting the terms corresponding to the cubes $Q \in \mathscr{Q}_{d+1} \setminus \mathscr{Q}^+_{d+1}$ and $Q' \in \mathscr{Q}_{d+1} \setminus \mathscr{Q}^+_{d+1}$.

Remark 2.4. One routinely checks that $\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^d,\mu)$ are quasi-Banach spaces (Banach spaces for $q \geq 1$). Fubini's theorem implies that

$$\mathcal{F}^s_{p,p}(\mathbb{R}^d,\mu) = \mathcal{B}^s_{p,p}(\mathbb{R}^d,\mu)$$

with equivalent norms for $p \in [1,\infty)$, and the monotonicity of the ℓ^q-norms shows that

$$\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu) \subset \mathcal{B}^s_{p,q'}(\mathbb{R}^d,\mu) \quad \text{and} \quad \mathcal{F}^s_{p,q}(\mathbb{R}^d,\mu) \subset \mathcal{F}^s_{p,q'}(\mathbb{R}^d,\mu)$$

with continuous embeddings when $q' > q$. All this of course holds with $\mathbb{R}^{d+1}_+$ in place of $\mathbb{R}^d$.

In case μ is the standard Lebesgue measure on $\mathbb{R}^d$, we shall omit μ from the notation of the three function spaces above and simply write $W^{1,p}(\mathbb{R}^d)$, $\mathcal{B}^s_{p,q}(\mathbb{R}^d)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^d)$ where appropriate.

Remark 2.5. (i) A Besov quasinorm that is perhaps more standard in the literature is given by

$$f \mapsto \|f\|_{L^p(\mathbb{R}^d,\mu)} + \Big(\int_0^\infty t^{-sq}\Big(\int_{\mathbb{R}^d}\fint_{B(x,t)} |f(x)-f(y)|^p d\mu(y)d\mu(x)\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}. \tag{10}$$

A straightforward calculation using Fubini's theorem shows that if $q=p$ and $\mu = m_d$, then then the pth power of this this quasinorm is comparable to

$$\|f\|^p_{L^p(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} dxdy,$$

which is of the same form as the quantity (1) in the introduction.

(ii) To justify the Definitions 2.2 and 2.3 above, let us point out that if the measure μ is doubling with respect to the Euclidean metric, i.e. if there exists a constant $c \geq 1$ such that

$$\mu\big(B(x, 2r)\big) \leq c\mu\big(B(x, r)\big) \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0,$$

then the quasi-norm (8) is comparable to the quasi-norm (10) above. We refer to Subsection 7.2 of the Appendix for details.

(iii) Quasinorms similar to (8) and (9) in the setting of metric measure spaces were also considered in [43, Definition 5.1] in terms of a *hyperbolic filling* of $\mathbb{R}^d$. Another similar variant in the weighted Euclidean setting has been considered in [54].

We now give the definitions corresponding to the Whitney extensions discussed in the introduction. To this end, we have to define a partition of unity corresponding to the standard Whitney decomposition of the half-space $\mathbb{R}^{d+1}_+$. For $Q \in \mathscr{Q}_{d,k}$, $k \in \mathbb{Z}$, write $\mathscr{W}(Q) := Q \times (2^{-k}, 2^{-k+1}] \in \mathscr{Q}^+_{d+1,k}$. To simplify the notation in the sequel, we further define $\mathscr{Q}^0_d := \cup_{k \geq 0} \mathscr{Q}_{d,k}$.

It is then easy to see that $\{\mathscr{W}(Q) : Q \in \mathscr{Q}_d\}$ is a Whitney decomposition of $\mathbb{R}^d \times (0, \infty)$ with respect to the boundary $\mathbb{R}^d \times \{0\}$. For all $Q \in \mathscr{Q}^0_d$, define a smooth function $\psi_Q \colon \mathbb{R}^{d+1}_+ \to [0, 1]$ such that $\operatorname{Lip} \psi_Q \lesssim 1/\ell(Q)$, $\inf_{x \in \mathscr{W}(Q)} \psi_Q(x) > 0$ uniformly in $Q \in \mathscr{Q}^0_d$, $\operatorname{supp} \psi_Q$ is contained in an $\frac{\ell(Q)}{4}$-neighborhood of $\mathscr{W}(Q)$ and

$$\sum_{Q \in \mathscr{Q}^0_d} \psi_Q \equiv 1 \quad \text{in} \quad \bigcup_{Q \in \mathscr{Q}^0_d} \mathscr{W}(Q).$$

Let us point out that the sum above is locally finite – more precisely, it follows from the definition that

$$\operatorname{supp} \psi_Q \cap \operatorname{supp} \psi_{Q'} \neq \emptyset \quad \text{if and only if} \quad Q \sim Q'. \tag{11}$$

Definition 2.6. *(i) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then the Whitney extension $\mathcal{E}f \colon \mathbb{R}^{d+1}_+ \to \mathbb{C}$ is defined by*

$$\mathcal{E}f(x) = \sum_{Q \in \mathscr{Q}^0_d} \Big(\fint_Q f \, dm_d \Big) \psi_Q(x).$$

This definition gives rise in the obvious way to the linear operator $\mathcal{E} \colon L^1_{\mathrm{loc}}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^{d+1}_+)$.

(ii) Let $X \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a quasinormed function space on $\mathbb{R}^d$, and let Y be a quasinormed function space on the weighted half-space $(\mathbb{R}^{d+1}_+, \mu)$. We say that (X, Y) is a Whitney trace-extension pair *if $\mathcal{E}$ maps X continuously into Y, if the trace function $\mathcal{R}f$ defined by*

$$\mathcal{R}f(x) = \lim_{r \to 0} \fint_{B((x,0),r) \cap \mathbb{R}^{d+1}_+} f(y) \, d\mu(y), \tag{12}$$

is for all $f \in Y$ well-defined almost everywhere and belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, if $\mathcal{R}$ maps Y continuously into X and if

$$\mathcal{R}(\mathcal{E}f) = f$$

pointwise almost everywhere for all $f \in X$.

For the proofs of our main results, let us recall some basic facts about the weights w_α and measures μ_α defined in (6). First, it is well-known that for $\alpha > -1$, the weight w_α belongs to the Muckenhoupt class A_r for all $r > \max(\alpha+1, 1)$, which implies that the measure μ_α satisfies the doubling property with respect to the standard Euclidean metric (see e.g. [21, Chapter 15] or [9]). This in particular means that

$$\mu_\alpha(Q) \approx \mu_\alpha(Q') \quad \text{if} \quad Q \sim Q'.$$

A straightforward calculation also shows that

$$\mu_\alpha\big(B(x,r)\big) \approx r^{d+1+\alpha} \tag{13}$$

for all $x \in \mathbb{R}^d \times \{0\}$ and $0 < r \leq 1$.

Finally, let us recall the standard $(1,1)$-Poincaré inequality satisfied by the functions that are locally $W^{1,1}$-regular in the upper half-space. If Q is a cube in $\mathbb{R}^{d+1}_+$ such that $\operatorname{dist}(Q, \mathbb{R}^d \times \{0\}) > 0$ and $f \in W^{1,1}(Q)$, we have

$$\fint_Q |f - f_Q|\, dm_{d+1} \leq C\ell(Q) \fint_Q |\nabla f| dm_{d+1} \tag{14}$$

for some constant C independent of Q and f.

3. Proof of Theorem 1.1

Proof. (i) Let us first prove the desired norm inequality for the Whitney extension $\mathscr{E}f$ of $f \in \mathcal{B}^{1-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)$. We begin by noting that if $Q \in \mathscr{Q}^0_d$, it follows directly from the definitions that $w_\alpha \approx \ell(Q)^\alpha$ in $\mathscr{W}(Q)$, and hence we have $\mu_\alpha(\mathscr{W}(Q)) \approx \ell(Q)^\alpha m_{d+1}(\mathscr{W}(Q)) \approx \ell(Q)^{d+1+\alpha}$. Since the supports of the functions ψ_Q have bounded overlap, the $L^p(\mathbb{R}^{d+1}_+, \mu_\alpha)$-norm of $\mathscr{E}f$ is thus easy to estimate:

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}_+} |\mathscr{E}f|^p d\mu_\alpha &\lesssim \sum_{Q \in \mathscr{Q}^0_d} \mu_\alpha\big(\mathscr{W}(Q)\big) \fint_Q |f|^p dm_d \approx \sum_{Q \in \mathscr{Q}^0_d} \ell(Q)^{\alpha+1} \int_Q |f|^p dm_d \\
&= \sum_{k \geq 0} 2^{-k(\alpha+1)} \sum_{Q \in \mathscr{Q}_{d,k}} \int_Q |f|^p dm_d = \sum_{k \geq 0} 2^{-k(\alpha+1)} \int_{\mathbb{R}^d} |f|^p\, dm_d \\
&= \sum_{k \geq 0} 2^{-k(\alpha+1)} \|f\|^p_{L^p(\mathbb{R}^d)} \approx \|f\|^p_{L^p(\mathbb{R}^d)}.
\end{aligned} \tag{15}$$

In order to estimate the $L^p(\mathbb{R}^{d+1}_+, \mu_\alpha)$-norm of $|\nabla(\mathscr{E}f)|$, we divide the half-space $\mathbb{R}^{d+1}_+$ into two parts: $X_1 := \bigcup_{P \in \mathscr{Q}^0_d} \mathscr{W}(P)$ and $X_2 := \mathbb{R}^{d+1}_+ \setminus X_1$. Now if $x \in X_1$, i.e. $x \in \mathscr{W}(P)$ for some $P \in \mathscr{Q}^0_d$, we have that $\sum_{Q \in \mathscr{Q}^0_d} \psi_Q(x) = 1$, and as noted in (11), the terms in this sum are nonzero at most for the cubes Q such that $Q \sim P$. Hence

$$\begin{aligned}
\mathscr{E}f(x) - \fint_P f\, dm_d &= \sum_{Q \in \mathscr{Q}^0_d} \Big(\fint_Q f\, dm_d\Big)\psi_Q(x) - \fint_P f\, dm_d \\
&= \sum_{Q \sim P} \Big(\fint_Q f\, dm_d - \fint_P f\, dm_d\Big)\psi_Q(x) = \sum_{Q \sim P} (f_Q - f_P)\psi_Q(x),
\end{aligned}$$

and the Lipschitz continuity of the functions ψ_Q yields

$$|\nabla(\mathcal{E}f)(x)| \le |\mathrm{Lip}\,(\mathcal{E}f)(x)| = \Big|\mathrm{Lip}\,\Big(\mathcal{E}f(\cdot) - \fint_P f\,dm_d\Big)(x)\Big|$$
$$\le \sum_{Q\sim P} |f_Q - f_P||\mathrm{Lip}\,(\psi_Q)(x)| \lesssim \sum_{Q\sim P} \frac{1}{\ell(Q)}|f_Q - f_P|. \tag{16}$$

This means that

$$\int_{X_1} |\nabla(\mathcal{E}f)|^p d\mu_\alpha = \sum_{P\in\mathscr{Q}_d^0} \int_{\mathscr{W}(P)} |\nabla(\mathcal{E}f)|^p d\mu_\alpha \lesssim \sum_{P\in\mathscr{Q}_d^0} \mu_\alpha\big(\mathscr{W}(P)\big) \sum_{Q\sim P} \frac{1}{\ell(Q)^p}|f_Q - f_P|^p$$
$$\approx \sum_{P\in\mathscr{Q}_d^0} \ell(P)^{d+1+\alpha} \sum_{Q\sim P} \frac{1}{\ell(Q)^p}|f_Q - f_P|^p$$
$$\approx \sum_{P\in\mathscr{Q}_d^0} \ell(P)^{-(1-\frac{\alpha+1}{p})p} m_d(P) \sum_{Q\sim P} |f_Q - f_P|^p$$
$$\lesssim \|f\|^p_{\mathcal{B}^{1-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}. \tag{17}$$

If on the other hand $x \in X_2$, we can have $\psi_Q(x) \ne 0$ only for $Q \in \mathscr{Q}_{d,0}$. Thus,

$$\mathcal{E}f(x) = \sum_{Q\in\mathscr{Q}_{d,0}} \Big(\fint_Q f\,dm_d\Big)\psi_Q(x) = \sum_{\substack{Q\in\mathscr{Q}_{d,0}\\ \mathrm{supp}\,\psi_Q\ni x}} f_Q\psi_Q(x),$$

and using the Lipschitz continuity of the functions ψ_Q as above, we get

$$|\nabla(\mathcal{E}f)(x)| \le |\mathrm{Lip}\,(\mathcal{E}f)(x)| \le \sum_{\substack{Q\in\mathscr{Q}_{d,0}\\ \mathrm{supp}\,\psi_Q\ni x}} |f_Q||\mathrm{Lip}\,(\psi_Q)(x)| \lesssim \sum_{Q\in\mathscr{Q}_{d,0}} |f_Q|\chi_{\mathrm{supp}\,\psi_Q}(x).$$

Since $\mu_\alpha(\mathrm{supp}\,\psi_Q) \approx \mu_\alpha(\mathscr{W}(Q)) \approx 1$ for all $Q \in \mathscr{Q}_{d,0}$, the estimate above yields

$$\int_{X_2} |\nabla(\mathcal{E}f)|^p d\mu_\alpha \lesssim \sum_{Q\in\mathscr{Q}_{d,0}} |f_Q|^p \le \sum_{Q\in\mathscr{Q}_{d,0}} \int_Q |f|^p dm_d = \|f\|^p_{L^p(\mathbb{R}^d)}. \tag{18}$$

Combining (15), (17) and (18), we arrive at

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}^{d+1}{}_+,\mu_\alpha)} + \|\nabla(\mathcal{E}f)\|_{L^p(\mathbb{R}^{d+1}{}_+,\mu_\alpha)} \lesssim \|f\|_{\mathcal{B}^{1-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)},$$

which is the desired norm inequality.

(ii) Let us now consider the existence and norm of the trace function $\mathcal{R}f$ of a function $f \in W^{1,p}(\mathbb{R}^{d+1}_+, \mu_\alpha)$. For $k \in \mathbb{N}_0$, define the function $\mathscr{T}_k f : \mathbb{R}^d \to \mathbb{C}$ by

$$\mathscr{T}_k f := \sum_{Q\in\mathscr{Q}_{d,k}} \Big(\fint_{\mathscr{N}(Q)} f\,dm_{d+1}\Big)\chi_Q,$$

where $\mathscr{N}(Q) := \frac{5}{4}\mathscr{W}(Q) := \{y \in \mathbb{R}^{d+1}_+ : \mathrm{dist}\,(y, \mathscr{W}(Q)) < \frac{1}{4}\ell(Q)\}$ – note that the functions $\mathscr{T}_k f$ are well-defined, since $f \in L^1(\mathscr{N}(Q), \mu_\alpha)$ implies $f \in L^1(\mathscr{N}(Q), m_{d+1})$ for all $Q \in \mathscr{Q}_d^0$. We first show that the limit $\lim_{k\to\infty} \mathscr{T}_k f$ exists pointwise m_d-almost everywhere in $\mathbb{R}^d$ (and, in fact, in $L^p(\mathbb{R}^d)$). The limit function will be called $\mathcal{R}f$ for now even though it is not of the same form as in Definition 2.6 – we shall return to this point in part (iii) below.

To verify the existence of the limit in question, it suffices to show that the function

$$f^* := \sum_{k\geq 0} \big|\mathscr{T}_{k+1} f - \mathscr{T}_k f\big| + \big|\mathscr{T}_0 f\big|$$

belongs to $L^p(\mathbb{R}^d)$.

Let $P \in \mathscr{Q}_{d,0}$. Because $m_{d+1}(\mathscr{N}(P)) \approx 1$ and $\mu_\alpha \approx 1$ in $\mathscr{N}(P)$, we get

$$\begin{aligned}
\int_P |f^*|^p dm_d &\lesssim \int_P \Big|\sum_{k\geq 0}\Big(\mathscr{T}_{k+1} f(x) - \mathscr{T}_k f(x)\Big)\Big|^p dm_d(x) + \int_{\mathscr{N}(P)} |f|^p dm_{d+1} \\
&\approx \int_P \Big|\sum_{k\geq 0}\Big(\mathscr{T}_{k+1} f(x) - \mathscr{T}_k f(x)\Big)\Big|^p dm_d(x) + \int_{\mathscr{N}(P)} |f|^p d\mu_\alpha \\
&= \int_P \Big(\sum_{k\geq 0} 2^{-k\epsilon/p}\big|2^{k\epsilon/p}\big(\mathscr{T}_{k+1} f(x) - \mathscr{T}_k f(x)\big)\big|\Big)^p dm_d(x) + \int_{\mathscr{N}(P)} |f|^p d\mu_\alpha \\
&\lesssim \sum_{k\geq 0} 2^{k\epsilon} \int_P \big|\mathscr{T}_{k+1} f(x) - \mathscr{T}_k f(x)\big|^p dm_d(x) + \int_{\mathscr{N}(P)} |f|^p d\mu_\alpha, \qquad (19)
\end{aligned}$$

where $\epsilon := p - (\alpha + 1) > 0$ and the last estimate uses Hölder's inequality.

In order to estimate the kth integral above, recall that for $x \in \mathbb{R}^d$, Q_k^x stands for unique cube in $\mathscr{Q}_{d,k}$ that contains x. By the definition of the $\mathscr{N}(Q)$'s, the intersection of $\mathscr{N}(Q_k^x)$ and $\mathscr{N}(Q_{k+1}^x)$ contains a cube $\tilde{Q}$ with edge length comparable to 2^{-k}. We thus have the estimate

$$\begin{aligned}
|\mathscr{T}_k f(x) - \mathscr{T}_{k+1} f(x)| &= \left|\fint_{\mathscr{N}(Q_k^x)} f\, dm_{d+1} - \fint_{\mathscr{N}(Q_{k+1}^x)} f\, dm_{d+1}\right| \\
&\leq \left|\fint_{\mathscr{N}(Q_k^x)} f\, dm_{d+1} - \fint_{\tilde{Q}} f\, dm_{d+1}\right| \\
&\qquad + \left|\fint_{\tilde{Q}} f\, dm_{d+1} - \fint_{\mathscr{N}(Q_{k+1}^x)} f\, dm_{d+1}\right| \\
&\lesssim \fint_{\mathscr{N}(Q_k^x)} |f - f_{\mathscr{N}(Q_k^x)}|\, dm_{d+1} + \fint_{\mathscr{N}(Q_{k+1}^x)} |f - f_{\mathscr{N}(Q_{k+1}^x)}|\, dm_{d+1}.
\end{aligned}$$

We have $w_\alpha(y) \approx 2^{-k\alpha}$ for all $y \in \mathscr{N}(Q_k^x)$, and hence also $\mu_\alpha(\mathscr{N}(Q_k^x)) \approx 2^{-k\alpha} m_{d+1}(\mathscr{N}(Q_k^x))$ as in part (i) above. We may therefore use the Poincaré inequality (14) in conjunction with Hölder's inequality to estimate the first integral from above by

$$2^{-k}\fint_{\mathscr{N}(Q_k^x)} |\nabla f|\, dm_{d+1} \approx 2^{-k}\fint_{\mathscr{N}(Q_k^x)} |\nabla f|\, d\mu_\alpha \leq 2^{-k}\left(\fint_{\mathscr{N}(Q_k^x)} |\nabla f|^p d\mu_\alpha\right)^{1/p}.$$

A similar estimate obviously holds for the second integral. We thus get

$$|\mathscr{T}_k f(x) - \mathscr{T}_{k+1} f(x)| \lesssim 2^{-k}\left(\fint_{\mathscr{N}(Q_k^x)} |\nabla f|^p d\mu_\alpha\right)^{1/p} + 2^{-k}\left(\fint_{\mathscr{N}(Q_{k+1}^x)} |\nabla f|^p d\mu_\alpha\right)^{1/p}, \quad (20)$$

and hence

$$\int_P \big|\mathscr{T}_{k+1}f(x) - \mathscr{T}_k f(x)\big|^p dm_d(x) = \sum_{\substack{Q\in\mathscr{Q}_{d,k}\\ Q\subset P}} \int_Q \big|\mathscr{T}_{k+1}f(x) - \mathscr{T}_k f(x)\big|^p dm_d(x)$$

$$\lesssim \sum_{\substack{Q\in\mathscr{Q}_{d,k}\\ Q\subset P}} m_d(Q) \sum_{\substack{Q'\in\mathscr{Q}_{d,k+1}\\ Q'\subset Q}} \left(\ell(Q)^p \fint_{\mathscr{N}(Q)} |\nabla f|^p \, d\mu_\alpha + \ell(Q')^p \fint_{\mathscr{N}(Q')} |\nabla f|^p \, d\mu_\alpha\right)$$

$$\lesssim \ 2^{-k(d+p)} \sum_{\substack{Q\in\mathscr{Q}_{d,k}\cup\mathscr{Q}_{d,k+1}\\ Q\subset P}} \fint_{\mathscr{N}(Q)} |\nabla f|^p \, d\mu_\alpha = 2^{-k\epsilon} \sum_{\substack{Q\in\mathscr{Q}_{d,k}\cup\mathscr{Q}_{d,k+1}\\ Q\subset P}} \int_{\mathscr{N}(Q)} |\nabla f|^p \, d\mu_\alpha.$$

Plugging this into (19) and summing over $P \in \mathscr{Q}_{d,0}$, we arrive at

$$\|f^*\|^p_{L^p(\mathbb{R}^d,m_d)} \lesssim \sum_{P\in\mathscr{Q}_{d,0}} \sum_{k\geq 0} \sum_{\substack{Q\in\mathscr{Q}_{d,k}\cup\mathscr{Q}_{d,k+1}\\ Q\subset P}} \int_{\mathscr{N}(Q)} |\nabla f|^p \, d\mu_\alpha + \sum_{P\in\mathscr{Q}_{d,0}} \int_{\mathscr{N}(P)} |f|^p d\mu_\alpha$$

$$\approx \sum_{Q\in\mathscr{Q}_d^0} \int_{\mathscr{N}(Q)} |\nabla f|^p \, d\mu_\alpha + \sum_{P\in\mathscr{Q}_{d,0}} \int_{\mathscr{N}(P)} |f|^p d\mu_\alpha$$

$$\lesssim \|f\|^p_{W^{1,p}(\mathbb{R}^{d+1},\mu_\alpha)}.$$

Here the last inequality follows from the fact that $\sum_{Q\in\mathscr{Q}_d^0} \chi_{\mathscr{N}(Q)} \leq 2$.

Hence $f^*(x) < \infty$ for m_d-almost every $x \in \mathbb{R}^d$, so the limit $\mathcal{R}f(x) := \lim_{k\to\infty} \mathscr{T}_k f(x)$ exists at these points. In the remainder of this proof, we shall abuse notation by writing simply f for $\mathcal{R}f$. Since $|f| \leq |f^*|$ almost everywhere in $\mathbb{R}^d$, the estimate above immediately gives

$$\|f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{W^{1,p}(\mathbb{R}^{d+1},\mu_\alpha)}.$$

Now to estimate the $\mathcal{B}^{1-(1+\alpha)/p}_{p,p}$-energy of f, let $Q \in \mathscr{Q}_{d,k}$ with $k \geq 0$ and write $Q^* := Q \cup \bigcup_{Q'\sim Q} Q'$. We get

$$\sum_{Q'\sim Q} \big|f_Q - f_{Q'}\big|^p \lesssim \sum_{Q'\sim Q} \left(\big|f_Q - f_{\mathscr{N}(Q)}\big|^p + \big|f_{Q'} - f_{\mathscr{N}(Q')}\big|^p + \big|f_{\mathscr{N}(Q)} - f_{\mathscr{N}(Q')}\big|^p\right)$$

$$\lesssim \fint_{Q^*} \big|f(x) - \mathscr{T}_k f(x)\big|^p dm_d(x) + \sum_{Q'\sim Q} \big|f_{\mathscr{N}(Q)} - f_{\mathscr{N}(Q')}\big|^p.$$

Note that $m_d(Q^*) \approx m_d(Q)$, that the collection of cubes $\{Q^* : Q \in \mathscr{Q}_{d,k}\}$ has bounded overlap (uniformly in k) and that $m_d(Q)/\mu_\alpha(\mathscr{N}(Q)) \approx 2^{k(\alpha+1)}$. Using these facts together with an estimate similar to (20), we get

$$\sum_{Q\in\mathscr{Q}_{d,k}} m_d(Q) \sum_{Q'\sim Q} \big|f_Q - f_{Q'}\big|^p$$

$$\lesssim \int_{\mathbb{R}^d} \big|f(x) - \mathscr{T}_k f(x)\big|^p dm_d(x)$$

$$+ 2^{k(\alpha+1)} \sum_{Q\in\mathscr{Q}_{d,k}} \mu_\alpha\big(\mathscr{N}(Q)\big) \sum_{Q'\sim Q} \big|f_{\mathscr{N}(Q)} - f_{\mathscr{N}(Q')}\big|^p \tag{21}$$

$$\lesssim \int_{\mathbb{R}^d} \left|f(x) - \mathscr{T}_k f(x)\right|^p dm_d(x) + 2^{k(\alpha+1-p)} \int_{\cup_{2^{-k-1}\le \ell(Q')\le 2^{-k+1}} \mathscr{N}(Q')} |\nabla f|^p d\mu_\alpha$$
$$=: I_k + 2^{k(\alpha+1-p)} I'_k,$$

so that

$$\sum_{k\ge 0} 2^{k(1-\frac{\alpha+1}{p})p} \sum_{Q\in\mathscr{Q}_{d,k}} m_d(Q) \sum_{Q'\sim Q} \left|f_Q - f_{Q'}\right|^p \lesssim \sum_{k\ge 0} 2^{k(1-\frac{\alpha+1}{p})p} I_k + \sum_{k\ge 0} I'_k. \tag{22}$$

We have

$$\sum_{k\ge 0} I'_k \lesssim \|f\|^p_{W^{1,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)} \tag{23}$$

because the domains of integration in the definition of the I'_k's have bounded overlap. To estimate the terms I_k, we may take $\epsilon \in (0, p-\alpha-1)$ and proceed as in the estimates following (19):

$$\begin{aligned} I_k &\lesssim \sum_{n\ge k} 2^{(n-k)\epsilon} \int_{\mathbb{R}^d} \left|\mathscr{T}_{n+1} f(x) - \mathscr{T}_n f(x)\right|^p dm_d(x) \\ &\lesssim \sum_{n\ge k} 2^{(n-k)\epsilon} 2^{-n(d+p)} \sum_{Q\in\mathscr{Q}_{d,n}\cup\mathscr{Q}_{d,n+1}} \fint_{\mathscr{N}(Q)} |\nabla f|^p \, d\mu_\alpha \\ &\approx \sum_{n\ge k} 2^{(n-k)\epsilon} 2^{-n(p-\alpha-1)} \sum_{Q\in\mathscr{Q}_{d,n}\cup\mathscr{Q}_{d,n+1}} \int_{\mathscr{N}(Q)} |\nabla f|^p \, d\mu_\alpha \\ &=: \sum_{n\ge k} 2^{(n-k)\epsilon} 2^{-n(p-\alpha-1)} O'_n, \end{aligned}$$

so that

$$\begin{aligned} \sum_{k\ge 0} 2^{k(1-\frac{\alpha+1}{p})p} I_k &\lesssim \sum_{n\ge 0} 2^{n(\alpha+1-p+\epsilon)} O'_n \sum_{0\le k\le n} 2^{k(p-\alpha-1-\epsilon)} \approx \sum_{n\ge 0} O'_n \\ &\lesssim \|f\|_{W^{1,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)} \end{aligned}$$

where the last estimate follows from the definition of the norm. Plugging this and (23) into (22), we get the desired energy estimate for $\mathcal{R}f$.

(iii) Let $\mathcal{R}$ be as in part (ii) above. Since m_d-almost all points of $\mathbb{R}^d$ are Lebesgue points of a function $f \in \mathcal{B}^{1-(1+\alpha)/p}_{p,p}$, it is evident from the definition of $\mathcal{R}$ that $\mathcal{R}(\mathcal{E}f) = f$ pointwise m_d-almost everywhere.

We are now done with the proof of the Theorem, with the exception that the trace operator $\mathcal{R}$ considered in part (ii) is not of the form required by Definition 2.6. This is in fact a cosmetic difference – by a well-known argument, if $f \in W^{1,p}(\mathbb{R}^{d+1}_+, \mu_\alpha)$, then the point $(x, 0)$ is for m_d-almost all $x \in \mathbb{R}^d$ in a sense a μ_α-Lebesgue point of f. We refer to Subsection 7.1 for details. Keeping this fact in mind, it is easily seen that the function $\mathcal{R}f$ considered in part (ii) coincides almost everywhere with the function in (12) (with $\mu = \mu_\alpha$). □

4. Proof of Theorem 1.2

Proof. For simplicity, we only consider the case $q = p < \infty$. The cases where $q \in (0, \infty]$ and/or $p = \infty$ can be proven by simple modifications of the arguments below.

(i) We first establish the desired norm inequality for the function $\mathcal{E}f$ for $f \in \mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)$. To begin with, since the parameters p, s and α are also admissible for Theorem 1.1, the estimate (15) therein tells us that

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{24}$$

Now to estimate the $\mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+, \mu_\alpha)$-energy of $\mathcal{E}f$, we divide the dyadic cubes in $\mathbb{R}^{d+1}_+$ into three classes that will be considered separately. For $k \geq 0$, write $\mathscr{Q}^1_k$ for the collection of dyadic cubes Q in $\mathscr{Q}^+_{d+1}$ with edge length 2^{-k} such that $\operatorname{dist}(Q, \mathbb{R}^d \times \{0\}) \geq 2$, $\mathscr{Q}^2_k$ for the collection of dyadic cubes Q in $\mathscr{Q}^+_{d+1}$ with edge length 2^{-k} such that $2^{-k} \leq \operatorname{dist}(Q, \mathbb{R}^d \times \{0\}) < 2$ and $\mathscr{Q}^3_k$ for the collection of dyadic cubes in $\mathscr{Q}^+_{d+1}$ with edge length 2^{-k} whose closures intersect $\mathbb{R}^d \times \{0\}$. Also write $\mathscr{Q}^{2,*}_k$ for the collection of cubes in $\cup_{i=\max(k-1,0)}^{k+1} \mathscr{Q}^2_i$ that are contained in $\cup_{Q \in \mathscr{Q}^2_k} Q$.

We thus want to estimate

$$\begin{aligned}\sum_{Q\in\mathscr{Q}^1_k} \mu_\alpha(Q) \sum_{Q'\sim Q} \left|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\right|^p + \sum_{Q\in\mathscr{Q}^2_k} \mu_\alpha(Q) \sum_{\substack{Q'\sim Q\\ Q'\in\mathscr{Q}^{2,*}_k}} \left|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\right|^p \\ + \sum_{Q\in\mathscr{Q}^3_k} \mu_\alpha(Q) \sum_{Q'\sim Q} \left|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\right|^p =: O^{(1)}_k + O^{(2)}_k + O^{(3)}_k \end{aligned} \tag{25}$$

at each level $k \geq 0$ – the reason why we can omit the terms corresponding to $Q' \notin \mathscr{Q}^{2,*}_k$ in the middle sum is that a comparable term is contained in $O^{(1)}_k$, $O^{(3)}_k$, $O^{(2)}_{k+1}$ or $O^{(1)}_{k-1}$.

We first note that $O^{(1)}_k$ can for $k \in \{0, 1\}$ be simply estimated by

$$\|\mathcal{E}f\|^p_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)} \lesssim \|f\|^p_{L^p(\mathbb{R}^d)}.$$

Now suppose that $Q \in \mathscr{Q}^1_k$ with $k \geq 2$ and $Q' \sim Q$. Using the Lipschitz continuity of the bump functions ψ_P and noting that we can only have $\operatorname{supp}\psi_P \cap (Q \cup Q') \neq \emptyset$ if $P \in \mathscr{Q}_{d,0}$, we get

$$\begin{aligned}\left|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\right|^p &\lesssim \fint_Q \fint_{Q'} |\mathcal{E}f(x) - \mathcal{E}f(y)|^p d\mu_\alpha(x) d\mu_\alpha(y) \\ &\lesssim \Big(2^{-k} \sum_{\substack{P\in\mathscr{Q}_{d,0}\\ \operatorname{supp}\psi_P\cap(Q\cup Q')\neq\emptyset}} \fint_P |f| dm_d\Big)^p \\ &\lesssim 2^{-kp} \sum_{\substack{P\in\mathscr{Q}_{d,0}\\ \operatorname{supp}\psi_P\cap(Q\cup Q')\neq\emptyset}} \int_P |f|^p dm_d.\end{aligned}$$

Since the admissible cubes Q above are relatively far away from $\mathbb{R}^d$, we have $\mu_\alpha(Q) \approx 2^{-k(d+1)}$, so

$$O_k^{(1)} \lesssim 2^{-k(d+1+p)} \sum_{Q\in\mathscr{Q}_k^1} \sum_{Q'\sim Q} \sum_{\substack{P\in\mathscr{Q}_{d,0} \\ \operatorname{supp}\psi_P\cap(Q\cup Q')\neq\emptyset}} \int_P |f|^p dm_d,$$

and since each $P \in \mathscr{Q}_{d,0}$ appears at most some constant times $2^{(d+1)k}$ times in the above triple sum, we get

$$O_k^{(1)} \lesssim 2^{-kp} \sum_{P\in\mathscr{Q}_{d,0}} \int_P |f|^p dm_d = 2^{-kp} \|f\|^p_{L^p(\mathbb{R}^d,m_d)}.$$

Thus,

$$\sum_{k\geq 0} 2^{ksp} O_k^{(1)} \lesssim \sum_{k\geq 0} 2^{k(s-1)p} \|f\|^p_{L^p(\mathbb{R}^d,m_d)} \approx \|f\|^p_{L^p(\mathbb{R}^d,m_d)}. \tag{26}$$

Now suppose that $Q \in \mathscr{Q}_k^2$, $Q' \in \mathscr{Q}_k^{2,*}$ and $Q \sim Q'$. Let P and P' be the (unique) cubes in $\mathscr{Q}_d^0$ such that $Q \subset \mathscr{W}(P)$ and $Q' \subset \mathscr{W}(P')$. We evidently have $\ell(\mathscr{W}(P)) \geq 2^{-k}$ and $\ell(\mathscr{W}(P')) \approx \ell(\mathscr{W}(P))$. Using the Lipschitz continuity of the bump functions in the definition of $\mathcal{E}f$ in conjunction with the fact that the bump functions form a partition of unity in $Q \cup Q'$, we get

$$\begin{aligned} \big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p &\lesssim \fint_Q \fint_{Q'} \big|\mathcal{E}\big(f-f_P\big)(x) - \mathcal{E}\big(f-f_P\big)(y)\big|^p d\mu_\alpha(x) d\mu_\alpha(y) \\ &\lesssim \frac{2^{-kp}}{\ell(P)^p} \sum_{\substack{R\in\mathscr{Q}_d^0 \\ \overline{\mathscr{W}(R)}\cap(\overline{\mathscr{W}(P)}\cup\overline{\mathscr{W}(P')})\neq\emptyset}} \big|f_P - f_R\big|^p \\ &\lesssim \frac{2^{-kp}}{\ell(P)^p} \Big(\sum_{\substack{R\in\mathscr{Q}_d^0 \\ \overline{\mathscr{W}(R)}\cap\overline{\mathscr{W}(P)}\neq\emptyset}} \big|f_P - f_R\big|^p + \sum_{\substack{R\in\mathscr{Q}_d^0 \\ \overline{\mathscr{W}(R)}\cap\overline{\mathscr{W}(P')}\neq\emptyset}} \big|f_{P'} - f_R\big|^p \Big). \end{aligned} \tag{27}$$

Since $w_\alpha \approx \ell(P)^\alpha$ in $\mathscr{W}(P)$, we have $\mu_\alpha(Q) \approx 2^{-k(d+1)}\ell(P)^\alpha$, so

$$\begin{aligned} \mu_\alpha(Q)\big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p \lesssim 2^{-k(d+1+p)} \Big(&\ell(P)^{\alpha-p} \sum_{\substack{R\in\mathscr{Q}_d^0 \\ \overline{\mathscr{W}(R)}\cap\overline{\mathscr{W}(P)}\neq\emptyset}} \big|f_P - f_R\big|^p \\ &+ \ell(P')^{\alpha-p} \sum_{\substack{R\in\mathscr{Q}_d^0 \\ \overline{\mathscr{W}(R)}\cap\overline{\mathscr{W}(P')}\neq\emptyset}} \big|f_{P'} - f_R\big|^p \Big). \end{aligned}$$

Now summing over admissible Q and Q', geometric considerations imply that the terms $P \in \mathscr{Q}_d^0$ and $P' \in \mathscr{Q}_d^0$ (with $\ell(P) \geq 2^{-k}$ and $\ell(P') \geq 2^{-k}$) will appear at most a constant times $(2^k\ell(P))^{d+1}$ times in the resulting triple sum, so

$$\sum_{Q\in\mathscr{Q}_k^2} \mu_\alpha(Q) \sum_{\substack{Q'\sim Q \\ Q'\in\mathscr{Q}_k^{2,*}}} \big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p$$

$$\begin{aligned}
&\lesssim 2^{-kp} \sum_{\substack{P\in\mathscr{Q}_d^0\\ \ell(P)\geq 2^{-k}}} \ell(P)^{d+1+\alpha-p} \sum_{\substack{R\in\mathscr{Q}_d^0\\ \overline{\mathscr{W}(R)}\cap\overline{\mathscr{W}(P)}\neq\emptyset}} \big|f_P - f_R\big|^p \\
&\lesssim 2^{-kp} \sum_{0\leq n\leq k} 2^{-n(d+1+\alpha-p)} \sum_{P\in\mathscr{Q}_{d,n}} \sum_{\substack{R\in\mathscr{Q}_d^0\\ \overline{\mathscr{W}(R)}\cap\overline{\mathscr{W}(P)}\neq\emptyset}} \big|f_P - f_R\big|^p \\
&= 2^{-kp} \sum_{0\leq n\leq k} 2^{-n(1+\alpha-p)} \sum_{P\in\mathscr{Q}_{d,n}} m_d(P) \sum_{\substack{R\in\mathscr{Q}_d^0\\ \overline{\mathscr{W}(R)}\cap\overline{\mathscr{W}(P)}\neq\emptyset}} \big|f_P - f_R\big|^p \\
&=: 2^{-kp} \sum_{0\leq n\leq k} 2^{-n(1+\alpha-p)} O_n'.
\end{aligned}$$

Multiplying this by 2^{ksp} and summing over $k\geq 0$, we get

$$\sum_{k\geq 0} 2^{ksp} O_k^{(2)} \lesssim \sum_{n\geq 0} 2^{-n(1+\alpha-p)} O_n' \sum_{k\geq n} 2^{k(s-1)p} \approx \sum_{n\geq 0} 2^{n(s-\frac{\alpha+1}{p})p} O_n' \lesssim \|f\|^p_{\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}, \tag{28}$$

where the last estimate follows from the definition of the norm.

Finally, let us consider the terms in the sum $O_k^{(3)}$. Let $Q\in\mathscr{Q}_k^3$ and $Q'\sim Q$. Define $P := P_Q \in \mathscr{Q}_{d,k}$ as the projection of Q on $\mathbb{R}^d$, and let P' be a neighbor of P in $\mathscr{Q}_d$ – we will specify the choice of P' later. We have

$$\begin{aligned}
&\mu_\alpha(Q)\big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p \\
&\qquad \lesssim \int_Q \big|\mathcal{E}f - f_P\big|^p d\mu_\alpha + \int_{Q'} \big|\mathcal{E}f - f_{P'}\big|^p d\mu_\alpha + \mu_\alpha(Q)\big|f_P - f_{P'}\big|^p.
\end{aligned}\tag{29}$$

To estimate the first integral above, note that

$$\begin{aligned}
\int_Q \big|\mathcal{E}f - f_P\big|^p d\mu_\alpha &= \sum_{\substack{R\in\mathscr{Q}_d\\ R\subset P}} \int_{\mathscr{W}(R)} \big|\mathcal{E}f - f_P\big|^p d\mu_\alpha \\
&= \sum_{n\geq k} \sum_{\substack{R\in\mathscr{Q}_{d,n}\\ R\subset P}} \int_{\mathscr{W}(R)} \big|\mathcal{E}f - f_P\big|^p d\mu_\alpha.
\end{aligned}$$

For $R\in\mathscr{Q}_{d,n}$ as in the sum above, denote by $R^{(j)}$, $k\leq j\leq n$, the (unique) cube in $\mathscr{Q}_{d,j}$ that contains R. Taking $\epsilon\in(0,1+\alpha)$, we get

$$\begin{aligned}
\big|\mathcal{E}f(x) - f_P\big|^p &\lesssim \big|\mathcal{E}f(x) - f_R\big|^p + \Big(\sum_{j=k+1}^{n} \big|f_{R^j} - f_{R^{j-1}}\big|\Big)^p \\
&\lesssim \sum_{j=k}^{n} 2^{(n-j)\epsilon} \sum_{\substack{R'\in\mathscr{Q}_{d,j}\\ R'\supset R}} \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p \\
&\approx \sum_{j=k}^{n} 2^{(n-j)\epsilon} 2^{jd} \sum_{\substack{R'\in\mathscr{Q}_{d,j}\\ R'\supset R}} m_d(R') \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p
\end{aligned}$$

$$= \ell(R)^{-\epsilon} \sum_{\substack{R'\in\mathscr{Q}_d\\ R\subset R'\subset P}} \ell(R')^{-d+\epsilon} m_d(R') \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p$$

and since $\mu_\alpha(\mathscr{W}(R)) \approx \ell(R)^{d+1+\alpha}$, we arrive at

$$\begin{aligned}\int_Q \big|\mathcal{E}f - f_P\big|^p d\mu_\alpha &\lesssim \sum_{\substack{R\in\mathscr{Q}_d\\ R\subset P}} \ell(R)^{d+1+\alpha-\epsilon} \sum_{\substack{R'\in\mathscr{Q}_d\\ R\subset R'\subset P}} \ell(R')^{-d+\epsilon} m_d(R') \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p \\ &= \sum_{\substack{R'\in\mathscr{Q}_d\\ R'\subset P}} \Big(\ell(R')^{-d+\epsilon} m_d(R') \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p\Big) \sum_{\substack{R\in\mathscr{Q}_d\\ R\subset R'}} \ell(R)^{d+1+\alpha-\epsilon}.\end{aligned}$$

Geometric considerations again imply that every $\ell(R) \in \{\ell(R'), \ell(R')/2, \ell(R')/4, \cdots\}$ in the innermost appears $(\ell(R')/\ell(R))^d$ times, and since $1+\alpha-\epsilon > 0$, the sum in question is comparable to $\ell(R')^{d+1+\alpha-\epsilon}$. Thus,

$$\int_Q \big|\mathcal{E}f - f_P\big|^p d\mu_\alpha \lesssim \sum_{\substack{R'\in\mathscr{Q}_d\\ R'\subset P}} \ell(R')^{1+\alpha} m_d(R') \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p. \tag{30}$$

Now to estimate the second term in (29), we have to specify the choice of P'. If $\overline{Q'} \cap \mathbb{R}^d \times \{0\} \neq \emptyset$, we define P analogously to P', and the integral in question can be estimated by the right-hand side of (30), with Q replaced Q' and P replaced by P'. If on the other hand $\overline{Q'} \cap \mathbb{R}^d \times \{0\} = \emptyset$, we can take $P' \in \mathscr{Q}_{d,k} \cup \mathscr{Q}_{d,k+1}$ so that $Q' = \mathscr{W}(P')$, which yields

$$\int_{Q'} \big|\mathcal{E}f - f_{P'}\big|^p d\mu_\alpha \lesssim \mu_\alpha(Q') \sum_{P''\sim P'} \big|f_{P'} - f_{P''}\big|^p \approx \ell(P')^{1+\alpha} m_d(P') \sum_{P''\sim P'} \big|f_{P'} - f_{P''}\big|^p.$$

Finally, the estimate for the third term in (29) is obvious:

$$\mu_\alpha(Q)\big|f_P - f_{P'}\big|^p \approx \ell(P)^{1+\alpha} m_d(P)\big|f_P - f_{P'}\big|^p. \tag{31}$$

Putting together (30), (31) and a suitable estimate for the second term in (29), we get

$$\mu_\alpha(Q) \sum_{Q'\sim Q} \big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p \lesssim \sum_{\substack{R'\in\mathscr{Q}_d\\ R'\subset P_Q^*}} \ell(R')^{1+\alpha} m_d(R') \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p,$$

where $P_Q^* := P \cup \bigcup_{P'\sim P} P'$. Since each $R' \in \mathscr{Q}_d$ (with $\ell(R') \leq \min(2^{-k+1}, 1)$) is contained in a finite number of admissible cubes P_Q^*, we thus have

$$O_k^{(3)} \lesssim \sum_{n\geq(k-1)_+} 2^{-n(1+\alpha)} \sum_{R'\in\mathscr{Q}_{d,n}} m_d(R') \sum_{R''\sim R'} \big|f_{R'} - f_{R''}\big|^p =: \sum_{n\geq(k-1)_+} 2^{-n(1+\alpha)} O_n'',$$

and so

$$\sum_{k\geq 0} 2^{ksp} O_k^{(3)} \lesssim \sum_{n\geq 0} 2^{-n(1+\alpha)} O_n'' \sum_{0\leq k\leq n+1} 2^{ksp} \approx \sum_{n\geq 0} 2^{n(s-\frac{1+\alpha}{p})p} O_n'' \approx \|f\|^p_{\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}. \tag{32}$$

Combining the estimates (24), (26), (28) and (32), we finally get

$$\|\mathcal{E}f\|^p_{\mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)} \lesssim \|\mathcal{E}f\|^p_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)} + \sum_{k\geq 0} 2^{ksp}\big(O_k^{(1)} + O_k^{(2)} + O_k^{(3)}\big) \lesssim \|f\|^p_{\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}.$$

(ii) Now let $f \in \mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+, \mu_\alpha)$, and for $k \in \mathbb{N}_0$ write

$$T_k f := \sum_{Q \in \mathscr{Q}_{d,k}} \Big(\fint_{N(Q)} f \, d\mu_\alpha \Big) \chi_Q,$$

where $N(Q) = Q \times (0, \ell(Q)] \in \mathscr{Q}_{d+1,k}$ for all $Q \in \mathscr{Q}^0_d$. The operators T_k will play a role similar to that of the operators $\mathscr{T}_k$ in the proof of Theorem 1.1.

We first show that the limit $\lim_{k\to\infty} T_k f$ exists pointwise m_d-almost everywhere in $\mathbb{R}^d$ by estimating the $L^p(\mathbb{R}^d)$-norm of the function

$$f^* := \sum_{k \geq 0} \big|T_{k+1} f - T_k f\big| + \big|T_0 f\big|.$$

Now if $P \in \mathscr{Q}_{d,0}$, the definition of T_0 shows that

$$\begin{aligned} \int_P |f^*(x)|^p dm_d &\leq \int_P \Big(\sum_{k\geq 0} \Big|T_{k+1} f(x) - T_k f(x)\Big| \Big)^p dm_d(x) + \int_{N(P)} |f|^p d\mu_\alpha \\ &\lesssim \sum_{k\geq 0} 2^{k\epsilon} \int_P \big|T_{k+1} f(x) - T_k f(x)\big|^p dm_d(x) + \int_{N(P)} |f|^p d\mu_\alpha, \end{aligned} \tag{33}$$

where $\epsilon := sp - \alpha - 1 > 0$. To estimate the k-th integral above, note that

$$\begin{aligned} \int_P \big|T_{k+1} f(x) - T_k f(x)\big|^p dm_d(x) &= \sum_{\substack{Q \in \mathscr{Q}_{d,k} \\ Q \subset P}} \int_Q \big|T_{k+1} f(x) - T_k f(x)\big|^p dm_d(x) \\ &\lesssim \sum_{\substack{Q \in \mathscr{Q}_{d,k} \\ Q \subset P}} m_d(Q) \sum_{\substack{Q' \in \mathscr{Q}^+_{d+1} \\ Q' \sim N(Q)}} \big|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\big|^p \\ &= 2^{-kd} \sum_{\substack{Q \in \mathscr{Q}_{d,k} \\ Q \subset P}} \sum_{\substack{Q' \in \mathscr{Q}^+_{d+1} \\ Q' \sim N(Q)}} \big|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\big|^p. \end{aligned}$$

By this and (33), we can estimate $\|f^*\|^p_{L^p(\mathbb{R}^d)}$ by

$$\begin{aligned} &\sum_{k\geq 0} 2^{k(\epsilon - d)} \sum_{Q \in \mathscr{Q}_{d,k}} \sum_{\substack{Q' \in \mathscr{Q}^+_{d+1} \\ Q' \sim N(Q)}} \big|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\big|^p + \|f\|^p_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)} \\ \approx &\sum_{k\geq 0} 2^{k(\alpha+1+\epsilon)} \sum_{Q \in \mathscr{Q}_{d,k}} \mu_\alpha\big(N(Q)\big) \sum_{\substack{Q' \in \mathscr{Q}^+_{d+1} \\ Q' \sim N(Q)}} \big|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\big|^p + \|f\|^p_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)} \\ = &\sum_{k\geq 0} 2^{ksp} \sum_{Q \in \mathscr{Q}_{d,k}} \mu_\alpha\big(N(Q)\big) \sum_{\substack{Q' \in \mathscr{Q}^+_{d+1} \\ Q' \sim N(Q)}} \big|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\big|^p + \|f\|^p_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)} \qquad (34) \\ \lesssim &\|f\|^p_{\mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)}. \end{aligned}$$

This shows that $f^* < \infty$ pointwise m_d-almost everywhere, so that the limit $\mathcal{R}f := \lim_{k\to\infty} T_k f$ exists at these points. We may abuse notation by writing f for $\mathcal{R}f$ in the remainder of this proof. Since $|f| \le f^*$ pointwise m_d-almost everywhere, the estimate above plainly implies

$$\|f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)}.$$

Now to estimate the $\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)$-energy of f, let $k \in \mathbb{N}_0$ and recall that, by a calculation similar to (13), $m_d(Q)/\mu_\alpha(N(Q)) \approx 2^{k(\alpha+1)}$ for all $Q \in \mathscr{Q}_{d,k}$. The estimate (21) in the proof of Theorem 1.1 (with T and N in place of $\mathscr{T}$ and $\mathscr{N}$ respectively) yields

$$\begin{aligned}
&\sum_{Q\in\mathscr{Q}_{d,k}} m_d(Q) \sum_{Q'\sim Q} \left|f_Q - f_{Q'}\right|^p \\
&\lesssim \int_{\mathbb{R}^d} \left|f(x) - T_k f(x)\right|^p dm_d(x) + 2^{k(\alpha+1)} \sum_{Q\in\mathscr{Q}_{d,k}} \mu_\alpha\big(N(Q)\big) \sum_{Q'\sim Q} \left|f_{N(Q),\mu_\alpha} - f_{N(Q'),\mu_\alpha}\right|^p \\
&=: I_k + 2^{k(\alpha+1)} O_k,
\end{aligned}$$

so that

$$\sum_{k\ge0} 2^{k(s-\frac{\alpha+1}{p})p} \sum_{Q\in\mathscr{Q}_{d,k}} m_d(Q) \sum_{Q'\sim Q} \left|f_Q - f_{Q'}\right|^p \lesssim \sum_{k\ge0} 2^{k(s-\frac{\alpha+1}{p})p} I_k + \sum_{k\ge0} 2^{ksp} O_k. \tag{35}$$

We have

$$\sum_{k\ge0} 2^{ksp} O_k \lesssim \|f\|^p_{\mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)} \tag{36}$$

by definition. To estimate the terms I_k, take $\epsilon \in (0, sp-\alpha-1)$ and proceed as in the estimates following (33) to obtain

$$\begin{aligned}
I_k &\lesssim \sum_{n\ge k} 2^{(n-k)\epsilon} \int_{\mathbb{R}^d} \left|T_{n+1}f(x) - T_n f(x)\right|^p dm_d(x) \\
&\lesssim \sum_{n\ge k} 2^{(n-k)\epsilon} 2^{-nd} \sum_{Q\in\mathscr{Q}_{d,n}} \sum_{\substack{Q'\in\mathscr{Q}^+_{d+1} \\ Q'\sim N(Q)}} \left|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\right|^p \\
&\approx \sum_{n\ge k} 2^{(n-k)\epsilon} 2^{n(\alpha+1)} \sum_{Q\in\mathscr{Q}_{d,n}} \mu_\alpha\big(N(Q)\big) \sum_{\substack{Q'\in\mathscr{Q}^+_{d+1} \\ Q'\sim N(Q)}} \left|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\right|^p \\
&=: \sum_{n\ge k} 2^{(n-k)\epsilon} 2^{n(\alpha+1)} O'_n,
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{k\ge0} 2^{k(s-\frac{\alpha+1}{p})p} I_k &\lesssim \sum_{n\ge0} 2^{n(\alpha+1+\epsilon)} O'_n \sum_{0\le k\le n} 2^{k(sp-\alpha-1-\epsilon)} \approx \sum_{n\ge0} 2^{nsp} O'_n \\
&\lesssim \|f\|^p_{\mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)},
\end{aligned} \tag{37}$$

where the last estimate again follows from the definition of the norm. Plugging (36) and (37) into (35) leads to the desired energy estimate.

