JYU DISSERTATIONS 264

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# Unraveling Intrinsic Geometry of Sets and Functions in Carnot groups

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS
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Jyväskylä, August 12, 2020
Department of Mathematics and Statistics
Terhi Moisala

## List of included articles

This dissertation consists of an introductory part and the following three articles:

[A] Enrico Le Donne, Sean Li and Terhi Moisala, *Infinite-Dimensional Carnot Groups and Gâteaux Differentiability*, J. Geom. Anal. (2019). https://doi.org/10.1007/s12220-019-00324-x

[B] Sebastiano Don, Enrico Le Donne, Terhi Moisala and Davide Vittone, *A rectifiability result for finite-perimeter sets in Carnot groups*, arXiv e-prints (2019), arXiv:1912.00493. To be published in Indiana U. Math. J.

[C] Enrico Le Donne and Terhi Moisala, *Semigenerated Carnot algebras and applications to sub-Riemannian perimeter*, arXiv e-prints (2020), arXiv:2004.08619.

The author of this dissertation has actively taken part in the research of the articles.

## Contents



# 1. Background

Analysis in metric spaces is a rapidly developing area of study in mathematics, which was initiated in its modern form a couple of decades ago. Pioneers of the field include Pansu [46], Heinonen and Koskela [32], Cheeger [15], and Ambrosio and Kirchheim [4, 5], who successfully brought fundamental concepts of Geometric Analysis into non-Euclidean metric measure spaces. Analysis is, by its very definition, first order calculus and needs a notion of differential in order to exist. Various forms of differentiability have been introduced in metric measure spaces of different generality. However, a notion of differential can give rise to a meaningful theory only if it respects the geometry of the ambient space. Studying this interplay of differential and metric structure is the core of this thesis.

More specifically, we study differentiability of both sets and functions on metric spaces of sub-Riemannian nature that are equipped with group structure and dilation automorphisms. Combining algebraic and metric-measure theoretic points of view is crucial in our methods. In the first two articles [B] and [C] we consider Carnot groups, which are sub-Riemannian Lie groups with rich, Euclidean-like metric-measure structure. Due to the Euclidean-like properties, many notions and questions of Geometric Measure Theory generalize there naturally when the intrinsic sub-Riemannian geometry is taken into account. Carnot groups appear in many different fields of mathematics and physics, like in Control Theory, Mechanics and Robotics to mention some. In Analysis, besides of their independent interest, they serve as valuable examples of metric measure spaces of fractal nature. In the last article [A] we work in a class of metric spaces that generalize Carnot groups into infinite dimensions.

## 2. Carnot groups

We start with a brief introduction to Carnot groups. For a more comprehensive presentation of Carnot groups as metric groups, we refer to [37]. A Carnot group $\mathbb{G}$ is, by definition, a simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification. A Lie algebra $\mathfrak{g}$ is said to be *stratified* or *Carnot* if it admits a decomposition

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$$

into vector subspaces satisfying $V_{i+1} = [V_1, V_i]$ and $V_s \neq \{0\}$. Stratification of a Lie algebra is unique up to an isomorphism, and so Carnot groups are in one-to-one correspondence to stratified Lie algebras. A Carnot group $\mathbb{G}$ and its Lie algebra $\mathfrak{g}$ can be identified via a diffeomorphic exponential map $\exp\colon \mathfrak{g} \to \mathbb{G}$, and we often talk about them interchangeably.

The natural number $s$ indexing the last layer of the stratification is also the nilpotency step of $\mathfrak{g}$, which we simply call the *step* of $\mathfrak{g}$. The subspace $V_1$ (also called the *first layer* or the *horizontal layer*) plays a special role in theory of Carnot groups. It is crucial that, by definition of stratification, the horizontal layer generates $\mathfrak{g}$ as a Lie algebra. The dimension of $V_1$ is called the *rank* of $\mathfrak{g}$.

Due to the existence of stratification, every Carnot group can be equipped with a unique one-parameter family of dilation automorphisms $(\delta_\lambda)_{\lambda \in \mathbb{R}}$, which are defined on the Lie algebra level by

$$\delta_\lambda(X) = \lambda^k X \quad \text{for} \quad X \in V_k \,.$$

The horizontal layer can, therefore, also be characterized in terms of its dilations. Indeed, the horizontal layer is the distinguished subspace of $\mathfrak{g}$ where the intrinsic dilations agree with the linear vector-space scalings.

Here we have chosen an abstract viewpoint to Carnot groups, following [37], and defined them as Lie groups with certain algebraic properties. This starting point is also natural for purposes of Section 4. However, in the literature another route is often chosen (see e.g. [54, 12]), which has its own benefits and is often more easily applied, too. Since a Carnot group $\mathbb{G}$ can be identified with its Lie algebra through the diffeomorphic exponential map, it can be represented by $\mathbb{R}^n$, where $n$ is the topological dimension of $\mathbb{G}$, equipped with a suitable group product. Hence, every Carnot group also has a Euclidean structure that comes with global coordinates, Euclidean metric and Lebesgue measure. Especially when the group product can explicitly be expressed in terms of coordinates, like in the case of Heisenberg groups, this approach provides a priceless toolbox. However, it will be evident also in the context of this introduction that often Euclidean concepts describe poorly the geometry of Carnot groups. This obstruction leads to a need of finding suitable intrinsic counterparts, which are invariant under group translations and dilations, regardless of the point of view chosen. The rest of this section is devoted to defining the building blocks of intrinsic Geometric Measure Theory in Carnot groups.

2.1. **Carnot groups as metric measure spaces.** For an introduction to Differential Calculus and Geometric Measure Theory in Carnot groups, see [14], and [54] for a cross-section of the state of the art. Every Carnot group can be equipped with a sub-Riemannian structure by fixing a scalar product $\langle \cdot, \cdot \rangle$ on the horizontal layer $V_1$. As opposed to Riemannian geometry where the scalar product is defined on the whole tangent space, on a sub-Riemannian Carnot group the geometry is restricted such that the only allowed directions of travel are those lying in $V_1$. Therefore, a curve $\gamma \colon [0,1] \to \mathbb{G}$ has finite length if and only if it is *horizontal*, i.e., it is absolutely continuous and $\dot\gamma(t) \in V_1$ for almost every $t \in [0,1]$. The length of a horizontal curve is defined as the integral of its speed, and the obtained length structure induces a distance function

$$d_{CC}(x,y) := \inf\{\int_0^1 \|\dot\gamma(t)\| \,\mathrm{d}t : \gamma \text{ is horizontal, } \gamma(0) = x \text{ and } \gamma(1) = y\}.$$

The generating property of $V_1$ ensures by Chow-Rashevsky theorem [18, 51] that the distance function $d_{CC}$ is finite and induces the manifold topology of $\mathbb{G}$. Hence $d_{CC}$ defines a metric (called the *Carnot-Caratheodory-metric* or *CC-metric* for short) on $\mathbb{G}$. This metric is geodesic and invariant under left-translations of the group. Moreover, it turns the dilation automorphisms of $\mathbb{G}$ into metric scalings by

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{CC}(x,y) \quad \forall x, y \in \mathbb{G},\ \lambda \in \mathbb{R}. \tag{1}$$

We call left-invariant metrics satisfying equation (1) *invariant.* Also other invariant metrics (often explicitly defined in coordinates and hence more convenient in computations) are used on Carnot groups in the literature. They are, however, all bi-Lipschitz equivalent and therefore often interchangeable when considering problems in Geometric Measure Theory.

In fact, admitting geodesic left-invariant distance and metric dilations is characteristic for Carnot groups. In [36] it was proven that Carnot groups are the only metric spaces $(X,d)$ that are locally compact, geodesic, isometrically homogeneous, and self-similar. Here $X$ is said to be isometrically homogeneous if for every two elements $x, y \in X$, there exists an isometry $f\colon X \to X$ mapping $f(x) = y$, and self-similar if there exists a $\lambda > 1$ such that $(X, d)$ is isometric to $(X, \lambda d)$. Observe that if $\mathbb{G}$ is a Carnot group and $p, q \in \mathbb{G}$, then the left-translation $L_{qp^{-1}}$ is an isometry mapping $p$ to $q$. Moreover, the dilation $\delta_\lambda\colon \mathbb{G} \to \mathbb{G}$ is an isometry between $(\mathbb{G}, d)$ and $(\mathbb{G}, \lambda d)$ for every invariant metric $d$ on $\mathbb{G}$ and for every $\lambda > 0$.

Recall that every topological group admits a Haar measure, which is unique up to a multiplicative constant. The Haar measure of a Carnot group is the $Q$-dimensional Hausdorff measure $\mathcal{H}^Q$ built with respect to an invariant distance, where $Q$ is defined by

$$Q = \sum_{k=1}^{s} k \dim V_k \,.$$

The quantity $Q$ is called the *homogeneous dimension* of $\mathbb{G}$. Notice that $Q$ is strictly larger than the topological dimension of $\mathbb{G}$ for every nonabelian Carnot group, giving them a fractal nature. The homogeneity of the distance passes down to the measure; it is indeed straightforward to show that, for every ball $B(x, r)$ in the group with center $x$ and radius $r$, it holds

$$\mathcal{H}^Q(B(x, r)) = r^Q \mathcal{H}^Q(B(0, 1)).$$

In particular, $(\mathbb{G}, d_{CC}, \mathcal{H}^Q)$ is an Ahlfors Q-regular metric measure space, equipped with a homogeneous group structure and dilation automorphisms.

2.2. **Pansu-differentiability.** The question of differentiability of Lipschitz maps in Carnot groups is fully solved by Pansu [46] with a class of functions that we nowadays call Pansu-differentiable. A map $f\colon \mathbb{G} \to \mathbb{H}$ between Carnot groups $\mathbb{G}$ and $\mathbb{H}$ is *Pansu-differentiable* at a point $p \in \mathbb{G}$ if the maps $f_{p,\lambda}$,

$$f_{p,\lambda}(v) := \delta_{1/\lambda}(f(p)^{-1} f(p\delta_\lambda(v)))$$

converge uniformly on compact sets as $\lambda \to 0$ and the limiting function $\mathrm{d}f_p\colon \mathbb{G} \to \mathbb{H}$ is a continuous homomorphism. The celebrated Pansu-Rademacher theorem states that every function between Carnot groups that is Lipschitz with respect to the intrinsic distances is Pansu-differentiable almost everywhere.

Recall that the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is, by definition, the space of left-invariant vector fields on $\mathbb{G}$. Seeing vector fields as first order differential operators on $\mathbb{G}$ and the horizontal layer $V_1$ of $\mathfrak{g}$ as a subbundle of the tangent bundle of $\mathbb{G}$, Pansu differentiability of a continuous function $f\colon \mathbb{G} \to \mathbb{R}$ can be described via the action of horizontal vector fields on $f$. Indeed, given a basis $\{X_1, \ldots, X_m\}$ of the first layer $V_1$ of the

Lie algebra and assuming that each one of the partial derivatives $X_i f$ exists, one defines the *horizontal gradient* of $f$ as

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m} (X_i f) X_i,$$

which is a section of the horizontal bundle. Then $f$ is continuously Pansu-differentiable if and only if the component functions $X_1 f, \ldots, X_m f$ of the horizontal gradient are continuous. We denote by $C^1_{\mathbb{G}}(U)$ the class of continuously Pansu-differentiable real valued functions on an open subset $U \subseteq \mathbb{G}$, as opposed to the family of continuously differentiable functions $C^1(U)$ on $U$ with respect to the ambient Euclidean structure of $\mathbb{G}$. From the characterization of continuously Pansu-differentiable functions in terms of horizontal gradient, it is immediate that every function in $C^1(\Omega)$ is in the class $C^1_{\mathbb{G}}(\Omega)$. The inclusion is, however, strict.

## 3. Rectifiabilty in Carnot groups

Rectifiability is a classical and unquestionably important notion in Geometric Measure Theory and in Calculus of Variations. Indeed, rectifiable sets are a natural relaxation of smooth surfaces in the context of Geometric Measure Theory, being the weakest notion of sets having Lusin property with $C^1$-surfaces and a notion of tangent bundle. Classically, a set $E \subseteq \mathbb{R}^n$ is said to be $(n-1)$-*rectifiable* if it can be covered, up to measure zero, by a countable collection of $C^1$-hypersurfaces. Due to Rademacher's theorem, this definition remains unchanged if we replace '$C^1$' by 'Lipschitz', or even by *a priori* weaker geometric notion, that we call the cone property and which will be defined in Section 3.3. For a self-contained description of rectifiability and Geometric Measure Theory in Euclidean spaces, we refer to [43].

The celebrated rectifiability theorem of De Giorgi [19, 20] states that the (reduced) boundary of a set of finite perimeter $E \subseteq \mathbb{R}^n$ is $(n-1)$-rectifiable. Generalizing De Giorgi's theorem into Carnot groups is a difficult problem, which has been under intense investigation since the seminal work of Franchi, Serapioni and Serra-Cassano in [25, 26], as part of the general program of developing Geometric Measure Theory in Carnot groups. This work is vastly motivated by a new notion of differentiability for $L^1$-valued maps on the Heisenberg group introduced by Cheeger and Kleiner [16, 17].

It is not yet, however, fully understood what is the correct notion of rectifiability in Carnot groups. There exists a general definition for rectifiability in metric spaces that goes back to Federer [23], which involves covering the set by Lipschitz images of a Euclidean space.

However, in [5], Ambrosio and Kirchheim showed that every non-abelian Carnot group is purely $Q$-unrectifiable in the sense of Federer, where $Q$ stands for the homogeneous dimension of the group. Since one cannot meaningfully compare the size of Carnot-subsets to Euclidean ones, the question of rectifiability in Carnot groups becomes more subtle.

At the moment we are after an intrinsic definition for rectifiability in Carnot groups. In the Euclidean setting, rectifiable sets are those that can be countably covered by regular surfaces of certain dimension. To this end, we need to find a suitable invariant notion of a regular submanifold of a Carnot group. It is nowadays evident that Pansu's regularity theory for functions between Carnot groups fits poorly to functions defined within a Carnot group. Currently there exists a well established notion of $\mathbb{G}$-*regular hypersurface*, and the corresponding *intrinsic* $C^1$*-rectifiability* in Carnot groups due to Franchi, Serapioni and Serra Cassano [25, 26]. Also Lipschitz rectifiability and the cone property mentioned in the beginning of this section have their corresponding natural counterparts in Carnot groups. However, we do not yet know if these different definitions of rectifiability in Carnot groups agree. Among these the $C^1$-rectifiability is supposedly most restrictive, and the geometric cone property the broadest. We shall make a more detailed analysis of different notions of regular Carnot-subsets and their mutual connections in Sections 3.3 and 3.4.

Also other suggestions for the definition of rectifiability exist in Carnot groups: in [47] Pauls introduced a notion of rectifiability in the spirit of Federer, where the model Euclidean space is replaced by a (subset of a) Carnot group. Recently, progress in this direction was made in [8], where they study the relation of rectifiability in the sense of Pauls and the intrinsic $C^1$-rectifiability after [25] by proving that there exist $C^1$-regular hypersurfaces that are not Pauls rectifiable.

### 3.1. **Sets of Finite Perimeter and Blow-up.**

The study of sets of finite perimeter was initiated by Caccioppoli and later developed and deeply studied by De Giorgi. The classical notion of finite-perimeter sets builds on theory of functions of bounded variation, which are a natural generalization of Sobolev functions. Therefore, sets of finite perimeter enjoy good compactness and approximation properties, which makes them useful e.g. in the study of minimal surfaces. For a comprehensive introduction to the Euclidean theory of bounded-variation functions, see for instance [3].

In Carnot groups functions of bounded variation and sets of finite perimeter were for the first time defined in [13]. Our discussion on finite-perimeter sets in Carnot groups follows [6]. Given a Carnot group $\mathbb{G}$ and a function $u \in L^1_{loc}(\mathbb{G})$, the *distributional derivative* of $u$ in the direction of $X \in \mathfrak{g}$ is defined as

$$\langle Xu, f\rangle := -\int_{\mathbb{G}} uXf \,\mathrm{d}\mathcal{H}^Q, \qquad \text{where } f \in C_c^\infty(\mathbb{G}).$$

A function $u \in L^1_{loc}(\mathbb{G})$ is said to be of *locally bounded variation* if the distributional derivatives $Xu$ are representable by Radon measures for all $X \in V_1$; that is, if for every $X \in V_1$ there exists a Radon measure $\mu$ such that

$$\langle Xu, f\rangle = \int_{\mathbb{G}} f \,\mathrm{d}\mu,$$

for all functions $f \in C_c^\infty(\mathbb{G})$. A set $E \subseteq \mathbb{G}$ is said to have *locally finite perimeter* if its characteristic function $\mathbb{1}_E$ is of locally bounded variation. We stress that in the definition of finite-perimeter sets we only require regularity along the horizontal directions, as opposed to the sets with finite Euclidean perimeter.

Let us then fix a scalar product and an orthonormal basis $\{X_1, \dots, X_m\}$ for $V_1$. If $E \subseteq \mathbb{G}$ has locally finite perimeter, then we denote by $|D\mathbb{1}_E|$ the total variation of the vector-valued Radon measure $(X_1\mathbb{1}_E, \dots, X_m\mathbb{1}_E)$. The measure $|D\mathbb{1}_E|$ plays then the role of perimeter measure of $E$. Indeed, following the approach of De Giorgi, one defines the *reduced boundary* $\mathcal{F}E$ of $E$ as the set of points $p \in \mathbb{G}$ such that $|\mathrm{D}\mathbb{1}_E|(B(p,r)) > 0$ for all $r > 0$, and there exists

$$\lim_{r\to 0} \frac{\mathrm{D}\mathbb{1}_E(B(p,r))}{|\mathrm{D}\mathbb{1}_E|(B(p,r))} =: \nu_E(p) \tag{2}$$

with $|\nu_E(p)| = 1$. The function $\nu_E$ is called the measure theoretical *(horizontal) inner normal* of $E$. It is immediate that the reduced boundary is contained in the topological boundary of $E$ and also in the support of $|D\mathbb{1}_E|$. On the other side, by a result of Ambrosio (see [2]), the perimeter measure is concentrated on the reduced boundary, i.e., $|D\mathbb{1}_E|(\mathbb{G} \setminus \mathcal{F}E) = 0$. In fact, the result of Ambrosio is even stronger: it states that for $|D\mathbb{1}_E|$-a.e. $p \in \mathbb{G}$ and for sufficiently small scales $r = r(p)$, the measure $|D\mathbb{1}_E|(B(p,r))$ of a ball $B(p,r)$ is comparable to $r^{Q-1}$, up to multiplicative constants that only depend on the group $\mathbb{G}$. Hence, the perimeter measure of a finite-perimeter set is equivalent to the $(Q-1)$-dimensional Hausdorff measure $\mathcal{H}^{Q-1}$ for subsets of the reduced boundary.

As mentioned above, De Giorgi proved in [19, 20] that every subset $E$ of $\mathbb{R}^n$ with locally finite perimeter has $(n-1)$-rectifiable reduced boundary. We briefly recall here main steps of his blow-up method (see [3] for more details). The first step is to show that for all $x \in \mathcal{F}E$, as $r \to 0^+$, the translated and scaled sets $(E-x)/r$ converge in $L^1_{\mathrm{loc}}$ to a half-space $H_{\nu_E(x)}$ defined by the normal $\nu_E(x)$. Then by theorems of

Lusin and Egoroff, the reduced boundary is divided (up to measure zero) into sets $K_h$, $h \in \mathbb{N}$, where $\nu_E(x)$ is continuous and the convergence of the blow-up sets is uniform. By careful estimates on the convergence, one can prove that the normal $\nu_E$ satisfies assumptions of Whitney extension theorem on each $K_h$, providing $f_h \in C^1(\mathbb{R}^n)$ such that

$$f_h|_{K_h} = 0 \quad \text{and} \quad \nabla f_h|_{K_h} = \nu_E(x) \neq 0.$$

Consequently, each $K_h$ is contained in a zero level set of some $f_h \in C^1(\mathbb{R}^n)$, which defines a $C^1$-regular hypersurface in $\mathbb{R}^n$. So the reduced boundary of $E$ is covered up to a null set by graphs of smooth surfaces and is therefore $(n-1)$-rectifiable.

In their seminal work [25, 26] on rectifiability in Carnot groups, Franchi, Serapioni and Serra-Cassano studied the generalization of De Giorgi's approach into Carnot groups. In particular, they proved a version of Whitney's extension theorem and studied the structure of intrinsic blow-up sets at points of the reduced boundary of a finite-perimeter set. Regarding the latter, they proved the following theorem.

**Theorem 3.1** (Franchi-Serapioni-Serra Cassano)**.** *Let $\mathbb{G}$ be a Carnot group and let $E \subseteq \mathbb{G}$ have locally finite perimeter. Then for every $p \in \mathcal{F}E$ and for every sequence $(r_n)_n$, $r_n \to 0^+$, there exists a subsequence $(s_n)_n \subseteq (r_n)_n$ and a finite-perimeter set $F \subseteq \mathbb{G}$ such that*

$$\mathbb{1}_{\delta_{1/s_n}(p^{-1}\cdot E)} \to \mathbb{1}_F \quad \text{in } L^1_{loc}(\mathbb{G}), \text{ as } n \to \infty.$$

*Moreover, the set $F$ has constant horizontal normal $\nu_F(x) = \nu_E(p)$ for $|D\mathbb{1}_F|$-a.e. $x$.*

The authors discovered in [26] that the only obstruction for applying De Giorgi's method in Carnot groups lies in the fact that, in general, sets having constant horizontal normal may fail to be (vertical) half-spaces, as they show by an example in the Engel group. Hence, the convergence to a constant-normal set $F$ does not provide sufficient control on the normal $\nu_E(x)$ for applying the Whitney extension theorem.

Based on the work of Franchi, Serapioni and Serra-Cassano, there are a few ways to proceed with the study of rectifiability of finite-perimeter sets in Carnot groups. In [6] it was proved that at $|D\mathbb{1}_E|$-a.e. point, there exists a half-space in the set of all possible blow-ups of $E$ at $x$. Hence we are missing a uniqueness result for the blow-up sets. This approach seems, however, very difficult to follow.

Another possibility is to study in which Carnot groups the only constant-normal sets are the vertical half-spaces, and where the proof of De Giorgi can therefore be applied. In [26] it was shown that this is the case in all step-2 Carnot groups and later, in [42], in Carnot groups of so called type $(\star)$. In article [C] we have chosen this approach. Finally, one can concentrate on studying regularity of sets of constant horizontal normal in Carnot groups and, using them, prove some possibly weaker form of rectifiability for finite-perimeter sets in a broader class of Carnot groups. This is the starting point in article [B]. We take next a closer look at properties of constant normal sets in Carnot groups, which will be crucial in both of our rectifiability results.

3.2. **Sets with Constant Horizontal Normal.** Recall that a set $E$ in a Carnot group $\mathbb{G}$ is said to have constant (horizontal) normal if the measure-theoretic normal $\nu_E(x)$ defined in (2) is constant as a function of $x \in \mathcal{F}E$. Equivalently, $E$ has constant normal if there exists a half-space $W$ in $V_1$ (i.e., the closure of either of the two parts into which a hyperplane divides $V_1$) such that $E$ is monotone along all directions in $W$. In this case, we say that $E$ is $W$-*monotone.* Formally one requires, in the sense of distributions,

$$X\mathbb{1}_E \geq 0 \quad \forall\, X \in W.$$

In [11], properties of such sets are studied. The consideration boils down to certain kinds of $W$-monotone sets with respect to a horizontal half-space $W$, which are the semigroups generated by $\exp(W)$. The semigroup generated by $\exp(W)$ is denoted by $S_W$. By definition, semigroup is a set that is closed under group multiplication, so $S_W$ has the form

$$S_W := \bigcup_{k=1}^{\infty} \exp(W)^k.$$

The key point is that semigroups generated by horizontal half-spaces are the minimal constant normal sets with respect to set inclusion. The fact that the set $S_W$ is $W$-monotone and that every $W$-monotone set contains a translated copy of $S_W$ as a subset can be heuristically explained as follows. Recall that $p \cdot \exp(tX)$ is the flow of the left-invariant vector field $X$ starting from $p$ at time $t$. Let us think for a moment, for the sake of the argument, that $\mathbb{1}_E$ is a smooth function. Then $\mathbb{1}_E$ is increasing in the direction of $X$ (i.e., we have $X\mathbb{1}_E \geq 0$) if and only if the flow line of $X$ does not exit the set $E$ once it has entered. In other words, if $E$ is $X$-monotone, then $p \cdot \exp(tX) \subset E$ for every $p \in E$. Since $S_W$ is just a union of composed flow lines in the directions in $W$, we immediately have the containment of a translated copy of $S_W$ in a $W$-monotone set $E$. Vice versa, since $S_W$ contains all flow lines of $W$ by construction, it is increasing with respect to all directions in $W$, so $W$-monotone. In [11] the authors prove that for every $W$-monotone set $E$ there exists a representative (a set $\widetilde{E}$ such that the symmetric difference $E \Delta \widetilde{E}$ has measure zero) for which the previous argument can be made formal, even with respect to the closure of $S_W$:

**Theorem 3.2** (Bellettini-Le Donne)**.** *A subset $E$ of a Carnot group has constant normal with respect to a horizontal half-space $W$ if and only if $E$ has a representative $\widetilde{E}$ satisfying*

$$p \cdot \mathrm{Cl}(S_W) \subseteq \widetilde{E} \quad \forall p \in \widetilde{E}.$$

The above theorem proves fruitful in the study of finite-perimeter sets in Carnot groups. The reason is that the semigroup $S_W$ is an intrinsic object with explicit representation, and has therefore several useful properties. First, since $W$ is invariant under intrinsic dilations, also the set $S_W$ satisfies $\delta_\lambda(S_W) = S_W$, for all $\lambda > 0$. Secondly, the set $S_W$ has some convenient topological properties; most importantly, it has nonempty interior. This last fact is a consequence of the classical Krener's theorem in Geometric Control Theory, as $W$ is a generating subset of the Lie algebra $\mathfrak{g}$ (see [1, Theorem 8.1] for more details). These two features of the semigroups are in the core of the weak rectifiability results in [B].

In addition to the geometric viewpoint to semigroups, they can be seen as algebraic objects. In this approach it is convenient to work on the Lie algebra side: given a semigroup $S$ in $\mathbb{G}$, we shall denote by $\mathfrak{s}$ the set $\log(S) \subseteq \mathfrak{g}$. There are two geometric subsets of $\mathfrak{s}$ which play an important role in our study. Namely, the largest Euclidean cone in $\mathfrak{s}$ is called the *wedge* of $\mathfrak{s}$, and the largest vector subspace in $\mathfrak{s}$ is called the *edge* of $\mathfrak{s}$. We denote these subsets by $\mathfrak{w}(\mathfrak{s})$ and $\mathfrak{e}(\mathfrak{s})$, respectively. They have the following expressions:

$$\begin{aligned} \mathfrak{w}(\mathfrak{s}) &= \{X \in \mathfrak{g} \;:\; \mathbb{R}_+ X \subseteq \mathfrak{s}\}; \\ \mathfrak{e}(\mathfrak{s}) &= \mathfrak{w}(\mathfrak{s}) \cap (-\mathfrak{w}(\mathfrak{s})) = \mathfrak{w}(\mathfrak{s}) \cap \mathfrak{w}(-\mathfrak{s}) . \end{aligned}$$

Lie semigroups and their tangent wedges are a classical object of study in Lie group theory, and we refer to [33] for more information. One can verify that $\mathfrak{e}(\mathfrak{s})$ is a Lie subalgebra of $\mathfrak{g}$ and that, if $\mathfrak{s}$ is closed, then $\mathfrak{w}(\mathfrak{s})$ is closed and convex with respect to the vector space structure of $\mathfrak{g}$. We exploit repeatedly these two attributes of semigroups in [C], which contributes to the $C^1$-rectifiability problem in Carnot groups.

3.3. **Cone property and Intrinsic Lipschitz rectifiability.** In this and the following section we focus on different forms of intrinsic rectifiability in Carnot groups. Our approach differs from the customary introductions to the subject and proceeds from the weak and most recent notion of cone property, introduced in [B], to the classical, strong rectifiability in the sense of [25].

In what follows, we call a set $C \subseteq \mathbb{G}$ a *cone* if $C$ has non-empty interior and it is invariant under intrinsic dilations, i.e., $\delta_\lambda(C) = C$ for all $\lambda > 0$. We say that a set $\Gamma \subseteq \mathbb{G}$ satisfies the *$C$-cone property* if there exists a cone $C \subseteq \mathbb{G}$ such that

$$\Gamma \cap p \cdot C = \emptyset, \qquad \text{for every } p \in \Gamma.$$

In article [B] it is shown that, in every Carnot group $\mathbb{G}$ and for every set of locally finite perimeter $E \subseteq \mathbb{G}$, there exists a family $\{C_h : h \in \mathbb{N}\}$ of open cones in $\mathbb{G}$ and a family $\{\Gamma_h : h \in \mathbb{N}\}$ of subsets of $\mathbb{G}$ such that each $\Gamma_h$ satisfies the $C_h$-cone property and

$$\mathcal{F}E = \bigcup_{h \in \mathbb{N}} \Gamma_h.$$

The main ingredients behind this result are the blow-up theorem (Theorem 3.1) and the minimality of semigroups as constant-normal sets (Theorem 3.2). Namely, by Theorem 3.1, in small scales the finite-perimeter set $E$ is well approximated around a point $p \in \mathcal{F}E$ by a set $F$, which has constant normal equal to $\nu := \nu_E(p)$. By Theorem 3.2, up to changing representative, the cone $p \cdot S_{\nu^\perp}$ is contained in $F$. Then it is possible to find an open cone $C$, that is compactly contained in the interior of $S_{\nu^\perp}$, for which $\mathcal{F}E$ satisfies the $C$-cone property in some small neighborhood of $p$ (see Figure 1). The final result is then obtained by a compactness argument and a careful division of $\mathcal{F}E$ into suitable subsets $\Gamma_h$ that satisfy the cone property for some fixed cone $C_h$.

We stress that the result is valid without any restriction on the Carnot group $\mathbb{G}$. Notice that the cone property is indeed a regularity condition for the set $\Gamma$. In fact, if $\mathbb{G} = \mathbb{R}^n$ and $\Gamma$ has the cone property with some cone $C$, then $\Gamma$ is locally a Lipschitz graph where the Lipschitz constant is determined by the opening of the cone $C$; it follows from basic algebra that if $\Gamma$ has the $C$-cone property, then it satisfies $\Gamma \cap p \cdot (C \cup C^{-1}) = \emptyset$ for every $p \in \Gamma$. Here $p \cdot (C \cup C^{-1})$ represents an hour-glass shape whose closure intersects $\Gamma$ only at its vertex $p$. Since any Lipschitz map defined

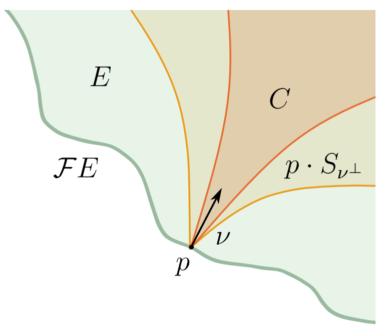



FIGURE 1. Illustrative picture for the cone property of finite-perimeter sets.

on an open subset $\Omega \subseteq \mathbb{R}^{n-1}$ can be extended to the whole of $\mathbb{R}^{n-1}$, we deduce that every Euclidean set with the cone property is contained in a graph of a Lipschitz map. However, in non-abelian Carnot groups this implication is not so straightforward, the reason being that the notion of graph is more convoluted. Indeed, existence of graphs depends on the possibility of dividing the group into complementary subgroups, mimicking the division of $\mathbb{R}^n$ into Cartesian product of its subspaces. We next define graphs in Carnot groups and investigate under which conditions on the Carnot group $\mathbb{G}$ the geometric cone property can be enhanced to the property of being contained in a(n intrinsic) Lipschitz graph.

Graphs in Carnot groups were first, implicitly, exploited in [25] and then formally introduced in [28, 29]. For an exhaustive introduction to intrinsic (Lipschitz) graphs in Carnot groups, see [30]. Two subgroups $\mathbb{W}, \mathbb{L} \subseteq \mathbb{G}$ are said to be *complementary subgroups* of $\mathbb{G}$ if they are homogeneous (i.e., invariant under dilations) and if $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$ with $\mathbb{W} \cap \mathbb{L} = \{0\}$. Under these conditions, every element $g \in \mathbb{G}$ has a unique expression as $g = w \cdot l$ for some $w \in \mathbb{W}$ and $l \in \mathbb{L}$. For purposes of this thesis, we are only interested in graphs of codimension 1, that is, in the case when $\mathbb{L}$ is one-dimensional. Then $\mathbb{L}$ is necessarily a horizontal line. This observation will be crucial later in this section.

Given complementary subgroups $\mathbb{W}$ and $\mathbb{L}$ in $\mathbb{G}$ and a function $f\colon U \subseteq \mathbb{W} \to \mathbb{L}$, we define its graph as the set

$$\operatorname{graph}(f) := \{p \cdot f(p) : p \in U\}.$$

It would be natural to define a Lipschitz graph in a Carnot group to be a graph of a function that is Lipschitz with respect to an invariant distance on $\mathbb{G}$. This approach does not, however, lead to an intrinsic concept, since this notion is not invariant under group translations. An intrinsic definition is obtained by a more geometric alternative: we say that $\Sigma \subseteq \mathbb{G}$ is an *intrinsic Lipschitz graph* if there exist complementary subgroups $\mathbb{W}$ and $\mathbb{L}$ in $\mathbb{G}$ and $\beta > 0$ such that

(i) for every $p \in \Sigma$ one has

$$\Sigma \cap p \cdot \bigcup_{\ell \in \mathbb{L} \setminus \{0\}} B(\ell, \beta d(0, \ell)) = \emptyset;$$

(ii) $\pi_{\mathbb{W}}(\Sigma) = \mathbb{W}$.

Naturally, if $Q$ is the homogeneous dimension of $\mathbb{G}$, we say that $E$ is *intrinsically Lipschitz* $(Q-1)$*-rectifiable* if there exists a countable family $\{\Sigma_h : h \in \mathbb{N}\}$ of intrinsic Lipschitz graphs such that

$$\mathcal{H}^{Q-1}\left(E \setminus \bigcup_{h \in \mathbb{N}} \Sigma_h\right) = 0.$$

Also other equivalent definitions for an intrinsic Lipschitz graph exist in the literature, see e.g. [30, 56] as well as [45, 52] for different notions in the Heisenberg groups. This definition suits our purposes as it shows clearly the connection with the cone property defined above. Indeed, on the one hand, since the set $\cup_{\ell\in\mathbb{L}\setminus\{0\}}B(\ell,\beta d(0,\ell))$ is obviously dilation invariant and has nonempty interior, it is a cone and so every intrinsic Lipschitz graph has the cone property by (i). On the other hand, if the set $\Gamma$ has the cone property with some cone $C$, then we claim that $\Gamma$ is contained in an intrinsic Lipschitz graph if there exists some $X \in V_1$ for which $\exp(X) \in \mathrm{int}(C)$.

To prove the claim, notice that $\exp(\mathbb{R}X) =: \mathbb{L}$ is a one-dimensional horizontal subgroup which is complementary to the subgroup $\exp(X^\perp \oplus [\mathfrak{g},\mathfrak{g}]) =: \mathbb{W}$. Being $\exp(X)$ in the interior of $C$, there exists $r > 0$ such that $B(\exp(X), r) \subset C$. Since $C$ is invariant under dilations, condition (i) follows now from the $C$-cone property of $\Gamma$. By the extension theorem [56, Proposition 3.4] (see also [30]), the set $\Gamma$ is then contained in some entire intrinsic Lipschitz graph $\Sigma$.

We recall that in [B] it was shown that in every Carnot group $\mathbb{G}$, the reduced boundary of a finite-perimeter set can be countably covered by sets $\Gamma_h$, where each $\Gamma_h$ has $C_h$-cone property and where each $C_h$ is an open subset of some semigroup $S_{W_h}$ generated by a horizontal half-space $W_h$. In fact, it is pointed out in [B] that if the Carnot group $\mathbb{G}$ satisfies: *for every semigroup $S_W$ generated by a horizontal half-space $W$ it holds* $\mathrm{int}(S_W) \cap \exp(V_1) \neq \emptyset$, then the reduced boundary of every finite-perimeter set is intrinsically Lipschitz rectifiable.

It may indeed happen that $\mathrm{int}(S_W)\cap\exp(V_1) = \emptyset$ for every horizontal half-space $W$, as shown in [11, Section 5] by the free Lie algebra of rank 2 and step 3. In [B] the property $\mathrm{int}(S_W) \cap \exp(V_1) \neq \emptyset$ is related to some mild regularity properties of the end-point map, which associates to a horizontal curve $\gamma\colon [0,1] \to \mathbb{G}$ its end point $\gamma(1)$. In particular, it is shown that if $\mathbb{G}$ admits a non-abnormal horizontal line (a condition on the end-point map that can be verified algebraically), then every semigroup $S_W$ satisfies $\mathrm{int}(S_W) \cap \exp(V_1) \neq \emptyset$ and the Lipschitz rectifiability for finite-perimeter sets is achieved. As a corollary, the Lipschitz-rectifiability result is obtained e.g. in filiform groups (see [B] for definition and properties). In addition, it is verified in [C] that every Engel-type group, which will be introduced in Section 3.4, admits a large collection of non-abnormal horizontal lines.

It would be interesting to see under which conditions on $\mathbb{G}$ the end-point map has sufficient regularity to ensure that the semigroups have horizontal lines in their interior. Admitting a non-abnormal horizontal line is indeed only a sufficient condition: if $\mathbb{G}$ has step 2 and $\dim V_2 \geq \dim V_1$, then every horizontal line of $\mathbb{G}$ is abnormal, but still every step-2 Carnot group is known to have strong rectifiability properties, as we shall discuss in Section 3.4. In [34] some milder condition on the end-point map (called pliability) is connected to the Whitney extension property on curves (see also [53]). In fact, also pliability of the Carnot group is enough to provide the Lipschitz rectifiability result. Resembling "deformability" properties of curves are related also to universal differentiability sets in [48].

3.4. **Intrinsic $C^1$-rectifiability and semigenerated Lie algebras.** We define next $\mathbb{G}$-regular hypersurfaces and intrinsic $C^1$-rectifiability following [26]. A subset $S$ of a Carnot group $\mathbb{G}$ is called a $\mathbb{G}$-*regular hypersurface* if it is (locally) a non-critical level set of a continuously Pansu-differentiable function, i.e., if there exists a neighborhood $U \subseteq \mathbb{G}$ and a function $f \in C^1_{\mathbb{G}}(U)$ such that

$$S \cap U = \{x \in U : f(x) = 0\} \quad \text{and} \quad \nabla_{\mathbb{G}} f \neq 0.$$

A subset $\Gamma$ of a Carnot group is said to be intrinsically ($(Q-1)$-dimensional) $C^1$-rectifiable if there exists a countable union of intrinsic $C^1$-hypersurfaces $(S_j)_{j\in\mathbb{N}}$ such that

$$\mathcal{H}^{Q-1}\left(\Gamma \setminus \bigcup_{j=1}^{\infty} S_j\right) = 0.$$

Notice that, unlike for sets with a cone property or for intrinsic Lipschitz graphs introduced in the previous section, the definition of a $\mathbb{G}$-regular hypersurface is analytic. An implicit function theorem for $\mathbb{G}$-regular hypersurfaces was already shown in [27], according to which every $\mathbb{G}$-regular surface is locally a graph of a function $\phi\colon \mathbb{M} \to \mathbb{N}$ between complementary subgroups $\mathbb{M}$ and $\mathbb{N}$. It follows also from [27] that the map $\phi$ is intrinsically Lipschitz and that the metric tangents of graph($\phi$) are homogeneous subgroups of $\mathbb{G}$. The latter geometric condition is nowadays called intrinsic differentiability and it was introduced in [24]. Only recently, in [21] (see also [7]) it was proven that being a graph of an intrinsic differentiable function in the sense of [24] is a characterizing property of $\mathbb{G}$-regular hypersurfaces. For more information and references on regular submanifolds in Carnot groups, see e.g. [39].

Until now the intrinsic $C^1$-rectifiability of reduced boundaries of finite-perimeter sets has been positively answered in step-2 Carnot groups and in so called type ($\star$) Carnot groups introduced by Marchi in [42], the reason being that in these groups every set with constant horizontal normal is a vertical half-space and so De Giorgi's proof by blow-up can be applied. A vertical half-space is, by definition, the exponential of the direct sum of a horizontal half-space and the derived algebra $[\mathfrak{g},\mathfrak{g}]$. In exponential coordinates, these sets are exactly the half-spaces in $\mathfrak{g}$ defined by a normal in $V_1$, assuming $[\mathfrak{g},\mathfrak{g}] \perp V_1$.

In article [C] we continue, from an algebraic point of view, the classification of Carnot groups in which every constant-normal set is a vertical half-space. Since the semigroups $S_W$ generated by the exponentials of the horizontal half-spaces $W$ are the minimal constant-normal sets with respect to set inclusion (see Theorem 3.2), a Carnot group has this property if and only if, for each horizontal half-space $W$, the set $\mathfrak{s}_W := \mathrm{Cl}(\log(S_W))$ is a vertical half-space. We call a horizontal half-space $W$ for which $\mathfrak{s}_W$ is a vertical half-space *semigenerating*. A stratified Lie algebra is *semigenerated* if every horizontal half-space in it is semigenerating. Respectively, a Carnot group is semigenerated if its Lie algebra is semigenerated. We note that, even if the notion of semigenerated Carnot group is vastly motivated by the rectifiability problem, the study of semigroups is also of its own independent interest, with wide range of possible applications.

On the Lie algebra side, we may write the condition of being semigenerated as

$$[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{s}_W, \text{ for every horizontal half-space } W.$$

Our main results are twofold. First, we introduce a class of semigenerated Carnot groups, called *type* ($\Diamond$), that is defined by an algebraic condition and that strictly generalizes the class of type ($\star$) algebras due to Marchi. A stratified Lie algebra is said to be of type ($\star$) if there exists a basis $\{X_1,\dots,X_m\}$ of $V_1$ such that

$$\mathrm{ad}^2_{X_i} X_j = 0, \quad \forall\, i,j = 1,\dots,m.$$

A stratified Lie algebra is said to be of type ($\Diamond$) if, for every subalgebra $\mathfrak{h}$ for which $\mathfrak{h} \cap V_1$ has codimension 1 in $V_1$, there exists a basis $\{X_1,\dots,X_m\}$ of $V_1$ such that

$$\mathrm{ad}^2_{X_i} X_j \in \mathfrak{h} \quad \text{and} \quad \mathrm{ad}^2_{\mathrm{ad}^k_{X_i} X_j}(X_i) \in \mathfrak{h}, \tag{3}$$

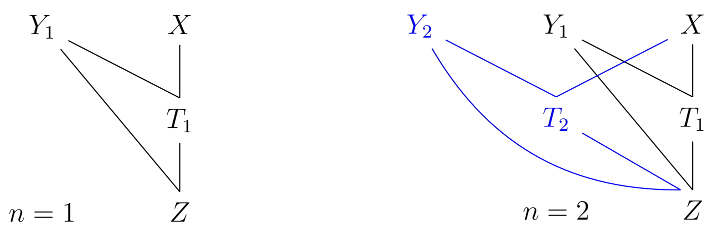
for all $i,j = 1,\dots,m$ and $k \geq 2$. Therefore, every type-($\star$) algebra trivially satisfies (3) for every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Our proof for the fact that type ($\Diamond$) algebras are indeed semigenerated relies on some algebraic properties of the wedge $\mathfrak{w}(\mathfrak{s}_W)$ and edge $\mathfrak{e}(\mathfrak{s}_W)$ of $\mathfrak{s}_W$, for a horizontal half-space $W$. Being $\mathfrak{e}(\mathfrak{s}_W)$ the largest Lie subalgebra of $\mathfrak{g}$ in $\mathfrak{s}_W$, we have that $W$ is semigenerating if and only if $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{e}(\mathfrak{s}_W)$. Using the geometric and algebraic properties of $\mathfrak{w}(\mathfrak{s}_W)$ and $\mathfrak{e}(\mathfrak{s}_W)$ and considering a suitable Hall basis of the Lie algebra, we are able to deduce that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{s}_W$ if and only if there exists a basis $\{X_1,\dots,X_m\}$ of $V_1$ such that the terms in (3) are in $\mathfrak{e}(\mathfrak{s}_W)$. If $\mathfrak{g}$ is of type ($\Diamond$), then choosing $\mathfrak{h} = \mathfrak{e}(\mathfrak{s}_W)$ in (3) gives the latter containment, and so $W$ is semigenerating.

In the second part of [C] we adopt a different strategy for characterizing semigenerated algebras. On the one hand, projections of semigenerated Carnot algebras by homogeneous ideals are semigenerated; indeed, projections of half-spaces are again half-spaces, or possibly the whole space. On the other hand, we know by [26] that every step-2 Carnot algebra is semigenerated, and therefore every non-semigenerated Carnot group has a non-trivial semigenerated quotient. Hence, one may aim to describe the minimal non-semigenerated Carnot algebras, in the sense that every non-semigenerated Carnot algebra has one such an algebra as a quotient and that every non-trivial quotient of such an algebra is semigenerated.

We achieve this objective within step-3 Carnot algebras. Namely, we construct a family of stratified algebras, called *Engel-type* algebras, that play the role of minimal non-semigenerated stratified algebras with respect to quotient. For each $n \in \mathbb{N}$, we define the $n-$th Engel-type algebra to be the $2(n+1)$-dimensional Lie algebra (of step 3 and rank $n+1$) with basis $\{X, Y_i, T_i, Z\}_{i=1}^n$ where the only non-trivial bracket relations are given by

$$[Y_i, X] = T_i \quad \text{and} \quad [Y_i, T_i] = Z, \quad \forall\, i \in \{1, \ldots, n\}.$$

Notice that the first Engel-type algebra is the classical 4-dimensional Engel algebra, that has proven to be non-semigenerated already in [26]. The diagrams of the first two Engel-type algebras are presented below.




Considering a suitable coordinate system on each Engel-type algebra, it is not difficult to see that none of them is semigenerated. They also have the property that every non-trivial quotient of them has lower step (we call Lie algebras with this property *trimmed*). Consequently, they are not quotients of each others and every non-trivial quotient of theirs is of step 2, so semigenerated. A more subtle result is that the Engel-type algebras are the only occurences of trimmed and non-semigenerated Lie algebras of step 3, which is proven in [C] by a careful induction argument. Combining this result with the (non-trivial) fact that, at least when the step is 3, every non-semigenerated algebra has a trimmed, non-semigenerated quotient, we achieve the classification result for step 3 Carnot algebras. It is possible that this characterization result holds also for higher nilpotency step. However, the algebraic properties get much more complicated as the step increases, and we are not able to go beyond step 3. Nonetheless, our result gives a necessary condition for arbitrary step: if a Carnot algebra has an Engel-type algebra as a quotient, then it is not semigenerated.

As a final remark on article [C] we point out that, in every semigenerated group, a Rademacher-type theorem holds for intrinsic Lipschitz functions. Indeed, in [24, Theorem 4.3.5] it was shown for groups of type $(\star)$ that if $f\colon \mathbb{M} \to \mathbb{L}$ is an intrinsic Lipschitz function and $\mathbb{L}$ is 1-dimensional, then $f$ is intrinsically differentiable $\mathcal{L}^{n-1} \llcorner \mathbb{M}$ -a.e, where $\mathcal{L}^{n-1} \llcorner \mathbb{M}$ stands for the $(n-1)$-dimensional Lebesgue measure on $\mathbb{M}$. Nevertheless, the argument is valid in every Carnot group where the $C^1$-rectifiability result for finite-perimeter sets holds. Moreover, in every semigenerated Carnot group, notions of intrinsic Lipschitz rectifiability and intrinsic $C^1$-rectifiability coincide (see [24, Proposition 4.4.4]).

## 4. Infinite-dimensional Carnot groups

It is natural to ask whether there exists an infinite-dimensional generalization of Carnot groups that could play the role of non-commutative Banach spaces. In the last article [A] of this thesis, we introduce our notion of infinite-dimensional Carnot groups and prove a differentiability result for Lipschitz functions defined on such spaces.

In physics and in pure mathematics, Lie groups lie at the foundation of a great deal of theories, and in many cases these groups are of infinite dimension. Examples can be found in differential and algebraic geometry, knot theory, fluid dynamics, cosmology and quantum mechanics [35]. A suitable generalization of Carnot groups into infinite dimensions is currently under investigation, and various approaches to the question have appeared in the last years. A notion of Infinite-dimensional Heisenberg groups based on an abstract Wiener space was introduced in [22, 10] and Lie groups generalizing those in [44]. Form of infinite-dimensional sub-Riemannian geometry from control theoretic viewpoint was suggested in [31]. Recently, in [41, 40], a Rademacher-type theorem has been proved when the target is a so called Banach homogeneous group, which is a Banach space equipped with a suitable non-abelian group structure.

Our starting point is the metric characterization of Carnot groups (proven in [36] and briefly discussed in Section 2), according to which

Carnot groups are the only metric spaces that are locally compact, geodesic, isometrically homogeneous, and self-similar. In this characterization, the property which associates to some sort of finite-dimensionality is local compactness. Hence, our goal is to find a concept that is an immediate non-locally compact generalization of Carnot groups. At the same time, these groups should be a direct non-abelian generalization of Banach spaces. We expect them also to work as an abstract completion to the class of Carnot groups, similarly as Banach spaces occur as direct limits of Euclidean spaces. In article [A] the notion of infinite-dimensional Carnot group is built in an axiomatic way. A main difficulty arising is that there is not an obvious notion of Lie algebra, and so almost all of the conventional tools that exist in the classical Lie group theory are not available in our setting. Consequently, everything must be defined intrinsically.

4.1. **Definition and examples.** The underlying structure of our construction is a topological group $G$ equipped with a continuous map $\delta\colon \mathbb{R}\times G \to G$ such that $\delta_\lambda := \delta(\lambda,\cdot)$ is a group automorphisms of $G$ for all $\lambda \in \mathbb{R}\setminus\{0\}$,

$$\delta_\lambda \circ \delta_\mu = \delta_{\lambda\mu} \quad \forall\, \lambda,\mu \in \mathbb{R}, \tag{4}$$

and $\delta_0 \equiv e_G$, where $e_G$ is the identity element of $G$. The pair $(G,\delta)$ is called a *scalable group*. Scalable subgroups and scalable homomorphisms in this category are defined in an obvious way. Notice that, due to property (4), the map $\delta_{(\cdot)}\colon (\mathbb{R}\setminus\{0\},\cdot) \to \mathrm{Aut}(G)$, $\lambda \mapsto \delta_\lambda$, is a homomorphism, and therefore in a perfect accordance with Carnot group dilations. Indeed, every Carnot group naturally has a (unique) structure of a scalable group. Vice versa, we say that a scalable group $(G,\delta)$ has Carnot group structure if there exists a Carnot group that is isomorphic to $(G,\delta)$ as a scalable group.

Scalable groups are, most likely, far too general to be a successful framework for non-commutative functional analysis. In order to provide more structure, we introduce the notion of filtrations in scalable groups: we say that a scalable group $G$ is *filtrated by Carnot subgroups* if there exists a sequence $(N_m)_m$, $m\in\mathbb{N}$, of scalable subgroups of $G$ such that each $N_m$ has a Carnot group structure, $N_m$ is a scalable subgroup of $N_{m+1}$, and $G$ is the closure of $\cup_{m\in\mathbb{N}} N_m$. In this case, we say that the sequence $(N_m)_m$ is *a filtration by Carnot subgroups of the scalable group $G$*. We stress that scalable groups admitting filtrations are not necessarily nilpotent.

The notion of filtration might seem, at the first glance, somewhat arbitrary. Nonetheless, we shall argue that it leads to very natural analogues to both classical Carnot groups and separable Banach spaces. Observe that every separable Banach space is filtrated by finite-dimensional Banach spaces, and that every scalable group admitting a filtration is separable.

Recall that, in Carnot groups, the horizontal layer can be detected as the unique subspace of the Lie algebra, in which the intrinsic dilations act as linear scalings. On the group side, one characterizes the set $\exp(V_1)$ as the set of points $p$, where the maps $t \in \mathbb{R} \mapsto \delta_t(p)$ are one-parameter subgroups. Formally this means that, for all $t, s \in \mathbb{R}$, the dilations satisfy $\delta_{t+s}(p) = \delta_t(p)\delta_s(p)$. Since we are expecting a correspondence to Carnot groups, we define for a scalable group $G$ its *first layer* as

$$V_1(G) \coloneqq \{p \in G : t \in \mathbb{R} \mapsto \delta_t(p) \text{ is a one-parameter subgroup}\},$$

We next examine the correspondence of generating first layer and existence of a filtration by Carnot subgroups. In the context of scalable groups, we say that set $A$ *generates* $G$ if $G$ is the closure of the group generated by $\{\delta_t(a) : a \in A,\, t \in \mathbb{R}\}$. It turns out that filtrations and generating one-parameter subgroups are closely related. Indeed, it is rather straightforward to show that if $G$ admits a filtration by Carnot subgroups, then $V_1(G)$ generates $G$ as a scalable group. The opposite implication is less obvious. However, in [A] it is shown that if $G$ is nilpotent, $V_1(G)$ is separable, and $V_1(G)$ generates $G$ as a scalable group, then $G$ admits a filtration by Carnot subgroups. This result is based on the following algebraic characterization of Carnot groups proven in [A]: *if $G$ is a nilpotent scalable group that is generated by finitely many elements of $V_1(G)$, then it has structure of a Carnot group.*

The proof of this characterization relies on the work of Siebert [55], from where it follows that any locally compact topological group admitting a one-parameter family of dilations is a positively graded Lie group, where positive grading is an algebraic condition slightly more general than stratification. In fact, every positively graded Lie group with a generating layer is a Carnot group. Since we are assuming a generating property of the first layer $V_1(G)$, to prove the algebraic characterization result of Carnot groups, one is left to show that any nilpotent scalable group $G$ generated by finitely many one-parameter subgroups is locally compact. We now give the key points of the proof of local compactness.

If $G$ is nilpotent of step $s$, then there exists an abelian subgroup $G^{(s)}$ of $G$ that is generated by group commutators of length $s$. The subgroup $G^{(s)}$ should be seen as an analogue to the last layer of a Carnot group. One then shows, using the dilations on $G$, that this abelian subgroup has structure of a finite-dimensional topological vector space. Consequently, it is locally compact. The argument is finished by induction on step together with the fact that if a topological group $G$ has a locally compact normal subgroup $N$ such that also $G/N$ is locally compact, then $G$ is locally compact itself.

After carefully investigating scalable groups and filtrations, let us add metric to our data. Having the invariant metrics of Carnot groups in our minds, we define *metric scalable group* as a triple $(G, \delta, d)$ where $(G, \delta)$ is a scalable group and $d$ is a left-invariant distance on $G$, which induces the topology of $G$ and satisfies

$$d(\delta_t(p), \delta_t(q)) = |t| d(p, q) \quad \forall\, t \in \mathbb{R}.$$

Observe that every Carnot group can be metrized as a metric scalable group, and that the metric is unique up to bi-Lipschitz equivalence. Finally, we suggest the following definition.

**Definition 4.1.** An *infinite-dimensional Carnot group* is a complete metric scalable group that admits a filtration by Carnot subgroups.

With this interpretation, separable Banach spaces are the only abelian infinite-dimensional Carnot groups, and Carnot groups are the only locally compact infinite-dimensional Carnot groups.

Natural examples of metric scalable groups appear as $\ell_p$-sequences on classical Carnot groups. For a countable family $(G_n, \delta^n, d_n)_{n \in \mathbb{N}}$ of metric scalable groups and for $p \in [1, \infty)$, we define the set $\ell_p((G_n)_n)$ as

$$\ell_p((G_n)_n) := \{(x_n)_{n \in \mathbb{N}} : x_n \in G_n, \sum_{n \in \mathbb{N}} d_n(x_n, e_n)^p < \infty\}.$$

We equip it with the metric

$$d((x_n)_n, (y_n)_n) = \left( \sum_{n \in \mathbb{N}} d_n(x_n, y_n)^p \right)^{1/p}$$

and define group operation and a scaling $\delta \colon \mathbb{R} \times \ell_p((G_n)_n) \to \ell_p((G_n)_n)$ element wise, using the operations on individual scalable groups $(G_n, \delta^n)$. In [A] it is shown that if $(G_n)_{n \in \mathbb{N}}$ is a sequence of infinite-dimensional Carnot groups and $p \in [1, \infty)$, then $\ell_p((G_n)_n)$ is an infinite-dimensional Carnot group as well. Using this $\ell_p$-construction, in [A] we provide an example of an infinite-dimensional Carnot group that is not a Banach Lie group, and also an infinite-dimensional Carnot group that is not nilpotent.

As the last observation before discussing a Rademacher's theorem on infinite-dimensional Carnot groups, we point out an obstruction to the analogy with the metric characterization of Carnot groups. Indeed, by [36], any locally compact, geodesic metric scalable group is a Carnot group. That is, in the locally compact setting, any metric scalable group is a Lie group, and admitting a (scalable) geodesic distance implies existence of a generating subbundle of the tangent bundle. It would be natural to expect that also in infinite dimensions, a (scalable) geodesic distance gives rise to a large family of one-parameter subgroups. However, we now give an example of a geodesic complete metric scalable group $G$ for which $V_1(G) = \{0\}$. Such an example is missing from the published paper [A].

Consider the space $L^1(\mathbb{R})$ seen as an abelian topological group equipped with the usual $L^1$-norm $\|\cdot\|_1$. In this setting, straight lines $t \mapsto tg + (1-t)f$ between two elements $f, g \in L^1(\mathbb{R})$ are geodesics. Instead of the usual vector space scalings on $L^1(\mathbb{R})$, we shall equip this group with a dilation map $\delta\colon \mathbb{R} \times L^1(\mathbb{R}) \to L^1(\mathbb{R})$ defined by

$$(\delta_\lambda f)(x) := f\left(\frac{x}{\lambda}\right), \quad \forall\, \lambda \in \mathbb{R} \setminus \{0\}, \quad \text{and} \quad \delta_0 \equiv 0. \tag{5}$$

It is readily checked that $(L^1(\mathbb{R}), \delta, \|\cdot\|_1)$ is a geodesic complete metric scalable group; the facts that $\delta_\lambda$ is an automorphism of the abelian group $L^1(\mathbb{R})$ for every $\lambda \neq 0$ and property (4) are immediate, and continuity of $\delta$ follows from the basic fact that compactly supported, continuous functions are dense in $L^1(\mathbb{R})$. The identity $\|\delta_\lambda f\|_1 = \lambda \|f\|_1$ is the change of variables formula.

Nevertheless, we claim that $V_1(L^1(\mathbb{R})) = \{0\}$. Indeed, observe that, for every $f \in V_1(L^1(\mathbb{R}))$, it holds

$$\delta_4 f(x) = \delta_2 f(x) + \delta_2 f(x) = 2f\left(\frac{x}{2}\right), \quad \text{for a.e. } x \in \mathbb{R}. \tag{6}$$

Assume then, aiming for contradiction, that there exists some $f \in V_1(L^1(\mathbb{R}))$ such that $\|f\|_1 > 0$. Let $a > 0$ be large enough so that $\int_{(-a,a)} |f| =: c > 0$. Since the first layer of a scalable group is invariant under dilations, we may set $g := \delta_{\frac{1}{4}} f \in V_1(L^1(\mathbb{R}))$. Then

$$\begin{aligned} \int_{(-a,a)} |f(x)| \mathrm{d}x &= \int_{(-a,a)} |\delta_4 g(x)| \mathrm{d}x \overset{(5)}{=} \int_{(-a,a)} |g((x/2)/2)| \mathrm{d}x \\ &= \int_{(-a/2,a/2)} |2g(x/2)| \mathrm{d}x \overset{(6)}{=} \int_{(-a/2,a/2)} |\delta_4 g(x)| \mathrm{d}x \\ &= \int_{(-a/2,a/2)} |f(x)| \mathrm{d}x. \end{aligned}$$

By iterating this argument we see that $f$ has constant positive mass $c$ on an arbitrarily small interval, which contradicts the fact that $f \in L^1(\mathbb{R})$.

4.2. **Rademacher's theorem.** Hans Rademacher proved in 1919 that Lipschitz maps between Euclidean spaces are differentiable almost everywhere with respect to the Lebesgue measure [50]. This theorem has, obviously, far-reaching consequences in Analysis and Geometric Measure Theory, as well as in Metric Geometry. It has accordingly been generalized in various ways over the last century, as was done by Pansu in [46] in the context of Carnot groups.

In [A] a version of Rademacher's theorem is formulated for functions defined on infinite-dimensional Carnot groups. Already in the abelian setting there are two immediate issues that one has to settle when the ambient space has infinite dimensions. First, in Banach spaces there are several nonequivalent notions of differential. In a Banach space $X$, derivatives of real valued functions are linear maps $T\colon X \to \mathbb{R}$ that come as limits of the different quotients

$$T_x(v) \coloneqq \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \quad \text{for } x, v \in X.$$

The map $f$ is said to be Gâteaux differentiable if the map $T$ exists and is linear. If, in addition, the map $T$ satisfies

$$f(x+v) = f(x) + T_x(v) + o(\|v\|) \text{ as } \|v\| \to 0,$$

then it is the Fréchet differential of $f$. In finite dimensions, these two notions agree. Presumably the first strong results on Gâteaux differentiability of Lipschitz maps on separable Banach spaces were proven by Aronszajn [9], whose approach our work closely follows. Results on Fréchet differentiability have proven much harder, see [38, 49] and references therein.

Given group translations and dilations, one can define different quotients, and due to existence of topology, limits thereof. Accordingly, we mimic the idea of Pansu [46] and define our notion of derivative as follows. Denoting by $L_g : G \to G$ the left multiplication by an element $g \in G$, we say that a map $f : G \to H$ between two scalable groups $G$ and $H$ is *Gâteaux differentiable* at a point $p \in G$ if, as $\lambda \to 0$, the maps

$$\delta_{\frac{1}{\lambda}} \circ L^{-1}_{f(p)} \circ f \circ L_p \circ \delta_\lambda$$

point wise converge to a continuous homomorphism from $G$ to $H$.

Another difficulty to overcome is the meaning of "almost everywhere". Indeed, it follows from basic measure theory that every translation invariant, locally finite Borel measure on a separable infinite-dimensional Banach space is identically zero. Therefore, one has to find a suitable collection of "exceptional sets" that would play the role of the $\sigma$-ideal of Lebesgue-null sets in the Rademacher's theorem. The objective in [A] is to follow the idea of [9] and exploit the Haar measures on elements of the filtration of an infinite-dimensional Carnot group: given a filtration $(N_m)_m$, $m \in \mathbb{N}$, by Carnot subgroups of a scalable group $G$, we say that a Borel set $\Omega \subseteq G$ is $(N_m)_m$-*negligible* if $\Omega$ is the countable union of Borel sets $\Omega_m$ such that

$$\mathrm{vol}_{N_m}(N_m \cap (g\Omega_m)) = 0, \quad \forall\, m \in \mathbb{N},\ \forall\, g \in G,$$

where $\mathrm{vol}_{N_m}$ denotes any Haar measure on $N_m$. This class of negligible sets has the natural properties to be hereditary (i.e., if $B$ is negligible and $A \subseteq B$, then $A$ is negligible), and closed under countable unions. Moreover, in [A] it is proven that every filtration-negligible subset of an infinite-dimensional Carnot group has empty interior. The proof is by contradiction: assume that there exists a $(N_m)_m$-negligible set that contains an open set $\Omega$. Then, using the charts on each $N_m$, one can construct a product probability measure $\mu$ with support in $\Omega$. In particular, we have $\mu(\Omega) = 1$. However, by Fubini, every $(N_m)_m$-negligible set has $\mu$-measure zero, leading to a contradiction by monotonicity of $\mu$.

The Rademacher's theorem presented in [A] states that, for every Lipschitz function on an infinite-dimensional Carnot group $G$, there exists a Borel subset $\Omega \subseteq G$ that is $(N_m)_m$-negligible for every filtration $(N_m)_{m\in\mathbb{N}}$ by Carnot subgroups of $G$ such that the map $f$ is Gâteaux differentiable at every $p \notin \Omega$. After carefully building the setting, the proof of the theorem follows the one of Aronszajn [9]. Applying the Pansu-Rademacher theorem on each element of the filtration gives differentiability of the Lipschitz map up to null sets with respect to the respective Haar measures. Differentiability on finite-dimensional subspaces leads to the notion of Gâteaux differential, and the non-differentiability points are by construction contained in a filtration-negligible set.

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# Included articles

# [A]

# Infinite-Dimensional Carnot Groups and Gâteaux Differentiability

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# Infinite-Dimensional Carnot Groups and Gâteaux Differentiability

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**Abstract**
This paper contributes to the generalization of Rademacher's differentiability result for Lipschitz functions when the domain is infinite dimensional and has nonabelian group structure. We introduce an infinite-dimensional analogue of Carnot groups that are metric groups equipped with dilations (which we call metric scalable groups) admitting a dense increasing sequence of finite-dimensional Carnot subgroups. For such groups, we show that every Lipschitz function has a point of Gâteaux differentiability. As a step in the proof, we show that a certain $\sigma$-ideal of sets that are null with respect to this sequence of subgroups cannot contain open sets. We also give a geometric criterion for when such Carnot subgroups exist in metric scalable groups and provide examples of such groups. The proof of the main theorem follows the work of Aronszajn (Stud Math 57(2):147–190, 1976) and Pansu (Ann Math 129(1):1–60, 1989).

**Keywords** Carnot groups · Differentiability · Rademacher · Gateaux derivative

**Mathematics Subject Classification** 28A15 · 53C17 · 58C20 · 46G05

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## 1 Introduction

Rademacher's theorem states that Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}$ are differentiable almost everywhere. This result has far-reaching consequences in Geometric Measure Theory and has been generalized in many ways over the past few decades. In the case of considering domains more general than $\mathbb{R}^n$, there have been two distinct branches. On the one side, extensions of Rademacher's theorem have been studied in infinite-dimensional vector spaces, where there does not exist a Lebesgue-like measure. On the other side, there has been interest in removing the vector-space assumption but preserving the structure of metric measure space. Extensions to more general target spaces have also been considered, but is not the focus of this paper.

Our goal is to extend the theorem to domains that are nonabelian and infinite dimensional. We will concentrate on $\mathbb{R}$-valued functions, although the results will hold for more general targets like RNP Banach spaces. We now quickly review previous results, discuss the issues present in both branches, and provide some references.

For the case of Banach space domain $X$, derivatives of a function $f$ are linear mappings. However, in infinite-dimensional case there are two ways this may be interpreted. A function is Gâteaux differentiable at $x_0 \in X$ if there exists a linear function $T : X \to \mathbb{R}$ satisfying

$$T(v) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

for every $v \in X$. Instead, a function is Fréchet differentiable at $x_0$ if the map $T$ satisfies

$$f(x_0 + v) = f(x_0) + T(v) + o(\|v\|) \text{ as } \|v\| \to 0.$$

Thus, for Gâteaux differentiability, the rate of convergence as $t \to 0$ can depend on $v$ whereas it only depends on $\|v\|$ for Fréchet differentiability. Fréchet differentiability clearly implies Gâteaux differentiability, but the opposite does not hold in general in the infinite dimensional setting. In fact, Lipschitz functions $f : X \to \mathbb{R}$ always have points of Gâteaux differentiability whereas they may lack any point of Fréchet differentiability [1].

In infinite-dimensional settings, one also needs to find a good notion of "almost everywhere". One can reinterpret Rademacher's theorem as stating that the nondifferentiability points lie in the $\sigma$-ideal of Lebesgue null sets. Thus, one aims to prove that the nondifferentiability points of Lipschitz functions lie in some suitable $\sigma$-ideal $\mathcal{N}$. To guarantee at least one point of differentiability, the $\sigma$-ideal $\mathcal{N}$ should not contain open sets. Results of this type have been found for both Gâteaux differentiability and Fréchet differentiability, although the Fréchet differentiability results are far harder and less broad [1,9,10,16].

When considering domains without a linear structure, one typically works in a metric measure space where "almost everywhere" has natural meaning. But resolving what a derivative means becomes more involved, and one requires some additional structure on the domain. For general metric spaces, one needs a collection of Lipschitz charts —which may not, in general, exist— to differentiate the given function $f$

against as it was done by Cheeger in [3]. Other works expanding on this theory of differentiation include [2,4,5,18]. In the special case of Carnot group domains, there are in addition group structure as well as a family of scaling automorphism. This allows us to define a derivative as the limit of rescaled difference ratios converging to a homomorphism as it was done by Pansu in [15].

This paper seeks to bridge the infinite-dimensional framework with the Carnot group setting. Specifically, we will define infinite-dimensional variants of Carnot groups and consider Gâteaux differentiability in this context. We will show that if $G$ is a metric group with a family of dilations (or metric scalable group, as we will call them) and has a dense collection of finite-dimensional Carnot subgroups, then there is a nontrivial $\sigma$-ideal $\mathcal{N}$ so that the Gâteaux non-differentiability points of any Lipschitz function $f : G \to \mathbb{R}$ form an element in $\mathcal{N}$.

We remark that there have been previous studies of infinite-dimensional variants of Carnot groups. Notably, in [13], the authors defined so-called Banach homogeneous groups and showed that Lipschitz functions from $\mathbb{R}$ to these groups are almost everywhere differentiable in the notion of Pansu. Metric scalable groups include these groups as special cases, but they also contain other examples.

Our investigations leave open several natural questions. Most notably, one can ask how small the points of Fréchet nondifferentiability of Lipschitz functions are for metric scalable groups. Even in Banach space domains, this problem is very hard and depends on fine geometric properties of the norm, and so we leave this problem for the future. One can also ask if there are infinite-dimensional variants of Cheeger differentiability. Here, the question becomes more subtle as the differentiability charts must take value in an infinite-dimensional Banach space for which there is no canonical choice. Finally, one would like to know when a metric scalable group is generated by its finite-dimensional subgroups. Specifically, are there geometric properties (geodicity, for example) that tell us when this is the case?

We begin by introducing the notion of scalable group that is the underlying structure of the metric groups with which we will be concerned.

**Definition 1.1** (*Scalable group*) A *scalable group* is a pair $(G, \delta)$, where $G$ is a topological group and $\delta \colon \mathbb{R} \times G \to G$ is a continuous map such that $\delta_\lambda := \delta(\lambda, \cdot) \in \mathrm{Aut}(G)$ for all $\lambda \in \mathbb{R}\backslash\{0\}$,

$$\delta_\lambda \circ \delta_\mu = \delta_{\lambda\mu}, \qquad \forall\, \lambda, \mu \in \mathbb{R}, \tag{1.2}$$

and $\delta_0 \equiv e_G$, where $e_G$ is the identity element of $G$.

Property (1.2) can be rephrased as follows: for every $p \in G$, the map $\delta_{(\cdot)}(p) \colon (\mathbb{R}\backslash\{0\}, \cdot) \to \mathrm{Aut}(G)$ is a homomorphism. Hence it follows that $\delta_1$ is the identity map of $G$.

In an obvious way, in the setting of scalable groups, one can consider the notion of scalable subgroups; a subgroup $H$ of a scalable group $(G, \delta)$ is called a *scalable subgroup* if $G$ if $\delta_\lambda(H) = H$ for all $\lambda \in \mathbb{R}\backslash\{0\}$. We denote then $H < G$. In order to talk about Lipschitz functions, we will endow these groups with metrics that make the dilation automorphisms $\delta_\lambda$ metric scalings in the following sense.

**Definition 1.3** (*Metric scalable group*) A *metric scalable group* is a triple $(G, \delta, d)$ where $(G, \delta)$ is a scalable group and $d$ is an admissible left-invariant distance on $G$ such that

$$d(\delta_t(p), \delta_t(q)) = |t| d(p, q), \qquad \forall t \in \mathbb{R}.$$

By *admissible*, we mean that the metric induces the given topology.