(iii) We plainly have $\mathcal{R}(\mathcal{E}f) = f$ for all $f \in \mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)$.

As in the proof of Theorem 1.1, the remaining question is whether the trace operator $\mathcal{R}$ constructed above is of the correct form. We again refer to Subsection 7.1 in the Appendix for details on this. □

5. Proof of Theorem 1.3

Let us recall that in the proof of Theorem 1.2, $N(Q)$ for $Q \in \mathscr{Q}_d^0$ was defined as $Q \times (0, \ell(Q)] \in \mathscr{Q}_{d+1}^0$. Before giving the proof of Theorem 1.3, let us introduce the auxiliary seminorm $[f]_{s,p,q,\alpha}$, defined by

$$[f]_{s,p,q,\alpha}^p = \int_{\mathbb{R}_+^{d+1}} \Big(\sum_{k=0}^{\infty} 2^{ksq} \sum_{P \in \mathscr{Q}_{d,k}} \sum_{\substack{Q' \in \mathscr{Q}_{d+1}^0 \\ Q' \sim N(P)}} \big| f_{N(P),\mu_\alpha} - f_{Q',\mu_\alpha} \big|^q \chi_{N(P)}(x) \Big)^{p/q} d\mu_\alpha(x),$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}, \mu_\alpha)$ and the parameters p, q, s and α as in the statement of Theorem 1.3. We obviously have $[f]_{s,p,q,\alpha} \le \|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1},\mu_\alpha)}$ and

$$[f]_{s,p,p,\alpha}^p = \sum_{k=0}^{\infty} 2^{ksp} \sum_{P \in \mathscr{Q}_{d,k}} \mu_\alpha\big(N(P)\big) \sum_{\substack{Q' \in \mathscr{Q}_{d+1}^0 \\ Q' \sim N(P)}} \big| f_{N(P),\mu_\alpha} - f_{Q',\mu_\alpha} \big|^p$$

for all admissible values of the parameters. We shall omit α from the notation and write $[f]_{s,p,q}$ if there is no risk of confusion.

For the proof of Theorem 1.3, we shall need the following lemma concerning the seminorms $[f]_{s,p,q}$.

Lemma 5.1. *Suppose that $0 < s < 1$, $1 \le p < \infty$, $0 < q$, $q' \le \infty$ and $\alpha > -1$. Then for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+^{d+1}, \mu_\alpha)$, we have*

$$[f]_{s,p,q,\alpha} \approx [f]_{s,p,q',\alpha}$$

with the implied constants independent of f.

Proof. It suffices to consider the case $q' = p$. First, in order to estimate $[f]_{s,p,q}$ from above, write

$$D(P) := D(f,P) := \sum_{Q' \sim N(P)} \big| f_{N(P),\mu_\alpha} - f_{Q',\mu_\alpha} \big|$$

for $P \in \mathscr{Q}_d^0$, so that

$$[f]_{s,p,q}^p \approx \int_{\mathbb{R}_+^{d+1}} \Big(\sum_{k=0}^{\infty} 2^{ksq} \sum_{P \in \mathscr{Q}_{d,k}} D(P)^q \chi_{N(P)}(x) \Big)^{p/q} d\mu_\alpha(x)$$

(because the sum defining $D(P)$ is uniformly finite).

Note that $\bigcup_{P \in \mathscr{Q}_d^0} N(P) = \mathbb{R}^d \times (0,1] = \bigcup_{j \ge 1} \bigcup_{P \in \mathscr{Q}_{d,j}} \mathscr{W}(P)$. Moreover, from the definitions it is easily seen that for R, $P \in \mathscr{Q}_d^0$, we have $N(P) \cap \mathscr{W}(R) \ne \emptyset$ if and only if R is a proper subset of P, and in this case also $\mathscr{W}(R) \subset N(P)$. Thus, taking $\epsilon \in (0, 1+\alpha)$ and using Hölder's inequality (or the subadditivity of $t \mapsto t^{p/q}$ if $p \le q$) leads to

$$[f]_{s,p,q}^p = \sum_{j \ge 1} \sum_{R \in \mathscr{Q}_{d,j}} \int_{\mathscr{W}(R)} \Big(\sum_{k=0}^{j-1} 2^{ksq} \sum_{\substack{P \in \mathscr{Q}_{d,k} \\ P \supset R}} D(P)^q \chi_{N(P)}(x) \Big)^{p/q} d\mu_\alpha(x)$$

$$
\begin{aligned}
&= \sum_{j\ge 1} \sum_{R\in \mathscr{Q}_{d,j}} \mu_\alpha\big(\mathscr{W}(R)\big)\Big(\sum_{k=0}^{j-1} 2^{ksq} \sum_{\substack{P\in\mathscr{Q}_{d,k}\\ P\supset R}} D(P)^q\Big)^{p/q} \\
&\lesssim \sum_{j\ge 1} \sum_{R\in \mathscr{Q}_{d,j}} \mu_\alpha\big(\mathscr{W}(R)\big) \sum_{k=0}^{j-1} 2^{(j-k)\epsilon} 2^{ksp} \sum_{\substack{P\in\mathscr{Q}_{d,k}\\ P\supset R}} D(P)^p \\
&= \sum_{k\ge 0} 2^{k(sp-\epsilon)}\Big(\sum_{P\in\mathscr{Q}_{d,k}} D(P)^p \sum_{j>k} 2^{j\epsilon} \sum_{\substack{R\in\mathscr{Q}_{d,j}\\ R\subset P}} \mu_\alpha\big(\mathscr{W}(R)\big)\Big).
\end{aligned}
$$

As in the previous proofs, each term $\mu_\alpha(\mathscr{W}(R))$ in the innermost sum above is comparable to $2^{-j(d+1+\alpha)}$, and the sum has $2^{(j-k)d}$ such terms. This together with the choice of ϵ yields

$$
\begin{aligned}
[f]^p_{s,p,q} &\lesssim \sum_{k\ge 0} 2^{k(sp-d-\epsilon)}\Big(\sum_{P\in\mathscr{Q}_{d,k}} D(P)^p \sum_{j>k} 2^{j(\epsilon-\alpha-1)}\Big) \\
&\approx \sum_{k\ge 0} 2^{k(sp-d-\alpha-1)} \sum_{P\in\mathscr{Q}_{d,k}} D(P)^p \\
&\approx \sum_{k\ge 0} 2^{ksp} \sum_{P\in\mathscr{Q}_{d,k}} \mu_\alpha\big(N(P)\big) D(P)^p \\
&\approx [f]^p_{s,p,p}.
\end{aligned}
$$

For the other direction, write $W(P) := P \times (\frac{1}{2}\ell(P), \ell(P)]$ for all $P \in \mathscr{Q}^0_d$. Note that $W(P) \subset N(P)$ and $\mu_\alpha(W(P)) \approx \mu_\alpha(N(P))$ for all P, and that the cubes $W(P)$ are pairwise disjoint. We get

$$
\begin{aligned}
[f]^p_{s,p,p} &\approx \sum_{k\ge 0} 2^{ksp} \sum_{P\in\mathscr{Q}_{d,k}} \mu_\alpha\big(W(P)\big) D(P)^p \\
&= \sum_{k\ge 0} \sum_{P\in\mathscr{Q}_{d,k}} \int_{W(P)} \Big(2^{ksq} D(P)^q\Big)^{p/q} d\mu_\alpha \\
&\le \sum_{k\ge 0} \sum_{P\in\mathscr{Q}_{d,k}} \int_{W(P)} \Big(\sum_{j\ge 0} 2^{jsq} \sum_{Q\in\mathscr{Q}_{d,j}} D(Q)^q \chi_{N(Q)}(x)\Big)^{p/q} d\mu_\alpha(x) \\
&\le \int_{\mathbb{R}^{d+1}_+} \Big(\sum_{j\ge 0} 2^{jsq} \sum_{Q\in\mathscr{Q}_{d,j}} D(Q)^q \chi_{N(Q)}(x)\Big)^{p/q} d\mu_\alpha(x) \\
&= [f]^p_{s,p,q}.
\end{aligned}
$$

□

Proof of Theorem 1.3. (i) Let us first establish the relevant norm inequality for the Whitney extension of a function $f \in \mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)$. By Theorem 1.2 and Remark 2.4, it suffices to consider the case $q < p$. As in (24), we again have

$$
\|\mathcal{E}f\|_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.
$$

Now for the $\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu_\alpha)$-energy of $\mathcal{E}f$, it suffices to estimate

$$\int_{\mathbb{R}^d}\Big(\sum_{k=k_0}^{\infty}2^{ksq}\sum_{Q'\sim Q^x_k}\big|(\mathcal{E}f)_{Q^x_k,\mu_\alpha}-(\mathcal{E}f)_{Q',\mu_\alpha}\big|^q\Big)^{p/q}\,d\mu_\alpha(x), \tag{38}$$

where $k_0\geq 4$ is a fixed integer (we will specify the choice of k_0 later), since the corresponding integral with $\sum_{k=k_0}^{\infty}$ replaced by $\sum_{k=0}^{k_0-1}$ is easily estimated by

$$\|\mathcal{E}f\|^p_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)}\lesssim\|f\|^p_{L^p(\mathbb{R}^d)}.$$

To this end, we divide the cubes in $\mathscr{Q}^+_{d+1}$ into several classes as in the proof of Theorem 1.2, but this time we need to consider four different cases. More precisely, for $k\geq k_0$ write $\mathscr{Q}^1_k$ for the dyadic cubes Q in $\mathscr{Q}^+_{d+1}$ with edge length 2^{-k} such that $\operatorname{dist}(Q,\mathbb{R}^d\times\{0\})>2-2^{-k+2}$, $\mathscr{Q}^2_k$ for the cubes Q with edge length 2^{-k} such that $2^{-k+1}<\operatorname{dist}(Q,\mathbb{R}^d\times\{0\})\leq 2-2^{-k+2}$, $\mathscr{Q}^3_k$ for the cubes with edge length 2^{-k} such that $2^{-k}\leq\operatorname{dist}(Q,\mathbb{R}^d\times\{0\})\leq 2^{-k+1}$ and $\mathscr{Q}^4_k$ for the cubes with edge length 2^{-k} whose closures intersect $\mathbb{R}^d\times\{0\}$. With these choices, the quantity (38) is comparable to

$$\sum_{j=1}^{4}\int_{\mathbb{R}^{d+1}_+}\Big(\sum_{k=k_0}^{\infty}2^{ksq}\sum_{Q\in\mathscr{Q}^j_k}\sum_{Q'\sim Q}\big|(\mathcal{E}f)_{Q,\mu_\alpha}-(\mathcal{E}f)_{Q',\mu_\alpha}\big|^q\chi_Q(x)\Big)^{p/q}d\mu_\alpha(x)=:\sum_{j=1}^{4}O^j.$$

The necessary estimates for the term O^4 are already contained in Lemma 5.1 and Theorem 1.2:

$$O^4=[\mathcal{E}f]^p_{s,p,q}\approx[\mathcal{E}f]^p_{s,p,p}\lesssim\|\mathcal{E}f\|^p_{\mathcal{B}^s_{p,p}(\mathbb{R}^{d+1}_+,\mu_\alpha)}\lesssim\|f\|^p_{\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}.$$

The term O^3 can be estimated in a similar manner as O^4, since the quantity O^3 is also essentially independent of the parameter q. This is because the cubes in $\bigcup_{k\geq k_0}\mathscr{Q}^2_k$ have bounded overlap.

In order to estimate O^1, let us specify k_0: it can be taken such that whenever $Q\in\mathscr{Q}^1_k$ with $k\geq k_0$ and $Q'\sim Q$, $\operatorname{supp}\psi_P\cap(Q\cup Q')\neq\emptyset$ can only hold for $P\in\mathscr{Q}_{d,0}$. Using this property together with the Lipschitz continuity of the bump functions ψ_P, we get

$$\begin{aligned}
\big|(\mathcal{E}f)_{Q,\mu_\alpha}-(\mathcal{E}f)_{Q',\mu_\alpha}\big|^q&\leq\Big(\fint_Q\fint_{Q'}|\mathcal{E}f(x)-\mathcal{E}f(y)|d\mu_\alpha(x)d\mu_\alpha(y)\Big)^q\\
&\lesssim\Big(2^{-k}\sum_{\substack{P\in\mathscr{Q}_{d,0}\\ \operatorname{supp}\psi_P\cap(Q\cup Q')\neq\emptyset}}\fint|f|dm_d\Big)^q\\
&\lesssim 2^{-kq}\Big(\sum_{\substack{P\in\mathscr{Q}_{d,0}\\ \operatorname{supp}\psi_P\cap(Q\cup Q')\neq\emptyset}}\int_P|f|dm_d\Big)^q=:2^{-kq}J^q_{Q,Q'}.
\end{aligned}$$

Take $\epsilon\in(0,1-s)$ and $q^*>1$ so that $1/q^*+q/p=1$. Using the estimate above together with Hölder's inequality yields

$$\Big(\sum_{k\geq k_0}2^{ksq}\sum_{Q\in\mathscr{Q}^1_k}\sum_{Q'\sim Q}\big|(\mathcal{E}f)_{Q,\mu_\alpha}-(\mathcal{E}f)_{Q',\mu_\alpha}\big|^q\chi_Q(x)\Big)^{p/q}$$

$$\lesssim\Big(\sum_{k\ge k_0}2^{k(s-1+\epsilon)p}\sum_{Q\in\mathscr{Q}_k^1}\sum_{Q'\sim Q}J^p_{Q,Q'}\chi_Q(x)\Big)\Big(\sum_{k\ge k_0}2^{-k\epsilon qq^*}\Big)^{p/(qq^*)}$$

$$\approx\sum_{k\ge k_0}2^{k(s-1+\epsilon)}\sum_{Q\in\mathscr{Q}_k^1}\sum_{Q'\sim Q}J^p_{Q,Q'}\chi_Q(x).$$

Hence we have

$$O^1\lesssim\sum_{k\ge k_0}2^{k(s-1+\epsilon)p}\sum_{Q\in\mathscr{Q}_k^1}\mu_\alpha(Q)\sum_{Q'\sim Q}J^p_{Q,Q'}$$

$$\lesssim\sum_{k\ge k_0}2^{k(s-1-\epsilon)p-k(d+1)}\sum_{Q\in\mathscr{Q}_k^1}\sum_{Q'\sim Q}\sum_{\substack{P\in\mathscr{Q}_{d,0}\\ \operatorname{supp}\psi_P\cap(Q\cup Q')\neq\emptyset}}\int_P|f|^p dm_d,$$

and since each $P\in\mathscr{Q}_{d,0}$ appears at most some constant times $2^{(d+1)k}$ times in the above triple sum, we arrive at

$$O^1\lesssim\sum_{k\ge k_0}2^{k(s-1+\epsilon)}\sum_{P\in\mathscr{Q}_{d,0}}\int_P|f|^p dm_d\approx\sum_{k\ge k_0}2^{k(s-1+\epsilon)}\|f\|^p_{L^p(\mathbb{R}^d,m_d)}\approx\|f\|^p_{L^p(\mathbb{R}^d,m_d)}.$$

Finally let us estimate O^2. Suppose that $Q\in\mathscr{Q}_k^2$ and $Q\sim Q'$. Since $\operatorname{dist}(Q,\mathbb{R}^d\times\{0\})>2^{-k+1}$, $\ell(Q')\le 2\ell(Q)=2^{-k+1}$ and $\overline{Q'}\cap\overline{Q}\neq\emptyset$, we have $Q'\cap\mathbb{R}^d\times\{0\}=\emptyset$. As in the proof of Theorem 1.2, we can therefore take $P:=P_Q$ and P' to be the cubes in $\mathscr{Q}_d^0$ such that $Q\subset\mathscr{W}(P)$ and $Q'\subset\mathscr{W}(P')$. Moreover, the definition of $\mathscr{Q}_k^2$ implies that $Q\cup Q'\subset\bigcup_{R\in\mathscr{Q}_d^0}\mathscr{W}(R)$, and the bump functions ψ_P form a partition of unity of the latter set. As in (27), we thus get

$$\big|(\mathcal{E}f)_{Q,\mu_\alpha}-(\mathcal{E}f)_{Q',\mu_\alpha}\big|^q\lesssim\frac{2^{-kq}}{\ell(P)^q}\sum_{\substack{R\in\mathscr{Q}_d^0\\ \overline{\mathscr{W}(R)}\cap(\overline{\mathscr{W}(P)\cup\mathscr{W}(P')})\neq\emptyset}}\big|f_P-f_R\big|^q,$$

and hence

$$\sum_{Q'\sim Q}\big|(\mathcal{E}f)_{Q,\mu_\alpha}-(\mathcal{E}f)_{Q',\mu_\alpha}\big|^q\lesssim\frac{2^{-kq}}{\ell(P_Q)^q}\Big(\sum_{\substack{R\in\mathscr{Q}_d^0\\ R\sim\sim P_Q}}\big|f_{P_Q}-f_R\big|\Big)^q,$$

where the notation $R\sim\sim P_Q$ means that there exists $R'\in\mathscr{Q}_d^0$ such that $R\sim R'$ and $R'\sim P_Q$. The latter sum obviously has a uniformly finite number of terms $|f_{P_Q}-f_R|$.

In order to apply this estimate to O^2, note that by the definition of the $\mathscr{Q}_k^2$'s, we have

$$\bigcup_{k\ge k_0}\mathscr{Q}_k^2\subset\bigcup_{P\in\mathscr{Q}_d^0}\mathscr{W}(P),$$

and that if a point x belongs to one of the $\mathscr{W}(P)$'s above, we can have $\chi_Q(x)\neq 0$ for some $Q\in\bigcup_{k\ge k_0}\mathscr{Q}_k^2$ only if $Q\subset\mathscr{W}(P)$, and in this case also $\ell(Q)\le\ell(P)$. Using these facts and Hölder's inequality (with $\epsilon\in(0,1-s)$ as in the estimate for O^1 above), we get

$$
\begin{aligned}
O^2 &\le \sum_{j\ge 0}\sum_{P\in\mathscr{Q}_{d,j}}\int_{\mathscr{W}(P)}\Big(\sum_{k\ge j}2^{ksq}\sum_{\substack{Q\in\mathscr{Q}_{d+1,k}\\ Q\subset\mathscr{W}(P)}}\sum_{Q'\in\mathscr{Q}^2_Q}\big|(\mathcal{E}f)_{Q,\mu_\alpha}-(\mathcal{E}f)_{Q',\mu_\alpha}\big|^q\chi_Q(x)\Big)^{p/q}\,d\mu_\alpha(x)\\
&\lesssim \sum_{j\ge 0}\sum_{P\in\mathscr{Q}_{d,j}}\int_{\mathscr{W}(P)}\Big(\sum_{k\ge j}2^{k(s-1)q}\sum_{\substack{Q\in\mathscr{Q}_{d+1,k}\\ Q\subset\mathscr{W}(P)}}\frac{1}{\ell(P)^q}\Big(\sum_{\substack{R\in\mathscr{Q}^0_d\\ R\sim\sim P}}\big|f_P-f_R\big|\Big)^q\chi_Q(x)\Big)^{p/q}\,d\mu_\alpha(x)\\
&= \sum_{j\ge 0}2^{jp}\sum_{P\in\mathscr{Q}_{d,j}}\int_{\mathscr{W}(P)}\Big(\sum_{k\ge j}2^{k(s-1)q}\sum_{\substack{Q\in\mathscr{Q}_{d+1,k}\\ Q\subset\mathscr{W}(P)}}\Big(\sum_{\substack{R\in\mathscr{Q}^0_d\\ R\sim\sim P}}\big|f_P-f_R\big|\Big)^q\chi_Q(x)\Big)^{p/q}\,d\mu_\alpha(x)\\
&\lesssim \sum_{j\ge 0}2^{j(1-\epsilon)p}\sum_{P\in\mathscr{Q}_{d,j}}\sum_{k\ge j}2^{k(s-1+\epsilon)p}\sum_{\substack{Q\in\mathscr{Q}_{d+1,k}\\ Q\subset\mathscr{W}(P)}}\mu_\alpha(Q)\sum_{\substack{R\in\mathscr{Q}^0_d\\ R\sim\sim P}}\big|f_P-f_R\big|^p\\
&= \sum_{j\ge 0}2^{j(1-\epsilon)p}\sum_{P\in\mathscr{Q}_{d,j}}\mu_\alpha\big(\mathscr{W}(P)\big)\Big(\sum_{\substack{R\in\mathscr{Q}^0_d\\ R\sim\sim P}}\big|f_P-f_R\big|^p\Big)\sum_{k\ge j}2^{k(s-1+\epsilon)p}\\
&\approx \sum_{j\ge 0}2^{jsp}\sum_{P\in\mathscr{Q}_{d,j}}\mu_\alpha\big(\mathscr{W}(P)\big)\sum_{\substack{R\in\mathscr{Q}^0_d\\ R\sim\sim P}}\big|f_P-f_R\big|^p\\
&\approx \sum_{j\ge 0}2^{j(s-\frac{\alpha+1}{p})p}\sum_{P\in\mathscr{Q}_{d,j}}m_d(P)\sum_{\substack{R\in\mathscr{Q}^0_d\\ R\sim\sim P}}\big|f_P-f_R\big|^p.
\end{aligned}
$$

Finally, since for each P above we have $R\sim\sim P$ for a (uniformly) finite number of cubes R, the above quantity is easily estimated by $\|f\|^p_{\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}$.

Combining the estimates for O^1, O^2, O^3 and O^4 with the L^p-estimate for $\mathcal{E}f$, we conclude that

$$\|\mathcal{E}f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu_\alpha)}\lesssim\|f\|_{\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}$$

(ii) In order to establish the existence of the trace of a function $f\in\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu_\alpha)$, we proceed as in the proof of Theorem 1.2 (ii). Let $f\in\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu_\alpha)$, define T_kf for $k\in\mathbb{N}_0$ as in that proof and put

$$f^*:=\sum_{k\ge 0}\big|T_{k+1}f-T_kf\big|+\big|T_0f\big|.$$

By the estimate (34) and Lemma 5.1, we have

$$\|f^*\|_{L^p(\mathbb{R}^d)}\lesssim\|f\|_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)}+[f]_{s,p,p}\approx\|f\|_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)}+[f]_{s,p,q}\le\|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu_\alpha)}<\infty,$$

so the trace $\mathcal{R}f:=\lim_{k\to\infty}T_kf$ is well-defined m_d-almost everywhere in $\mathbb{R}^d$. The estimates (34), (35) and (37) then imply

$$\|\mathcal{R}f\|_{\mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)}\lesssim\|f\|_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)}+[f]_{s,p,p}\approx\|f\|_{L^p(\mathbb{R}^{d+1}_+,\mu_\alpha)}+[f]_{s,p,q}\lesssim\|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+,\mu_\alpha)}$$

which is the desired norm estimate.

(iii) That $\mathcal{R}(\mathcal{E}f) = f$ for all $f \in \mathcal{B}^{s-(\alpha+1)/p}_{p,p}(\mathbb{R}^d)$ again follows plainly from the definition of $\mathcal{R}$. Concerning the fact that $\mathcal{R}$ is actually of the form required by Definition 2.6, we again refer to Subsection 7.1 of the Appendix. □

6. The trace of a weighted Hardy-Sobolev space

In this section we present a refinement of the case $p = 1$ of Theorem 1.1, where $W^{1,1}(\mathbb{R}^{d+1}_+, \mu_\alpha)$ is replaced by a *weighted Hardy-Sobolev space* on $\mathbb{R}^{d+1}_+$.

The real-variable Hardy spaces $H^p(\mathbb{R}^d)$, $0 < p \leq 1$, were defined for a general dimension d and exponent p in the seminal paper by Fefferman and Stein [10]. They have since been studied extensively, as many results of harmonic analysis that fail for $p \leq 1$ work for these spaces. We refer to [47] for an extensive treatment of these spaces.

A localized version of the space H^p, better suited e.g. for studying functions on domains, was introduced by Goldberg [16]. A variety of similar spaces, including spaces on domains, weighted spaces on domains and Sobolev-type spaces based on the H^p norm, have since been studied e.g. in [34, 48, 35, 36, 57, 6, 26].

Let us now define the Hardy-Sobolev space relevant to us. Fix a function $\Phi \in C^\infty(\mathbb{R}^{d+1})$ such that $\operatorname{supp}\Phi \subset B(0,1)$ and $\int \Phi \, dm_{d+1} = 1$. Following Miyachi [34, 35], for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+, m_{d+1})$, define the radial maximal function $f^+\colon \mathbb{R}^{d+1}_+ \to [0,\infty]$ by

$$f^+(x) = \sup_{0<t<\min(x_{d+1},1)} \big|(f * \Phi_t)(x)\big|,$$

where x_{d+1} is the $(d+1)$-th coordinate of x and $\Phi_t := t^{-(d+1)}\Phi(\cdot/t)$. If μ is a Borel regular and absolutely continuous measure on $\mathbb{R}^{d+1}_+$, define the localized Hardy space $h^1(\mathbb{R}^{d+1}_+, \mu)$ as the space of locally m_{d+1}-integrable functions f on $\mathbb{R}^{d+1}_+$ such that

$$\|f\|_{h^1(\mathbb{R}^{d+1}_+,\mu)} := \|f^+\|_{L^1(\mathbb{R}^{d+1}_+,\mu)}$$

is finite. We clearly have $|f(x)| \leq |f^+(x)|$ for almost all x, so $h^1(\mathbb{R}^{d+1}_+, \mu) \subset L^1(\mathbb{R}^{d+1}_+, \mu)$ with a continuous embedding.

It follows from Miyachi's results (see also (41) below) that for the measures μ relevant to us, the space defined above is independent of Φ in the sense that two admissible choices yield the same space with equivalent norms. In fact, it will be convenient for us to choose Φ so that $\operatorname{supp}\Phi \subset B(0,1/8)$.

Now the Hardy-Sobolev space $h^{1,1}(\mathbb{R}^{d+1}_+, \mu)$ is defined as the space of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+, m_{d+1})$ such that the first-order distributional derivatives $\partial_j f$, $1 \leq j \leq d+1$, also belong to $L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+, m_{d+1})$ and

$$\|f\|_{h^{1,1}(\mathbb{R}^{d+1}_+,\mu)} := \|f\|_{L^1(\mathbb{R}^{d+1}_+,\mu)} + \sum_{j=1}^{d+1} \|\partial_j f\|_{h^1(\mathbb{R}^{d+1}_+,\mu)}$$

is finite.

The trace theorem for these spaces then reads as follows.

Theorem 6.1. *Let $\alpha \in (-1,0)$ Then $\big(\mathcal{B}^{-\alpha}_{1,1}(\mathbb{R}^d), h^{1,1}(\mathbb{R}^{d+1}_+, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

Before proving this Theorem, let us formulate a sampling lemma which is essentially folklore. For the convenience of the reader, a proof is presented in Subsection 7.3 of the Appendix.

Lemma 6.2. *Suppose that Ω is an open subset of $\mathbb{R}^d$, that μ is a doubling measure on Ω such that every Euclidean ball (restricted to Ω) has positive and finite μ-measure and that $0 < \lambda < 1$. Then there is a constant C depending only on the dimension d, the doubling constant of μ and λ such that the following statement holds.*

For every cube $Q \subset \Omega$ and $f \in L^1(Q)$, there exists a cube $\tilde{Q} \subset Q$ with $\ell(\tilde{Q}) = \lambda\ell(Q)$ such that

$$\fint_Q |f - f_{Q,\mu}| d\mu \leq C \fint_{\tilde{Q}} |f - f_{\tilde{Q},\mu}| d\mu.$$

Proof of Theorem 6.1. (i) In order to estimate the $h^{1,1}(\mathbb{R}^{d+1}_+, \mu_\alpha)$-norm of the Whitney extension of a function $f \in B^{-\alpha}_{1,1}(\mathbb{R}^d)$, we proceed as in the proof of Theorem 1.1. First, the L^1-norm of $\mathcal{E}f$ can be estimated as in (15). In order to estimate the $h^1(\mathbb{R}^{d+1}_+, \mu_\alpha)$-norm of a partial derivative $\partial_j f$, write $X_1 := \cup_{Q \in \mathscr{Q}^0_d} \mathscr{W}(Q)$ and $X_2 := \mathbb{R}^{d+1}_+ \setminus X_1$.

Suppose first that $x \in X_1$, i.e. $x \in \mathscr{W}(P)$ for some Q, and $0 < t < \min(x_{d+1}, 1)$. We plainly have

$$\big(\partial_j(\mathcal{E}f)\big) * \Phi_t(x) = \Big(\partial_j\big(\mathcal{E}f - f_P\big)\Big) * \Phi_t(x).$$

Now since $x_{d+1} < 2\ell(Q)$ and we assumed the support of Φ to be contained in $B(0, 1/8)$, we see that

$$\operatorname{supp} \Phi_t(x - \cdot) \subset \frac{5}{4}\mathscr{W}(Q),$$

and hence $\operatorname{supp}\psi_Q \cap \operatorname{supp}\Phi_t(x - \cdot) \neq \emptyset$ can only hold if $Q \sim P$. Since also the L^1-norm of $\Phi_t(x - \cdot)$ does not depend on t, we get

$$\begin{aligned}\big(\partial_j(\mathcal{E}f)\big)^+(x) &\leq \sup_{0<t<\min(x_{d+1},1)} \int_{\frac{5}{4}\mathscr{W}(Q)} \sum_{Q\sim P} |f_Q - f_P||\partial_j(\psi_Q)(y)||\Phi_t(x-y)| dm_{d+1}(y) \\ &\lesssim \sum_{Q\sim P} \frac{1}{\ell(Q)} |f_Q - f_P|.\end{aligned}$$

This is estimate corresponds to (16) in the proof of Theorem 1.1, so $\|(\partial_j(\mathcal{E}f))^+\|_{L^1(X_1,\mu_\alpha)}$ can be estimated in the same way as in that proof.

Now if $x \in X_2$ and $0 < t < \min(x_{d+1}, 1)$, we can only have $\operatorname{supp}\psi_Q \cap \operatorname{supp}\Phi_t(x - \cdot) \neq \emptyset$ if $Q \in \mathscr{Q}_{0,d}$. Thus,

$$\begin{aligned}\big|\big(\partial_j(\mathcal{E}f)\big) * \Phi_t(x)\big| &\leq \int_{\mathbb{R}^{d+1}_+} \sum_{Q\in\mathscr{Q}_{d,0}} |f_Q||\partial_j(\psi_Q)(y)||\Phi_t(x-y)| dy m_{d+1} \\ &\lesssim \sum_{\substack{Q\in\mathscr{Q}_{d,0} \\ \operatorname{supp}\psi_Q\cap\operatorname{supp}\Phi_t(x-\cdot)\neq\emptyset}} |f_Q| \leq \sum_{\substack{Q\in\mathscr{Q}_{d,0} \\ \operatorname{supp}\psi_Q\cap B(x,1/8)\neq\emptyset}} |f_Q|\end{aligned}$$

Since the μ_α-measures of the $\frac{1}{8}$-neighborhoods of the supports of ψ_Q above are comparable to 1, we get

$$\|(\partial_j(\mathcal{E}f))^+\|_{L^1(X_2,\mu_\alpha)} \lesssim \sum_{Q\in\mathscr{Q}_{d,0}} |f_Q| \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

That $\mathcal{R}(\mathcal{E}f) = f$ is then checked as in the previous proofs. This finishes the proof of part (ii).

(ii) Let us recall some notation from the proof of Theorem 1.1. For $Q \in \mathscr{Q}_d^0$, write $\mathscr{W}(Q) := Q \times [\ell(Q), 2\ell(Q)) \in \mathscr{Q}_{d+1}^0$ and $\mathscr{N}(Q) := \frac{5}{4}\mathscr{W}(Q)$.

Now in order to verify the existence of the trace of a Hardy-Sobolev function and estimate its norm, the argument in part (ii) of the proof of Theorem 1.1 applies here as well, as long as we can verify that every $f \in h^{1,1}(\mathbb{R}_+^{d+1}, \mu_\alpha)$ satisfies a suitable Poincaré-type inequality on cubes that are relatively far away from the boundary $\mathbb{R}^d$. More precisely, it suffices to show that there exists a measurable function $g\colon \mathbb{R}_+^{d+1} \to [0,\infty]$ such that

$$\fint_{\mathscr{N}(Q)} |f - f_{\mathscr{N}(Q)}| dm_{d+1} \lesssim \ell(Q) \fint_{\mathscr{N}(Q)} g dm_{d+1} \tag{39}$$

for all $Q \in \mathscr{Q}_d^0$ (with the implied constant independent of f) and

$$\|g\|_{L^1(\mathbb{R}_+^{d+1}),\mu} \lesssim \|f\|_{h^{1,1}(\mathbb{R}_+^{d+1},\mu_\alpha)}. \tag{40}$$

To this end, let us recall the definition of the *grand maximal function* related to the space h^1. For $h \in L^1_{\mathrm{loc}}(\mathbb{R}_+^{d+1})$ and $N \in \mathbb{N}$, define the function $\mathcal{M}_N^* h\colon \mathbb{R}_+^{d+1} \to [0,\infty]$ by

$$\mathcal{M}_N^* h(x) = \sup_{\psi \in \mathcal{F}_N(x)} \Big| \int_{\mathbb{R}_+^{d+1}} h(y)\psi(y) dm_{d+1}(y) \Big|,$$

where

$$\begin{aligned}\mathcal{F}_N(x) = \Big\{ \psi \in C^\infty(\mathbb{R}_+^{d+1}) \,:\, &\text{there exist } y \in \mathbb{R}_+^{d+1} \text{ and } r \in (0,1) \text{ such that} \\ &x \in B(y,r) \subset \mathbb{R}_+^{d+1},\ \operatorname{supp}\psi \subset B(y,r) \text{ and } |\partial^\beta \psi| \le r^{-(d+1)-|\beta|} \\ &\text{for all multi-indices } \beta \text{ such that } |\beta| \le N \Big\}.\end{aligned}$$

We claim that

$$g := \sum_{j=1}^{d+1} \mathcal{M}_1^*(\partial_j f)$$

satisfies (39) and (40).

Now by [26, Theorem 7], there exists a constant c depending only on the dimension d such that

$$|f(x) - f(y)| \le c|x-y|\big(g(x) + g(y)\big)$$

for all $x,\ y \in \mathbb{R}_+^{d+1}$ such that $|x - y| < \min(x_{d+1}, y_{d+1}, 1)$. We can apply this estimate in a cube $\mathscr{N}(Q)$ as follows. Since $\operatorname{dist}(\mathscr{N}(Q), \mathbb{R}^d) \approx \ell(\mathscr{N}(Q))$, we can use Lemma 6.2 to find a cube $\tilde{Q} \subset \mathscr{N}(Q)$ such that $\ell(\tilde{Q}) \approx \ell(\mathscr{N}(Q))$,

$$\fint_{\mathscr{N}(Q)} |f - f_{\mathscr{N}(Q)}| dm_{d+1} \lesssim \fint_{\tilde{Q}} |f - f_{\tilde{Q}}| dm_{d+1}$$

and $|x - y| < \min(x_{d+1}, y_{d+1}, 1)$ for all $x,\ y \in \tilde{Q}$. Thus,

$$\fint_{\mathscr{N}(Q)} |f - f_{\mathscr{N}(Q)}| dm_{d+1} \lesssim \fint_{\tilde{Q}} \fint_{\tilde{Q}} |x-y|\big(g(x) + g(y)\big) dy dx \lesssim \ell(Q) \fint_{\mathscr{N}(Q)} g dm_{d+1},$$

which is (39).

As for (40), we denote by $\tilde{g}_j$, $1 \le j \le d+1$, the function on $\mathbb{R}^{d+1}$ that is obtained by extending $(\partial_j f)^+$ as zero on $\mathbb{R}^{d+1} \setminus \mathbb{R}^{d+1}_+$. Then by [34, Corollary 2], there exists an exponent $q \in (0,1)$ and a constant C independent of $f \in h^{1,1}(\mathbb{R}^{d+1}_+, \mu_\alpha)$ such that

$$\mathcal{M}_1^*(\partial_j f)(x) \le C\Big(\mathcal{M}(\tilde{g}_j^q)(x)\Big)^{1/q} \tag{41}$$

for all $x \in \mathbb{R}^{d+1}_+$, where $\mathcal{M}$ stands for the standard Hardy-Littlewood maximal operator on $\mathbb{R}^{d+1}$. Because $\alpha \in (-1,0)$, w_α can be extended in a natural way as an $A_{1/q}$-weight on $\mathbb{R}^{d+1}$, which in particular means that $\mathcal{M}$ is bounded on $L^{1/q}(\mathbb{R}^{d+1}, \mu_\alpha)$. Thus,

$$\|\mathcal{M}_1^*(\partial_j f)\|_{L^1(\mathbb{R}^{d+1}_+,\mu_\alpha)} \lesssim \|\tilde{g}_j\|_{L^1(\mathbb{R}^{d+1},\mu_\alpha)} \approx \|\partial_j f\|_{h^1(\mathbb{R}^{d+1}_+,\mu_\alpha)},$$

and summing up over j yields (40).

(iii) As in the previous proofs, we have $\mathcal{R}(\mathcal{E}f) = f$ for all $f \in \mathcal{B}^{-\alpha}_{1,1}(\mathbb{R}^d)$. The discussion concerning the form of the trace operator $\mathcal{R}$ is again postponed until Subsection 7.1 of the Appendix. ☐

7. Appendix

In this section we present some details which were, for the sake of presentation, omitted in the previous sections.

7.1. Coincidence of trace operators. Recall that it was not a priori obvious that the trace operators constructed in the proofs of Theorems 1.1, 1.2, 1.3 and 6.1 are of the form required by Definition 2.6. In this subsection we explain why this is the case.

Suppose that $f \in \mathcal{B}^s_{p,q}(\mathbb{R}^{d+1}_+, \mu_\alpha)$ or $f \in \mathcal{F}^s_{p,q}(\mathbb{R}^{d+1}_+, \mu_\alpha)$ with the parameters p, q and α admissible for the trace theorems concerning these spaces. Then, because of (13) and the fact that the measure μ_α is doubling on $\mathbb{R}^{d+1}_+$, we have that for m_d-almost all $x \in \mathbb{R}^d$, there exists a number $c \in \mathbb{C}$ such that

$$\lim_{r\to 0} \fint_{B((x,0),r)} |f(y) - c| d\mu_\alpha = 0. \tag{42}$$

In fact, the set of points x for which this does not hold has Hausdorff dimension at most $\max(d+1+\alpha-sp, 0) < d$. This follows from a well-known covering argument and a Poincaré-type inequality for the function spaces in question; we refer to e.g. [43, Lemma 3.1 and Remark 3.2] for details. By the same argument and the Poincaré inequality established e.g. in [2, Theorem 4], the same holds if $f \in W^{1,p}(\mathbb{R}^{d+1}_+, \mu_\alpha)$ and s above is replaced by 1. Finally, the aforementioned argument in [43, Lemma 3.1 and Remark 3.2] also applies for functions $f \in h^{1,1}(\mathbb{R}^{d+1}_+, \mu_\alpha)$, since by a modification of the proof of [26, Theorem 16], f has a local Hajłasz gradient in $L^1(\mathbb{R}^{d+1}_+, \mu_\alpha)$, which yields a suitable $(1,1)$-Poincaré inequality for f.

From (42), it is then easy to see that the limits defining each trace operator in the above-mentioned proofs can be rewritten in the form (12).

7.2. Equivalence of norms. Here we present a direct proof of the equivalence of the (quasi-)norm (8) with the standard Besov quasi-norm (10).

Proposition 7.1. *Let μ be a Borel regular measure on $\mathbb{R}^d$ such that every Euclidean ball has positive and finite measure, and such that μ is doubling with respect to the Euclidean metric. If $0<s<1$, $1\le p<\infty$ and $0<q\le\infty$, then*

$$\|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)} \approx \|f\|_{L^p(\mathbb{R}^d,\mu)} + \left(\int_0^\infty t^{-sq}\Big(\int_{\mathbb{R}^d}\fint_{B(x,r)}|f(x)-f(y)|^p d\mu(y)d\mu(x)\Big)^{q/p}\frac{dt}{t}\right)^{1/q}$$

for all $f\in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mu)$, where the implied constants are independent of f.

Proof. Let us denote the standard Besov quasi-norm (10) by $\|f\|_{B^s_{p,q}(\mathbb{R}^d,\mu)}$. We first prove that $\|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)} \lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^d,\mu)}$. To simplify the notation, write dx for $d\mu(x)$ for the rest of this proof.

The doubling property of μ implies that $\mu(Q)\approx\mu(Q')$ if Q and Q' are cubes in $\mathscr{Q}_d$ with $Q\sim Q'$. Thus,

$$\begin{aligned}
\sum_{Q\in\mathscr{Q}_{d,k}}\mu(Q)\sum_{Q'\sim Q}\big|f_{Q,\mu}-f_{Q',\mu}\big|^p &\le \sum_{Q\in\mathscr{Q}_{d,k}}\mu(Q)\sum_{Q'\sim Q}\fint_Q\fint_{Q'}|f(x)-f(y)|^p dydx\\
&\lesssim \sum_{Q\in\mathscr{Q}_{d,k}}\sum_{Q'\sim Q}\frac{1}{\mu(Q)}\int_Q\int_{Q'}|f(x)-f(y)|^p dydx\\
&\le \sum_{Q\in\mathscr{Q}_{d,k}}\sum_{Q'\sim Q}\frac{1}{\mu(Q)}\int_Q\int_{B(x,C\cdot2^{-k})}|f(x)-f(y)|^p dydx\\
&\lesssim \sum_{Q\in\mathscr{Q}_{d,k}}\int_Q\fint_{B(x,C\cdot2^{-k})}|f(x)-f(y)|^p dydx\\
&= \int_{\mathbb{R}^d}\fint_{B(x,C\cdot2^{-k})}|f(x)-f(y)|^p dydx,
\end{aligned}$$

where $C=4\sqrt{d}$ and the doubling property of μ was again used in the second-to-last line. This leads to

$$\begin{aligned}
&\sum_{k\ge0}2^{ksq}\Big(\sum_{Q\in\mathscr{Q}_{d,k}}\mu(Q)\sum_{Q'\sim Q}\big|f_{Q,\mu}-f_{Q',\mu}\big|^p\Big)^q\\
&\quad\lesssim \sum_{k=0}^\infty 2^{ksq}\Big(\int_{\mathbb{R}^d}\fint_{B(x,C\cdot2^{-k})}|f(x)-f(y)|^p dydx\Big)^{q/p}\\
&\quad\lesssim \sum_{k=0}^\infty\int_{C\cdot2^{-k}}^{C\cdot2^{-k+1}}\Big(\int_{\mathbb{R}^d}\fint_{B(x,t)}|f(x)-f(y)|^p dydx\Big)^{q/p}\frac{dt}{t^{1+sq}}\\
&\quad\le \int_0^\infty\Big(\int_{\mathbb{R}^d}\fint_{B(x,t)}|f(x)-f(y)|^p dydx\Big)^{q/p}\frac{dt}{t^{1+sq}},
\end{aligned}$$

which implies that $\|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)} \lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^d,\mu)}$.

In order to prove that $\|f\|_{B^s_{p,q}(\mathbb{R}^d,\mu)} \lesssim \|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)}$, we first note that a straightforward application of Fubini's theorem in conjunction with the doubling property of μ yields

$$\int_1^\infty\Big(\int_{\mathbb{R}^d}\fint_{B(x,t)}|f(x)-f(y)|^p dydx\Big)^{q/p}\frac{dt}{t^{1+sq}}$$

$$\lesssim \int_1^\infty \Big(\int_{\mathbb{R}^d} |f(x)|^p dx + \int_{\mathbb{R}^d} |f(y)|^p \Big(\fint_{B(y,t)} \frac{dx}{\mu(B(x,t))} \Big) dy \Big)^{q/p} \frac{dt}{t^{1+sq}}$$
$$\approx \|f\|_{L^p(\mathbb{R}^d,\mu)}.$$

To estimate the corresponding integral from 0 to 1, use the doubling property of μ to get

$$\begin{aligned}
&\int_0^1 \Big(\int_{\mathbb{R}^d} \fint_{B(x,t)} |f(x)-f(y)|^p dy dx \Big)^{q/p} \frac{dt}{t^{1+sq}} \\
&\quad \lesssim \sum_{k\geq 0} \int_{2^{-k-1}}^{2^{-k}} \Big(\int_{\mathbb{R}^d} \fint_{B(x,2^{-k})} |f(x)-f(y)|^p \, dy \, dx \Big)^{q/p} \frac{dt}{t^{1+sq}} \\
&\quad \lesssim \sum_{k\geq 0} 2^{ksq} \Big(\int_{\mathbb{R}^d} \fint_{B(x,2^{-k})} |f(x)-f(y)|^p \, dy \, dx \Big)^{q/p} \\
&\quad = \sum_{k\geq 0} 2^{ksq} \Big(\sum_{Q\in\mathscr{Q}_{d,k}} \int_Q \fint_{B(x,2^{-k})} |f(x)-f(y)|^p \, dy \, dx \Big)^{q/p}.
\end{aligned}$$

Let $Q \in \mathscr{Q}_{d,k}$ for some $k \geq 0$. For $x \in Q$ we obviously have $B(x,2^{-k}) \subset \bigcup_{Q'\sim Q} Q'$ and $\mu(B(x,2^{-k})) \approx \mu(Q)$. Thus,

$$\begin{aligned}
&\int_Q \fint_{B(x,2^{-k})} |f(x)-f(y)|^p \, dy \, dx \\
&\lesssim \sum_{Q'\sim Q} \frac{1}{\mu(Q)} \int_Q \int_{Q'} |f(x)-f(y)|^p \, dydx. \\
&\lesssim \sum_{Q'\sim Q} \frac{1}{\mu(Q)} \int_Q \int_{Q'} |f(x)-f_{Q,\mu}|^p \, dydx + \sum_{Q'\sim Q} \frac{1}{\mu(Q)} \int_Q \int_{Q'} |f_{Q,\mu}-f_{Q',\mu}|^p \, dydx \\
&\qquad + \sum_{Q'\sim Q} \frac{1}{\mu(Q)} \int_Q \int_{Q'} |f_{Q',\mu}-f(y)|^p \, dydx \\
&=: O_Q^1 + O_Q^2 + O_Q^3,
\end{aligned}$$

so that

$$\begin{aligned}
\|f\|_{B^s_{p,q}(\mathbb{R}^d,\mu)} &\lesssim \|f\|_{L^p(\mathbb{R}^d,\mu)} + \Big(\sum_{k\geq 0} 2^{ksq} \Big(\sum_{Q\in\mathscr{Q}_{d,k}} (O_Q^1+O_Q^2+O_Q^3) \Big)^{q/p} \Big)^{1/q} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^d,\mu)} + \sum_{j=1,2,3} \Big(\sum_{k\geq 0} 2^{ksq} \Big(\sum_{Q\in\mathscr{Q}_{d,k}} O_Q^j \Big)^{q/p} \Big)^{1/q} \\
&=: \|f\|_{L^p(\mathbb{R}^d,\mu)} + H_1 + H_2 + H_3. \qquad (43)
\end{aligned}$$

We first estimate the quantity H_2. For each $Q \in \mathscr{Q}_{d,k}$ the doubling property yields

$$O_Q^2 = \sum_{Q'\sim Q} \mu(Q')|f_{Q,\mu}-f_{Q',\mu}|^p \approx \mu(Q) \sum_{Q'\sim Q} |f_{Q,\mu}-f_{Q',\mu}|^p,$$

and hence

$$H_2 \lesssim \Big(\sum_{k\geq 0} 2^{ksq}\Big(\sum_{Q\in\mathscr{Q}_{d,k}} \mu(Q) \sum_{Q'\sim Q} |f_{Q,\mu} - f_{Q',\mu}|^p\Big)^{q/p}\Big)^{1/q} \lesssim \|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)}. \tag{44}$$

Next we estimate H_1. For any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$, define Q^x_n as the (unique) cube in $\mathscr{Q}_{d,n}$ that contains x. By the Lebesgue differentiation theorem for doubling measures [20, Theorem 1.8], we have $\lim_{n\to} f_{Q^x_n,\mu} = f(x)$ for μ-almost every $x \in \mathbb{R}^d$. Hence, if $Q \in \mathscr{Q}_{d,k}$ and $x \in Q$, we have

$$|f(x) - f_{Q,\mu}|^p \leq \Big(\sum_{n=k}^{\infty} \big|f_{Q^x_n,\mu} - f_{Q^x_{n+1},\mu}\big|\Big)^p \lesssim 2^{-k\epsilon} \sum_{n=k}^{\infty} 2^{n\epsilon}\big|f_{Q^x_n,\mu} - f_{Q^x_{n+1},\mu}\big|^p,$$

where $\epsilon > 0$ is chosen so that $\epsilon < sp/2$. Applying this estimate to O^1_Q and using the fact that every cube has a (uniformly) finite number of neighbors, we get

$$\begin{aligned}
O^1_Q &\lesssim \sum_{Q'\sim Q} \frac{\mu(Q')}{\mu(Q)} 2^{-k\epsilon} \sum_{n=k}^{\infty} 2^{n\epsilon} \int_Q \big|f_{Q^x_n,\mu} - f_{Q^x_{n+1},\mu}\big|^p \,dx \\
&\lesssim 2^{-k\epsilon} \sum_{n=k}^{\infty} 2^{n\epsilon} \sum_{\substack{Q''\in\mathscr{Q}_{d,n}\\ Q''\subset Q}} \int_{Q''} \big|f_{Q^x_n,\mu} - f_{Q^x_{n+1},\mu}\big|^p \,dx \\
&\lesssim \sum_{n=k}^{\infty} 2^{\epsilon(n-k)} \sum_{\substack{Q''\in\mathscr{Q}_{d,n}\\ Q''\subset Q}} \mu(Q'') \sum_{Q'''\sim Q''} \big|f_{Q'',\mu} - f_{Q''',\mu}\big|^p.
\end{aligned}$$

In order to use this to estimate H_1, we consider two possible cases for the parameter q. First, if $0 < q \leq p$, the subadditivity of the function $t \mapsto t^{q/p}$ and the fact that $s - \epsilon/p > 0$ yield

$$\begin{aligned}
H_1^q &\lesssim \sum_{k\geq 0} 2^{ksq} \sum_{n=k}^{\infty} 2^{\epsilon(n-k)q/p} \Big(\sum_{Q\in\mathscr{Q}_{d,k}} \sum_{\substack{Q''\in\mathscr{Q}_{d,n}\\ Q''\subset Q}} \mu(Q'') \sum_{Q'''\sim Q''} \big|f_{Q'',\mu} - f_{Q''',\mu}\big|^p\Big)^{q/p} \\
&\leq \sum_{n=0}^{\infty} 2^{\epsilon nq/p} \Big(\sum_{k=0}^{n} 2^{kq(s-\epsilon/p)}\Big) \Big(\sum_{Q''\in\mathscr{Q}_{d,n}} \mu(Q'') \sum_{Q'''\sim Q''} \big|f_{Q'',\mu} - f_{Q''',\mu}\big|^p\Big)^{q/p} \\
&\approx \sum_{n=0}^{\infty} 2^{nsq} \Big(\sum_{Q\in\mathscr{Q}_{d,n}} \sum_{Q'\sim Q} \mu(Q)|f_{Q,\mu} - f_{Q',\mu}|^p\Big)^{q/p} \leq \|f\|^q_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)}.
\end{aligned}$$

If on the other hand $p < q \leq \infty$, we may use Hölder's inequality and the fact that $s - 2\epsilon/p > 0$, to obtain

$$H_1^q \lesssim \sum_{k\geq 0} 2^{ksq} \Big(\sum_{n=k}^{\infty} 2^{-\epsilon(n-k)q/p} 2^{2\epsilon(n-k)} \sum_{Q\in\mathscr{Q}_{d,k}} \sum_{\substack{Q''\in\mathscr{Q}_{d,n}\\ Q''\subset Q}} \mu(Q'') \sum_{Q'''\sim Q''} \big|f_{Q'',\mu} - f_{Q''',\mu}\big|^p\Big)^{q/p}$$

$$\le \sum_{k\ge 0} 2^{ksq} \sum_{n=k}^{\infty} 2^{2\epsilon(n-k)q/p} \Bigg(\sum_{Q\in\mathscr{Q}_{d,k}} \sum_{\substack{Q''\in\mathscr{Q}_{d,n}\\ Q''\subset Q}} \mu(Q'') \sum_{Q'''\sim Q''} \big|f_{Q'',\mu} - f_{Q''',\mu}\big|^p \Bigg)^{q/p}$$

$$\le \sum_{n=0}^{\infty} 2^{2\epsilon nq/p} \Big(\sum_{k=0}^{n} 2^{kq(s-2\epsilon/p)} \Big) \Bigg(\sum_{Q''\in\mathscr{Q}_{d,n}} \mu(Q'') \sum_{Q'''\sim Q''} \big|f_{Q'',\mu} - f_{Q''',\mu}\big|^p \Bigg)^{q/p}$$

$$\approx \sum_{n\ge 0} 2^{nsq} \Bigg(\sum_{Q\in\mathscr{Q}_{d,n}} \sum_{Q'\sim Q} \mu(Q)|f_{Q,\mu} - f_{Q',\mu}|^p \Bigg)^{q/p} \le \|f\|^q_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)},$$

which is the desired estimate for H_1. Finally, the terms O_Q^3 are essentially symmetric to with the terms O_Q^1, so H_3 can be estimated using the same argument as H_1. Combining these estimates with (44) and applying them to (43), we arrive at

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d,\mu)} \lesssim \|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}^d,\mu)}. \qquad \square$$

7.3. **Proof of Lemma 6.2.** Here we present the proof of the sampling lemma that was used in the proof of Theorem 6.1.