Every Carnot group naturally has structure of a scalable group, where by Carnot group $G$ we mean a simply connected Lie group whose Lie algebra $\mathrm{Lie}(G)$ is equipped with a stratification $\mathrm{Lie}(G) = V_1 \oplus \cdots \oplus V_s$. The stratification is unique up to an isomorphism, see [8], and it defines a family of dilations on $G$. Indeed, one considers the Lie group homomorphisms corresponding to the Lie algebra scalings defined by $\delta^*_\lambda(X) = \lambda^k X$ for $X \in V_k$ and $\lambda \in \mathbb{R}\backslash\{0\}$. Such a group can be metrized as a metric scalable group, and the metric is unique up to biLipschitz equivalence. Vice versa, we say that a scalable group $(G, \delta)$ has a Carnot group structure if there exists a Carnot group that is isomorphic to $G$ as a topological group and whose dilations given by the stratification coincide with $\delta$.

Given a group structure with a dilation, we can define derivatives as done by Pansu [15]. First, for any $g \in G$, let $L_g : G \to G$ be the left multiplication operator. As mentioned before, in the infinite-dimensional case, we need to take care of the distinction between Gâteaux and Fréchet differentiability. Here, we define Gâteaux differentiability.

**Definition 1.4** (*Gâteaux differentiability*) Given two scalable groups $G$ and $H$, a map $f : G \to H$ is *Gâteaux differentiable* at a point $p \in G$ if, as $\lambda \to 0$, the maps $\hat{f}_{p,\lambda} := \delta_{\frac{1}{\lambda}} \circ L^{-1}_{f(p)} \circ f \circ L_p \circ \delta_\lambda$ pointwise converge to a continuous homomorphism from $G$ to $H$. We denote this map by $Df_p$ and it is called the *Gâteaux differential* of $f$ at $p$.

Notice that if $Df_p$ exists, then it is 1-homogeneous in the sense that $Df_p(\delta_\lambda(u)) = \delta_\lambda(Df_p(u))$ for all $\lambda \in \mathbb{R}$ and $u \in G$.

We now introduce a notion requiring that our groups, which are possibly infinite dimensional, are generated by finite-dimensional Carnot subgroups. This will be needed to show that the $\sigma$-ideal we define later is not trivial.

**Definition 1.5** (*Filtration by Carnot subgroups*) We say that a scalable group $G$ is *filtrated by Carnot subgroups* if there exists a sequence $(N_m)_{m \in \mathbb{N}}$ of scalable subgroups of $G$ such that each $N_m$ has a Carnot group structure, $N_m < N_{m+1}$, and $G$ is the closure of $\cup_{m \in \mathbb{N}} N_m$. In this case, we say that the sequence $(N_m)_{m \in \mathbb{N}}$ is *a filtration by Carnot subgroups of the scalable group $G$*.

We have now all the necessary data to give the definition of infinite-dimensional Carnot group.

**Definition 1.6** We call a complete metric scalable group that admits a filtration by Carnot subgroups an *infinite-dimensional Carnot group*.

Necessarily, a metric scalable group that admits a filtration by Carnot subgroups is separable. Note that an infinite-dimensional Carnot group $G$ cannot be equal to its filtration $\cup_{m\in\mathbb{N}} N_m$ unless $G = N_m$ for some $m \in \mathbb{N}$. Indeed, each $N_m \subseteq G$ is nowhere dense and hence the union $\cup_{m\in\mathbb{N}} N_m$ is of first category in $G$. We define next what it means for a set to be null (that is, it lies in our $\sigma$-ideal).

**Definition 1.7** (*Filtration-negligible*) Given a filtration $(N_m)_{m\in\mathbb{N}}$ by Carnot subgroups of a scalable group $G$, we say that a Borel set $\Omega \subseteq G$ is $(N_m)_m$*-negligible* if $\Omega$ is the countable union of Borel sets $\Omega_m$ such that

$$\mathrm{vol}_{N_m}(N_m \cap (g\Omega_m)) = 0, \qquad \forall m \in \mathbb{N}, \forall g \in G,$$

where $\mathrm{vol}_{N_m}$ denotes any Haar measure on $N_m$.

We can now state the main theorem of this paper, which is the following generalization of Aronszajn's differentiability result [1], and the one of Pansu [15].

**Theorem 1.8** *Let $G$ be an infinite-dimensional Carnot group. If $f : G \to \mathbb{R}$ is a Lipschitz map, then there exists a Borel subset $\Omega \subseteq G$ that is $(N_m)_m$-negligible for every filtration $(N_m)_{m\in\mathbb{N}}$ by Carnot subgroups of $G$ and such that for every $p \notin \Omega$ the map $f$ is Gâteaux differentiable at $p$.*

Notice that the above statement is meaningful already within the class of metric scalable groups. However, be aware that a scalable group may not admit any filtration (for example, if the group is not separable), in which case the above theorem has no content, e.g., one can take $\Omega = G$. Nonetheless, there are large classes of scalable groups that admit filtrations (see Proposition 1.10 below for a general criterion and Section 5 for more examples). The first thing to clarify is that, as soon as there is one filtration, the whole scalable group cannot be negligible, as the next proposition states.

**Proposition 1.9** *If $(N_m)_{m\in\mathbb{N}}$ is a filtration by Carnot subgroups of an infinite-dimensional Carnot group $G$ and $\Omega \subseteq G$ is a Borel $(N_m)_m$-negligible set, then $\Omega$ has empty interior.*

As mentioned before, this allows us to conclude that, for groups admitting at least one filtration by Carnot subgroups, every Lipschitz function $f : G \to \mathbb{R}$ has at least one point of Gâteaux differentiability.

Finally, we would like to have geometric conditions that tell us when our group admits filtrations by Carnot subgroups. For a scalable group $G$ define its *first layer* as

$$V_1(G) := \{p \in G : t \in \mathbb{R} \mapsto \delta_t(p) \text{ is a one-parameter subgroup}\},$$

where by one-parameter subgroup we mean that for all $t, s \in \mathbb{R}$,

$$\delta_{t+s}(p) = \delta_t(p)\delta_s(p).$$

Note that if $p \in V_1(G)$, then $\delta_r(p) \in V_1(G)$ for all $r \in \mathbb{R}$, since

$$\delta_{t+s}(\delta_r(p)) = \delta_{tr+sr}(p) = \delta_{tr}(p)\delta_{sr}(p) = \delta_t(\delta_r(p))\delta_s(\delta_r(p)),$$

We say that a set $A \subseteq G$ *generates* $G$ *as a scalable group* or simply that $A$ *generates* $G$ if $G$ is the closure of the group generated by $\{\delta_t(a) : a \in A, t \in \mathbb{R}\}$. Note that $V_1(G)$ is completely analogous to the generating first layer of a finite-dimensional Carnot group. Moreover, the following proposition holds.

**Proposition 1.10** *Let $G$ be a scalable group. If $G$ admits a filtration by Carnot subgroups then $V_1(G)$ generates $G$ as a scalable group. Vice versa, if $G$ is nilpotent, $V_1(G)$ is separable, and $V_1(G)$ generates $G$ as a scalable group, then $G$ admits a filtration by Carnot subgroups.*

We point out that the nilpotency assumption in the previous proposition cannot be removed, since there exist scalable groups with generating first layer that do not admit filtrations (see Proposition 5.11). However, not every metric scalable group having filtrations is nilpotent, as shown in Proposition 5.10. We will discuss this relation in more detail in Sect. 2.

Relying on the result of Siebert, it is rather straightforward to show that scalable groups having Carnot group structure are exactly those scalable groups that are locally compact and have generating first layer (see Theorem 2.14 and the proof of Proposition 2.2). Therefore, keeping Proposition 1.10 in mind, our definition for infinite-dimensional Carnot groups (Definition 1.6) appears in this sense to be a natural non-locally compact generalization of Carnot groups.

We begin by proving Proposition 1.10 in Sect. 2. The crucial observation is that any nilpotent group generated by finitely many elements of $V_1(G)$ has structure of a Carnot group. In Sect. 3 we make a closer study of filtration-negligible sets and prove Proposition 1.9. Section 4 is devoted to the proof of Theorem 1.8 and finally in Sect. 5 we give examples and introduce a class of metric scalable groups that admit filtrations by Carnot subgroups.

## 2 Carnot Groups Generated

The aim of this section is to prove the following proposition, which easily implies Proposition 1.10.

**Proposition 2.1** *Let $G$ be a scalable group. The following are equivalent:*

(i) *$G$ admits a filtration by Carnot subgroups;*
(ii) *there exists a sequence $(a_n)_n \subseteq V_1(G)$ such that $\{a_n\}_{n\in\mathbb{N}}$ generates $G$ as a scalable group and the group generated by $\{a_1, \dots, a_m\}$ is nilpotent for every $m \in \mathbb{N}$.*

The challenging part is to prove that (ii) implies (i). In the core of the argument there is the following result, which we state as a proposition.

**Proposition 2.2** *Let $(G, \delta)$ be a scalable group that is generated by $x_1, \dots, x_r \in V_1(G)$, with $r \in \mathbb{N}$. If $G$ is nilpotent, then it has structure of a Carnot group.*

We give now a proof of Proposition 2.1 using Proposition 2.2 and devote the rest of the section for the proof of Proposition 2.2.

***Proof of Proposition 2.1*** Assume first that $(N_m)_m$ is a filtration by Carnot subgroups of $G$ and denote by $\mathfrak{n}_m$ the corresponding Lie algebras. Since the groups $(N_m)_m$ are Lie subgroups of each others, we may define inductively a basis $\{e_1, \dots, e_{i_m}\}$ for $V_1(\mathfrak{n}_m)$ as an extension of the basis for $V_1(\mathfrak{n}_{m-1})$. By Chow–Rashevskii theorem, the set $\{\exp(e_1), \dots, \exp(e_{i_m})\}$ generates $N_m$ as a scalable group, and since $\cup_m N_m$ is dense in $G$ we may take $(\exp(e_n))_n$ as the desired sequence.

Next, let $(a_n)_{n\in\mathbb{N}} \subseteq V_1(G)$ be the sequence given by (ii). This sequence generates a dense subgroup of $G$, and choosing $N_m$ to be the scalable group generated by $\{a_1, \dots, a_m\}$ gives $G$ a filtration by Carnot groups by Proposition 2.2. □

We begin by fixing the notation in Sect. 2.1. Analogously to Definition 1.1, one can consider $\mathbb{Q}$-*scalable groups* for which the dilation automorphism is defined on the rationals: $\delta\colon \mathbb{Q} \times G \to G$. In Sect. 2.2 we prove that if $G$ is a nilpotent $\mathbb{Q}$-scalable group of step $s$ that is generated by finitely many elements, then $G^{(s)}$ has structure of finite-dimensional $\mathbb{Q}$-vector space. Here $G^{(s)}$ is the last element of the lower central series of the nilpotent group $G$. Some of the simple commutator identities that we use are proved in Appendix A.

In Sect. 2.3 we use the result of Sect. 2.2 to show that under the assumption that $G$ is a nilpotent scalable group generated by finitely many elements, the last layer $G^{(s)}$ is a real finite-dimensional topological vector space, and in particular it is locally compact. Consequently, see Theorem 2.11, also $G$ is locally compact. The proof of Proposition 2.2 is concluded by the result of Siebert (Theorem 2.14), which says that any connected, locally compact, contractible group is a positively gradable Lie group. Namely, we find a gradation $\bigoplus_{t>0} V_t$ of the Lie algebra $\mathrm{Lie}(G)$ such that $V_1$ generates $\mathrm{Lie}(G)$, and hence $\bigoplus_{t>0} V_t$ is a stratification of $G$.

## 2.1 Notation

For a group $G$ and elements $g, h \in G$ we define the group commutator by

$$[g, h] := ghg^{-1}h^{-1}.$$

The elements of lower central series are defined by $G^{(1)} = G$ and $G^{(k)}$ is the group generated by $[G, G^{(k-1)}]$. We say that $G$ is nilpotent of step $s$ if $G^{(s+1)} = \{e\}$ but $G^{(s)} \neq \{e\}$. Notice that in this case $G^{(s)}$ is an abelian subgroup of $G$. We denote by $Z(G)$ the center of $G$.

We follow the terminology of [7] and define recursively *commutators of weight k* for $k \in \mathbb{N}$ in the variables $x_1, x_2, \dots$ as formal bracket expressions. The letters $x_1, x_2 \dots$ are commutators of length one; inductively, if $c_1$, $c_2$ are commutators of weight $k_1$ and $k_2$, then $[c_1, c_2]$ is a commutator of weight $k_1 + k_2$. We also call the commutator of the form $[x_1, [x_2, \dots, [x_{k-1}, x_k] \dots]]$ a *simple commutator* of $x_1, \dots, x_k$.

During this section, it is useful to keep in mind the following lemma. We remark that in [7] the definition of commutator is related to our notation by $[a, b]_{\mathrm{K}hu} = [a^{-1}, b^{-1}]$. However, since in the following lemma the generating set can equivalently be taken symmetric, it applies in our case without modifications.

**Lemma 2.3** (Lemma 3.6(c) in [7]) *Let $G$ be a group and $M \subseteq G$ a subset of $G$. If $M$ generates $G$ as a group, then $G^{(k)}$ is generated by simple commutators of weight $\geq k$ in the elements $m^{\pm 1}$, $m \in M$.*

We also write down the definition of vector space to ease the discussion later on.

**Definition 2.4** Let $\mathbb{K}$ be a field. A $\mathbb{K}$-*vector space* is an abelian group $G$ equipped with an operation $\sigma : \mathbb{K} \times G \to G$ satisfying

(i) $\sigma(q, \sigma(p, g)) = \sigma(qp, g)$,
(ii) $\sigma(q, g)\sigma(p, g) = \sigma(q + p, g)$,
(iii) $\sigma(1, g) = g$,
(iv) $\sigma(q, g)\sigma(q, h) = \sigma(q, gh)$,

for all $q, p \in \mathbb{K}$ and $g, h \in G$. We denote the map $\sigma(q, \cdot)$ by $\sigma_q$.

## 2.2 $\mathbb{Q}$-Scalable Groups

In this section, $G$ will always denote a nilpotent $\mathbb{Q}$-scalable group of step $s$ with dilations $\delta_t$, generated by $x_1, \ldots, x_r \in V_1(G)$. We will show that the last element $G^{(s)}$ of the lower central series admits a structure of finite-dimensional $\mathbb{Q}$-vector space.

**Lemma 2.5** *Let $m \in \mathbb{N}$ and $y \in G^{(k)}$ be a simple commutator of $k$ elements of $V_1(G)$ for some $k \in \{1, \ldots, s\}$. Then $\delta_m(y) = hy^{m^k}$ for some $h \in G^{(k+1)}$.*

***Proof*** The proof is by induction on $k$. If $k = 1$, then $\delta_m(y) = y^m$ since $t \mapsto \delta_t(y)$ is a one-parameter subgroup. Assume that the claim holds for $k - 1$ and let $y \in G^{(k)}$. Now $y = [x, w]$, where $x \in V_1(G)$ and $w \in G^{(k-1)}$ is a simple commutator of $k - 1$ elements of $V_1(G)$. Hence

$$\delta_m(y) = [\delta_m(x), \delta_m(w)] = [x^m, zw^{m^{k-1}}],$$

where $z \in G^{(k)}$. By Lemma A.1 and Corollary A.3, we get

$$\delta_m(y) = h_1[x^m, z][x^m, w^{m^{k-1}}] = h_1[x^m, z]h_2[x, w]^{mm^{k-1}} = h[x, w]^{m^k},$$

where $h = h_1[x^m, z]h_2 \in G^{(k+1)}$. □

**Lemma 2.6** *The abelian group $G^{(s)}$ is a $\mathbb{Q}$-vector space with the scalar multiplication $\sigma_{\frac{n}{m}}(z) := \delta_m^{-1}(z^{nm^{s-1}})$. Moreover, if $z = [x, w] \in G^{(s)}$ with $x \in V_1(G)$ and $w \in G^{(s-1)}$, then $\sigma_q(z) = [\delta_q(x), w]$.*

***Proof*** If the step $s = 1$, the group $G^{(s)} = G$ and the $\mathbb{Q}$-vector space structure is given by the dilation automorphisms $\delta : \mathbb{Q} \times G \to G$, as the maps $t \mapsto \delta_t(x_i)$ are one-parameter subgroups.

For step $s \geq 2$, let first $z \in G^{(s)}$ be a simple commutator of $s$ elements of $V_1(G)$. In particular, $z = [x, w]$, where $x \in V_1(G)$ and $w \in G^{(s-1)}$ is a simple commutator of $s - 1$ elements of $V_1(G)$. Define $\sigma : \mathbb{Q} \times G^{(s)} \to G^{(s)}$ for simple commutators by

$$\sigma_q([x, w]) = [\delta_q(x), w].$$

If $z$ is a product of simple commutators $z_1, \dots, z_k \in G^{(s)}$, we set

$$\sigma_q(z_1 \cdots z_k) = \sigma_q(z_1) \cdots \sigma_q(z_k).$$

By Lemma 2.3, this is enough to define the map $\sigma$ for all $z \in G^{(s)}$.

We show next that

$$\sigma_{\frac{n}{m}}(z) = \delta_m^{-1}(z^{nm^{s-1}}),$$

which proves that the map is well defined. Let first $z = [x, w]$, where $x \in V_1(G)$ and $w \in G^{(s-1)}$ is a simple commutator of $s-1$ elements of $V_1(G)$ and $q = \frac{n}{m} \in \mathbb{Q}_+$, $n, m \in \mathbb{N}$. Lemma 2.5 gives us that

$$\delta_m(\sigma_q([x, w])) = [\delta_m(\delta_{n/m}(x)), \delta_m(w)] = [x^n, hw^{m^{s-1}}] = [x^n, w^{m^{s-1}}],$$

where $h \in G^{(s)} \subseteq Z(G)$. Since $[x, w] \in Z(G)$ as well, we get by iterating Corollary A.2 that

$$\delta_m(\sigma_q([x, w])) = [x, w]^{nm^{s-1}} = z^{nm^{s-1}}.$$

If $q \in \mathbb{Q}_-$, we replace $x$ by $x^{-1}$ in the above calculation as $\delta_{-q}(x) = \delta_q(x^{-1})$ and use Lemma A.4, which gives

$$[x^{-1}, w] = [x, w]^{-1},$$

since now $[x^{-1}, [w, x]] = e_G$.

If $z \in G^{(s)}$ is a product of simple commutators $z_1, \dots, z_k \in G^{(s)}$,

$$\begin{aligned}
\delta_m(\sigma_q(z_1 \cdots z_k)) &= \delta_m(\sigma_q(z_1)) \cdots \delta_m(\sigma_q(z_k)) \\
&= z_1^{nm^{s-1}} \cdots z_k^{nm^{s-1}} \\
&= (z_1 \cdots z_k)^{nm^{s-1}} \\
&= z^{nm^{s-1}}
\end{aligned}$$

since $z_i \in Z(G)$ for all $i$.

Finally, let $z = [x, w]$ be such that $x \in V_1(G)$ and $w$ is an arbitrary element of $G^{(s-1)}$. Then, by Lemma 2.3 there exist simple commutators $v_1, \dots, v_l$ of length $s-1$ such that $w = v_1 \cdots v_l$. By Corollary A.2,

$$\begin{aligned}
\sigma_q([x, v_1 \cdots v_l]) &= \sigma_q([x, v_1] \cdots [x, v_l]) = \sigma_q([x, v_1]) \cdots \sigma_q([x, v_l]) \\
&= [\delta_q(x), v_1] \cdots [\delta_q(x), v_l] = [\delta_q(x), v_1 \cdots v_l].
\end{aligned}$$

It remains to check that the map $\sigma \colon \mathbb{Q} \times G^{(s)} \to G^{(s)}$ satisfies the conditions (i)–(iv) in the Definition 2.4. Condition (iv) is true by construction. The conditions (i) and (iii) follow from the fact that $\delta \colon (\mathbb{Q}^*, \cdot) \to \mathrm{Aut}(G)$ is a group homomorphism:

$$\delta_{qp} = \delta_q \circ \delta_p \quad \text{and} \quad \delta_1 = id,$$

so

$$\sigma_q(\sigma_p([x, w])) = [\delta_q \circ \delta_p(x), w] = [\delta_{qp}(x), w] = \sigma_{qp}([x, w])$$

and

$$\sigma_1([x, w]) = [\delta_1(x), w] = [x, w].$$

Condition (ii) holds by Corollary A.2 and because $t \mapsto \delta_t(x)$ is a one-parameter subgroup for all $x \in V_1(G)$, namely

$$\begin{aligned}\sigma_{q+p}([x, w]) &= [\delta_{q+p}(x), w] = [\delta_q(x)\delta_p(x), w] \\ &= [\delta_q(x), w][\delta_p(x), w] = \sigma_q([x, w])\sigma_p([x, w]).\end{aligned}$$

Hence the map $\sigma$ defines a $\mathbb{Q}$-vector space structure on $G^{(s)}$. □

**Lemma 2.7** *The group $G^{(s)}$ equipped with the $\mathbb{Q}$-vector space structure of Lemma 2.6 is finite dimensional.*

***Proof*** The proof is by induction on the step $s$. If step $s = 1$, $G = V_1(G)$ is commutative and the set $\{x_1, \dots, x_r\}$ is a basis for $V_1(G)$. Suppose that the claim holds for any $\mathbb{Q}$-scalable group of step $s - 1$. Let $K := G/G^{(s)}$ and define

$$\hat{\delta} : \mathbb{Q} \times K \to K, \quad \hat{\delta_q}(gG^{(s)}) := \delta_q(g)G^{(s)}.$$

This map is well defined since $\delta(G^{(s)}) = G^{(s)}$. Hence the group $K$ is a $\mathbb{Q}$-scalable group of step $s - 1$ and it is generated by $\{x_1G^{(s)}, \dots, x_rG^{(s)}\}$. Notice that

$$[xG^{(s)}, yG^{(s)}]_K = [x, y]_G G^{(s)}.$$

Let $\hat{\sigma} : \mathbb{Q} \times K \to K$ be the map from Lemma 2.6, which makes $K^{(s-1)}$ a $\mathbb{Q}$-vector space. By induction hypothesis, there exists a basis $\{k_1, \dots, k_l\}$ of $K^{(s-1)}$. Let $\pi : G \to K$ be the projection and choose $u_i \in \pi^{-1}(k_i) \subseteq G^{(s-1)}$ for all $1 \le i \le l$. We show that the set $\{[x_i, u_j] : 1 \le i \le r, 1 \le j \le l\}$ spans $G^{(s)}$. Since $G^{(s)}$ commutes, it is enough to show that $\{[x_i, u_j]\}$ spans all the elements of the form $[x, u]$, where $x \in V_1(G)$ and $u \in G^{(s-1)}$.

Fix $z = [x, u] \in G^{(s)}$ such that $x \in V_1(G)$ and $u \in G^{(s-1)}$. There exist $q_1, \dots, q_l \in \mathbb{Q}$, $q_i = \frac{n_i}{m_i}$, such that

$$\begin{aligned}\pi(u) &= \hat{\sigma}_{q_1}(k_1) \cdots \hat{\sigma}_{q_l}(k_l) \\ &= \hat{\delta}_{m_1}^{-1}((u_1G^{(s)})^{n_1 m_1^{s-2}}) \cdots \hat{\delta}_{m_l}^{-1}((u_lG^{(s)})^{n_l m_l^{s-2}}) \\ &= \delta_{m_1}^{-1}(u_1^{n_1 m_1^{s-2}}) \cdots \delta_{m_l}^{-1}(u_l^{n_l m_l^{s-2}})G^{(s)} \\ &=: vG^{(s)}.\end{aligned}$$

Hence there exists an element $h \in G^{(s)} \subseteq Z(G)$ such that $u = vh$. Therefore

$$\begin{aligned}
[x, u] &= [x, vh] = [x, v] \\
&= [x, \delta_{m_1}^{-1}(u_1^{n_1 m_1^{s-2}}) \cdots \delta_{m_l}^{-1}(u_l^{n_l m_l^{s-2}})] \\
&= [x, \delta_{m_1}^{-1}(u_1^{n_1 m_1^{s-2}})] \cdots [x, \delta_{m_l}^{-1}(u_l^{n_l m_l^{s-2}})] \\
&= \delta_{m_1}^{-1}([x^{m_1}, u_1^{n_1 m_1^{s-2}}]) \cdots \delta_{m_l}^{-1}([x^{m_l}, u_l^{n_l m_l^{s-2}}]) \\
&= \delta_{m_1}^{-1}([x, u_1]^{n_1 m_1^{s-1}}) \cdots \delta_{m_l}^{-1}([x, u_l]^{n_l m_l^{s-1}}) \\
&= \sigma_{q_1}([x, u_1]) \cdots \sigma_{q_l}([x, u_l]),
\end{aligned}$$

where we used Corollaries A.2 and A.3. Since $x \in V_1(G)$, there exist $q \in \mathbb{Q}$ and $i \in \{1, \dots, r\}$ such that $x = \delta_q(x_i)$. Thus, by the second part of Lemma 2.6,

$$\begin{aligned}
[x, u] &= \sigma_{q_1}([\delta_q(x_i), u_1]) \cdots \sigma_{q_l}([\delta_q(x_i), u_l]) \\
&= [\delta_{q_1 q}(x_i), u_1] \cdots [\delta_{q_l q}(x_i), u_l] \\
&= \sigma_{q_1 q}([x_i, u_1]) \cdots \sigma_{q_l q}([x_i, u_l]).
\end{aligned}$$

□

## 2.3 Proof of Proposition 2.2

Our first task is to prove that $G$ is locally compact. To show this, we consider the $\mathbb{Q}$-scalable subgroup $G_{\mathbb{Q}}$ of $G$ that by definition is generated as a group by $\{\delta_t(x_i) : t \in \mathbb{Q}, 1 \leq i \leq r\} =: V_{\mathbb{Q}}$. Let $\sigma : \mathbb{Q} \times G_{\mathbb{Q}}^{(s)} \to G_{\mathbb{Q}}^{(s)}$ be the continuous map from Lemma 2.7 which makes $G_{\mathbb{Q}}^{(s)}$ a $k$-dimensional $\mathbb{Q}$-vector space for some $k \in \mathbb{N}$. We use the following facts about topological groups to show that $G^{(s)}$ is a finite-dimensional real topological vector space.

**Theorem 2.8** (Theorem 1.22 in [17]) *A Hausdorff topological vector space is locally compact if and only if it is finite dimensional.*

**Lemma 2.9** *Every locally compact subgroup of a topological group is closed.*

***Proof*** This proof is adapted from a Mathematics Stack Exchange post by Eric Wofsey [6]. Let $H$ be a topological group and let $K$ be a locally compact subgroup of $H$. Then $\overline{K}$ is also a subgroup of $H$, and $K$ is dense in $\overline{K}$. We claim that every locally compact dense subset of a Hausdorff space is open. Indeed, let $S$ be a locally compact dense subset of a Hausdorff space $X$ and take $x \in S$. Let also $U$ be open in $S$ such that $x \in U$, $\overline{U} \subseteq S$, and $\overline{U}$ is compact. Take then an open set $V \subseteq X$ such that $V \cap S = U$. Since $X$ is Hausdorff, $\overline{U}$ is closed in $X$ and therefore $V \backslash \overline{U}$ is open in $X$. But

$$(V \backslash \overline{U}) \cap S = (V \cap S) \backslash \overline{U} = U \backslash \overline{U} = \emptyset,$$

and hence $V \setminus \overline{U} = \emptyset$ as $S$ is dense in $X$. We conclude that $V \subseteq S$, which proves the claim.

Hence, by the previous claim $K$ is open in $\overline{K}$. Recall that every open subgroup of a topological group is closed since the complement $K^c$ of an open subgroup $K$ is the union of open sets; $K^c = \cup_{x \in K^c} xK$. Hence $K$ is closed in $\overline{K}$ and therefore also in $H$. □

**Lemma 2.10** *$G^{(s)}$ equals to $\overline{G_{\mathbb{Q}}^{(s)}}$ and it is a k-dimensional real topological vector space.*

***Proof*** Let $\{v_1, \dots, v_k\}$ be a basis for $G_{\mathbb{Q}}^{(s)}$. We claim that since $G_{\mathbb{Q}}^{(s)} \subseteq Z(G)$, we may assume that each $v_i$ is of the form $[x_i, w_i]$ with $x_i \in V_{\mathbb{Q}}$ and $w_i \in G_{\mathbb{Q}}^{(s-1)}$. Indeed, recall that by Lemma 2.3 any element of $G_{\mathbb{Q}}^{(s)}$ is a product of simple commutators of elements of $V_{\mathbb{Q}}$ of weight $s$, which proves the claim. Let

$$W := \{[\delta_{t_1}(x_1), w_1] \cdots [\delta_{t_k}(x_k), w_k] \mid t_i \in \mathbb{R}\},$$

which is a group by Corollary A.2 and since $t \mapsto \delta_t(x_i)$ is a one-parameter subgroup for each $i \in \{1, \dots, k\}$. Now $G_{\mathbb{Q}}^{(s)} \subseteq W$ by definition of $\sigma$ and $W \subseteq \overline{G_{\mathbb{Q}}^{(s)}}$ by continuity of dilations. We define $\tilde{\sigma} : \mathbb{R} \times W \to W$ by

$$\tilde{\sigma}_\lambda([\delta_{t_1}(x_1), w_1] \cdots [\delta_{t_k}(x_k), w_k]) = [\delta_{\lambda t_1}(x_1), w_1] \cdots [\delta_{\lambda t_k}(x_k), w_k].$$

This map is continuous and it defines an $\mathbb{R}$-vector space structure on $W$: since $\tilde{\sigma}$ is a continuous extension of $\sigma$, it is easy to show that $\tilde{\sigma}$ fulfills the conditions in Definition 2.4. Hence $W$ is a $k$-dimensional real topological vector space. Therefore, by Theorem 2.8 and Lemma 2.9, $W$ is closed and so $W = \overline{G_{\mathbb{Q}}^{(s)}}$. We conclude the proof by noting that $G = \overline{G_{\mathbb{Q}}}$, and hence $G^{(s)} = \overline{G_{\mathbb{Q}}}^{(s)} = \overline{G_{\mathbb{Q}}^{(s)}}$, where the last equality follows form the continuity of the group operation. □

The following statement on topological groups will allow us to conclude that $G$ is locally compact.

**Theorem 2.11** ([14] p. 52) *If a topological group $G$ has a closed subgroup $H$ such that $H$ and the coset-space $G/H$ are locally compact, then $G$ is locally compact.*

**Lemma 2.12** *Let $(G, \delta)$ be a nilpotent scalable group that is generated by $x_1, \dots, x_r \in G$ as a scalable group over $\mathbb{R}$. Then $G$ is locally compact.*

***Proof*** The proof is again by induction on the step $s$. If $s = 1$, the group $G$ is a real topological vector space with basis $\{x_1, \dots, x_r\}$ and hence locally compact by Theorem 2.8. Assume that the claim holds for step $s - 1$ and consider $K := G/G^{(s)}$, which is generated by $x_1 G^{(s)}, \dots, x_r G^{(s)}$ with dilations $\hat{\delta}_t(x G^{(s)}) := \delta_t(x) G^{(s)}$. Now $K$ is indeed an $\mathbb{R}$-scalable topological group, since $G^{(s)}$ is a closed normal subgroup of $G$ by Lemma 2.10. Hence $K$ is locally compact by the induction hypothesis, and by Theorem 2.8 the group $G^{(s)}$ is locally compact as well. Finally Theorem 2.11 proves the claim. □

To prove Proposition 2.2, we use the result of Siebert below.

**Definition 2.13** Let $G$ be a topological group. A continuous automorphism $\zeta$ of $G$ is said to be *contractive* if $\lim_{n\to\infty} \zeta^n(x) = e_G$ for all $x \in G$. A group that admits a contractive continuous automorphism is called *contractible*.

**Theorem 2.14** (Corollary 2.4 in [19]) *A topological group $G$ is a positively gradable Lie group if and only if it is connected, locally compact and contractible. In particular, if $\zeta \in \mathrm{Aut}(G)$ is contractive, then the gradation $\bigoplus_{t>0} V_t$ given by $\zeta$ is such that*

$$\{X \in \mathrm{Lie}(G) \mid (\mathrm{d}\zeta - \alpha\,\mathrm{id})X = 0\} \subseteq V_{-\ln|\alpha|}$$

***Proof of Proposition 2.2*** We proved in Lemma 2.12 that the group $G$ is locally compact. It is also connected, since the map $\gamma_x : [0,1] \to G$, $\gamma_x(t) = \delta_t(x)$ is a continuous path between $e_G$ and $x$ for every $x \in G$. Additionally, the group $G$ is contractible as the automorphisms $\delta_t$ are contractive for all $t \in (0,1)$; for a fixed $t \in (0,1)$,

$$\lim_{n\to\infty} \delta_t^n(x) = \lim_{n\to\infty} \delta_{t^n}(x) = \delta_0(x) = e_G$$

for all $x \in G$. Hence by Theorem 2.14 the group $G$ is a Lie group and each $\delta_t$, $t \in (0,1)$, defines a positive gradation for $\mathrm{Lie}(G)$. We claim that, in order to prove that $G$ admits a structure of Carnot group, it is enough to find a gradation of $\mathrm{Lie}(G)$ such that $V_1$ generates the whole of $\mathrm{Lie}(G)$. Indeed, a stratification of a Lie algebra $\mathrm{Lie}(G)$ is equivalent to a positive gradation whose degree-one layer generates $\mathrm{Lie}(G)$ as a Lie algebra.

Let us consider the gradation given by $\delta_{1/e}$. By Theorem 2.14,

$$\left\{X \in \mathrm{Lie}(G) \mid \left(\mathrm{d}\delta_{1/e} - \frac{1}{e}\,\mathrm{id}\right) X = 0\right\} \subseteq V_{-\ln(1/e)} = V_1.$$

Let $x \in \{x_1, \ldots, x_r\}$. Since the map $t \mapsto \delta_t(x)$, $t \in \mathbb{R}$, is now a one-parameter subgroup of a Lie group, there exists $X \in \mathrm{Lie}(G)$ such that

$$\delta_t(x) = \exp(tX)$$

for all $t \in \mathbb{R}$. Additionally, on the one hand, since $\exp\colon \mathrm{Lie}(G) \to G$ is a global diffeomorphism,

$$\log(\delta_t(x)) = tX.$$

On the other hand

$$\log(\delta_t(x)) = \log(\delta_t(\exp(X))) = \log(\exp(\mathrm{d}\delta_t(X))) = \mathrm{d}\delta_t(X).$$

Hence

$$\mathrm{d}\delta_{1/e}(X) = \frac{1}{e}X$$

and $X \in V_1$. Therefore $\log(x_i) \in V_1$ for all $i \in \{1, \dots, r\}$. Notice that, for any $Y \in \mathrm{Lie}(G)$, we have for some $l \in \mathbb{N}$ and $i_l \in \{1, \dots, r\}$ that

$$\exp(Y) = \delta_{t_1}(x_{i_1}) \cdots \delta_{t_l}(x_{i_l}) = \exp(t_1 \log(x_{i_1})) \cdots \exp(t_l \log(x_{i_l})).$$

Hence $\{\log(x_1), \dots, \log(x_r)\}$ generates $\mathrm{Lie}(G)$ as a Lie algebra by the Baker–Campbell–Hausdorff formula and $V_1 = \mathrm{span}(\log(x_1), \dots, \log(x_r))$. Thus the gradation given by $\delta_{1/e}$ is a stratification and $G$ has structure of a Carnot group.

We still need to verify that the one-parameter family $(\delta_t)_{t\in\mathbb{R}}$ of Lie group automorphisms are the Carnot group dilations given by the stratification. The Carnot group dilation of factor $t \neq 0$ is by definition the unique map $\zeta_t \in \mathrm{Aut}(G)$ such that

$$\mathrm{d}\zeta_t(X) = t^k X \quad \text{for all } X \in V_k, \tag{2.15}$$

and $\mathrm{d}\zeta_0$ is the zero map. Obviously $\mathrm{d}\delta_0 = 0$, so consider the case $t \neq 0$. Recall that each $V_k$ is spanned by simple commutators of $\log(x_i)$, $i \in \{1, \dots, r\}$ that span the first layer. Since $\mathrm{d}\delta_t$ is a Lie algebra homomorphism, we get for these elements that

$$\begin{aligned}
&\mathrm{d}\delta_t([\log(x_{i_k}), \dots, [\log(x_{i_2}), \log(x_{i_1})]])\\
&\quad = [\mathrm{d}\delta_t(\log(x_{i_k})), \dots, [\mathrm{d}\delta_t(\log(x_{i_2})), \mathrm{d}\delta_t(\log(x_{i_1}))]]\\
&\quad = [t\log(x_{i_k}), \dots, [t\log(x_{i_2}), t\log(x_{i_1})]]\\
&\quad = t^k[\log(x_{i_k}), \dots, [\log(x_{i_2}), \log(x_{i_1})]].
\end{aligned}$$

By linearity of $\mathrm{d}\delta_t$ we conclude that the maps $\mathrm{d}\delta_t$ satisfy condition (2.15). Hence the scalable group $(G, \delta)$ is a Carnot group and the dilations $\delta_t$, $t \in \mathbb{R}$, are the unique Carnot group dilations given by the stratification. □

It would be interesting to find geometric conditions that allow us to conclude that $V_1(G)$ generates $G$. Indeed, in the case of simply connected nilpotent Lie groups admitting dilations, geodicity implies that the first layer generates the entire group since rectifiable curves can be approximated by horizontal line segments.

## 3 Negligible Sets of Metric Scalable Groups

In this section $(G, d, \delta)$ denotes a metric scalable group (according to Definition 1.3). We begin by giving some auxiliary lemmas and then prove Proposition 1.9, which states that filtration-negligible sets always have empty interior. After that we introduce another notion of null-sets following [1] and prove that it agrees with the definition of filtration-negligible sets. This result is formulated in Theorem 3.6.

### 3.1 Elementary Properties of Metric Scalable Groups

**Lemma 3.1** *For each $v \in V_1(G)$, $v \neq e$, the map $t \mapsto \delta_t(v)$ is a homothetic embedding from $\mathbb{R}$ to $G$.*

***Proof.*** Let $c = d(0, \delta_1(v)) > 0$. We claim that $d(\delta_\alpha(v), \delta_\beta(v)) = c|\alpha - \beta|$. Indeed, as $\delta_t(v)$ is a one parameter subgroup, we get by left-invariance that

$$d(\delta_\alpha(v), \delta_\beta(v)) = d(0, \delta_{\beta-\alpha}(v)) = |\beta - \alpha| d(0, v) = c|\beta - \alpha|.$$

□

**Lemma 3.2** *Let $K \subset G$ be a totally bounded set and $\epsilon > 0$. There exists $\delta > 0$ so that*

$$d(hk, k) < \epsilon, \qquad \forall h \in B(0, \delta), k \in K.$$

***Proof*** As $K$ is totally bounded, there is a finite number of points $\{y_1, \ldots, y_n\} \in G$ so that $K \subseteq \bigcup_{j=1}^{n} B(y_j, \epsilon/4)$. Choose $\delta$ small enough so that for any $h \in B(0, \delta)$, we have $\max_{1 \leq j \leq n} d(y_j, hy_j) < \epsilon/4$. Now let $k \in K$ and $y_i$ be so that $d(k, y_i) < \epsilon/4$. Then for any $h \in B(0, \delta)$, we get

$$d(hk, k) \leq d(hk, hy_i) + d(hy_i, y_i) + d(y_i, k) = 2d(y_i, k) + d(hy_i, y_i) < \epsilon,$$

where we used the left-invariance of the metric. □

**Lemma 3.3** *Let $G$ be a complete metric scalable group. For every $i \in \mathbb{N}$, let $\psi_i : \mathbb{R} \to G$ be continuous such that $\psi_i(0) = e_G$. Then for every non-empty open set $U$ containing $e_G$ there exists a sequence of positive numbers $\alpha_1, \alpha_2, \ldots > 0$ so that the map*

$$\phi : \prod_{i=1}^{\infty} [0, \alpha_i] \to G$$
$$(t_1, t_2, \ldots) \mapsto \cdots \psi_2(t_2)\psi_1(t_1)$$

*is well defined and has range in $U$.*

***Proof*** We may assume that $U$ contains the unit ball at $e_G$. Note that for each $k \in \mathbb{N}$, $K_k := \phi(\prod_{i=1}^{k} [0, \alpha_i] \times (0, 0, \ldots))$ is a compact set in $G$. We can construct $\alpha_i$ recursively. First choose $\alpha_1 > 0$ small enough so that $K_1 \subset B(e_G, 1/2)$. Now having chosen $\alpha_i$, we choose $\alpha_{i+1} > 0$ so that

$$\sup_{g \in K_i} \sup_{t \in [0, \alpha_{i+1}]} d(g, \psi_{i+1}(t)g) < 2^{-i-1}.$$

This is possible by Lemma 3.2 and the fact that $\psi_{i+1}(0) = e_G$ is continuous at 0. Then each sequence defining a $\phi(t_1, t_2, \ldots)$ is Cauchy and so the limit exists. The fact that the image is in $U$ also follows immediately. □

Here, $\psi_i$ can be anything, but in the application of Lemma 3.3, we will take each $\psi_i(t)$ to be $\exp(t \log(v_i))$ for some $v_i \in V_1(G)$.

### 3.2 Non-negligibility of Open Sets: Proof of Proposition 1.9

Let $(N_m)_{m\in\mathbb{N}}$ be a filtration of $G$ by Carnot groups. Assume for contradiction that an $(N_m)_m$-negligible set $\Omega$ contains an open non-empty set $U$.

For every $m \in \mathbb{N}$, choose $\{v_{k_{m-1}+1}, \dots, v_{k_m}\} \subset V_1(G)$ so that $\{v_1, \dots, v_{k_m}\}$ generate $N_m$ as a basis. Let $\psi_i : \mathbb{R} \to G$ be $\psi_i(t) := \exp(t \log(v_i))$. With the above choice of $U$, let $(\alpha_m)$ and $\phi$ be as in Lemma 3.3. Notice that the maps $\tilde{\phi}_m : \mathbb{R}^{k_m} \to N_m$, $\tilde{\phi}_m(t_1, t_2, \dots, t_{k_m}) := \psi_{k_m}(t_{k_m}) \cdots \psi_2(t_2)\psi_1(t_1)$, are diffeomorphisms.

Let then

$$\phi_m := \tilde{\phi}_m|_{\prod_{i=1}^{k_m}[0,\alpha_i]}.$$

Let $\mu$ be the measure on $G$ that is the pushforward via $\phi$ of the probability measure on $\prod_{i=1}^{\infty}[0, \alpha_i]$ that is the product of the rescaled probability Lebesgue measure on each of the $[0, \alpha_i]$.

Since $\phi$ has image contained in $U$, $\mu(U) = 1$ and hence $\mu(\Omega) = 1$. However, we shall show that $\mu(\Omega) = 0$, which will be our contradiction. Since the set $\Omega$ is $(N_m)_m$-negligible, then $\Omega = \cup_{m\in\mathbb{N}}\Omega_m$ for some $\Omega_m$ such that for each $m$,

$$\mathrm{vol}_{N_m}(N_m \cap g\Omega_m) = 0, \qquad \forall g \in G. \tag{3.4}$$

It is enough to show that $\mu(\Omega_m) = 0$ for any arbitrary $m$. For doing so, fix $m \in \mathbb{N}$ and let $\nu_1$ and $\nu_2$ denote the product probability measures (again with respect to the rescaled Lebsgue probability measures) on $C_1 = \prod_{i=1}^{k_m}[0, \alpha_i]$ and $C_2 = \prod_{i=k_m+1}^{\infty}[0, \alpha_i]$, respectively. Notice that $(\phi_m)_\#(\nu_1)$ is a smooth measure on some open set of $N_m$ and hence it is absolutely continuous with respect to $\mathrm{vol}_{N_m}$. In conjunction with (3.4), we get for any $\mathbf{t}_2 \in C_2$,

$$\begin{aligned}
\int_{C_1} \chi_{\phi^{-1}(\Omega_m)}(\mathbf{t}_1, \mathbf{t}_2)\, d\nu_1(\mathbf{t}_1) &= \int_{C_1} \chi_{\Omega_m}(\phi(\bar{0}, \mathbf{t}_2)\phi_m(\mathbf{t}_1))\, d\nu_1(\mathbf{t}_1) \\
&= \int_{C_1} \chi_{\phi_m^{-1}(\phi(\bar{0},\mathbf{t}_2)^{-1}\Omega_m)}(\mathbf{t}_1)\, d\nu_1(\mathbf{t}_1) \\
&= \nu_1(\phi_m^{-1}(\phi(\bar{0}, \mathbf{t}_2)^{-1}\Omega_m)) \\
&= (\phi_m)_\#(\nu_1)(N_m \cap \phi(\bar{0}, \mathbf{t}_2)^{-1}\Omega_m) \\
&\preceq \mathrm{vol}_{N_m}(N_m \cap \phi(\bar{0}, \mathbf{t}_2)^{-1}\Omega_m) = 0.
\end{aligned}$$

Thus, $\mu(\Omega_m) = \int_{C_2}\int_{C_1} \chi_{\phi^{-1}(\Omega_m)}\, d\nu_1\, d\nu_2 = 0$. □

***Remark 3.5*** Note that the statement of Proposition 1.9 makes sense for scalable groups without any metric. Indeed, the notion of filtrations (and thus also negligibility) only relies on the topology. Thus, it may be possible that the result is true for all scalable groups although we have not verified this.

In the rest of this section we make a closer study of filtration-negligible sets of metric scalable groups. Below we define an *exceptional class* of null sets analogously to [1] and prove that it is equivalent to our notion of filtration-negligible sets.

### 3.3 The Exceptional Class $\mathcal{U}$

Let $G$ be a scalable group with identity element denoted by $e_G$ and let $\mathcal{B}(G)$ be the Borel sets of $G$. For every $a \in V_1(G)$, with $a \neq e_G$ set

$$\mathcal{U}(a) := \{A \in \mathcal{B}(G) : \forall g \in G, |A \cap (g \cdot \mathbb{R}a)| = 0\},$$

where we denote by $\mathbb{R}a$ the image of the curve $t \in \mathbb{R} \mapsto \delta_t a$ and by $|\cdot|$ the 1-dimensional Lebesgue measure on the curve. In other words,

$$|A \cap (g \cdot \mathbb{R}a)| = |\{t \in \mathbb{R} : g\delta_t a \in A\}|.$$

For every countable set $I$ and $\{a_n\}_{n\in I} \subset V_1(G)\backslash\{e_G\}$, define

$$\mathcal{U}(\{a_n\}_n) := \{A \in \mathcal{B}(G) : A = \cup_{n\in I} A_n, A_n \in \mathcal{U}(a_n)\}.$$

Finally, set $\mathcal{U}$ to be the intersection of all $\mathcal{U}(\{a_n\}_n)$ among all dense sequences $\{a_n\}_n \subseteq V_1(G)\backslash\{e_G\}$.

Recall a class of sets $\mathcal{F}$ is hereditary if $A \subset B$ and $B \in \mathcal{F}$ implies that $A \in \mathcal{F}$. The classes $\mathcal{U}(a), \mathcal{U}(\{a_n\}_n)$, and $\mathcal{U}$ are $\sigma$-additive, hereditary, and do not contain any open non-empty set (see the theorem below). Moreover, we have the property:

$$\{a'_n\} \subseteq \{a_n\} \implies \mathcal{U}(\{a'_n\}) \subseteq \mathcal{U}(\{a_n\}).$$

**Theorem 3.6** *Let $G$ be a metric scalable group and let $\{a_n\} \subset V_1(G)\backslash\{e_G\}$ be a dense sequence such that the group $N_m$ generated by $\{a_1, \ldots, a_m\}$ is nilpotent for all $m \in \mathbb{N}$. Then a set $\Omega \subseteq G$ is in the class $\mathcal{U}(\{a_n\})$ if and only if it is $(N_m)_{m\in\mathbb{N}}$ -negligible.*

Note that by Proposition 2.2, each $N_m$ in the theorem above is a Carnot group and the statement makes sense. The proof of the theorem will be a straightforward consequence of Proposition 3.12. The proof of Proposition 3.12 needs some preparation, and we postpone it to the end of this section.

**Lemma 3.7** *Let $A \subset G$ be a bounded Borel set and choose a $v \in V_1(G)$. Then the function*

$$f_A(x) = |A \cap (x \cdot \mathbb{R}v)|$$

*is Borel.*

***Proof*** Let $R > 0$ be arbitrary and let $\mathcal{A}$ denote the set of all $A \subseteq B(0, R)$ that satisfy the conclusion. We will prove that $\mathcal{A}$ contains the Borel sets of $B(0, R)$. We first prove that the open sets in $B(0, R)$ are in $\mathcal{A}$. Indeed, let $A$ be open and $t \in \mathbb{R}$. We will show $A' = f_A^{-1}((t, \infty))$ is open.

As $f_A$ is nonnegative, we may suppose without loss of generality that $t \geq 0$. Let $g \in f_A^{-1}((t, \infty))$ and $\delta = f_A(g) - t > 0$. Let $E = \{s \in \mathbb{R} : g\delta_s(v) \in A\}$, which is a bounded set by boundedness of $A$ and Lemma 3.1. For each $s \in E$ define

$d(s) = d(g\delta_s(v), A^c)$, a positive continuous function on $E$. We can choose $\epsilon > 0$ small enough so that

$$E' = \{s \in \mathbb{R} : d(g\delta_s(v)) > \epsilon\},$$

satisfies $|E'| > f_A(g) - \frac{\delta}{2}$.

Note that $E'$ is totally bounded. By Lemma 3.1, the set $\{g\delta_s(v) : s \in E'\}$ is the isometric image of the totally bounded set $E'$ and so it also is totally bounded. Thus by Lemma 3.2, there exists $\eta_0 > 0$ so that

$$\sup_{h \in B(0,\eta_0)} \sup_{s \in E'} d(hg\delta_s(v), g\delta_s(v)) < \epsilon.$$

Since $G$ is topological, there exists some $\eta > 0$ small enough so that $B(g, \eta) \subseteq B(0, \eta_0)g$. This then gives that

$$\sup_{h \in B(g,\eta)} \sup_{s \in E'} d(h\delta_s(v), g\delta_s(v)) < \epsilon.$$

This shows that $h\delta_s(v) \in A$ when $h \in B(g, \eta)$ and $s \in E'$ and so

$$f_A(h) \geq |E'| > f_A(g) - \frac{\delta}{2} > t,$$

which proves $B(g, \eta) \subseteq f_A^{-1}((t, \infty))$ and so $f_A^{-1}((t, \infty))$ is open.

We now show that $\mathcal{A}$ is a monotone class of sets, which will prove that $\mathcal{A}$ contains all Borel sets. Let $\{E_i\}$ be an increasing sequence in $\mathcal{A}$ and $E = \bigcup_i E_i$. Then $E \cap (x \cdot \mathbb{R}v) = \bigcup_i (E_i \cap (x \cdot \mathbb{R}v))$, which is also an increasing family and so by monotone convergence theorem we get

$$f_E(x) = \lim_{i \to \infty} f_{E_i}(x).$$

Thus, $f_E$, the increasing pointwise limit of $f_{E_i}$, must be Borel and so $E \in \mathcal{A}$. Similarly, let $\{E_i\}$ be a decreasing sequence in $\mathcal{A}$ and let $E = \bigcap_i E_i$. Then $E \cap (x \cdot \mathbb{R}v) = \bigcap_i (E_i \cap (x \cdot \mathbb{R}v))$, which is another decreasing sequence. As $E_1$ is bounded, $f_{E_1}(x) < \infty$ and so by dominated convergence theorem we conclude $f_E(x) = \lim_{i \to \infty} f_{E_i}(x)$. Thus, $E \in \mathcal{A}$, which proves the monotonicity property of $\mathcal{A}$. □

**Lemma 3.8** *Let $A \subseteq G$ be any Borel set and $v \in V_1(G)$. Then the set*

$$\{g \in A : |A \cap (g \cdot \mathbb{R}v)| > 0\}$$

*is Borel.*

***Proof*** Let $A_n = A \cap B(0, n)$. By monotone convergence theorem, the set in question is equal to $\bigcup_{n=0}^{\infty} \{g \in A_n : |A_n \cap (g \cdot \mathbb{R}v)| > 0\} = \bigcup_{n=0}^{\infty} (f_{A_n}^{-1}((0, \infty)) \cap A_n)$, which, by the previous lemma, is a countable union of Borel sets. □

### 3.4 Null Decomposition

Let $G$ be a Carnot group with $\dim V_1 = n$ and suppose $G$ is homeomorphic to $\mathbb{R}^m$. We let $X_1, \ldots, X_n$ be the vector fields in $\mathbb{R}^m$ that are given by left translation of a basis in $V_1$.

**Lemma 3.9** *Let $M$ be an analytic manifold in $\mathbb{R}^m$ of dimension less than $m$. Then for $\mathrm{vol}_M$-almost every $p \in M$, there exists an open neighborhood $U \subseteq M$ of $p$ and an index $i \in \{1, \ldots, n\}$ for which $X_i(q) \notin T_q M$ for any $q \in U$.*

***Proof*** Let $k = \dim M < m$. For each $i \in \{1, \ldots, n\}$, let $A_i = \{p \in M : X_i(p) \in T_p M\}$, which are closed subsets of $M$. We claim that $A = \bigcap_i A_i$ has measure zero.

Suppose not. Let $f : U \to V$ be the inverse of an analytic chart map where $U \subset \mathbb{R}^k$ and $V \subset M$. We pushforward the basis vector fields of $\mathbb{R}^k$ via $f$ to get vector fields $Y_1, \ldots, Y_k$ that form a basis of $TV$. As $f$ is analytic, these are analytic vector fields.

Note that $A_i \cap V$ are precisely the points of $V$ for which $X_i(p)$ is in the span of the $Y_j(p)$'s. This is the same as the being in the zero set of the function $g_i(p) = |X_i(p) - P_{\langle Y_1(p), \ldots, Y_k(p)\rangle} X_i(p)|^2$ where $P$ is the orthogonal projection map onto the span of the $Y_j(p)$'s. Note that each $g_i$ is an analytic function as projection is a combination of matrix multiplication and inverses. Thus, $A$ is the zero set of the product function $g = g_1 \cdots g_k$, also an analytic function. Finally, we consider the function $g \circ f : U \to \mathbb{R}$, another analytic function. If $A$ has positive measure, then $f^{-1}(A)$ has positive measure and so $g \circ f$ is identically zero [11]. This means $A \cap U = U$. By definition of the $A_i$'s, this means $U$ is an integral manifold. We now derive a contradiction.

This means that $TU$ contains the vector fields $\{X_i|_U\}$. If vector fields $X, Y$ are tangent to $U$, then so is $[X, Y]$. As $\{X_i\}$ are tangent to $U$ and generate all of $\mathbb{R}^m$ under Lie brackets, this means that $T_x U = \mathbb{R}^m$ for all $x \in U$. However, this is a contradiction as $\dim U = k < m$.

We have established that $A^c$ is a full measure open set. Let $p \in A^c$. Then $p \in A_i^c$ for some $i$. As $A_i^c$ is open, there then exists an open neighborhood $p \in U \subseteq A_i^c$. This neighborhood satisfies the conclusion of the lemma with $X_i$. □

Given a Borel set $A \subseteq \mathbb{R}^m$ and $i \in \{1, \ldots, n\}$ we define

$$A_i := \{p \in A : |A \cap (p \cdot \mathbb{R}X_i)| > 0\}.$$

By $p \cdot \mathbb{R}X_i$, we mean the 1-dimensional $\mathbb{R}$-flow of the vector field $X_i$ that passes through $p \in \mathbb{R}^m$. Note that this is an analytic submanifold.

Given a word $w$ written in the alphabet $\{1, \ldots, n\}$, we define $A_w = (A_{w'})_i$ where $w = w'i$ and $A_\emptyset = A$. Note that $A_w \subseteq A_{w'}$. Let $w$ denote the word $123 \cdots n$, the concatenation of all the letters. Define the word $w^k$ to be the $k$-fold concatenation of $w$ (so $w^k$ is $kn$ letters long).

**Lemma 3.10** *If $A \subset \mathbb{R}^m$ is a measure zero set, then $A_{w^m} = \emptyset$.*

***Proof*** Suppose otherwise. There then exists a point $p \in A_{w^m} = (A_{w'})_n$ and so

$$|A_{w'} \cap (p \cdot \mathbb{R}X_n)| > 0.$$

We let $H_1$ denote the analytic manifold $p \cdot \mathbb{R}X_n$, which has dimension 1. As $A_{w'} \subseteq A_{w^{m-1}}$, $|A_{w^{m-1}} \cap H_1| > 0$.

Now suppose we have a $k$-dimensional analytic manifold $H_k$ that intersects $A_{w^{m-k}}$ in a positive measure set (based on the surface area of $H_k$). Thus, we can find a density point of $A_{w^{m-k}} \cap H_k$ satisfying the previous lemma, i.e., there exists a density point $p$ of $A_{w^{m-k}} \cap H_k$, an open neighborhood $U \subseteq H_k$ of $p$, and an index $i \in \{1, \ldots, n\}$ so that $X_i \notin T_q H_k$ for all $q \in U$.

Since $A_{w^{m-k}} \subseteq (A_{w^{m-k-1}})_{1\cdots(i+1)}$, by definition of the set $(A_{w^{m-k-1}})_{1\cdots(i+1)}$, for any $q \in A_{w^{m-k}} \cap H_k$,

$$|(A_{w^{m-k-1}})_{1\cdots i} \cap (q \cdot \mathbb{R}X_i)| > 0.$$

Since $(A_{w^{m-k-1}})_{1\cdots i} \subseteq A_{w^{m-k-1}}$, we get for all $q \in A_{w^{m-k}} \cap H_k$ that

$$|A_{w^{m-k-1}} \cap (q \cdot \mathbb{R}X_i)| \geq |(A_{w^{m-k-1}})_{1\cdots i} \cap (q \cdot \mathbb{R}X_i)| > 0.$$

Let $H_{k+1} = \bigcup_{q \in U}(q \cdot \mathbb{R}X_i)$. As $X_i(q) \notin T_p U$, we conclude that $H_{k+1}$ is a $k+1$-dimensional analytic manifold and $|H_{k+1} \cap A_{w^{m-k-1}}| > 0$.

We repeat until we obtain an $m$-dimensional analytic manifold for which $|H_m \cap A_\emptyset| = |H_m \cap A| > 0$. But since $H_m$ has the same dimension as $\mathbb{R}^m$, it follows that $|A| > 0$, contradicting our assumption. □

**Proposition 3.11** *Let $A \subset \mathbb{R}^m$ be a Borel set of zero measure. Then there exists a decomposition $A = C_1 \cup \cdots \cup C_n$ into Borel sets*

$$C_i = \bigcup_{k=0}^{m-1} (A_{w^k 1\cdots(i-1)} \backslash A_{w^k 1\cdots i})$$

*so that for each $i \in \{1, \ldots, n\}$,*

$$|C_i \cap (x \cdot \mathbb{R}X_i)| = 0, \qquad \forall x \in \mathbb{R}^m.$$

***Proof*** Let $B_1 = \{p \in A_\emptyset : |A_\emptyset \cap (p \cdot \mathbb{R}X_1)| = 0\}$. Then $A = B_1 \cup A_1$ where $A_1$ and $B_1$ are both Borel by Lemma 3.8, and so

$$|B_1 \cap (p \cdot \mathbb{R}X_1)| = 0, \qquad \forall p \in \mathbb{R}^m.$$

By induction, we obtain a Borel decomposition

$$A = B_1 \cup B_{12} \cup B_{123} \cup \cdots \cup B_{w^{m-1}1\cdots(n-1)} \cup A_{w^m} = B_1 \cup \cdots \cup B_{w^{m-1}1\cdots(n-1)}.$$

Note that for every $B_{w'i}$,

$$|B_{w'i} \cap (p \cdot \mathbb{R}X_i)| = 0, \qquad \forall p \in \mathbb{R}^m.$$

We take $C_i = \bigcup_{k=0}^{m-1} B_{w^k 1\cdots i}$ to finish the proof of the proposition. □

**Proposition 3.12** *Let $H$ be a subgroup of $G$ with a Carnot structure generated by $a_1, \ldots, a_k \in V_1(G)$ and $A \subset G$ be Borel. Then $\mathrm{vol}_H(H \cap gA) = 0$ for all $g \in G$ if and only if $A \in \mathcal{U}(\{a_1, \ldots, a_k\})$.*

***Proof*** The backwards direction is clear by Fubini. We will prove the forwards direction. Let $H$ be homeomorphic to $\mathbb{R}^m$. We will reuse the notation of the previous section where for each Borel set $E \subset G$ and word $w$, we define $E_{wi} = \{g \in E_w : |E_w \cap (g \cdot \mathbb{R}a_i)| > 0\}$. Lemma 3.8 yields that these are Borel sets whenever $E$ is. By construction, $E_{wi} \in \mathcal{U}(a_i)$.

We claim that $A = C_1 \cup \cdots \cup C_k$ where $C_i = \bigcup_{j=0}^{k-1} \left(A_{w^j 1\cdots(i-1)} \backslash A_{w^j 1\cdots i}\right)$, from which the proposition easily follows. To prove the claim, we observe that for any $g \in A$, we have by assumption that $\mathrm{vol}_H(H \cap g^{-1}A) = 0$. As $e_H \in H \cap g^{-1}A$, by Proposition 3.11, there exists some $i$ so that

$$
\begin{aligned}
e_H &\in \bigcup_{j=0}^{k-1} \left((H \cap g^{-1}A)_{w^j 1\cdots(i-1)} \backslash (H \cap g^{-1}A)_{w^j 1\cdots i}\right) \\
&\subseteq \bigcup_{j=0}^{k-1} \left(g^{-1}A_{w^j 1\cdots(i-1)} \backslash g^{-1}A_{w^j 1\cdots i}\right).
\end{aligned}
$$

This means that $g \in C_i$. □

## 4 Differentiability of Lipschitz Maps

We prove now our main result, Theorem 1.8. Notice that if $f : G \to H$ is a Lipschitz map between metric scalable groups for which $Df_g$ exists for some $g \in G$, then it is Lipschitz as a function from $G$ to $H$, with the same Lipschitz constant as $f$.

**Lemma 4.1** *Let $f : G \to \mathbb{R}$ be Lipschitz and $(N_m)_{m\in\mathbb{N}}$ be a filtration of G by Carnot subgroups. Then there exists an $(N_m)_m$-negligible set $\Omega \subset G$ so that if $p \notin \Omega$ then for every $N_m$, the limit $\lim_{\lambda\to 0} \hat{f}_{p,\lambda}(u)$ exists for all $u \in N_m$ and the resulting map on $N_m$ is a homomorphism.*

***Proof*** Fix $N_m$ and let $A$ denote the set of points $p \in G$ for which the limit $\lim_{\lambda\to 0} \hat{f}_{p,\lambda}(u)$ does not exist or the limit map is not a homomorphism. We will show $\mathrm{vol}_{N_m}(gA \cap N_m) = 0$ for all $g$. This will prove the lemma.

Fix a $g \in G$ and let $p \in gA \cap N_m$. If $F_g(u) := f(g^{-1}u)$ as a map defined on $N_m$, then $gp \in N_m$ is a nondifferentiability point of $F$. However, by Pansu's theorem [15], $F$ is differentiable almost everywhere with respect to the Haar measure on $N_m$. Thus, $\mathrm{vol}_{N_m}(gA \cap N_m) = 0$, which proves the lemma. □

With the previous lemma we get our differentiability result.

***Proof of Theorem 1.8*** As the theorem is vacuous if $G$ does not admit a filtration by Carnot subgroups, we may assume that there is a filtration $(N_m)_m$. By the previous lemma, $Df_g$ exists and is a homomorphism when restricted to any $N_m$ for $g$ outside of an $(N_m)$-negligible set. Take such a $g$. We first claim that $Df_g$ exists on all of $G$. Indeed, this follows from the fact that the maps

$$u \mapsto n(f(g\delta_{1/n}(u)) - f(g))$$

are uniformly Lipschitz and converge, by assumption, on the dense subset $\bigcup_m N_m$.

As $Df_g$ is a homomorphism when restricted to every $N_m$, an easy density argument yields that $Df_g$ is also a homomorphism. This proves the theorem. $\square$

## 5 Examples

In this final section we show that our derivative existence result does not generalize to general metric scalable groups and provide a constructive way to define infinite-dimensional Carnot groups. We start by constructing a metric scalable group $G$ that does not admit filtrations by Carnot groups. We also construct a Lipschitz function $f : G \to \mathbb{R}^2$ that is nowhere differentiable. We then introduce $L^p$-sums of metric spaces when the indexing set is an abstract measure space. After that we restrict the discussion to $\ell_p$-sequences of topological groups equipped with left-invariant metrics, and prove that this object is a topological group whenever $p \in [1, \infty)$ (see Proposition 5.3). Finally, we prove that an $\ell_p$-sum of Carnot groups is an infinite-dimensional Carnot group for every $p \in [1, \infty)$ and give some detailed examples.

### 5.1 A Nowhere Differentiable Function on a Metric Scalable Group

An example of metric scalable groups not admitting filtrations by Carnot subgroups is the group $(\mathbb{R}, +)$ endowed with the metric $d^\gamma(x, y) = |x - y|^\gamma$ and scaling $\delta_\lambda(x) = \lambda^{1/\gamma} x$ where $\gamma = \frac{\log 3}{\log 4}$. One can also construct a Lipschitz function $f : G \to \mathbb{R}^2$ that is not differentiable anywhere. Indeed, let $K \subset \mathbb{R}^2$ be the Koch snowflake built from an equilateral triangle of sidelength 1. We can first define a map $g$ from $([0, 1], d^\gamma)$ to one side of the Koch snowflake so that, for each $k \in \{0, 1, 2, 3\}$, $g|_{[0,1]}$ is equivalent to $3g_{[k/4,(k+1)/4]}$ up to postcomposition with an affine isometry.

We now prove nowhere differentiability of $g$. Recall that the derivative of $g$ at $x_0$ is the pointwise limit of

$$h_r(y) := \frac{g(x_0 + r^{1/\gamma} y) - g(x_0)}{r} \tag{5.1}$$

as $r \to 0$. If $g$ were differentiable, then $h_r$ must converge to a homomorphism $\mathbb{R} \to \mathbb{R}^2$.

For every $n \geq 0$, there exists some $k \in \{0, \dots, 4^{-n} - 1\}$ such that $x_0$ resides in $[k4^{-n}, (k + 1)4^{-n}]$. Then $h_{3^{-n}}|_{[k-4^n x_0, k+1-4^n x_0]}$ is equivalent to $g$ up to postcomposition by an affine isometry as $3^n g|_{[k4^{-n},(k+1)4^{-n}]}$ is from self-similarity. Note that the length 1 interval $[k - 4^n x_0, k + 1 - 4^n x_0]$ lies in $[-1, 1]$. As $g$ is not affine, we then get that $h_r$ cannot converge to a homomorphism and so $g$ is not differentiable anywhere.

To extend $g$ to a nowhere differentiable function $f$ on all of $G$, one can simply "wrap" $g$ periodically around $K$ so that $f|_{[n,n+1]} = f|_{[n+3,n+4]}$ for all $n \in \mathbb{N}$.

## 5.2 $L^p$- and $\ell_p$-Sums

Let $\Omega = (\Omega, \mu)$ be a measure space, e.g., the natural numbers $\mathbb{N}$ with the counting measure. Fix $p \in [1, \infty)$. For each $\omega \in \Omega$ fix a pointed metric space $X_\omega = (X_\omega, d_\omega, \star_\omega)$. We first define the collection $\mathcal{M}(\Omega, (X_\omega)_\omega)$ of *measurable sequences* as the set of those sequences $(x_\omega)_{\omega\in\Omega}$ with $x_\omega \in X_\omega$ such that the function $\omega \in \Omega \mapsto d(x_\omega, \star_\omega) \in \mathbb{R}$ is measurable. Then we define

$$\mathcal{L}^p((X_\omega)_\omega) := \{(x_\omega)_\omega \in \mathcal{M}(\Omega, (X_\omega)_\omega) : \int d_\omega(x_\omega, \star_\omega)^p \mathrm{d}\mu(\omega) < \infty\}.$$

and further $L^p((X_\omega)_\omega) := \mathcal{L}^p/N$ with

$$N := \{\{(x_\omega)_\omega \in \mathcal{M}(\Omega, (X_\omega)_\omega) : \int d_\omega(x_\omega, \star_\omega)^p \mathrm{d}\mu(\omega) = 0\}.$$

We write $L^p(\Omega; X)$ for $L^p((X_\omega)_\omega)$ if $X_\omega = X$ for all $\omega \in \Omega$.

The *distance function* on $L^p((X_\omega)_\omega)$ between $(x_\omega)_{\omega\in\Omega}, (y_\omega)_{\omega\in\Omega} \in L^p((X_\omega)_\omega)$ is

$$d((x_\omega)_{\omega\in\Omega}, (y_\omega)_{\omega\in\Omega}) := \left(\int d_\omega(x_\omega, y_\omega)^p \mathrm{d}\mu(\omega)\right)^{1/p}.$$

**Proposition 5.2** *The set $L^p((X_\omega)_\omega)$ is naturally a pointed metric space, which is geodesic if all $X_\omega$ are geodesic.*

***Proof*** The fact that $d$ is a metric for $L^p((X_\omega)_\omega)$ follows from the usual proof of Minkowski inequality for the norm $\|(x_\omega)_{\omega\in\Omega}\|_p := d((x_\omega)_{\omega\in\Omega}, (\star_\omega)_{\omega\in\Omega})$.

Let us then show that $L^p((X_\omega)_\omega)$ is geodesic if $X_\omega$ is geodesic for each $\omega \in \Omega$. Let $(x_\omega)_\omega, (y_\omega)_\omega \in L^p((X_\omega)_\omega)$. Now for all $\omega \in \Omega$ there exists a curve $\gamma_\omega \colon [0, 1] \to X_\omega$ taking $x_\omega$ to $y_\omega$ such that $d(x_\omega, y_\omega) = L(\gamma_\omega)$. We may assume that $\gamma_\omega$ are parametrized by constant speed.

Let $\gamma \colon [0, 1] \to L^p((X_\omega)_\omega)$, $\gamma(t) = (\gamma_\omega(t))_\omega$. The curve $\gamma$ is well defined, since for all $t \in [0, 1]$,

$$\begin{aligned} d(\gamma(t), (x_\omega)_\omega)^p &= \int d(\gamma_\omega(t), x_\omega)^p \mathrm{d}\mu(\omega) \le \int d(y_\omega, x_\omega)^p \mathrm{d}\mu(\omega) \\ &= d((y_\omega)_\omega, (x_\omega)_\omega)^p \end{aligned}$$

and so

$$\begin{aligned} d(\gamma(t), (\star_\omega)_\omega) &\le d(\gamma(t), (x_\omega)_\omega) + d((x_\omega)_\omega, (\star_\omega)_\omega) \\ &\le d((y_\omega)_\omega, (\star_\omega)_\omega) + 2d((x_\omega)_\omega, (\star_\omega)_\omega) < \infty. \end{aligned}$$

Let then $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $[0, 1]$. Since $\gamma_\omega$ are geodesics with constant speed,

$$d(\gamma_\omega(t_{i-1}), \gamma_\omega(t_i)) = (t_i - t_{i-1})d(x_\omega, y_\omega)$$

for all $\omega \in \Omega$ and $i \in \{1, \ldots, n\}$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) &= \sum_{i=1}^{n} \left( \int d(\gamma_\omega(t_{i-1}), \gamma_\omega(t_i))^p \mathrm{d}\mu(\omega) \right)^{1/p} \\ &= \sum_{i=1}^{n} \left( \int (t_i - t_{i-1})^p d(x_\omega, y_\omega)^p \mathrm{d}\mu(\omega) \right)^{1/p} \\ &= \sum_{i=1}^{n} (t_i - t_{i-1})\, d((x_\omega)_\omega, (y_\omega)_\omega) \\ &= d((x_\omega)_\omega, (y_\omega)_\omega).\end{aligned}$$

Hence

$$L(\gamma) = \inf_{\mathcal{P}} \left( \sum_{t_i \in \mathcal{P}} d(\gamma(t_{i-1}), \gamma(t_i)) \right) = d((x_\omega)_\omega, (y_\omega)_\omega),$$

where the infimum is taken over all partitions $\mathcal{P}$ of $[0, 1]$. The proof is complete. □

Notice that if each $X_\omega$ admits a group structure we may define a group operation for $L^p((X_\omega)_\omega)$ element wise. We focus now on $\ell_p$-sums of groups. For a countable family $\{G_n\}_{n\in\mathbb{N}}$ of groups we define $\ell_p((G_n)_n)$ by

$$\begin{aligned}&\ell_p((G_n)_n) := \left\{ (x_n)_{n\in\mathbb{N}} : x_n \in G_n, \sum_{n\in\mathbb{N}} d(x_n, e_n)^p < \infty \right\}, \\ &(x_n)_n \cdot (y_n)_n := (x_n y_n)_n.\end{aligned}$$

We write $\ell_p(G)$ for $\ell_p((G_n)_n)$ if $G_n = G$ for all $n \in \mathbb{N}$.