Proof. Let Q and f be as in the statement. Let us first consider the case $\lambda = 3/4$. Let $Q_i \subset Q$, $1 \le i \le 2^d$, be the cubes with edge length $\frac{3}{4}\ell(Q)$ that are situated at the corners of Q. Then $Q^* := \bigcap_{1\le i\le 2^d} Q_i$ is a cube with edge length $\frac{1}{2}\ell(Q)$. By doubling, we get

$$\begin{aligned} \fint_Q |f - f_{Q,\mu}|d\mu &\lesssim \fint_Q |f - f_{Q^*,\mu}|d\mu \approx \max_{1\le i\le 2^d} \fint_{Q_i} |f - f_{Q^*,\mu}|d\mu \\ &\le \max_{1\le i\le 2^d} \Big(\fint_{Q_i} |f - f_{Q_i,\mu}|d\mu + \big|f_{Q_i,\mu} - f_{Q^*,\mu}\big| \Big), \end{aligned}$$

and again using the doubling property of μ to estimate the latter term in the parentheses, we arrive at

$$\fint_Q |f - f_{Q,\mu}|d\mu \le c \max_{1\le i\le 2^d} \fint_{Q_i} |f - f_{Q_i,\mu}|d\mu,$$

where the constant c depends only on d and the doubling constant of μ.

Now suppose that $\lambda \in (0,1)$ as in the statement of the Lemma. Write k_λ for the positive integer such that $(3/4)^{k_\lambda} \le \lambda < (3/4)^{k_\lambda - 1}$. Iterating the argument above k_λ times yields a cube $Q^{k_\lambda} \subset Q$ such that $\ell(Q^{k_\lambda}) = (3/4)^{k_\lambda}\ell(Q)$ and

$$\fint_Q |f - f_{Q,\mu}|d\mu \le c^{k_\lambda} \fint_{Q^{k_\lambda}} |f - f_{Q^{k_\lambda},\mu}|d\mu.$$

Now one can simply take a cube $\tilde{Q} \subset Q$ that contains Q^{k_λ} and has edge length $\lambda\ell(Q)$. By doubling, the integral on the right-hand side above can then be estimated by a constant times

$$\fint_{\tilde{Q}} |f - f_{\tilde{Q},\mu}|d\mu. \qquad \square$$

7.4. **Extending functions from $\mathbb{R}^d$ to $\mathbb{R}^{d+n}$.** Here we present the generalizations of Theorems 1.1 through 1.3 for Euclidean codimensions higher than 1. The dimensions $d \in \mathbb{N}$ and $d+n$, $n \in \mathbb{N}$, will be fixed in the sequel. For convenience we also write $\mathbb{R}^d$ for $\mathbb{R}^d \times \{0\}^n \subset \mathbb{R}^{d+n}$ when there is no risk of confusion.

The spaces $W^{1,p}(\mathbb{R}^{d+n}, \mu)$, $\mathcal{B}^s_{p,q}(\mathbb{R}^{d+n}, \mu)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^{d+n}, \mu)$ are as in the Defintions 2.1 through 2.3. In what follows, we consider the measures μ_α, $\alpha > -n$, on $\mathbb{R}^{d+n}$, defined by

$$\mu_\alpha(E) = \int_E w_\alpha dm_{d+n},$$

where $w_\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+n})$ stands for the weight $x \mapsto \min(1, \mathrm{dist}\,(x, \mathbb{R}^d))^\alpha$.

In order to define the Whitney extension of a function on $\mathbb{R}^d$ to $\mathbb{R}^{d+n}$, we introduce some additional notation. For $Q \in \mathscr{Q}_{d,k}$, $k \in \mathbb{Z}$, define

$$\mathscr{A}_Q := \Big\{ P \in \mathscr{Q}_{d+n,k} : P \subset \big(Q \times [-2^{-k+1}, 2^{-k+1}]^n\big) \setminus \big(Q \times (-2^{-k}, 2^{-k})^n\big) \Big\}$$

It is then evident that $\#\mathscr{A}_Q = 4^n - 2^n \approx 1$, and that

$$\bigcup_{Q \in \mathscr{Q}_d} \mathscr{A}_Q$$

is a Whitney decomposition of the the space $\mathbb{R}^{d+n} \setminus \mathbb{R}^d$ with respect to the boundary $\mathbb{R}^d$. We define the bump functions $\psi_P \colon \mathbb{R}^{d+n} \to [0,1]$ for all $P \in \bigcup_{Q \in \mathscr{Q}^0_d} \mathscr{A}_Q$ so that $\mathrm{Lip}\, \psi_P \lesssim 1/\ell(P)$, $\inf_{x \in P} \psi_P(x) > 0$ uniformly in P, $\mathrm{supp}\, \psi_P$ is contained in an $\ell(P)/4$-neighborhood of P and

$$\sum_{Q \in \mathscr{Q}^0_d} \sum_{P \in \mathscr{A}_Q} \psi_P \equiv 1 \quad \text{in} \quad \bigcup_{Q \in \mathscr{Q}^0_d} \bigcup_{P \in \mathscr{A}_Q} P.$$

Definition 7.2. *(i) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then the Whitney extension $\mathcal{E}f \colon \mathbb{R}^{d+n} \to \mathbb{C}$ is defined by*

$$\mathcal{E}f(x) = \sum_{Q \in \mathscr{Q}^0_d} \sum_{P \in \mathscr{A}_Q} \Big(\fint_Q f dm_d \Big) \psi_P(x).$$

This definition gives rise in the obvious way to the linear operator $\mathcal{E} \colon L^1_{\mathrm{loc}}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^{d+n})$.

(ii) Let $X \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a quasinormed function space on $\mathbb{R}^d$, and let Y be a quasinormed function space on the weighted space $(\mathbb{R}^{d+n}, \mu)$. We say that (X, Y) is a Whitney trace-extension pair *if they satisfy the conditions in Definition 2.6 with $\mathbb{R}^{d+n}$ in place of $\mathbb{R}^{d+1}_+$ and with $\mathcal{E}$ as defined above.*

We then have the following trace theorems.

Theorem 7.3. *Let $1 \le p < \infty$ and $-n < \alpha < p-n$. Then $\big(\mathcal{B}^{1-(\alpha+n)/p}_{p,p}(\mathbb{R}^d), W^{1,p}(\mathbb{R}^{d+n}, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

Theorem 7.4. *Let $0 < s < 1$, $1 \le p \le \infty$, $0 < q \le \infty$ and $-n < \alpha < sp - n$. Then $\big(\mathcal{B}^{s-(\alpha+n)/p}_{p,q}(\mathbb{R}^d), \mathcal{B}^s_{p,q}(\mathbb{R}^{d+n}, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

Theorem 7.5. *Let $0 < s < 1$, $1 \le p < \infty$, $0 < q \le \infty$ and $-n < \alpha < sp - n$. Then $\big(\mathcal{B}^{s-(\alpha+n)/p}_{p,p}(\mathbb{R}^d), \mathcal{F}^s_{p,q}(\mathbb{R}^{d+n}, \mu_\alpha)\big)$ is a Whitney trace-extension pair.*

These results can be proven by suitable modifications of the arguments in the proofs of Theorems 1.1 through 1.3. For the reader's convenience, we sketch the modified arguments below.

Proof of Theorem 7.3. (i) Let us estimate the weighted Sobolev norm of the Whitney extension of a function $f \in \mathcal{B}^{1-(\alpha+n)/p}_{p,p}(\mathbb{R}^d)$. First, if $P \in \mathscr{A}_Q$ for some $Q \in \mathscr{Q}^0_d$, it is easily seen that $\mu_\alpha(P) \approx \ell(Q)^\alpha m_{d+n}(P) \approx \ell(Q)^{d+n+\alpha}$. Since the supports of the bump functions ψ_P in the definition of $\mathcal{E}$ above have bounded overlap and $\#\mathscr{A}_Q \approx 1$ for all $Q \in \mathscr{Q}_d$, we get

$$\int_{\mathbb{R}^{d+n}} |\mathcal{E}f|^p d\mu_\alpha \lesssim \sum_{Q\in\mathscr{Q}^0_d} \sum_{P\in\mathscr{A}_Q} \mu_\alpha(P) \fint_Q |f|^p dm_d \approx \sum_{Q\in\mathscr{Q}^0_d} \ell(Q)^{\alpha+n} \int_Q |f|^p dm_d \lesssim \int_{\mathbb{R}^d} |f|^p dm_d.$$

Now to estimate the weighted L^p-norm of $|\nabla(\mathcal{E}f)|$, write $X_1 := \bigcup_{Q\in\mathscr{Q}^0_d} \cup_{P\in\mathscr{A}(Q)} P$ and $X_2 := \mathbb{R}^{d+n} \setminus X_1$. If $x \in X_1$, i.e. $x \in \cup_{P\in\mathscr{A}(Q)} P$ for some $Q \in \mathscr{Q}^0_d$, we have $\sum_{Q'\in\mathscr{Q}^0_d} \sum_{P\in\mathscr{A}_{Q'}} \psi_P(x) = 1$, and the inner sum can only be nonzero for $Q' \sim Q$. Thus,

$$|\nabla(\mathcal{E}f)(x)| \le \sum_{Q'\sim Q} \sum_{P\in\mathscr{A}'_Q} |f_Q - f_{Q'}||\mathrm{Lip}\,(\psi_P)(x)| \lesssim \sum_{Q'\sim Q} \frac{1}{\ell(Q)} |f_Q - f_{Q'}|.$$

Since $\mu_\alpha(\cup_{P\in\mathscr{A}(Q)}) \approx \ell(Q)^{n+\alpha} m_d(Q)$, we arrive at

$$\int_{X_1} |\nabla(\mathcal{E}f)|^p d\mu_\alpha \lesssim \sum_{Q\in\mathscr{Q}^0_d} \ell(Q)^{n+\alpha-p} m_d(Q) \sum_{Q'\sim Q} |f_Q - f_{Q'}|^p \lesssim \|f\|^p_{\mathcal{B}^{1-(\alpha+n)/p}_{p,p}(\mathbb{R}^d)}.$$

If on the other hand $x \in X_2$, we can only have $\psi_P(x) \neq 0$ if $P \in \mathscr{A}_Q$ for some $Q \in \mathscr{Q}_{d,0}$, so estimating as in the part (i) of the proof of Theorem 1.1, we get

$$\int_{X_2} |\nabla(\mathcal{E}f)(x)| d\mu_\alpha \lesssim \sum_{Q\in\mathscr{Q}_{d,0}} \int_Q |f|^p dm_d = \|f\|^p_{L^p(\mathbb{R}^d)}.$$

Combining these estimates yields the desired norm inequality for the function $\mathcal{E}f$.

(ii) Let us now show that the trace of a function $f \in W^{1,p}(\mathbb{R}^{d+n}, \mu_\alpha)$ exists and estimate its Besov norm. To this end, write

$$\mathscr{W}(Q) := Q \times \big(\ell(Q), 2\ell(Q)\big]^n \in \mathscr{A}_Q \quad \text{and} \quad \mathscr{N}(Q) := \frac{5}{4}\mathscr{W}(Q)$$

for all $Q \in \mathscr{Q}_d$, and for $k \in \mathbb{N}_0$ write

$$\mathscr{T}_k f := \sum_{Q\in\mathscr{Q}_{d,k}} \Big(\fint_{\mathscr{N}(Q)} f\, dm_{d+n}\Big) \chi_Q.$$

To establish the existence of the trace function, we thus want to estimate the L^p-norm of the function

$$f^* := \sum_{k\ge 0} \big|\mathscr{T}_k f - \mathscr{T}_{k+1} f\big| + \big|\mathscr{T}_0 f\big|.$$

Then, since $\mu_\alpha(\mathscr{N}(Q)) \approx \ell(Q)^\alpha m_{d+n}(\mathscr{N}(Q)) \approx \ell(Q)^{d+n+\alpha}$ for all $Q \in \mathscr{Q}^0_d$, an estimate similar to the one in the part (ii) of the proof of Theorem 1.1 yields

$$\big|\mathscr{T}_k f(x) - \mathscr{T}_{k+1} f(x)\big| \lesssim 2^{-k} \Big(\fint_{\mathscr{N}(Q^x_k)} |\nabla f|^p d\mu_\alpha\Big)^{1/p} + 2^{-k} \Big(\fint_{\mathscr{N}(Q^x_{k+1})} |\nabla f|^p d\mu_\alpha\Big)^{1/p},$$

and since $p-(n+\alpha)>0$, an estimate similar to the one in the part (ii) of the proof of Theorem 1.1 again yields

$$\int_{\mathbb{R}^d} |f^*|^p dm_d \lesssim \sum_{Q\in\mathscr{Q}_d^0} \int_{\mathscr{N}(Q)} |\nabla f|^p d\mu_\alpha + \sum_{P\in\mathscr{Q}_{d,0}} \int_{\mathscr{N}(P)} |f|^p d\mu_\alpha \lesssim \|f\|^p_{W^{1,p}(\mathbb{R}^{d+n},\mu_\alpha)}.$$

Hence the trace function $\mathcal{R}f\in L^p(\mathbb{R}^d)$ exists in a suitable sense and has the correct bound for its L^p norm. In the sequel, we shall simply write f for $\mathcal{R}f$.

Now to estimate the $\mathcal{B}^{1-(\alpha+n)/p}_{p,p}$-energy of f, recall that $m_d(Q)/\mu_\alpha(\mathscr{N}(Q))\approx \ell(Q)^{-(\alpha+n)}$. Hence, replacing $\alpha+1$ by $\alpha+n$ in (22), we get

$$\begin{aligned}&\sum_{k\geq 0} 2^{k(1-\frac{\alpha+n}{p})p} \sum_{Q\in\mathscr{Q}_{d,k}} m_d(Q) \sum_{Q'\sim Q} \left|f_Q - f_{Q'}\right|^p \\ \lesssim &\sum_{k\geq 0} 2^{k(1-\frac{\alpha+n}{p})p} \int_{\mathbb{R}^d} \left|f(x)-\mathscr{T}_k f(x)\right|^p dm_d(x) + \sum_{k\geq 0} \int_{\cup_{2^{-k-1}\leq \ell(Q')\leq 2^{-k+1}}\mathscr{N}(Q')} |\nabla f|^p d\mu_\alpha \\ \lesssim &\|f\|^p_{W^{1,p}(\mathbb{R}^{d+n},\mu_\alpha)},\end{aligned}$$

which is the desired estimate.

(iii) As in the proofs of Theorems 1.1 through 1.3, it remains to verify that the trace operator $\mathcal{R}f$ above coincides with the one in Definition 7.2. This again follows from the discussion in Subsection 7.1. □

Proof of Theorem 7.4 (sketch). Again, we only consider the case $q=p<\infty$. In the following proof, we shall use the notation

$$\mathscr{U}_Q := \bigcup_{R\in\mathscr{A}_Q} R \subset \mathbb{R}^{d+n}$$

for all $Q\in\mathscr{Q}_d$.

(i) We first establish the desired norm inequality for the extension of a function $f\in\mathcal{B}^{s-(\alpha+n)/p}_{p,q}(\mathbb{R}^d)$. As in the proof of Theorem 7.3 above, we have

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}^{d+n})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

To estimate the $\mathcal{B}^{s-(\alpha+n)/p}_{p,q}(\mathbb{R}^d)$-energy of $\mathcal{E}f$, we divide the cubes in $\mathscr{Q}_{d+n}$ into three separate classes according to their distances to $\mathbb{R}^d$. For $Q\in\mathscr{Q}_{d+n}$, define

$$\operatorname{dist}^*(Q,\mathbb{R}^d) := \inf\left\{ \max_{1\leq i\leq d+n} |x_i-y_i| \,:\, x\in Q,\, y\in\mathbb{R}^d\times\{0\}^n\right\},$$

where x_i and y_i stand for the ith coordinates of x and y respectively. For $k\geq 0$, write $\mathscr{Q}^1_k$ for the collection of dyadic cubes in $\mathscr{Q}_{d+n,k}$ such that $\operatorname{dist}^*(Q,\mathbb{R}^d)\geq 2$, $\mathscr{Q}^2_k$ for the collection of dyadic cubes such that $2^{-k}\leq \operatorname{dist}^*(Q,\mathbb{R}^d)<2$ and $\mathscr{Q}^3_k$ for the collection of dyadic cubes whose closures intersect $\mathbb{R}^d$. Also write $\mathscr{Q}^{2,*}_k$ for the collectino of cubes in $\bigcup_{i=\max(k-1,0)}^{k+1}\mathscr{Q}^2_i$ that are contained in $\bigcup_{Q\in\mathscr{Q}^2_k} Q$. With these definitions, it suffices to estimate the quantity in (25) at each level $k\geq 0$.

We then have $O_k^{(1)} \lesssim \|f\|^p_{L^p(\mathbb{R}^d)}$ for $k \in \{0,1\}$, and for $k \geq 2$ we may estimate $O_k^{(1)}$ essentially as in part (i) of the proof of Theorem 1.2. One gets

$$\big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p \lesssim 2^{-kp} \sum_{P \in \mathscr{Q}_{d,0}} \sum_{\substack{P' \in \mathscr{A}_P \\ \operatorname{supp}\psi_{P'} \cap (Q \cup Q') \neq \emptyset}} \int_P |f|^p dm_d$$

for all cubes $Q \in \mathscr{Q}_k^1$ and $Q' \sim Q$. Now $\mu_\alpha(Q) \approx 2^{-k(d+n)}$ and summing the previous estimate over Q, each term P' will appear in the resulting triple sum at most a constant times $2^{(d+n)k}$ times, so

$$\sum_{k\geq 0} 2^{ksp} O_k^{(1)} \lesssim \sum_{k \geq 0} 2^{k(s-1)p} \|f\|^p_{L^p(\mathbb{R}^d)} \approx \|f\|^p_{L^p(\mathbb{R}^d)}.$$

Now to estimate the terms $O_k^{(2)}$, suppose that $Q \in \mathscr{Q}_k^2$ and $Q' \in \mathscr{Q}_k^{2,*}$ for some $k \geq 0$ and that $Q' \sim Q$. Denoting by P and P' the unique cubes in $\mathscr{Q}_d^0$ such that $Q \in \mathscr{U}_P$ and $Q' \in \mathscr{U}_{P'}$, the argument used in (27) yields.

$$\big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p \lesssim \frac{2^{-kp}}{\ell(P)^p}\Big(\sum_{\substack{R \in \mathscr{Q}_d^0 \\ \overline{\mathscr{U}_R} \cap \overline{\mathscr{U}_P} \neq \emptyset}} \big|f_P - f_R\big|^p + \sum_{\substack{R \in \mathscr{Q}_d^0 \\ \overline{\mathscr{U}_R} \cap \overline{\mathscr{U}_{P'}} \neq \emptyset}} \big|f_{P'} - f_R\big|^p\Big)$$

Now multiplying this estimate by $\mu_\alpha(Q) \approx 2^{-k(d+n)}\ell(P)^\alpha$ and summing over admissible Q and Q', it can be seen that the terms P and P' will appear in the resulting sum at most a constant times $(2^k\ell(P))^{d+n}$ times. Thus, the estimates for the terms $O_k^{(2)}$ in the proof of Theorem 1.2 apply here as well, with $\alpha+1$ replaced by $\alpha+n$.

Finally, let $Q \in \mathscr{Q}_k^3$ and $Q' \sim Q$. Write $P := P_Q$ for the projection of Q on $\mathbb{R}^d$, and let P' be a neighbor of P (to be specified later). We have

$$\begin{aligned}\mu_\alpha(Q)&\big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^p \\ &\lesssim \int_Q |\mathcal{E}f - f_P|^p d\mu_\alpha + \int_{Q'} |\mathcal{E}f - f_{P'}|^p d\mu_\alpha + \mu_\alpha(Q)\big|f_P - f_{P'}\big|^p. \end{aligned} \tag{45}$$

The first integral can be written as

$$\sum_{n \geq k} \sum_{\substack{R \in \mathscr{Q}_{d,n} \\ R \subset P}} \sum_{\substack{Q^* \in \mathscr{A}_R \\ Q^* \subset Q}} \int_{Q^*} \big|\mathcal{E}f - f_P\big|^p d\mu_\alpha,$$

and this sum can be estimated like the corresponding sum in the proof of Theorem 1.2, again with $1+\alpha$ replaced by $n+\alpha$, so

$$\int_Q |\mathcal{E}f - f_P|^p d\mu_\alpha \lesssim \sum_{\substack{R' \in \mathscr{Q}_d \\ R' \subset P}} \ell(R')^{n+\alpha} m_d(R') \sum_{R'' \sim R'} \big|f_{R'} - f_{R''}\big|^p.$$

The second term in (45) can (with an appropriate choice of P') be estimated either like the first term, or by

$$\ell(P')^{n+\alpha} m_d(P') \sum_{P'' \sim P'} |f_{P'} - f_{P''}|^p.$$

Putting together these estimates and recalling (for the third term in (45)) that $\mu_\alpha(Q) \approx \ell(P)^{n+\alpha} m_d(P)$, we get

$$\mu_\alpha(Q) \sum_{Q' \sim Q} \left|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\right|^p \lesssim \sum_{\substack{R' \in \mathscr{Q}_d \\ R' \subset P_Q^*}} \ell(R')^{n\alpha} m_d(R') \sum_{R'' \sim R'} \left|f_{R'} - f_{R''}\right|^p,$$

where $P_Q^* := P \cup \bigcup_{P' \sim P} P'$. Part (i) of the proof can then be finished as in the proof of Theorem 1.2.

(ii) Now for $f \in \mathcal{B}^s_{p,p}(\mathbb{R}^{d+n}_+, \mu_\alpha)$ and $k \in \mathbb{N}_0$, write

$$T_k := \sum_{Q \in \mathscr{Q}_{d,k}} \Big(\fint_{N(Q)} f \, d\mu_\alpha \Big) \chi_Q,$$

where $N(Q) := Q \times (0, \ell(Q)]^n$, and

$$f^* := \sum_{k \geq 0} \left|T_{k+1} f - T_k f\right| + \left|T_0 f\right|.$$

Repeating the corresponding argument in the proof of Theorem 1.2 (with $\epsilon = sp - \alpha - n$ instead of $\epsilon = sp - \alpha - 1$), we get

$$\begin{aligned} &\|f^*\|^p_{L^p(\mathbb{R}^d)} \\ &\lesssim \sum_{k \geq 0} 2^{ksp} \sum_{Q \in \mathscr{Q}_{d,k}} \mu_\alpha\big(N(Q)\big) \sum_{\substack{Q' \in \mathscr{Q}_{d+n} \\ Q' \sim N(Q)}} \left|f_{N(Q),\mu_\alpha} - f_{Q',\mu_\alpha}\right|^p + \|f\|^p_{L^p(\mathbb{R}^{d+n},\mu_\alpha)} \qquad (46) \\ &\lesssim \|f\|^p_{\mathbb{R}^{d+n}_+,\mu_\alpha}, \end{aligned}$$

so the trace $\mathcal{R}f := \lim_{k \to \infty} T_k f$ exists in $L^p(\mathbb{R}^d)$ and pointwise m_d-almost everywhere, with the correct bound for its L^p-norm. For the energy estimate, recall that $m_d(Q)/\mu_\alpha(N(Q)) \approx \ell(Q)^{-(\alpha+n)}$, and proceed as in the proof of Theorem 1.2 (with $1 + \alpha$ replaced by $n + \alpha$).

(iii) To see that the trace operator $\mathcal{R}$ constructed above can be written in the form required by Definition 7.2, we again refer to Subsection 7.1. □

For the proof of Theorem 7.5, let us introduce the sets

$$\mathscr{N}_P := \big\{ Q \in \mathscr{Q}_{d+n,k} : \overline{Q} \cap P \neq \emptyset \big\}$$

for all $P \in \mathscr{Q}_{d,k}$, $k \in \mathbb{N}_0$, and the quantities

$$\langle f \rangle^p_{s,p,q} := \langle f \rangle^p_{s,p,q,\alpha} := \int_{\mathbb{R}^{d+n}} \Big(\sum_{k \geq 0} 2^{ksq} \sum_{P \in \mathscr{Q}_{d,k}} \sum_{Q \in \mathscr{N}_P} \sum_{Q' \sim Q} \left|f_{Q,\mu_\alpha} - f_{Q',\mu_\alpha}\right|^q \chi_Q(x) \Big)^{p/q} d\mu_\alpha(x)$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+n}, \mu_\alpha)$. We then have $\langle f \rangle_{s,p,q,\alpha} \leq \|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^{d+n}),\mu_\alpha}$ and

$$\langle f \rangle^p_{s,p,p,\alpha} = \sum_{k=0}^{\infty} \sum_{P \in \mathscr{Q}_{d,k}} \sum_{Q \in \mathscr{N}_P} \mu_\alpha(Q) \sum_{Q' \sim Q} \left|f_{Q,\mu_\alpha} - f_{Q',\mu_\alpha}\right|^p$$

for all admissible values of the parameters. We also have

$$\langle f \rangle_{s,p,q,\alpha} \approx \langle f \rangle_{s,p,q',\alpha} \qquad (47)$$

for all admissible values of the parameters, with the implied constants independent of f, which can be proven like Lemma 5.1.

Proof of Theorem 7.5 (sketch). (i) In order to estimate the Triebel-Lizorkin norm of the extension of a function $f \in \mathcal{B}_{p,p}^{s-(\alpha+n)/p}(\mathbb{R}^d)$, recall first that

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}_+^{d+n},\mu_\alpha)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

For the energy estimate, it suffices to consider the quantity

$$\int_{\mathbb{R}^d} \Big(\sum_{k=k_0}^{\infty} 2^{ksq} \sum_{Q\in\mathscr{Q}_{d+n,k}} \sum_{Q'\sim Q} \big|(\mathcal{E}f)_{Q,\mu_\alpha} - (\mathcal{E}f)_{Q',\mu_\alpha}\big|^q \chi_Q(x)\Big)^{p/q} d\mu_\alpha(x) \tag{48}$$

with a suitably chosen $k_0 \in \mathbb{N}$ (independent of f). To this end, recall that the distance $\operatorname{dist}^*(Q,\mathbb{R}^d)$ for $Q \in \mathscr{Q}_{d,n}$ was defined in the proof of Theorem 7.4 above. Now for $k \geq k_0$, write $\mathscr{Q}_k^1$ for the collection of cubes Q in $\mathscr{Q}_{d+n,k}$ such that $\operatorname{dist}^*(Q,\mathbb{R}^d) > 2-2^{-k+2}$, $\mathscr{Q}_k^2$ for the collection of cubes Q in $\mathscr{Q}_{d+n,k}$ with $2^{-k+1} < \operatorname{dist}^*(Q,\mathbb{R}^d) \leq 2^{-k+2}$, $\mathscr{Q}_k^3$ for the collection of cubes Q in $\mathscr{Q}_{d+n,k}$ with $2^{-k} \leq \operatorname{dist}^*(Q,\mathbb{R}^d) \leq 2^{-k+1}$ and $\mathscr{Q}_k^4$ for the collection of dyadic Q in $\mathscr{Q}_{d+n,k}$ such that $\overline{Q} \cap \mathbb{R}^d \neq \emptyset$. Then (48) can be estimated from above by $O^1+O^2+O^3+O^4$, where each O^j is defined as the quantity (48) with $\mathscr{Q}_k^j$ in place of $\mathscr{Q}_{d+n,k}$ in the middle sum. As in the proof of 1.3, it turns out that by (47), the quantities O^4 and O^3 are essentially independent of the parameter q, so the desired norm estimate for them follows from Theorem 7.4. The quantities O^1 and O^2 can be estimated by a suitable modification of the argument in the proof of Theorem 1.3, the details being omitted.

(ii) To obtain the existence and norm inequality for the trace function of $f \in \mathcal{F}_{p,q}^s(\mathbb{R}^{d+n},\mu_\alpha)$, one defines $\mathcal{R} := \lim_{k\to\infty} T_k f$, where $T_k f$ is as in the proof of Theorem 7.4, and the limit exists in $L^p(\mathbb{R}^d)$ with the correct norm bound. From the proof of Theorem 7.4 and (47), one further deduces that

$$\|\mathcal{R}f\|_{\mathcal{B}_{p,p}^{s-(\alpha+n)/p}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^{d+n},\mu_\alpha)} + \langle f\rangle_{s,p,p} \approx \|f\|_{L^p(\mathbb{R}^{d+n},\mu_\alpha)} + \langle f\rangle_{s,p,q} \lesssim \|f\|_{\mathcal{F}_{p,q}^s(\mathbb{R}^{d+n},\mu_\alpha)}.$$

(iii) To see that the trace operator $\mathcal{R}$ constructed above can be written in the form required by Definition 7.2, we again refer to Subsection 7.1. □

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Dyadic Norm Besov-Type Spaces as Trace Spaces on Regular Trees

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Abstract
In this paper, we study function spaces defined via dyadic energies on the boundaries of regular trees. We show that correct choices of dyadic energies result in Besov-type spaces that are trace spaces of (weighted) first order Sobolev spaces.

Keywords Besov-type space · Regular tree · Trace space · Dyadic norm

Mathematics Subject Classification (2010) 46E35 · 30L99

1 Introduction

Over the past two decades, analysis on general metric measure spaces has attracted a lot of attention, e.g., [2, 4, 12, 13, 15–17]. Especially, the case of a regular tree and its Cantor-type boundary has been studied in [3]. Furthermore, Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces on metric measure spaces have been studied in [5, 25, 26] via hyperbolic fillings. A related approach was used in [23], where the trace results of Sobolev spaces and of related fractional smoothness function spaces were recovered by using a dyadic norm and the Whitney extension operator.

Dyadic energy has also been used to study the regularity and modulus of continuity of space-filling curves. One of the motivations for this paper is the approach in [20]. Given a continuous $g : S^1 \to \mathbb{R}^n$, consider the dyadic energy

$$\mathcal{E}(g; p, \lambda) := \sum_{i=1}^{+\infty} i^{\lambda} \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{\widehat{I}_{i,j}}|^p. \tag{1.1}$$

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Here, $\{I_{i,j} : i \in \mathbb{N}, j = 1, \cdots, 2^i\}$ is a dyadic decomposition of S^1 such that for every fixed $i \in \mathbb{N}$, $\{I_{i,j} : j = 1, \cdots, 2^i\}$ is a family of arcs of length $2\pi/2^i$ with $\bigcup_j I_{i,j} = S^1$. The next generation is constructed in such a way that for each $j \in \{1, \cdots, 2^{i+1}\}$, there exists a unique number $k \in \{1, \cdots, 2^i\}$, satisfying $I_{i+1,j} \subset I_{i,k}$. We denote this parent of $I_{i+1,j}$ by $\widehat{I}_{i+1,j}$ and set $\widehat{I}_{1,j} = S^1$ for $j = 1, 2$. By g_A, $A \subset S^1$, we denote the mean value $g_A = \fint_A g\, d\mathcal{H}^1 = \frac{1}{\mathcal{H}^1(A)} \int_A g\, d\mathcal{H}^1$. One could expect to be able to use the energy Eq. 1.1 to characterize the trace spaces of some Sobolev spaces (with suitable weights) on the unit disk. On the contrary, the results in [23] suggest that the trace spaces of Sobolev spaces (with suitable weights) on the unit disk should be characterized by the energy

$$\mathbb{E}(g; p, \lambda) := \sum_{i=1}^{+\infty} i^{\lambda} \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{I_{i,j-1}}|^p, \tag{1.2}$$

where $I_{i,0} = I_{i,2^i}$, and the example $g(x) = \chi_{I_{1,1}}$ shows that $\mathcal{E}(g; p, \lambda)$ is not comparable to $\mathbb{E}(g; p, \lambda)$.

Notice that the energies (1.1) and (1.2) can be viewed as dyadic energies on the boundary of a binary tree (2-regular tree). More precisely, for a 2-regular tree X in Section 2.1 with $\epsilon = \log 2$ in the metric (2.1), the measure ν on the boundary ∂X is the Hausdorff 1-measure by Proposition 2.10. Furthermore, there is a one-to-one map h from the dyadic decomposition of S^1 to the dyadic decomposition of ∂X defined in Section 2.4, which preserves the parent relation, i.e., $h(\widehat{I}) = \widehat{h(I)}$ for all dyadic intervals I of S^1. Since every point in S^1 is the limit of a sequence of dyadic intervals, we can define a map $\tilde{h}$ from S^1 to ∂X by mapping any point $x = \bigcap_{k\in\mathbb{N}} I_k$ in S^1 to the limit of $\{h(I_k)\}_{k\in\mathbb{N}}$ (if the limit is not unique for different choices of sequence $\{I_k\}$ for x, then just pick one of them). It follows from the definition of ∂X that the map $\tilde{h}$ is an injective map. Since the measure ν is the Hausdorff 1-measure and $\partial X \setminus \tilde{h}(S^1)$ is a set of countably many points, it follows from the definition of Hausdorff measure that $\nu(\partial X \setminus \tilde{h}(S^1)) = 0$. Since $\operatorname{diam}(I) \approx \operatorname{diam}(h(I))$ for any dyadic interval I of S^1 and we can use dyadic intervals to cover a given set in the definition of a Hausdorff measure, there is a constant $C \geq 1$ such that

$$\frac{1}{C}\mathcal{H}^1(A) \leq \nu(\tilde{h}(A)) \leq C\mathcal{H}^1(A)$$

for any measurable set $A \subset S^1$. Then one could expect to be able to use an energy similar to Eq. 1.2, the $\dot{\mathbb{B}}^{1/p,\lambda}_p$-energy given by

$$\|g\|^p_{\dot{\mathbb{B}}^{1/p,\lambda}_p} := \sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} \left|g_{h(I_{j,i})} - g_{h(I_{j,i-1})}\right|^p, \tag{1.3}$$

to characterize the trace spaces of suitable Sobolev spaces of the 2-regular tree. This turns out to hold in the sense that any function in $L^p(\partial X)$ with finite $\dot{\mathbb{B}}^{1/p,\lambda}_p$-energy can be extended to a function in a certain Sobolev class.

However, there exists a Sobolev function whose trace function has infinite $\dot{\mathbb{B}}^{1/p,\lambda}_p$-energy. More precisely, let 0 be the root of the tree X and let x_1, x_2 be the two children of 0. We define a function u on X by setting $u(x) = 0$ if the geodesic from 0 to x passes through x_1, $u(x) = 1$ if the geodesic from 0 to x passes through x_2 and define u to be linear on the geodesic $[x_1, x_2] = [0, x_1] \cup [0, x_2]$. Then u is a Sobolev function on X with the trace function $g = \chi_{h(I_{1,1})}$ whose $\dot{\mathbb{B}}^{1/p,\lambda}_p$-energy is not finite for any $\lambda \geq -1$, since the energy (1.2) of the function $\chi_{I_{1,1}}$ is not finite for any $\lambda \geq -1$. But the energy (1.1) of the function

$\chi_{I_{1,1}}$ is finite. Hence, rather than studying the energy (1.3), we shall work with an energy similar to Eq. 1.1. We define the dyadic $\dot{\mathcal{B}}_p^{1/p,\lambda}$ energy by setting

$$\|g\|^p_{\dot{\mathcal{B}}_p^{1/p,\lambda}} := \sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} \left| g_{h(I_{i,j})} - g_{h(\widehat{I}_{i,j})} \right|^p = \sum_{i=1}^{\infty} i^{\lambda} \sum_{I \in \mathscr{Q}_i} \left| g_I - g_{\widehat{I}} \right|^p ,$$

where $\mathscr{Q} = \cup_{j\in\mathbb{N}} \mathscr{Q}_j$ is a dyadic decomposition on the boundary of the 2-regular tree in Section 2.4.

Instead of only considering the above dyadic energy on the boundary of a 2-regular tree, we introduce a general dyadic energy $\dot{\mathcal{B}}_p^{\theta,\lambda}$ in Definition 2.12, defined on the boundary of any regular tree and for any $0 \le \theta < 1$. It is natural to ask whether the Besov-type space $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ in Definition 2.12 defined via the $\dot{\mathcal{B}}_p^{\theta,\lambda}$-energy is a trace space of a suitable Sobolev space defined on the regular tree. We refer to [1, 9, 10, 14, 18, 19, 23, 24, 27–30] for trace results on Euclidean spaces and to [3, 21, 25] for trace results on metric measure spaces.

In [3], the trace spaces of the Newtonian spaces $N^{1,p}(X)$ on regular trees were shown to be Besov spaces defined via double integrals. Our first result is the following generalization of this theorem.

Theorem 1.1 *Let X be a K-ary tree with $K \ge 2$. Fix $\beta > \log K$, $\epsilon > 0$ and $\lambda \in \mathbb{R}$. Suppose that $p \ge 1$ and $p > (\beta - \log K)/\epsilon$. Then the Besov-type space $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ is the trace space of $N^{1,p}(X, \mu_\lambda)$ whenever $\theta = 1 - (\beta - \log K)/\epsilon p$.*

The measure μ_λ above is defined in Eq. 2.2 by

$$d\mu_\lambda(x) = e^{-\beta|x|}(|x| + C)^{\lambda}\, d|x|,$$

and the space $N^{1,p}(X, \mu_\lambda)$ is a Newtonian space defined in Section 2.3. If $\lambda = 0$, then $N^{1,p}(X, \mu_\lambda) = N^{1,p}(X)$ and Theorem 1.1 recovers the trace results from [3] for the Newtonian spaces $N^{1,p}(X)$. Here and throughout this paper, for given Banach spaces $\mathbb{X}(\partial X)$ and $\mathbb{Y}(X)$, we say that the space $\mathbb{X}(\partial X)$ is a trace space of $\mathbb{Y}(X)$ if and only if there is a bounded linear operator $T : \mathbb{Y}(X) \to \mathbb{X}(\partial X)$ and there exists a bounded linear extension operator $E : \mathbb{X}(\partial X) \to \mathbb{Y}(X)$ that acts as a right inverse of T, i.e., $T \circ E = \mathrm{Id}$ on the space $\mathbb{X}(\partial X)$.

We required in Theorem 1.1 that $p > (\beta - \log K)/\epsilon > 0$. The assumption that $\beta - \log K > 0$ is necessary in the sense that we need to make sure that the measure μ_λ on X is doubling; see Section 2.2. The requirement that $p > (\beta - \log K)/\epsilon$ will ensure that $\theta > 0$. So it is natural to consider the case $p = (\beta - \log K)/\epsilon \ge 1$.

Theorem 1.2 *Let X be a K-ary tree with $K \ge 2$. Fix $\beta > \log K$, $\epsilon > 0$ and $\lambda \in \mathbb{R}$. Suppose that $p = (\beta - \log K)/\epsilon \ge 1$ and $\lambda > p - 1$ if $p > 1$ or $\lambda \ge 0$ if $p = 1$. Then there is a bounded linear trace operator $T : N^{1,p}(X, \mu_\lambda) \to L^p(\partial X)$, defined via limits along geodesic rays. Here, $\lambda > p - 1$ is sharp in the sense that for any $p > 1$, $\delta > 0$ and $\lambda = p - 1 - \delta$, there exists a function $u \in N^{1,p}(X, \mu_\lambda)$ so that $Tu(\xi) = \infty$ for every $\xi \in \partial X$.*

Moreover, for any $p = (\beta - \log K)/\epsilon \ge 1$, there exists a bounded nonlinear extension operator $E : L^p(\partial X) \to N^{1,p}(X)$ so that the trace operator $\widehat{T}$ defined via limits of $E(f)$ along geodesic rays for $f \in L^p(\partial X)$ satisfies $\widehat{T} \circ E = Id$ on $L^p(\partial X)$.

A result similar to Theorem 1.2 for the weighted Newtonian space $N^{1,p}(\Omega, \omega\, d\mu)$ with a suitable weight ω has been established in [21] provided that Ω is a bounded domain that admits a p-Poincaré inequality and whose boundary $\partial\Omega$ is endowed with a p-co-dimensional Ahlfors regular measure. In Theorem 1.2, for the case $p = (\beta - \log K)/\epsilon > 1$, we require that $\lambda > p - 1$ to ensure the existence of limits along geodesic rays. In the case $p = (\beta - \log K)/\epsilon = 1$, these limits exist even for $\lambda = 0$, and there is a nonlinear extension operator that acts as a right inverse of the trace operator, similarly to the case of $W^{1,1}$ in Euclidean setting; see [10, 24].

However, except for the case $p = 1$ and $\lambda = 0$, Theorem 1.2 does not even tell whether the trace operator T is surjective or not: $N^{1,p}(X, \mu_\lambda)$ is a strict subset of $N^{1,p}(X)$ when $\lambda > 0$. In the case $p = (\beta - \log K)/\epsilon = 1$ and $\lambda > 0$, the trace operator T is actually not surjective, and we can find a Besov-type space $\mathcal{B}_1^\alpha(\partial X)$ (see Definition 2.14) which is the trace space of the Newtonian space $N^{1,1}(X, \mu_\lambda)$. We stress that $\mathcal{B}_1^\alpha(\partial X)$ and $\mathcal{B}_1^{0,\lambda}(\partial X)$ are different spaces. More precisely, $\mathcal{B}_1^{0,\lambda}(\partial X)$ is a strict subspace of $\mathcal{B}_1^\alpha(\partial X)$, see Proposition 3.8 and Example 3.9.

Theorem 1.3 *Let X be a K-ary tree with $K \geq 2$. Fix $\beta > \log K$, $\epsilon > 0$ and $\lambda > 0$. Suppose that $p = 1 = (\beta - \log K)/\epsilon$. Then the trace space of $N^{1,1}(X, \mu_\lambda)$ is the Besov-type space $\mathcal{B}_1^\alpha(\partial X)$.*

Trace results similar to Theorem 1.3 in the Euclidean setting can be found in [11, 30]. The second part of Theorem 1.2 asserts the existence of a bounded nonlinear extension operator from $L^p(\partial X)$ to $N^{1,p}(X)$ whenever $p = (\beta - \log K)/\epsilon \geq 1$. Nonlinearity is natural here since results due to Peetre [24] (also see [8]) indicate that, for $p = 1$ and $\lambda = 0$, one can not find a bounded linear extension operator that acts as a right inverse of the trace operator in Theorem 1.2. On the other hand, the recent work [22] gives the existence of a bounded linear extension operator E from a certain Besov-type space to BV or to $N^{1,1}$ such that $T \circ E$ is the identity operator on this Besov-type space, under the assumption that the domain satisfies the co-dimension 1 Ahlfors-regularity. The extension operator in [22] is a version of the Whitney extension operator. This motivates us to further analyze the operator E from Theorem 1.1: it is also of Whitney type. The co-dimension 1 Ahlfors-regularity does not hold for our regular tree (X, μ_λ), but we are still able to establish the following result for $N^{1,p}(X, \mu_\lambda)$ with $p \geq 1$ for our fixed extension operator E.

Theorem 1.4 *Let X be a K-ary tree with $K \geq 2$. Fix $\beta > \log K$, $\epsilon > 0$ and $\lambda \in \mathbb{R}$. Suppose that $p = (\beta - \log K)/\epsilon \geq 1$ and $\lambda > p - 1$ if $p > 1$ or $\lambda \geq 0$ if $p = 1$. Then the operator E from Theorem 1.1 is a bounded linear extension operator from $\mathcal{B}_p^{0,\lambda}(\partial X)$ to $N^{1,p}(X, \mu_\lambda)$ and acts as a right inverse of T, i.e., $T \circ E$ is the identity operator on $\mathcal{B}_p^{0,\lambda}(\partial X)$, where T is the trace operator in Theorem 1.2.*

Moreover, the space $\mathcal{B}_p^{0,\lambda}(\partial X)$ is the optimal space for which E is both bounded and linear, i.e., if $\mathbb{X} \subset L^1_{loc}(\partial X)$ is a Banach space so that the extension operator $E : \mathbb{X} \to N^{1,p}(X, \mu_\lambda)$ is bounded and linear and so that $T \circ E$ is the identity operator on $\mathbb{X}$, then $\mathbb{X}$ is a subspace of $\mathcal{B}_p^{0,\lambda}(\partial X)$.

The optimality of the space $\mathcal{B}_p^{0,\lambda}(\partial X)$ is for the explicit extension operator E in Theorem 1.4. The space $\mathcal{B}_p^{0,\lambda}(\partial X)$ may not be the optimal space unless we consider this particular extension operator. For example, for $p = 1$ and $\lambda > 0$, the optimal space is $\mathcal{B}_1^\alpha(\partial X)$ rather than $\mathcal{B}_1^{0,\lambda}$ by Theorem 1.3. This splitting happens since the two extension operators from Theorems 1.3 and 1.4 are very different: the latter one is of Whitney type in the sense that the extension to an edge is based on the average of the boundary function over the dyadic "shadow" of size comparable to that of the edge, while the former one uses the average over a dyadic boundary element for the definition of the extension to several edges of different sizes.

The paper is organized as follows. In Section 2, we give all the preliminaries for the proofs. More precisely, we introduce regular trees in Section 2.1 and we consider the doubling condition on a regular tree X and the Hausdorff dimension of its boundary ∂X. We introduce the Newtonian spaces on X and the Besov-type spaces on ∂X in Sections 2.3 and 2.4, respectively. In Section 3, we give the proofs of all the above mentioned theorems, one by one.

In what follows, the letter C denotes a constant that may change at different occurrences. The notation $A \approx B$ means that there is a constant C such that $1/C \cdot A \le B \le C \cdot A$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a constant C such that $A \le C \cdot B$ ($A \ge C \cdot B$).

2 Preliminaries

2.1 Regular Trees and Their Boundaries

A *graph* G is a pair (V, E), where V is a set of vertices and E is a set of edges. We call a pair of vertices $x, y \in V$ neighbors if x is connected to y by an edge. The degree of a vertex is the number of its neighbors. The graph structure gives rise to a natural connectivity structure. A *tree* is a connected graph without cycles. A graph (or tree) is made into a metric graph by considering each edge as a geodesic of length one.

We call a tree X a *rooted tree* if it has a distinguished vertex called the *root*, which we will denote by 0. The neighbors of a vertex $x \in X$ are of two types: the neighbors that are closer to the root are called *parents* of x and all other neighbors are called *children* of x. Each vertex has a unique parent, except for the root itself that has none.

A *K-ary tree* is a rooted tree such that each vertex has exactly K children. Then all vertices except the root of a K-ary tree have degree $K + 1$, and the root has degree K. In this paper we say that a tree is *regular* if it is a K-ary tree for some $K \ge 1$.

For $x \in X$, let $|x|$ be the distance from the root 0 to x, that is, the length of the geodesic from 0 to x, where the length of every edge is 1 and we consider each edge to be an isometric copy of the unit interval. The geodesic connecting two vertices $x, y \in V$ is denoted by $[x, y]$, and its length is denoted $|x - y|$. If $|x| < |y|$ and x lies on the geodesic connecting 0 to y, we write $x < y$ and call the vertex y a descendant of the vertex x. More generally, we write $x \le y$ if the geodesic from 0 to y passes through x, and in this case $|x-y| = |y|-|x|$.

Let $\epsilon > 0$ be fixed. We introduce a *uniformizing metric* (in the sense of Bonk-Heinonen-Koskela [6], see also [3]) on X by setting

$$d_X(x, y) = \int_{[x,y]} e^{-\epsilon|z|}\, d\,|z|. \tag{2.1}$$

Here $d\,|z|$ is the measure which gives each edge Lebesgue measure 1, as we consider each edge to be an isometric copy of the unit interval and the vertices are the end points of this interval. In this metric, $\operatorname{diam} X = 2/\epsilon$ if X is a K-ary tree with $K \ge 2$.

Next we construct the boundary of the regular K-ary tree by following the arguments in [3, Section 5]. We define the boundary of a tree X, denoted ∂X, by completing X with respect to the metric d_X. An equivalent construction of ∂X is as follows. An element ξ in ∂X is identified with an infinite geodesic in X starting at the root 0. Then we may denote $\xi = 0x_1x_2\cdots$, where x_i is a vertex in X with $|x_i| = i$, and x_{i+1} is a child of x_i. Given two points $\xi, \zeta \in \partial X$, there is an infinite geodesic $[\xi, \zeta]$ connecting ξ and ζ. Then the distance of ξ and ζ is the length (with respect to the metric d_X) of the infinite geodesic $[\xi, \zeta]$. More precisely, if $\xi = 0x_1x_2\cdots$ and $\zeta = 0y_1y_2\cdots$, let k be an integer with $x_k = y_k$ and $x_{k+1} \neq y_{k+1}$. Then by Eq. 2.1

$$d_X(\xi, \zeta) = 2\int_k^{+\infty} e^{-\epsilon t}\, dt = \frac{2}{\epsilon} e^{-\epsilon k}.$$

The restriction of d_X to ∂X is called the *visual metric* on ∂X in Bridson-Haefliger [7].

The metric d_X is thus defined on $\bar{X}$. To avoid confusion, points in X are denoted by Latin letters such as x, y and z, while for points in ∂X we use Greek letters such as ξ, ζ and ω. Moreover, balls in X will be denoted $B(x, r)$, while $B(\xi, r)$ stands for a ball in ∂X.

Throughout the paper we assume that $1 \leq p < +\infty$ and that X is a K-ary tree with $K \geq 2$ and metric d_X defined as in Eq. 2.1.