**Proposition 5.3** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of topological groups metrized by left-invariant metrics and let $p \in [1, \infty)$. Then $\ell_p((G_n)_{n\in\mathbb{N}})$ is a topological group.*

***Proof*** We first show that the right translations are continuous. Fix $(b_n)_n \in \ell_p((G_n)_{n\in\mathbb{N}})$, that is, $b_n \in G_n$ and $\sum_{n=1}^{\infty} |b_n|^p < \infty$, where $|b_n| := d(b_n, e)$ and $d$ is the distance on $G_n$. Let $(a_{n,j})_n$ be a sequence in $\ell_p((G_n)_{n\in\mathbb{N}})$ converging to some $(a_n)_n$. Fix some $\epsilon > 0$. We take $N$ large enough so that $\sum_{n=N+1}^{\infty} |b_n|^p < \epsilon$. Then, being $N$ fixed and being the right translations $R_{b_1}, \ldots, R_{b_N}$ continuous, we take $J$ large enough so that for all $j > J$ and all $n = 1, \ldots, N$

$$d((a_{n,j})_n, (a_n)_n) < \epsilon, \tag{5.4}$$

$$d(R_{b_n}(a_{n,j}), R_{b_n}(a_n))^p < \epsilon/N. \tag{5.5}$$

Notice that consequently

$$\begin{aligned}\sum_{n=N+1}^{\infty} d(a_{n,j}b_n, a_nb_n)^p &\leq \sum_{n=N+1}^{\infty} (d(a_{n,j}b_n, a_{n,j}) + d(a_{n,j}, a_n) + d(a_n, a_nb_n))^p \\ &= \sum_{n=N+1}^{\infty} (|b_n| + d(a_{n,j}, a_n) + |b_n|)^p \\ &\leq 2^p d((a_{n,j})_n, (a_n)_n) + 2^{p+1} \sum_{n=N+1}^{\infty} |b| \\ &\leq 2^p \cdot 3\epsilon,\end{aligned}$$

where we used the trick

$$\sum (a+b)^p \leq \sum 2^p \max\{a, b\}^p \leq 2^p \left(\sum a^p + \sum b^p\right).$$

Then for all $j > J$,

$$\begin{aligned}d(R_{(b_n)_n}(a_{n,j})_n, R_{(b_n)_n}(a_n)_n) &= \sum_{n=1}^{\infty} d(a_{n,j}b_n, a_nb_n)^p \\ &= \left(\sum_{n=1}^{N} d(R_{b_n}(a_{n,j}), R_{b_n}(a_n))^p\right) \\ &\quad + \sum_{n=N+1}^{\infty} d(a_{n,j}b_n, a_nb_n)^p \\ &\leq N\epsilon/N + 2^p \cdot 3\epsilon = (1 + 3 \cdot 2^p)\epsilon.\end{aligned}$$

Therefore, the multiplication in $\ell_p((G_n)_{n\in\mathbb{N}})$ is continuous since, if $(a_{n,j})_n \to (a_n)_n$ and $(b_{n,j})_n \to (b_n)_n$, as $j \to \infty$, then using left invariance we have

$$\begin{aligned}&d((a_{n,j})_n(b_{n,j})_n, (a_n)_n(b_n)_n) \\ &\quad \leq d((a_{n,j})_n(b_{n,j})_n, (a_{n,j})_n(b_n)_n) + d((a_{n,j})_n(b_n)_n, (a_n)_n(b_n)_n) \\ &\quad \leq d((b_{n,j})_n, (b_n)_n) + d(R_{(b_n)_n}(a_{n,j})_n, R_{(b_n)_n}(a_n)_n) \to 0.\end{aligned}$$

We then show that the inversion is also continuous. Let $(a_{n,j})_n \to (a_n)_n$. Take $N$ large so that $\sum_{n=N+1}^{\infty} |a_n|^p < \epsilon$. Since the inversions in $G_1, \ldots, G_N$ are continuous, there exists $J$ such that for all $j > J$ and all $n = 1, \ldots, N$,

$$d((a_{n,j})_n, (a_n)_n) < \epsilon, \tag{5.6}$$

$$d(a_{n,j}^{-1}, a_n^{-1})^p < \epsilon/N. \tag{5.7}$$

Then for all $j > J$,

$$\begin{aligned} d((a_{n,j}^{-1})_n, (a_n^{-1})_n) &= \sum_{n=1}^{N} d(a_{n,j}^{-1}, a_n^{-1})^p + \sum_{n=N+1}^{\infty} d(a_{n,j}a_n^{-1}, e)^p \\ &\le N\epsilon/N + \sum_{n=N+1}^{\infty} (d(a_{n,j}a_n^{-1}, a_{n,j}) + d(a_{n,j}, a_n) + d(a_n, e))^p \\ &\le \epsilon + 2^p d((a_{n,j})_n, (a_n)_n) + 2^{p+1} \sum_{n=N+1}^{\infty} |a_n|^p \\ &\le (1 + 3 \cdot 2^p)\epsilon. \end{aligned}$$

□

***Remark 5.8*** In a similar manner as for Proposition 5.3, one can also show that

$$c_0((G_n)_{n\in\mathbb{N}}) := \left\{ (x_n)_n \in \ell_\infty((G_n)_{n\in\mathbb{N}}) : \lim_{n\to\infty} d(x_n, e_{G_n}) = 0 \right\}$$

is a topological group.

### 5.3 Examples of Metric Scalable Groups

Using the previous subsection, we can build examples of metric scalable groups starting with arbitrary sequences of Carnot groups equipped with homogeneous distances.

**Proposition 5.9** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of metric scalable groups and let $p \in [1, \infty)$. Then $\ell_p((G_n)_{n\in\mathbb{N}})$ is a metric scalable group. Moreover, if each $G_n$ is complete and admits a filtration by Carnot subgroups, then $\ell_p((G_n)_{n\in\mathbb{N}})$ is complete and admits a filtration by Carnot subgroups.*

***Proof*** We define the scaling map $\delta\colon \mathbb{R} \times \ell_p((G_n)_n) \to \ell_p((G_n)_n)$ element wise using the scalings of each scalable group $G_n$. By the previous proposition, $\ell_p((G_n)_n)$ is a topological group. Hence it remains to see that $\delta$ satisfies the conditions of a scalable group as in Definition 1.1 and that the metric is homogeneous with respect to $\delta$, which is straightforward to check. The proof for the fact that $\ell_p((G_n)_{n\in\mathbb{N}})$ is complete assuming that each $G_n$ is complete, is analogous to the proof of completeness of the classical $\ell_p$ spaces. Assume then that $(N_m^n)_m$ is a filtration by Carnot subgroups for each $G_n$. Then letting

$$N_m = N_m^1 \times N_{m-1}^2 \times \cdots \times N_1^m \times \{e\}^{\mathbb{N}}$$

for each $m \in \mathbb{N}$ defines a filtration by Carnot subgroups for $\ell_p((G_n)_{n\in\mathbb{N}})$. Indeed, each $N_m$ is isomorphic to a finite product of Carnot groups, and the union $\cup_m N_m$ is dense in $\ell_p((G_n)_{n\in\mathbb{N}})$ as the set of finite sequences is dense in $\ell_p((G_n)_{n\in\mathbb{N}})$. □

Proposition 5.9 gives us a simple way to construct many different noncommutative and infinite-dimensional metric scalable groups that admit filtrations by Carnot subgroups. Indeed, we may consider examples where each $G_n$ is a Carnot group, like $G_n = \mathbb{H}^1$ or $G_n = \mathbb{H}^n$, where $\mathbb{H}^n$ is the $n$-th Heisenberg group equipped with a homogeneous distance. We stress that the last result does not require any bound on the nilpotency step of $(G_n)_{n\in\mathbb{N}}$, in case they are Carnot groups. In fact, an interesting example is when $G_n$ is the free Carnot group of step $n$ and rank 2, which we denote by $\mathbb{F}_{2,n}$. We state this example as a result.

**Proposition 5.10** *Even though $\ell_2((\mathbb{F}_{2,n})_n)$ is not nilpotent, it is a metric scalable group that is complete and admits a filtration by Carnot groups. Moreover, the subset $V_1(\ell_2((\mathbb{F}_{2,n})_n))$ generates $\ell_2((\mathbb{F}_{2,n})_n)$ and is separable.*

***Proof*** The space $\ell_2((\mathbb{F}_{2,n})_n)$ is a metric scalable group by Proposition 5.9 and the filtration is simply given by

$$N_m = \mathbb{F}_{2,1} \times \cdots \times \mathbb{F}_{2,m}.$$

The first layer $V_1(\ell_2((\mathbb{F}_{2,n})_n))$ is given by $\ell_2((V_1(\mathbb{F}_{2,n}))_n)$ as we defined the dilation map on $\ell_2((\mathbb{F}_{2,n})_n)$ component wise. Indeed, a sequence $(x_n)_n \in \ell_2((\mathbb{F}_{2,n})_n)$ is a one-parameter subgroup if and only if each $x_n \in \mathbb{F}_{2,n}$ is a one-parameter subgroup. The fact that $V_1(\ell_2((\mathbb{F}_{2,n})_n))$ generates follows from Proposition 1.10. Moreover, $V_1(\ell_2((\mathbb{F}_{2,n})_n)) = \ell_2((V_1(\mathbb{F}_{2,n}))_n)$ is separable since now each $V_1(\mathbb{F}_{2,n})$ is separable. □

The property of having a filtration by Carnot subgroups is not, however, stable under taking subgroups, as shown by the following example in $\ell_2((\mathbb{F}_{2,n})_n)$. It also proves that the assumption of nilpotency cannot be removed in Proposition 1.10.

**Proposition 5.11** *There exists a scalable subgroup of $\ell_2((\mathbb{F}_{2,n})_n)$ that is generated by its first layer but which does not admit a filtration by Carnot subgroups.*

***Proof*** Denote for every $n \in \mathbb{N}$ by $X_1^{(n)}, X_2^{(n)}$ the two generators of $\mathbb{F}_{2,n}$. Let $x = (\frac{1}{n}X_1^{(n)})_n$ and $y = (\frac{1}{n}X_2^{(n)})_n$ and consider the (non-nilpotent) scalable group $H$ generated by $x$ and $y$. Now both $x, y \in V_1(H)$ but $H$ does not admit a filtration by Carnot groups. Indeed, any scalable group having Carnot group structure is generated by its first layer, but the only nilpotent subgroups of $H$ generated by one-parameter subgroups are one-dimensional. □

If $\mathbb{H}^1$ is the first Heisenberg group, then by Proposition 5.9, for all $p \in [1, \infty)$, the space $\ell_p(\mathbb{H}^1)$ is a metric scalable group admitting filtration by Carnot subgroups. The space $\ell_2(\mathbb{H}^1)$ has the extra property of being a Banach Lie group. Indeed, it can be modelled on $\ell_2(\mathbb{R}^2) + \ell_1(\mathbb{R})$, following [13]. However, we shall show that there are metric scalable groups, e.g. $\ell_1(\mathbb{H}^1)$, admitting filtrations by Carnot groups that are not Banach manifolds. Hence the notion of metric scalable group strictly extends the one of Banach homogeneous group as defined in [13] and studied later in [12].

**Proposition 5.12** *The topological group $\ell_1(\mathbb{H}^1)$ is not a Banach Lie group.*

***Proof*** Suppose by contradiction that $\ell_1(\mathbb{H}^1)$ is a Banach Lie group and let $Z$ be the center of $\mathbb{H}$. One sees that

$$Z = \{(\exp(\alpha_i Z_i))_i : \alpha_i \in \mathbb{R}\},$$

where $Z_i$ is the center of the $i$th Heisenberg Lie algebra. As $Z$ is a closed subgroup of a Banach Lie group, $Z$ is a Banach Lie group as well. However, recall that the center of the Heisenberg group is isometric to $(\mathbb{R}, \sqrt{d_E})$ and therefore

$$\ell_1(Z) = \left\{(a_n)_{n\in\mathbb{N}} : \sum_n \sqrt{|a_n|} < \infty\right\} = \ell_{1/2}(\mathbb{R}).$$

Hence also $\ell_{1/2}(\mathbb{R})$ has a structure of a Banach Lie group and its Lie algebra is a Banach space. Since $\ell_{1/2}(\mathbb{R})$ is a vector space, the exponential map $\exp\colon \mathrm{Lie}(\ell_{1/2}(\mathbb{R})) \to \ell_{1/2}(\mathbb{R})$ is a linear isomorphism. But this is a contradiction as $\ell_{1/2}(\mathbb{R})$ is not even locally convex (to be proven) and so not a normed space.

That $\ell_{1/2}(\mathbb{R})$ is not locally convex follows simply from the fact that the convex hull of any ball is unbounded. Indeed, consider any ball around the origin

$$B(0,r) = \left\{(a_1, a_2, \ldots) : \sum_n \sqrt{|a_n|} < r\right\}.$$

Then $x_n = (r/2n, r/2n, \ldots, r/2n, 0, \ldots)$ is in the convex hull of $B(0,r)$ (where the first $n$ coordinates are nonzero), but

$$d(x_n, 0) = n\sqrt{\frac{r}{2n}} = \sqrt{\frac{nr}{2}}$$

diverges as $n \to \infty$. □

## Appendix A: Some Useful Commutator Identities

**Lemma A.1** *Let $G$ be a group and $x, y, z \in G$. Then*

$$[xy, z] = [x, [y, z]][y, z][x, z] \quad \textit{and} \quad [z, xy] = [z, x][z, y][[y, z], x] = h[z, x][z, y],$$

*where $h$ is a product of commutators of $x, y, z$ of weight $\geq 3$.*

***Proof*** For the first equation,

$$[y, z][x, z] = [y, z]xzx^{-1}z^{-1} = [[y, z], x]xyzy^{-1}z^{-1}zx^{-1}z^{-1} = [[y, z], x][xy, z].$$

Since $[a, b] = [b, a]^{-1}$,

$$[xy, z] = [x, [y, z]][y, z][x, z].$$

Using this and the identity $[a, b] = [b, a]^{-1}$,

$$[z, xy] = [z, x][z, y][[y, z], x].$$

The last equation follows by reordering the terms, which produces some higher order commutators into $h$. $\square$

**Corollary A.2** *If $[y, z] \in Z(G)$, then*

$$[xy, z] = [x, z][y, z] \quad \textit{and} \quad [z, xy] = [z, x][z, y].$$

**Corollary A.3** *Let $n, m \in \mathbb{N}$. Then*

$$[x^n, y^m] = h[x, y]^{nm},$$

*where $h$ is a product of commutators of $x$ and $y$ of weight $\geq 3$.*

***Proof*** The proof is by iterating Lemma A.1 for $nm$ times and reordering the terms, which produces some additional higher order commutators into $h$. $\square$

**Lemma A.4** *Let $G$ be a group, $x, y \in G$. Then*

$$[x^{-1}, y] = [x^{-1}, [y, x]][x, y]^{-1}.$$

***Proof*** The statement follows from

$$[x^{-1}, [y, x]] = x^{-1}yxy^{-1}x^{-1}x[y, x]^{-1} = [x^{-1}, y][x, y].$$

$\square$

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# [B]

## A rectifiability result for finite-perimeter sets in Carnot groups

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# A RECTIFIABILITY RESULT FOR FINITE-PERIMETER SETS IN CARNOT GROUPS

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Abstract. In the setting of Carnot groups, we are concerned with the rectifiability problem for subsets that have finite sub-Riemannian perimeter. We introduce a new notion of rectifiability that is, possibly, weaker than the one introduced by Franchi, Serapioni, and Serra Cassano. Namely, we consider subsets $\Gamma$ that, similarly to intrinsic Lipschitz graphs, have a cone property: there exists an open dilation-invariant subset $C$ whose translations by elements in $\Gamma$ don't intersect $\Gamma$. However, a priori the cone $C$ may not have any horizontal directions in its interior. In every Carnot group, we prove that the reduced boundary of every finite-perimeter subset can be covered by countably many subsets that have such a cone property. The cones are related to the semigroups generated by the horizontal half-spaces determined by the normal directions. We further study the case when one can find horizontal directions in the interior of the cones, in which case we infer that finite-perimeter subsets are countably rectifiable with respect to intrinsic Lipschitz graphs. A sufficient condition for this to hold is the existence of a horizontal one-parameter subgroup that is not an abnormal curve. As an application, we verify that this property holds in every filiform group, of either first or second kind.

## 1. Introduction

The celebrated rectifiability theorem by De Giorgi, see [DG54, DG55], states that the reduced boundary of a set of finite perimeter in the Euclidean space $\mathbb{R}^n$ is $C^1$-rectifiable, i.e., it can be covered, up to a negligible set with respect to the Hausdorff measure $\mathscr{H}^{n-1}$, by a countable union of $C^1$ hypersurfaces. The proof of this theorem relies on the fact that the blow-up of a set of finite perimeter at a point of its reduced boundary is a set with constant normal, and each constant-normal set in $\mathbb{R}^n$ is a half-space. The importance of having sufficiently regular sets of finite perimeter is

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evident on many classical key problems in Geometric Measure Theory and underlines the relevance of the notion of rectifiability in this context. A wide impact can be, for example, detected in developing a sufficiently rich theory for functions of bounded variation; see e.g. the monographs [Fed69, GMS98, AFP00, Mag12, EG15].

In the more general context of metric measure spaces, the regularity of finite-perimeter sets and the structure of some suitable notions of their boundaries has been object of several studies in the last decades. We refer to this task as the *rectifiability problem*.

In the current paper, we study the rectifiability problem in the setting of Carnot groups. A Carnot group $\mathbb{G}$ of step $s \in \mathbb{N}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ is stratified into $s$ layers, i.e., it is linearly decomposed as $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ with

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1} \text{ for } i = 1, \dots, s-1, \quad \mathfrak{g}_s \neq \{0\} \quad \text{and} \quad [\mathfrak{g}_1, \mathfrak{g}_s] = \{0\}.$$

We refer the reader to [FS82, LD17] for an introduction to Carnot groups. Canonically, every Carnot group has a one-parameter family of dilations that we denote by $\{\delta_t : t \geq 0\}$. We further fix a homogeneous distance $d$ on $\mathbb{G}$, which is unique up to biLipschitz equivalence. In this setting the notions of perimeter and constant-normal set can be naturally defined (see Definitions 2.1 and 2.3). The original result of De Giorgi has been generalized by Franchi, Serapioni and Serra Cassano first in Heisenberg groups [FSSC01] and then in all step-2 Carnot groups [FSSC03]. The same result holds in the so-called type $\star$ Carnot groups, which are Carnot groups satisfying a suitable algebraic condition, see [Mar14], which generalizes the step-2 condition but may hold in arbitrary step. We also mention the recent [**?**], where the authors provide a class of Carnot groups that generalizes the type $\star$ class for which De Giorgi's rectifiability result holds. These are the only classes of Carnot groups in which the rectifiability problem has been solved in a satisfactory way, so far.

In general, only few partial results are known. In arbitrary groups, the first delicate issue concerns the blow-up analysis of the reduced boundary of a set of finite perimeter. By [FSSC01] every tangent set at a point of the reduced boundary is a set with constant horizontal normal, but this is not enough to prove that the tangent is a half-space, in general. Indeed, in the Engel group (the simplest Carnot group of step 3) there are examples of sets with constant horizontal normal that are not half-spaces; see [FSSC03, Example 3.2]. However, in the paper [AKLD09], the authors prove that, among all the possible blow-ups of a set of constant horizontal normal in a Carnot group, there is always a half-space.

Another issue is to understand what is the correct notion of rectifiable set in Carnot groups. Namely, at the moment it is not known which kind of rectifiability property one should expect for finite-perimeter sets in these spaces. It is well-known that in the Euclidean setting, three equivalent notions of codimension-1 rectifiability are available: the countable covering family can be composed by $C^1$ hypersurfaces, Lipschitz images of sets in $\mathbb{R}^{n-1}$ or Lipschitz codimension-1 graphs; see e.g. [Mat75, Mat95] for an account of rectifiability theory in the Euclidean spaces. Actually, a very natural notion of rectifiability, via Lipschitz images of open subsets of Euclidean spaces, was given in the setting of metric spaces already by Federer in [Fed69] (and later by Ambrosio and Kirchheim in [AK00a]). Unfortunately, this notion does not fit in the geometric structure of a Carnot group since, according to this definition, already the Heisenberg group is purely unrectifiable (see [AK00b]).

Nonetheless, a definition of rectifiability using a suitable notion of Lipschitz graphs or $C^1$ hypersurfaces can still be fruitful in the setting of Carnot groups. For this purpose, Franchi Serapioni and Serra Cassano introduced the notions of intrinsic Lipschitz graphs (see Definition 1.1) and of intrinsic $C^1$ hypersurfaces. We know that the notion of rectifiability with respect to intrinsic $C^1$ hypersurfaces implies the one with intrinsic Lipschitz graphs; see e.g. [Vit12, Theorem 3.2]. This stronger notion of rectifiability was used in [FSSC01, FSSC03, Mar14]. More precisely, the authors proved that the reduced boundary of a set of finite perimeter in a type $\star$ group $\mathbb{G}$ can be covered, up to a set of $\mathscr{H}^{Q-1}$-measure zero, by a countable union of intrinsic $C^1$ hypersurfaces, where $Q$ is the Hausdorff dimension of $\mathbb{G}$.

It is not known whether the possibly weaker notion of rectifiability, obtained by replacing the intrinsic $C^1$ hypersurfaces by intrinsic Lipschitz graphs, leads to an equivalent definition. In fact, the validity of an intrinsic Rademacher-type theorem is still an unsolved problem. One of the aim of this paper is to discuss another form of rectifiability that may be a priori even weaker than the intrinsic Lipschitz rectifiability.

The notion of intrinsic Lipschitz graph appeared in different equivalent forms in [FSSC06, FSSC11, FS16] and we briefly recall it here (see also Section 5 for a more complete discussion) in a way that is suitable for our purposes.

**Definition 1.1.** Let $\mathbb{G}$ be a Carnot group and let $\mathbb{W}, \mathbb{L} \subseteq \mathbb{G}$ be homogeneous subgroups of $\mathbb{G}$ such that $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$ and $\mathbb{W} \cap \mathbb{L} = \{0\}$. We say that $\Sigma \subseteq \mathbb{G}$ is an (entire) *intrinsic Lipschitz graph* if there exists $\beta > 0$ such that

(i) for every $p \in \Sigma$ one has

$$\Sigma \cap p \cdot \bigcup_{\ell \in \mathbb{L} \setminus \{0\}} B(\ell, \beta d(0, \ell)) = \emptyset; \tag{1.1}$$

(ii) $\pi_{\mathbb{W}}(\Sigma) = \mathbb{W}$.

The set $\bigcup_{\ell \in \mathbb{L}} B(\ell, \beta d(0, \ell))$ is a homogeneous cone around $\mathbb{L}$ of aperture given by parameter $\beta$, while $\pi_{\mathbb{W}}$ is the canonical projection associated with the decomposition $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$. The cones introduced above are not the ones that are often used in the literature of Carnot groups, but they produce the same notion of intrinsic Lipschitz graph (see e.g. [DMV19, Remark A.2])

One of the goals of the current paper is to analyze the geometry of sets that possess a "cone-behavior" that resembles property (1.1) in the following weaker sense.

**Definition 1.2.** A non-empty set $C \subseteq \mathbb{G}$ is said to be a *cone* if

$$\delta_r C = C, \qquad \text{for any } r > 0.$$

**Definition 1.3.** Let $\mathbb{G}$ be a Carnot group. We say that a set $\Gamma \subseteq \mathbb{G}$ satisfies an *(outer) cone property* if there exists a cone $C \subseteq \mathbb{G}$ such that

$$\Gamma \cap pC = \emptyset, \qquad \text{for every } p \in \Gamma.$$

A dilation-invariant set with nonempty interior, which turns out to be related with sets of locally finite perimeter, is the following semigroup. This notion has been introduced in [BLD13, BLD19] in the study of constant-normal sets in Carnot groups. (We point out that in [BLD19] the authors are mainly interested in an *inner cone property* of sets as opposed to the outer cone property.)

**Definition 1.4.** Let $\mathbb{G}$ be a Carnot group. For any $\nu \in \mathfrak{g}_1 \setminus \{0\}$, the *semigroup of horizontal normal* $\nu$ is defined by

$$S_\nu := S(\exp(\nu^\perp + \mathbb{R}^+\nu)),$$

where for any $A \subseteq \mathbb{G}$, the set $S(A) := \bigcup_{k=1}^\infty A^k$ is the semigroup generated by $A$. Here $\nu^\perp$ denotes the orthogonal space to $\nu$ within $\mathfrak{g}_1$ with respect to some scalar product that we fixed on $\mathfrak{g}_1$.

Semigroups with horizontal normal are cones and, by a standard argument of Geometric Control Theory (see [AS13]), they have non-empty interior. It can be also proved (see Proposition 3.1) that any semigroup of horizontal direction $\nu$ has $\nu$ as constant horizontal normal. We first point out that semigroups of horizontal normal $\nu$ are minimal in the following sense: $S_\nu$ is contained in every set with $\nu$ as constant horizontal normal and for which the identity element 0 of $\mathbb{G}$ has positive density, see [BLD19, Corollary 2.31 or Theorem 2.37] . This property, together with the dilation-invariance of such semigroups, allows us to perform a fruitful blow-up procedure and get our main result; see Theorem 4.6.

**Theorem 1.5.** *Let $\mathbb{G}$ be a Carnot group and let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then there exists a family $\{C_h : h \in \mathbb{N}\}$ of open cones in $\mathbb{G}$ and a family $\{\Gamma_h : h \in \mathbb{N}\}$ of subsets of $\mathbb{G}$ such that each $\Gamma_h$ satisfies the $C_h$-cone property (as in Definition 1.3) and*

$$\mathcal{F}E = \bigcup_{h\in\mathbb{N}} \Gamma_h,$$

*where $\mathcal{F}E$ denotes the reduced boundary of $E$.*

Notice that the previous result is obtained without requiring any assumption on the Carnot group $\mathbb{G}$. The first natural question one may ask is when a set satisfying a cone property is also an intrinsic Lipschitz graph. We can point out a sufficient condition. Indeed, since each $C_h$ in the theorem above comes as a small shrinking of some semigroup $S_\nu$, it is enough that $S_\nu$ is such that there exists $X \in \mathfrak{g}_1 \setminus \{0\}$ with $\exp(X) \in \operatorname{int}(S_\nu)$ (see Remark 5.1 for more details). In Section 5 we find some conditions on the group $\mathbb{G}$ that are sufficient to conclude that the reduced boundary of every set of finite perimeter in $\mathbb{G}$ is intrinsically Lipschitz rectifiable (see Definition 5.2). In particular, we notice that, whenever the group possesses a non-abnormal horizontal direction (see Definition 5.9), then each set $\Gamma_h$ appearing in Theorem 1.5 can be chosen to be an intrinsic Lipschitz graph (see Proposition 5.3 and Corollary 5.7). The notions of normal and abnormal curve naturally appeared in Geometric Control Theory (see [Mon02]), and in the study of regularity of geodesics in sub-Riemannian manifolds (see [Vit14]). In the paper [LDMO+16], it is proved that the one-parameter subgroup generated by $X \in \mathfrak{g}_1$ is non-abnormal if and only if

$$\operatorname{span}\{\operatorname{ad}_X^k(\mathfrak{g}_1) : k = 0, \dots, s-1\} = \mathfrak{g}, \tag{1.2}$$

where the adjoint of $X$ is defined by $\operatorname{ad}_X^0(Y) \coloneqq Y$ and $\operatorname{ad}_X^k(Y) \coloneqq \operatorname{ad}_X^{k-1}([X,Y])$, for every $k \geq 1$ and every $Y \in \mathfrak{g}$. This gives us a purely algebraic sufficient condition on the group $\mathbb{G}$ for the intrinsic Lipschitz rectifiability of reduced boundaries of sets of finite perimeter.

**Theorem 1.6.** *Let $\mathbb{G}$ be a Carnot group and assume there exists $X \in \mathfrak{g}_1$ such that* (1.2) *holds. Then, the reduced boundary of every set of finite perimeter in $\mathbb{G}$ is intrinsically Lipschitz rectifiable. In particular, this result can be applied to all filiform Carnot groups.*

For the second part of Theorem 1.6 see Section 6. Among filiform groups, we can find the Engel group, which is the simplest Carnot group of step 3. We also point out that possessing a non-abnormal horizontal direction is stable under direct products and quotients. More precisely, if $\mathbb{G}_1$ and $\mathbb{G}_2$ are two Carnot groups that possess a non-abnormal horizontal direction, and $N$ is a normal subgroup of $\mathbb{G}_1$, then both $\mathbb{G}_1 \times \mathbb{G}_2$ and $\mathbb{G}_1/N$ possess a non-abnormal horizontal direction (see Propositions 5.13 and 5.14).

On the negative side, we point out that for example in the free group $\mathbb{F}_{2,3}$ of rank 2 and step 3, also known as Cartan group, all horizontal directions are abnormal and

$$\exp(\mathfrak{f}_1) \cap \operatorname{int}(S_\nu) = \emptyset, \quad \text{for every } \nu \in \mathfrak{f}_1 \setminus \{0\},$$

where $\mathfrak{f}_1$ denotes the horizontal layer of the Lie algebra of $\mathbb{F}_{2,3}$ (see [BLD19]).

As an example of application, we remark that in [DV19], to study some properties of functions of bounded variation, the authors consider sub-Riemannian structures in which sets of finite perimeter have reduced boundary that is intrinsically Lipschitz rectifiable.

The outline of the paper is the following. Section 2 is devoted to the basic notions and facts related to Carnot groups. Section 3 is devoted to studying properties of closed semigroups of horizontal direction that will be crucial for the main theorem. Section 4 contains the proof of Theorem 1.5. Section 5 contains the definitions of intrinsic Lipschitz graphs, intrinsic Lipschitz rectifiability and the proof of Theorem 1.6 together with applications to some classes of Carnot groups. In Section 6 we introduce filiform groups and determine their abnormal lines and their graded Lie algebra automorphisms.

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## 2. Preliminaries

For an introduction to Carnot groups we refer the reader to [FS82, LD17], while for a theory of sets of finite perimeter in Carnot groups, we refer to [AKLD09]. In what follows, let $\mathbb{G}$ be a Carnot group of dimension $n$, let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be its stratified Lie algebra and denote by $m := \dim \mathfrak{g}_1$ the rank of $\mathbb{G}$. We fix a scalar product $\langle \cdot,\cdot \rangle$ on the horizontal layer $\mathfrak{g}_1$ of $\mathfrak{g}$ and a left-invariant Haar measure $\mu$ on $\mathbb{G}$. We endow $\mathbb{G}$ with the usual Carnot-Carathéodory metric $d$ and we denote by $B_r$ the metric ball of radius $r$ at the identity element of $\mathbb{G}$.

It is well-known that there exists a family $\{\delta_r \in \mathbb{G}^{\mathbb{G}} : r \geq 0\}$ such that $\delta_0 := 0$ and, for any $r > 0$, $\delta_r$ is a graded diagonalizable automorphism of $\mathbb{G}$ satisfying the following properties. For any $r, s \geq 0$, $\delta_{rs} = \delta_r \circ \delta_s$ and, for every $x, y \in \mathbb{G}$ and $r \geq 0$

$$d(\delta_r(x), \delta_r(y)) = rd(x, y).$$

**Definition 2.1.** Let $\Omega$ be an open set in $\mathbb{G}$. We say that a measurable set $E \subseteq \mathbb{G}$ has *locally finite perimeter* in $\Omega$, if, for every $Y \in \mathfrak{g}_1$, there exists a Radon measure, denoted by $Y\mathbb{1}_E$, on $\Omega$ such that

$$\int_{A\cap E} Y\varphi \, d\mu = -\int_A \varphi \, d(Y\mathbb{1}_E), \quad \text{for every open set } A \Subset \Omega \text{ and every } \varphi \in C^1_c(A).$$

We say that $E$ has *finite perimeter* in $\Omega$ if $E$ has locally finite perimeter in $\Omega$ and, for every basis $(X_1, \ldots, X_m)$ of $\mathfrak{g}_1$, the total variation $|D\mathbb{1}_E|(\Omega)$ of the measure $D\mathbb{1}_E := (X_1\mathbb{1}_E, \ldots, X_m\mathbb{1}_E)$ is finite.

**Definition 2.2.** Let $E \subseteq \mathbb{G}$ be a set with locally finite perimeter in $\mathbb{G}$. We define the *reduced boundary* $\mathcal{F}E$ of $E$ to be the set of points $p \in \mathbb{G}$ such that $|\mathrm{D}\mathbb{1}_E|(B(p, r)) > 0$ for all $r > 0$ and there exists

$$\lim_{r\to 0} \frac{\mathrm{D}\mathbb{1}_E(B(p, r))}{|\mathrm{D}\mathbb{1}_E|(B(p, r))} =: \nu_E(p)$$

with $|\nu_E(p)| = 1$.

We denote by $\mathbb{S}(\mathbb{G})$ the unit sphere of $\mathbb{G}$ and by $\mathbb{S}(\mathfrak{g}_1)$ the unit sphere in $\mathfrak{g}_1$.

**Definition 2.3.** Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter in $\mathbb{G}$ and let $\nu \in \mathbb{S}(\mathfrak{g}_1)$. We say that $E$ has $\nu$ as *constant horizontal normal* if $\nu \mathbb{1}_E \geq 0$ in the sense of distributions and, for every $X \in \nu^\perp := \{Y \in \mathfrak{g}_1 : \langle \nu, Y \rangle = 0\}$, one has $X\mathbb{1}_E = 0$.

## 3. Cones in Carnot groups

In this section, we show that semigroups of horizontal normal $\nu$ represent the "minimal" sets having $\nu$ as constant horizontal normal (see Proposition 3.1 and Lemma 3.5). Moreover, by Lemma 3.2, they are attainable sets of halfspaces in $\mathfrak{g}_1$. By a standard argument of Geometric Control Theory (see e.g. [AS13, Theorem 8.1]), this implies that every $S_\nu$ has non-empty interior.

An important result of this section is Proposition 3.7, which proves that the semigroups $S_\nu$ are continuous with respect to the horizontal direction $\nu$. This fact, resumed in Remark 3.8, will be used in the proof of Theorem 4.5.

The following proposition has been proven in [BLD19, Proposition 2.29]. We, however, write its short proof for the sake of completeness.