2.2 Doubling Condition on *X* and Hausdorff Dimension of *∂X*

The first aim of this section is to show that the weighted measure

$$d\mu_\lambda(x) = e^{-\beta|x|}(|x| + C)^\lambda\, d|x| \tag{2.2}$$

is doubling on X, where $\beta > \log K$, $\lambda \in \mathbb{R}$ and $C \geq \max\{2|\lambda|/(\beta - \log K), 2(\log 4)/\epsilon\}$ are fixed from now on. Here the lower bound of the constant C will make the estimates below simpler. If $\lambda = 0$, then

$$d\mu_0(x) = e^{-\beta|x|}\, d|x| = d\mu(x),$$

which coincides with the measure used in [3]. If $\beta \leq \log K$, then $\mu_\lambda(X) = \infty$ for the regular K-ary tree X by Eq. 2.4 below. Hence X would not be doubling as X is bounded.

Next we estimate the measures of balls in X and show that our measure is doubling. Let

$$B(x, r) = \{y \in X : d_X(x, y) < r\}$$

denote an open ball in X with respect to the metric d_X. Also let

$$F(x, r) = \{y \in X : y \geq x \text{ and } d_X(x, y) < r\}$$

denote the downward directed "half ball".

The following algebraic lemma and the relation between a ball and a "half ball" come from [3, Lemma 3.1 and 3.2].

Lemma 2.1 *Let $\sigma > 0$ and $t \in [0, 1]$. Then*

$$\min\{1, \sigma\}t \leq 1 - (1 - t)^\sigma \leq \max\{1, \sigma\}t.$$

Lemma 2.2 *For every $x \in X$ and $r > 0$ we have*

$$F(x, r) \subset B(x, r) \subset F(z, 2r),$$

where $z \leq x$ and

$$|z| = \max\left\{|x| - \frac{1}{\epsilon}\log(1 + \epsilon r e^{\epsilon|x|}), 0\right\}. \tag{2.3}$$

In the above lemma, z is the largest (in the $\leq$ relationship) common ancestor of $B(x, r)$, i.e., we have $z \leq y$ for any $y \in B(x, r)$.

We begin to estimate the measure of the ball $B(x, r)$ and of the half ball $F(z, r)$.

Lemma 2.3 *If* $0 < r \le e^{-\epsilon|z|}/\epsilon$, *then*

$$\mu_\lambda(F(z, r)) \approx e^{(\epsilon-\beta)|z|} r(|z| + C)^\lambda.$$

Proof Let $\rho > 0$ be such that

$$\int_{|z|}^{|z|+\rho} e^{-\epsilon t}\, dt = \frac{1}{\epsilon} e^{-\epsilon|z|}(1 - e^{-\epsilon\rho}) = r.$$

Note that for each $|z| \le t \le |z| + \rho$, the number of points $y \in F(z, r)$ with $|y| = t$ is approximately $K^{t-|z|}$. Hence

$$\mu_\lambda(F(z, r)) \approx \int_{|z|}^{|z|+\rho} K^{t-|z|} e^{-\beta t}(t+C)^\lambda\, dt = K^{-|z|} \int_{|z|}^{|z|+\rho} e^{(\log K-\beta)t}(t+C)^\lambda\, dt. \quad (2.4)$$

Since

$$\left(\frac{1}{\log K - \beta} e^{(\log K-\beta)t}(t + C)^\lambda\right)' = e^{(\log K-\beta)t}(t + C)^\lambda \left(1 + \frac{\lambda}{(t + C)(\log K - \beta)}\right),$$

then for $C \ge 2|\lambda|/(\beta - \log K)$, we have

$$\left|\frac{\lambda}{(t + C)(\log K - \beta)}\right| \le \frac{1}{2} \quad \forall\, t > 0.$$

Hence we obtain that

$$\mu_\lambda(F(z, r)) \approx \frac{K^{-|z|}}{\beta - \log K} e^{(\log K-\beta)|z|}(|z| + C)^\lambda \left(1 - e^{(\log K-\beta)\rho}\left(\frac{|z| + \rho + C}{|z| + C}\right)^\lambda\right). \quad (2.5)$$

It is easy to check that for any $\rho > 0$ and $z \in X$, we have that

$$1 \le \frac{|z| + \rho + C}{|z| + C} \le \frac{\rho + C}{C} \le e^{\rho/C}.$$

Therefore,

$$e^{-\frac{|\lambda|}{C}\rho} \le \left(\frac{|z| + \rho + C}{|z| + C}\right)^\lambda \le e^{\frac{|\lambda|}{C}\rho} \quad \forall\, z \in X, \rho > 0.$$

Since $C \ge 2|\lambda|/(\beta - \log K)$, we obtain that

$$e^{\frac{1}{2}(\log K-\beta)\rho} \le \left(\frac{|z| + \rho + C}{|z| + C}\right)^\lambda \le e^{-\frac{1}{2}(\log K-\beta)\rho} \quad \forall\, z \in X, \rho > 0. \quad (2.6)$$

Then for any $z \in X$ and $\rho > 0$,

$$e^{(\log K-\beta)\rho}\left(\frac{|z| + \rho + C}{|z| + C}\right)^\lambda \approx e^{c(\log K-\beta)\rho}, \text{ for some } \frac{1}{2} \le c \le \frac{3}{2}.$$

Hence we obtain that

$$\begin{aligned}\mu_\lambda(F(z, r)) &\approx \tfrac{K^{-|z|}}{\beta-\log K} e^{(\log K-\beta)|z|}(|z| + C)^\lambda \left(1 - e^{c(\log K-\beta)\rho}\right)\\ &= \tfrac{e^{-\beta|z|}}{\beta-\log K}(|z| + C)^\lambda \left(1 - (1 - \epsilon r e^{\epsilon|z|})^{c(\beta-\log K)/\epsilon}\right)\end{aligned}$$

for some $c \in [1/2, 3/2]$. Lemma 2.1 with $t = \epsilon r e^{\epsilon|z|}$ implies that

$$\mu_\lambda(F(z, r)) \approx e^{-\beta|z|}(|z| + C)^\lambda \epsilon r e^{\epsilon|z|} \approx e^{(\epsilon-\beta)|z|} r(|z| + C)^\lambda.$$

□

Corollary 2.4 *If* $0 < r \le e^{-\epsilon|x|}/\epsilon$, *then*

$$\mu_\lambda(B(x,r)) \approx e^{(\epsilon-\beta)|x|} r(|x|+C)^\lambda \approx e^{(\epsilon-\beta)|x|} r(|z|+C)^\lambda.$$

Proof For any $x \in X$ and $0 < r \le e^{-\epsilon|x|}/\epsilon$, let z be as in Lemma 2.2. If $z = 0$, then $B(x,r) \subset F(0, r+\rho)$, where

$$\rho = d_X(0,x) = \frac{1}{\epsilon}(1-e^{-\epsilon|x|}) \le r$$

and $r + \rho \le 1/\epsilon = e^{-\epsilon|z|}/\epsilon$. For $z > 0$ we have

$$2r \le \frac{e^{-\epsilon|x|}(1+\epsilon r e^{\epsilon|x|})}{\epsilon} = \frac{e^{-\epsilon|z|}}{\epsilon}.$$

Moreover, in both cases, since $r < e^{-\epsilon|x|}/\epsilon$, by Lemma 2.2, we have

$$|z| \le |x| \le |z| + \frac{1}{\epsilon}\log(1+\epsilon r e^{\epsilon|x|}) \le |z| + \frac{1}{\epsilon}\log 2,$$

which implies

$$\left(\frac{|x|+C}{|z|+C}\right)^\lambda \approx 1. \tag{2.7}$$

Combing Eq. 2.7 with the fact that in both cases $1 \le e^{|x|-|z|} \le (1+\epsilon r e^{\epsilon|x|})^{1/\epsilon} \approx 1$, the result follows by applying Lemma 2.3 to $F(x,r)$ and $F(z,2r)$ (or $F(0, r+\rho)$ for $z = 0$). □

Lemma 2.5 *Let* $z \in X$ *and* $x \in \overline{X}$ *with* $z \le x$. *Then*

$$\mu_\lambda([z,x]) \approx \mu_\lambda(F(z, d_X(z,x))).$$

where $[z,x]$ *denotes the geodesic in the tree* X *joining* x *and* z.

Proof Since $[z,x]$ is a subset of $F(z, d_X(z,x))$ by definition, we have $\mu_\lambda([z,x]) \le \mu_\lambda(F(z,d_X(z,x)))$. Hence it remains to show that

$$\mu_\lambda([z,x]) \gtrsim \mu_\lambda(F(z,d_X(z,x))).$$

For any $z \in X$ and $x \in \overline{X}$ with $z \le x$, we have that

$$\mu_\lambda([z,x]) = \int_{|z|}^{|x|} e^{-\beta t}(t+C)^\lambda\, dt,$$

where $|x| = \infty$ if $x \in \partial X$. Then by using an argument similar to the estimate in Lemma 2.3, since $C \ge 2|\lambda|/(\beta - \log K) \ge 2|\lambda|/\beta$, we have that

$$\left|\frac{\lambda}{(t+C)\beta}\right| \le \frac{1}{2} \quad \forall\, t \ge 0,$$

which implies that for any $t \ge 0$,

$$\left(-\frac{1}{\beta}e^{-\beta t}(t+C)^\lambda\right)' = e^{-\beta t}(t+C)^\lambda\left(1 - \frac{\lambda}{\beta(t+C)}\right) \approx e^{-\beta t}(t+C)^\lambda.$$

Hence we obtain that

$$\int_{|z|}^{|x|} e^{-\beta t}(t+C)^\lambda\, dt \approx \frac{e^{-\beta|z|}}{\beta}(|z|+C)^\lambda \left(1 - e^{-\beta(|x|-|z|)}\left(\frac{|x|+C}{|z|+C}\right)^\lambda\right). \tag{2.8}$$

Comparing the estimate (2.8) with the estimate (2.5), since $\rho = |x| - |z|$, $e^{\rho \log K} \geq 1$ and $K^{-|z|}e^{(\log K - \beta)|z|} = e^{-\beta|z|}$, we have that

$$\int_{|z|}^{|x|} e^{-\beta t}(t+C)^{\lambda}\, dt \gtrsim \mu_\lambda(F(z,r)) \text{ with } r = d_X(z,x),$$

which induces that

$$\mu_\lambda([z,x]) \gtrsim \mu_\lambda(F(z,r)) = \mu_\lambda(F(z, d_X(z,x))).$$

□

Corollary 2.6 *Let $x \in X$ and z be as in Lemma 2.2. Then if*

$$\frac{e^{-\epsilon|x|}}{\epsilon} \leq r \leq \frac{1}{\epsilon}(1 - e^{-\epsilon|x|}), \tag{2.9}$$

we obtain

$$\mu_\lambda(B(x,r)) \approx e^{-\beta|z|}(|z|+C)^{\lambda} \approx r^{\beta/\epsilon}(|z|+C)^{\lambda}.$$

Proof Since $r \geq e^{-\epsilon|x|}/\epsilon$, by Lemma 2.2, we have

$$B(x,r) \subset F(z,\infty) = F\left(z, \frac{e^{-\epsilon|z|}}{\epsilon}\right).$$

Then Lemma 2.3 implies

$$\mu_\lambda(B(x,r)) \leq \mu_\lambda(F(z,\infty)) \lesssim e^{(\epsilon-\beta)|z|}e^{-\epsilon|z|}(|z|+C)^{\lambda} \approx e^{-\beta|z|}(|z|+C)^{\lambda} \tag{2.10}$$

Towards the another direction, by Eq. 2.3 and Lemma 2.5, we have that

$$\mu_\lambda(B(x,r)) \geq \mu_\lambda([x,z]) \gtrsim \mu(F(z,r)) = e^{(\epsilon-\beta)|z|}r(|z|+C)^{\lambda} = e^{-\beta|z|}(|z|+C)^{\lambda}e^{\epsilon|z|}r.$$

Moreover, we have

$$e^{\epsilon|z|}r = e^{\epsilon|x|}r \cdot e^{-\epsilon(|x|-|z|)} = e^{\epsilon|x|}r(1+\epsilon r e^{\epsilon|x|})^{-1} = \frac{t}{\epsilon(1+t)} \geq \frac{1}{2\epsilon},$$

where $t = \epsilon r e^{\epsilon|x|}$. Here in the last inequality we used the fact that $\epsilon r e^{\epsilon|x|} \geq 1$. Hence we obtain that

$$\mu_\lambda(B(x,r)) \gtrsim e^{-\beta|z|}(|z|+C)^{\lambda}.$$

Combing the above inequality with Eq. 2.10, we finish the proof of

$$\mu_\lambda(B(x,r)) \approx e^{-\beta|z|}(|z|+C)^{\lambda}.$$

Since $\epsilon r e^{\epsilon|x|} \geq 1$, we know that

$$\epsilon r e^{\epsilon|x|} \leq 1 + \epsilon r e^{\epsilon|x|} \leq 2\epsilon r e^{\epsilon|x|}.$$

It then follows from Eq. 2.3 that

$$e^{-\beta|z|} = e^{-\beta|x|}(1+\epsilon r e^{\epsilon|x|})^{\beta/\epsilon} \approx r^{\beta/\epsilon}.$$

Hence we obtain that

$$e^{-\beta|z|}(|z|+C)^{\lambda} \approx r^{\beta/\epsilon}(|z|+C)^{\lambda},$$

which finishes the proof. □

Lemma 2.7 *Let $x \in X$ and $(1 - e^{-\epsilon|x|})/\epsilon \leq r \leq 2\,\mathrm{diam}X$. Then*

$$\mu_\lambda(B(x,r)) \approx r.$$

In particular, if $x = 0$, then this estimate holds for all $r \geq 0$.

Proof We have $0 \in \overline{B(x,r)}$ by assumption, and hence

$$B(x,r) \subset F(0,2r).$$

From Lemma 2.3, we have that

$$\mu_\lambda(B(x,r)) \le \mu_\lambda(F(0,2r)) \lesssim r.$$

As for the lower bound, if $r < 1/\epsilon$, since $0 \in \overline{B(x,r)}$, letting

$$\rho = -\frac{\log(1-\epsilon r)}{\epsilon}$$

and $x \le x'$ with $|x'| = \rho$, then the estimate (2.5) and Lemma 2.3 imply

$$\mu_\lambda(B(x,r)) \ge \mu_\lambda([0,x']) \gtrsim \mu_\lambda(F(0,r)) \approx r.$$

If $1/\epsilon \le r \le 2\,\mathrm{diam}X = 4/\epsilon$, then by Lemma 2.5, we have that

$$\mu_\lambda(B(x,r)) \ge \mu_\lambda(F(0,1/\epsilon)) \approx \frac{1}{\epsilon} \approx r.$$

□

Proposition 2.8 *Let $x \in X$, $0 < r \le 2\,\mathrm{diam}X$, $R_0 = e^{-\epsilon|x|}/\epsilon$ and z be as in Lemma 2.2. If $|x| \le (\log 2)/\epsilon$, then*

$$\mu_\lambda(B(x,r)) \approx r.$$

If $|x| \ge (\log 2)/\epsilon$, then

$$\mu_\lambda(B(x,r)) \approx \begin{cases} e^{(\epsilon-\beta)|x|}(|x|+C)^\lambda, & r \le R_0; \\ r^{\beta/\epsilon}(|z|+C)^\lambda, & r \ge R_0. \end{cases}$$

Proof If $|x| \le (\log 2)/\epsilon$, then $e^{(\epsilon-\beta)|x|} \approx 1$, $(|x|+C)^\lambda \approx 1$ and the result follows from directly from Corollary 2.4 and Lemma 2.7.

If $|x| \ge (\log 2)/\epsilon$ and $r \le (1-e^{-\epsilon|x|})/\epsilon$, then the estimate follows directly from Corollaries 2.4 and 2.6. For $r \ge (1-e^{-\epsilon|x|})/\epsilon \ge 1/2\epsilon$, since $|z| = 0$, we have by Lemma 2.7 that

$$\mu_\lambda(B(x,r)) \approx r \approx 1 \approx r^{\beta/\epsilon}(|z|+C)^\lambda.$$

□

Corollary 2.9 *The measure μ_λ is doubling, i.e., $\mu_\lambda(B(x,2r)) \lesssim \mu_\lambda(B(x,r))$.*

Proof In the case $|x| \le (\log 2)/\epsilon$ and the case $|x| \ge (\log 2)/\epsilon$ with $2r \le R_0$, the result follows directly from Proposition 2.8.

In the case $|x| \ge (\log 2)/\epsilon$ with $2r \ge R_0$, if $r \ge R_0$, then

$$r^{\beta/\epsilon} \approx (2r)^{\beta/\epsilon};$$

if $r \le R_0$, then

$$\frac{e^{(\epsilon-\beta)|x|}r}{(2r)^{\beta/\epsilon}} \approx \left(\frac{R_0}{r}\right)^{\beta/\epsilon-1} \approx 1.$$

Let z_r and z_{2r} be defined as in Lemma 2.2 with respect to r and $2r$. From Corollary 2.4 and the above estimates, the doubling condition of μ_λ follows once we prove that

$$\frac{|z_r| + C}{|z_{2r}| + C} \approx 1. \tag{2.11}$$

If $r \geq (1 - e^{-\epsilon|x|})/\epsilon$, then $|z_r| = |z_{2r}| = 0$ give Eq. 2.11. If $2r \geq (1 - e^{-\epsilon|x|})/\epsilon \geq r$, then $r \geq (1 - e^{-\epsilon|x|})/2\epsilon$ implies that

$$\begin{aligned} |z_r| + C &= |x| - \frac{1}{\epsilon}\log(1 + \epsilon r e^{\epsilon|x|}) + C \leq |x| - \frac{1}{\epsilon}\log\left(\frac{1}{2}(1 + e^{\epsilon|x|})\right) + C \\ &= |x| + C + \frac{\log 2}{\epsilon} - \frac{1}{\epsilon}\log(1 + e^{\epsilon|x|}) \leq C + \frac{\log 2}{\epsilon} \approx C = |z_{2r}| + C, \end{aligned}$$

which gives Eq. 2.11. If $2r \leq (1 - e^{-\epsilon|x|})/\epsilon$, for $C \geq 2(\log 2)/\epsilon$, we obtain that

$$\begin{aligned} 2(|z_{2r}| + C) - (|z_r| + C) &= |x| + C + \frac{1}{\epsilon}\log(1 + \epsilon r e^{\epsilon|x|}) - \frac{2}{\epsilon}\log(1 + 2\epsilon r e^{\epsilon|x|}) \\ &\geq |x| + C + \frac{1}{\epsilon}\log(1 + \epsilon r e^{\epsilon|x|}) - \frac{2}{\epsilon}\log(2(1 + \epsilon r e^{\epsilon|x|})) \\ &= |x| + C - \frac{2\log 2}{\epsilon} - \frac{1}{\epsilon}\log(1 + \epsilon r e^{\epsilon|x|}) \\ &= |z_r| + C - \frac{2\log 2}{\epsilon} \geq 0, \end{aligned}$$

which gives that $|z_r| + C \leq 2(|z_{2r}| + C)$. Combining with the fact that $|z_{2r}| \leq |z_r|$, Eq. 2.11 is obtained. Therefore we finish the proof of this corollary. □

The following result is given by [3, Lemma 5.2].

Proposition 2.10 *The boundary ∂X is an Ahlfors Q-regular space with Hausdorff dimension*

$$Q = \frac{\log K}{\epsilon}.$$

Hence we have an Ahlfors Q-regular measure ν on ∂X with

$$\nu(B(\xi, r)) \approx r^Q = r^{\log K/\epsilon},$$

for any $\xi \in \partial X$ and $0 < r \leq \operatorname{diam}\partial X$.

2.3 Newtonian Spaces on *X*

Let $u \in L^1_{\text{loc}}(X, \mu_\lambda)$. We say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if

$$|u(z) - u(y)| \leq \int_\gamma g\, ds_X \tag{2.12}$$

whenever $z, y \in X$ and γ is the geodesic from z to y, where ds_X denotes the arc length measure with respect to the metric d_X. In the setting of a tree any rectifiable curve with end points z and y contains the geodesic connecting z and y, and therefore the upper gradient defined above is equivalent to the definition which requires that inequality (2.12) holds for all rectifiable curves with end points z and y.

The notion of upper gradients is due to Heinonen and Koskela [16]; we refer interested readers to [12, 17] for a more detailed discussion on upper gradients.

The *Newtonian space* $N^{1,p}(X, \mu_\lambda)$, $1 \le p < \infty$, is defined as the collection of all the functions for which

$$\|u\|_{N^{1,p}(X,\mu_\lambda)} := \left(\int_X |u|^p \, d\mu_\lambda + \inf_g \int_X g^p \, d\mu_\lambda \right)^{1/p} < \infty,$$

where the infimum is taken over all upper gradients of u.

Throughout the paper, we use $N^{1,p}(X)$ to denote $N^{1,p}(X, \mu_\lambda)$ if $\lambda = 0$.

2.4 Besov-Type Spaces on ∂X via Dyadic Norms

We first recall the Besov space $B^\theta_{p,p}(\partial X)$ defined in [3].

Definition 2.11 For $0 < \theta < 1$ and $p \ge 1$, The Besov space $B^\theta_{p,p}(\partial X)$ consists of all functions $f \in L^p(\partial X)$ for which the seminorm $\|f\|_{\dot{B}^\theta_p(\partial X)}$ defined as

$$\|f\|^p_{\dot{B}^\theta_p(\partial X)} := \int_{\partial X} \int_{\partial X} \frac{|f(\zeta)| - f(\xi)|^p}{d_X(\zeta,\xi)^{\theta p} \nu(B(\zeta, d_X(\zeta,\xi)))} d\nu(\xi)\, d\nu(\zeta)$$

is finite. The corresponding norm for $B^\theta_{p,p}(\partial X)$ is

$$\|f\|_{B^\theta_{p,p}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{\dot{B}^\theta_p(\partial X)}.$$

Next, we give a dyadic decomposition on the boundary ∂X of the K-ary tree X: Let $V_n = \{x^n_j : j = 1, 2, \cdots, K^n\}$ be the set of all n-level vertices of the tree X for any $n \in \mathbb{N}$, where a vertex x is *n-level* if $|x| = n$. Then we have that

$$V = \bigcup_{n\in\mathbb{N}} V_n$$

is the set containing all the vertices of the tree X. For any vertex $x \in V$, denote by I_x the set

$$\{\xi \in \partial X : \text{the geodesic } [0, \xi) \text{ passes through } x\}.$$

We denote by $\mathcal{Q}$ the set $\{I_x : x \in V\}$ and $\mathcal{Q}_n$ the set $\{I_x : x \in V_n\}$ for any $n \in \mathbb{N}$. Then $\mathcal{Q}_0 = \{\partial X\}$ and we have

$$\mathcal{Q} = \bigcup_{n\in\mathbb{N}} \mathcal{Q}_n.$$

Then the set $\mathcal{Q}$ is a dyadic decomposition of ∂X. Moreover, for any $n \in \mathbb{N}$ and $I \in \mathcal{Q}_n$, there is a unique element $\widehat{I}$ in $\mathcal{Q}_{n-1}$ such that I is a subset of it. It is easy to see that if $I = I_x$ for some $x \in V_n$, then $\widehat{I} = I_y$ with y the unique parent of x in the tree X. Hence the structure of the tree X gives a corresponding structure of the dyadic decomposition of ∂X which we defined above.

Since we want to characterize the trace spaces of the Newtonian spaces with respect to our measure μ_λ, we introduce the following Besov-type spaces $\mathcal{B}^{\theta,\lambda}_p(\partial X)$.

Definition 2.12 For $0 \le \theta < 1$ and $p \ge 1$, the Besov-type space $\mathcal{B}^{\theta,\lambda}_p(\partial X)$ consists of all functions $f \in L^p(\partial X)$ for which the dyadic $\dot{\mathcal{B}}^{\theta,\lambda}_p$-energy of f defined as

$$\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} := \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^\lambda \sum_{I\in\mathcal{Q}_n} \nu(I) \left| f_I - f_{\widehat{I}} \right|^p$$

is finite. The norm on $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ is

$$\|f\|_{\mathcal{B}_p^{\theta,\lambda}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)}.$$

Here and throughout this paper, the measure ν on the boundary ∂X is the Ahlfors regular measure in Proposition 2.10 and f_I is the mean value $⨍_I f\,d\nu = \frac{1}{\nu(I)}\int_I f\,d\nu$.

The following proposition states that the Besov space $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ coincides with the Besov space $B_{p,p}^{\theta}(\partial X)$ whenever $0<\theta<1$ and $\lambda=0$. The proof of this proposition follows by using [3, Lemma 5.4] and a modification of the proof of [23, Proposition A.1]. We omit the details.

Proposition 2.13 *Let* $0<\theta<1$ *and* $p\geq 1$. *For any* $f\in L^1_{loc}(\partial X)$, *we have*

$$\|f\|_{B_{p,p}^{\theta}(\partial X)} \approx \|f\|_{\mathcal{B}_p^{\theta,0}(\partial X)}.$$

For $\lambda>0$, we next define special Besov-type spaces with $\theta=0$ and $p=1$. Before the definition, we first fix a sequence $\{\alpha(n): n\in\mathbb{N}\}$ such that there exist constants $c_1\geq c_0>1$ satisfying

$$c_0 \leq \frac{\alpha(n+1)}{\alpha(n)} \leq c_1, \quad \forall\, n\in\mathbb{N}. \tag{2.13}$$

A simple example of such a sequence is obtained by letting $\alpha(n)=2^n$.

Definition 2.14 For $\lambda>0$, the Besov-type space $\mathcal{B}_1^{\alpha}(\partial X)$ consists of all functions $f\in L^1(\partial X)$ for which the $\dot{\mathcal{B}}_1^{\alpha}$-dyadic energy of f defined as

$$\|f\|_{\dot{\mathcal{B}}_1^{\alpha}(\partial X)} = \sum_{n=1}^{\infty}\alpha(n)^{\lambda}\sum_{I\in\mathcal{Q}_{\alpha(n)}}\nu(I)|f_I - f_{\widetilde{I}}|$$

is finite. Here for any $I=I_x\in\mathcal{Q}_{\alpha(n)}$ with $x\in V_{\alpha(n)}$ and $n\geq 1$, we denote $\widetilde{I}=I_y$ where $y\in V_{\alpha(n-1)}$ is the ancestor of x in X. The norm on $\mathcal{B}_1^{\alpha}(\partial X)$ is

$$\|f\|_{\mathcal{B}_1^{\alpha}(\partial X)} := \|f\|_{L^1(\partial X)} + \|f\|_{\dot{\mathcal{B}}_1^{\alpha}(\partial X)}.$$

Remark 2.15 Actually, the choice of the sequence $\{\alpha(n)\}_{n\in\mathbb{N}}$ will not affect the definition of $\mathcal{B}_1^{\alpha}(\partial X)$: by Theorem 1.3 we obtain that any two choices of the sequences $\{\alpha(n)\}_{n\in\mathbb{N}}$ lead to comparable norms, for more details see Corollary 3.7.

It is easy to check that $\mathcal{B}_1^{\alpha}(\partial X)=\mathcal{B}_1^{0,\lambda}(\partial X)$ if we let $\alpha(n)=n$. But the sequence $\{\alpha(n)\}$ with $\alpha(n)=n$ does not satisfy Eq. 2.13. Actually, we show in Proposition 3.8 and Example 3.9 that $\mathcal{B}_1^{0,\lambda}(\partial X)$ is a strict subset of $\mathcal{B}_1^{\alpha}(\partial X)$ whenever Eq. 2.13 holds.

3 Proofs

3.1 Proof of Theorem 1.1

Proof **Trace Part:** Let $f\in N^{1,p}(X,\mu_\lambda)$. We first define the trace operator as

$$\mathrm{Tr}\, f(\xi) := \tilde{f}(\xi) = \lim_{[0,\xi)\ni x\to\xi} f(x), \quad \xi\in\partial X, \tag{3.1}$$

where the limit is taken along the geodesic ray $[0, \xi)$. Then our task is to show that the above limit exists for ν-a.e. $\xi \in \partial X$ and that the trace $\mathrm{Tr}\, f$ satisfies the norm estimates.

Let $\xi \in \partial X$ be arbitrary and let $x_j = x_j(\xi)$ be the ancestor of ξ with $|x_j| = j$. To show that the limit in Eq. 3.1 exists for ν-a.e. $\xi \in \partial X$, it suffices to show that the function

$$\tilde{f}^*(\xi) = |f(0)| + \int_{[0,\xi)} g_f \, ds \tag{3.2}$$

is in $L^p(\partial X)$, where $[0, \xi)$ is the geodesic ray from 0 to ξ and g_f is an upper gradient of f. To be more precise, if $\tilde{f}^* \in L^p(\partial X)$, we have $|\tilde{f}^*| < \infty$ for ν-a.e. $\xi \in \partial X$, and hence the limit in Eq. 3.1 exists for ν-a.e. $\xi \in \partial X$.

Set $r_j = 2e^{-j\epsilon}/\epsilon$. Then on the edge $[x_j, x_{j+1}]$ we have the relations

$$ds \approx e^{(\beta-\epsilon)j} j^{-\lambda} \, d\mu_\lambda \approx r_j^{1-\beta/\epsilon} j^{-\lambda} \, d\mu \quad \text{and} \quad \mu_\lambda([x_j, x_{j+1}]) \approx r_j^{\beta/\epsilon} j^\lambda, \tag{3.3}$$

where the comparison constants depend on ϵ, β. Then we obtain the estimate

$$\begin{aligned} \tilde{f}^*(\xi) &= |f(0)| + \int_{[0,\xi)} g_f \, ds = |f(0)| + \sum_{j=0}^{+\infty} \int_{[x_j,x_{j+1}]} g_f \, ds \\ &\lesssim |f(0)| + \sum_{j=0}^{+\infty} r_j^{1-\beta/\epsilon} j^{-\lambda} \int_{[x_j,x_{j+1}]} g_f \, d\mu_\lambda \approx |f(0)| + \sum_{j=0}^{+\infty} r_j ⨍_{[x_j,x_{j+1}]} g_f \, d\mu_\lambda. \end{aligned} \tag{3.4}$$

Since $\theta = 1 - (\beta - \log K)/(p\epsilon) > 0$, we may choose $0 < \kappa < \theta$. Then for $p > 1$, by the Hölder inequality and Eq. 3.3, we have that

$$\begin{aligned} |\tilde{f}^*(\xi)|^p &\lesssim |f(0)|^p + \sum_{j=0}^{+\infty} r_j^{p(1-\kappa)} ⨍_{[x_j,x_{j+1}]} g_f{}^p \, d\mu_\lambda \\ &\approx |f(0)|^p + \sum_{j=0}^{+\infty} r_j^{p(1-\kappa)-\beta/\epsilon} j^{-\lambda} \int_{[x_j,x_{j+1}]} g_f{}^p \, d\mu_\lambda. \end{aligned}$$

For $p = 1$, the above estimates are also true without using the Hölder inequality. It follows that for $p \geq 1$,

$$|\tilde{f}^*(\xi)|^p \lesssim |f(0)|^p + \sum_{j=0}^{+\infty} r_j^{p(1-\kappa)-\beta/\epsilon} j^{-\lambda} \int_{[x_j,x_{j+1}]} g_f{}^p \, d\mu_\lambda.$$

Integrating over all $\xi \in \partial X$, since $\nu(\partial X) \approx 1$, we obtain by means of Fubini's theorem that

$$\begin{aligned} \int_{\partial X} |\tilde{f}^*(\xi)|^p \, d\nu &\lesssim |f(0)|^p + \int_{\partial X} \sum_{j=0}^{+\infty} r_j^{p(1-\kappa)-\beta/\epsilon} j^{-\lambda} \int_{[x_j(\xi),x_{j+1}(\xi)]} g_f{}^p \, d\mu_\lambda \, d\nu(\xi) \\ &= |f(0)|^p + \int_X g_f(x)^p \int_{\partial X} \sum_{j=0}^{+\infty} r_j^{p(1-\kappa)-\beta/\epsilon} j^{-\lambda} \chi_{[x_j(\xi),x_{j+1}(\xi)]}(x) \, d\nu(\xi) \, d\mu_\lambda(x). \end{aligned}$$

Notice that $\chi_{[x_j(\xi),x_{j+1}(\xi)]}(x)$ is nonzero only if $j \leq |x| \leq j+1$ and $x < \xi$. Thus the last estimate can be rewritten as

$$\int_{\partial X} |\tilde{f}^*(\xi)|^p \, d\nu \lesssim |f(0)|^p + \int_X g_f(x)^p r_{j(x)}^{p(1-\kappa)-\beta/\epsilon} j(x)^{-\lambda} \nu(E(x)) \, d\mu_\lambda(x),$$

where $E(x) = \{\xi \in \partial X : x < \xi\}$ and $j(x)$ is the largest integer such that $j(x) \leq |x|$.

It follows from [3, Lemma 5.1] that $E(x) = B(\xi, r)$ for any $\xi \in E(x)$ and $r \approx e^{-\epsilon j(x)}$. Hence we obtain from Proposition 2.10 that $\nu(E(x)) \approx r_{j(x)}^{Q}$. Since $p(1-\kappa) > \beta/\epsilon - \log K/\epsilon = \beta/\epsilon - Q$, then for any $j(x) \in \mathbb{N}$, we have that

$$r_{j(x)}^{p(1-\kappa)-\beta/\epsilon+Q} j(x)^{-\lambda} \lesssim 1,$$

which induces the estimate

$$\int_{\partial X} |\tilde{f}^*(\xi)|^p \, d\nu \lesssim |f(0)|^p + \int_X g_f(x)^p \, d\mu_\lambda(x).$$

Hence we obtain that $\tilde{f}^*$ is in $L^p(\partial X)$, which gives the existence of the limit in Eq. 3.1 for ν-a.e. $\xi \in \partial X$. In particular, since $|\tilde{f}| \le \tilde{f}^*$, we have the estimate

$$\int_{\partial X} |\tilde{f}|^p \, d\nu \lesssim \int_X |f|^p \, d\mu_\lambda + \int_X {g_f}^p \, d\mu_\lambda,$$

and hence the norm estimate

$$\|\tilde{f}\|_{L^p(\partial X)} \lesssim \left(\int_X |f|^p \, d\mu_\lambda + \int_X {g_f}^p \, d\mu_\lambda \right)^{1/p} = \|f\|_{N^{1,p}(X,\mu_\lambda)}. \tag{3.5}$$

To estimate the dyadic energy $\|\tilde{f}\|^p_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)}$, for any $I \in \mathcal{Q}_n$, $\xi \in I$ and $\zeta \in \widehat{I}$, we have that

$$|\tilde{f}(\xi) - \tilde{f}(\zeta)| \le \sum_{j=n-1}^{+\infty} |f(x_j) - f(x_{j+1})| + \sum_{j=n-1}^{+\infty} |f(y_j) - f(y_{j+1})|,$$

where $x_j = x_j(\xi)$ and $y_j = y_j(\zeta)$ are the ancestors of ξ and ζ with $|x_j| = |y_j| = j$, respectively. In the above inequality, we used the fact that $x_{n-1}(\xi) = y_{n-1}(\eta)$. By using Eq. 3.3 and an argument similar to Eq. 3.4, we obtain that

$$|\tilde{f}(\xi) - \tilde{f}(\zeta)| \lesssim \sum_{j=n-1}^{+\infty} r_j \fint_{[x_j(\xi), x_{j+1}(\xi)]} g_f \, d\mu_\lambda + \sum_{j=n-1}^{+\infty} r_j \fint_{[y_j(\zeta), y_{j+1}(\zeta)]} g_f \, d\mu_\lambda.$$

Choose $0 < \kappa < \theta$ and insert $r_j^{\kappa} r_j^{-\kappa}$ into the above sum. If $p > 1$, then the Hölder inequality and Eq. 3.3 imply that

$$\begin{aligned} |\tilde{f}(\xi) - \tilde{f}(\zeta)|^p &\lesssim r_{n-1}^{\kappa p} \sum_{j=n-1}^{+\infty} r_j^{p(1-\kappa)} \fint_{[x_j(\xi), x_{j+1}(\xi)]} {g_f}^p \, d\mu_\lambda + r_{n-1}^{\kappa p} \sum_{j=n-1}^{+\infty} r_j^{p(1-\kappa)} \fint_{[y_j(\zeta), y_{j+1}(\zeta)]} {g_f}^p \, d\mu_\lambda \\ &\approx r_{n-1}^{\kappa p} \sum_{j=n-1}^{+\infty} r_j^{p(1-\kappa)-\beta/\epsilon} j^{-\lambda} \left(\int_{[x_j(\xi), x_{j+1}(\xi)]} {g_f}^p \, d\mu_\lambda + \int_{[y_j(\zeta), y_{j+1}(\zeta)]} {g_f}^p \, d\mu_\lambda \right). \end{aligned}$$

For $p = 1$ the estimates above is also true without using the Hölder inequality. It follows from Fubini's theorem and from $\nu(I) \approx \nu(\widehat{I})$ that

$$\begin{aligned} \sum_{I \in \mathcal{Q}_n} \nu(I) |\tilde{f}_I - \tilde{f}_{\widehat{I}}|^p &\le \sum_{I \in \mathcal{Q}_n} \nu(I) \fint_I \fint_{\widehat{I}} |\tilde{f}(\xi) - \tilde{f}(\zeta)|^p \, d\nu(\xi) \, d\nu(\zeta) \\ &\lesssim \int_{\partial X} r_{n-1}^{\kappa p} \sum_{j=n-1}^{+\infty} r_j^{p(1-\kappa)-\beta/\epsilon} j^{-\lambda} \int_{[x_j(\xi), x_{j+1}(\xi)]} {g_f}^p \, d\mu_\lambda \, d\nu(\xi) \\ &= r_{n-1}^{\kappa p} \int_{X \cap \{|x| \ge n-1\}} {g_f}^p \int_{\partial X} \sum_{j=n-1}^{+\infty} r_j^{p(1-\kappa)-\beta/\epsilon} j^{-\lambda} \chi_{[x_j(\xi), x_{j+1}(\xi)]}(x) \, d\nu(\xi) \, d\mu_\lambda(x). \end{aligned}$$

Using the notation $E(x)$ and $j(x)$ defined before, the above estimate can be rewritten as

$$\sum_{I\in\mathscr{Q}_n} \nu(I)|\tilde{f}_I - \tilde{f}_{\widehat{I}}| \lesssim r_{n-1}^{\kappa p} \int_{X\cap\{|x|\geq n-1\}} {g_f}^p\, r_{j(x)}^{p(1-\kappa)-\beta/\epsilon} j(x)^{-\lambda} \nu(E(x))\, d\mu_\lambda$$
$$\lesssim r_{n-1}^{\kappa p} \int_{X\cap\{|x|\geq n-1\}} {g_f}^p\, r_{j(x)}^{p(1-\kappa)-\beta/\epsilon+Q} j(x)^{-\lambda}\, d\mu_\lambda.$$

Since $e^{-\epsilon n} \approx r_{n-1}$ and $p-\beta/\epsilon+Q=\theta p$, we obtain the estimate

$$\begin{aligned}
\|\tilde{f}\|^p_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)} &\lesssim \sum_{n=1}^{+\infty} r_{n-1}^{\kappa p-\theta p} n^\lambda \int_{X\cap\{|x|\geq n-1\}} {g_f}^p\, r_{j(x)}^{p(1-\kappa)-\beta/\epsilon+Q} j(x)^{-\lambda}\, d\mu_\lambda \\
&= \sum_{n=0}^{+\infty} r_n^{\kappa p-\theta p}(n+1)^\lambda \sum_{j=n}^{+\infty} \int_{X\cap\{j+1>|x|\geq j\}} {g_f}^p\, r_j^{(\theta-\kappa)p} j^{-\lambda}\, d\mu_\lambda \\
&= \sum_{j=0}^{+\infty} \int_{X\cap\{j+1>|x|\geq j\}} {g_f}^p\, r_j^{(\theta-\kappa)p} j^{-\lambda}\, d\mu_\lambda \left(\sum_{n=0}^{j} r_n^{\kappa p-\theta p}(n+1)^\lambda\right) \\
&\lesssim \sum_{j=0}^{+\infty} \int_{X\cap\{j+1>|x|\geq j\}} {g_f}^p\, d\mu_\lambda = \int_X {g_f}^p\, d\mu_\lambda.
\end{aligned}$$

Here the last inequality employed the estimate

$$\sum_{n=0}^{j} r_n^{\kappa p-\theta p}(n+1)^\lambda \lesssim r_j^{\kappa p-\theta p}(j+1)^\lambda \approx r_j^{(\kappa-\theta)p} j^\lambda,$$

which comes from the facts $r_n = 2e^{-\epsilon n}/\epsilon$ and $\kappa p - \theta p < 0$. Thus, we obtain the estimate

$$\|\tilde{f}\|_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)} \lesssim \|g_f\|_{L^p(X,\mu_\lambda)} \leq \|f\|_{N^{1,p}(X,\mu_\lambda)},$$

which together with Eq. 3.5 finishes the proof of Trace Part.

Extension Part: Let $u \in \mathcal{B}_p^{\theta,\lambda}(\partial X)$. For $x \in X$ with $|x| = n \in \mathbb{N}$, let

$$\tilde{u}(x) = \fint_{I_x} u\, d\nu, \tag{3.6}$$

where $I_x \in \mathscr{Q}_n$ is the set of all the points $\xi \in \partial X$ such that the geodesic $[0,\xi)$ passes through x, that is, I_x consists of all the points in ∂X that have x as an ancestor. By Eqs. 3.1 and 3.6 we notice that $\mathrm{Tr}\,\tilde{u}(\xi) = u(\xi)$ whenever $\xi \in \partial X$ is a Lebesgue point of u.

If y is a child of x, then $|y| = n+1$ and I_x is the parent of I_y. We extend $\tilde{u}$ to the edge $[x,y]$ as follows: For each $t \in [x,y]$, set

$$g_{\tilde{u}}(t) = \frac{\tilde{u}(y)-\tilde{u}(x)}{d_X(x,y)} = \frac{\epsilon(u_{I_y}-u_{I_x})}{(1-e^{-\epsilon})e^{-\epsilon n}} = \frac{\epsilon(u_{I_y}-u_{\widehat{I_y}})}{(1-e^{-\epsilon})e^{-\epsilon n}} \tag{3.7}$$

and

$$\tilde{u}(t) = \tilde{u}(x) + g_{\tilde{u}}(t) d_X(x,t). \tag{3.8}$$

Then we define the extension of u to be $\tilde{u}$.

Since $g_{\tilde{u}}$ is a constant and $\tilde{u}$ is linear with respect to the metric d_X on the edge $[x, y]$, it follows that $|g_{\tilde{u}}|$ is an upper gradient of $\tilde{u}$ on the edge $[x, y]$. We have that

$$\int_{[x,y]} |g_{\tilde{u}}|^p \, d\mu_\lambda \approx \int_n^{n+1} |u_{I_y} - u_{\widehat{I_y}}|^p e^{-\beta\tau+\epsilon np}(\tau + C)^\lambda \, d\tau$$
$$\approx e^{(-\beta+\epsilon p)(n+1)}(n+1)^\lambda |u_{I_y} - u_{\widehat{I_y}}|^p. \tag{3.9}$$

Now sum up the above integrals over all the edges on X to obtain that

$$\int_X |g_{\tilde{u}}|^p \, d\mu_\lambda \approx \sum_{n=1}^{+\infty} \sum_{I \in \mathcal{Q}_n} e^{(-\beta+\epsilon p)n} n^\lambda |u_I - u_{\widehat{I}}|^p.$$

For $I \in \mathcal{Q}_n$, the estimate

$$e^{\epsilon n\theta p}\nu(I) \approx e^{\epsilon n(p-(\beta-\log K)/\epsilon)-\epsilon nQ} \approx e^{n(\epsilon p-\beta)}$$

implies that

$$\int_X |g_{\tilde{u}}|^p \, d\mu_\lambda \approx \sum_{n=1}^{+\infty} e^{\epsilon n\theta p} n^\lambda \sum_{I \in \mathcal{Q}_n} \nu(I)|u_I - u_{\widehat{I}}|^p = \|u\|^p_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)}. \tag{3.10}$$

To estimate the L^p-norm of $\tilde{u}$, we first observe that

$$|\tilde{u}(t)| \le |\tilde{u}(x)| + |g_{\tilde{u}}| d_X(x, y) = |\tilde{u}(x)| + |\tilde{u}(y) - \tilde{u}(x)| \lesssim |u_{I_x}| + |u_{I_y}| \tag{3.11}$$

for any $t \in [x, y]$. Then we obtain the estimate

$$\int_{[x,y]} |\tilde{u}(t)|^p \, d\mu_\lambda \lesssim \mu_\lambda([x, y]) \left(|u_{I_x}|^p + |u_{I_y}|^p\right) \lesssim e^{-\beta n+\epsilon nQ} n^\lambda \int_{I_x} |u|^p \, d\nu. \tag{3.12}$$

Here the last inequality used the facts $\nu(I_x) \approx \nu(I_y) \approx e^{\epsilon nQ}$ and $\mu_\lambda([x, y]) \approx e^{-\beta n} n^\lambda$. Now sum up the above integrals over all the edges on X to obtain that

$$\int_X |\tilde{u}(t)|^p \, d\mu_\lambda \lesssim \sum_{n=0}^{+\infty} \sum_{I \in \mathcal{Q}_n} e^{-\beta n+\epsilon nQ} n^\lambda \int_I |u|^p \, d\nu = \sum_{n=0}^{+\infty} e^{-\beta n+\epsilon nQ} n^\lambda \int_{\partial X} |u|^p \, d\nu.$$

Since $\beta - \epsilon Q = \beta - \log K > 0$, the sum of $e^{-\beta n+\epsilon nQ} n^\lambda$ converges. Hence we obtain the L^p-estimate

$$\int_X |\tilde{u}|^p \, d\mu_\lambda \lesssim \int_{\partial X} |u|^p \, d\nu. \tag{3.13}$$

Combing Eq. 3.10 with Eq. 3.13, we obtain the norm estimate

$$\|\tilde{u}\|_{N^{1,p}(X,\mu_\lambda)} \lesssim \|u\|_{\mathcal{B}_p^{\theta,\lambda}(\partial X)}.$$

□

3.2 Proof of Theorem 1.2

Proposition 3.1 *Let $p = (\beta - \log K)/\epsilon$ and $\lambda > p - 1$ if $p > 1$ or $\lambda \ge 0$ if $p = 1$. Then the trace operator Tr defined in* Eq. 3.1 *is a bounded linear operator from $N^{1,p}(X, \mu_\lambda)$ to $L^p(\partial X)$.*

Proof Let $f \in N^{1,p}(X, \mu_\lambda)$. We first show that the limit in Eq. 3.1 exists for ν-a.e. $\xi \in \partial X$. It suffices to show that the function $\tilde{f}^*$ defined by Eq. 3.2 is in $L^p(\partial X)$. By estimates (3.3) and (3.4), we obtain that

$$\tilde{f}^*(\xi) \lesssim |f(0)| + \sum_{j=0}^{+\infty} r_j \fint_{[x_j, x_{j+1}]} g_f \, d\mu_\lambda.$$

Insert $j^{-\lambda/p} j^{\lambda/p}$ into the above sum. If $p > 1$, the Hölder inequality gives us that

$$\begin{aligned} |\tilde{f}^*(\xi)|^p &\lesssim |f(0)|^p + \left(\sum_{j=0}^{+\infty} j^{\frac{-\lambda}{p} \cdot \frac{p}{p-1}} \right)^{p-1} \left(\sum_{j=0}^{+\infty} r_j^p j^\lambda \fint_{[x_j, x_{j+1}]} g_f{}^p \, d\mu_\lambda \right) \\ &\lesssim |f(0)|^p + \sum_{j=0}^{+\infty} r_j^{p-\beta/\epsilon} \int_{[x_j, x_{j+1}]} g_f{}^p \, d\mu_\lambda, \end{aligned}$$

since $\mu([x_j, x_{j+1}]) \approx r_j^{\beta/\epsilon} j^\lambda$ and for $\lambda > p-1$, the sum $j^{-\lambda/(p-1)}$ converges. If $p = 1$, then the Hölder inequality is not needed and the estimate is simpler. It follows that

$$|\tilde{f}^*(\xi)|^p \lesssim |f(0)|^p + \sum_{j=0}^{+\infty} r_j^{p-\beta/\epsilon} \int_{[x_j, x_{j+1}]} g_f{}^p \, d\mu_\lambda$$

for any $\lambda > p-1$ if $p = 1$ or for $\lambda \geq 0$ if $p = 1$. Integrating over all $\xi \in \partial X$ we obtain by means of Fubini's theorem that

$$\begin{aligned} \int_{\partial X} |\tilde{f}^*(\xi)|^p \, d\nu &\lesssim |f(0)|^p + \int_{\partial X} \sum_{j=0}^{+\infty} r_j^{p-\beta/\epsilon} \int_{[x_j(\xi), x_{j+1}(\xi)]} g_f{}^p \, d\mu_\lambda \, d\nu(\xi) \\ &= |f(0)|^p + \int_X g_f(x)^p \int_{\partial X} \sum_{j=0}^{+\infty} r_j^{p-\beta/\epsilon} \chi_{[x_j(\xi), x_{j+1}(\xi)]}(x) \, d\nu(\xi) \, d\mu_\lambda(x) \\ &\lesssim |f(0)|^p + \int_X g_f(x)^p r_{j(x)}^{p-\beta/\epsilon} \nu(E(x)) \, d\mu_\lambda(x) \\ &\lesssim |f(0)|^p + \int_X g_f(x)^p r_{j(x)}^{p-\beta/\epsilon+Q} \, d\mu_\lambda(x) = |f(0)|^p + \int_X g_f(x)^p \, d\mu_\lambda(x). \end{aligned}$$

Here in the above estimates, the notations $E(x)$ and $j(x)$ are the same ones as those we used in the proof of Theorem 1.1. It follows that $\tilde{f}^*$ is in $L^p(\partial X)$ with the estimate

$$\int_{\partial X} |\tilde{f}|^p \, d\nu \lesssim \int_X |f|^p \, d\mu_\lambda + \int_X g_f{}^p \, d\mu_\lambda.$$

Hence the limit in the definition of our trace operator exists, i.e., the trace operator is well-defined, and we also have the estimate

$$\|\tilde{f}\|_{L^p(\partial X)} \lesssim \left(\int_X |f|^p \, d\mu_\lambda + \int_X g_f{}^p \, d\mu_\lambda \right)^{1/p} = \|f\|_{N^{1,p}(X,\mu_\lambda)},$$

which finishes the proof. □

Example 3.2 Let f be the continuous function on X given by $f(x) = \log(|x| + 1)$. Then the function $g_f(x) = e^{\epsilon|x|}/(|x| + 1)$ is an upper gradient of f on X with respect to the metric d_X. For $p = (\beta - \log K)/\epsilon > 1$ and $\lambda = p - 1 - \delta$ with $\delta > 0$ arbitrary, we have the estimates

$$\int_X {g_f}^p \, d\mu_\lambda \approx \sum_{n=0}^{+\infty} \frac{e^{p\epsilon n}}{(n+1)^p} K^n e^{-\beta n} n^\lambda \approx \sum_{n=0}^{+\infty} \frac{e^{(p\epsilon - \beta + \log K)n}}{(n+1)^{1+\delta}} = \sum_{n=1}^{+\infty} \frac{1}{n^{1+\delta}} < \infty$$

and

$$\int_X |f|^p \, d\mu_\lambda \approx \sum_{n=0}^{+\infty} \log^p(n+1) K^n e^{-\beta n} n^\lambda \approx \sum_{n=0}^{+\infty} e^{(-\beta+\log K)n} n^\lambda \log^p(n+1) < \infty.$$

Hence we have $f \in N^{1,p}(X, \mu_\lambda)$. On the other hand, $f(x) \to \infty$ as $x \to \partial X$.