**Proposition 3.1.** *Let $\mathbb{G}$ be a Carnot group, let $\nu \in \mathbb{S}(g_1)$ and let $S_\nu$ be the semigroup with normal $\nu$. Then $S_\nu$ has $\nu$ as constant horizontal normal.*

*Proof.* By definition, $\mathbb{1}_{S_\nu \cdot \exp(sX)} \leq \mathbb{1}_{S_\nu}$ for all $s \geq 0$ and $X \in \{\nu\} \cup \nu^\perp$. Let $\varphi \in C_c^\infty(\mathbb{G}; [0, +\infty[)$ and denote by $\Phi_X(p, s) := p\exp(sX)$ the flow of $X$ at time $s$ starting from point $p \in \mathbb{G}$. Then

$$\begin{aligned}\int_{\mathbb{G}} \mathbb{1}_{S_\nu} X\varphi \,\mathrm{d}\mu &= \int_{\mathbb{G}} \mathbb{1}_{S_\nu} \cdot \lim_{s\to 0} \frac{1}{s}(\varphi(\Phi_X(\cdot, s)) - \varphi) \,\mathrm{d}\mu \\ &= \lim_{s\to 0} \frac{1}{s}\left(\int_{\mathbb{G}} \mathbb{1}_{S_\nu \cdot \exp(sX)}\varphi \,\mathrm{d}\mu - \int_{\mathbb{G}} \mathbb{1}_{S_\nu}\varphi \,\mathrm{d}\mu\right) \leq 0\end{aligned}$$

and therefore

$$-\int_G \mathbb{1}_{S_\nu} X\varphi \,\mathrm{d}\mu \geq 0\,.$$

Hence $\langle X\mathbb{1}_{S_\nu}, \cdot\rangle$ is a positive linear functional and by Riesz's Theorem $X\mathbb{1}_{S_\nu}$ is a Radon measure. Moreover, since for each $X \in \nu^\perp$ also $-X \in \nu^\perp$, we get that $X\mathbb{1}_{S_\nu} = 0$ for all $X \in \nu^\perp$. Consequently, $S_\nu$ has $\nu$ as constant horizontal normal. ☐

**Lemma 3.2.** *Let $\mathbb{G}$ be a Carnot group, $T > 0$, and let $\gamma : [0, T] \to \mathbb{G}$ be a horizontal curve such that $\gamma(0) = 0$. If $\langle \dot\gamma(t), \nu\rangle \geq 0$ for almost every $t$, then $\gamma(T) \in \overline{S}_\nu$.*

*Proof.* Fix $T > 0$, define $X_1 := \nu$ and let $X_2, \ldots, X_m$ be such that $(X_1, X_2, \ldots, X_m)$ is an orthonormal basis for $\mathfrak{g}_1$. We fix $T > 0$. We can assume without loss of generality that $\dot\gamma \in L^\infty([0, T]; \mathfrak{g}_1)$. Let $u_1, \ldots, u_m \in L^\infty([0, T])$ be such that

$$\dot\gamma(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)),$$

for $\mathscr{L}^1$-almost every $t \in [0,T]$. Then, by assumption, we know that $u_1(t) \geq 0$ for $\mathscr{L}^1$-almost every $t \in [0,T]$. Consider piecewise constant sequences $(u_1^h), \ldots, (u_m^h)$ in $L^\infty([0,T])$, with $h \in \mathbb{N}$, such that

$$u_i^h \to u_i \qquad \text{in } L^1([0,T]), \quad \text{as } h \to \infty,$$

for any $i = 1, \ldots, m$ and such that $u_1^h(t) \geq 0$ for $\mathscr{L}^1$-almost every $t \in [0,T]$ . We can also assume that $\sup_{h\in\mathbb{N}} \sum_{i=1}^m \|u_i^h\|_\infty \leq M$, for some $M > 0$. According to to the definition of $S_\nu$ and since $u_i^h$ are piecewise constant, the curves defined by

$$\begin{cases} \dot\gamma^h(t) = \sum_{i=1}^m u_i^h(t) X_i(\gamma^h(t)) & \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,T], \\ \gamma^h(0) = 0, \end{cases}$$

are such that $\gamma^h(t) \in S_\nu$ for any $t \in [0,T]$. Since $d(\gamma^h(t), 0) \leq Mt$ for every $t \in [0,T]$, there exists a compact set $K \subseteq \mathbb{G}$ for which

$$\bigcup_{h\in\mathbb{N}} \gamma^h([0,T]) \cup \gamma([0,T]) \subseteq K.$$

We prove that

$$\lim_{h\to\infty} d(\gamma^h(T), \gamma(T)) = 0.$$

It is not restrictive to work in coordinates and compute, for every $t \in [0,T]$,

$$\begin{aligned} |\gamma^h(t) - \gamma(t)| &= \left| \int_0^t \sum_{i=1}^m \left( u_i^h(\tau) X_i(\gamma^h(\tau)) - u_i(\tau) X_i(\gamma(\tau)) \right) \, d\tau \right| \\ &\leq \int_0^t \sum_{i=1}^m |u_i^h(\tau)| \left| X_i(\gamma^h(\tau)) - X_i(\gamma(\tau)) \right| \, d\tau \\ &\quad + \int_0^t \sum_{i=1}^m \left| u_i^h(\tau) - u_i(\tau) \right| |X_i(\gamma(\tau))| \, d\tau. \end{aligned}$$

Notice that, by the choice of $u_i^h$, the term

$$\alpha_h(t) := \int_0^t \sum_{i=1}^m \left| u_i^h(\tau) - u_i(\tau) \right| |X_i(\gamma(\tau))| \, d\tau$$

is infinitesimal as $h \to \infty$, and that, by the smoothness of $X_1, \ldots, X_n$ and letting

$$C := \sup_{i=1,\ldots,m} \operatorname{Lip}(X_i)(K)$$

and recalling the definition of $M$, we have

$$|\gamma^h(t) - \gamma(t)| \leq \alpha_h(t) + CM \int_0^t \left| \gamma^h(\tau) - \gamma(\tau) \right| \, d\tau,$$

for all $t \in [0,T]$. We are then in a position to apply Grönwall Lemma to get

$$|\gamma^h(T) - \gamma(T)| \leq \alpha_h(T) e^{CMT},$$

and letting $h \to \infty$ and by the arbitrariness of $T$, we conclude the proof. ☐

Before proving Lemma 3.5, we make the following remark.

**Remark 3.3.** Let $\mathbb{G}$ be a Carnot group and let $d_L$ and $d_R$ be the left- and right-invariant Carnot-Carathéodory distances on $\mathbb{G}$ built with respect to the same scalar product on $\mathfrak{g}_1$. Then the inversion is an isometry between $(\mathbb{G}, d_L)$ and $(\mathbb{G}, d_R)$. In particular, for every $p \in \mathbb{G}$ we have that $d_L(0,p) = d_L(p^{-1}, 0) = d_R(p,0) = d_R(0,p)$.

Given $p \in \mathbb{G}$ and a measurable set $F \subseteq \mathbb{G}$, we define the *lower and upper densities* $\theta_*(p,F)$ and $\theta^*(p,F)$ of $F$ at $p$ letting

$$\theta_*(p,F) := \liminf_{r\to 0} \frac{\mu(F\cap B(p,r))}{\mu(B(p,r))} \qquad \theta^*(p,F) := \limsup_{r\to 0} \frac{\mu(F\cap B(p,r))}{\mu(B(p,r))}.$$

The *measure theoretic boundary* $\partial^* F$ of $F$ is defined by

$$\partial^* F := \{p \in \mathbb{G} : \theta_*(p,F) > 0 \text{ and } \theta^*(p,F) < 1\}.$$

The Lebesgue representative $\widetilde{F}$ of $F$ is defined by

$$\widetilde{F} := \{p \in \mathbb{G} : \theta_*(p,F) = 1\}. \tag{3.1}$$

Notice that, by the Lebesgue's theorem, we know that $\widetilde{F} = F$ up to a set of $\mu$-measure zero.
The following proposition is proved in [BLD19, Proposition 3.6]

**Proposition 3.4.** *Let $F \subseteq \mathbb{G}$ be a set with $\nu$ as constant horizontal normal. Then*

$$\mathcal{F}\widetilde{F} = \partial\widetilde{F} = \partial^*\widetilde{F}.$$

Lemma 3.5 below will be used in the proof of Lemma 4.2.

**Lemma 3.5.** *Let $\mathbb{G}$ be a Carnot group, let $F \subseteq \mathbb{G}$ be a set with $\nu$ as constant horizontal normal and assume that $\theta^*(0,F) > 0$. Then $\mathbb{1}_{S_\nu} \le \mathbb{1}_F$ $\mu$-almost everywhere.*

*Proof.* Recall first that every left-invariant Haar measure of a Carnot group is right-invariant, being Carnot groups nilpotent and therefore unimodular.
Since $F$ has $\nu$ as constant horizontal normal, by [BLD13, Lemma 3.1] we have that,

$$\mathbb{1}_{F\cdot p} \le \mathbb{1}_F \quad \text{for every } p \in S_\nu, \quad \mu\text{-almost everywhere.}$$

If $B^L(p,r)$ and $B^R(p,r)$ denote, respectively, the metric balls built with respect to the left-invariant and right-invariant metrics, then for all $p \in S_\nu$

$$\begin{aligned}\mu(F\cap B^L(0,r)) &\overset{\text{Rem. 3.3}}{=} \mu(F\cap B^R(0,r)) = \mu(F\cdot p\cap B^R(p,r)) \\ &\le \mu(F\cap B^R(p,r)).\end{aligned}$$

Since $\theta^*(0, F) > 0$, we deduce that for all $p \in S_\nu$,

$$\theta_F^R(p) := \limsup_{r\to 0} \frac{\mu(F \cap B^R(p,r))}{\mu(B^R(p,r))} > 0. \tag{3.2}$$

On the other hand, by Lebesgue Theorem we have that $\theta_F^R(p) \in \{0,1\}$ for $\mu$-almost every $p \in \mathbb{G}$. By (3.2) we then get that $\mu$-almost every $p \in S_\nu$ is a Lebesgue point for $F$, hence for almost every $p \in S_\nu$ one also has $p \in F$, as required. ☐

In the following lemma, we denote by $\mathbb{R}^*O(n)$ the set of $n \times n$ matrices $A$ that can be written as

$$A = \lambda B,$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$ and some $B \in O(n)$.

**Lemma 3.6.** *Let $n \in \mathbb{N}$. Then the following facts hold.*

(i) *Let $H$ be a subgroup of $GL(n)$. Then for every $p \in \mathbb{R}^n$ and every open neighborhood $U$ of $p$, there exists a neighborhood $M \subseteq H$ of the identity matrix such that*

$$\bigcap_{\ell \in M} \ell U$$

*is a neighborhood of $p$.*

(ii) *For every $p \in \mathbb{R}^n \setminus \{0\}$ and every open neighborhood $M \subseteq \mathbb{R}^*O(n)$ of the identity matrix, the set $Mp$ is an open neighborhood of $p$.*

*Proof.* $(i)$ Consider $r, R \in (0, +\infty)$ and $p \in \mathbb{R}^n$ such that $B(p, 2r) \subseteq U$ and $B(p, r) \subseteq B(0, R)$. We consider in $H$ the distance coming from the usual operator norm. We define $M := B(\mathrm{Id}, r/R)^{-1} \subseteq H$. Then, take any $\ell \in M$, so that $\|\ell^{-1} - \mathrm{Id}\| < r/R$. Then we have that for all $x \in B(0, R)$

$$|\ell^{-1}(x) - x| \leq \|\ell^{-1} - \mathrm{Id}\| \, |x| \leq \frac{r}{R} R = r.$$

Consequently, by triangle inequality if $x \in B(p, r)$, and so also $x \in B(0, R)$, we have that

$$|\ell^{-1}(x) - p| \leq |\ell^{-1}(x) - x| + |x - p| \leq 2r.$$

Therefore, we showed that $\ell^{-1}(B(p, r)) \subseteq B(p, 2r) \subseteq U$, that is, $B(p, r) \subseteq \ell U$ for every $\ell \in M$. Hence, we infer that $B(p, r) \subseteq \bigcap_{\ell \in M} \ell U$.

$(ii)$ Consider the scaled orthogonal transformations $\mathbb{R}^*O(n)$ acting continuously and transitively on $\mathbb{R}^n \setminus \{0\}$. Fix $p \in \mathbb{R}^n \setminus \{0\}$ and let us denote by $(\mathbb{R}^*O(n))_p$ the stabilizer subgroup of $p$. By [Hel01, Theorem 3.2], the mapping

$$\begin{aligned} \psi \colon \mathbb{R}^*O(n)/(\mathbb{R}^*O(n))_p &\to \mathbb{R}^n \setminus \{0\} \\ [\ell] &\longmapsto \ell(p) \end{aligned}$$

is a well-defined homeomorphism. Hence, since the projection $\pi \colon \mathbb{R}^*O(n) \to \mathbb{R}^*O(n)/(\mathbb{R}^*O(n))_p$ is open, the map $\ell \mapsto \ell(p)$ obtained as $\psi \circ \pi$ is open. ☐

**Proposition 3.7.** *Let $\mathbb{G}$ be a Carnot group. Then, for every $\nu \in \mathbb{S}(\mathfrak{g}_1)$, there exists an open neighborhood $U$ of $\nu$ in $\mathbb{S}(\mathfrak{g}_1)$ such that*

$$\bigcap_{\mu \in U} S_\mu$$

*has non-empty interior. Moreover, if $\nu \in \mathbb{S}(\mathfrak{g}_1)$ and $X \in \mathfrak{g}_1$ are such that $\exp(X) \in \operatorname{int}(S_\nu)$, then one can choose $U$ in such a way that $X \in \operatorname{int}(\bigcap_{\mu \in U} S_\mu)$.*

*Proof.* Denote by $\mathbb{F}$ the free Carnot group of both the same rank and step of $\mathbb{G}$ (for a definition of free Lie algebra, see e.g. [VSCC92, p. 45] or [Var84, p. 174]). Then, the canonical projection $\pi\colon \mathbb{F} \to \mathbb{G}$ induces a surjective isometry between the horizontal layers of the Lie algebras $\pi_*\colon \mathfrak{f}_1 \to \mathfrak{g}_1$[1]. Define $\widetilde{\nu} \in \mathfrak{f}_1$ such that $\pi_*(\widetilde{\nu}) = \nu$, and consider a non-empty open set $W \subseteq S_{\widetilde{\nu}}$, which exists by [BLD19, Proposition 2.26]. The canonical action of $\mathbb{R}O(m)$ on $\mathfrak{f}_1$ can be extended linearly to a map in $\operatorname{Aut}(\mathfrak{f})$, since in free groups all non-horizontal left-invariant vector fields can be written in a (essentially) unique way as commutators of horizontal ones. We denote by $H < \operatorname{Aut}(\mathfrak{f})$ this group of automorphisms of $\mathfrak{f}$ induced by the action of $\mathbb{R}O(m)$ on $\mathfrak{f}_1$. Since automorphisms of $\mathfrak{f}$ are linear bijections, we may interpret $H$ as a subgroup of $GL(n)$, where $n$ is the topological dimension of $\mathfrak{f}$.

Considering the corresponding Lie group automorphisms, with abuse of notation, we then identify the actions $\mathbb{R}O(m) \times \mathfrak{f}_1 \to \mathfrak{f}_1$ and $\mathbb{R}O(m) \times \mathbb{F} \to \mathbb{F}$. Notice that these actions are continuous and open. By (i) of Lemma 3.6 there exists an open neighborhood $M$ of the identity in $H$ such that

$$\bigcap_{\ell \in M} \ell W$$

has non-empty interior. Moreover, since $H$ restricted to $\mathfrak{f}_1$ is isomorphic to $\mathbb{R}^* O(m)$, by (ii) of Lemma 3.6 the set $M\widetilde{\nu}$ is an open neighborhood of $\widetilde{\nu}$ in $\mathfrak{f}_1$. Set now $U \coloneqq \pi_*(M\widetilde{\nu})$. Since $\pi_*$ is open, $U$ is open and for every $\mu \in U$ we can find $m_0 \in M$ such that $\mu = \pi_*(m_0\widetilde{\nu})$. By properties of orthogonal matrices, notice that we also have $\pi_*((m_0\widetilde{\nu})^\perp) = \mu^\perp$. Since $\pi$ is open as well, the inclusions

$$\begin{aligned} \pi\left(\operatorname{int} \bigcap_{\ell \in M} \ell W\right) &\subseteq \pi\left(m_0 W\right) \subseteq \pi\left(m_0 S_{\widetilde{\nu}}\right) = \pi(m_0(S(\exp(\widetilde{\nu}^\perp + \mathbb{R}^+ \widetilde{\nu})))) \\ &= \pi S(\exp((m_0\widetilde{\nu})^\perp + \mathbb{R}^+ m_0 \widetilde{\nu})) = S(\exp(\mu^\perp + \mathbb{R}^+ \mu)) = S_\mu \end{aligned} \tag{3.3}$$

conclude the first part of the proof.

Assume now that $\exp(X) \in \operatorname{int}(S_\nu)$. Then, choosing $\widetilde{X} \in \mathfrak{f}_1$ and an open set $W \subseteq S_{\widetilde{\nu}}$ such that $\pi_*(\widetilde{X}) = X$ and $\exp(\widetilde{X}) \in W$, one can repeat the argument of the

[1] It is understood that the metric on $\mathfrak{g}_1(\mathbb{F})$ is the pull-back metric of the metric on $\mathfrak{g}_1(\mathbb{G})$

previous part of the proof with the additional condition that, by (i) of Lemma 3.6,

$$\exp(\widetilde{X}) \in \operatorname{int}\left(\bigcap_{\ell \in M} \ell W\right).$$

The proof is again finished by (3.3). □

For any subset $A \subseteq \mathbb{S}(\mathbb{G})$, we define

$$C(A) := \{\delta_r(a) : a \in A,\, r > 0\}. \tag{3.4}$$

Notice that $C(A)$ is a cone.

**Remark 3.8.** As an immediate consequence of Proposition 3.7, we notice that there exist $N \in \mathbb{N}$ and a finite family of triples $(U_i, \Omega_i, K_i)_{i=1}^N$ such that

(1) $(U_i)_{i=1}^N$ is an open covering of $\mathbb{S}(\mathfrak{g}_1)$;
(2) for every $i = 1, \ldots, N$, $\Omega_i$ is a nonempty open subset of $\mathbb{S}(\mathbb{G})$ such that
$$C(\Omega_i) \subseteq \bigcap_{\nu \in U_i} S_\nu;$$
(3) for every $i = 1, \ldots, N$, $K_i \subseteq \Omega_i$ is a compact set with nonempty interior in $\mathbb{S}(\mathbb{G})$.

Moreover, if for every $\nu \in \mathbb{S}(\mathfrak{g}_1)$, there exists $X \in \mathfrak{g}_1$ such that $\exp(X) \in \operatorname{int}(S_\nu)$, then each compact set $K_i$ can be chosen in such a way that there exists $X_i \in \mathfrak{g}_1$ with $\exp(X_i) \in \operatorname{int}(C(K_i))$.

## 4. Proof of Theorem 1.5

In this section we show that, in Carnot groups, the reduced boundary of sets of locally finite perimeter can be decomposed into countably many pieces satisfying a cone property. This is precisely stated in Theorem 4.6, whose proof and statement complete and combine the previous Lemmata 4.2, 4.3, and 4.5.

**Remark 4.1.** If a subset $\Gamma$ of a Carnot group satisfies the $C$-cone property, for some cone $C$, then $\Gamma$ also satisfies the $C^{-1}$-cone property. Indeed, assume by contradiction that there exist $p \in \Gamma$ and $q \in \Gamma \cap pC^{-1}$. Since $q \in pC^{-1}$, then $p \in qC$. But $\Gamma$ satisfies a $C$-cone property and, since $q \in \Gamma$ we have $\Gamma \cap qC = \emptyset$. This is in contradiction with $p \in \Gamma \cap qC$.

**Lemma 4.2.** *Let $\mathbb{G}$ be a Carnot group. If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter in $\mathbb{G}$ and $\varepsilon > 0$, then, for every $p \in \mathcal{F}E$, there exists $r_p > 0$ such that for any $r \in (0, r_p)$ we have*

$$\mu(B_r \cap S_{\nu_E(p)} \cap p^{-1}(\mathbb{G} \setminus E)) \leq \varepsilon r^Q$$

*and*

$$\mu(B_r \cap S^{-1}_{\nu_E(p)} \cap p^{-1}E) \leq \varepsilon r^Q.$$

*Proof.* Fix $p \in \mathcal{F}E$ and notice that $E$ admits a tangent at $p$ that has, by the same argument of [FSSC03, Theorem 3.1], $\nu_E(p)$ as constant horizontal normal. This means that, for any sequence $r_h \to 0$, we can find a set $F$ with constant horizontal normal $\nu_F \equiv \nu_E(p)$ and a subsequence $r_{h_k}$ such that

$$\mathbb{1}_{\delta_{r_{h_k}^{-1}}(p^{-1}E)} \to \mathbb{1}_F \text{ in } L^1_{\mathrm{loc}}(\mathbb{G}), \qquad \text{as } k \to +\infty.$$

It is not restrictive to assume that $\widetilde{F} = F$, where $\widetilde{F}$ is as in (3.1). By Proposition 3.4, since $0 \in \mathcal{F}\widetilde{F}$ then $0 \in \partial^*\widetilde{F}$ and by Lemma 3.5, one has

$$\mathbb{1}_{\delta_{r_{h_k}^{-1}}(p^{-1}E)\cap S_{\nu_E(p)}} \to \mathbb{1}_{S_{\nu_E(p)}} \text{ in } L^1_{\mathrm{loc}}(\mathbb{G}), \qquad \text{as } k \to +\infty.$$

Consequently, we have

$$\mathbb{1}_{\delta_{r^{-1}}(p^{-1}E)\cap S_{\nu_E(p)}} \to \mathbb{1}_{S_{\nu_E(p)}} \text{ in } L^1_{\mathrm{loc}}(\mathbb{G}), \qquad \text{as } r \to 0.$$

This completes the proof of the first statement. The proof of the second inequality follows from the first one by replacing $E$ with $\mathbb{G} \setminus E$ and recalling that $S_{-\nu} = S_\nu^{-1}$. ☐

Regarding next lemma, recall the notation $C(\Omega)$ introduced in (3.4).

**Lemma 4.3.** *Let $\mathbb{G}$ be a Carnot group, let $K \subseteq \Omega \subseteq \mathbb{S}(\mathbb{G})$ be such that $K$ is compact and $\Omega$ is open. Then*

$$\eta(C(\Omega), C(K)) := \inf\left\{\mu(C(\Omega) \cap B_2 \cap \xi C(\Omega)^{-1} \cap \xi B_2) : \xi \in K\right\} > 0.$$

*Proof.* Since $K$ is a compact set, there exists $\delta \in (0,1)$ such that

$$0 < \delta < d(K, \mathbb{G} \setminus C(\Omega)).$$

Hence for any $\xi \in K$ one has $\xi B_\delta \subseteq C(\Omega) \cap B_2$. Therefore

$$\begin{aligned}\mu(C(\Omega) \cap B_2 \cap \xi C(\Omega)^{-1} \cap \xi B_2) &\geq \mu(C(\Omega) \cap B_2 \cap \xi C(\Omega)^{-1} \cap \xi B_\delta)\\ &= \mu(\xi C(\Omega)^{-1} \cap \xi B_\delta)\\ &= \mu(C(\Omega)^{-1} \cap B_\delta),\end{aligned}$$

which is a positive lower bound independent of $\xi \in K$. ☐

By combining Remark 3.8 and Lemmata 4.2 and 4.3 we get the following corollary.

**Corollary 4.4.** *Let $\mathbb{G}$ be a Carnot group, let $(U_i, \Omega_i, K_i)_{i=1}^N$ be as in Remark 3.8, and let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then, for every $p \in \mathcal{F}E$, there exists $r_p > 0$ such that, if $\nu_E(p) \in U_i$ and $r \in (0, r_p)$, then*

$$\mu(B_{2r} \cap C_i \cap p^{-1}(\mathbb{G} \setminus E)) \leq \frac{\eta_i}{3} r^Q \tag{4.1}$$

*and*

$$\mu(B_{2r} \cap C_i^{-1} \cap p^{-1}E) \leq \frac{\eta_i}{3} r^Q, \tag{4.2}$$

*where $C_i := C(\Omega_i)$ and $\eta_i := \eta(C_i, C(K_i)) = \inf\{\mu(C_i \cap B_2 \cap \xi C_i^{-1} \cap \xi B_2) : \xi \in K_i\}$.*

**Lemma 4.5.** *Let $\mathbb{G}$ be a Carnot group and let $E$ be a set of locally finite perimeter in $\mathbb{G}$. Consider a family $(U_i, \Omega_i, K_i)_{i=1}^N$ as in Remark 3.8 and, for every $p \in \mathcal{F}E$, define $r_p > 0$ as in Corollary 4.4. Then, the sets*

$$F_{i,\ell} := \{p \in \mathcal{F}E \,:\, \nu_E(p) \in U_i, r_p > \tfrac{1}{\ell}\}, \tag{4.3}$$

*defined for $i \in \{1, \dots, N\}$ and $\ell \in \mathbb{N}$, satisfy*

$$F_{i,\ell} \cap pB_{1/\ell} \cap pC(K_i) = \emptyset \qquad \forall\ p \in F_{i,\ell}. \tag{4.4}$$

*Proof.* Denote for shortness $\widetilde{C}_i := C(K_i)$. Suppose by contradiction that there exist $x \in F_{i,\ell}$ and $y \in F_{i,\ell} \cap xB_{1/\ell} \cap x\widetilde{C}_i$ with $y \neq x$. Let $\eta_i$ be as in Corollary 4.4 and define

$$r := d(x, y) \quad \text{and} \quad I := xC_i \cap yC_i^{-1} \cap xB_{2r} \cap yB_{2r}.$$

By construction, we have that

$$r = d(x, y) < \tfrac{1}{\ell} < \min\{r_x, r_y\}$$

and using the facts that $I \subseteq xC_i \cap xB_{2r}$ and $I \subseteq yC_i^{-1} \cap yB_{2r}$, by applying (4.1) and (4.2) we have

$$\mu(I \setminus E) \le \mu(xC_i \cap xB_{2r} \cap (\mathbb{G} \setminus E)) \le \tfrac{\eta_i}{3} r^Q$$

and

$$\mu(I \cap E) \le \mu(yC_i^{-1} \cap yB_{2r} \cap E) \le \tfrac{\eta_i}{3} r^Q.$$

This contradicts the fact that by Lemma 4.3, we have $\mu(I) \ge \eta_i r^Q$. □

**Theorem 4.6.** *Let $\mathbb{G}$ be a Carnot group and let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter in $\mathbb{G}$. Then there exist a countable family $\{C_h : h \in \mathbb{N}\}$ of open cones in $\mathbb{G}$ and a countable family $\{\Gamma_h : h \in \mathbb{N}\}$ of subsets of $\mathbb{G}$ such that each $\Gamma_h$ satisfies the $C_h$-cone property and*

$$\mathcal{F}E = \bigcup_{h \in \mathbb{N}} \Gamma_h.$$

*Proof.* It is enough to show that, for any $i, \ell \in \mathbb{N}$, the set $F_{i,\ell}$ defined in (4.3) can be covered by a countable union of sets satisfying a cone property. This is simply done by covering $F_{i,\ell}$ by a countable family of balls $\{B(p_j, 1/\ell) : p_j \in F_{i,\ell}, j \in \mathbb{N}\}$ and decomposing $F_{i,\ell} = \bigcup_{j \in \mathbb{N}} (F_{i,\ell} \cap B(p_j, 1/\ell))$. We now set

$$\Gamma_{i,\ell}^j := F_{i,\ell} \cap B(p_j, 1/\ell) \tag{4.5}$$

Then by (4.4), we have that, for every $i, j \in \mathbb{N}$, the set $\Gamma_{i,\ell}^j$ has the $C(\mathrm{int}(K_i))$-cone property, where $K_i$ are the compact subsets of the sphere introduced in Remark 3.8. Up to relabeling the family $\{\Gamma_{i,\ell}^j : i = 1, \dots, N, \ell, j \in \mathbb{N}\}$ and renaming $C_i := C(\mathrm{int}(K_i))$, we have then a countable family $\{C_h : h \in \mathbb{N}\}$ of open cones and a countable family $\{\Gamma_h : h \in \mathbb{N}\}$ of sets such that each $\Gamma_h$ satisfies the $C_h$-cone property and $\mathcal{F}E = \bigcup_{h \in \mathbb{N}} \Gamma_h$. □

**Remark 4.7.** Notice that the family of cones $\{C_h : h \in \mathbb{N}\}$ appearing in Theorem 4.6 is indeed finite. This comes from the construction $C_i = C(\mathrm{int}(K_i))$, and the fact that, by Remark 3.8, the family $\{K_i : i = 1, \dots, N\}$ is finite.

## 5. Intrinsic Lipschitz graphs

In this section we follow [FSSC06] to recall the notion of intrinsic Lipschitz graph in Carnot groups. The construction of an intrinsic Lipschitz graph requires the space to have a decomposition into complementary subgroups (see Definition 1.1). As observed in Remark 5.1 below, in certain cases Theorem 4.6 can be strengthened to deduce that the reduced boundary of sets of finite perimeter is intrinsically Lipschitz-rectifiable, i.e., it can be covered by a countable union of intrinsic Lipschitz graphs.
Since we only deal with the notion of codimension-one rectifiable sets, Definition 1.1 will be used only in case $\dim \mathbb{L} = 1$. In this situation, according to [Vit12, Proposition 3.4] (see also [FS16]), one can see that, up to a modification of the parameter $\beta > 0$, sets $\Sigma$ satisfying point (i) of Definition 1.1 can be extended to sets $\Sigma \subseteq \widetilde{\Sigma}$ that satisfy (ii).

**Remark 5.1.** Assume $\Gamma \subseteq \mathbb{G}$ is a set with the $C$-cone property such that $C$ is open and there exists $X \in \mathfrak{g}_1 \setminus \{0\}$ with the property that $\exp(X) \in C$. Then $\Gamma$ is an intrinsic Lipschitz graph. Indeed, according to Definition 1.1, we may choose $\mathbb{L} := \{\exp(tX) : t \in \mathbb{R}\}$ and $\mathbb{W} := \exp(X^{\perp} \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_s)$. Recall that, by Remark 4.1, the set $\Gamma$ satisfies also $C^{-1}$-cone property. Since both $C$ and $C^{-1}$ are open, there exists $\varepsilon > 0$ such that $B(\exp(X), \varepsilon) \subseteq C$ and $B(\exp(-X), \varepsilon) \subseteq C^{-1}$. By scaling we may assume that $\|X\| = 1$, and so with the choice $\beta = \varepsilon$ condition (1.1) is satisfied.

In particular, Lemma 4.5 shows that, if for some $i = 1, \dots, N$, the set $K_i$ defined in Remark 3.8 is such that $\mathrm{int}(K_i) \cap \exp(\mathfrak{g}_1) \neq \emptyset$, then, for every $j, \ell \in \mathbb{N}$, the set $\Gamma^j_{i,\ell}$ defined in (4.5) is an intrinsic Lipschitz graph.

**Definition 5.2.** Let $\mathbb{G}$ be a Carnot group of homogeneous dimension $Q$ and let $E \subseteq \mathbb{G}$. We say that $E$ is *intrinsically Lipschitz rectifiable* if there exists a countable family $\{\Sigma_h : h \in \mathbb{N}\}$ of intrinsic Lipschitz graphs such that

$$\mathscr{H}^{Q-1}\left(E \setminus \bigcup_{h \in \mathbb{N}} \Sigma_h\right) = 0.$$

An immediate consequence of Remark 5.1 and Theorem 4.6 is given by Corollary 5.3 below.

**Corollary 5.3.** *Let $\mathbb{G}$ be a Carnot group and assume that for all $\nu \in \mathbb{S}(\mathfrak{g}_1)$ there exists $X \in \mathbb{S}(\mathfrak{g}_1)$ such that $\exp(X) \in \mathrm{int}(S_\nu)$. Then the reduced boundary of every set of locally finite perimeter in $\mathbb{G}$ is intrinsically Lipschitz rectifiable.*

To describe some conditions on the group that guarantee the validity of the assumptions of Corollary 5.3, we introduce the definition of end-point map.

**Definition 5.4.** Let $\mathbb{G}$ be a Carnot group. The *end-point map* $\mathrm{End}\colon L^\infty([0,1];\mathfrak{g}_1) \to \mathbb{G}$ is defined by letting

$$\mathrm{End}(h) = \gamma(1),$$

where $\gamma : [0,1] \to \mathbb{G}$ is the horizontal curve that is the unique solution of

$$\begin{cases} \dot\gamma(t) = h(t) \\ \gamma(0) = 0. \end{cases}$$

With abuse of notation we also write $\mathrm{End}(\gamma)$ meaning $\mathrm{End}(h)$ for the defining control $h$.

In what follows, we say that a map $F\colon M \to N$ between topological spaces $M$ and $N$ is *locally open at* $p \in M$ if, for every neighborhood $U$ of $p$, the set $F(U)$ is a neighborhood of $F(p)$. If $\mathbb{G}$ is a Carnot group and $X \in \mathfrak{g}_1$ is a horizontal direction, we also say that the end-point map $\mathrm{End}\colon L^\infty([0,1];\mathfrak{g}_1) \to \mathbb{G}$ is locally open at $X$, if it is locally open at $h(t) \equiv X$.
Before proving Lemma 5.6, we point out some topological properties of the semigroups $S_\nu$.

**Lemma 5.5.** *Let $\mathbb{G}$ be a Carnot group and let $\nu \in \mathfrak{g}_1 \setminus \{0\}$. Then* $\mathrm{int}(S_\nu) = \mathrm{int}(\overline{S}_\nu)$.

*Proof.* Since $\mathrm{int}(S_\nu)$ has the (inner) $\mathrm{int}(S_\nu)$-cone property, then by [BLD19, Lemma 2.36] the set $\mathrm{int}(S_\nu)$ is regularly open, i.e., we have that

$$\mathrm{int}(S_\nu) = \mathrm{int}(\overline{\mathrm{int}(S_\nu)}). \tag{5.1}$$

On the other hand, by [AS13, Theorem 8.1], we also have that

$$\overline{S}_\nu = \overline{\mathrm{int}(S_\nu)}. \tag{5.2}$$

The result follows combining (5.1) and (5.2). □

**Lemma 5.6.** *Let $\mathbb{G}$ be a Carnot group, let $X \in \mathfrak{g}_1 \setminus \{0\}$ and assume that the end-point map* $\mathrm{End}\colon L^\infty([0,1];\mathfrak{g}_1) \to \mathbb{G}$ *is locally open at $X$. Then, for every $\nu \in \mathbb{S}(\mathfrak{g}_1)$ satisfying $\langle \nu, X\rangle > 0$, we have* $\exp(X) \in \mathrm{int}(S_\nu)$.

*Proof.* Let $\varepsilon \coloneqq \langle \nu, X\rangle > 0$ and

$$B_\epsilon(X) \coloneqq \{v \in L^\infty([0,1];\mathfrak{g}_1) : \|X - v\|_\infty < \varepsilon\}.$$

Since End is open at $X$, by Lemma 5.5 it suffices to show that $\mathrm{End}(B_\varepsilon(X)) \subseteq \overline{S}_\nu$. On the other hand, by Lemma 3.2, if $v \in L^\infty([0,1];\mathfrak{g}_1)$ satisfies $\langle v(t), \nu\rangle > 0$ for almost every $t \in [0,1]$, then $\mathrm{End}(v) \in \overline{S}_\nu$. The proof is then achieved by noticing that, for every $v \in B_\varepsilon(X)$, we have

$$\langle v(t), \nu\rangle = \langle \nu, X\rangle - \langle \nu, X - v(t)\rangle \geq \varepsilon - \|\nu\|\|X - v\|_\infty > 0$$

for almost every $t \in [0,1]$. □

**Corollary 5.7.** *Let $\mathbb{G}$ be a Carnot group and assume there exists a basis $\{X_i : i = 1, \dots, m\}$ of $\mathfrak{g}_1$ such that the end-point map* End: $L^\infty([0,1];\mathfrak{g}_1) \to \mathbb{G}$ *is locally open at $X_i$, for every $i = 1, \dots, m$. Then the reduced boundary of every set of locally finite perimeter in $\mathbb{G}$ is intrinsically Lipschitz rectifiable.*

*Proof.* It is enough to combine Corollary 5.3, Lemma 5.6 and the following fact. If $X \in \mathfrak{g}_1$ and $\exp(X) \in \mathrm{int}(S_\nu)$, then $\exp(-X) \in \mathrm{int}(S_{-\nu})$. □

**Remark 5.8.** Every Carnot group possessing a spanning set of pliable or strongly pliable vectors in the sense of [JS17] and [SS18], respectively, has the property that the reduced boundary of any set of locally finite perimeter is intrinsically Lipschitz rectifiable.

We next give a sufficient condition that has an equivalent algebraic formulation, and that can be more easily verified. This will be used to deduce that, for example, Corollary 5.7 applies to *filiform groups* (see Section 6).

**Definition 5.9.** Let $\mathbb{G}$ be a Carnot group and let $\gamma : [0,1] \to \mathbb{G}$ be a horizontal curve. We say that $\gamma$ is *non-abnormal* if $\mathrm{dEnd}(\gamma)$ has full rank. We also say that a horizontal direction $X \in \mathfrak{g}_1 \setminus \{0\}$ is non-abnormal, if $t \mapsto \exp(tX)$ is non-abnormal.

As pointed out in [ABB19] and in [Mon02], the fact that a curve is abnormal does not depend on its parametrization and one can develop a theory considering the end-point map defined on any $L^p$ space, $1 \le p \le \infty$. Moreover, the Volterra expansion (see e.g. [ABB19, Formula (6.9)]), allows to compute the differential of the End-point map with respect to any variation in $L^p$. In particular, if the differential of End: $L^2([0,T];\mathbb{R}^m) \to \mathbb{G}$ has full-rank at $X \in \mathfrak{g}_1$, then also the differential of End: $L^\infty([0,T];\mathbb{R}^m) \to \mathbb{G}$ has full rank at $X$. This observation allows us to consider the formula for the differential of the end-point map in $L^2$ developed by [LDMO+16].

**Proposition 5.10.** *Let $\mathbb{G}$ be a Carnot group and let $X \in \mathfrak{g}_1 \setminus \{0\}$. Then the curve $t \mapsto \exp(tX)$ is non-abnormal if and only if*

$$\mathrm{span}\{\mathrm{ad}_X^k(\mathfrak{g}_1) : k = 0, \dots, s-1\} = \mathfrak{g}. \tag{5.3}$$

*Proof.* Denote by $\gamma(t) := \exp(tX)$. It is enough to notice that, by [LDMO+16, Proposition 2.3]

$$\mathrm{Im}(\mathrm{dEnd}(\gamma)) = \mathrm{d}R_{\gamma(1)}\mathrm{span}\{\mathrm{Ad}_{\gamma(t)}\mathfrak{g}_1 : t \in (0,1)\} = \mathrm{d}R_{\gamma(1)}\mathrm{span}\{\mathrm{ad}_X^k\mathfrak{g}_1 : k \in \mathbb{N}\}.$$ □

**Remark 5.11.** Condition (5.3) is clearly open in $X$. In particular, if $\mathbb{G}$ contains a non-abnormal curve, then there exists an open set $U \subseteq \mathbb{S}(\mathfrak{g}_1)$ such that $U = -U$ and $t \mapsto \exp(tX')$ is non-abnormal for all $X' \in U$.

**Corollary 5.12.** *Let $\mathbb{G}$ be a Carnot group and assume there exists a non-abnormal $X \in \mathfrak{g}_1 \setminus \{0\}$. Then the reduced boundary of any set of locally finite perimeter in $\mathbb{G}$ is intrinsically Lipschitz rectifiable.*

*Proof.* Since $\gamma(t) := \exp(tX)$ is non-abnormal, then $\mathrm{dEnd}(\gamma)$ has full rank and, in particular, End is locally open at $X$. The proof then follows by combining Corollary 5.7 and Remark 5.11. □

**Proposition 5.13.** *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two Carnot groups possessing non-zero non-abnormal horizontal directions. Then the Carnot group $\mathbb{G}_1 \times \mathbb{G}_2$ possesses a non-zero non-abnormal horizontal direction.*

*Proof.* Denote by $\mathfrak{g}(\mathbb{G}_i)$ the Lie algebra of $\mathbb{G}_i$ and by $\mathfrak{g}_1(\mathbb{G}_i)$ the related horizontal layer, for $i = 1, 2$. Recall that the Lie bracket of the product algebra $\mathfrak{g}(\mathbb{G}_1) \times \mathfrak{g}(\mathbb{G}_2)$ is defined by

$$[(Y_1, Y_2), (Z_1, Z_2)] = ([Y_1, Z_1], [Y_2, Z_2]),$$

for every $Y_1, Z_1 \in \mathfrak{g}(\mathbb{G}_1)$ and every $Y_2, Z_2 \in \mathfrak{g}(\mathbb{G}_2)$. Then, by induction on $k$, one can check that

$$\mathrm{ad}^k_{(Y_1,Y_2)}(Z_1, Z_2) = (\mathrm{ad}^k_{Y_1}(Z_1), \mathrm{ad}^k_{Y_2}(Z_2)),$$

for every $k \in \mathbb{N}$, every $Y_1, Z_1 \in \mathfrak{g}(\mathbb{G}_1)$ and every $Y_2, Z_2 \in \mathfrak{g}(\mathbb{G}_2)$.

Let $X_1 \in \mathfrak{g}_1(\mathbb{G}_1)$ and $X_2 \in \mathfrak{g}_1(\mathbb{G}_2)$ be non-zero non-abnormal directions for $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. Then $(X_1, X_2)$ is non-abnormal for $\mathbb{G}_1 \times \mathbb{G}_2$. To prove this it is enough to notice that for any $k \in \mathbb{N}$ one has

$$\mathrm{ad}^k_{(X_1,X_2)}(\mathfrak{g}_1(\mathbb{G}_1) \times \mathfrak{g}_1(\mathbb{G}_2)) = \mathrm{ad}^k_{X_1}(\mathfrak{g}_1(\mathbb{G}_1)) \times \mathrm{ad}^k_{X_2}(\mathfrak{g}_1(\mathbb{G}_2))$$ □

**Proposition 5.14.** *Let $\mathbb{G}$ be a Carnot group possessing a non-zero non-abnormal horizontal direction and assume that $N \trianglelefteq \mathbb{G}$ is a normal subgroup of $\mathbb{G}$. Then, the Carnot group $\mathbb{G}/N$ possesses a non-zero non-abnormal horizontal direction.*

*Proof.* Let $X \in \mathfrak{g}_1 \setminus \{0\}$ be a non-abnormal direction for $\mathbb{G}$. Then, if $\pi : \mathbb{G} \to \mathbb{G}/N$ is the canonical projection, the push-forward vector $\pi_* X$ is non-abnormal for $\mathbb{G}/N$. This is true by the fact that

$$\pi \mathfrak{g} = \mathrm{span}\{\pi_* \mathrm{ad}^k_X \mathfrak{g}_1 : k \in \mathbb{N}\} = \mathrm{span}\{\mathrm{ad}^k_{\pi_* X} \pi_* \mathfrak{g}_1 : k \in \mathbb{N}\},$$

and by noticing that $\pi_* \mathfrak{g}_1 = \mathfrak{g}_1(\mathbb{G}/N)$. If $\pi_* X \neq 0$, the proof is concluded. If $\pi_* X = 0$, then $0 \in \mathfrak{g}_1(\mathbb{G}/N)$ is non-abnormal and, since non-abnormality is an open and dilation-invariant condition, all the elements in $\mathfrak{g}_1(\mathbb{G}/N)$ are non-abnormal. □

## 6. Filiform groups

In this section we briefly introduce filiform groups and study their abnormal horizontal lines and automorphisms. As a corollary, we obtain that, in all filiform groups, the reduced boundary of any set of locally finite perimeter is intrinsically Lipschitz rectifiable.

**Definition 6.1.** We say that a Carnot group $\mathbb{G}$ is a *filiform group* of step $s$ if the stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ of its Lie algebra satisfies $\dim \mathfrak{g}_1 = 2$ and $\dim \mathfrak{g}_i = 1$, for every $i = 2, \dots, s$.

**Definition 6.2.** Let $\mathbb{G}$ be a filiform group of step $s$. If its Lie algebra $\mathfrak{g}$ has a basis $\{X_0, \dots, X_s\}$ with the only nonzero bracket relations

(i) $[X_0, X_i] = X_{i+1}$ for $i = 1, \dots, s-1$, then $\mathbb{G}$ is said to be *of the first kind.*
(ii) $[X_0, X_i] = X_{i+1}$ for $i = 1, \dots, s-2$ and $[X_i, X_{s-i}] = (-1)^i X_s$ for $i = 1, \dots, s-1$, then $\mathbb{G}$ is said to be *of the second kind.*

In what follows, any basis of the Lie algebra $\{X_0, \dots, X_s\}$ satisfying relations (i) or (ii) will be called *filiform basis.* The classification of filiform groups below follows from [Ver70, Proposition 5].

**Proposition 6.3** (Vergne)**.** *Let $\mathbb{G}$ be a filiform group of step $s$. If $s = 2n+1$ for $n \geq 2$, then $\mathbb{G}$ is either of the first kind or of the second kind. Otherwise $\mathbb{G}$ is of the first kind.*

We point out that our choice of basis for the filiform groups of the second kind differs from the one of Vergne. Indeed, in [Ver70] the author considers a basis $\{Y_0, \dots, Y_s\}$ for which $[Y_0, Y_i] = Y_{i+1}$ and $[Y_i, Y_{s-i}] = (-1)^i Y_s$ for $i = 1, \dots, s-1$. One sees that the basis of Vergne has an extra nonzero bracket $[Y_0, Y_{s-1}] = Y_s$ and the basis in Definition 6.2 (ii) is obtained by choosing $X_0 = Y_0 + Y_1$ and $Y_i = X_i$ for $i = 1, \dots, s$. In addition to having fewer nontrivial bracket relations, our choice of basis has the benefit that it is adapted to the abnormal lines, as we will see next.

**Proposition 6.4.** *Let $\mathbb{G}$ be a filiform group of step at least 3 and let $\{X_0, \dots, X_s\}$ be a filiform basis for $\mathfrak{g}$. Then the line $t \mapsto \exp(tX_1)$ is abnormal. If $\mathbb{G}$ is of the first kind, then this is the only abnormal horizontal line of $\mathbb{G}$. Otherwise there exists exactly one other abnormal horizontal line, namely $t \mapsto \exp(tX_0)$.*

*Proof.* We begin by proving the following claim: if $\mathbb{G}$ is a filiform group of step $s \geq 3$ and $X \in \mathfrak{g}_1$, then the line $t \mapsto \exp(tX)$ is abnormal if and only if $[X, X_i] = 0$ for some $i = 2, \dots, s-1$. Assume first that $[X, X_i] = 0$ for some $i = 2, \dots, s-1$. Then, since $\mathfrak{g}_i$ is one-dimensional, $[X, \mathfrak{g}_i] = 0$ and in particular

$$\mathfrak{g}_{i+1} \cap \operatorname{span}\{\operatorname{ad}_X^k(\mathfrak{g}_1) : k = 0, \dots, s-1\} = \operatorname{ad}_X^i(\mathfrak{g}_1) = [X, \operatorname{ad}_X^{i-1}(\mathfrak{g}_1)] = 0.$$

Hence, by Proposition 5.10, the line $t \mapsto \exp(tX)$ is abnormal. To prove the other implication, suppose that, for each $i = 2, \ldots, s-1$, one has $[X, X_i] \neq 0$. Since the first layer $\mathfrak{g}_1$ does not contain elements of the center, we also find $Y_1 \in \mathfrak{g}_1$ for which $[X, Y_1] \neq 0$. Observe now that, since $\mathfrak{g}_i = \operatorname{span}([X, X_{i-1}])$, one has $\mathfrak{g}_i = \operatorname{span}(\operatorname{ad}_X^{i-1}(Y_1))$ for all $i = 2, \ldots, s$. By Proposition 5.10, the claim is proved.

As a consequence of the claim, since any filiform basis satisfies $[X_1, X_2] = 0$, the line $t \mapsto \exp(tX_1)$ is abnormal. If $\mathbb{G}$ is of the second kind, then the relation $[X_0, X_{s-1}] = 0$ shows that the line $t \mapsto \exp(tX_0)$ is abnormal as well. We now show that, if $\mathbb{G}$ is of the first kind, the line $t \mapsto \exp(tX_1)$ is the only abnormal horizontal line of $\mathbb{G}$. Indeed, for every $a \in \mathbb{R}$, the element $X := X_0 + aX_1 \in \mathfrak{g}_1$ satisfies $[X, X_i] = X_{i+1}$ for all $i = 1, \ldots, s-1$. By the previous observations, the line $t \mapsto \exp(tX)$ is non-abnormal.

We are left to prove that a filiform group cannot possess more than two abnormal horizontal lines. This fact would imply that $t \mapsto \exp(tX_0)$ and $t \mapsto \exp(tX_1)$ are the only horizontal abnormal lines of the filiform groups of the second kind. First, notice that a line $t \mapsto \exp(tX)$ is abnormal if and only if $X \in \bigcup_{i=2}^{s-1} \ker \phi_i$, where $\phi_i \colon \mathfrak{g}_1 \to \mathfrak{g}_{i+1}$ is defined by $\phi_i(X) = [X, X_i]$. According to Proposition 6.3, the algebra $\mathfrak{g}/\mathfrak{g}_s$ is filiform of step $s-1$ and therefore is of the first kind. By the previous part of this proof, the set $\bigcup_{i=2}^{s-2} \ker \phi_i \subseteq \mathfrak{g}/\mathfrak{g}_s$ is one-dimensional. Since $\phi_{s-1}$ is surjective, also $\dim \ker \phi_{s-1} = 1$ and we conclude that $\bigcup_{i=2}^{s-1} \ker \phi_i$ cannot contain more than two linearly independent lines. □

**Corollary 6.5.** *Every filiform group has a horizontal non-abnormal line. In particular, Corollary 5.12 applies to all filiform groups.*

**Remark 6.6.** Applying [LDMO+16, Proposition 2.21], one can check that in filiform groups of the first kind the only horizontal injective abnormal curve from the origin is, up to reparametrization, the line $t \mapsto \exp(tX_1)$. On the other hand, in filiform groups of the second kind there are also horizontal abnormal curves that are not lines. For example, the curve defined by

$$t \mapsto \begin{cases} \exp(tX_1) & \text{for } t \in [0,1], \\ \exp(X_1)\exp((t-1)X_0) & \text{for } t \in [1,2] \end{cases}$$

is abnormal.

For the sake of completeness, we end by describing the graded Lie algebra automorphisms (i.e., the stratification preserving Lie automorphisms) of filiform Lie algebras. By the following two propositions, we observe that any linear bijection on the horizontal layer that fixes the abnormal lines extends uniquely to a Lie algebra automorphism.

**Proposition 6.7.** *Let $\mathfrak{g}$ be a filiform Lie algebra of the first kind of step at least 3 equipped with a filiform basis $\{X_0, \ldots, X_s\}$. The linear transformation on $\mathfrak{g}_1$ that in* *basis* $\{X_0, X_1\}$ *is given by the matrix*

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$$

*induces a (graded) Lie algebra automorphism for every* $a, b \in \mathbb{R} \setminus \{0\}$ *and* $c \in \mathbb{R}$. *Moreover, every graded Lie algebra automorphism of* $\mathfrak{g}$ *is of this form.*

*Proof.* Notice that the only horizontal vectors that commute with $\mathfrak{g}_2$ are those parallel to $X_1$. Therefore, any $\psi \in \mathrm{Aut}(\mathfrak{g})$ maps $\psi(X_1) = bX_1$ with $b \in \mathbb{R}\setminus\{0\}$. Then mapping $\psi(X_0) = aX_0 + cX_1$ defines a Lie algebra automorphism for any choice of $a, b \in \mathbb{R}\setminus\{0\}$ and $c \in \mathbb{R}$ by $\psi(X_i) := \mathrm{ad}^{i-1}_{\psi_1(X_0)}(\psi(X_1)) = a^i b X_i$. □

**Proposition 6.8.** *Let* $\mathfrak{g}$ *be a filiform Lie algebra of the second kind with a filiform basis* $\{X_0, \dots, X_s\}$. *The linear transformation on* $\mathfrak{g}_1$ *that in basis* $\{X_0, X_1\}$ *is given by the matrix*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

*induces a (graded) Lie algebra automorphism for every* $a, b \in \mathbb{R}\setminus\{0\}$. *Moreover, every graded Lie algebra automorphism of* $\mathfrak{g}$ *is of this form.*

*Proof.* Similarly to the filiform groups of the first kind, here $X_1$ is the unique direction commuting with $\mathfrak{g}_2$, and hence the line $bX_1$, $b \in \mathbb{R}$, must be fixed by every $\psi \in \mathrm{Aut}(\mathfrak{g})$. Since in addition a graded Lie algebra automorphism maps abnormal horizontal lines into abnormal horizontal lines, necessarily $\psi(X_0) = aX_0$ for some $a \in \mathbb{R}\setminus\{0\}$ according to Proposition 6.4. Let us verify that the linear map defined by $\psi(X_0) = aX_0$ and $\psi(X_1) = bX_1$ on $\mathfrak{g}_1$ induces a Lie algebra automorphism for all $a, b \in \mathbb{R} \setminus \{0\}$ by explicitly calculating the bracket relations. Indeed, the extension

$$\psi(X_i) = \psi(\mathrm{ad}^{i-1}_{X_0} X_1) := \mathrm{ad}^{i-1}_{\psi(X_0)} \psi(X_1) = a^{i-1} b X_i \quad \forall\, i = 1, \dots, s-1,$$
$$\psi(X_s) = \psi(-[X_1, X_{s-1}]) := -[\psi(X_1), \psi(X_{s-1})] = a^{s-2} b^2 X_s$$

satisfies

$$[\psi(X_i), \psi(X_{s-i})] = [a^{i-1} b X_i, a^{s-i-1} b X_{s-i}] = a^{s-2} b^2 (-1)^i X_s = \psi([X_i, X_{s-i}]) \quad \forall\, i = 2, \dots, s-2$$

and all the other brackets are zero, as required. □

**Remark 6.9.** To the best of our knowledge, in the literature (see e.g. [Ver70, page 93] or [Pan89, page 49]) there is no clear motivation why the two types of filiform groups are not isomorphic; let us provide some evidence here. First of all, one can check that, in dimension 4, there is only one filiform group, known as the *Engel group*. Starting from dimension 6, we provided two ways to distinguish the two classes. A first reason is that the spaces of graded automorphisms are different: in filiform groups of the first kind this class is 3 dimensional, while for the second kind this class is 2 dimensional. A second reason, which has a control-theoretic flavor, is that the two classes have different number of abnormal one-parameter subgroups: in filiform groups of the first kind there is only one, while for the second kind there are two. Recall that the stratification of a stratifiable group is unique up to isomorphisms, see [LD17, Proposition 2.17]. Hence, since we showed that the two classes are different as stratified groups, they are different as Lie groups.

**Remark 6.10.** We stress that the type $\star$ condition, see [Mar14], and the non-abnormality condition introduced in Corollary 5.12 are independent. Indeed, all step-2 Carnot groups are of type $\star$, but not all step-2 Carnot groups have horizontal non-abnormal lines. Consider for example the free Carnot group $\mathbb{F}_{3,2}$ of step 2 and rank 3. One can easily check that, in this case, all horizontal lines are abnormal. On the other hand, as shown in Proposition 6.4, all filiform groups have nontrivial horizontal non-abnormal lines but already the Engel group $\mathbb{E}$ does not satisfy the type $\star$ condition.

**Remark 6.11.** We stress that no filiform group is Pansu-rigid, as instead stated in [Pan89, page 49]. Indeed, in both types of filiform groups, there are more graded automorphisms than just the homotheties, in which case we would have said, by definition, that the Carnot group is Pansu-rigid, see [LDOW14]. It is however true that, in filiform groups of the second kind, the only graded automorphisms that are unipotent are the dilations.

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**[C]**

# Semigenerated Carnot algebras and applications to sub-Riemannian perimeter

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# SEMIGENERATED CARNOT ALGEBRAS AND APPLICATIONS TO SUB-RIEMANNIAN PERIMETER

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Abstract. This paper contributes to the study of sets of finite intrinsic perimeter in Carnot groups. Our intent is to characterize in which groups the only sets with constant intrinsic normal are the vertical half-spaces. Our viewpoint is algebraic: such a phenomenon happens if and only if the semigroup generated by each horizontal half-space is a vertical half-space. We call *semigenerated* those Carnot groups with this property. For Carnot groups of nilpotency step 3 we provide a complete characterization of semigeneration in terms of whether such groups do not have any Engel-type quotients. Engel-type groups, which are introduced here, are the minimal (in terms of quotients) counterexamples. In addition, we give some sufficient criteria for semigeneration of Carnot groups of arbitrary step. For doing this, we define a new class of Carnot groups, which we call type ($\Diamond$) and which generalizes the previous notion of type ($\star$) defined by M. Marchi. As an application, we get that in type ($\Diamond$) groups and in step 3 groups that do not have any Engel-type algebra as a quotient, one achieves a strong rectifiability result for sets of finite perimeter in the sense of Franchi, Serapioni, and Serra-Cassano.

## Contents



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## 1. Introduction

Carnot groups, which are by definition simply connected Lie groups with stratified Lie algebras, raised attention because of their natural occurrences in Geometric Measure Theory and Metric Geometry. In particular, subsets of Carnot groups whose intrinsic normal is constantly equal to a left-invariant vector field appear both in the development of a theory à la De Giorgi for sets of locally finite perimeter in sub-Riemannian spaces [FSS01, FSS03, AGM15] and in the obstruction results for bi-Lipschitz embeddings into $L^1$ of non-abelian nilpotent groups [CK10]. The work [FSS03] by Franchi, Serapioni and Serra-Cassano provides complete understanding of sets with constant intrinsic normal in the case of Carnot groups with nilpotency step 2 by proving that they are half-spaces when read in exponential coordinates. However, in higher step the study appears to be much more challenging due to the more complex underlying algebraic structure, and only in the case of type $(\star)$ groups and of filiform groups we have a satisfactory understanding of sets with constant intrinsic normal, see [Mar14, BL13].

In a recent paper [BL19], C. Bellettini and the first-named author of this article related the property of having constant intrinsic normal to the containment of distinguished constant-normal sets, which are semigroups generated by the horizontal half-space defined by the normal, as we shall explain soon. We shall use the following terminology: a *horizontal half-space* of a stratified algebra $\mathfrak{g}$ with horizontal layer $V_1$ is the closure of either of the two parts into which a hyperplane divides $V_1$. A *vertical half-space* is defined as the direct sum of a horizontal half-space and the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$. By [BL19, Corollary 2.31], in exponential coordinates a Carnot group has the property that all its constant-normal sets are equivalent to vertical half-spaces if and only if the closure of the semigroup generated by each horizontal half-space is a vertical half-space. In Carnot groups with this property, one has the intrinsic $C^1$-rectifiability result for finite-perimeter sets à la De Giorgi. In arbitrary groups, the study of semigroups can still give some weaker rectifiability results, see [DLMV19].

In this paper, we continue the study of such semigroups from an algebraic viewpoint. In particular, we get to a complete characterization of those step-3 Carnot groups for which all constant-normal sets are vertical half-spaces. In addition, for Carnot groups of arbitrary nilpotency step, we give some sufficient criteria which generalize the previous work by M. Marchi [Mar14].

**Definition 1.1.** Given a Carnot group $G$ with exponential map $\exp : \mathfrak{g} \to G$, we say that a set $W \subseteq \mathfrak{g}$ is *semigenerating* if the closure of the semigroup generated by $\exp(W)$ in $G$ contains the commutator subgroup $[G, G]$. We say also that the Lie algebra $\mathfrak{g}$ is *semigenerated* if every horizontal half-space $W$ in $\mathfrak{g}$ is semigenerating.

We shall use the term *Carnot algebra* to denote the (stratified) Lie algebra of a Carnot group, which is completely determined by the Lie group, see [LD17]. By the work [FSS03] of Franchi, Serapioni and Serra-Cassano, we know that step-2 Carnot algebras are semigenerated. Their work has been then extended by Marchi to a class of Carnot algebras, called of type $(\star)$, which includes examples of arbitrarily large nilpotency step. However, the basic example given by the Engel Lie algebra is not semigenerated, see [FSS03, BL13], and also Proposition 5.13. From this example, it is easy to generate more examples of non-semigenerated algebras, because of the observation that each quotient of a semigenerated Lie algebra is semigenerated, see Proposition 2.29. Thus, for example we have that no stratified Lie algebra of rank 2 and step $\geq 3$ is semigenerated because each of them has the Engel Lie algebra as quotient as pointed out in Remark 2.30.

Here, we mostly focus on step-3 Lie algebras, in which we discover a class of Lie algebras that are not semigenerated. Since they are a generalization of the Engel Lie algebra we call them Engel-type algebras. Our main result is that these algebras are the only obstruction to semigeneration.

**Theorem 1.2.** *Let $\mathfrak{g}$ be a stratified Lie algebra of step at most 3. Then $\mathfrak{g}$ is not semigenerated if and only if it has one of the Engel-type algebras (as in Definition 1.3) as a quotient.*

**Definition 1.3.** For each $n \in \mathbb{N}$, we call *$n$-th Engel-type algebra* the $2(n+1)$-dimensional Lie algebra (of step 3 and rank $n+1$) with basis $\{X, Y_i, T_i, Z\}_{i=1}^{n}$, where the only non-trivial bracket relations are given by $[Y_i, X] = T_i$ and $[Y_i, T_i] = Z$ for all $i \in \{1, \dots, n\}$.

It is a challenge to understand how one can express in pure combinatorial terms the property of not having any Engel-type algebra as quotient. However, we have examples of step-3 Lie algebras that are not of type $(\star)$ but have no Engel-type quotients. Hence, our result is a strict generalization of [Mar14]. It is possible that Theorem 1.2 holds also in case the nilpotency step is arbitrary; we have no counterexample. However, the situation in step greater than 3 is more technical. For this reason, we are only able to give a sufficient condition to ensure semigeneration in arbitrary step. Such criterion is not necessary (see Example 4.7); however, as for the type $(\star)$ condition, it is computable in terms of brackets of some particular basis.

In the next result, we assume the existence of a basis with specific properties. We could restate the condition in other forms (see Lemma 3.1), which alas are just as technical.

**Definition 1.4.** Let $\mathfrak{g}$ be a stratified Lie algebra. If, for each subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ for which $\mathfrak{h} \cap V_1$ has codimension 1 in $V_1$, there exists a basis $\{X_1, \dots, X_m\}$ of $V_1$ such that

$$\mathrm{ad}^2_{X_i} X_j \in \mathfrak{h} \quad \text{and} \quad \mathrm{ad}^2_{\mathrm{ad}^k_{X_i} X_j}(X_i) \in \mathfrak{h}, \qquad \text{for all } i, j = 1, \dots, m \text{ and } k \geq 2,$$

then we say that $\mathfrak{g}$ is *of type* ($\lozenge$).

**Theorem 1.5.** *Every stratified Lie algebra that is of type* ($\lozenge$) *(as in Definition 1.4) is semigenerated.*

To put the results in perspective, we remind the reader that by [FSS03], we know that if a Carnot group has the property that every set with constant intrinsic normal is a vertical half-space, then every set of locally finite sub-Riemannian perimeter have a strong rectifiability property. Since semigroups generated by horizontal half-spaces are minimal constant-normal sets with respect to set inclusion according to [BL19], we obtain the following corollary.

**Corollary 1.6.** *If the Lie algebra of a Carnot group is semigenerated (e.g., if it is of type ($\lozenge$), see Theorem 1.5, or has step 3 and does not have any Engel-type algebra as a quotient, see Theorem 1.2), then the reduced boundary of every set of locally finite perimeter in $G$ is intrinsically $C^1$-rectifiable.*

The structure of the article is the following. In Section 2 we discuss some preliminaries. In addition to the notions of semigenerated and trimmed algebras, we introduce a useful set called the edge of a semigroup. In Section 3 we analyze Lie algebras of type ($\lozenge$) and prove Theorem 1.5, see Corollary 3.12. Section 4 is devoted to both a list of examples and of results valid for Carnot algebras of step at most 4. In Section 5 we study the Engel-type algebras. We show that they are the only non-semigenerated Carnot algebras with step 3 that are minimal with respect to quotient in a sense that will be made precise with a notion that we call trimmed (see Definition 3.4). We end with the proof of Theorem 1.2.

## 2. Preliminaries

We start with a small list of notations. Then, in Definition 2.4, we define the edge $\mathfrak{e}_{\mathfrak{s}}$ and the wedge $\mathfrak{w}_{\mathfrak{s}}$ of a semigroup $\mathfrak{s}$. The notion of edge will be in the core of the discussion, since understanding if a horizontal half-space is semigenerating reduces to calculating the edge of its generated semigroup. We provide several preliminary results regarding the size of such edges. In particular, we consider Lemma 2.13 extremely useful and we shall exploit it repeatedly. In Proposition 2.31 we provide equivalent conditions for the definition of trimmed algebra, a notion that is fundamental in our arguments in Section 5.

In this paper, the Lie algebra $\mathfrak{g}$ will always be stratified with layers $V_i = V_i(\mathfrak{g})$. We denote by $\mathcal{Z}(\mathfrak{g})$ the center of a Lie algebra $\mathfrak{g}$. Given an ideal $\mathfrak{i}$ of $\mathfrak{g}$ we denote by $\pi = \pi_{\mathfrak{i}} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ the quotient map, and we shall interchangeably use the equivalent notations $A/\mathfrak{i} = A + \mathfrak{i} = \pi_{\mathfrak{i}}(A)$, for subsets $A$ of $\mathfrak{g}$. We denote for a subset $A$ of $\mathfrak{g}$ by $\mathfrak{I}_{\mathfrak{g}}(A)$ the ideal generated by $A$ within $\mathfrak{g}$, by $\mathrm{Lie}(A)$ the Lie algebra generated by $A$, and by $\mathrm{Cl}(A)$ or $\bar{A}$ the closure of $A$ in $\mathfrak{g}$.

We say that $W$ is a *horizontal half-space* of $\mathfrak{g}$ if there exists a non-zero element $\lambda$ in the dual of $V_1$ such that

$$W = \lambda^{-1}([0, +\infty)) \subseteq V_1. \tag{2.1}$$

If $W$ is a horizontal half-space defined by $\lambda \in (V_1)^*$ as in (2.1), then we define its (horizontal) boundary as

$$\partial W := \lambda^{-1}(\{0\}).$$

Notice that $W$ is a closed subset of $V_1$ and $\partial W$ is its boundary within $V_1$, which in our case will always contain $0 \in \mathfrak{g}$. Observe that $\partial W$ is a hyperplane in $V_1$.

Given a subset $W$ of $\mathfrak{g}$, which we shall usually assume to be a horizontal half-space, the semigroup $S_W$ generated by $\exp(W)$ is described as

$$S_W = \bigcup_{k=1}^{\infty} (\exp(W))^k, \tag{2.2}$$

where

$$(\exp(W))^k := \{\Pi_{i=1}^k \exp(w_i) \mid w_1, \ldots, w_k \in W\}.$$

Be aware that, even when $W$ is closed (within $V_1$), the set $S_W$ may not be closed within $\exp(\mathfrak{g})$.

A vector space $\mathfrak{h}$ of a stratified Lie algebra $\mathfrak{g}$ is said to be *homogeneous* if there exist subspaces $\mathfrak{h}_i$ of $V_i(\mathfrak{g})$ such that $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus h_s$. Equivalently, we have that $\mathfrak{h}$ is homogeneous if and only if $\delta_\lambda \mathfrak{h} = \mathfrak{h}$ for all $\lambda > 0$, where $\delta_\lambda$ is the Lie algebra automorphism such that $\delta_\lambda(v) = \lambda v$ for $v \in V_1(\mathfrak{g})$. We shall frequently use the fact that the center of a stratified Lie algebra is homogeneous:

$$\mathcal{Z}(\mathfrak{g}) = \big(V_1(\mathfrak{g}) \cap \mathcal{Z}(\mathfrak{g})\big) \oplus \ldots \oplus \big(V_s(\mathfrak{g}) \cap \mathcal{Z}(\mathfrak{g})\big). \tag{2.3}$$

2.1. **Lemmata in arbitrary algebras.** In this subsection, let $\mathfrak{g}$ be a Lie algebra of a simply connected Lie group $G$. We assume that $\mathfrak{g}$ is stratified with nilpotency step equal to $s$. Since $G$ is consequently nilpotent and simply connected, the exponential map $\exp : \mathfrak{g} \to G$ is a bijection. We then have a correspondence between subsets $\mathfrak{s} \subseteq \mathfrak{g}$ and subsets $S = \exp(\mathfrak{s}) \subseteq G$.

**Definition 2.4.** We associate with every subset $\mathfrak{s} \subseteq \mathfrak{g}$ the following two sets

$$\mathfrak{w}_\mathfrak{s} := \{X \in \mathfrak{g} \ : \ \mathbb{R}_+ X \subseteq \mathfrak{s}\}. \tag{2.5}$$

$$\mathfrak{e}_\mathfrak{s} := \mathfrak{w}_\mathfrak{s} \cap (-\mathfrak{w}_\mathfrak{s}) = \mathfrak{w}_\mathfrak{s} \cap \mathfrak{w}_{-\mathfrak{s}}. \tag{2.6}$$

The set $\mathfrak{w}_\mathfrak{s}$ is known as the *tangent wedge of* $\mathfrak{s}$ and $\mathfrak{e}_\mathfrak{s}$ as the *edge of the wedge* $\mathfrak{w}_\mathfrak{s}$, see [HN93, Page 2 and page 19]. For typographical reasons, we sometimes write $\mathfrak{e}(\mathfrak{s})$ instead of $\mathfrak{e}_\mathfrak{s}$. An equivalent definition for $\mathfrak{e}_\mathfrak{s}$ is

$$\mathfrak{e}_\mathfrak{s} = \{X \in \mathfrak{g} \ : \ \mathbb{R} X \subseteq \mathfrak{s}\}. \tag{2.7}$$

Regarding the next result, we claim very little originality. The arguments are mostly taken from [HN93] and [BL19]. Also, the notions of cone and convexity that we shall use are the usual ones with respect to the vector-space structure of the Lie algebra.

**Lemma 2.8.** *Let $G$ be a Lie group whose exponential map* $\exp : \mathfrak{g} \to G$ *is injective. Let $\mathfrak{s} \subseteq \mathfrak{g}$ be such that* $\exp(\mathfrak{s})$ *is a semigroup. Then the sets $\mathfrak{e}_\mathfrak{s}$ and $\mathfrak{w}_\mathfrak{s}$, defined in* (2.7) *and* (2.5)*, respectively, satisfy the following properties:*

*(1) $\mathfrak{w}_\mathfrak{s}$ is the largest cone in $\mathfrak{s}$;*
*(2) $\mathfrak{e}_\mathfrak{s}$ is the largest subalgebra of $\mathfrak{g}$ contained in $\mathfrak{s}$;*
*(3) for each $X \in \mathfrak{s} \cap (-\mathfrak{s})$, we have that $\mathfrak{s}$, $\mathfrak{w}_\mathfrak{s}$, and $\mathfrak{e}_\mathfrak{s}$ are invariant under $e^{\mathrm{ad}_X}$, i.e.,*

$$e^{\mathrm{ad}_X} \mathfrak{s} = \mathfrak{s}, \qquad \text{for all } X \text{ such that } \pm X \in \mathfrak{s}, \tag{2.9}$$

$$e^{\mathrm{ad}_X} \mathfrak{w}_\mathfrak{s} = \mathfrak{w}_\mathfrak{s}, \qquad \text{for all } X \text{ such that } \pm X \in \mathfrak{s}, \tag{2.10}$$

$$e^{\mathrm{ad}_X} \mathfrak{e}_\mathfrak{s} = \mathfrak{e}_\mathfrak{s}, \qquad \text{for all } X \text{ such that } \pm X \in \mathfrak{s}; \tag{2.11}$$

*(4) if* $\exp(\mathfrak{s})$ *is closed, then $\mathfrak{w}_\mathfrak{s}$ is closed and convex.*

*Proof.* Point (1) is immediate from the definition. Regarding (2), to see that $\mathfrak{e}_\mathfrak{s}$ is a Lie algebra, let $\mathfrak{h}$ be the Lie algebra generated by $\mathfrak{e}_\mathfrak{s}$. Let $\hat{S}$ be the semigroup generated by $\exp(\mathfrak{e}_\mathfrak{s})$. Since $\mathfrak{e}_\mathfrak{s}$ Lie generates $\mathfrak{h}$, then by [AS13, Theorem 8.1] the set $\hat{S}$ has nonempty interior in $\exp(\mathfrak{h})$. Since $\mathfrak{e}_\mathfrak{s}$ is symmetric, then $\hat{S}$ is closed under inversion, hence a group. Being a group with nonempty interior, $\hat{S}$ is an open subgroup of $\exp(\mathfrak{h})$. Since $\hat{S}$ is an open subgroup of the connected group $\exp(\mathfrak{h})$, then $\exp(\mathfrak{h})$ equals $\hat{S}$, which is a subset of $\exp(\mathfrak{s})$. Therefore, $\mathfrak{h}$ is a subset of $\mathfrak{s}$, being exp injective. Since in addition $\mathfrak{h}$ is symmetric, we infer that $\mathfrak{h} \subseteq \mathfrak{e}_\mathfrak{s}$, which tells us that $\mathfrak{e}_\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ contained in $\mathfrak{s}$. It is the largest since, if a Lie algebra $\mathfrak{h}$ is contained in $\mathfrak{s}$, then from $\mathbb{R}\mathfrak{h} = \mathfrak{h}$ we deduce that $\mathfrak{h} \subseteq \mathfrak{e}_\mathfrak{s}$.