Lemma 3.3 *Let $u \in L^1(\partial X)$ and $\tilde{u}$ be defined by* Eqs. 3.6, 3.7 *and* 3.8. *Then*

$$\int_{X \cap \{|x| \ge n\}} |\tilde{u}|^p \, d\mu \lesssim r_n^{(\beta - \log K)/\epsilon} \int_{\partial X} |u|^p \, d\nu,$$

where $n \in \mathbb{N}$, $p \ge 1$ and $r_n = 2^{-n\epsilon}/\epsilon$.

Proof By using the estimate (3.11), for $x, y \in X$ with y a child of x and $|x| = j$, we obtain that

$$\int_{[x,y]} |\tilde{u}(t)|^p \, d\mu \lesssim \mu([x,y])(|u_{I_x}|^p + |u_{I_x}|^p) \lesssim e^{-\beta j + \epsilon j Q} \int_{I_x} |u|^p \, d\nu.$$

Summing up the integrals over all edges of $X \cap \{|x| \ge n\}$, we obtain that

$$\int_{X \cap \{|x| \ge n\}} |\tilde{u}|^p \, d\mu \lesssim \sum_{j=n}^{+\infty} \sum_{I \in \mathcal{Q}_j} e^{-\beta j + \epsilon j Q} \int_I |u|^p \, d\nu = \sum_{j=n}^{+\infty} e^{-\beta j + \epsilon j Q} \int_{\partial X} |u|^p \, d\nu$$
$$\approx e^{-(\beta - \log K)n} \int_{\partial X} |u|^p \, d\nu \approx r_n^{(\beta - \log K)/\epsilon} \int_{\partial X} |u|^p \, d\nu.$$

□

Lemma 3.4 *Let u be Lipschitz continuous on ∂X and $\tilde{u}$ be defined by* Eqs. 3.6, 3.7 *and* 3.8. *Then*

$$\int_{X \cap \{|x| \ge n\}} |g_{\tilde{u}}|^p \, d\mu \lesssim r_n^{(\beta - \log K)/\epsilon} LIP(u, \partial X)^p,$$

where $r_n = 2e^{-n\epsilon}/\epsilon$, $p \ge 1$ and

$$LIP(u, \partial X) = \sup_{\xi, \zeta \in \partial X : \xi \ne \zeta} \frac{|u(\xi) - u(\zeta)|}{d_X(\xi, \zeta)}.$$

Proof For $x, y \in X$ with y a child of x and $|x| = j$, since $g_{\tilde{u}}$ is a constant on the edge $[x, y]$, we obtain the estimate

$$\int_{[x,y]} |g_{\tilde{u}}|^p \, d\mu \approx \int_j^{j+1} \frac{|u_{I_y} - u_{\widehat{I_y}}|^p}{e^{-\epsilon j p}} e^{-\beta \tau} \, d\tau \approx e^{-\beta j + \epsilon j p} |u_{I_y} - u_{\widehat{I_y}}|^p.$$

Summing up the above integrals over all edges of $X \cap \{|x| \geq n\}$, we obtain that

$$\int_{X\cap\{|x|\geq n\}} |g_{\tilde{u}}|^p \, d\mu \approx \sum_{j=n+1}^{+\infty} \sum_{I \in \mathcal{Q}_j} e^{(-\beta+\epsilon p)j} |u_I - u_{\widehat{I}}|^p.$$

Since u is Lipschitz on ∂X, then for any $\xi, \zeta \in \partial X$,

$$|f(\xi) - f(\zeta)| \leq \mathrm{LIP}\,(u, \partial X) d_X(\xi, \zeta).$$

Hence, for any $I \in \mathcal{Q}_j$, we have that

$$\begin{aligned} |u_I - u_{\widehat{I}}|^p &\lesssim \fint_I \fint_{\widehat{I}} |f(\xi) - f(\zeta)|^p \, d\nu(\xi) \, d\nu(\zeta) \leq \fint_I \fint_{\widehat{I}} \mathrm{LIP}\,(u, \partial X)^p d_X(\xi, \zeta)^p \, d\nu(\xi) \, d\nu(\zeta) \\ &\leq \mathrm{LIP}\,(u, \partial X)^p \, \mathrm{diam}\big(\widehat{I}\big)^p \approx e^{-j\epsilon p} \mathrm{LIP}\,(u, \partial X)^p. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{X\cap\{|x|\geq n\}} |g_{\tilde{u}}|^p \, d\mu &\lesssim \sum_{j=n+1}^{+\infty} K^j e^{(-\beta+\epsilon p)j} e^{-j\epsilon p} \mathrm{LIP}\,(u, \partial X)^p \\ &= \sum_{j=n+1}^{+\infty} e^{-(\beta - \log K)j} \mathrm{LIP}\,(u, \partial X)^p \\ &\approx e^{-(\beta - \log K)n} \mathrm{LIP}\,(u, \partial X)^p \approx r_n^{(\beta - \log K)/\epsilon} \mathrm{LIP}\,(u, \partial X)^p. \end{aligned}$$

□

Proposition 3.5 *Let $p = (\beta - \log K)/\epsilon \geq 1$. Then there exists a bounded non-linear extension operator Ex from $L^p(\partial X)$ to $N^{1,p}(X)$ that acts as a right inverse of the trace operator Tr in* Eq. 3.1, *i.e., $Tr \circ Ex = Id$ on $L^p(\partial X)$.*

The construction of the extension operator is given by gluing the $N^{1,p}$ extensions in Lemma 3.4 of Lipschitz approximations of the boundary data with respect to a sequence of layers on the tree X. The main idea of the construction is inspired by [21, Section 7] and [22, Section 4] whose core ideas can be traced back to Gagliardo [10] who discussed extending functions in $L^1(\mathbb{R}^n)$ to $W^{1,1}(\mathbb{R}^{n+1}_+)$.

Proof of Proposition 3.5 Let $f \in L^p(\partial X)$. We approximate f in $L^p(\partial X)$ by a sequence of Lipschitz functions $\{f_k\}_{k=1}^{+\infty}$ such that $\|f_{k+1} - f_k\|_{L^p(\partial X)} \leq 2^{2-k}\|f\|_{L^p(\partial X)}$. Note that this requirement of rate of convergence of f_k to f ensures that $f_k \to f$ pointwise ν-a.e. in ∂X. For technical reasons, we choose $f_1 \equiv 0$.

Then we choose a decreasing sequence of real numbers $\{\rho_k\}_{k=1}^{+\infty}$ such that

- $\rho_k \in \{e^{-\epsilon n}/\epsilon : n \in \mathbb{N}\}$;
- $0 < \rho_{k+1} \leq \rho_k/2$;
- $\sum_k \rho_k \mathrm{LIP}\,(f_k, \partial X) \leq C\|f\|_{L^p(\partial X)}$.

These will now be used to define layers in X. Let

$$\psi_k(x) = \max\left\{0, \min\left\{1, \frac{\rho_k - \mathrm{dist}(x, \partial X)}{\rho_k - \rho_{k+1}}\right\}\right\}, \quad x \in X.$$

We denote $-\log(\epsilon\rho_k)/\epsilon$ by $[\rho_k]$. This is a integer satisfying $e^{-\epsilon[\rho_k]}/\epsilon = \rho_k$. Then we obtain $0 \le \psi_k \le 1$ and that

$$\psi_k(x) = \begin{cases} 0, & |x| \le [\rho_k]; \\ 1, & |x| \ge [\rho_{k+1}]. \end{cases} \tag{3.14}$$

For any Lipschitz function f_k, we can define the extension $\tilde{f}_k$ of f_k by using Eqs. 3.6, 3.7 and 3.8. Then we define the extension of f as

$$\tilde{f}(x) := \sum_{k=2}^{+\infty}(\psi_{k-1}(x) - \psi_k(x))\tilde{f}_k(x) = \sum_{k=1}^{+\infty}\psi_k(x)(\tilde{f}_{k+1}(x) - \tilde{f}_k(x)). \tag{3.15}$$

It follows from Eq. 3.14 that for any $x \in X$ with $|x| = [\rho_k]$, we have $\tilde{f}(x) = \tilde{f}_{k-1}(x)$. Since for the trace operator Tr defined in Eq. 3.1, Tr $\tilde{f}_k = f_k$ for ν-a.e. in ∂X, the pointwise convergence $f_k \to f$ ν-a.e. in ∂X implies that Tr $\tilde{f} = f$ for ν-a.e. in ∂X, since $\{[\rho_k]\}_{k=1}^{+\infty}$ is a subsequence of $\mathbb{N}$. Hence the extension operator defined by Eq. 3.15 is a right inverse of the trace operator Tr in Eq. 3.1.

It remains to show that $\tilde{f} \in N^{1,p}(X)$ with norm estimates. Lemma 3.3 allows us to obtain the L^p-estimate for $\tilde{f}$. Since the extension operator that we apply for each f_k is linear, we have that $\tilde{f}_{k+1} - \tilde{f}_k = \widetilde{f_{k+1} - f_k}$. Therefore, it follows from $(\beta - \log K)/\epsilon = p$ that

$$\begin{aligned}
\|\tilde{f}\|_{L^p(X)} &\le \sum_{k=1}^{+\infty}\|\psi_k(\tilde{f}_{k+1} - \tilde{f}_k)\|_{L^p(X)} \le \sum_{k=1}^{+\infty}\|\tilde{f}_{k+1} - \tilde{f}_k\|_{L^p(X\cap\{|x|\ge[\rho_k]\})} \\
&\lesssim \sum_{k=1}^{+\infty} r_{[\rho_k]}\|f_{k+1} - f_k\|_{L^p(\partial X)} \approx \sum_{k=1}^{+\infty}\rho_k\|f_{k+1} - f_k\|_{L^p(\partial X)} \\
&\lesssim \sum_{k=1}^{+\infty}\|f_{k+1} - f_k\|_{L^p(\partial X)} \lesssim \|f\|_{L^p(\partial X)}.
\end{aligned}$$

In order to obtain the L^p-estimate of an upper gradient of $\tilde{f}$, it suffices to consider the L^p-estimate of Lip $\tilde{f}$, where for any function u, Lip $u(x)$ is defined as

$$\mathrm{Lip}\, u(x) = \limsup_{y\to x}\frac{|u(y) - u(x)|}{d_X(x, y)}.$$

We first apply the product rule for locally Lipschitz function, which yields that

$$\begin{aligned}
\mathrm{Lip}\,\tilde{f} &= \sum_{k=1}^{+\infty}\left(|\widetilde{f_{k+1} - f_k}|\mathrm{Lip}\,\psi_k + \psi_k \mathrm{Lip}\,(\widetilde{f_{k+1} - f_k})\right) \\
&\le \sum_{k=1}^{+\infty}\left(\frac{|\widetilde{f_{k+1} - f_k}|\chi_{\{|x|\ge[\rho_k]\}}}{\rho_k - \rho_{k+1}} + \chi_{\{|x|\ge[\rho_k]\}}\mathrm{Lip}\,(\widetilde{f_{k+1} - f_k})\right).
\end{aligned}$$

Thus,

$$\|\mathrm{Lip}\,\tilde{f}\|_{L^p(\partial X)} \le \sum_{k=1}^{+\infty}\left(\left\|\frac{|\widetilde{f_{k+1} - f_k}|}{\rho_k - \rho_{k+1}}\right\|_{L^p(X\cap\{|x|\ge[\rho_k]\})} + \|\mathrm{Lip}\,(\widetilde{f_{k+1} - f_k})\|_{L^p(X\cap\{|x|\ge[\rho_k]\})}\right).$$

It follows from Lemma 3.3 that

$$\sum_{k=1}^{+\infty}\left\|\frac{|\widetilde{f_{k+1}-f_k}|}{\rho_k-\rho_{k+1}}\right\|_{L^p(X\cap\{|x|\ge[\rho_k]\})} \lesssim \sum_{k=1}^{+\infty}\frac{\rho_k}{\rho_k-\rho_{k+1}}\|f_{k+1}-f_k\|_{L^p(\partial X)}$$
$$\approx \sum_{k=1}^{+\infty}\|f_{k+1}-f_k\|_{L^p(\partial X)} \lesssim \|f\|_{L^p(\partial X)}.$$

Recall that $\tilde{u}$ is affine one any edge of X, with "slope" $g_{\tilde{u}}$, for the extension $\tilde{u}$ given via Eqs. 3.6, 3.7 and 3.8, for any function u. Hence $\mathrm{Lip}\,\tilde{u} = g_{\tilde{u}}$. Therefore, it follows from Lemma 3.4 that

$$\sum_{k=1}^{+\infty}\|\mathrm{Lip}\,(\widetilde{f_{k+1}-f_k})\|_{L^p(X\cap\{|x|\ge[\rho_k]\})} \lesssim \sum_{k=1}^{+\infty}\rho_k \mathrm{LIP}\,(f_{k+1}-f_k,\partial X)$$
$$\le \sum_{k=1}^{+\infty}\rho_k\,(\mathrm{LIP}\,(f_{k+1},\partial X)+\mathrm{LIP}\,(f_k,\partial X))$$
$$\lesssim \|f\|_{L^p(\partial X)}.$$

Here in the last inequality, we used the defining properties of $\{\rho_k\}_{k=1}^{+\infty}$. Thus, we have shown that

$$\|\mathrm{Lip}\,\tilde{f}\|_{L^p(\partial X)} \lesssim \|f\|_{L^p(\partial X)}.$$

Altogether, we obtain that

$$\|\tilde{f}\|_{N^{1,p}(X)} \le \|\tilde{f}\|_{L^p(\partial X)} + \|\mathrm{Lip}\,\tilde{f}\|_{L^p(\partial X)} \lesssim \|f\|_{L^p(\partial X)}.$$

□

Proof of Theorem 1.2 The boundedness and linearity of the trace operator follows from Proposition 3.1 and the sharpness of $\lambda > p-1$ follows from Example 3.2. The extension operator is given in Proposition 3.5. □

Remark 3.6 For $p = (\beta - \log K)/\epsilon > 1$ and $\lambda > p-1$, Theorem 1.2 only tells us that there exists a bounded linear trace operator (3.1) from $N^{1,p}(X,\mu_\lambda)$ to $L^p(\partial X)$. It is unknown whether this trace operator is surjective or not. All we know is that there exists a nonlinear bounded extension operator from $L^p(\partial X)$ to $N^{1,p}(X)$ that acts as a right inverse of the trace operator (3.1). Since $\lambda > p-1 > 0$ implies $N^{1,p}(X,\mu_\lambda) \subsetneq N^{1,p}(X)$, we have an open question: Which space does the bounded linear trace operator (3.1) map $N^{1,p}(X,\mu_\lambda)$ surjectively onto?

3.3 Proof of Theorem 1.3

Proof of Theorem 1.3 **Trace Part:** Let $f \in N^{1,1}(X,\mu_\lambda)$ with $\lambda > 0$ and let g_f be an upper gradient of f. By Proposition 3.1, we know that the trace operator $\mathrm{Tr}\, f = \tilde{f}$ defined in Eq. 3.1 is well-defined and that $\tilde{f}$ satisfies the norm estimate

$$\|\tilde{f}\|_{L^1(\partial X)} \lesssim \|f\|_{N^{1,1}(X,\mu_\lambda)}.$$

Then the remaining task is to establish the estimate on the dyadic energy $\|\tilde{f}\|_{\dot{\mathcal{B}}^{\alpha}_1(\partial X)}$. For any $I \in \mathscr{Q}_{\alpha(n)}$, $\xi \in I$ and $\zeta \in \widetilde{I} \in \mathscr{Q}_{\alpha(n-1)}$, we obtain that

$$
\begin{aligned}
|\tilde{f}(\xi) - \tilde{f}(\zeta)| &\le \sum_{j=\alpha(n-1)}^{+\infty} |f(x_j) - f(x_{j+1})| + \sum_{j=\alpha(n-1)}^{+\infty} |f(y_j) - f(y_{j+1})| \\
&\lesssim \sum_{j=\alpha(n-1)}^{+\infty} r_j \fint_{[x_j, x_{j+1}]} g_f \, d\mu_\lambda + \sum_{j=\alpha(n-1)}^{+\infty} r_j \fint_{[y_j, y_{j+1}]} g_f \, d\mu_\lambda,
\end{aligned}
$$

where $x_j = x_j(\xi)$ and $y_j = y_j(\zeta)$ are the ancestors of ξ and ζ with $|x_j| = |y_j| = j$, respectively. For any $I \in \mathscr{Q}_{\alpha(n)}$ and any function $h \in L^1(\partial X)$, we have

$$
\frac{\nu(I)}{\nu(\widetilde{I})} \approx \left(\frac{r_{\alpha(n)}}{r_{\alpha(n-1)}} \right)^Q \approx e^{(\alpha(n-1)-\alpha(n)) \log K} \approx K^{\alpha(n-1)-\alpha(n)}
$$

and

$$
\sum_{I \in \mathscr{Q}_{\alpha(n)}} \int_{\widetilde{I}} h(\zeta) \, d\nu(\zeta) = K^{\alpha(n)-\alpha(n-1)} \int_{\partial X} h(\zeta) \, d\nu(\zeta). \tag{3.16}
$$

Hence it follows from the fact that $\mu_\lambda([x_j, x_{j+1}]) \approx r_j^{\beta/\epsilon} j^\lambda$ and Fubini's theorem that

$$
\begin{aligned}
\sum_{I \in \mathscr{Q}_{\alpha(n)}} \nu(I) |\tilde{f}_I - \tilde{f}_{\widetilde{I}}| &\le \sum_{I \in \mathscr{Q}_{\alpha(n)}} \nu(I) \fint_I \fint_{\widetilde{I}} |\tilde{f}(\xi) - \tilde{f}(\zeta)| \, d\nu(\xi) \, d\nu(\zeta) \\
&\lesssim \sum_{I \in \mathscr{Q}_{\alpha(n)}} \int_I \sum_{j=\alpha(n-1)}^{+\infty} r_j \fint_{[x_j(\xi), x_{j+1}(\xi)]} g_f \, d\mu_\lambda \, d\nu(\xi) \\
&\quad + \sum_{I \in \mathscr{Q}_{\alpha(n)}} K^{\alpha(n-1)-\alpha(n)} \int_{\widetilde{I}} \sum_{j=\alpha(n-1)}^{+\infty} r_j \fint_{[y_j(\zeta), y_{j+1}(\zeta)]} g_f \, d\mu_\lambda \, d\nu(\zeta) \\
&\approx \int_{\partial X} \sum_{j=\alpha(n-1)}^{+\infty} r_j \fint_{[x_j(\xi), x_{j+1}(\xi)]} g_f \, d\mu_\lambda \, d\nu(\xi) \\
&\approx \int_{X \cap \{|x| \ge \alpha(n-1)\}} g_f \int_{\partial X} \sum_{j=\alpha(n-1)}^{+\infty} r_j^{1-\beta/\epsilon} j^{-\lambda} \chi_{[x_j(\xi), x_{j+1}(\xi)]}(x) \, d\nu(\xi) \, d\mu_\lambda(x).
\end{aligned}
$$

Using the notation $E(x)$ and $j(x)$ defined in the proof of Theorem 1.1, the above estimate can be rewritten as

$$
\begin{aligned}
\sum_{I \in \mathscr{Q}_{\alpha(n)}} \nu(I) |\tilde{f}_I - \tilde{f}_{\widetilde{I}}| &\lesssim \int_{X \cap \{|x| \ge \alpha(n-1)\}} g_f r_{j(x)}^{1-\beta/\epsilon} j(x)^{-\lambda} \nu(E(x)) \, d\mu_\lambda \\
&\lesssim \int_{X \cap \{|x| \ge \alpha(n-1)\}} g_f r_{j(x)}^{1-\beta/\epsilon+Q} j(x)^{-\lambda} \, d\mu_\lambda \\
&= \int_{X \cap \{|x| \ge \alpha(n-1)\}} g_f j(x)^{-\lambda} \, d\mu_\lambda.
\end{aligned}
$$

It follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\alpha(n)^{\lambda}\sum_{I\in\mathcal{Q}_{\alpha(n)}}\nu(I)|f_I-f_{\tilde{I}}| &\lesssim \sum_{n=1}^{\infty}\alpha(n)\sum_{j=\alpha(n-1)}^{+\infty}\int_{X\cap\{j+1>|x|\geq j\}} g_f\, j^{-\lambda}\,d\mu_\lambda\\ &= \sum_{n=0}^{\infty}\alpha(n+1)\sum_{j=\alpha(n)}^{+\infty}\int_{X\cap\{j+1>|x|\geq j\}} g_f\, j^{-\lambda}\,d\mu_\lambda\\ &\leq \sum_{j=0}^{+\infty}\int_{X\cap\{j+1>|x|\geq j\}} g_f\, j^{-\lambda}\,d\mu_\lambda\left(\sum_{n=0}^{\alpha^{-1}(j)}\alpha(n+1)^{\lambda}\right),\end{aligned}$$

where $\alpha^{-1}(j)$ is the largest integer m such that $\alpha(m)\leq j$. Since $\lambda>0$ and

$$1<c_0\leq\frac{\alpha(n+1)}{\alpha(n)}\leq c_1,$$

we obtain the estimate

$$\sum_{n=0}^{\alpha^{-1}(j)}\alpha(n+1)^{\lambda}\approx\sum_{n=0}^{\alpha^{-1}(j)}\alpha(n)^{\lambda}\leq\sum_{k=0}^{+\infty}j^{\lambda}c_0^{-\lambda k}\lesssim j^{\lambda}.$$

Hence we obtain the estimate

$$\begin{aligned}\|\tilde{f}\|_{\dot{\mathcal{B}}_1^{\alpha}(\partial X)} = \sum_{n=1}^{\infty}\alpha(n)^{\lambda}\sum_{I\in\mathcal{Q}_{\alpha(n)}}\nu(I)|f_I-f_{\tilde{I}}| &\lesssim \sum_{j=0}^{+\infty}\int_{X\cap\{j+1>|x|\geq j\}} g_f\,d\mu_\lambda\\ &= \int_X g_f\,d\mu_\lambda=\|g_f\|_{L^1(X,\mu_\lambda)}.\end{aligned}$$

Thus, we obtain the norm estimate

$$\|f\|_{\mathcal{B}_1^{\alpha}(\partial X)}=\|f\|_{L^1(\partial X)}+\|f\|_{\dot{\mathcal{B}}_1^{\alpha}(\partial X)}\lesssim\|f\|_{N^{1,1}(X,\mu_\lambda)},$$

which finishes the proof of the Trace Part.

Extension Part: Let $u\in\mathcal{B}_1^{\alpha}(\partial X)$. Since $\alpha(0)$ is not necessarily zero, we let $\alpha(-1)=0$. For any $x\in X$ with $|x|=\alpha(n)$ and $-1\leq n\in\mathbb{Z}$, let

$$\tilde{u}(x)=\fint_{I_x}u\,d\nu,$$

where $I_x\in\mathcal{Q}$ is the set of all the points $\xi\in\partial X$ such that the geodesic $[0,\xi)$ passes through x, that is, I_x consists of all the points in ∂X that have x as an ancestor.

If y is a descendant of x with $|y|=\alpha(n+1)$, then there exists $\tilde{y}\in X$ which is the parent of y. We extend $\tilde{u}$ to the edge $[x,y]$ as follows: For each $t\in[x,\tilde{y}]$, set $\tilde{u}(t)=\tilde{u}(x)$ and $g_{\tilde{u}}(t)=0$; for each $t\in[\tilde{y},y]$, set

$$g_{\tilde{u}}(t)=\frac{\tilde{u}(y)-\tilde{u}(x)}{d_X(\tilde{y},y)}=\frac{\epsilon(u_{I_y}-u_{I_x})}{(e^{\epsilon}-1)e^{-\epsilon\alpha(n+1)}}=\frac{\epsilon(u_{I_y}-u_{\tilde{I}_y})}{(e^{\epsilon}-1)e^{-\epsilon\alpha(n+1)}}$$

and

$$\tilde{u}(t)=\tilde{u}(x)+g_{\tilde{u}}(t)d_X(\tilde{y},t).$$

Then we define $\tilde{u}$ to be the extension of u. Notice that $\operatorname{Tr}\tilde{u}(\xi)=u(\xi)$ whenever ξ is a Lebesgue point of u.

Now on the geodesic $[x,\tilde{y}]$, $g_{\tilde{u}}$ is zero and $\tilde{u}$ is a constant; on the edge $[\tilde{y},y]$, $g_{\tilde{u}}$ is a constant and $\tilde{u}$ is linear with respect to the metric on the edge $[\tilde{y},x]$. It follows that $|g_{\tilde{u}}|$ is an upper gradient of $\tilde{u}$ on the geodesic $[x, y]$. Then for $x \in X$ with $|x| = \alpha(n)$, $n \geq 0$, we obtain the estimate

$$\int_{[x,y]} |g_{\tilde{u}}|\, d\mu_\lambda = \int_{[\tilde{y},y]} |g_{\tilde{u}}|\, d\mu_\lambda \approx \int_{\alpha(n+1)-1}^{\alpha(n+1)} \frac{|u_{I_y} - u_{\widetilde{I_y}}|}{e^{-\epsilon\alpha(n+1)}} e^{-\beta\tau}(t+C)^\lambda\, d\tau$$
$$\approx e^{(\epsilon-\beta)\alpha(n+1)}\alpha(n+1)^\lambda |u_{I_y} - u_{\widetilde{I_y}}|. \tag{3.17}$$

For $x = 0$ and $|y| = \alpha(0)$, since $\nu(I_0) \approx \nu(I_y) \approx 1$, we have the estimate

$$\int_{[0,y]} |g_{\tilde{u}}|\, d\mu_\lambda = \int_{[\tilde{y},y]} |g_{\tilde{u}}|\, d\mu_\lambda \approx |u_{I_0} - u_{I_y}| \leq |u_{I_0}| + |u_{I_y}| \lesssim \int_{\partial X} |u|\, d\nu. \tag{3.18}$$

Now sum up the estimates (3.17) and (3.18) over all edges of X to obtain that

$$\begin{aligned}\int_X |g_{\tilde{u}}|\, d\mu_\lambda &= \int_{X\cap\{|x|\leq\alpha(0)\}} |g_{\tilde{u}}|\, d\mu_\lambda + \int_{X\cap\{|x|\geq\alpha(0)\}} |g_{\tilde{u}}|\, d\mu_\lambda \\ &\lesssim \sum_{y\in V_{\alpha(0)}} \int_{[0,y]} |g_{\tilde{u}}|\, d\mu_\lambda + \sum_{n=1}^{+\infty} \sum_{y\in V_{\alpha(n)}} \int_{[x,y]} |g_{\tilde{u}}|\, d\mu_\lambda \\ &\lesssim K^{\alpha(0)} \int_{\partial X} |u|\, d\nu + \sum_{n=1}^{+\infty} \sum_{I\in\mathcal{Q}_{\alpha(n)}} e^{(\epsilon-\beta)\alpha(n)}\alpha(n)^\lambda |u_I - u_{\tilde{I}}|.\end{aligned}$$

Since for any $I \in \mathcal{Q}_{\alpha(n)}$, we have that

$$\nu(I) \approx r^{Q}_{\alpha(n)} \approx e^{-\epsilon\alpha(n)\log K/\epsilon} = e^{-\alpha(n)\log K} = e^{(\epsilon-\beta)\alpha(n)}.$$

Hence we obtain the estimate

$$\begin{aligned}\int_X |g_{\tilde{u}}|\, d\mu_\lambda &\lesssim \int_{\partial X} |u|\, d\nu + \sum_{n=1}^{\infty} \alpha(n)^\lambda \sum_{I\in\mathcal{Q}_{\alpha(n)}} \nu(I)|f_I - f_{\tilde{I}}| \\ &= \|u\|_{L^1(\partial X)} + \|u\|_{\dot{\mathcal{B}}^\alpha_1(\partial X)} = \|u\|_{\mathcal{B}^\alpha_1(\partial X)}.\end{aligned} \tag{3.19}$$

Towards the L^1-estimate for $\tilde{u}$, by the construction, we know that $|\tilde{u}(t)| = |\tilde{u}(x)|$ on the geodesic $[x, \tilde{y}]$ and that $|\tilde{u}(t)| \lesssim |\tilde{u}(x)| + |\tilde{u}(y)|$ on the edge $[\tilde{y}, y]$. Then for $n \geq -1$, we have the estimate

$$\begin{aligned}\int_{X\cap\{\alpha(n)\leq|x|\leq\alpha(n+1)\}} |\tilde{u}|\, d\mu_\lambda &= \int_{X\cap\{\alpha(n)\leq|x|\leq\alpha(n+1)-1\}} |\tilde{u}|\, d\mu_\lambda \\ &\quad + \int_{X\cap\{\alpha(n+1)-1\leq|x|\leq\alpha(n+1)\}} |\tilde{u}|\, d\mu_\lambda \\ &\leq \sum_{x\in V_{\alpha(n)}} |u(x)|\mu_\lambda(F(x, d_X(x, \partial X))) \\ &\quad + \sum_{y\in V_{\alpha(n+1)}} (|\tilde{u}(x)| + |\tilde{u}(y)|)\mu_\lambda([\tilde{y}, y]) =: H^n_1 + H^n_2.\end{aligned}$$

By Lemma 2.3, we obtain the estimate

$$H^n_1 \lesssim \sum_{x\in V_{\alpha(n)}} e^{(-\beta+\log K)\alpha(n)}\alpha(n)^\lambda \int_{I_x} |u|\, d\nu = e^{(-\beta+\log K)\alpha(n)}\alpha(n)^\lambda \int_{\partial X} |u|\, d\nu.$$

For H_2^n, by Eq. 3.16 and relation Eq. 3.3, we have that

$$H_2^n \lesssim \sum_{y\in V_{\alpha(n+1)}} e^{(-\beta+\log K)\alpha(n+1)}\alpha(n+1)^\lambda \left(\int_{I_y} |u|\,d\nu + K^{\alpha(n)-\alpha(n+1)} \int_{\widetilde{I}_y} |u|\,d\nu\right)$$
$$\lesssim e^{(-\beta+\log K)\alpha(n+1)}\alpha(n+1)^\lambda \int_{\partial X} |u|\,d\nu.$$

Sum up the above estimate with respect to n to obtain via $\epsilon = \beta - \log K$ that

$$\int_X |\tilde{u}|\,d\mu_\lambda = \sum_{n=-1}^{+\infty} \int_{X\cap\{\alpha(n)\le|x|\le\alpha(n+1)\}} |\tilde{u}|\,d\mu_\lambda = \sum_{n=-1}^{+\infty} H_1^n + H_2^n$$
$$\lesssim \sum_{n=-1}^{+\infty} e^{(-\beta+\log K)\alpha(n)}\alpha(n)^\lambda \int_{\partial X} |u|\,d\nu$$
$$= \sum_{n=-1}^{+\infty} e^{-\epsilon\alpha(n)}\alpha(n)^\lambda \int_{\partial X} |u|\,d\nu \lesssim \int_{\partial X} |u|\,d\nu = \|u\|_{L^1(\partial X)}. \tag{3.20}$$

By the estimates (3.19) and (3.20), we obtain the norm estimate

$$\|\tilde{u}\|_{N^{1,1}(X,\mu_\lambda)} \lesssim \|u\|_{\mathcal{B}_1^\alpha(\partial X)}.$$

□

Corollary 3.7 *For given sequences $\{\alpha_1(n)\}_{n\in\mathbb{N}}$ and $\{\alpha_1(n)\}_{n\in\mathbb{N}}$ satisfying the relation (2.13) with respect to different pairs of (c_0, c_1), the Banach spaces $\mathcal{B}_1^{\alpha_1}(\partial X)$ and $\mathcal{B}_1^{\alpha_2}(\partial X)$ coincide.*

Proof For any function $u \in \mathcal{B}_1^{\alpha_1}(\partial X)$, by the Extension part in the proof of Theorem 1.3, there is an extension $Eu = \tilde{u}$ such that

$$\|\tilde{u}\|_{N^{1,1}(X,\mu_\lambda)} \lesssim \|u\|_{\mathcal{B}_1^{\alpha_1}(\partial X)}.$$

Since $u = T \circ Eu = T(\tilde{u})$, it follows from the trace part in the proof of Theorem 1.3 that we have the estimate

$$\|u\|_{\mathcal{B}_1^{\alpha_2}(\partial X)} \lesssim \|\tilde{u}\|_{N^{1,1}(X,\mu_\lambda)}.$$

Thus, we obtain

$$\|u\|_{\mathcal{B}_1^{\alpha_2}(\partial X)} \lesssim \|u\|_{\mathcal{B}_1^{\alpha_1}(\partial X)}.$$

The opposite inequality follows analogously and the claim follows. □

Next, we compare the function spaces $\mathcal{B}_1^\alpha(\partial X)$ and $\mathcal{B}_1^{0,\lambda}(\partial X)$.

Proposition 3.8 *Let $\lambda > 0$. The space $\mathcal{B}_1^{0,\lambda}(\partial X)$ is a subset of $\mathcal{B}_1^\alpha(\partial X)$, i.e., for any $f \in L^1(\partial X)$, we have*

$$\|f\|_{\dot{\mathcal{B}}_1^\alpha(\partial X)} \lesssim \|f\|_{\dot{\mathcal{B}}_1^{0,\lambda}(\partial X)}.$$

Proof Let $f \in L^1(\partial X)$. For any $I \in \mathcal{Q}_{\alpha(n)}$ with $n \in \mathbb{R}$, define the set

$$\mathcal{J}_I := \{I' \in \mathcal{Q} : I \subset I' \subsetneq \widetilde{I}\}.$$

Then it follows from the triangle inequality that

$$|f_I - f_{\widetilde{I}}| \leq \sum_{I' \in \mathcal{J}_I} |f_{I'} - f_{\widehat{I'}}|.$$

Hence, by using Fubini's theorem, we have that

$$\begin{aligned}\sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I)|f_I - f_{\widetilde{I}}| &\leq \sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I) \sum_{I' \in \mathcal{J}_I} |f_{I'} - f_{\widehat{I'}}| \\ &= \sum_{m=\alpha(n-1)+1}^{\alpha(n)} \sum_{I' \in \mathcal{Q}_m} |f_{I'} - f_{\widehat{I'}}| \left(\sum_{I \in \mathcal{Q}_{\alpha(n)}} \sum_{I' \in \mathcal{J}_I} \nu(I) \right).\end{aligned}$$

Notice that for any $I \in \mathcal{Q}_{\alpha(n)}$, we have $\nu(I) \approx e^{-\epsilon\alpha(n)Q} = K^{-\alpha(n)}$ and that for any $I' \in \mathcal{Q}_m$, the number of the dyadic elements $I \in \mathcal{Q}_{\alpha(n)}$ with $I' \in \mathcal{J}_I$ is $K^{\alpha(n)-m}$. Therefore,

$$\sum_{I \in \mathcal{Q}_{\alpha(n)}} \sum_{I' \in \mathcal{J}_I} \nu(I) \approx K^{\alpha(n)-m-\alpha(n)} = K^{-m} = e^{-\epsilon\alpha(n)Q} \approx \nu(I').$$

Hence, we have the estimate

$$\sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I)|f_I - f_{\widetilde{I}}| \lesssim \sum_{m=\alpha(n-1)+1}^{\alpha(n)} \sum_{I' \in \mathcal{Q}_m} \nu(I')|f_{I'} - f_{\widehat{I'}}|,$$

and therefore the estimate

$$\begin{aligned}\|f\|_{\dot{\mathcal{B}}_1^{\alpha}(\partial X)} &= \sum_{n=1}^{+\infty} \alpha(n)^{\lambda} \sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I)|f_I - f_{\widetilde{I}}| \\ &\lesssim \sum_{n=1}^{+\infty} \alpha(n)^{\lambda} \sum_{m=\alpha(n-1)+1}^{\alpha(n)} \sum_{I' \in \mathcal{Q}_m} \nu(I')|f_{I'} - f_{\widehat{I'}}| \\ &\lesssim \sum_{m=1}^{+\infty} m^{\lambda} \sum_{I' \in \mathcal{Q}_m} \nu(I')|f_{I'} - f_{\widehat{I'}}| = \|f\|_{\dot{\mathcal{B}}_1^{0,\lambda}(\partial X)}.\end{aligned}$$

Here in the last inequality, we used the fact that $m^{\lambda} > \alpha(n-1)^{\lambda} \geq \alpha(n)^{\lambda}/c_1^{\lambda}$ whenever $m > \alpha(n-1)$, where the constant c_1 is from the condition (2.13). □

Example 3.9 Let X be a 2-regular tree. We may identify each vertex of X with a finite sequence formed by 0 and 1. For example, the children of the root can be denoted by 00 and 01. The children of the vertex $x = 0\tau_1 \cdots \tau_k$ is $0\tau_1 \cdots \tau_k 0$ and $0\tau_1 \cdots \tau_k 1$, where $\tau_i \in \{0, 1\}$. Moreover, each element ξ of the boundary ∂X can be identified with an infinite sequence formed by 0 and 1. We denote $\xi = 0\tau_1\tau_2 \cdots$ with $\tau_i \in \{0, 1\}$ when the geodesic from 0 to ξ passes through all the vertices $x_k = 0\tau_1 \cdots \tau_k$, $k \in \mathbb{R}$.

We define a function f on ∂X as follows: for $\xi = 0\tau_1\tau_2 \cdots \in \partial X$ where $\tau_i \in \{0, 1\}$, we define

$$f(\xi) = \sum_{i=1}^{+\infty} \frac{(-1)^{\tau_i}}{i^{\lambda+1}}.$$

Since the sum of $1/i^{\lambda+1}$ converges for $\lambda > 0$, f is well defined for all $\xi \in \partial X$ and is bounded. Moreover, for any vertex $x = 0\tau_1 \cdots \tau_k$, it follows from the definition of f that

$$f_{I_x} = \fint_{I_x} f(\zeta)\, d\nu(\zeta) = \sum_{i=1}^{k} \frac{(-1)^{\tau_i}}{i^{\lambda+1}}. \tag{3.21}$$

Therefore, for the vertex x above, we have

$$|f_{I_x} - f_{\widehat{I_x}}| = \frac{1}{k^{\lambda+1}}.$$

Hence the $\dot{\mathcal{B}}_1^{0,\lambda}$-energy of f is

$$\begin{aligned}\|f\|_{\dot{\mathcal{B}}_1^{0,\lambda}(\partial X)} &= \sum_{n=1}^{+\infty} n^{\lambda} \sum_{I \in \mathcal{Q}_n} \nu(I) |f_I - f_{\widehat{I}}| \\ &= \sum_{n=1}^{+\infty} n^{\lambda} \sum_{I \in \mathcal{Q}_n} \nu(I) \frac{1}{n^{\lambda+1}} = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty.\end{aligned}$$

On the other hand, for any $I \in \mathcal{Q}_{\alpha(n)}$, we have

$$|f_I - f_{\widetilde{I}}| = \left| \sum_{i=\alpha(n-1)+1}^{\alpha(n)} \frac{(-1)^{\tau_i}}{i^{\lambda+1}} \right|, \tag{3.22}$$

where $\tau_i \in \{0, 1\}$ depends on I. We define a random series $\mathcal{X}_{\alpha(n)}$ by setting

$$\mathcal{X}_{\alpha(n)} = \sum_{i=\alpha(n-1)+1}^{\alpha(n)} \frac{\sigma_i}{i^{\lambda+1}},$$

where $(\sigma_i)_i$ are independent random variables with common distribution $P(\sigma_i = 1) = P(\sigma_i = -1) = 1/2$. Since the measure ν is a probability measure which is uniformly distributed on ∂X, it follows from Eq. 3.22 that

$$\sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I) |f_I - f_{\widetilde{I}}| = \mathbb{E}(|\mathcal{X}_{\alpha(n)}|).$$

Here $\mathbb{E}(|\mathcal{X}_{\alpha(n)}|)$ is the expected value of $|\mathcal{X}_{\alpha(n)}|$. By the Cauchy-Schwarz inequality, $\mathbb{E}(|\mathcal{X}_{\alpha(n)}|) \le (\mathbb{E}(\mathcal{X}^2_{\alpha(n)}))^{1/2}$, we have that

$$\begin{aligned}\sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I) |f_I - f_{\widetilde{I}}| &\le (\mathbb{E}(\mathcal{X}^2_{\alpha(n)}))^{1/2} = \left(\sum_{i,j=\alpha(n-1)+1}^{\alpha(n)} \frac{\mathbb{E}(\sigma_i \sigma_j)}{i^{\lambda+1} j^{\lambda+1}} \right)^{1/2} \\ &= \left(\sum_{i=\alpha(n-1)+1}^{\alpha(n)} \frac{\mathbb{E}({\sigma_i}^2)}{i^{2\lambda+2}} \right)^{1/2} = \left(\sum_{i=\alpha(n-1)+1}^{\alpha(n)} \frac{1}{i^{2\lambda+2}} \right)^{1/2}.\end{aligned}$$

Here the second to last equality holds since σ_i and σ_j are independent for $i \neq j$ and $\mathbb{E}(\sigma_i \sigma_j) = \mathbb{E}(\sigma_i)\mathbb{E}(\sigma_j) = 0$ for $i \neq j$. Define $\alpha(n) = 2^n$. Then we obtain that

$$\sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I) |f_I - f_{\widetilde{I}}| \le \left(\sum_{i=2^{n-1}+1}^{2^n} \frac{1}{i^{2\lambda+2}} \right)^{1/2} \le \left(\sum_{i=2^{n-1}+1}^{2^n} \frac{1}{2^{(n-1)(2\lambda+2)}} \right)^{1/2} = \frac{1}{2^{(n-1)(\lambda+1/2)}}.$$

Therefore the $\dot{\mathcal{B}}_1^{\alpha}(\partial X)$-energy of f is estimated by

$$\|f\|_{\dot{\mathcal{B}}_1^{\alpha}(\partial X)} = \sum_{n=1}^{+\infty} \alpha(n)^{\lambda} \sum_{I \in \mathcal{Q}_{\alpha(n)}} \nu(I)|f_I - f_{\tilde{I}}|$$

$$\leq \sum_{n=1}^{+\infty} 2^{n\lambda} \frac{1}{2^{(n-1)(\lambda+1/2)}} = \sum_{n=0}^{+\infty} \frac{2^{\lambda}}{2^{n/2}} < +\infty.$$

Hence $f \in \mathcal{B}_1^{\alpha}(\partial X)$ while $f \notin \mathcal{B}_1^{0,\lambda}(\partial X)$, and it follows that $\mathcal{B}_1^{0,\lambda}(\partial X)$ is a strict subset of $\mathcal{B}_1^{\alpha}(\partial X)$.

3.4 Proof of Theorem 1.4

Proof Let $p = (\beta - \log K)/\epsilon$ and $\lambda > p - 1$ if $p > 1$ or $\lambda \geq 0$ if $p = 1$. From Proposition 3.1, the trace operator $T : N^{1,p}(X, \mu_\lambda) \to L^p(\partial X)$ in Theorem 1.2 is bounded and linear. Now we define an extension operator E by using Eqs. 3.6, 3.7 and 3.8. It is easy to see that the extension Eu is well defined for any function $u \in L^1_{\text{loc}}(\partial X)$ and that $T \circ E$ is the identity operator on $L^1_{\text{loc}}(\partial X)$.

Repeating the estimates in Extension Part of the proof of Theorem 1.1, for $\theta = 1 - (\beta - \log K)/(p\epsilon) = 0$, we also have the following estimates:

$$\int_X |g_{\tilde{u}}|^p \, d\mu_\lambda \approx \|u\|^p_{\dot{\mathcal{B}}_p^{0,\lambda}(\partial X)} \tag{3.23}$$

and

$$\int_X |\tilde{u}|^p \, d\mu \lesssim \int_{\partial X} |u|^p \, d\nu. \tag{3.24}$$

Hence the extension operator E is bounded and linear from $\mathcal{B}_p^{0,\lambda}(\partial X)$ to $N^{1,p}(X, \mu_\lambda)$.

Moreover, since u is the trace of $\tilde{u}$, by Theorem 1.2 and Proposition 3.1, we have

$$\|u\|_{L^p(\partial X)} \lesssim \|\tilde{u}\|_{N^{1,p}(X,\mu_\lambda)}.$$

Combining the above inequality with Eqs. 3.23 and 3.24, we obtain the estimate

$$\|u\|_{\mathcal{B}_p^{0,\lambda}(\partial X)} \approx \|\tilde{u}\|_{N^{1,p}(X,\mu_\lambda)}. \tag{3.25}$$

Hence the $\mathcal{B}_p^{0,\lambda}(\partial X)$-norm of u is comparable to the $N^{1,p}(X, \mu_\lambda)$-norm of $\tilde{u} = Eu$. Thus $\mathcal{B}_p^{0,\lambda}(\partial X)$ is the optimal space for which E is both bounded and linear. □

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Traces of Newton-Sobolev, Hajłasz-Sobolev, and BV functions on metric spaces

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Traces of Newton-Sobolev, Hajłasz-Sobolev, and BV functions on metric spaces *

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Abstract

We study the boundary traces of Newton-Sobolev, Hajłasz-Sobolev, and BV (bounded variation) functions. Assuming less regularity of the domain than is usually done in the literature, we show that all of these function classes achieve the same "boundary values", which in particular implies that the trace spaces coincide provided that they exist. Many of our results seem to be new even in Euclidean spaces but we work in a more general complete metric space equipped with a doubling measure and supporting a Poincaré inequality.

1 Introduction

Boundary traces for various function classes, especially functions of bounded variation (BV functions), have been studied in recent years in the setting of metric measure spaces (X, d, μ). In [28], the authors studied the boundary traces, or traces for short, of BV functions in suitably regular domains. Typically, the boundary trace Tu of a function u in a domain Ω is defined by the condition

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - Tu(x)|\, d\mu = 0 \tag{1.1}$$

for a.e. $x \in \partial\Omega$ with respect to the codimension 1 Hausdorff measure $\mathcal{H}$. In [30] (see also references therein for previous works in Euclidean spaces) the authors considered the corresponding extension problem, that is, the problem of finding a function whose trace is a prescribed L^1-function on the boundary. They showed that in sufficiently regular domains, the trace operator of BV functions is surjective, and that in fact the extension can always be taken to be a Newton-Sobolev function. This implies that the trace space of both $\mathrm{BV}(\Omega)$ and $N^{1,1}(\Omega)$ is $L^1(\partial\Omega)$. This trace and extension problem is motivated by Dirichlet problems for functions of least gradient, in which one minimizes the total variation among BV functions with prescribed boundary data, see [5, 11, 22, 31, 36].

In the current paper, we consider boundary traces from a different viewpoint. Unlike in the existing literature, we assume very little regularity of the domain, meaning that traces

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need not always exist. We are nonetheless able to show in various cases that for a given function, it is possible to find a more regular function that "achieves the same boundary values". In particular, if the original function has a boundary trace, then the more regular function has the same trace. This sheds further light on the extension problem. To prove our results, we apply some existing approximation results for BV and Newton-Sobolev functions, and develop some new ones.

We will always assume that (X,d,μ) is a complete metric space equipped with a doubling measure μ and supporting a $(1,1)$-Poincaré inquality. Let $\Omega\subset X$ be a nonempty open set. For BV functions we prove the following three theorems. The exponent s is sometimes called the homogeneous dimension of the space. $N^{1,1}(\Omega)$ is a generalization of the Sobolev class $W^{1,1}(\Omega)$ to metric spaces; see Section 2 for definitions.

Theorem 1.2. *Let $u\in \mathrm{BV}(\Omega)$. Then there exists $v\in N^{1,1}(\Omega)\cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that*

$$\fint_{B(x,r)\cap\Omega}|v-u|^{s/(s-1)}\,d\mu\to 0\quad \text{as } r\to 0^+$$

uniformly for all $x\in\partial\Omega$.

In particular, *whenever* there exists a BV extension of a given function defined on the boundary, it is possible to also find a Newton-Sobolev extension. If we give up the requirement that v is locally Lipschitz, we can replace $s/(s-1)$ by an arbitrarily large exponent.

Theorem 1.3. *Let $u\in \mathrm{BV}(\Omega)$ and let $1\le q<\infty$. Then there exists $v\in N^{1,1}(\Omega)$ such that*

$$\fint_{B(x,r)\cap\Omega}|v-u|^{q}\,d\mu\to 0\quad \text{as } r\to 0^+$$

uniformly for all $x\in\partial\Omega$.

If we also allow v to have a small (approximate) *jump set* S_v, then we can include the case $q=\infty$. The class of *special functions of bounded variation*, denoted by $\mathrm{SBV}(\Omega)$, is defined as those BV functions whose variation measure only has an absolutely continuous part (like Sobolev functions) and a jump part. The class was introduced by De Giorgi and Ambrosio [2] as a natural class in which to solve various variational problems, e.g. the minimization of Mumford–Shah functional.

Theorem 1.4. *Let $u\in \mathrm{BV}(\Omega)$ and let $\varepsilon>0$. Denote $\Omega(r):=\{x\in\Omega:\ \mathrm{dist}(x,X\setminus\Omega)>r\}$ for $r>0$. Then there exists $v\in\mathrm{SBV}(\Omega)$ such that $\mathcal{H}(S_v)<\varepsilon$ and*

$$\|v-u\|_{L^\infty(\Omega\setminus\Omega(r))}\to 0\quad \text{as } r\to 0^+.$$

Note that $v\in\mathrm{SBV}(\Omega)$ belongs to $N^{1,1}(\Omega)$ if and only if $\mathcal{H}(S_v)=0$ (see [21, Theorem 4.1], (2.13), and [16, Theorem 4.6]). Thus we could equivalently require

- $v\in\mathrm{SBV}(\Omega)\cap\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ (in particular, $S_v=\emptyset$) in Theorem 1.2,
- $v\in\mathrm{SBV}(\Omega)$ with $\mathcal{H}(S_v)=0$ in Theorem 1.3, and

- $v \in \mathrm{SBV}(\Omega)$ with $\mathcal{H}(S_v) < \varepsilon$ in Theorem 1.4,

illustrating how we get better boundary approximation by relaxing the regularity requirements on v.

From Theorem 1.2 (or Theorem 1.3), we obtain the following corollary.

Corollary 1.5. *The trace spaces of* $\mathrm{BV}(\Omega)$ *and* $N^{1,1}(\Omega)$ *are the same.*

The definitions of trace and trace space are given in Definition 2.16 and Definition 2.18. Here and throughout this paper, for two function spaces $\mathbb{X}(\Omega)$ and $\mathbb{Y}(\Omega)$, that the trace spaces of $\mathbb{X}(\Omega)$ and $\mathbb{Y}(\Omega)$ are the same means that if the function space $\mathbb{Z}(\partial\Omega)$ is the trace space of $\mathbb{X}(\Omega)$, then it is also the trace space of $\mathbb{Y}(\Omega)$, and vice versa.

Corollary 1.5 is stronger than we expected; it says that we can obtain the existence of the trace and the trace space of $\mathrm{BV}(\Omega)$ by only knowing the existence of the trace and the trace space of $N^{1,1}(\Omega)$, which is nontrivial, since $N^{1,1}(\Omega)$ is a strict subset of $\mathrm{BV}(\Omega)$.

The so-called Hajłasz-Sobolev space $M^{1,p}(\Omega)$, $p \geq 1$, introduced in [12], is a subspace of $N^{1,p}(\Omega)$. For $p > 1$ and Ω supporting a $(1,p)$-Poincaré inequality and a doubling measure, we have $N^{1,p}(\Omega) = M^{1,p}(\Omega)$ with equivalent norms, see [13], and hence the traces of $M^{1,p}(\Omega)$ and $N^{1,p}(\Omega)$ will be the same. But for $p = 1$, even under these strong assumptions, $M^{1,1}(\Omega)$ is only a strict subspace of $N^{1,1}(\Omega)$ and it seems that trace results for $M^{1,1}$ are lacking in the literature. One can also define a local version $M^{1,1}_{c_H}(\Omega)$, see Section 2 and Remark 4.9 for more information. For these classes, we prove the following results.

Theorem 1.6. *Suppose Ω satisfies the measure density condition* (2.4)*. Then there exists $0 < c_H < 1$ such that for any $u \in N^{1,1}(\Omega)$, there is $v \in M^{1,1}_{c_H}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ satisfying $\|v\|_{M^{1,1}_{c_H}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}$ and*

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |v - u| \, d\mu = 0$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$, where $\mathcal{H}$ is the codimension 1 Hausdorff measure.

If additionally Ω is a uniform domain, then v can be chosen in $M^{1,1}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

With the exception of this theorem, our results are not written in terms of the codimension 1 Hausdorff measure $\mathcal{H}$ (defined in (2.2) and (2.3)) which is used in most existing literature. In Theorems 1.2–1.4, the results hold for every point on the boundary. On the other hand, the space or domain may be endowed with a measure μ for which the codimension 1 Hausdorff measure is not σ-finite on the boundary of the domain (see Example 5.7). More precisely, in Example 5.7 we define a weighted measure on the Euclidean half-space $\mathbb{R}^2_+$ whose codimension 1 Hausdorff measure is infinity for any open interval of $\partial\mathbb{R}^2_+ = \mathbb{R}$. But on $\mathbb{R}^2_+$, it is natural to study instead the trace with respect to the 1-dimensional Lebesgue measure on $\mathbb{R}$, which we do in Example 5.9. Another motivation for us is that in certain Dirichlet problems one needs to consider the trace with respect to a measure different from $\mathcal{H}$, see [22, Definition 4.1].

More generally, instead of only studying the codimension 1 Hausdorff measure, we may study any arbitrary boundary measure $\widetilde{\mathcal{H}}$ on $\partial\Omega$. In order to study such problems, we first replace the codimension 1 Hausdorff measure $\mathcal{H}$ with $\widetilde{\mathcal{H}}$ in the previous definition of trace to give the definition of trace with respect to $\widetilde{\mathcal{H}}$, see Definition 2.19. Then we prove the following result.

Theorem 1.7. *Suppose Ω satisfies the measure doubling condition (2.5). Let $\widetilde{\mathcal{H}}$ be any Radon measure on $\partial\Omega$. Suppose that for a given $u \in N^{1,1}(\Omega)$, there exists a function Tu such that*

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - Tu(x)|\, d\mu = 0$$

for $\widetilde{\mathcal{H}}$-a.e. $x \in \partial\Omega$. Then there exist $0 < c_H < 1$ and $v \in M^{1,1}_{c_H}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $\|v\|_{M^{1,1}_{c_H}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}$ and

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |v - Tu(x)|\, d\mu = 0$$

for $\widetilde{\mathcal{H}}$-a.e. $x \in \partial\Omega$.

If additionally Ω is a uniform domain, then v can be chosen in $M^{1,1}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

Similarly to Corollary 1.5, from Theorem 1.6 and Theorem 1.7 we obtain the following corollary.

Corollary 1.8. *Let $\Omega \subset X$ be a uniform domain and suppose that Ω satisfies the measure doubling condition (2.5). Then for any given boundary measure $\widetilde{\mathcal{H}}$, the trace spaces of $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$ with respect to any boundary measure $\widetilde{\mathcal{H}}$ on $\partial\Omega$ are the same.*

The paper is organized as follows. In Section 2, we give the necessary preliminaries. In Section 3, we study the traces of $N^{1,1}$ and BV and give the proofs of Theorems 1.2–1.4 and Corollary 1.5. In Section 4, we study the traces of $N^{1,1}$ and $M^{1,1}$ and give the proofs of Theorem 1.6, Theorem 1.7, and Corollary 1.8. Finally, in Section 5, apart from giving several examples that we refer to in Section 3 and Section 4, we also discuss some trace results and examples obtained as applications of Corollary 1.5 and Corollary 1.8.

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2 Preliminaries

In this section we introduce the notation, definitions, and assumptions used in the paper.

Throughout this paper, (X, d, μ) is a complete metric space that is equipped with a metric d and a Borel regular outer measure μ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x,r) := \{y \in X : d(y,x) < r\}$. By iterating the doubling condition, for every $0 < r \le R$ and $y \in B(x,R)$, we have

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge 4^{-s}\left(\frac{r}{R}\right)^s, \tag{2.1}$$

for any $s \ge \log_2 C_d$. See [13, Lemma 4.7] or [6] for a proof of this. We fix such an $s > 1$ and call it the *homogeneous dimension*.

The letters c, C (sometimes with a subscript) will denote positive constants that usually depend only on the space and may change at different occurrences; if C depends on $a, b, \ldots$, we write $C = C(a, b, \ldots)$. The notation $A \approx B$ means that there is a constant C such that $1/C \cdot A \le B \le C \cdot A$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a constant C such that $A \le C \cdot B$ ($A \ge C \cdot B$).

All functions defined on X or its subsets will take values in $[-\infty, \infty]$. A complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact. For an open set $\Omega \subset X$, a function is in the class $L^1_{\mathrm{loc}}(\Omega)$ if and only if it is in $L^1(\Omega')$ for every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω. Other local spaces of functions are defined similarly.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R(A) := \inf\left\{\sum_{j\in I}\frac{\mu(B(x_j,r_j))}{r_j} : A \subset \bigcup_{j\in I} B(x_j,r_j),\, r_j \le R,\, I \subset \mathbb{N}\right\}. \tag{2.2}$$

The codimension 1 Hausdorff measure of $A \subset X$ is then defined as

$$\mathcal{H}(A) := \lim_{R\to 0^+} \mathcal{H}_R(A). \tag{2.3}$$

Given an open set $\Omega \subset X$, we can regard it as a metric space in its own right, equipped with the metric induced by X and the measure $\mu|_\Omega$ which is the restriction of μ to subsets of Ω. This restricted measure $\mu|_\Omega$ is a Radon measure, see [20, Lemma 3.3.11].

We say that an open set Ω satisfies a *measure density condition* if there is a constant $c_m > 0$ such that

$$\mu(B(x,r) \cap \Omega) \ge c_m \mu(B(x,r)) \tag{2.4}$$

for every $x \in \overline{\Omega}$ and every $r \in (0, \operatorname{diam}(\Omega))$. We say that Ω satisfies a *measure doubling condition* if the measure $\mu|_\Omega$ is a doubling measure, i.e., there is a constant $c_d > 0$ such that

$$0 < \mu(B(x,2r) \cap \Omega) \le c_d \mu(B(x,r) \cap \Omega) < \infty \tag{2.5}$$

for every $x \in \overline{\Omega}$ and every $r > 0$. Notice that if Ω satisfies the measure density condition, then it satisfies the measure doubling condition.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into X. A nonnegative Borel function g on X is an upper gradient of a function u on X if for all nonconstant curves γ, we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds, \tag{2.6}$$

where x and y are the end points of γ and the curve integral is defined by using an arc-length parametrization, see [19, Section 2] where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite.

We say that a family of curves Γ is of zero 1-modulus if there is a nonnegative Borel function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho\, ds$ is infinite. A property is said to hold for 1-almost every curve if it fails only for a curve family with zero 1-modulus. If g is a nonnegative μ-measurable function on X and (2.6) holds for 1-almost every curve, we say that g is a 1-weak upper gradient of u. By only considering curves γ in $A \subset X$, we can talk about a function g being a (1-weak) upper gradient of u in A.

Given a μ-measurable set $H \subset X$, we let

$$\|u\|_{N^{1,1}(H)} := \|u\|_{L^1(H)} + \inf \|g\|_{L^1(H)},$$

where the infimum is taken over all 1-weak upper gradients g of u in H. The substitute for the Sobolev space $W^{1,1}$ in the metric setting is the Newton-Sobolev space

$$N^{1,1}(H) := \{u : \|u\|_{N^{1,1}(H)} < \infty\},$$

which was first introduced in [35]. It is known that for any $u \in N^{1,1}_{\mathrm{loc}}(H)$ there exists a minimal 1-weak upper gradient of u in H, always denoted by g_u, satisfying $g_u \le g$ μ-a.e. in H, for any 1-weak upper gradient $g \in L^1_{\mathrm{loc}}(H)$ of u in H, see [6, Theorem 2.25].

Next we present the basic theory of functions of bounded variation on metric spaces. This was first developed in [1, 32]; see also the monographs [3, 9, 10, 11, 42] for the classical theory in Euclidean spaces. We will always denote by Ω an open subset of X. Given a function $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of u in Ω by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu :\ u_i \in N^{1,1}_{\mathrm{loc}}(\Omega), u_i \to u \text{in } L^1_{\mathrm{loc}}(\Omega) \right\}, \tag{2.7}$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i in Ω. (In [32], local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in \mathrm{BV}(\Omega)$, if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf\{\|Du\|(W) :\ A \subset W,\ W \subset X \text{is open}\}.$$

Proposition 2.8 ([32, Theorem 3.4]). *If $u \in L^1_{\mathrm{loc}}(\Omega)$, then $\|Du\|(\cdot)$ is a Borel measure on Ω.*

For any $u, v \in L^1_{\mathrm{loc}}(\Omega)$, it is straightforward to show that

$$\|D(u+v)\|(\Omega) \le \|Du\|(\Omega) + \|Dv\|(\Omega). \tag{2.9}$$

The BV norm is defined by

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

We will assume throughout the paper that X supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_P r \fint_{B(x,\lambda r)} g\, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u\, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u\, d\mu.$$

Recall the exponent $s > 1$ from (2.1). The $(1,1)$-Poincaré inequality implies the so-called Sobolev-Poincaré inequality, see e.g. [6, Theorem 4.21], and by applying the latter to approximating locally Lipschitz functions in the definition of the total variation, we get the following Sobolev-Poincaré inequality for BV functions. For every ball $B(x,r)$ and every $u \in L^1_{\mathrm{loc}}(X)$, we have

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^{s/(s-1)}\, d\mu \right)^{(s-1)/s} \leq C_{SP} r \frac{\|Du\|(B(x,2\lambda r))}{\mu(B(x,2\lambda r))}, \tag{2.10}$$

where $C_{SP} = C_{SP}(C_d, C_P, \lambda) \geq 1$ is a constant.

For an open set $\Omega \subset X$ and a μ-measurable set $E \subset X$ with $\|D\chi_E\|(\Omega) < \infty$, we know that for any Borel set $A \subset \Omega$,

$$\|D\chi_E\|(A) = \int_{\partial^* E \cap A} \theta_E\, d\mathcal{H}, \tag{2.11}$$

where $\theta_E \colon X \to [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [4, Theorem 4.6]. The following coarea formula is given in [32, Proposition 4.2]: if $\Omega \subset X$ is an open set and $u \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega)\, dt. \tag{2.12}$$

The lower and upper approximate limits of a function u on Ω are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(\{u < t\} \cap B(x,r))}{\mu(B(x,r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(\{u > t\} \cap B(x,r))}{\mu(B(x,r))} = 0 \right\}.$$

Then the jump set S_u is defined as the set of points $x \in \Omega$ for which $u^\wedge(x) < u^\vee(x)$. It is straightforward to check that $u^\wedge$ and $u^\vee$ are Borel functions.

By [4, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set $\Omega \subset X$ and $u \in \mathrm{BV}(\Omega)$, we have for any Borel set $A \subset \Omega$

$$\begin{aligned} \|Du\|(A) &= \|Du\|^a(A) + \|Du\|^s(A) \\ &= \|Du\|^a(A) + \|Du\|^c(A) + \|Du\|^j(A) \\ &= \int_A a\, d\mu + \|Du\|^c(A) + \int_{A\cap S_u} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\, dt\, d\mathcal{H}(x), \end{aligned} \tag{2.13}$$

where $a \in L^1(\Omega)$ is the density of the absolutely continuous part $\|Du\|^a(A)$ of $\|Du\|(A)$ and the functions $\theta_{\{u>t\}} \in [\alpha, C_d]$ are as in (2.11).

Next, we introduce the Hajłasz-Sobolev space. Let $0 < p < \infty$. Given a μ-measurable set $K \subset X$, we define $M^{1,p}(K)$ to be the set of all functions $u \in L^p(K)$ for which there exists $0 \le g \in L^p(K)$ and a set $A \subset K$ of measure zero such that for all $x, y \in K \setminus A$ we have the estimate

$$|u(x) - u(y)| \le d(x,y)(g(x) + g(y)). \tag{2.14}$$

The corresponding norm (when $p \ge 1$) is obtained by setting

$$\|u\|_{M^{1,p}(K)} = \|u\|_{L^p(K)} + \inf \|g\|_{L^p(K)},$$

where the infimum is taken over all admissible functions g in (2.14). We refer to [12, 13] for more properties of the Hajłasz-Sobolev space $M^{1,p}$. The space $M^{1,p}_{cH}(K)$ is defined exactly in the same manner as the space $M^{1,p}(K)$ except for one difference: in the definition of $M^{1,p}_{cH}(K)$, the condition (2.14) is assumed to hold only for points $x, y \in K \setminus A$ that satisfy the condition

$$d(x,y) \le c_H \cdot \min\{d(x, X \setminus K), d(y, X \setminus K)\}, \tag{2.15}$$

where $0 < c_H < 1$ is a constant.

We give the following definitions for the boundary trace, or trace for short, of a function defined on an open set Ω.

Definition 2.16. Let $\Omega \subset X$ be an open set and let u be a μ-measurable function on Ω. A number $Tu(x)$ is the trace of u at $x \in \partial\Omega$ if we have

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - Tu(x)|\, d\mu = 0. \tag{2.17}$$

We say that u has a trace Tu in $\partial\Omega$ if $Tu(x)$ exists for $\mathcal{H}$-almost every $x \in \partial\Omega$.

Moreover, we give the following definitions for the trace space of a function space defined on an open set Ω.

Definition 2.18. Let Ω be an open set and let $\mathbb{X}(\Omega)$ be a function space on Ω. A function space $\mathbb{Y}(\partial\Omega, \mathcal{H})$ on $\partial\Omega$ is the trace space of $\mathbb{X}(\Omega)$ if the trace operator $u \mapsto Tu$ defined in Definition 2.16 is a bounded linear surjective operator from $\mathbb{X}(\Omega)$ to $\mathbb{Y}(\partial\Omega, \mathcal{H})$.

Definition 2.19. Let Ω be an open set and $\widetilde{\mathcal{H}}$ be a measure on $\partial\Omega$. Let $\mathbb{X}(\Omega)$ be a function function space on Ω. A function space $\mathbb{Y}(\partial\Omega, \widetilde{\mathcal{H}})$ on $\partial\Omega$ is the trace space of $\mathbb{X}(\Omega)$ with respect to $\widetilde{\mathcal{H}}$, if the trace operator $u \mapsto Tu$ defined in Definition 2.16 by replacing $\mathcal{H}$ by $\widetilde{\mathcal{H}}$ is a bounded linear surjective operator from $\mathbb{X}(\Omega)$ to $\mathbb{Y}(\partial\Omega, \widetilde{\mathcal{H}})$.

3 Traces of $N^{1,1}(\Omega)$ and $\mathrm{BV}(\Omega)$

In this section, let $\Omega \subset X$ be an arbitrary nonempty open set. Recall the definition of the number $s > 1$ from (2.1).

Lemma 3.1. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then there exists a sequence $(u_i) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $u_i \to u$ in $L^{s/(s-1)}_{\mathrm{loc}}(\Omega)$ and*

$$\|Du\|(\Omega) = \lim_{i\to\infty} \int_\Omega g_{u_i}\, d\mu.$$

Proof. By the Sobolev-Poincaré inequality (2.10), we have $u \in L^{s/(s-1)}_{\mathrm{loc}}(\Omega)$. Take open sets $\Omega_1 \Subset \Omega_2 \Subset \ldots \Subset \Omega = \bigcup_{j=1}^\infty \Omega_j$. Now $u \in L^{s/(s-1)}(\Omega_j)$ for each $j \in \mathbb{N}$. Define the truncations

$$u_M := \min\{M, \max\{-M, u\}\}, \quad M > 0.$$

For each $j \in \mathbb{N}$ we find a number $M_j > 0$ such that $\|u_{M_j} - u\|_{L^{s/(s-1)}(\Omega_j)} < 1/j$. From the definition of the total variation, take a sequence $(v_i) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $v_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$\|Du\|(\Omega) = \lim_{i\to\infty} \int_\Omega g_{v_i}\, d\mu.$$

Then also $(v_i)_{M_j} \to u_{M_j}$ in $L^{s/(s-1)}(\Omega_j)$ for all $j \in \mathbb{N}$. Thus we can pick indices $i(j) \ge j$ such that $\|(v_{i(j)})_{M_j} - u_{M_j}\|_{L^{s/(s-1)}(\Omega_j)} < 1/j$ for each $j \in \mathbb{N}$. Defining $u_j := (v_{i(j)})_{M_j}$, we now have

$$\|u_j - u\|_{L^{s/(s-1)}(\Omega_j)} < 2/j \quad \text{for all } j \in \mathbb{N}$$

and so $u_j \to u$ in $L^{s/(s-1)}_{\mathrm{loc}}(\Omega)$. Moreover, since truncation does not increase energy,

$$\limsup_{j\to\infty} \int_\Omega g_{u_j}\, d\mu \le \|Du\|(\Omega).$$

But by lower semicontinuity, also $\|Du\|(\Omega) \le \liminf_{j\to\infty} \int_\Omega g_{u_j}\, d\mu$. □

We have the following standard fact; for a proof see e.g. [16, Proposition 3.8].

Lemma 3.2. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$ and let $(u_i) \subset N^{1,1}_{\mathrm{loc}}(\Omega)$ with $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and*

$$\|Du\|(\Omega) = \lim_{i\to\infty} \int_\Omega g_{u_i}\, d\mu.$$

Then we also have the weak convergence $g_{u_i}\, d\mu \overset{*}{\rightharpoonup} d\|Du\|$.*

Lemma 3.3. *Let $\Omega_1 \Subset \Omega_2 \Subset \ldots \Subset \bigcup_{j=1}^\infty \Omega_j = \Omega$ be open sets, let $\Omega_0 := \emptyset$, and let $\eta_j \in \mathrm{Lip}_c(\Omega_j)$ such that $0 \le \eta_j \le 1$ on X and $\eta_j = 1$ in Ω_{j-1} for each $j \in \mathbb{N}$, with $\eta_1 \equiv 0$. Let $1 \le q < \infty$. Moreover, let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$, and for each $j \in \mathbb{N}$ let $(u_{j,i}) \subset N^{1,1}(\Omega_j)$ such that $u_{j,i} - u \to 0$ in $L^q(\Omega_j)$ and*

$$\lim_{i\to\infty} \int_{\Omega_j} g_{u_{j,i}}\, d\mu = \|Du\|(\Omega_j),$$

where each $g_{u_{j,i}}$ is the minimal 1-weak upper gradient of $u_{j,i}$ in Ω_j. Finally, let $\delta_j > 0$ for each $j \in \mathbb{N}$, and let $\varepsilon > 0$. Then for each $j \in \mathbb{N}$ we find an index $i(j)$ such that letting $u_j := u_{j,i(j)}$ and

$$v := \sum_{j=2}^{\infty} (\eta_j - \eta_{j-1}) u_j,$$

we have

$$\max\{\|v - u\|_{L^1(\Omega_j \setminus \Omega_{j-1})}, \|v - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})}\} < \delta_j \quad \textit{for all } j \in \mathbb{N},$$

and $\int_\Omega g_v \, d\mu < \|Du\|(\Omega) + \varepsilon$.

Note that neither u nor the functions $u_{j,i}$ need to be in $L^q(\Omega_j)$, only in $L^1(\Omega_j)$, but still we can have $u_{j,i} - u \to 0$ in $L^q(\Omega_j)$ for each $j \in \mathbb{N}$. We can also see that in $\Omega_j \setminus \Omega_{j-1}$, the function v can be written as the finite sum (let $\eta_0 \equiv 0$)

$$\sum_{i=2}^{\infty} (\eta_i - \eta_{i-1}) u_i = (\eta_j - \eta_{j-1}) u_j + (\eta_{j+1} - \eta_j) u_{j+1} = \eta_j u_j + (1 - \eta_j) u_{j+1}. \tag{3.4}$$

Proof. By Lemma 3.2, for each $j \in \mathbb{N}$ we have $g_{u_{j,i}} \, d\mu \overset{*}{\rightharpoonup} d\|Du\|$ as $i \to \infty$ in Ω_j. For each $j \in \mathbb{N}$, let $L_j > 0$ denote a Lipschitz constant of η_j; we can take this to be an increasing sequence. Set $\delta_0 := 1$, $L_0 := 1$. Letting $u_j := u_{j,i(j)}$ for suitable indices $i(j) \in \mathbb{N}$, we get

$$\max\{\|u_j - u\|_{L^1(\Omega_j)}, \|u_j - u\|_{L^q(\Omega_j)}\} < \min\{\delta_{j-1}, \delta_j, 2^{-j-1}\varepsilon / L_j\}/2 \tag{3.5}$$

for all $j \in \mathbb{N}$, and

$$\int_{\Omega_j} (\eta_j - \eta_{j-1}) g_{u_j} \, d\mu < \int_{\Omega_j} (\eta_j - \eta_{j-1}) \, d\|Du\| + 2^{-j}\varepsilon \tag{3.6}$$

for all $j = 2, 3, \ldots$. We get for all $j \in \mathbb{N}$

$$\begin{aligned}
\|v - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} &= \Big\| \sum_{i=2}^{\infty} (\eta_i - \eta_{i-1}) u_i - u \Big\|_{L^q(\Omega_j \setminus \Omega_{j-1})} \\
&\overset{(3.4)}{=} \|\eta_j u_j + (1 - \eta_j) u_{j+1} - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} \\
&= \|\eta_j u_j + (1 - \eta_j) u_{j+1} - \eta_j u - (1 - \eta_j) u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} \\
&\le \|u_j - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} + \|u_{j+1} - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} \\
&< \delta_j
\end{aligned}$$

by (3.5) as desired, and similarly for the L^1-norm. Let $v_2 := u_2$ in Ω_2, and recursively $v_{i+1} := \eta_i v_i + (1 - \eta_i) u_{i+1}$ in Ω_{i+1}. We see that $v = \lim_{i \to \infty} v_i$ (at every point in Ω). By the proof of the Leibniz rule in [6, Lemma 2.18], the minimal 1-weak upper gradient of v_3 in Ω_3 satisfies

$$g_{v_3} \le g_{\eta_2} |u_2 - u_3| + \eta_2 g_{u_2} + (1 - \eta_2) g_{u_3}.$$

Inductively, we get for $i = 3, 4, \ldots$

$$g_{v_i} \le \sum_{j=2}^{i-1} g_{\eta_j} |u_j - u_{j+1}| + \sum_{j=2}^{i-1} (\eta_j - \eta_{j-1}) g_{u_j} + (1 - \eta_{i-1}) g_{u_i} \quad \text{in } \Omega_i;$$

to prove this, assume that it holds for the index i. Then we have by applying a Leibniz rule as above, and noting that g_{η_i} can be nonzero only in $\Omega_i \setminus \Omega_{i-1}$ (see [6, Corollary 2.21]), where $v_i = u_i$,

$$\begin{aligned} g_{v_{i+1}} &\le g_{\eta_i}|v_i - u_{i+1}| + \eta_i g_{v_i} + (1-\eta_i)g_{u_{i+1}} \\ &= g_{\eta_i}|u_i - u_{i+1}| + \eta_i g_{v_i} + (1-\eta_i)g_{u_{i+1}} \\ &\overset{\text{Induction}}{\le} g_{\eta_i}|u_i - u_{i+1}| + \sum_{j=2}^{i-1} g_{\eta_j}|u_j - u_{j+1}| \\ &\quad + \sum_{j=2}^{i-1}(\eta_j - \eta_{j-1})g_{u_j} + (\eta_i - \eta_{i-1})g_{u_i} + (1-\eta_i)g_{u_{i+1}} \\ &= \sum_{j=2}^{i} g_{\eta_j}|u_j - u_{j+1}| + \sum_{j=2}^{i}(\eta_j - \eta_{j-1})g_{u_j} + (1-\eta_i)g_{u_{i+1}} \quad \text{in } \Omega_{i+1}. \end{aligned}$$

This completes the induction. In each $\Omega_j \setminus \Omega_{j-1}$, by (3.4) we have

$$v = \eta_j u_j + (1-\eta_j)u_{j+1} = \eta_j v_j + (1-\eta_j)u_{j+1} = v_{j+1},$$

and so in fact $v = v_{j+1}$ in Ω_j, for each $j \in \mathbb{N}$. Thus the minimal 1-weak upper gradient of v in Ω_i satisfies

$$g_v = g_{v_{i+1}} \le \sum_{j=2}^{\infty} g_{\eta_j}|u_j - u_{j+1}| + \sum_{j=2}^{\infty}(\eta_j - \eta_{j-1})g_{u_j}.$$

Thus

$$\begin{aligned} \int_{\Omega_i} g_v \, d\mu &\le \sum_{j=2}^{\infty} \int_{\Omega_j} g_{\eta_j}|u_j - u_{j+1}| \, d\mu + \sum_{j=2}^{\infty} \int_{\Omega_j} (\eta_j - \eta_{j-1}) g_{u_j} \, d\mu \\ &\le \sum_{j=2}^{\infty} L_j \|u_j - u_{j+1}\|_{L^1(\Omega_j \setminus \Omega_{j-1})} + \sum_{j=2}^{\infty} \left(\int_{\Omega_j} (\eta_j - \eta_{j-1}) \, d\|Du\| + 2^{-j}\varepsilon \right) \quad \text{by (3.6)} \\ &\le \varepsilon/2 + \|Du\|(\Omega) + \varepsilon/2 \quad \text{by (3.5), (3.6)} \\ &= \|Du\|(\Omega) + \varepsilon. \end{aligned}$$

Note that g_v does not depend on i, see [6, Lemma 2.23], and so it is well defined on Ω. Since g_v is the minimal 1-weak upper gradient of v in each Ω_i, it is clearly also (the minimal) 1-weak upper gradient of v in Ω. Then by Lebesgue's monotone convergence theorem,

$$\int_{\Omega} g_v \, d\mu \le \|Du\|(\Omega) + \varepsilon.$$

□

Theorem 1.2 of the introduction follows from the following theorem.

Theorem 3.7. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$ and let $\varepsilon > 0$. Then there exists $v \in N^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $\|v-u\|_{L^1(\Omega)} < \varepsilon$, $\|v-u\|_{L^{s/(s-1)}(\Omega)} < \varepsilon$, $\int_\Omega g_v\,d\mu < \|Du\|(\Omega) + \varepsilon$, and*

$$\fint_{B(x,r)\cap\Omega} |v-u|^{s/(s-1)}\,d\mu \to 0 \quad \text{as } r \to 0^+$$

uniformly for all $x \in \partial\Omega$.

Note that if $u \in \mathrm{BV}(\Omega)$ as in the formulation of Theorem 1.2, then $v \in L^1(\Omega)$ and so $v \in N^{1,1}(\Omega)$.

Proof. Fix $x_0 \in X$. Define $\Omega_0 := \emptyset$ and pick numbers $d_j \in (2^{-j}, 2^{-j+1})$, $j \in \mathbb{N}$, such that the sets

$$\Omega_j := \{x \in \Omega :\ d(x, X \setminus \Omega) > d_j\} \cap B(x_0, d_j^{-1})$$

satisfy $\|Du\|(\partial\Omega_j) = 0$. For each $j \in \mathbb{N}$, take $\eta_j \in \mathrm{Lip}_c(\Omega_j)$ such that $0 \le \eta_j \le 1$ on X and $\eta_j = 1$ in Ω_{j-1}, and $\eta_1 \equiv 0$. Note that for a fixed $r > 0$, the function

$$x \mapsto \mu(B(x,r)\cap\Omega), \quad x \in \partial\Omega,$$

is lower semicontinuous and strictly positive. Since $\partial\Omega \cap \overline{B}(x_0, d_j^{-1})$ is compact for every $j \in \mathbb{N}$, the numbers

$$\beta_j := \inf\{\mu(B(x, 2^{-j})\cap\Omega) :\ x \in \partial\Omega \cap \overline{B}(x_0, d_{j+2}^{-1})\}, \quad j \in \mathbb{N},$$

are strictly positive. Set

$$\delta_j := 2^{-j}\min\left\{\varepsilon, \beta_j^{s/(s-1)}\right\}.$$

By Lemma 3.1 we find functions $(u_i) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $u_i \to u$ in $L^{s/(s-1)}_{\mathrm{loc}}(\Omega)$ and

$$\lim_{i\to\infty} \int_\Omega g_{u_i}\,d\mu = \|Du\|(\Omega).$$

Then also $u_i \to u$ in $L^{s/(s-1)}(\Omega_j)$ for every $j \in \mathbb{N}$, and by Lemma 3.2 and the fact that $\|Du\|(\partial\Omega_j) = 0$ we get

$$\lim_{i\to\infty} \int_{\Omega_j} g_{u_i}\,d\mu = \|Du\|(\Omega_j).$$

Then apply Lemma 3.3 to obtain a function $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. By the lemma, we have $\int_\Omega g_v\,d\mu < \|Du\|(\Omega) + \varepsilon$ as desired, and from the condition

$$\max\{\|v-u\|_{L^1(\Omega_j\setminus\Omega_{j-1})}, \|v-u\|_{L^{s/(s-1)}(\Omega_j\setminus\Omega_{j-1})}\} < \delta_j \le 2^{-j}\varepsilon \quad \text{for all } j \in \mathbb{N}$$

we easily get $\|v-u\|_{L^1(\Omega)} < \varepsilon$ and $\|v-u\|_{L^{s/(s-1)}(\Omega)} < \varepsilon$. In particular, $v \in N^{1,1}_{\mathrm{loc}}(\Omega)$ as desired.

Fix $x \in \partial\Omega$. Choose the smallest $l \in \mathbb{N}$ such that $x \in B(x_0, d_{l+2}^{-1})$. Note that then $B(x,1) \cap B(x_0, d_{l-1}^{-1}) = \emptyset$ (if $l \ge 2$) and so for any $k \in \mathbb{N}$,

$$B(x, 2^{-k+1}) \cap \Omega = B(x, 2^{-k+1}) \cap \left(\bigcup_{j=\max\{k,l\}}^{\infty} (\Omega_j \setminus \Omega_{j-1}) \right).$$

Now

$$
\begin{aligned}
&\frac{1}{\mu(B(x,2^{-k})\cap\Omega)}\int_{B(x,2^{-k+1})\cap\Omega}|v-u|^{s/(s-1)}\,d\mu\\
&\qquad=\frac{1}{\mu(B(x,2^{-k})\cap\Omega)}\sum_{j=\max\{k,l\}}^{\infty}\int_{B(x,2^{-k+1})\cap\Omega_j\setminus\Omega_{j-1}}|v-u|^{s/(s-1)}\,d\mu\\
&\qquad\le\frac{1}{\mu(B(x,2^{-k})\cap\Omega)}\sum_{j=\max\{k,l\}}^{\infty}\int_{\Omega_j\setminus\Omega_{j-1}}|v-u|^{s/(s-1)}\,d\mu\\
&\qquad\le\frac{1}{\mu(B(x,2^{-k})\cap\Omega)}\sum_{j=\max\{k,l\}}^{\infty}\delta_j^{(s-1)/s}\\
&\qquad\le\sum_{j=\max\{k,l\}}^{\infty}\frac{2^{-j}\beta_j}{\mu(B(x,2^{-j})\cap\Omega)}\\
&\qquad\le\sum_{j=\max\{k,l\}}^{\infty}2^{-j}\le 2^{-k+1}.
\end{aligned}
$$

Now it clearly follows that

$$
\fint_{B(x,r)\cap\Omega}|v-u|^{s/(s-1)}\,d\mu\to 0\quad\text{as } r\to 0^+
$$

uniformly for all $x\in\partial\Omega$. □

We have the following approximation result for BV functions in the L^q-norm.

Theorem 3.8. *Let $u\in L^1_{\mathrm{loc}}(\Omega)$ with $\|Du\|(\Omega)<\infty$ and let $1\le q<\infty$. Then there exists a sequence $(u_i)\subset N^{1,1}_{\mathrm{loc}}(\Omega)$ such that $u_i-u\to 0$ in $L^1(\Omega)\cap L^q(\Omega)$ and*

$$
\int_\Omega g_{u_i}\,d\mu\to\|Du\|(\Omega).
$$

Proof. For each $k=0,1,\ldots$ define the truncation of u at levels k and $k+1$

$$
u_k:=\min\{1,(u-k)_+\}.
$$

Then $u_k\in L^1_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$ for each $k=0,1,\ldots$ and $u_+=\sum_{k=0}^{\infty}u_k$. Also note that by the coarea formula (2.12),

$$
\|Du_k\|(\Omega)=\int_{-\infty}^{\infty}P(\{u_k>t\},\Omega)\,dt=\int_k^{k+1}P(\{u>t\},\Omega)\,dt.
$$

For each $k=0,1,\ldots$, from the definition of the total variation we get a sequence $(v_{k,i})\subset N^{1,1}_{\mathrm{loc}}(\Omega)$ with $v_{k,i}\to u_k$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$
\int_\Omega g_{v_{k,i}}\,d\mu\to\|Du_k\|(\Omega)\quad\text{as } i\to\infty.
$$

In the proof of Theorem 3.7 we saw that in fact we can get $v_{k,i} - u_k \to 0$ in $L^1(\Omega)$. Since $0 \le u_k \le 1$, by truncation we can assume that also $0 \le v_{k,i} \le 1$. Then also $v_{k,i} - u_k \to 0$ in $L^q(\Omega)$. Let $\varepsilon > 0$. For a suitable choice of indices $i = i(k)$, for $v_k := v_{k,i(k)}$ we have $\|v_k - u_k\|_{L^1(\Omega)} < 2^{-k-2}\varepsilon$, $\|v_k - u_k\|_{L^q(\Omega)} < 2^{-k-2}\varepsilon$, and

$$\int_\Omega g_{v_k}\, d\mu < \|Du_k\|(\Omega) + 2^{-k-1}\varepsilon = \int_k^{k+1} P(\{u > t\}, \Omega)\, dt + 2^{-k-1}\varepsilon.$$

Then for $v := \sum_{k=0}^\infty v_k$ we have $\|v - u_+\|_{L^1(\Omega)} < \varepsilon/2$ and $\|v - u_+\|_{L^q(\Omega)} < \varepsilon/2$. Moreover, using e.g. [6, Lemma 1.52] we get $g_v \le \sum_{k=0}^\infty g_{v_k}$ and then

$$\begin{aligned}\int_\Omega g_v\, d\mu \le \sum_{k=0}^\infty \int_\Omega g_{v_k}\, d\mu &\le \sum_{k=0}^\infty \left(\int_k^{k+1} P(\{u > t\}, \Omega)\, dt + 2^{-k-1}\varepsilon\right)\\ &= \int_0^\infty P(\{u > t\}, \Omega)\, dt + \varepsilon/2\\ &= \|Du_+\|(\Omega) + \varepsilon/2\end{aligned}$$

again by the coarea formula. Similarly we find a function $w \in N^{1,1}_{\mathrm{loc}}(\Omega)$ with $\|w - u_-\|_{L^1(\Omega)} < \varepsilon/2$, $\|w - u_-\|_{L^q(\Omega)} < \varepsilon/2$, and $\int_\Omega g_w\, d\mu < \|Du_-\|(\Omega) + \varepsilon/2$. Then for $h := v - w$ we have $\|h - u\|_{L^1(\Omega)} < \varepsilon$, $\|h - u\|_{L^q(\Omega)} < \varepsilon$, and

$$\int_\Omega g_h\, d\mu < \|Du_+\|(\Omega) + \varepsilon/2 + \|Du_-\|(\Omega) + \varepsilon/2 = \|Du\|(\Omega) + \varepsilon$$

using the coarea formula once more. In this way we get the desired sequence. □

Theorem 1.3 of the introduction follows from the following theorem. In Example 5.1 we will show that here we cannot take u to be continuous or even locally bounded in Ω.

Theorem 3.9. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$, let $1 \le q < \infty$, and let $\varepsilon > 0$. Then there exists $v \in N^{1,1}_{\mathrm{loc}}(\Omega)$ such that $\|v - u\|_{L^1(\Omega)} < \varepsilon$, $\|v - u\|_{L^q(\Omega)} < \varepsilon$, $\int_\Omega g_v\, d\mu < \|Du\|(\Omega) + \varepsilon$, and*

$$\fint_{B(x,r)\cap\Omega} |v - u|^q\, d\mu \to 0 \quad \text{as } r \to 0^+$$

uniformly for all $x \in \partial\Omega$.

Proof. The proof is essentially the same as for Theorem 3.7; the difference is that here we apply Theorem 3.8 to find sequences $(u_{j,i})_i \subset N^{1,1}(\Omega_j)$, $j \in \mathbb{N}$, such that $\|u_{j,i} - u\|_{L^q(\Omega_j)} \to 0$ and $\lim_{i\to\infty} \int_{\Omega_j} g_{u_{j,i}}\, d\mu = \|Du\|(\Omega_j)$ as $i \to \infty$. □

We say that $w \in \mathrm{SBV}(\Omega)$ if $w \in \mathrm{BV}(\Omega)$ and $\|Dw\|^c(\Omega) = 0$ (recall the decomposition (2.13)). Recall also that the jump set S_u is the set of points $x \in \Omega$ for which $u^\wedge(x) < u^\vee(x)$. Denote $\Omega(r) := \{x \in \Omega :\, \mathrm{dist}(x, X \setminus \Omega) > r\}$. We have the following approximation result for BV functions by SBV functions.

Theorem 3.10. *Let $u \in \mathrm{BV}(\Omega)$ and let $\varepsilon > 0$. Then there exists $w \in \mathrm{SBV}(\Omega)$ such that $\|w - u\|_{L^1(\Omega)} < \varepsilon$, $\|w - u\|_{L^\infty(\Omega)} < \varepsilon$, $\|Dw\|(\Omega) < \|Du\|(\Omega) + \varepsilon$, $\mathcal{H}(S_w \setminus S_u) = 0$, and*

$$\lim_{r\to 0^+} \|w - u\|_{L^\infty(\Omega\setminus\Omega(r))} = 0.$$

Proof. This is given in [26, Corollary 5.15]; for the above limit see [26, Eq. (3.7), (3.10)]. □

The following approximation result for BV functions by means of functions with a jump set of finite Hausdorff measure is given as part of [27, Theorem 5.3].

Theorem 3.11. *Let $u \in \mathrm{BV}(\Omega)$ and let $\varepsilon, \delta > 0$. Then we find $w \in \mathrm{BV}(\Omega)$ such that $\|w - u\|_{L^1(\Omega)} < \varepsilon$,*

$$\|D(w-u)\|(\Omega) \le 2\|Du\|(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon,$$

$\|w - u\|_{L^\infty(\Omega)} \le 10\delta$, and $\mathcal{H}(S_w \setminus \{u^\vee - u^\wedge \ge \delta\}) = 0$.

We apply this theorem first to obtain the following proposition.

Proposition 3.12. *Let $u \in \mathrm{BV}(\Omega)$ and let $\varepsilon > 0$. Then we find $v \in \mathrm{BV}(\Omega)$ such that $\|v - u\|_{\mathrm{BV}(\Omega)} < \varepsilon$, $\|v - u\|_{L^\infty(\Omega)} < \varepsilon$, $\mathcal{H}(S_v) < \infty$, and*

$$\lim_{r \to 0^+} \|v - u\|_{L^\infty(\Omega \setminus \Omega(r))} = 0.$$

Proof. Take numbers $\delta_j \searrow 0$, $0 < \delta_j < \varepsilon/20$, such that

$$\sum_{j=2}^{\infty} \|Du\|(\{0 < u^\vee - u^\wedge < \delta_j\}) < \frac{\varepsilon}{4}. \tag{3.13}$$

Note that by the decomposition (2.13), $\mathcal{H}(\{u^\vee - u^\wedge > t\}) < \infty$ for all $t > 0$. Thus we can take a strictly decreasing sequence of $r_j \searrow 0$ so that the sets $\Omega_j := \Omega(r_j)$ satisfy (let $\Omega_0 := \emptyset$)

$$\mathcal{H}((\Omega_j \setminus \Omega_{j-2}) \cap \{u^\vee - u^\wedge \ge \delta_j\}) \le \mathcal{H}((\Omega \setminus \Omega_{j-2}) \cap \{u^\vee - u^\wedge \ge \delta_j\}) < 2^{-j}\varepsilon$$

for all $j = 3, 4, \ldots$. Then

$$\sum_{j=2}^{\infty} \mathcal{H}((\Omega_j \setminus \Omega_{j-2}) \cap \{u^\vee - u^\wedge \ge \delta_j\}) < \mathcal{H}(\{u^\vee - u^\wedge \ge \delta_2\}) + \varepsilon. \tag{3.14}$$

Also choose functions $\eta_j \in \mathrm{Lip}(X)$ supported in Ω_j, $j \in \mathbb{N}$, such that $0 \le \eta_j \le 1$ on X and $\eta_j = 1$ in Ω_{j-1}, with $\eta_1 \equiv 0$. For each $j \in \mathbb{N}$, apply Theorem 3.11 to find a function $v_j \in \mathrm{BV}(\Omega)$ satisfying

$$\max\{\|g_{\eta_j} + g_{\eta_{j-1}}\|_{L^\infty(\Omega)}, 1\} \cdot \|v_j - u\|_{L^1(\Omega)} < 2^{-j-1}\varepsilon \tag{3.15}$$

as well as

$$\|D(v_j - u)\|(\Omega) \le 2\|Du\|(\{0 < u^\vee - u^\wedge < \delta_j\}) + 2^{-j-1}\varepsilon, \tag{3.16}$$

$\|v_j - u\|_{L^\infty(\Omega)} \le 10\delta_j$, and $\mathcal{H}(S_{v_j} \setminus \{u^\vee - u^\wedge \ge \delta_j\}) = 0$. Let

$$v := \sum_{j=2}^{\infty} (\eta_j - \eta_{j-1}) v_j. \tag{3.17}$$

Then

$$\|v-u\|_{L^1(\Omega)} = \|\sum_{j=2}^{\infty}(\eta_j-\eta_{j-1})(v_j-u)\|_{L^1(\Omega)} \le \sum_{j=2}^{\infty}\|v_j-u\|_{L^1(\Omega)} \le \sum_{j=2}^{\infty}2^{-j-1}\varepsilon = \varepsilon/4.$$

Since $\|v_j-u\|_{L^\infty(\Omega)} \le 10\delta_j < \varepsilon/2$, also $\|v-u\|_{L^\infty(\Omega)} < \varepsilon$. It is also easy to check that $\lim_{r\to 0^+}\|v-u\|_{L^\infty(\Omega\setminus\Omega(r))} = 0$.

Clearly $\sum_{j=2}^{k}(\eta_j-\eta_{j-1})(v_j-u) \to v-u$ in $L^1_{\mathrm{loc}}(\Omega)$ as $k\to\infty$. Thus by lower semicontinuity and a Leibniz rule (see [17, Lemma 3.2]),

$$\begin{aligned}
\|D(v-u)\|(\Omega) &\le \liminf_{k\to\infty}\left\|D\sum_{j=2}^{k}(\eta_j-\eta_{j-1})(v_j-u)\right\|(\Omega)\\
&\le \sum_{j=2}^{\infty}\|D((\eta_j-\eta_{j-1})(v_j-u))\|(\Omega) \quad \text{by (2.9)}\\
&\le \sum_{j=2}^{\infty}\left(\|D(v_j-u)\|(\Omega)+\int_\Omega (g_{\eta_j}+g_{\eta_{j-1}})|v_j-u|\,d\mu\right)\\
&< \sum_{j=2}^{\infty}\left(2\|Du\|(\{0<u^\vee-u^\wedge<\delta_j\})+2^{-j-1}\varepsilon\right)+\sum_{j=2}^{\infty}2^{-j-1}\varepsilon \quad \text{by (3.16), (3.15)}\\
&< \varepsilon/2+\varepsilon/4+\varepsilon/4 \quad \text{by (3.13)}\\
&= \varepsilon.
\end{aligned}$$

Finally we want to show that $\mathcal{H}(S_v)<\infty$. Note that (3.17) is a locally finite sum. If $x\in S_{(\eta_j-\eta_{j-1})v_j}$, then we get $x\in S_{v_j}$, and so $S_v\subset\bigcup_{j=2}^{\infty}\left(S_{v_j}\cap(\Omega_j\setminus\Omega_{j-2})\right)$. By the fact that $\mathcal{H}(S_{v_j}\setminus\{u^\vee-u^\wedge\ge\delta_j\})=0$ for all $j\in\mathbb{N}$ and by (3.14), we find that

$$\begin{aligned}
\mathcal{H}(S_v) &\le \sum_{j=2}^{\infty}\mathcal{H}(S_{v_j}\cap(\Omega_j\setminus\Omega_{j-2})) \le \sum_{j=2}^{\infty}\mathcal{H}(\{u^\vee-u^\wedge\ge\delta_j\}\cap(\Omega_j\setminus\Omega_{j-2}))\\
&< \mathcal{H}(\{u^\vee-u^\wedge\ge\delta_2\})+\varepsilon<\infty,
\end{aligned}$$

as desired. □

Now we can prove Theorem 1.4 of the introduction. In Example 5.2 we will show that here we cannot have $\mathcal{H}(S_v)=0$.

Proof of Theorem 1.4. First apply Proposition 3.12 to find $\widehat{w}\in\mathrm{BV}(\Omega)$ such that $\|\widehat{w}-u\|_{\mathrm{BV}(\Omega)}<\varepsilon/4$, $\mathcal{H}(S_{\widehat{w}})<\infty$, and

$$\lim_{r\to 0^+}\|\widehat{w}-u\|_{L^\infty(\Omega\setminus\Omega(r))}=0.$$

Then apply Theorem 3.10 to find $w\in\mathrm{SBV}(\Omega)$ such that $\|w-\widehat{w}\|_{L^1(\Omega)}<\varepsilon/4$, $\|Dw\|(\Omega)<\|D\widehat{w}\|(\Omega)+\varepsilon/4$, $\mathcal{H}(S_w\setminus S_{\widehat{w}})=0$, and

$$\lim_{r\to 0^+}\|w-\widehat{w}\|_{L^\infty(\Omega\setminus\Omega(r))}=0.$$

In total, we have $w \in \mathrm{SBV}(\Omega)$ such that $\|w-u\|_{L^1(\Omega)} < \varepsilon/2$, $\|Dw\|(\Omega) < \|Du\|(\Omega) + \varepsilon/2$, $\mathcal{H}(S_w) < \infty$, and

$$\lim_{r\to 0^+} \|w-u\|_{L^\infty(\Omega\setminus\Omega(r))} = 0.$$

Take $\Omega' \Subset \Omega$ such that $\|Dw\|(\Omega\setminus\Omega') < \varepsilon/2$ and $\mathcal{H}(S_w \setminus \Omega') < \varepsilon$, and take a function $\eta \in \mathrm{Lip}_c(\Omega)$ with $0 \le \eta \le 1$ on X and $\eta = 1$ in Ω'. From the definition of the total variation, take a sequence $(w_i) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $w_i \to w$ in $L^1_{\mathrm{loc}}(\Omega)$ and $\lim_{i\to\infty} \|Dw_i\|(\Omega) = \|Dw\|(\Omega)$. Define for each $i \in \mathbb{N}$

$$v_i := \eta w_i + (1-\eta) w.$$

Then clearly $\lim_{i\to\infty} \|v_i - w\|_{L^1(\Omega)} = 0$ and by a Leibniz rule (see [17, Lemma 3.2]) and since g_η is bounded,

$$\begin{aligned}\|Dv_i\|(\Omega) &\le \int_\Omega |w_i - w| g_\eta \, d\mu + \|Dw_i\|(\Omega) + \|Dw\|(\Omega\setminus\Omega') \\ &\to \|Dw\|(\Omega) + \|Dw\|(\Omega\setminus\Omega') < \|Du\|(\Omega) + \varepsilon.\end{aligned}$$

Thus if we choose $v := v_i$ for suitably large $i \in \mathbb{N}$, we have $\|v-u\|_{L^1(\Omega)} < \varepsilon$ and $\|Dv\|(\Omega) < \|Du\|(\Omega) + \varepsilon$, and so in particular $v \in \mathrm{BV}(\Omega)$. It is then easy to check that in fact $v \in \mathrm{SBV}(\Omega)$. Since $S_{w_i} = \emptyset$ for all $i \in \mathbb{N}$, we have $S_{v_i} \subset S_w \setminus \Omega'$ for all $i \in \mathbb{N}$, and since $\mathcal{H}(S_w \setminus \Omega') < \varepsilon$, in fact $\mathcal{H}(S_v) < \varepsilon$. Finally,

$$\lim_{r\to 0^+} \|v-u\|_{L^\infty(\Omega\setminus\Omega(r))} = \lim_{r\to 0^+} \|w-u\|_{L^\infty(\Omega\setminus\Omega(r))} = 0$$

as required. □

To complete this section, we give the proof of Corollary 1.5 by using Theorem 3.7 (or Theorem 3.9).

Proof of Corollary 1.5. Assume that $\mathbb{Z}(\partial\Omega, \mathcal{H})$ is the trace space of $\mathrm{BV}(\Omega)$, i.e., the trace operator $u \mapsto Tu$ in Definition 2.16 is a bounded linear surjective operator from $\mathrm{BV}(\Omega)$ to $\mathbb{Z}(\partial\Omega, \mathcal{H})$. From the definition of the total variation (2.7) we immediately get $N^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|\cdot\|_{\mathrm{BV}(\Omega)} \le \|\cdot\|_{N^{1,1}(\Omega)}$. Thus the trace operator $u \mapsto Tu$ is still a bounded linear operator from $N^{1,1}(\Omega)$ to $\mathbb{Z}(\partial\Omega, \mathcal{H})$. Hence it remains to show the surjectivity. For any $f \in \mathbb{Z}(\partial\Omega, \mathcal{H})$, we know that there is a function $u \in \mathrm{BV}(\Omega)$ such that $Tu = f$. It follows from Theorem 3.7 (or Theorem 3.9) that there is a function $v \in N^{1,1}(\Omega)$ such that $Tv = Tu = f$, since

$$\begin{aligned}\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |v - f(x)| \, d\mu &\le \lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u-v| + |u - f(x)| \, d\mu \\ &\le \lim_{r\to 0^+} \left(\fint_{B(x,r)\cap\Omega} |u-v|^{s/s-1} \, d\mu \right)^{(s-1)/s} + \lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - f(x)| \, d\mu \\ &= 0, \quad \text{for } \mathcal{H}\text{-a.e. } x \in \partial\Omega. \end{aligned} \tag{3.18}$$

This gives the surjectivity as desired.

Conversely, assume that $\mathbb{Z}(\partial\Omega,\mathcal{H})$ is the trace space of $N^{1,1}(\Omega)$, i.e., the trace operator $u \mapsto Tu$ in Definition 2.16 is a bounded linear surjective operator from $N^{1,1}(\Omega)$ to $\mathbb{Z}(\partial\Omega,\mathcal{H})$. Then for any $h \in \mathrm{BV}(\Omega)$, without loss of generality, we may assume that $\|h\|_{\mathrm{BV}(\Omega)} > 0$. By Theorem 3.7, choosing $\varepsilon = \|h\|_{\mathrm{BV}(\Omega)}/2$, there is a function $v \in N^{1,1}(\Omega)$ with $\|v\|_{N^{1,1}(\Omega)} \le 2\|h\|_{\mathrm{BV}(\Omega)}$ and

$$\fint_{B(x,r)\cap\Omega} |v-h|^{s/(s-1)}\,d\mu \to 0 \quad \text{as } r \to 0^+$$

uniformly for all $x \in \partial\Omega$. Then we have that $Th = Tv$ by a similar argument to (3.18), and that

$$\|Th\|_{\mathbb{Z}(\partial\Omega,\mathcal{H})} = \|Tv\|_{\mathbb{Z}(\partial\Omega,\mathcal{H})} \lesssim \|v\|_{N^{1,1}(\Omega)} \le 2\|h\|_{\mathrm{BV}(\Omega)}.$$

Hence the trace Th exists for any $h \in \mathrm{BV}(\Omega)$ and the trace operator $h \to Th$ is linear and bounded from $\mathrm{BV}(\Omega)$ to $\mathbb{Z}(\partial\Omega,\mathcal{H})$. Moreover, the surjectivity of the trace operator follows immediately from $N^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$. Thus $\mathbb{Z}(\partial\Omega,\mathcal{H})$ is also the trace space of $\mathrm{BV}(\Omega)$. □

Remark 3.19. The trace spaces of $\mathrm{BV}(\Omega)$ and $N^{1,1}(\Omega)$ are also the same with respect to any given boundary measure $\widetilde{\mathcal{H}}$ under Definition 2.19.

4 Traces of $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$

In this section, let $\Omega \subset X$ be an arbitrary nonempty open set with nonempty complement.

We will work with Whitney coverings of open sets. For a ball $B = B(x,r)$ and a number $a > 0$, we use the notation $aB := B(x,ar)$. We can choose a Whitney covering $\{B_j = B(x_j,r_j)\}_{j=1}^{\infty}$ of Ω such that:

1. for each $j \in \mathbb{N}$,
$$r_j = \mathrm{dist}(x_j, X \setminus \Omega)/100\lambda,$$
2. for each $k \in \mathbb{N}$, the ball $20\lambda B_k$ meets at most $C_0 = C_0(C_d)$ balls $20\lambda B_j$ (that is, a bounded overlap property holds),
3. if $20\lambda B_k$ meets $20\lambda B_j$, then $r_j \le 2r_k$;

see e.g. [20, Proposition 4.1.15] and its proof. Given such a covering of Ω, we find a partition of unity $\{\phi_j\}_{j=1}^{\infty}$ subordinate to the covering, that is, for each $j \in \mathbb{N}$ the function ϕ_j is c/r_j-Lipschitz, $c = c(C_d)$, with $\mathrm{spt}(\phi_j) \subset 2B_j$ and $0 \le \phi_j \le 1$, such that $\sum_j \phi_j = 1$ on Ω; see e.g. [20, p. 103]. We define a *discrete convolution* u_W of $u \in L^1_{\mathrm{loc}}(\Omega)$ with respect to the Whitney covering by

$$u_W := \sum_{j=1}^{\infty} u_{B_j}\phi_j.$$

In general, $u_W \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$.

Theorem 4.1. *For any function $u \in N^{1,1}(\Omega)$, there exists a constant $0 < c_H = c_H(\lambda) < 1$ such that the discrete convolution u_W of u with respect to the Whitney covering $\{B_j = B(x_j,r_j)\}_{j=1}^{\infty}$ is in $M^{1,1}_{c_H}(\Omega)$ with the norm estimate*

$$\|u_W\|_{M^{1,1}_{c_H}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}.$$

Proof. First we consider the L^1-norm of w_W. By the bounded overlap property of the Whitney covering $\{B_j = B(x_j, r_j)\}_{j=1}^{\infty}$, it follows from the facts $\mathrm{spt}(\phi_j) \subset 2B_j$ and $0 \le \phi_j \le 1$ that

$$\|u_W\|_{L^1(\Omega)} \le \sum_{j=1}^{\infty} \mu(2B_j) \fint_{B_j} |u|\, d\mu \le C_d \sum_{j=1}^{\infty} \int_{B_j} |u|\, d\mu \lesssim \|u\|_{L^1(\Omega)}.$$

Next, for the minimal 1-weak upper gradient g_u of u, we will give an admissible function g that satisfies (2.14) when the pair of points x, y satisfy (2.15) with $c_H = 1/50\lambda$. We claim that the admissible function g can be defined as follows: for any point $x \in \Omega$, we define

$$g(x) := C \sum_{j=1}^{\infty} \chi_{B_j}(x) \fint_{60\lambda B_j} g_u\, d\mu \tag{4.2}$$

with $C = C(C_d, C_P, \lambda)$. Indeed, for any pair of points $x, y \in \Omega$ satisfying (2.15), without loss of generality, we may assume that $\mathrm{dist}(x, X \setminus \Omega) \le \mathrm{dist}(y, X \setminus \Omega)$ and $x \in B_j$, $y \in B_i$ for some $i, j \in \mathbb{N}$. Recalling the properties of the Whitney covering, we have that

$$\mathrm{dist}(x, X \setminus \Omega) \le \mathrm{dist}(x_j, X \setminus \Omega) + r_j = (100\lambda + 1) r_j.$$

Hence we have

$$d(y, x_j) \le d(x, y) + r_j \le \frac{1}{50\lambda} \mathrm{dist}(x, X \setminus \Omega) + r_j < 4r_j,$$

which means $y \in 4B_j$. Hence $20\lambda B_i \cap 20\lambda B_j \ne \emptyset$, and so $r_i \le 2r_j$. Hence $B_i \subset 10B_j$. Moreover, if $2B_k \cap B_i \ne \emptyset$, then $r_k \le 2r_i$ and so $B_k \subset 6B_i \subset 20B_j$. Recall that the function ϕ_k is c/r_k-Lipschitz for any $k \in \mathbb{N}$ and that $\sum_k \phi_k = 1$ on Ω. Then by the bounded overlap property of the Whitney covering and the Poincaré inequality for u and g_u, we have that

$$\begin{aligned}
|u_W(x) - u_W(y)| &= \left| \sum_{k=1}^{\infty} u_{B_k} \phi_k(x) - \sum_{k=1}^{\infty} u_{B_k} \phi_k(y) \right| \\
&= \left| \sum_{k=1}^{\infty} (u_{B_k} - u_{B_j}) \phi_k(x) - \sum_{k=1}^{\infty} (u_{B_k} - u_{B_j}) \phi_k(y) \right| \\
&\le \sum_{k=1}^{\infty} |u_{B_k} - u_{B_j}| |\phi_k(x) - \phi_k(y)| \\
&\le d(x, y) \sum_{\{k:\, 2B_k \cap (B_j \cup B_i) \ne \emptyset\}} \frac{c}{r_k} |u_{B_k} - u_{B_j}| \\
&\lesssim d(x, y) \frac{c}{r_j} \fint_{20B_j} |u - u_{20B_j}|\, d\mu \qquad (4.3) \\
&\le C d(x, y) \fint_{20\lambda B_j} g_u\, d\mu,
\end{aligned}$$

where C is a constant depending on λ, c, C_d, C_P and C_0 only, and thus in fact only on C_d, C_P, λ. Thus, the function g defined in (4.2) is an admissible function for u_W.

At last, we show the L^1-norm estimate for g. It follows from the bounded overlap property of the Whitney covering that

$$\int_\Omega g(x)\,d\mu(x) \le \sum_{j=1}^{\infty} \int_{B_j} g(x)\,d\mu(x) \lesssim \sum_{j=1}^{\infty} \mu(B_j) \fint_{20\lambda B_j} g_u\,d\mu$$
$$\lesssim \sum_{j=1}^{\infty} \int_{20\lambda B_j} g_u(x)\,d\mu(x) \lesssim \int_\Omega g_u(x)\,d\mu(x) = \|g_u\|_{L^1(\Omega)}.$$

□

Recall the homogeneous dimension $s > 1$ from (2.1).

Theorem 4.4 ([13, Theorem 9.2])**.** *Let $\sigma > 1$ and let $B = B(x,r)$ be a ball in X. If $u \in M^{1,p}(\sigma B, d, \mu)$ and g is an admissible function in (2.14), where $p \ge s/(s+1)$, then*

$$\fint_B |u - u_B|\,d\mu \le Cr \left(\fint_{\sigma B} g^p\,d\mu\right)^{1/p}, \tag{4.5}$$

with C depending on C_d, p, and σ only.

Next we will consider the relationship between $M^{1,1}_{c_H}(\Omega)$ and $M^{1,1}(\Omega)$. The next theorem shows that when $\Omega \subset X$ is a uniform domain, $M^{1,1}_{c_H}(\Omega)$ and $M^{1,1}(\Omega)$ are the same. The case $X = \mathbb{R}^n$, i.e. the Euclidean case was proved in [23, Theorem 19]. Before stating the theorem, we first give the definition of uniform domain.

Definition 4.6. A domain $\Omega \subset X$ is called *uniform* if there is a constant $c_U \in (0,1]$ such that every pair of distinct points $x, y \in \Omega$ can be connected by a curve $\gamma\colon [0,\ell_\gamma] \to \Omega$ parametrized by arc-length such that $\gamma(0) = x$, $\gamma(\ell_\gamma) = y$, $\ell_\gamma \le c_U^{-1} d(x,y)$, and

$$\operatorname{dist}(\gamma(t), X \setminus \Omega) \ge c_U \min\{t, \ell_\gamma - t\} \quad \text{for all} \quad t \in [0,\ell_\gamma]. \tag{4.7}$$

Theorem 4.8. *Assume $\Omega \subset X$ is a uniform domain. Then for any $0 < c_H < 1$, we have $M^{1,1}_{c_H}(\Omega) = M^{1,1}(\Omega)$ with equivalent norms.*

Proof. Choose arbitrary $x, y \in \Omega$. By modifying the standard covering argument in uniform domains (see [14, 15, 23] for details), from the uniformity condition we deduce easily that there is a chain of balls B_k resembling a cigar that joins the points x and y. More precisely, there are balls $B_k := B(z_k, r_k)$ with $k \in \mathbb{Z}$ and $z_k \in \Omega$ such that for each k one has for some $c' = c'(\lambda, c_H, c_U)$

$$15\lambda/c_H B_k \subset \Omega \quad \text{and} \quad r_k \ge \frac{1}{c'} \min\{d(z_k,x), d(z_k,y)\},$$

with also $B_k \cap B_{k+1} \ne \emptyset$, and $r_k/2 \le r_{k+1} \le 2r_k$. In addition, $\lim_{k\to+\infty} d(x,B_k) = 0 = \lim_{k\to-\infty} d(y,B_k)$. Finally, we may assume that $\sum_{k\in\mathbb{Z}} r_k \le C' d(x,y)$.

Let $u \in M^{1,1}_{c_H}(\Omega)$ with admissible function $g \in L^1(\Omega)$. We can zero extend g outside Ω. Since $15\lambda/c_H B_k \subset \Omega$ and $c_H < 1$, then for any $x_0, y_0 \in 5\lambda B_k$, we have

$$d(x_0,y_0) \le 10\lambda r_k \le c_H(15\lambda/c_H - 5\lambda) r_k \le c_H \min\{\operatorname{dist}(x_0, X\setminus\Omega), \operatorname{dist}(y_0, X\setminus\Omega)\}.$$

Hence, for any $x_0, y_0 \in 5\lambda B_k$, the condition (2.15) is satisfied. Thus, $u \in M^{1,1}(5\lambda B_k)$ for any $k \in \mathbb{Z}$. It follows from the Poincaré inequality in Theorem 4.4 on the ball $5B_k$ with $\sigma = \lambda$ that

$$\begin{aligned}
|u_{B_k} - u_{B_{k+1}}| &\lesssim \fint_{5B_k} |u - u_{5B_k}| \lesssim r_k \left(\fint_{5\lambda B_k} g^{s/(s+1)}\, d\mu \right)^{(s+1)/s} \\
&\lesssim r_k \left(\fint_{(5\lambda + 2c')B_k} g^{s/(s+1)}\, d\mu \right)^{(s+1)/s} \\
&\lesssim r_k \left(\left(\mathcal{M} g^{s/(s+1)}(x) \right)^{(s+1)/s} + \left(\mathcal{M} g^{s/(s+1)}(y) \right)^{(s+1)/s} \right),
\end{aligned}$$

where s is the associated homogeneous dimension. Here the last inequality follows from the fact that either x or y is contained in $2c'B_k \subset (5\lambda + 2c')B_k$.

If x, y are Lebesgue points of u, we have $|u(x) - u(y)| \leq \sum_{k \in \mathbb{Z}} |u_{B_k} - u_{B_{k+1}}|$. By summing over k, it follows that

$$|u(x) - u(y)| \leq d(x, y)(\tilde{g}(x) + \tilde{g}(y)),$$

where $\widetilde{g}(x) = 2C\big(\mathcal{M} g^{s/(s+1)}(x)\big)^{(s+1)/s}$. The conclusion follows from the Hardy-Littlewood maximal inequality. □

Remark 4.9. From the proof of Theorem 4.8, we know that if X is a geodesic space, i.e., for any $x, y \in X$, there exists a curve γ in X such that $\ell_\gamma = d(x, y)$, then $M^{1,1}_{c_1}(\Omega) = M^{1,1}_{c_2}(\Omega)$ with equivalent norms for any two constants $0 < c_1, c_2 < 1$. This fact coincides with the case $\Omega \subset \mathbb{R}^n$, where $\mathbb{R}^n$ is a geodesic space. When $\Omega \subset \mathbb{R}^n$, for any $0 < c_H < 1$, we obtain $M^{1,1}_{c_H}(\Omega) = M^{1,1}_{ball}(\Omega)$. Here we refer to [23, 41] for more details about the space $M^{1,1}_{ball}(\Omega)$.

To "achieve" the boundary values, we need the following proposition.

Proposition 4.10 ([28, Proposition 6.5])**.** *Let $u \in \mathrm{BV}(\Omega)$. Then the discrete convolution u_W of u satisfies*

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x,r) \cap \Omega} |u_W - u|\, d\mu = 0$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$.

The above proposition considers the measure $\mathcal{H}$ on $\partial\Omega$, that is, the codimension 1 Hausdorff measure. But this may not be the measure we really want to study. For example, a classical problem is to study the trace spaces of weighted Sobolev spaces on Euclidean spaces. For the half plane $\Omega = \mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and the measure $d\mu(x) = w_\lambda(x)\, dm_2(x)$ with m_2 the 2-dimensional Lebesgue measure and $w_\lambda(x) := \log^\lambda(\max\{e, e/|x_2|\})$, $\lambda > 0$, the codimension 1 Hausdorff measure on $\partial\mathbb{R}^2_+ = \mathbb{R}$ is not even σ-finite and hence is not the 1-dimensional Lebesgue measure that we usually study, see Example 5.7. Thus, it is reasonable to consider the equivalence of the traces of $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$ under any general boundary measure $\widetilde{\mathcal{H}}$ on $\partial\Omega$. Thus, we introduce the following lemma.

Lemma 4.11. *Assume Ω satisfies a measure doubling condition (2.5), i.e., $\mu_{|\Omega}$ is doubling. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $z \in \partial\Omega$. Assume that there is $a \in \mathbb{R}$ such that*

$$\lim_{r\to 0+} \fint_{B(z,r)\cap\Omega} |u-a|\,d\mu = 0.$$

Then the discrete convolution u_W of u satisfies

$$\lim_{r\to 0+} \fint_{B(z,r)\cap\Omega} |u_W-a|\,d\mu = 0.$$

Proof. In the Whitney covering $\{B_k\}_{k=1}^{\infty}$, recall that for any $B_k = B(x_k, r_k)$ we have $r_k = \mathrm{dist}(x_k, X\setminus\Omega)/100\lambda$. If $2B_k \cap B(z,r) \neq \emptyset$, then

$$2r_k + r \geq d(x_k, z) \geq \mathrm{dist}(x_k, X\setminus\Omega) = 100\lambda r_k,$$

which implies

$$\bigcup_{\{k\in\mathbb{N}:\, 2B_k\cap B(z,r)\neq\emptyset\}} B_k \subset B(z,2r).$$

Then we have

$$\begin{aligned}
\int_{B(z,r)\cap\Omega} |u_W-a|\,d\mu &= \int_{B(z,r)\cap\Omega} \left|\sum_{k=1}^{\infty}(\phi_k u_{B_k} - \phi_k a)\right| d\mu \\
&\leq \int_{B(z,r)\cap\Omega} \sum_{k=1}^{\infty} |\phi_k||u_{B_k}-a|\,d\mu \\
&\leq \int_{B(z,r)\cap\Omega} \sum_{k=1}^{\infty} \chi_{2B_k}|u_{B_k}-a|\,d\mu \\
&\leq \int_{B(z,r)\cap\Omega} \sum_{k=1}^{\infty} \chi_{2B_k} \fint_{B_k} |u-a|\,d\mu\,d\mu \\
&\leq C_d \sum_{\{k\in\mathbb{N}:\, 2B_k\cap B(z,r)\neq\emptyset\}} \int_{B_k} |u-a|\,d\mu \\
&\lesssim \int_{B(z,2r)\cap\Omega} |u-a|\,d\mu
\end{aligned}$$

by the bounded overlap property. Thus, the doubling property of $\mu_{|\Omega}$ gives the estimate

$$\fint_{B(z,r)\cap\Omega} |u_W-a|\,d\mu \lesssim \fint_{B(z,2r)\cap\Omega} |u-a|\,d\mu.$$

The result follows by passing to the limit. □

Proof of Theorem 1.6, Theorem 1.7, and Corollary 1.8. Theorem 1.6 follows immediately by combining Theorem 4.1, Theorem 4.8 and Proposition 4.10, while Theorem 1.7 follows immediately by combining Theorem 4.1, Theorem 4.8 and Lemma 4.11.

For Corollary 1.8, by adapting the proof of Corollary 1.5, we obtain the result using Theorem 1.7. Note that $M^{1,1}(\Omega) \subset N^{1,1}(\Omega)$ with $\|\cdot\|_{N^{1,1}(\Omega)} \lesssim \|\cdot\|_{M^{1,1}(\Omega)}$, see [13, Theorem 8.6]. □

5 Examples and applications

The following example shows that in Theorem 1.3 we cannot take a function $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, or even $v \in L^\infty_{\mathrm{loc}}(\Omega)$.

Example 5.1. Let $X = \mathbb{R}^2$ (unweighted) and let $\Omega := B(0,1)$. We find a sequence $\{x_k\}$ that is dense in $B(0,1)$. Take

$$u_k(x) := |x - x_k|^{-1+1/k}, \quad k \in \mathbb{N}.$$

Then $\|u_k\|_{L^1(\Omega)} < \infty$ and the minimal 1-weak upper gradient satisfies (see [6, Proposition A.3])

$$g_{u_k}(x) = |\nabla u_k(x)| = (-1 + 1/k)|x - x_k|^{-2+1/k}$$

and so

$$\int_{B(0,1)} g_{u_k}\, dx \lesssim \int_{B(0,1)} |x - x_k|^{-2+1/k}\, dx \le \int_{B(0,2)} |x|^{-2+1/k}\, dx < \infty.$$

Let

$$u(x) := \sum_k 2^{-k} \frac{u_k}{\|u_k\|_{N^{1,1}(B(0,1))}}.$$

Then using e.g. [6, Lemma 1.52] we see that u has a 1-weak upper gradient

$$\sum_k 2^{-k} \frac{g_{u_k}}{\|u_k\|_{N^{1,1}(B(0,1))}},$$

which implies $u \in N^{1,1}(B(0,1))$. We know that the homogeneous dimension s of $\mathbb{R}^2$ is 2, and then $\frac{s}{s-1} = 2$. On the other hand, we can see that for any $q > 2$, we have for all sufficiently large $k \in \mathbb{N}$

$$\int_{B(x_k,r)\cap B(0,1)} |u_k|^q\, dx = \infty \quad \text{for all } r > 0,$$

and then for all balls $B \cap B(0,1) \ne \emptyset$ we have $\int_{B\cap B(0,1)} |u|^q\, dx = \infty$. Given $v \in \mathrm{Lip}_{\mathrm{loc}}(B(0,1))$, we know that $v \in L^q_{\mathrm{loc}}(B(0,1))$. Therefore we have $\|v - u\|_{L^q(B\cap B(0,1))} = \infty$ for all balls $B \cap B(0,1) \ne \emptyset$, which contradicts the desired conclusion in Theorem 1.3.

The following example shows that in Theorem 1.4 we cannot take a function v with $\mathcal{H}(S_v) = 0$.

Example 5.2. Let $X = \mathbb{R}^2$ (unweighted) and let $\Omega := (-1,1) \times (0,1)$. Define $u \in \mathrm{BV}(\Omega)$ by

$$u(x_1, x_2) := \begin{cases} 0 & \text{when } x_1 < 0 \\ 1 & \text{when } x_1 \ge 0. \end{cases}$$

Let $v \in \mathrm{BV}(\Omega)$ with $\mathcal{H}(S_v) = 0$. Since $\mathcal{H}(\{0\} \times (0,1)) > 0$, it is now easy to check that $\|v - u\|_{L^\infty(\Omega\setminus\Omega(r))} \ge 1/2$ for all $r > 0$.

A direct consequence of Corollary 1.5 and Corollary 1.8 is that under a proper setting, the trace spaces of the BV space, Newton-Sobolev space, and Hajłasz-Sobolev space are the same. Hence we can obtain many trace results for the BV and Hajłasz-Sobolev space directly from trace results for the Newton-Sobolev space obtained in the literature. In particular, from [29, Theorem 1.1] we are able to obtain the following result.

Theorem 5.3. *Let $\Omega \subset X$ be a bounded uniform domain satisfying the measure doubling condition (2.5). Assume also that $(\Omega, d, \mu|_\Omega)$ admits a $(1,1)$-Poincaré inequality. Let $\partial\Omega$ be endowed with an Ahlfors codimension θ-regular measure ν for some $0 < \theta < 1$. Then the trace spaces of $N^{1,1}(\Omega, \mu)$, $\mathrm{BV}(\Omega, \mu)$ and $M^{1,1}(\Omega, \mu)$ are the same, namely the Besov space $B^{1-\theta}_{1,1}(\partial\Omega, \nu)$.*

We say that $\partial\Omega$ is endowed with an Ahlfors codimension θ-regular measure ν if there is a σ-finite Borel measure ν on $\partial\Omega$ and a constant $c_\theta > 0$ such that

$$c_\theta{}^{-1} \frac{\mu(B(x,r) \cap \Omega)}{r^\theta} \le \nu(B(x,r) \cap \partial\Omega) \le c_\theta \frac{\mu(B(x,r) \cap \Omega)}{r^\theta} \tag{5.4}$$

for all $x \in \partial\Omega$ and $0 < r < 2 \operatorname{diam} \Omega$. The Besov space $B^{1-\theta}_{1,1}(\partial\Omega, \nu)$ consists of L^1-functions of finite Besov norm that is given by

$$\|u\|_{B^{1-\theta}_{1,1}(\partial\Omega,\nu)} = \|u\|_{L^1(\partial\Omega,\nu)} + \int_0^\infty \int_{\partial\Omega} \fint_{B(y,t)} \frac{|u(x) - u(y)|}{t^{1-\theta}} \, d\nu(x) \, d\nu(y) \frac{dt}{t}.$$

The above theorem seems to be new even for BV and $M^{1,1}$ functions in the (weighted) Euclidean setting. As an illustration, we give an example in weighted Euclidean spaces.

Example 5.5. Let $\Omega = \mathbb{D} \subset \mathbb{R}^2$ be the unit disk with $\partial\Omega = \mathbb{S}^1$ the unit circle. Take the measure $d\mu(x) = \operatorname{dist}(x, \mathbb{S}^1)^{-\alpha} \, dm_2(x)$ with $0 < \alpha < 1$ and m_2 two-dimensional Lebesgue measure. Then by a direct computation, $\operatorname{dist}(x, \mathbb{S}^1)^{-\alpha}$ with $0 < \alpha < 1$ is an A_1-weight and hence μ supports a $(1,1)$-Poincaré inequality, see [18, Chapter 15]. Moreover, it is easy to check that the 1-dimensional Hausdorff measure $\mathcal{H}^1$ on $\mathbb{S}^1$ is an Ahlfors codimension $(1-\alpha)$-regular measure, i.e., $\mathcal{H}^1$ on $\mathbb{S}^1$ satisfies (5.4) with $\theta = 1 - \alpha$. Hence we obtain from Theorem 5.3 that the trace spaces of $N^{1,1}(\mathbb{D}, \mu)$, $\mathrm{BV}(\mathbb{D}, \mu)$, and $M^{1,1}(\mathbb{D}, \mu)$ are $B^\alpha_{1,1}(\mathbb{S}^1, \mathcal{H}^1)$. It is also known from the classical trace results of weighted Sobolev spaces that the trace space of $N^{1,1}(\mathbb{D}, \mu)$ is the classical Besov space $B^\alpha_{1,1}(\mathbb{S}^1, \mathcal{H}^1)$. Here we refer to [33, 38, 39] for the trace results for weighted Sobolev spaces on Euclidean spaces and refer to the seminal monographs by Triebel [37] for more information on Besov spaces.

On the other hand, using our theory it is also possible to obtain new trace results for Hajłasz-Sobolev or Newton-Sobolev functions from the known trace results for BV functions. In particular, from [30, Corollary 1.4] we are able to obtain the following trace results.

Theorem 5.6. *Let $\Omega \subset X$ be a bounded uniform domain that satisfies the measure density condition (2.4) and admits a $(1,1)$-Poincaré inequality. Assume also that the codimension 1 Hausdorff measure $\mathcal{H}$ is Ahlfors codimension 1-regular. Then we have that the trace spaces of $\mathrm{BV}(\Omega, \mu)$, $N^{1,1}(\Omega, \mu)$ and $M^{1,1}(\Omega, \mu)$ are the same, namely the space $L^1(\partial\Omega, \mathcal{H})$.*

When $\Omega = \mathbb{D}$, $\partial\Omega = \mathbb{S}^1$, $\mu = m_2$ the 2-dimension Lebesgue measure and $\mathcal{H} \approx \mathcal{H}^1$ the 1-dimension Hausdorff measure, the above theorem coincides with the classical results that the trace spaces of $\mathrm{BV}(\mathbb{D})$ and $N^{1,1}(\mathbb{D})$ are both $L^1(\mathbb{S}^1)$. Moreover, the above theorem gives that $L^1(\mathbb{S}^1)$ is also the trace space of $M^{1,1}(\mathbb{D})$, which seems to be new even in this case.

The above Theorem 5.3 and Theorem 5.6 both require that the boundaries are endowed with some codimension Ahlfors regular measure. In the following, we will give an example where the measure on the boundary do not satisfy any codimension Ahlfors regularity.

Example 5.7. Let $\Omega = \mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and take the measure $d\mu(x) = w_\lambda(x)\, dm_2(x)$ with m_2 the 2-dimensional Lebesgue measure and $w_\lambda(x) = \log^\lambda(\max\{e, e/|x_2|\})$, $\lambda > 0$. For any $x \in \mathbb{R} = \partial\Omega$ and $0 < r < e^{-2\lambda}$, let $Q(x, r)$ denote the cube parallel to the coordinate axes with center x and sidelength r. Then we have the estimate

$$\mu(Q(x,r)) = 2\int_0^r \int_0^{r/2} \log^\lambda(e/|x_2|)\, dx_2\, dx_1 = 2r \int_0^{r/2} \log^\lambda(e/t)\, dt \approx r^2 \log^\lambda(e/r). \quad (5.8)$$

Here the last equality holds since we have

$$\left(t \log^\lambda(e/t)\right)' = \log^\lambda(e/t)\left(1 - \frac{\lambda}{\log(e/t)}\right) \approx \log^\lambda(e/t), \quad \text{for} \;\; 0 < t \le r < e^{-2\lambda}.$$

By using the estimate (5.8), it follows from the definition of the codimension 1 Hausdorff measure (2.3) that for any nonempty interval $[a, b]$ in $\mathbb{R} = \partial\mathbb{R}^2_+$, we have that

$$\mathcal{H}([a,b]) = \lim_{R \to 0^+} \mathcal{H}_R([a,b]) \approx \lim_{R \to 0^+} |a - b| \log^\lambda(e/R) = \infty.$$

Hence the codimension 1 Hausdorff measure $\mathcal{H}$ on $\mathbb{R}$ is not even σ-finite and is not the 1-dimensional Lebesgue measure that we usually study.

Moreover, the weight w_λ defined above is a Muckenhoupt A_1-weight, since it is easy to check from estimate (5.8) that

$$\frac{\mu(B(z,r))}{r^2} \lesssim \inf_{x \in B(z,r)} w_\lambda(x), \quad \text{for any } z \in \mathbb{R}^2_+ \text{ and } r > 0.$$

We refer to [8] and [18, Chapter 15] for definitions, properties and examples of Muckenhoupt class weights.

Example 5.9. Let Ω, μ be as in the above example. Then it is easy to check from estimate (5.8) that the 1-dimensional Lebesgue measure on $\mathbb{R}$ does not satisfy the condition (5.4) for any θ. We denote by $\mathcal{Q}$ the collection of dyadic semi-open intervals in $\mathbb{R}$, i.e. the intervals of the form $I := 2^{-k}\big((0,1] + m\big)$, where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$. Write $\ell(I)$ for the edge length of $I \in \mathcal{Q}$, i.e. 2^{-k} in the preceding representation, and $\mathcal{Q}_k$ for the cubes $Q \in \mathcal{Q}$ such that $\ell(Q) = 2^{-k}$. For any $I \in \mathcal{Q}_{2^j}$, denote by $\widetilde{I}$ the interval in $\mathcal{Q}_{2^{j-1}}$ containing the interval I. By applying the methods used in [39] and [25, Theorem 1.3], we are able to use the dyadic norm similar with the ones used in [24] and [25] to characterize the trace space of

$N^{1,1}(\mathbb{R}^2_+, \mu)$, which is the Besov-type space $\mathcal{B}^\lambda_1(\mathbb{R})$. The Besov-type space $\mathcal{B}^\lambda_1(\mathbb{R})$ consists of functions in $L^1(\mathbb{R})$ of finite dyadic norm that is given by

$$\|u\|_{\mathcal{B}^\lambda_1(\mathbb{R})} = \|u\|_{L^1(\mathbb{R})} + \sum_{j=1}^{+\infty} 2^{-\lambda j} \sum_{I \in \mathcal{Q}_{2^j}} 2^{-2^j} |u_I - u_{\tilde{I}}|.$$

We omit the detailed proof here. Since $\mathbb{R}^2_+$ is uniform domain and satisfies the measure doubling condition (2.5), hence we obtain that the trace spaces of $\mathrm{BV}(\mathbb{R}^2_+, \mu)$, $N^{1,1}(\mathbb{R}^2_+, \mu)$ and $M^{1,1}(\mathbb{R}^2_+, \mu)$ are the same, the Besov-type space $\mathcal{B}^\lambda_1(\mathbb{R})$.

Example 5.10. The recent papers [7, 25, 40] studied trace results on regular trees. We refer to [7, Section 2] or [25, Section 2.1] for the definition of regular trees. It is easy to check that a regular tree is uniform and that it supports $(1,1)$-Poincaré inequality by modifying the proof in [7, Theorem 4.2] under the setting in [7, 25]. Even the definition of trace in [7, 25, 40] looks different from the one we used here, but [34] shows that they are equivalent. Hence the trace results of $N^{1,1}$ in [7, 25] can be immediately applied to BV and $M^{1,1}$. We omit the detail here and leave it to the interested reader.

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[D]

Characterization of trace spaces on regular trees via dyadic norms

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Characterization of trace spaces on regular trees via dyadic norms

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Abstract

In this paper, we study the traces of Orlicz-Sobolev spaces on a regular rooted tree. After giving a dyadic decomposition of the boundary of the regular tree, we present a characterization on the trace spaces of those first order Orlicz-Sobolev spaces whose Young function is of the form $t^p \log^\lambda(e+t)$, based on integral averages on dyadic elements of the dyadic decomposition.

1 Introduction

The problem of the characterization of the trace spaces (on the boundary of a domain) of Sobolev spaces has a long history. It was first studied in the Euclidean setting by Gagliardo [13], who proved that the trace operator $T : W^{1,p}(\mathbb{R}^{n+1}_+) \to B^{1-1/p}_{p,p}(\mathbb{R}^n)$, where $B^{1-1/p}_{p,p}(\mathbb{R}^n)$ stands for the classical Besov space, is linear and bounded for every $p > 1$ and that there exists a bounded linear extension operator that acts as a right inverse of T. Moreover, he proved that the trace operator $T : W^{1,1}(\mathbb{R}^{n+1}_+) \to L^1(\mathbb{R}^n)$ is a bounded linear surjective operator with a non-linear right inverse. Peetre [40] showed that one can not find a bounded linear extension operator that acts as a right inverse of $T : W^{1,1}(\mathbb{R}^{n+1}_+) \to L^1(\mathbb{R}^n)$. We refer to the seminal monographs by Peetre [41] and Triebel [47,48] for extensive treatments of the Besov spaces and related smoothness spaces. In potential theory, certain types of Dirichlet problem are guaranteed to have solutions when the boundary data belongs to a trace space corresponding to the Sobolev class on the domain. In the Euclidean setting, we refer to [1, 33, 36, 45, 50, 51] for more information on the traces of (weighted) Sobolev spaces and [8–10,12,29,30,38,39] for results on traces of (weighted) Orlicz-Sobolev spaces.

Analysis on metric measure spaces has recently been under active study, e.g., [2, 4, 16–20]. Especially the trace theory in the metric setting has been under development. Malý [34] proved that the trace space of the Newtonian space $N^{1,p}(\Omega)$ is the Besov space $B^{1-\theta/p}_{p,p}(\partial\Omega)$ provided that Ω is a John domain for $p > 1$ (uniform domain for $p \geq 1$) that admits a p-Poincaré inequality and whose boundary $\partial\Omega$ is endowed with a codimensional-θ Ahlfors regular measure with $\theta < p$. We also refer to the paper [43] for studies on the traces of Hajłasz-Sobolev functions to porous Ahlfors regular closed subsets via a method based

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on hyperbolic fillings of a metric space, see [6, 46]. For the trace result of BV (bounded variation) functions, we refer to [31, 32, 35].

The recent paper [3] dealt with geometric analysis on Cantor-type sets which are uniformly perfect totally disconnected metric measure spaces, including various types of Cantor sets. Cantor sets embedded in Euclidean spaces support a fractional Sobolev space theory based on Besov spaces. Indeed, suitable Besov functions on such a set are traces of the classical Sobolev functions on the ambient Euclidean spaces, see Jonsson-Wallin [21,22]. The paper [3, 25] established similar trace and extension theorems for Sobolev and Besov spaces on regular trees and their Cantor-type boundaries. Indeed, for a K-regular tree X with $K \geq 2$ and its Cantor-type boundary ∂X (see Section 2.1 for the definitions), if we give the uniformizing metric (see (2.1))

$$d_X(x, y) = \int_{[x,y]} e^{-\epsilon|z|}\, d\,|z|$$

and the weighted measure (see (2.2))

$$d\mu_\lambda(x) = e^{-\beta|x|}(|x| + C)^\lambda\, d\,|x| \tag{1.1}$$

on X, then the Besov space $\mathcal{B}^{\theta,\lambda}_p(\partial X)$ in Definition 2.4 below is exactly the trace of the Newton-Sobolev space $N^{1,p}(X, \mu_\lambda)$ defined in Section 2.3, see [25, Theorem 1.1] and [3, Theorem 6.5]. Here the smoothness exponent of the Besov space is

$$\theta = 1 - \frac{\beta/\epsilon - Q}{p}, \quad 0 < \theta < 1,$$

where $Q = \log K/\epsilon$ is the Hausdorff dimension of the Cantor-type boundary and $\beta/\epsilon - Q$ is a "codimension" determined by the uniformizing metric d_X and the measure μ on the tree.

In Euclidean spaces, the classical Besov norm is equivalent to a dyadic norm, and the trace spaces of the Sobolev spaces can be characterized by the Besov spaces defined via dyadic norms, see e.g. [24, Theorem 1.1]. Inspired by this, we give a dyadic decomposition of the boundary ∂X and define a Besov space $\mathcal{B}^\theta_p(\partial X)$ on the boundary ∂X by using a dyadic norm, see Section 2.4 and Definition 2.5. We show in Proposition 2.7 that the dyadic Besov spaces $\mathcal{B}^\theta_p(\partial X)$ coincide with the Besov space $B^\theta_{p,p}(\partial X)$ and the Hajłasz-Besov space $N^\theta_{p,p}(\partial X)$, see Definition 2.3 and Definition 2.6 for definitions of $B^\theta_{p,p}(\partial X)$ and $N^\theta_{p,p}(\partial X)$. We refer to [3, 14, 15, 23, 26, 27] for more information about Besov spaces $B^\theta_{p,p}(\cdot)$ and Hajłasz-Besov spaces $N^\theta_{p,p}(\cdot)$ on metric measure spaces.

By relying on dyadic norms, we define the Orlicz-Besov space $\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$, $\lambda_2 \in \mathbb{R}$ for the Young function $\Phi(t) = t^p \log^{\lambda_1}(e+t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \geq 0$, see Definition 2.8. Our first result shows that the Orlicz-Besov space $\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$ is the trace space of the Orlicz-Sobolev space $N^{1,\Phi}(X, \mu_{\lambda_2})$ defined in Section 2.3.

Theorem 1.1. *Let X be a K-regular tree with $K \geq 2$ and let $\Phi(t) = t^p \log^{\lambda_1}(e + t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \geq 0$. Fix $\lambda_2 \in \mathbb{R}$ and let μ_{λ_2} be the weighted measure given by (1.1). Assume that $p > (\beta - \log K)/\epsilon > 0$. Then the trace space of $N^{1,\Phi}(X, \mu_{\lambda_2})$ is the space $\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$ where $\theta = 1 - (\beta - \log K)/\epsilon p$.*

In this paper, for given function spaces $\mathbb{X}(\partial X)$ and $\mathbb{Y}(X)$, we call the space $\mathbb{X}(\partial X)$ a trace space of $\mathbb{Y}(X)$ if and only if there exist a bounded linear operator $T : \mathbb{Y}(X) \to \mathbb{X}(\partial X)$ and a bounded linear extension operator $E : \mathbb{X}(\partial X) \to \mathbb{Y}(X)$ such that $T \circ E = \mathrm{Id}$ on the space $\mathbb{X}(\partial X)$.

Our next result identifies the Orlicz-Besov space $\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$ as the Besov space $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$.

Proposition 1.2. *Let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$. Let $\Phi(t) = t^p \log^{\lambda_1}(e+t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \geq 0$. Assume that $\lambda_1 + \lambda_2 = \lambda$. Then the Banach spaces $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$ and $\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$ coincide, i.e., $\mathcal{B}^{\theta,\lambda}_{p}(\partial X) = \mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$.*

By combining Theorem 1.1 and Proposition 1.2, we obtain the following result.

Corollary 1.3. *Let X be a K-regular tree with $K \geq 2$. Let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$. Assume that $p > (\beta - \log K)/\epsilon > 0$ and let $\theta = 1 - (\beta - \log K)/\epsilon p$. Let $\Phi(t) = t^p \log^{\lambda_1}(e+t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \geq 0$. Then the Besov-type space $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$ is the trace space of $N^{1,\Phi}(X, \mu_{\lambda_2})$ whenever $\lambda_1 + \lambda_2 = \lambda$.*

When $\lambda_1 = 0$ and $\lambda_2 = \lambda$, the above result coincides with [25, Theorem 1.1], which states that the Besov-type space $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$ is the trace space of $N^{1,p}(X, \mu_\lambda)$ for a suitable θ. The above result shows that the Besov-type space $\mathcal{B}^{\theta,\lambda}_{p}(\partial X)$ is not only the trace space of $N^{1,p}(X, \mu_\lambda)$ but actually the trace space of all these Orlicz-Sobolev spaces $N^{1,\Phi}(X, \mu_{\lambda_2})$ (including $N^{1,p}(X, \mu_\lambda)$) for suitable θ, λ_2 and Φ. It may be worth to point out here that these Orlicz-Sobolev spaces $N^{1,\Phi}(X, \mu_{\lambda_2})$ are different from each other.

The paper is organized as follows. In Section 2, we give all the necessary preliminaries. More precisely, we introduce regular trees in Section 2.1 and we consider a doubling property of the measure μ on a regular tree X and the Ahlfors regularity of its boundary ∂X. The definition of Young functions is given in Section 2.2. We introduce the Newtonian and Orlicz-Sobolev spaces on X and the Besov-type spaces on ∂X in Section 2.3 and Section 2.4, respectively. In Section 3, we give the proofs of Theorem 1.1 and Proposition 1.2.

2 Preliminaries

Throughout this paper, the letter C denotes a constant that may change at different occurrences. The notation $A \approx B$ means that there is a constant C such that $1/C \cdot A \leq B \leq C \cdot A$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a constant C such that $A \leq C \cdot B$ ($A \geq C \cdot B$).

2.1 Regular trees and their boundaries

A *graph* G is a pair (V, E), where V is a set of vertices and E is a set of edges. Given vertices $x, y \in V$ are neighbors if x is connected to y by an edge. The number of the neighbors of a vertex x is referred to as the degree of x. A *tree* G is a connected graph without cycles.

Let us fix a vertex that we refer to by 0. The neighbors of 0 will be called children of 0 and 0 is called their mother. If x is one of the children of 0, then the neighbors of x

different from 0 are called children of x and we say that x is their mother. We continue in the obvious manner to define the children and the mother for all $y \neq 0$. We then call G a rooted tree with root 0 and say that G is K-regular if additionally each vertex has precisely K children.

Let G be a K-regular tree with a set of vertices V and a set of edges E for some $K \geq 1$. For simplicity of notation, we let $X = V \cup E$ and call it a K-regular tree. We consider each edge as a geodesic of length one and further consider each edge to be an isometric copy of the unit interval. More precisely, for any edge $\mathcal{E} \in E$ and a homeomorphism $\phi : \mathcal{E} \to [0, 1]$, the distance of two point $x, y \in \mathcal{E}$ is the Euclidean distance of $\phi(x)$ and $\phi(y)$. For any $x \in X$, let $|x|$ be the length of the geodesic from 0 to x. The geodesic connecting $x, y \in V$ is unique. We refer to it by $[x, y]$, and to its length by $|x - y|$. We write $x < y$ if $x \in [0, y]$. Then $|x - y| = |y| - |x|$. We say that a vertex $y \neq x$ is a descendant of the vertex x if $x < y$.

Towards defining the metric of X, let $\epsilon > 0$, and set

$$d_X(x, y) = \int_{[x,y]} e^{-\epsilon|z|}\, d\,|z|. \tag{2.1}$$

Here $d\,|z|$ is the natural measure that gives each edge Lebesgue measure 1; recall that each edge is an isometric copy of the unit interval. Notice that $\operatorname{diam} X = 2/\epsilon$ if X is a K-regular tree with $K \geq 2$.

The boundary ∂X of a tree X is obtained by completing X with respect to the metric d_X. An element $\xi \in \partial X$ is identified with an infinite geodesic starting at the root 0. Equivalently we employ the labeling $\xi = 0x_1x_2\cdots$, where x_i is a vertex in X with $|x_i| = i$, and x_{i+1} is a child of x_i. The extension of the metric to ∂X can be realized in the following manner. Given $\xi, \zeta \in \partial X$, if $\xi = 0x_1x_2\cdots$ and $\zeta = 0y_1y_2\cdots$, let k be the integer with $x_k = y_k$ and $x_{k+1} \neq y_{k+1}$. Then

$$d_X(\xi, \zeta) = 2\int_k^{+\infty} e^{-\epsilon t}\, dt = \frac{2}{\epsilon} e^{-\epsilon k}.$$

For any $\xi \in \partial X$, if $\xi = 0x_1x_2\cdots$, let

$$[0, \xi) = \bigcup_{i=1}^{\infty} [0, x_i],$$

where $[0, x_i]$ is the geodesic connecting 0 and x_i. We call $[0, \xi)$ the geodesic ray from 0 to ξ. We write $x < \xi$ if $x \in [0, \xi)$. For more details, see [3, 5, 7]. For clarity, we use ξ, ζ, ω to denote points in ∂X and x, y, z points in X.

On the K-regular tree X, we use the weighted measure μ_λ introduced in [25, Section 2.2], defined by

$$d\mu_\lambda(x) = e^{-\beta|x|}(|x| + C)^\lambda\, d\,|x|, \tag{2.2}$$

where $\beta > \log K$, $\lambda \in \mathbb{R}$ and $C \geq \max\{2|\lambda|/(\beta - \log K), 2(\log 4)/\epsilon\}$. For $\lambda = 0$, this is the measure used in [3].

It is proven in proposition below that μ_λ is doubling, see [25, Corollary 2.9].

Proposition 2.1. *For any $\lambda \in \mathbb{R}$, the measure μ_λ is doubling, i.e., $\mu_\lambda(B(x,2r)) \lesssim \mu_\lambda(B(x,r))$.*

A metric space $\tilde{X}$ is called Ahlfors Q-regular for some $Q > 0$ if it admits an Ahlfors Q-regular measure $\tilde{\mu}$, i.e., there is a constant $C > 0$ such that

$$C^{-1}R^Q \le \tilde{\mu}(B_R) \le CR^Q$$

for all closed balls B_R of radius $0 < R < \operatorname{diam}(\tilde{X})$. If a metric space $\tilde{X}$ is Ahlfors Q-regular, then $\tilde{X}$ has Hausdorff dimension precisely Q. We refer to [18, Section 8.3] for more information about Hausdorff dimension and Ahlfors regularities of measures and metric spaces.

The result in [3, Lemma 5.2] shows that the boundary ∂X of the K-regular tree X is Ahlfors regular with the regularity exponent depending only on K and on the metric density exponent ϵ of the tree.

Proposition 2.2. *The boundary ∂X is an Ahlfors Q-regular space with Hausdorff dimension*

$$Q = \frac{\log K}{\epsilon}.$$

Hence ∂X is equipped with an Ahlfors Q-regular measure ν:

$$\nu(B_{\partial X}(\xi, r)) \approx r^Q = r^{\log K/\epsilon},$$

for any $\xi \in \partial X$ and $0 < r \le \operatorname{diam}\partial X$.

Throughout the paper we assume that $1 \le p < +\infty$ and that X is a K-regular tree with $K \ge 2$.

2.2 Young functions and Orlicz spaces

In the standard definition of an Orlicz space, the function t^p of an L^p-space is replaced with a more general convex function, a Young function. We recall the definition of a Young function. We refer to [49, section 2.2] and [42] for more details about Young functions and we also warn the reader of slight differences between the definitions in various references.

A function $\Phi : [0,\infty) \to [0,\infty)$ is a *Young function* if it is a continuous, increasing and convex function satisfying $\Phi(0) = 0$,

$$\lim_{t\to 0+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to +\infty} \frac{\Phi(t)}{t} = +\infty.$$

A Young function Φ can be expressed as

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0,\infty) \to [0,\infty)$ is an increasing, right-continuous function with $\phi(0) = 0$ and $\lim\limits_{t\to+\infty} \phi(t) = +\infty$.

A Young function Φ is said to satisfy the Δ_2-condition if there is a constant $C_\Phi > 0$, called a *doubling constant* of Φ, such that

$$\Phi(2t) \le C_\Phi \Phi(t), \ \forall \ \ t \ge 0.$$

If Young function Φ satisfies the Δ_2-condition, then for any constant $c > 0$, there exist $c_1, c_2 > 0$ such that

$$c_1\Phi(t) \le \Phi(ct) \le c_2\Phi(t) \quad \text{for all} \quad t \ge 0,$$

where c_1 and c_2 depend only on c and the doubling constant C_Φ. Therefore, we obtain that if $A \approx B$, then $\Phi(A) \approx \Phi(B)$. This property will be used frequently in the rest of this paper.

Let Φ_1, Φ_2 be two Young functions. If there exist two constants $k > 0$ and $C \ge 0$ such that

$$\Phi_1(t) \le \Phi_2(kt) \quad \text{for} \quad t \ge C,$$

we write

$$\Phi_1 \prec \Phi_2.$$

The function $\Phi(t) = t^p \log^\lambda(e+t)$ with $p > 1, \lambda \in \mathbb{R}$ or $p = 1, \lambda \ge 0$ is a Young function and it satisfies the Δ_2-condition. Moreover, it also satisfies that

$$t^{\max\{p-\delta,1\}} \prec \Phi(t) \prec t^{p+\delta} \tag{2.3}$$

for any $\delta > 0$.

Let Φ be a Young function. Then the *Orlicz space* $L^\Phi(X)$ is defined by setting

$$L^\Phi(X, \mu_\lambda) = \left\{ u : X \to \mathbb{R} : u \text{ measurable}, \int_X \Phi(\alpha|u|)\, d\mu_\lambda < +\infty \text{ for some } \alpha > 0 \right\}.$$

As in the theory of L^p-spaces, the elements in $L^\Phi(X, \mu_\lambda)$ are actually equivalence classes consisting of functions that differ only on a set of measure zero. The Orlicz space $L^\Phi(X, \mu_\lambda)$ is a vector space and, equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi(X,\mu_\lambda)} = \inf \left\{ k > 0 : \int_X \Phi(|u|/k)\, d\mu_\lambda \le 1 \right\},$$

a Banach space, see [42, Theorem 3.3.10]. If $\Phi(t) = t^p$ with $p \ge 1$, then $L^\Phi(X, \mu_\lambda) = L^p(X, \mu_\lambda)$. We refer to [37, 42, 49] for more detailed discussions and properties of Orlicz spaces.

2.3 Newtonian spaces and Orlicz-Sobolev spaces on X

We call a Borel function $g : X \to [0, \infty]$ an *upper gradient* of $u \in L^1_{\text{loc}}(X, \mu_\lambda)$ if

$$|u(z) - u(y)| \le \int_\gamma g\, ds_X \tag{2.4}$$

whenever $z, y \in X$ and γ is the geodesic from z to y, where ds_X denotes the arc length measure with respect to the metric d_X. Since any rectifiable curve with end points z and

y in our tree contains the corresponding geodesic, the above definition is equivalent to the usual definition which requires that inequality (2.4) holds for all rectifiable curves with end points z and y. See [2, 16, 19, 20, 44] for a more detailed discussion on upper gradients.

The *Newtonian space* $N^{1,p}(X, \mu_\lambda)$, $1 \le p < \infty$, is the collection of all functions u for which the norm of u defined as

$$\|u\|_{N^{1,p}(X,\mu_\lambda)} := \left(\int_X |u|^p \, d\mu_\lambda + \inf_g \int_X g^p \, d\mu_\lambda \right)^{1/p}$$

is finite. Here the infimum is taken over all upper gradients of u.

For any Young function Φ, the *Orlicz-Sobolev space* $N^{1,\Phi}(X, \mu_\lambda)$ is defined as the collection of all functions u for which the norm of u defined as

$$\|u\|_{N^{1,\Phi}(X,\mu_\lambda)} = \|u\|_{L^\Phi(X,\mu_\lambda)} + \inf_g \|g\|_{L^\Phi(X,\mu_\lambda)}$$

is finite, where the infimum is taken over all upper gradients of u.

For the Young function $\Phi(t) = t^p$, $1 \le p < \infty$, the Orlicz-Sobolev space $N^{1,\Phi}(X, \mu_\lambda)$ is exactly the Newtonian space $N^{1,p}(X, \mu_\lambda)$. We refer to [49] for further results on Orlicz-Sobolev spaces on metric measure spaces. If $u \in N^{1,p}(X, \mu_\lambda)$ ($u \in N^{1,\Phi}(X, \mu_\lambda)$ with Φ doubling), then it has a minimal p-weak upper gradient (Φ-weak upper gradient) g_u, which in our case is an upper gradient. The minimal upper gradient is minimal in the sense that if $g \in L^p(X, \mu_\lambda)$ ($g \in L^\Phi(X, \mu_\lambda)$) is any upper gradient of u, then $g_u \le g$ a.e. We refer the interested reader to [16, Theorem 7.16] ($p \ge 1$) and [49, Corollary 6.9](Φ doubling) for proofs of the existence of such a minimal upper gradient.

2.4 Besov-type spaces on ∂X

Towards the definition of our Besov-type spaces, we recall a definition from [3].

Definition 2.3. For $0 < \theta < 1$ and $p \ge 1$, The Besov space $B^\theta_{p,p}(\partial X)$ consists of all functions $f \in L^p(\partial X)$ for which the seminorm $\|f\|_{\dot{B}^\theta_p(\partial X)}$ defined as

$$\|f\|^p_{\dot{B}^\theta_p(\partial X)} := \int_{\partial X} \int_{\partial X} \frac{|f(\zeta)| - f(\xi)|^p}{d_X(\zeta,\xi)^{\theta p} \nu(B(\zeta, d_X(\zeta,\xi)))} d\nu(\xi) \, d\nu(\zeta)$$

is finite. The corresponding norm for $B^\theta_{p,p}(\partial X)$ is

$$\|f\|_{B^\theta_{p,p}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{\dot{B}^\theta_p(\partial X)}.$$

We base our definition on a dyadic decomposition on the boundary ∂X of the K-regular tree X, see also [25, Section 2.4]. Let $V_n = \{x^n_j : j = 1, 2, \cdots, K^n\}$ be the set of all n-level vertices of the tree X for each $n \in \mathbb{N}$, where a vertex x is of *n-level* if $|x| = n$. Then

$$V = \bigcup_{n\in\mathbb{N}} V_n.$$

Given a vertex $x \in V$, set

$$I_x := \{\xi \in \partial X : \text{the geodesic ray } [0, \xi) \text{ passes through } x\}.$$

Let $\mathscr{Q} = \{I_x : x \in V\}$ and $\mathscr{Q}_n = \{I_x : x \in V_n\}$ for each $n \in \mathbb{N}$. Then $\mathscr{Q}_0 = \{\partial X\}$ and our dyadic decomposition $\mathscr{Q}$ satisfies

$$\mathscr{Q} = \bigcup_{n \in \mathbb{N}} \mathscr{Q}_n.$$

Given $I \in \mathscr{Q}_n$, there is a unique element $\widehat{I}$ in $\mathscr{Q}_{n-1}$ such that $I \subset \widehat{I}$. If $I = I_x$ for some $x \in V_n$, then $\widehat{I} = I_y$ where y is the unique mother of x in the tree X. Hence the structure of the dyadic decomposition of ∂X is uniquely determined by the structure of the K-regular tree X.

We recall a definition from [25].

Definition 2.4. For $0 \le \theta < 1$, $p \ge 1$ and $\lambda \in \mathbb{R}$, the Besov-type space $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ consists of all functions $f \in L^p(\partial X)$ for which the $\dot{\mathcal{B}}_p^{\theta,\lambda}$-dyadic energy of f defined as

$$\|f\|^p_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)} := \sum_{n=1}^{\infty} e^{\epsilon n \theta p} n^{\lambda} \sum_{I \in \mathscr{Q}_n} \nu(I) \left| f_I - f_{\widehat{I}} \right|^p$$

is finite. The norm on $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ is

$$\|f\|_{\mathcal{B}_p^{\theta,\lambda}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)}.$$

The measure ν above is the Ahlfors regular measure given by Proposition 2.2 and $f_I := ⨍_I f\, d\nu = \frac{1}{\nu(I)} \int_I f\, d\nu$ is the usual mean value.

Definition 2.5. For $0 < \theta < 1$ and $p \ge 1$, The Besov space $\mathcal{B}_p^{\theta}(\partial X)$ consists of all the functions $f \in L^p(\partial X)$ for which the $\dot{\mathcal{B}}_p^{\theta}$-dyadic energy of f defined as

$$\|f\|^p_{\dot{\mathcal{B}}_p^{\theta}(\partial X)} := \sum_{n=1}^{\infty} e^{\epsilon n \theta p} \sum_{I \in \mathscr{Q}_n} \nu(I) \left| f_I - f_{\widehat{I}} \right|^p$$

is finite. The norm of $\mathcal{B}_p^{\theta}(\partial X)$ is

$$\|f\|_{\mathcal{B}_p^{\theta}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{\dot{\mathcal{B}}_p^{\theta}(\partial X)}.$$

The Besov-type spaces $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ and $\mathcal{B}_p^{\theta}(\partial X)$ were first introduced in [25]. Notice that $\mathcal{B}_p^{\theta}(\partial X)$ coincides with $\mathcal{B}_p^{\theta,\lambda}(\partial X)$ when $\lambda = 0$. Next we introduce the *Hajłasz-Besov spaces* $N_{p,p}^{\theta}(\partial X)$ first introduced by [27] on the boundary ∂X.

Definition 2.6. (i) Let $0 < \theta < \infty$ and let u be a measurable function on ∂X. A sequence of nonnegative measurable functions, $\vec{g} = \{g_k\}_{k \in \mathbb{Z}}$, is called a *fractional θ-Hajłasz gradient* of u if there exists $Z \subset \partial X$ with $\nu(Z) = 0$ such that for all $k \in \mathbb{Z}$ and $\zeta, \xi \in \partial X \setminus Z$ satisfying $2^{-k-1} \le d_X(\zeta, \xi) < 2^{-k}$,

$$|u(\zeta) - u(\xi)| \le [d_X(\zeta, \xi)]^{\theta} [g_k(\zeta) + g_k(\xi)].$$

Denote by $\mathbb{D}^{\theta}(u)$ the *collection of all fractional θ-Hajłasz gradients of u.*

(ii) Let $0<\theta<\infty$ and $0<p<\infty$. The *Hajłasz-Besov space* $N^{\theta}_{p,p}(\partial X)$ consists of all functions $u\in L^p(\partial X)$ for which the seminorm $\|u\|_{\dot N^{\theta}_{p,p}(\partial X)}$ defined as

$$\|u\|_{\dot N^{\theta}_{p,p}(\partial X)} := \inf_{\vec g\in\mathbb{D}^{\theta}(u)} \|(\|g_k\|_{L^p(\partial X)})_{k\in\mathbb{Z}}\|_{l^p} = \inf_{\vec g\in\mathbb{D}^{\theta}(u)} \left(\sum_{k\in\mathbb{Z}}\int_{\partial X}[g_k(\xi)]^p\,d\nu(\xi)\right)^{1/p}$$

is finite. The norm of $N^{\theta}_{p,p}(\partial X)$ is

$$\|u\|_{N^{\theta}_{p,p}(\partial X)} := \|u\|_{L^p(\partial X)} + \|u\|_{\dot N^{\theta}_{p,p}(\partial X)}.$$

The following proposition states that these three Besov-type spaces $\mathcal{B}^{\theta}_p(\partial X)$, $B^{\theta}_{p,p}(\partial X)$ and $N^{\theta}_{p,p}(\partial X)$ coincide with each other.

Proposition 2.7. *(i) Let $0<\theta<1$ and $p\geq 1$. For any $f\in L^1_{\rm loc}(\partial X)$, we have*

$$\|f\|_{\dot B^{\theta}_p(\partial X)} \approx \|f\|_{\dot{\mathcal{B}}^{\theta}_p(\partial X)} \approx \|f\|_{\dot N^{\theta}_{p,p}(\partial X)}.$$

(ii) Let $0<s<\theta<1$, $\lambda\in\mathbb{R}$ and $p\geq 1$. For any $f\in L^1_{\rm loc}(\partial X)$, we have

$$\|f\|_{\dot{\mathcal{B}}^{s}_p(\partial X)} \lesssim \|f\|_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)}.$$

Proof. (i): The first part $\|f\|_{\dot B^{\theta}_p(\partial X)} \approx \|f\|_{\dot{\mathcal{B}}^{\theta}_p(\partial X)}$ follows by [25, Proposition 2.13]. The second part $\|f\|_{\dot B^{\theta}_p(\partial X)} \approx \|f\|_{\dot N^{\theta}_{p,p}(\partial X)}$ is given by [3, Lemma 5.4] and [15, Theorem 1.2].

(ii): From the definitions of the Besov-types norms, we have

$$\|f\|_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} = \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda} \sum_{I\in\mathcal{Q}_n} \nu(I)\left|f_I - f_{\hat I}\right|^p$$

and

$$\|f\|_{\dot{\mathcal{B}}^{s}_p(\partial X)} = \sum_{n=1}^{\infty} e^{\epsilon nsp} \sum_{I\in\mathcal{Q}_n} \nu(I)\left|f_I - f_{\hat I}\right|^p.$$

For $0<s<\theta<1$, we have $e^{\epsilon nsp}\lesssim e^{\epsilon n\theta p}n^{\lambda}$ for all $n\in\mathbb{N}$. Hence the result $\|f\|_{\dot{\mathcal{B}}^{s}_p(\partial X)} \lesssim \|f\|_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)}$ follows. □

The dyadic norms give an easy way to introduce Orlicz-Besov spaces by replacing t^p with some Zygmund function (logarithmic Orlicz function) $\Phi(t)$.

Definition 2.8. Let Φ be the Young function $\Phi(t)=t^p\log^{\lambda_1}(e+t)$ with $p>1, \lambda_1\in\mathbb{R}$ or $p=1, \lambda_1\geq 0$. Then the Orlicz-Besov space $\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$ consists of all $f\in L^{\Phi}(\partial X)$ whose norm generally defined as

$$\|f\|_{\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)} := \|f\|_{L^{\Phi}(\partial X)} + \inf\left\{k>0 : |f/k|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_{\Phi}(\partial X)}\leq 1\right\}$$

is finite, where for any $g\in L^1_{\rm loc}(\partial X)$, the $\dot{\mathcal{B}}^{\theta,\lambda_2}_{\Phi}$-dyadic energy is defined as

$$|g|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_{\Phi}(\partial X)} := \sum_{n=1}^{\infty} e^{\epsilon n(\theta-1)p} n^{\lambda_2} \sum_{I\in\mathcal{Q}_n} \nu(I)\Phi\left(\frac{\left|g_I - g_{\hat I}\right|}{e^{-\epsilon n}}\right).$$

In this paper, we are only interested in the Young functions in the above definition. Hence in the rest of this paper, we always assume that the Young function is $\Phi(t) = t^p \log^{\lambda_1}(e+t)$ with $p>1, \lambda_1 \in \mathbb{R}$ or $p=1, \lambda_1 \geq 0$.

3 Proofs

3.1 Proof of Theorem 1.1

We prove Theorem 1.1 by two parts: trace part and extension part. In the trace part, we give the definition of trace $\operatorname{Tr} f$ in (3.1) via limits along geodesic rays for any function $f \in N^{1,\Phi}(X, \mu_{\lambda_2})$. Then we prove the existence of the trace function $\operatorname{Tr} f$ and prove the norm estimate $\|\operatorname{Tr} f\|_{\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)} \lesssim \|f\|_{N^{1,\Phi}(X,\mu_{\lambda_2})}$. For the extension part, we give the definition of the extension Eu in (3.14)-(3.16) for any function $u \in \mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$. Then we show that $\operatorname{Tr}(Eu) = u$ (i.e., $\operatorname{Tr} \circ E = \operatorname{Id}$) and prove the norm estimate $\|Eu\|_{N^{1,\Phi}(X,\mu_{\lambda_2})} \lesssim \|u\|_{\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)}$.

Proof. **Trace Part:** Let $f \in N^{1,\Phi}(X)$. We follow an idea from [25] and set

$$\operatorname{Tr} f(\xi) := \tilde{f}(\xi) = \lim_{[0,\xi) \ni x \to \xi} f(x), \quad \xi \in \partial X, \tag{3.1}$$

provided that the limit taken along the geodesic ray $[0,\xi)$ exists. We begin by showing that the above limit exists for ν-a.e. $\xi \in \partial X$.

Since g_f is an upper gradient of f, it suffices to show that the function $\tilde{f}^*$ defined by setting

$$\tilde{f}^*(\xi) = |f(0)| + \int_{[0,\xi)} g_f \, ds \tag{3.2}$$

belongs to $L^\Phi(\partial X)$, where $[0,\xi)$ is the geodesic ray from 0 to ξ. Indeed, if $\tilde{f}^* \in L^\Phi(\partial X)$, we have $|\tilde{f}^*| < \infty$ for ν-a.e. $\xi \in \partial X$, and hence the limit in (3.1) exists for ν-a.e. $\xi \in \partial X$.

Fix $\xi \in \partial X$. Set $r_j = 2e^{-j\epsilon}/\epsilon$ and $x_j = x_j(\xi)$ be the ancestor of ξ with $|x_j| = j$ for $j \in \mathbb{N}$. Recall from (2.1) and (2.2) that

$$ds(x) = e^{-\epsilon|x|}\, d\,|x|, \quad d\mu_{\lambda_2}(x) \approx e^{\beta}|x|^{\lambda_2}\, d\,|x|.$$

Then for any $y \in [x_j, x_{j+1}]$, we have that

$$ds(y) \approx e^{(\beta-\epsilon)j} j^{-\lambda_2}\, d\mu_{\lambda_2}(y) \approx r_j^{1-\beta/\epsilon} j^{-\lambda_2}\, d\mu_{\lambda_2}(y), \qquad \mu_{\lambda_2}([x_j, x_{j+1}]) \approx r_j^{\beta/\epsilon} j^{\lambda_2}, \tag{3.3}$$

where $[x_j, x_{j+1}]$ is the edge connecting $x_j = x_j(\xi)$ and $x_{j+1} = x_{j+1}(\xi)$. Thus

$$\begin{aligned} \tilde{f}^*(\xi) &= |f(0)| + \sum_{j=0}^{+\infty} \int_{[x_j,x_{j+1}]} g_f \, ds \\ &\approx |f(0)| + \sum_{j=0}^{+\infty} r_j^{1-\beta/\epsilon} j^{-\lambda_2} \int_{[x_j,x_{j+1}]} g_f \, d\mu_{\lambda_2} \end{aligned}$$

$$\approx |f(0)| + \sum_{j=0}^{+\infty} r_j \fint_{[x_j,x_{j+1}]} g_f \, d\mu_{\lambda_2}. \tag{3.4}$$

Since $\theta = 1 - (\beta - \log K)/(p\epsilon) > 0$, we may choose $1 \le q < \infty$ such that $\max\{(\beta - \log K)/\epsilon, 1\} < q < p$ if $p > 1$ or $q = 1 = p$. Let $\Psi(t) := t^{p/q} \log^{\lambda_1/q}(e + t)$. Then $\Psi^q = \Phi$ and Ψ is a Young function satisfying the Δ_2-condition. By the Jensen inequality and the Δ_2 property of Ψ, since $\sum_{j=0}^{+\infty} r_j \approx 1$, we have that

$$\begin{aligned}\Psi(\tilde{f}^*(\xi)) &\lesssim \Psi(|f(0)|) + \Psi\left(\sum_{j=0}^{+\infty} r_j \fint_{[x_j,x_{j+1}]} g_f \, d\mu_{\lambda_2}\right) \\ &\lesssim \Psi(|f(0)|) + \sum_{j=0}^{+\infty} r_j \fint_{[x_j,x_{j+1}]} \Psi(g_f) \, d\mu_{\lambda_2}.\end{aligned}$$

Choose $0 < \kappa < 1 - (\beta - \log K)/(q\epsilon)$. If $q > 1$, by the Hölder inequality, we obtain the estimate

$$\Phi(\tilde{f}^*(\xi)) = \Psi(\tilde{f}^*(\xi))^q \lesssim \Phi(|f(0)|) + \left(\sum_{j=0}^{+\infty} r_j^{\kappa} r_j^{(1-\kappa)} \fint_{[x_j,x_{j+1}]} \Psi(g_f) \, d\mu_{\lambda_2}\right)^q \tag{3.5}$$

$$\lesssim \Phi(|f(0)|) + \sum_{j=0}^{+\infty} r_j^{(1-\kappa)q} \left(\fint_{[x_j,x_{j+1}]} \Psi(g_f) \, d\mu_{\lambda_2}\right)^q \tag{3.6}$$

$$\lesssim \Phi(|f(0)|) + \sum_{j=0}^{+\infty} r_j^{q-\kappa q-\beta/\epsilon} j^{-\lambda_2} \int_{[x_j,x_{j+1}]} \Phi(g_f) \, d\mu_{\lambda_2}. \tag{3.7}$$

Second inequality follows from the fact that

$$\sum_{j=0}^{+\infty} r_j^{\kappa q/(q-1)} \approx 1.$$

If $q = 1$, then $\Psi = \Phi$, and hence the estimates (3.5)-(3.7) are not needed. We conclude that

$$\Phi(\tilde{f}^*(\xi)) \lesssim \Phi(|f(0)|) + \sum_{j=0}^{+\infty} r_j^{q-\kappa q-\beta/\epsilon} j^{-\lambda_2} \int_{[x_j,x_{j+1}]} \Phi(g_f) \, d\mu_{\lambda_2}.$$

Since $\nu(\partial X) \approx 1$, integration of this estimate over ∂X together with Fubini's theorem gives

$$\int_{\partial X} \Phi(\tilde{f}^*(\xi)) \, d\nu \lesssim \Phi(|f(0)|) + \int_{\partial X} \sum_{j=0}^{+\infty} r_j^{q-\kappa q-\beta/\epsilon} j^{-\lambda_2} \int_{[x_j,x_{j+1}]} \Phi(g_f) \, d\mu_{\lambda_2} \, d\nu(\xi)$$

$$= \Phi(|f(0)|) + \int_X \Phi(g_f(x)) \int_{\partial X} \sum_{j=0}^{+\infty} r_j^{q-\kappa q-\beta/\epsilon} j^{-\lambda_2} \chi_{[x_j,x_{j+1}]}(x) \, d\nu(\xi) \, d\mu_{\lambda_2}(x). \tag{3.8}$$

Since $\chi_{[x_j,x_{j+1}]}(x)$ is nonzero only if $j \le |x| \le j+1$ and $x < \xi$, our estimate (3.8) can be reformulated as

$$\int_{\partial X} \Phi(\tilde{f}^*(\xi))\, d\nu \lesssim \Phi(|f(0)|) + \int_X \Phi(g_f(x)) r_{j(x)}^{q-\kappa q-\beta/\epsilon} j(x)^{-\lambda_2} \nu(E(x))\, d\mu_{\lambda_2}(x), \tag{3.9}$$

where $E(x) = \{\xi \in \partial X : x < \xi\}$ and $j(x)$ is the largest integer such that $j(x) \le |x|$.

By Proposition 2.2, we have $\nu(E(x)) \lesssim r_{j(x)}^Q$, since $E(x) \subset \overline{B(\xi, r)}$ for any $\xi \in E(x)$ and $r = \operatorname{diam}(E(x)) \lesssim e^{-\epsilon j(x)} \approx r_{j(x)}$, see [3, Lemma 5.21]. This together with $q - \kappa q - \beta/\epsilon + Q > 0$ gives

$$r_{j(x)}^{p(1-\kappa)-\beta/\epsilon+Q} j(x)^{-\lambda_2} \lesssim 1.$$

Consequently, (3.9) implies that

$$\begin{aligned}\int_{\partial X} \Phi(\tilde{f}^*(\xi))\, d\nu &\lesssim \Phi(|f(0)|) + \int_X \Phi(g_f(x)) r_{j(x)}^{q-\kappa q-\beta/\epsilon+Q} j(x)^{-\lambda_2}\, d\mu_{\lambda_2}(x)\\ &\lesssim \Phi(|f(0)|) + \int_X \Phi(g_f(x))\, d\mu_{\lambda_2}(x).\end{aligned}$$

Actually, the value $|f(0)|$ is not essential. For any $y \in \{x \in X : |x| < 1\}$, a neighborhood of 0, we could modify the definition of $\tilde{f}^*(\xi)$ as

$$\tilde{f}^{**}(\xi) = |f(y)| + |f(y) - f(0)| + \int_{[0,\xi)} g_f\, ds.$$

Since $\mu_{\lambda_2}(X) \approx 1$, we have that

$$\Phi(|f(y) - f(0)|) \le \Phi\left(\int_{[0,y]} g_f\, ds\right) \le \Phi\left(\int_X g_f\, ds\right) \lesssim \int_X \Phi(g_f)\, d\mu_{\lambda_2}.$$

By the same argument as above, we obtain the estimate

$$\int_{\partial X} \Phi(\tilde{f}^{**}(\xi))\, d\nu(\xi) \lesssim \Phi(|f(y)|) + \int_X \Phi(g_f)\, d\mu_{\lambda_2},$$

for any $y \in \{x \in X : |x| < 1\}$. The fact that $f \in L^\Phi(X, \mu_{\lambda_2})$ gives us that $\Phi(|f(y)|) < \infty$ for μ_{λ_2}-a.e. $y \in X$. This shows that $\tilde{f}^{**}(\xi)$ is L^Φ-integrable on ∂X, which finishes the proof of the existence of the limit in (3.1).

We continue towards norm estimates. Since $|\tilde{f}| \le \tilde{f}^*$ for any modified $\tilde{f}^*$, the above arguments also show that for any $y \in \{x \in X : |x| < 1\}$, we have that

$$\int_{\partial X} \Phi(\tilde{f}(\xi))\, d\nu(\xi) \lesssim \Phi(|f(y)|) + \int_X \Phi(g_f)\, d\mu_{\lambda_2}.$$

Integrating over all $y \in \{x \in X : |x| < 1\}$, since $\mu_{\lambda_2}(\{x \in X : |x| < 1\}) \approx 1$, we arrive at the estimate

$$\int_{\partial X} \Phi(\tilde{f}(\xi))\, d\nu(\xi) \lesssim \int_X \Phi(|f|)\, d\mu_{\lambda_2} + \int_X \Phi(g_f)\, d\mu_{\lambda_2}. \tag{3.10}$$

Assume that $\|f\|_{L^\Phi(X,\mu_{\lambda_2})} = t_1$ and $\|g_f\|_{L^\Phi(X,\mu_{\lambda_2})} = t_2$. By the definition of Luxemburg norms, we know that

$$\int_X \Phi(f/t_1)\, d\mu_{\lambda_2} \le 1 \quad \text{and} \quad \int_X \Phi(g_f/t_2)\, d\mu_{\lambda_2} \le 1.$$

By estimate (3.10), there exists a constant $C > 0$ such that

$$\int_{\partial X} \Phi(\tilde f(\xi))\, d\nu(\xi) \le C \left(\int_X \Phi(|f|)\, d\mu_{\lambda_2} + \int_X \Phi(g_f)\, d\mu_{\lambda_2} \right).$$

We may assume $C \ge 1$, since if $C < 1$, we choose $C = 1$. Then we obtain that

$$\begin{aligned}\int_{\partial X} \Phi\left(\frac{\tilde f(\xi)}{2C(t_1+t_2)}\right) d\nu &\le C \left(\int_X \Phi\left(\frac{f}{2Ct_1}\right) d\mu_{\lambda_2} + \int_X \Phi\left(\frac{g_f}{2Ct_2}\right) d\mu_{\lambda_2} \right) \\ &\le \frac{1}{2} \left(\int_X \Phi(f/t_1)\, d\mu_{\lambda_2} + \int_X \Phi(g_f/t_2)\, d\mu_{\lambda_2} \right) \le 1,\end{aligned}$$

which implies

$$\|\tilde f\|_{L^\Phi(\partial X)} \le 2C(t_1+t_2) \approx \|f\|_{L^\Phi(X,\mu_{\lambda_2})} + \|g_f\|_{L^\Phi(X,\mu_{\lambda_2})} = \|f\|_{N^{1,\phi}(X,\mu_{\lambda_2})}. \tag{3.11}$$

Next, we estimate the dyadic energy $|\tilde f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_{\Phi}(\partial X)}$. Given $I \in \mathscr{Q}_n$, $\xi \in I$ and $\zeta \in \widehat{I}$, we have $x_{n-1} = y_{n-1}$, where $x_j = x_j(\xi)$ and $y_j = y_j(\zeta)$ are the ancestors of ξ and ζ with $|x_j| = |y_j| = j$, and therefore

$$|\tilde f(\xi) - \tilde f(\zeta)| \le \sum_{j=n-1}^{+\infty} |f(x_j) - f(x_{j+1})| + \sum_{j=n-1}^{+\infty} |f(y_j) - f(y_{j+1})|. \tag{3.12}$$

By (3.3) and an argument similar to (3.4), we infer from (3.12) that

$$|\tilde f(\xi) - \tilde f(\zeta)| \lesssim \sum_{j=n-1}^{+\infty} r_j \fint_{[x_j,x_{j+1}]} g_f\, d\mu_{\lambda_2} + \sum_{j=n-1}^{+\infty} r_j \fint_{[y_j,y_{j+1}]} g_f\, d\mu_{\lambda_2}.$$

It follows from the Jensen inequality that

$$\Psi\left(\frac{|\tilde f(\xi) - \tilde f(\zeta)|}{e^{-\epsilon n}}\right) \lesssim \sum_{j=n-1}^{+\infty} r_{n-1}^{-1} r_j \fint_{[x_j,x_{j+1}]} \Psi(g_f)\, d\mu_{\lambda_2} + \sum_{j=n-1}^{+\infty} r_{n-1}^{-1} r_j \fint_{[y_j,y_{j+1}]} \Psi(g_f)\, d\mu_{\lambda_2},$$

since we have the estimate

$$r_{n-1} \approx e^{-\epsilon n} \approx \sum_{j=n-1}^{+\infty} r_j.$$

By using the fact $\Phi = \Psi^q$ and the Hölder inequality if $q > 1$, we get that

$$\Phi\left(\frac{|\tilde f(\xi) - \tilde f(\zeta)|}{e^{-\epsilon n}}\right) = \Psi\left(\frac{|\tilde f(\xi) - \tilde f(\zeta)|}{e^{-\epsilon n}}\right)^q$$

$$\lesssim r_{n-1}^{-q+\kappa q}\sum_{j=n-1}^{+\infty} r_j^{q-\beta/\epsilon-\kappa q} j^{-\lambda_2}\left(\int_{[x_j,x_{j+1}]}\Phi(g_f)\,d\mu_{\lambda_2}+\int_{[y_j,y_{j+1}]}\Phi(g_f)\,d\mu_{\lambda_2}\right).$$

If $q=1$, then $\Phi=\Psi$ and it is easy to check that the above estimate still holds. Since $\nu(I)\approx\nu(\widehat{I})$ and $\widehat{I}$ is the mother of I, it follows from Fubini's theorem that

$$\begin{aligned}\sum_{I\in\mathscr{Q}_n}\nu(I)\Phi\left(\frac{|\tilde{f}_I-\tilde{f}_{\widehat{I}}|}{e^{-\epsilon n}}\right) &\le \sum_{I\in\mathscr{Q}_n}\nu(I)⨍_I⨍_{\widehat{I}}\Phi\left(\frac{|\tilde{f}(\xi)-\tilde{f}(\zeta)|}{e^{-\epsilon n}}\right)d\nu(\zeta)\,d\nu(\xi)\\ &\lesssim \int_{\partial X} r_{n-1}^{-q+\kappa q}\sum_{j=n-1}^{+\infty} r_j^{q-\beta/\epsilon-\kappa q} j^{-\lambda_2}\int_{[x_j,x_{j+1}]}\Phi(g_f)\,d\mu_{\lambda_2}\,d\nu(\xi)\\ &= \int_{X\cap\{|x|\ge n-1\}}\Phi(g_f)r_{n-1}^{-q+\kappa q}\int_{\partial X}\sum_{j=n-1}^{+\infty} r_j^{q-\beta/\epsilon-\kappa q} j^{-\lambda_2}\chi_{[x_j,x_{j+1}]}(x)\,d\nu(\xi)\,d\mu_{\lambda_2}(x).\end{aligned}\tag{3.13}$$

Note again that $\chi_{[x_j,x_{j+1}]}(x)$ is nonzero only if $j\le|x|\le j+1$ and $x<\xi$. Recall that $E(x)=\{\xi\in\partial X: x<\xi\}$, that $j(x)$ is the largest integer such that $j(x)\le|x|$ and that $\nu(E(x))\lesssim r_{j(x)}^{Q}$. Hence (3.13) gives

$$\begin{aligned}\sum_{I\in\mathscr{Q}_n}\nu(I)\Phi\left(\frac{|\tilde{f}_I-\tilde{f}_{\widehat{I}}|}{e^{-\epsilon n}}\right) &\lesssim \int_{X\cap\{|x|\ge n-1\}}\Phi(g_f)r_{n-1}^{-q+\kappa q}r_{j(x)}^{q-\beta/\epsilon-\kappa q} j(x)^{-\lambda_2}\nu(E(x))\,d\mu_{\lambda_2}(x)\\ &\lesssim \int_{X\cap\{|x|\ge n-1\}}\Phi(g_f)r_{n-1}^{-q+\kappa q}r_{j(x)}^{q-\beta/\epsilon-\kappa q+Q} j(x)^{-\lambda_2}\,d\mu_{\lambda_2}(x).\end{aligned}$$

Since $e^{-\epsilon n}\approx r_{n-1}$, we conclude the estimate

$$\begin{aligned}|\tilde{f}|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_{\Phi}(\partial X)} &\lesssim \sum_{n=1}^{+\infty} r_{n-1}^{(1-\theta)p-q+\kappa q} n^{\lambda_2}\int_{X\cap\{|x|\ge n-1\}}\Phi(g_f)r_{j(x)}^{q-\beta/\epsilon-\kappa q+Q} j(x)^{-\lambda_2}\,d\mu_{\lambda_2}(x)\\ &= \sum_{n=0}^{+\infty} r_n^{(1-\theta)p-q+\kappa q}(n+1)^{\lambda_2}\sum_{j=n}^{+\infty}\int_{X\cap\{j\le|x|<j+1\}}\Phi(g_f)r_j^{q-\beta/\epsilon-\kappa q+Q} j^{-\lambda_2}\,d\mu_{\lambda_2}(x)\\ &= \sum_{j=0}^{+\infty}\int_{X\cap\{j\le|x|<j+1\}}\Phi(g_f)r_j^{q-\beta/\epsilon-\kappa q+Q} j^{-\lambda_2}\,d\mu_{\lambda_2}(x)\left(\sum_{n=0}^{j} r_n^{(1-\theta)p-q+\kappa q}(n+1)^{\lambda_2}\right).\end{aligned}$$

Recall that $r_n=2e^{-n\epsilon}/\epsilon$ and

$$(1-\theta)p-q+\kappa q=\kappa q-(q-(\beta-\log K)/\epsilon)=\kappa q+\beta/\epsilon-q-\log K/\epsilon<0.$$

Hence we obtain that

$$\sum_{n=0}^{j} r_n^{(1-\theta)p-q+\kappa q}(n+1)^{\lambda_2}\approx r_j^{\kappa q+\beta/\epsilon-q-\log K/\epsilon}(j+1)^{\lambda_2}\approx r_j^{\kappa q+\beta/\epsilon-q-Q} j^{\lambda_2}.$$

Therefore, our estimate above for the dyadic energy can be rewritten as

$$|\tilde{f}|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_{\Phi}(\partial X)} \lesssim \sum_{j=0}^{+\infty} \int_{X\cap\{j\le|x|<j+1\}} \Phi(g_f)\, d\mu_{\lambda_2}(x) = \int_X \Phi(g_f)\, d\mu_{\lambda_2}(x).$$

By an argument similar to the one that we used to prove (3.11) after getting (3.10), we have that

$$\inf\left\{k>0 : |\tilde{f}/k|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_{\Phi}(\partial X)} \le 1\right\} \lesssim \|g_f\|_{L^{\Phi}(X,\mu_{\lambda_2})},$$

which together with (3.11) gives the norm estimate

$$\|\tilde{f}\|_{\mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)} \lesssim \|f\|_{N^{1,\Phi}(X,\mu_{\lambda_2})}.$$

Extension Part: Fix $u \in \mathcal{B}^{\theta,\lambda_2}_{\Phi}(\partial X)$. Given $x \in X$ with $|x| = n \in \mathbb{N}$, set

$$Eu(\xi) = \tilde{u}(x) = \fint_{I_x} u\, d\nu, \tag{3.14}$$

where $I_x \in \mathscr{Q}_n$ is the set of all the points $\xi \in \partial X$ such that the geodesic ray $[0,\xi)$ passes through x.

Let y be a child of x. Then $|y| = n+1$ and I_x is the mother of I_y. We define $\tilde{u}$ on the edge $[x,y]$ by setting

$$g_{\tilde{u}}(t) := \frac{\tilde{u}(y)-\tilde{u}(x)}{d_X(x,y)} = \frac{\epsilon(u_{I_y}-u_{I_x})}{(1-e^{-\epsilon})e^{-\epsilon n}} = \frac{\epsilon(u_{I_y}-u_{\widehat{I_y}})}{(1-e^{-\epsilon})e^{-\epsilon n}} \tag{3.15}$$

and

$$\tilde{u}(t) := \tilde{u}(x) + g_{\tilde{u}}(t) d_X(x,t). \tag{3.16}$$

By repeating this procedure for all edges, we obtain an extension $\tilde{u}$ of u. Then (3.1) and (3.14) imply that $\operatorname{Tr}\tilde{u}(\xi) = u(\xi)$ whenever $\xi \in \partial X$ is a Lebesgue point of u.

Simple integration shows that $|g_{\tilde{u}}|$ is an upper gradient of $\tilde{u}$. Clearly

$$\begin{aligned}\int_{[x,y]} \Phi(|g_{\tilde{u}}|)\, d\mu_{\lambda_2} &\approx \int_n^{n+1} \Phi\left(\frac{|u_{I_y}-u_{\widehat{I_y}}|}{e^{-\epsilon(n+1)}}\right) e^{-\beta\tau}(\tau+C)^{\lambda_2}\, d\tau \\ &\approx e^{-\beta(n+1)}(n+1)^{\lambda_2}\Phi\left(\frac{|u_{I_y}-u_{\widehat{I_y}}|}{e^{-\epsilon(n+1)}}\right).\end{aligned}$$

By summing over all the edges of X, we conclude that

$$\int_X \Phi(|g_{\tilde{u}}|)\, d\mu_{\lambda_2} \approx \sum_{n=1}^{+\infty}\sum_{I\in\mathscr{Q}_n} e^{-\beta n} n^{\lambda_2} \Phi\left(\frac{|u_I - u_{\widehat{I}}|}{e^{-\epsilon n}}\right). \tag{3.17}$$

We have that

$$\nu(I) \approx e^{-\epsilon n Q} = e^{-nK}$$

whenever $I \in \mathscr{Q}_n$, which implies that

$$e^{\epsilon n(\theta-1)p}\nu(I) \approx e^{-\epsilon n((\beta-\log K)/\epsilon+Q)} \approx e^{-\beta n}. \tag{3.18}$$

The above estimates (3.17) and (3.18) give

$$\int_X \Phi(|g_{\tilde{u}}|)\, d\mu_{\lambda_2} \approx \sum_{n=1}^{\infty} e^{\epsilon n(\theta-1)p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I)\Phi\left(\frac{|u_I - u_{\widehat{I}}|}{e^{-\epsilon n}}\right) = |u|_{\dot{\mathcal{B}}_{\Phi}^{\theta,\lambda_2}(\partial X)}. \tag{3.19}$$

When obtaining the L^{Φ}-estimate of $\tilde{u}$, notice that when $|x| = n$ and y is the child of x,

$$|\tilde{u}(t)| \le |\tilde{u}(x)| + |g_{\tilde{u}}| d_X(x,y) = |\tilde{u}(x)| + |\tilde{u}(y) - \tilde{u}(x)| \lesssim |u_{I_x}| + |u_{I_y}| \tag{3.20}$$

for any $t \in [x,y]$. Since $\mu_{\lambda_2}([x,y]) \approx e^{-\beta n} n^{\lambda_2}$ and $\nu(I_x) \approx \nu(I_y) \approx e^{-\epsilon n Q}$, this gives us

$$\int_{[x,y]} \Phi(|\tilde{u}(t)|)\, d\mu_{\lambda_2} \lesssim \mu_{\lambda_2}([x,y])\big(\Phi(|u_{I_x}|) + \Phi(|u_{I_y}|)\big) \lesssim e^{-\beta n+\epsilon n Q} n^{\lambda_2} \int_{I_x} \Phi(|u|)\, d\nu.$$

By summing over all the edges of X, we arrive at

$$\begin{aligned}\int_X \Phi(|\tilde{u}(t)|)\, d\mu_{\lambda_2} &\lesssim \sum_{n=0}^{+\infty} \sum_{I\in\mathscr{Q}_n} e^{-\beta n+\epsilon n Q} n^{\lambda_2} \int_I \Phi(|u|)\, d\nu \\ &= \sum_{n=0}^{+\infty} e^{-\beta n+\epsilon n Q} n^{\lambda_2} \int_{\partial X} \Phi(|u|)\, d\nu.\end{aligned}$$

The sum of $e^{-\beta n+\epsilon n Q} n^{\lambda_2}$ converges, because $\beta - \epsilon Q = \beta - \log K > 0$. It follows that

$$\int_X \Phi(|\tilde{u}(t)|)\, d\mu_{\lambda_2} \lesssim \int_{\partial X} \Phi(|u|)\, d\nu. \tag{3.21}$$

Applying the very same arguments that we used in proving (3.11) after getting (3.10) to (3.19) and (3.21), we finally arrive at the desired estimate for the norms

$$\|\tilde{u}\|_{N^{1,\Phi}(X,\mu_{\lambda_2})} \lesssim \|u\|_{\mathcal{B}_{\Phi}^{\theta,\lambda_2}(\partial X)}.$$

□

3.2 Proof of Proposition 1.2

In this section, we always assume that $\Phi(t) = t^p \log^{\lambda_1}(e+t)$ with $p > 1, \lambda_1 \in \mathbb{R}$ or $p = 1, \lambda_1 \ge 0$.

Lemma 3.1. *Let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$. Assume that $\lambda_1 + \lambda_2 = \lambda$. For any $f \in L^1(\partial X)$, we have that the condition $\|f\|_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)} < \infty$ is equivalent to the condition $|f|_{\dot{\mathcal{B}}_{\Phi}^{\theta,\lambda_2}(\partial X)} < \infty$ whenever $0 < \theta < 1$.*

Proof. When $\lambda_1 = 0$, then the result is obvious since $\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} = |f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X)}$.

When $\lambda_1 > 0$, first we estimate the logarithmic term from above. Since $f \in L^1(\partial X)$, for any $I \in \mathscr{Q}_n$, it follows from $\nu(I) \approx \nu(\widehat{I}) \approx e^{-n\log K}$ that

$$\log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \le \log^{\lambda_1}\left(e + \frac{|f_I| + |f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \lesssim \log^{\lambda_1}\left(e + \frac{\|f\|_{L^1(\partial X)}}{e^{-(\epsilon+\log K)n}}\right) \le Cn^{\lambda_1},$$

where $C = C(\|f\|_{L^1(\partial X)}, \lambda_1, \epsilon, K)$. Hence we can estimate $|f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X)}$ as follows:

$$\begin{aligned}
|f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X)} &= \sum_{n=1}^{\infty} e^{\epsilon n(\theta-1)p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I)\Phi\left(\frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right)\\
&= \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I)|f_I - f_{\widehat{I}}|^p \log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right)\\
&\le C\sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2+\lambda_1} \sum_{I\in\mathscr{Q}_n} \nu(I)|f_I - f_{\widehat{I}}|^p = C\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)},
\end{aligned}$$

where $C = C(\|f\|_{L^1(\partial X)}, \lambda_1, \epsilon, K)$.

In order to estimate the logarithmic term from below, for any $I \in \mathscr{Q}_n$, we define

$$\chi(n, I) = \begin{cases} 1, & \text{if } |f_I - f_{\widehat{I}}| > e^{-\epsilon n(\theta+1)/2} \\ 0, & \text{otherwise.} \end{cases} \tag{3.22}$$

Then we have that

$$\begin{aligned}
\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} &= \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda} \sum_{I\in\mathscr{Q}_n} \nu(I)|f_I - f_{\widehat{I}}|^p\\
&= \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda} \sum_{I\in\mathscr{Q}_n} \nu(I)\chi(n,I)|f_I - f_{\widehat{I}}|^p\\
&\quad + \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda} \sum_{I\in\mathscr{Q}_n} \nu(I)(1-\chi(n,I))|f_I - f_{\widehat{I}}|^p\\
&=: P_1 + P_2.
\end{aligned}$$

If $|f_I - f_{\widehat{I}}| > e^{-\epsilon n(\theta+1)/2}$, since $\theta < 1$ and $\lambda_1 > 0$, we obtain that

$$\log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) > \log^{\lambda_1}\left(e + e^{\epsilon n(1-\theta)/2}\right) \ge Cn^{\lambda_1},$$

where $C = C(\epsilon, \theta, \lambda_1)$. Hence we have the estimate

$$P_1 \le C\sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} |f_I - f_{\widehat{I}}|^p \log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) = C|f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X)}.$$

For P_2, since $\sum_{I\in\mathscr{Q}_n}\nu(I)\approx 1$, we have that

$$P_2 \le \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda} \sum_{I\in\mathscr{Q}_n} \nu(I) e^{-\epsilon np(\theta+1)/2} \approx \sum_{n=1}^{\infty} e^{\epsilon np(\theta-1)/2} n^{\lambda} = C' < +\infty,$$

where $C' = C'(\theta, p, \lambda)$. Therefore, we obtain

$$\frac{1}{C}|f|_{\dot{\mathcal{B}}_{\Phi}^{\theta,\lambda_2}(\partial X)} \le \|f\|^p_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)} = P_1 + P_2 \le C|f|_{\dot{\mathcal{B}}_{\Phi}^{\theta,\lambda_2}(\partial X)} + C', \tag{3.23}$$

where C and C' are constants depending only on $\epsilon, \theta, \lambda_1, \lambda, p$ and $\|f\|_{L^1(\partial X)}$.

When $\lambda_1 < 0$, in order to estimate the logarithmic term from above, using definition (3.22), we obtain that

$$\begin{aligned}
|f|_{\dot{\mathcal{B}}_{\Phi}^{\theta,\lambda_2}(\partial X)} &= \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I)|f_I - f_{\widehat{I}}|^p \log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \\
&= \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I)\chi(n,I)|f_I - f_{\widehat{I}}|^p \log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \\
&\quad + \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I)(1-\chi(n,I))|f_I - f_{\widehat{I}}|^p \log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \\
&=: P_1' + P_2'.
\end{aligned}$$

If $|f_I - f_{\widehat{I}}| > e^{-\epsilon n(\theta+1)/2}$, since $\theta < 1$ and $\lambda_1 < 0$, we have that

$$\log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) < \log^{\lambda_1}\left(e + e^{\epsilon n(1-\theta)/2}\right) \le Cn^{\lambda_1},$$

where $C = C(\epsilon, \theta, \lambda_1)$. Hence we have the estimate

$$P_1' \le C\sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2+\lambda_1} \sum_{I\in\mathscr{Q}_n} \nu(I)|f_I - f_{\widehat{I}}|^p = C\|f\|^p_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)}.$$

For P_2', since $\log^{\lambda_1}(e+t) \le 1$ for any $t \ge 0$ and $\sum_{I\in\mathscr{Q}_n}\nu(I)\approx 1$, we obtain that

$$P_2' \le \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2} \sum_{I\in\mathscr{Q}_n} \nu(I) e^{-\epsilon np(\theta+1)/2} = \sum_{n=1}^{\infty} e^{\epsilon np(\theta-1)/2} n^{\lambda_2} = C' < +\infty,$$

where $C' = C(\epsilon, \theta, \lambda_2)$.

Next, we estimate the logarithmic term from below. Since $f \in L^1(\partial X)$ and $\lambda_1 < 0$, for any $I \in \mathscr{Q}_n$, it follows from $\nu(I) \approx \nu(\widehat{I}) \approx e^{-n\log K}$ that

$$\log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \ge \log^{\lambda_1}\left(e + \frac{|f_I| + |f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \gtrsim \log^{\lambda_1}\left(e + \frac{\|f\|_{L^1(\partial X)}}{e^{-(\epsilon+\log K)n}}\right) \ge Cn^{\lambda_1},$$

where $C = C(\|f\|_{L^1(\partial X)}, \lambda_1, \epsilon, K)$. Now we get the estimate

$$\begin{aligned}\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} &= \sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2+\lambda_1} \sum_{I\in\mathcal{Q}_n} \nu(I)|f_I - f_{\widehat{I}}|^p \\ &\le C\sum_{n=1}^{\infty} e^{\epsilon n\theta p} n^{\lambda_2} \sum_{I\in\mathcal{Q}_n} \nu(I)|f_I - f_{\widehat{I}}|^p \log^{\lambda_1}\left(e + \frac{|f_I - f_{\widehat{I}}|}{e^{-\epsilon n}}\right) \\ &= C|f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X)}.\end{aligned}$$

Therefore, we obtain the estimate

$$\frac{1}{C}\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} \le |f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X)} = P_1' + P_2' \le C\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} + C', \tag{3.24}$$

where C and C' are constants depending only on $\epsilon, \theta, \lambda_1, \lambda_2$ and $\|f\|_{L^1(\partial X)}$.

Combining the inequalities (3.23) and (3.24) which are respect to $\lambda_1 > 0$ and $\lambda_1 < 0$ with the case $\lambda_1 = 0$, we obtain that $\|f\|^p_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} < +\infty$ is equivalent to $|f|_{\dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X)} < +\infty$. □

Let us recall the following result from functional analysis, see for example [11].

Lemma 3.2 (Closed graph theorem)**.** *Let X, Y be Banach spaces and let $T : X \to Y$ be a linear operator. Then T is continuous if and only if the graph $\sum := \{(x, T(x)) : x \in X\}$ is closed in $X \times Y$ with the product topology.*

Let $L^\Phi(\partial X) \cap \dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)$ be the Banach space equipped with the norm

$$\|f\|_{L^\Phi(\partial X)\cap\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)} := \|f\|_{L^\Phi(\partial X)} + \|f\|_{\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)}.$$

Using the same manner, we could define the space $X \cap Y$ for any two spaces X and Y.

Corollary 3.3. *Let $\lambda, \lambda_1, \lambda_2$ and Φ be as in Lemma 3.1. Then we have*

$$L^\Phi(\partial X) \cap \dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X) = \mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$$

with equivalent norms.

Proof. It directly follows from Lemma 3.1 that $L^\Phi(\partial X)\cap\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)$ and $\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$ are the same vector spaces. Next we use Lemma 3.2 (Closed graph theorem) to show that they are the same Banach spaces with equivalent norms.

Consider the identity map $\mathrm{Id} : L^\Phi(\partial X)\cap\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X) \to \mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$, i.e., $\mathrm{Id}\,(x) = x$ for any $x \in L^\Phi(\partial X) \cap \dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)$. Then the graph of Id is closed. Indeed, if (x_n, x_n) is a sequence in this graph that converges to (x, y) in $(L^\Phi(\partial X) \cap \dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)) \times (L^\Phi(\partial X) \cap \dot{\mathcal{B}}^{\theta,\lambda_2}_\Phi(\partial X))$ with product topology, then x_n converges to x in $\|\cdot\|_{L^\Phi(\partial X)\cap\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)}$ norm and hence in $L^\Phi(\partial X)$. In the same manner, x_n converges to y in $\|\cdot\|_{\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)}$ and hence in $L^\Phi(\partial X)$. But the limits are unique in $L^\Phi(\partial X)$, so $x = y$.

Applying Lemma 3.2 (Closed graph theorem), we see that the map Id is continuous from $L^\Phi(\partial X) \cap \dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)$ to $\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)$; similarly for the inverse. Thus the norms $\|\cdot\|_{L^\Phi(\partial X)\cap\dot{\mathcal{B}}^{\theta,\lambda}_p(\partial X)}$ and $\|\cdot\|_{\mathcal{B}^{\theta,\lambda_2}_\Phi(\partial X)}$ are equivalent and the claim follows. □

There is a not big difference between the results in Corollary 3.3 and Proposition 1.2, since $\mathcal{B}_p^{\theta,\lambda}(\partial X) = L^p(\partial X) \cap \dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)$. To get Proposition 1.2 from Corollary 3.3, we need some estimates between the L^p-norm and L^Φ-norm. Since $\nu(\partial X) \approx 1$, we have the following lemma, see [28, Theorem 3.17.1 and Theorem 3.17.5].

Lemma 3.4. *Let Φ_1, Φ_2 be two Young functions. If $\Phi_2 \prec \Phi_1$, then*

$$\|u\|_{L^{\Phi_2}(\partial X)} \lesssim \|u\|_{L^{\Phi_1}(\partial X)}$$

for all $u \in L^{\Phi_1}(\partial X)$.

By the relation (2.3), for any $\delta > 0$, we have

$$\|u\|_{L^{\max\{p-\delta,1\}}(\partial X)} \lesssim \|u\|_{L^\Phi(\partial X)} \lesssim \|u\|_{L^{p+\delta}(\partial X)} \tag{3.25}$$

for all $u \in L^{p+\delta}(\partial X)$.

Recall that $\nu(\partial X) \approx 1$ and $\operatorname{diam}(\partial X) \approx 1$. Since ∂X is Ahlfors Q-regular where $Q = \frac{\log K}{\epsilon}$, we obtain the following lemma immediately from [23, Theorem 4.2]

Lemma 3.5. *Let $0 < s < 1$ and $p \geq 1$. Let $u \in \dot{N}^s_{p,p}(\partial X)$. If $0 < sp < Q = \frac{\log K}{\epsilon}$, then $u \in L^{p^*}(\partial X)$, $p^* = \frac{Qp}{Q-sp}$ and*

$$\inf_{c\in\mathbb{R}} \left(⨍_{\partial X} |u-c|^{p^*}\, d\nu \right)^{1/p^*} \lesssim \|u\|_{\dot{N}^s_{p,p}(\partial X)}$$

Proof of Proposition 1.2. Let $s = \min\{\frac{\theta}{2}, \frac{Q}{2p}\}$, where $Q = \frac{\log K}{\epsilon}$. Then $sp < 2sp \leq Q$. Let $p^* = \frac{Qp}{Q-sp}$ and $\delta = p^* - p$. Since $s \leq \theta/2 < \theta$, it follows from Proposition 2.7 that

$$\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X) \subset \dot{\mathcal{B}}_p^{s}(\partial X) = \dot{N}^s_{p,p}(\partial X).$$

By Lemma 3.5 and triangle inequality, we obtain that

$$\begin{aligned}\left(⨍_{\partial X} |u-u_{\partial X}|^{p^*}\, d\nu \right)^{1/p^*} &\leq 2\inf_{c\in\mathbb{R}} \left(⨍_{\partial X} |u-c|^{p^*}\, d\nu \right)^{1/p^*} \\ &\lesssim \|u\|_{\dot{N}^s_{p,p}(\partial X)} \lesssim \|u\|_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)},\end{aligned}$$

for any $u \in \dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)$, where $u_{\partial X} = ⨍_{\partial X} u\, d\nu$. Since $|u| \leq |u-u_{\partial X}| + |u_{\partial X}|$ and $\nu(\partial X) \approx 1$, it follows from the Minkowski inequality that

$$\begin{aligned}\|u\|_{L^{p^*}(\partial X)} &\leq \|u-u_{\partial X}\|_{L^{p^*}(\partial X)} + \|u_{\partial X}\|_{L^{p^*}(\partial X)} \\ &= \left(⨍_{\partial X} |u-u_{\partial X}|^{p^*}\, d\nu \right)^{1/p^*} + \left| ⨍_{\partial X} u\, d\nu \right| \\ &\lesssim \|u\|_{L^1(\partial X)} + \|u\|_{\dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)},\end{aligned}$$

for any $u \in \dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X)$. Since $\|\cdot\|_{L^1(\partial X)} \leq \|\cdot\|_{L^p(\partial X)} \leq \|\cdot\|_{L^{p^*}(\partial X)}$ is trivial, we have that

$$L^1(\partial X) \cap \dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X) = \mathcal{B}_p^{\theta,\lambda}(\partial X) = L^{p^*}(\partial X) \cap \dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X).$$

Recall the relation (3.25) and $\delta = p^* - p$. Hence we have that

$$\|\cdot\|_{L^1}(\partial X) \lesssim \|\cdot\|_{L^\Phi(\partial X)} \lesssim \|\cdot\|_{L^{p^*}(\partial X)}.$$

Thus,

$$\mathcal{B}_p^{\theta,\lambda}(\partial X) = L^\Phi(\partial X) \cap \dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X).$$

Combining this with Corollary 3.3, i.e., the equivalences

$$L^\Phi(\partial X) \cap \dot{\mathcal{B}}_p^{\theta,\lambda}(\partial X) = L^\Phi(\partial X) \cap \dot{\mathcal{B}}_\Phi^{\theta,\lambda_2}(\partial X) = \mathcal{B}_\Phi^{\theta,\lambda_2}(\partial X),$$

we finally arrive at

$$\mathcal{B}_p^{\theta,\lambda}(\partial X) = \mathcal{B}_\Phi^{\theta,\lambda_2}(\partial X).$$

□

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