To prove (2.9), take $X$ such that $\pm X \in \mathfrak{s}$, so that for all $Y \in \mathfrak{s}$ we have

$$\begin{aligned} \exp(e^{\mathrm{ad}_X} Y) &= \exp(\mathrm{Ad}_{\exp(X)} Y) \\ &= \exp((C_{\exp(X)})_* Y) \\ &= C_{\exp(X)}(\exp(Y)) \\ &= \exp(X)\exp(Y)\exp(-X) \in S, \end{aligned}$$

where we have used that ad is the differential of Ad, that $\mathrm{Ad}_g$ is the differential of $C_g$, that exp intertwines this differential with $C_g$ and, finally, that $S$ is a semigroup. Hence, we have proved that $e^{\mathrm{ad}_X}\mathfrak{s} = \mathfrak{s}$. Consequently, since the map $e^{\mathrm{ad}_X}$ is linear, it sends half-lines to half-lines and lines to lines. Thus, we have (2.10) and (2.11).

We now prove (4). If $\exp(\mathfrak{s})$ is closed, then also $\mathfrak{s}$ is closed since exp is continuous and injective. Then the closure of $\mathfrak{w}_\mathfrak{s}$ is a cone in $\mathfrak{s}$. By maximality of $\mathfrak{w}_\mathfrak{s}$, we deduce that $\mathfrak{w}_\mathfrak{s}$ is closed. Since $\mathfrak{w}_\mathfrak{s}$ is a cone, to check that $\mathfrak{w}_\mathfrak{s}$ is convex it is enough to show that $X + Y \in \mathfrak{s}$ for all $X, Y \in \mathfrak{w}_\mathfrak{s}$. Indeed, noticing that also $\mathbb{R}_+X, \mathbb{R}_+Y \subseteq \mathfrak{w}_\mathfrak{s}$, this would imply that $\mathbb{R}_+(X + Y) \subseteq \mathfrak{s}$ and so $X + Y \in \mathfrak{w}_\mathfrak{s}$. To prove that $X + Y \in \mathfrak{s}$ for every $X, Y \in \mathfrak{w}_\mathfrak{s}$, recall the formula, which holds in all Lie groups,

$$\exp(X+Y) = \lim_{n\to\infty} \left(\exp\left(\tfrac{1}{n}X\right)\exp\left(\tfrac{1}{n}Y\right)\right)^n. \tag{2.12}$$

Set $S = \exp(\mathfrak{s})$. Since $\mathbb{R}_+X, \mathbb{R}_+Y \subseteq \mathfrak{s}$, then $\exp(\frac{1}{n}X), \exp(\frac{1}{n}Y) \in S$, for all $n \in \mathbb{N}$. Consequently, since $S$ is a semigroup, we have $\left(\exp(\frac{1}{n}X)\exp(\frac{1}{n}Y)\right)^n \in S$. Being $S$ closed by assumption, we get from (2.12) that $\exp(X + Y) \in S$. Since exp is injective, we infer that $X + Y \in \mathfrak{s}$. So the convexity of $\mathfrak{w}_\mathfrak{s}$ is proved. □

We prove next a useful lemma, which states that if $\mathbb{R}_+X \subseteq \mathfrak{s}$, $\mathbb{R}Y \subseteq \mathfrak{s}$, and $\mathbb{R}\,\mathrm{ad}_Y^2 X \subseteq \mathfrak{s}$ then also $\mathbb{R}[X, Y] \subseteq \mathfrak{s}$. Recall the notions of $\mathfrak{e}_\mathfrak{s}$ and $\mathfrak{w}_\mathfrak{s}$, defined in (2.7) and (2.5).

**Lemma 2.13.** *Let $\mathfrak{g}$ be a stratified Lie algebra. Let $\mathfrak{s} \subseteq \mathfrak{g}$ be a subset $\exp(\mathfrak{s})$ is a closed semigroup. If $X \in \mathfrak{w}_\mathfrak{s}$ and $Y \in \mathfrak{e}_\mathfrak{s}$ are such that $\mathrm{ad}_Y^2 X \in \mathfrak{e}_\mathfrak{s}$, then $[X, Y] \in \mathfrak{e}_\mathfrak{s}$.*

*Proof.* One the one hand, since $\mathfrak{e}_\mathfrak{s}$ is a Lie algebra by Lemma 2.8.(2) we have $\mathrm{ad}_Y^k X \in \mathfrak{e}_\mathfrak{s}$, for all $k \geq 2$. On the other hand, from (2.10) we have that $e^{\mathrm{ad}_{tY}} X \in \mathfrak{w}_\mathfrak{s}$, for all $t \in \mathbb{R}$. Hence, since $\mathfrak{w}_\mathfrak{s}$ is convex by Lemma 2.8.(4), for all $t \in \mathbb{R}$ we have

$$X + t[Y, X] = e^{\mathrm{ad}_{tY}} X - \sum_{k\geq 2} \frac{t^k}{k!}\,\mathrm{ad}_Y^k(X) \in \mathfrak{w}_\mathfrak{s}.$$

Hence $\frac{1}{|t|}(X + t[Y, X]) \in \mathfrak{w}_\mathfrak{s}$, for all $t \in \mathbb{R}$. Therefore, taking $t$ to $\pm\infty$ and using that $\mathfrak{w}_\mathfrak{s}$ is closed and convex by Lemma 2.8.(4), we get $[Y, X] \in \mathfrak{e}_\mathfrak{s}$. □

In the rest of the paper, we focus on semigroups generated by horizontal half-spaces in stratified Lie algebras. For every horizontal half-space $W$, see (2.1), in a stratified

Lie algebra $\mathfrak{g}$, we denote by $S_W$ the semigroup generated by $\exp(W)$ in $\exp(\mathfrak{g})$, see (2.2), and by $\mathfrak{s}_W \subset \mathfrak{g}$ the set such that $\exp(\mathfrak{s}_W) = S_W$, i.e.,

$$\mathfrak{s}_W := \log(S_W). \tag{2.14}$$

If $\mathfrak{s} := \mathfrak{s}_W$, we stress the following two immediate facts:

$$\text{for every } X \in V_1, \text{ either } X \in \mathfrak{w}_\mathfrak{s} \text{ or } -X \in \mathfrak{w}_\mathfrak{s}; \tag{2.15}$$

$$\partial W = \mathfrak{e}_\mathfrak{s} \cap V_1. \tag{2.16}$$

The semigeneration condition stated in the introduction can equivalently be defined as follows: A set $W$ in $\mathfrak{g}$ is *semigenerating* if

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Cl}(\mathfrak{s}_W), \tag{2.17}$$

and we say that $\mathfrak{g}$ is *semigenerated* if every horizontal half-space $W$ in $\mathfrak{g}$ is semigenerating. Observe that, by (2.7) a set $W \subseteq \mathfrak{g}$ is semigenerating if and only if $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{e}_\mathfrak{s}$ for $\mathfrak{s} = \mathrm{Cl}(\mathfrak{s}_W)$. We will exploit this fact several times.

**Remark 2.18.** For a horizontal half-space $W \subseteq \mathfrak{g}$ and for $\mathfrak{s}$ equal to $\mathfrak{s}_W$ or $\mathrm{Cl}(\mathfrak{s}_W)$, we have that $\mathfrak{e}_\mathfrak{s}$ is a homogeneous subalgebra of $\mathfrak{g}$ contained in $\mathfrak{s}$. Indeed, in Lemma 2.8.(2) we already proved everything except the homogeneity. In such a case, for all $\lambda > 0$ we have that $\delta_\lambda W = W$ and, hence, $\delta_\lambda \mathfrak{s} = \mathfrak{s}$. Thus we infer that $\delta_\lambda \mathbb{R}X \subseteq \mathfrak{s}$ if and only if $\mathbb{R}X \subseteq \mathfrak{s}$. Therefore $\mathfrak{e}_\mathfrak{s}$ is homogeneous.

**Lemma 2.19.** *Let $\mathfrak{g}$ be a stratified Lie algebra and $W \subseteq \mathfrak{g}$ a horizontal half-space. Then the set $\mathfrak{s} = \mathrm{Cl}(\mathfrak{s}_W)$ has the following two properties:*

$$X, Y \in V_1 \text{ with } \mathrm{ad}_X^2 Y = \mathrm{ad}_Y^2 X = 0 \text{ implies } [X, Y] \in \mathfrak{e}_\mathfrak{s}; \tag{2.20}$$

$$V_2 \cap \mathcal{Z}(\mathfrak{g}) \subseteq \mathfrak{e}_\mathfrak{s}. \tag{2.21}$$

*Proof.* Regarding (2.20), we have, up to changing signs, that $X, Y \in W$. Moreover, since $\partial W$ is a codimension 1 subspace of $V_1$, we have that, up to possibly swapping $X$ with $Y$, there exists some $a \in \mathbb{R}$ for which $Z := Y - aX \in \partial W$. Thus, we have $Z \in \partial W \subseteq \mathfrak{e}_\mathfrak{s}$ and $X \in W \subseteq \mathfrak{w}_\mathfrak{s}$. Moreover, by the assumptions on $X$ and $Y$, we have that

$$\mathrm{ad}_Z^2 X = [Y - aX, [Y, X]] = \mathrm{ad}_Y^2 X + a\, \mathrm{ad}_X^2 Y = 0 \in \mathfrak{e}_\mathfrak{s}.$$

By Lemma 2.13, we obtain $\mathfrak{e}_\mathfrak{s} \ni [X, Z] = [X, Y]$.

Regarding (2.21), we choose $\{X_1, \dots, X_m\}$ to be a basis of $V_1$ such that $X_1 \in W \subseteq \mathfrak{w}_\mathfrak{s}$ and $X_2, \dots, X_m \in \partial W \subseteq \mathfrak{e}_\mathfrak{s}$. Take $Z \in V_2 \cap \mathcal{Z}(\mathfrak{g})$ and express it, for some $a_i, b_{ij} \in \mathbb{R}$, as

$$Z = \sum_{i \geq 2} a_i [X_i, X_1] + \sum_{i,j \geq 2} b_{ij} [X_i, X_j] = [Y, X_1] + \tilde{Y}, \tag{2.22}$$

where

$$Y := \sum_{i\geq 2} a_i X_i \quad \text{and} \quad \tilde{Y} := \sum_{i,j\geq 2} b_{ij}[X_i, X_j].$$

Since $\mathfrak{e}_\mathfrak{s}$ is a Lie algebra by Lemma 2.8.2, we have that the elements $Y, \tilde{Y}, [Y, \tilde{Y}]$ belong to $\mathfrak{e}_\mathfrak{s}$. Since $Z \in \mathcal{Z}(\mathfrak{g})$, we get also

$$0 = [Y, Z] = \operatorname{ad}_Y^2 X_1 + [Y, \tilde{Y}],$$

which implies that $\operatorname{ad}_Y^2 X_1 \in \mathfrak{e}_\mathfrak{s}$. Since $X_1 \in \mathfrak{w}_\mathfrak{s}$, $Y \in \mathfrak{e}_\mathfrak{s}$, and $\operatorname{ad}_Y^2 X_1 \in \mathfrak{e}_\mathfrak{s}$ Lemma 2.13 tells us that $[Y, X_1] \in \mathfrak{e}_\mathfrak{s}$. Going back to (2.22), we finally infer that $Z \in \mathfrak{e}_\mathfrak{s}$, again because $\mathfrak{e}_\mathfrak{s}$ is a Lie algebra by Lemma 2.8.2. □

For the next lemma, recall that $\pi_\mathfrak{i} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ is the quotient map modulo an ideal $\mathfrak{i}$. We also recall the basic fact that in the Lie algebra $\mathfrak{g}$ of a simply connected nilpotent Lie group $G$, a subset $\mathfrak{i} \subseteq \mathfrak{g}$ is an ideal if and only if $N := \exp(\mathfrak{i})$ is a normal Lie subgroup of $G$; in this case, the quotient $\mathfrak{g}/\mathfrak{i}$ is canonically isomorphic to the Lie algebra of $G/N$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi_\mathfrak{i}} & \mathfrak{g}/\mathfrak{i} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\pi_N} & G/N. \end{array}$$

Moreover, if $\mathfrak{g}$ is stratified, then $\mathfrak{g}/\mathfrak{i}$ canonically admits a stratification if and only if $\mathfrak{i}$ is homogeneous.

We stress that we have the following fact for each subset $W \subseteq \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$:

$$\pi_\mathfrak{i}(\mathfrak{s}_W) = \mathfrak{s}_{\pi_\mathfrak{i}(W)}. \tag{2.23}$$

Indeed, setting $N := \exp(\mathfrak{i})$ and denoting by $S(A)$ the semigroup generated by $A$, we need to show that $\pi_N(S(\exp(W))) = S(\pi_N \exp(W))$. In fact, on the one hand, since the homomorphic image of a semigroup is a semigroup, we have that $\pi_N(S(\exp(W)))$ is a semigroup containing $\pi_N(\exp(W))$, so $S(\pi_N \exp(W)) \subseteq \pi_N(S(\exp(W)))$. On the other hand, the set $\pi_N(S(\exp(W))) = S(\exp(W))N$ is contained in the semigroup generated by $\exp(W)N = \pi_N(\exp(W))$, i.e., we have $\pi_N(S(\exp(W))) \subseteq S(\pi_N \exp(W))$.

**Lemma 2.24.** *Let $\mathfrak{i}$ be a homogeneous ideal of a stratified Lie algebra $\mathfrak{g}$ and let $W \subseteq \mathfrak{g}$.*

(i) *If $W$ is semigenerating, then $\pi_\mathfrak{i}(W)$ is semigenerating.*
(ii) *If $\mathfrak{i} \subseteq \mathrm{Cl}(\mathfrak{s}_W)$ and $\pi_\mathfrak{i}(W)$ is semigenerating, then $W$ is semigenerating.*

*Proof.* Assume first that $W$ is semigenerating. Then from (2.23) we obtain that $\pi_\mathfrak{i}(W)$ is semigenerating by the following calculation:

$$[\pi_\mathfrak{i}(\mathfrak{g}), \pi_\mathfrak{i}(\mathfrak{g})] = \pi_\mathfrak{i}([\mathfrak{g}, \mathfrak{g}]) \overset{(2.17)}{\subseteq} \pi_\mathfrak{i}(\mathrm{Cl}(\mathfrak{s}_W)) \subseteq \mathrm{Cl}(\pi_\mathfrak{i}(\mathfrak{s}_W)) \overset{(2.23)}{=} \mathrm{Cl}(\mathfrak{s}_{\pi_\mathfrak{i}(W)}).$$

Suppose then that $\pi_{\mathfrak{i}}(W)$ is semigenerating and that $\mathfrak{i} \subseteq \mathrm{Cl}(\mathfrak{s}_W)$. Then we also have the containment $N := \exp(\mathfrak{i}) \subseteq \mathrm{Cl}(S_W)$. Since $\mathrm{Cl}(S_W)$ is a semigroup, we have

$$\mathrm{Cl}(S_W) \cdot N = \mathrm{Cl}(S_W). \tag{2.25}$$

Therefore from (2.23) we get

$$S_{\pi_{\mathfrak{i}}(W)} = \exp(\mathfrak{s}_{\pi_{\mathfrak{i}}(W)}) \overset{(2.23)}{=} \exp(\pi_{\mathfrak{i}}(\mathfrak{s}_W)) = \pi_N(\exp(\mathfrak{s}_W)) = \pi_N(S_W). \tag{2.26}$$

Taking the closure and the preimage under $\pi_N$, from the fact that $\pi_N$ is an open map (and hence $\pi_N^{-1}$ and Cl commute) and from (2.25), we get that

$$\pi_N^{-1}\mathrm{Cl}(S_{\pi_{\mathfrak{i}}(W)}) \overset{(2.26)}{=} \pi_N^{-1}\mathrm{Cl}(\pi_N(S_W)) = \mathrm{Cl}(S_W \cdot N) = \mathrm{Cl}(S_W) \cdot N \overset{(2.25)}{=} \mathrm{Cl}(S_W). \tag{2.27}$$

Consequently, taking the logarithm,

$$\pi_{\mathfrak{i}}^{-1}\mathrm{Cl}(\mathfrak{s}_{\pi_{\mathfrak{i}}(W)}) = \log(\pi_N^{-1}\mathrm{Cl}(S_{\pi_{\mathfrak{i}}(W)})) \overset{(2.27)}{=} \log \mathrm{Cl}(S_W) = \mathrm{Cl}(\mathfrak{s}_W). \tag{2.28}$$

Hence, since $\pi_{\mathfrak{i}}(W)$ is semigenerating, we infer

$$[\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}] + \mathfrak{i} = \pi_{\mathfrak{i}}^{-1}[\mathfrak{g}/\mathfrak{i}, \mathfrak{g}/\mathfrak{i}] \subseteq \pi_{\mathfrak{i}}^{-1}\mathrm{Cl}(\mathfrak{s}_{\pi_{\mathfrak{i}}(W)}) \overset{(2.28)}{=} \mathrm{Cl}(\mathfrak{s}_W),$$

proving that $W$ is semigenerating. □

We keep reminding that a quotient algebra $\mathfrak{g}/\mathfrak{i}$ of a Carnot algebra $\mathfrak{g}$ is Carnot if and only if the ideal $\mathfrak{i}$ is homogeneous. In such a case, we say that $\mathfrak{g}/\mathfrak{i}$ is a *Carnot quotient* of $\mathfrak{g}$.

**Proposition 2.29.** *Carnot quotients and products of semigenerated algebras are semigenerated.*

*Proof.* Consider a quotient algebra $\mathfrak{g}/\mathfrak{i}$ of a semigenerated Carnot algebra $\mathfrak{g}$ by a homogeneous ideal $\mathfrak{i}$. Then, by Lemma 2.24.i, the Carnot algebra $\mathfrak{g}/\mathfrak{i}$ is semigenerated, since every horizontal half-space in $\mathfrak{g}/\mathfrak{i}$ is of the form $\pi_{\mathfrak{i}}(W)$ for some horizontal half-space $W \subseteq \mathfrak{g}$.

Regarding products, let $\mathfrak{g}$ be a Carnot algebra that is the direct product $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ of two of its Carnot subalgebras. Assume that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semigenerated. Let $W \subset \mathfrak{g}$ be a horizontal half-space. Then for each $i = 1, 2$ we have that the set $W_i := W \cap V_1(\mathfrak{g}_i)$ is a horizontal half-space in $V_1(\mathfrak{g}_i)$, or possibly the whole of $V_1(\mathfrak{g}_i)$. Since each $\mathfrak{g}_i$ is semigenerated, $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \bar{\mathfrak{s}}_{W_i}$ and hence $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{e}(\bar{\mathfrak{s}}_{W_i})$ as $\mathfrak{e}(\bar{\mathfrak{s}}_{W_i})$ is the largest subalgebra of $\bar{\mathfrak{s}}_{W_i}$ by Lemma 2.8.2. Consequently,

$$[\mathfrak{g}_1, \mathfrak{g}_1] \cup [\mathfrak{g}_2, \mathfrak{g}_2] \subseteq \mathfrak{e}(\bar{\mathfrak{s}}_{W_1}) \cup \mathfrak{e}(\bar{\mathfrak{s}}_{W_2}) \subseteq \mathfrak{e}(\bar{\mathfrak{s}}_W).$$

As $\mathfrak{e}(\bar{\mathfrak{s}}_W)$ is a vector space, we have that also

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1] \times [\mathfrak{g}_2, \mathfrak{g}_2] = \mathrm{span}\{[\mathfrak{g}_1, \mathfrak{g}_1] \cup [\mathfrak{g}_2, \mathfrak{g}_2]\} \subseteq \bar{\mathfrak{s}}_W.$$

Hence we infer that $W$ is semigenerating. □

**Remark 2.30.** As a direct consequence of Proposition 2.29, one observes that if a Carnot algebra has a non-semigenerated Carnot quotient, then the algebra cannot be semigenerated. In particular, we point out that every rank-2 Carnot algebra of step at least 3 is not semigenerated, since it has the Engel algebra as a quotient. Indeed, for every such algebra $\mathfrak{g}$, we have that $\mathfrak{g}/\mathfrak{g}^{(4)}$ with $\mathfrak{g}^{(4)} = V_4 \oplus \cdots \oplus V_s$ is a rank-2 Lie algebra of step exactly 3, i.e., either the Engel algebra or the free Lie algebra of rank 2 and step 3. Since the Engel algebra is a quotient of the free Lie algebra, the claim follows. Regarding the fact that the Engel algebra is not semigenerated, we refer to Section 5.1 and specifically to Proposition 5.13.

In the next proposition we verify that three conditions for a stratified Lie algebra are equivalent. In the rest of the paper, we shall call *trimmed* every such Lie algebra.

**Proposition 2.31** (Equivalent conditions for the definition of trimmed algebra)**.** *For a stratified Lie algebra* $\mathfrak{g}$ *the following are equivalent:*

(a) *every proper quotient of* $\mathfrak{g}$ *has lower step;*
(b) $V_s \subseteq \mathfrak{i}$ *for every nontrivial ideal* $\mathfrak{i}$ *of* $\mathfrak{g}$*, where* $s$ *is the step of* $g$*;*
(c) $\dim \mathcal{Z}(\mathfrak{g}) = 1$.

*Proof.* The fact that (a) and (b) are equivalent comes from the correspondence between ideals and kernels of homomorphisms. If every ideal contains the last layer, then any quotient has lower step. Vice versa, if there exists an ideal that does not contain the last layer, then the quotient modulo that ideal has still step $s$.

To see that (b) implies (c), suppose by contradiction that $\dim \mathcal{Z}(\mathfrak{g}) > 1$. We consider the two cases: $\dim V_s > 1$ or $\dim V_s = 1$. In the first case, we get a contradiction since every one-dimensional subspace $\mathfrak{i}$ of $V_s$ is a nontrivial ideal of $\mathfrak{g}$ for which $V_s \subseteq \mathfrak{i}$ is not true. In the case $\dim V_s = 1$, recalling that $\mathcal{Z}(\mathfrak{g})$ is graded by (2.3), we get that $\mathcal{Z}(\mathfrak{g}) \cap (V_1 \oplus \cdots \oplus V_{s-1})$ is a nontrivial ideal of $\mathfrak{g}$ for which $V_s \subseteq \mathfrak{i}$ is not true. These contradictions prove that (b) implies (c).

To see that (c) implies (b), let $\mathfrak{i} \subseteq \mathfrak{g}$ be a nontrivial ideal. Since $\mathfrak{g}$ is nilpotent, we have[1] that $\mathfrak{i} \cap \mathcal{Z}(\mathfrak{g}) \neq \{0\}$. Since $\dim \mathcal{Z}(\mathfrak{g}) = 1$, we have $\mathcal{Z}(\mathfrak{g}) \subseteq \mathfrak{i}$. Finally, since $V_s \subseteq \mathcal{Z}(\mathfrak{g}) \subseteq \mathfrak{i}$ we get the claim. □

**Definition 2.32.** If $\mathfrak{g}$ is a stratified Lie algebra that satisfies the equivalent conditions of Proposition 2.31, then we say that $\mathfrak{g}$ is *trimmed*.

We expect that every non-semigenerated algebra has a trimmed non-semigenerated quotient. However, we only prove the following weaker statement, which will suffice in the step-3 case.

**Proposition 2.33.** *Let* $\mathfrak{g}$ *be a stratified Lie algebra. If* $\mathfrak{g}$ *is not semigenerated, then there exists a quotient algebra* $\hat{\mathfrak{g}}$ *of* $\mathfrak{g}$ *that is not semigenerated such that* $\mathcal{Z}(\hat{\mathfrak{g}}) \cap V_j(\hat{\mathfrak{g}}) = \{0\}$ *for* $j = 1, 2$*, and* $\dim \mathcal{Z}(\hat{\mathfrak{g}}) \cap V_j(\hat{\mathfrak{g}}) \leq 1$ *for all* $j = 3, \ldots, s$.

[1]Looking at the sequence $\mathrm{ad}^j_{\mathfrak{g}}(\mathfrak{i})$ one finds a non trivial subset of $\mathfrak{i}$ that commutes with $\mathfrak{g}$.

*Proof.* Let $W \subseteq \mathfrak{g}$ be a non-semigenerating horizontal half-space. Replacing $\mathfrak{g}$ with some quotient of it, we may suppose that for every proper homogeneous ideal $\mathfrak{i}$ of $\mathfrak{g}$ the half-space $W/\mathfrak{i}$ is semigenerating in $\mathfrak{g}/\mathfrak{i}$. Indeed, if there exists a homogeneous ideal $\mathfrak{i}$ of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{i}$ is not semigenerated, we replace $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{i}$. We repeat this procedure until every homogeneous ideal $\mathfrak{i}$ of $\mathfrak{g}$ has the property that $\mathfrak{g}/\mathfrak{i}$ is semigenerated. This will happen, eventually, since every step-2 Carnot algebra is semigenerated.

We shall then show that $\mathfrak{g}$ has the required properties. First, we check that $\mathcal{Z}(\mathfrak{g})\cap V_1$ is trivial. Indeed, if this space is nontrivial, then, for some $n \geq 1$,

$$\mathfrak{g} \cong (\mathfrak{g}/(\mathcal{Z}(\mathfrak{g}) \cap V_1)) \times (\mathcal{Z}(\mathfrak{g}) \cap V_1) \cong (\mathfrak{g}/(\mathcal{Z}(\mathfrak{g}) \cap V_1)) \times \mathbb{R}^n.$$

Since the product of two semigenerated Lie algebras is semigenerated (see Proposition 2.29), we get a contradiction.

To prove that $\mathcal{Z}(\mathfrak{g})\cap V_2$ is trivial, recall that $\mathcal{Z}(\mathfrak{g})\cap V_2 \subseteq \mathfrak{e}(\mathrm{Cl}(\mathfrak{s}_W))$ by (2.21). Then, denoting by $\pi$ the projection $\pi\colon \mathfrak{g} \to \mathfrak{g}/(\mathcal{Z}(\mathfrak{g}) \cap V_2)$, by Lemma 2.24.ii we have that $\pi(W)$ is not semigenerating. Since we assumed that $W/\mathfrak{i}$ is semigenerating for every proper ideal of $\mathfrak{g}$, we deduce that $\mathcal{Z}(\mathfrak{g}) \cap V_2 = \{0\}$.

Fix any $j \geq 3$. Assume by contradiction that $\dim \mathcal{Z}(\mathfrak{g}) \cap V_j > 1$ and let $V$ be a 2-dimensional subspace of $\mathcal{Z}(\mathfrak{g}) \cap V_j$. Let us also fix a scalar product on $\mathfrak{g}$ and set $\tilde{\mathfrak{s}} := \mathrm{Cl}(\mathfrak{s}_W) \cap V$. Observe that, as $V$ is central, each line $\mathbb{R}v \in V$ is an ideal of $\mathfrak{g}$. Therefore, being $W + \mathbb{R}v$ semigenerating in $\mathfrak{g}/\mathbb{R}v$, we have for all $u, v \in V$ that, if $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathbb{R}v$ stands for the projection,

$$u+\mathbb{R}v \subset V \subseteq [\mathfrak{g},\mathfrak{g}] \subseteq \pi^{-1}([\mathfrak{g}/\mathbb{R}v, \mathfrak{g}/\mathbb{R}v]) \subseteq \pi^{-1}(\mathrm{Cl}(\mathfrak{s}_{\pi(W)})) = \mathrm{Cl}(\pi^{-1}(\mathfrak{s}_{\pi(W)})) \overset{(2.23)}{=} \mathrm{Cl}(\mathfrak{s}_W+\mathbb{R}v),$$

where we again used that $\pi^{-1}$ and the closure commute. Hence, denoting by $B_{1/n}(u + \mathbb{R}v)$ the $1/n$-neighborhood of the line $u + \mathbb{R}v$ within $\mathfrak{g}$, we obtain

$$\text{for every } n \in \mathbb{N} \text{ there exists } s_n \in B_{1/n}(u + \mathbb{R}v) \cap \mathfrak{s}_W. \tag{2.34}$$

We claim that

$$\tilde{\mathfrak{s}} \cap \mathcal{S}^1(V) \cap H \neq \emptyset \text{ for every closed half-space } H \subseteq V, \tag{2.35}$$

where $\mathcal{S}^1(V)$ stands for the unit circle of $V$ with respect the restricted norm on $V$. Indeed, denote by $\partial H$ the boundary of $H$, which is a line in $V$, and let $\nu$ be the inner unit normal of $H$. Consider the sequence $(s_n)_n$ given by (2.34) for the line $2\nu+\partial H \subseteq H$. Then, for every $n \in \mathbb{N}$, there exists $h_n \in 2\nu + \partial H$ such that $\|s_n - h_n\| < 1/n$. Observe that, since $\|\nu\| = 1$ and $\partial H$ is orthogonal to $\nu$, we have $\|h_n\| > 1$ for all $n \in \mathbb{N}$. Moreover, since $V$ is spanned by homogeneous elements of the same degree, each path $\{\delta_\lambda(h_n) \mid \lambda \in [0, 1]\}$ is a straight line segment between 0 and $h_n$ contained in $H$. Hence there exists $\lambda_n \in (0, 1)$ such that $\delta_{\lambda_n}(h_n) \in S^1(V) \cap H$. Since $\delta_{\lambda_n}$ is a contraction, from $\|s_n - h_n\| < 1/n$ we deduce that $\mathrm{dist}(\delta_{\lambda_n}(s_n), \mathcal{S}^1(V) \cap H) < 1/n$ for each $n$. Being $\mathcal{S}^1(V) \cap H$ compact and $\mathfrak{s}_W$ invariant under dilations, we find a converging subsequence of $(\delta_{\lambda_n}(s_n))_n \subseteq \mathfrak{s}_W$ with the limit in $\mathcal{S}^1(V) \cap H$. Hence $\tilde{\mathfrak{s}} \cap \mathcal{S}^1(V) \cap H \neq \emptyset$, proving the claim (2.35).

Notice that since the subalgebra $V$ is abelian, the set $\tilde{\mathfrak{s}}$ is a semigroup. Since, in addition, $V$ is spanned by elements of the same degree of homogeneity and $\tilde{\mathfrak{s}}$ is dilation invariant, we have that $\tilde{\mathfrak{s}}$ is a Euclidean convex cone. Namely, denoting by $\alpha$ the degree of homogeneity of $V$, we find for every $p_1, p_2 \in \tilde{\mathfrak{s}}$ that, in coordinates, the straight segment connecting them is in $\tilde{\mathfrak{s}}$, since

$$p_1 + t(p_2 - p_1) = p_1 + \delta_{t^{1/\alpha}}(p_2 - p_1) \in \tilde{\mathfrak{s}}, \quad \forall t \in [0,1].$$

We therefore deduce that either $\tilde{\mathfrak{s}}$ is contained in some closed half-space $H \subseteq V$ or $\tilde{\mathfrak{s}} = V$. In the latter case $V$ is an ideal of $\mathfrak{g}$ such that $V \subseteq \mathrm{Cl}(\mathfrak{s})$, which implies by Lemma 2.24.ii that $W/V$ is not semigenerating, contradicting our assumptions.

We may then assume that there exists some closed half-space $H \subseteq V$ such that $\tilde{\mathfrak{s}} \subseteq H$. Let $v$ denote one of the two intersection points of $\partial H$ and $\mathcal{S}^1(V)$. We are going to argue that $v \in \tilde{\mathfrak{s}}$. Consider a sequence $(H_n)_n$ of closed half-spaces in $V$ for which $\cap_n(H_n \cap H) = \mathbb{R}_+ v$. By (2.35) we find a sequence $(s_n)_n \subseteq \tilde{\mathfrak{s}} \cap \mathcal{S}^1(V)$ such that each $s_n \in H_n$. But since $\tilde{\mathfrak{s}} \subseteq H$, we have that $s_n \in H \cap H_n$ for every $n$. Hence $s_n \to v$ and $v \in \tilde{\mathfrak{s}}$. With a similar argument also $-v \in \tilde{\mathfrak{s}}$ and therefore $\{\delta_\lambda(v) \mid \lambda \in \mathbb{R}\} = \mathbb{R}v \subseteq \tilde{\mathfrak{s}}$. Now $\mathbb{R}v$ is again an ideal of $\mathfrak{g}$ contained in $\mathrm{Cl}(\mathfrak{s})$, leading to a contradiction by Lemma 2.24.ii and the fact that $W/\mathbb{R}v$ is semigenerating. □

The following lemma is an algebraic observation. It will be essential in our proof of Theorem 3.9, which is a refinement of Theorem 1.5.

**Lemma 2.36.** *Let $\mathfrak{g}$ be a stratified Lie algebra, let $W \subseteq \mathfrak{g}$ be a horizontal half-space, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ containing $\partial W$. Then, the following conditions are equivalent.*

*(1) There exists $X \in V_1 \setminus \partial W$ such that $\mathrm{ad}_X^k Y \in \mathfrak{h}$ for all $Y \in \partial W$ and $k \geq 1$;*
*(2) $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$.*

*Proof.* The fact that (2) implies (1) is trivial, since if $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$, then any choice of basis will satisfy the requirements. For the opposite direction, without loss of generality we may assume that $\mathfrak{g}$ is a free nilpotent Lie algebra. Indeed, if $\mathfrak{f}$ is the free Lie algebra of the same rank and step as $\mathfrak{g}$, then there exists an ideal $\mathfrak{i} \subseteq [\mathfrak{f}, \mathfrak{f}]$ of $\mathfrak{f}$ such that $\mathfrak{g} \cong \mathfrak{f}/\mathfrak{i}$. Namely, there is a surjective Carnot morphism $\pi\colon \mathfrak{f} \to \mathfrak{g}$. Assume that the lemma is shown for $\mathfrak{f}$ and that $\mathfrak{g}$ satisfies (1) for a horizontal half-space $W \in V_1(\mathfrak{g})$ and $X \in V_1(\mathfrak{g}) \setminus \partial W$. Since $\pi$ is injective on $V_1(\mathfrak{f})$, then $\pi^{-1}(W)$ is a horizontal half-space of $\mathfrak{f}$ for which $\pi^{-1}(X) \in V_1(\mathfrak{f}) \setminus \partial(\pi^{-1}(W))$, and

$$\mathrm{ad}^k_{\pi^{-1}(X)} \pi^{-1}(Y) \in \pi^{-1}(\mathrm{ad}^k_X Y) \subseteq \pi^{-1}(\mathfrak{h}), \quad \forall\, k \geq 1,\ Y \in \partial W.$$

Since $\pi^{-1}(\mathfrak{h}) = \mathfrak{h} + \mathfrak{i}$ is a subalgebra of $\mathfrak{f}$, by the lemma we have $[\mathfrak{f}, \mathfrak{f}] \subseteq \pi^{-1}(\mathfrak{h})$ and so $[\mathfrak{g}, \mathfrak{g}] = \pi([\mathfrak{f}, \mathfrak{f}]) \subseteq \mathfrak{h}$.

Let then $\mathfrak{g}$ be a free nilpotent Lie algebra of step $s$. We shall consider a basis for $\mathfrak{g}$ that is constructed by a well-known algorithm due to M. Hall [Hal50]. Below we say that a vector $Z$ has degree $k$ if $Z \in V_k$. Let $\{Y_1, \dots, Y_m\}$ be a basis for $\partial W$ and $X \in V_1 \setminus \partial W$, whence $\{X, Y_1, \dots, Y_m\}$ is a basis for $V_1$. To construct the Hall basis, first fix an ordering for $\{X, Y_1, \dots, Y_m\}$ so that $Y \geq X$ for all $Y \in \{X, Y_1, \dots, Y_m\}$. Suppose then that we have defined Hall basis elements of degree $1, \dots, k-1$ with an ordering satisfying $Y < Z$ if $\deg Y < \deg Z$. Then by Hall's construction $[Y, Z]$ is a basis element of degree $k$ if and only if $Y$ and $Z$ are elements of the Hall basis satisfying

(i) $Y < Z$;
(ii) $\deg Y + \deg Z = k$;
(iii) if $Z = [U, V]$, then $Y \geq U$.

Assuming that $\mathrm{ad}_X^k Y_i \in \mathfrak{h}$ for all basis elements $Y_i$ of $\partial W$ and $k \in \mathbb{N}$, we shall show, by induction on $m$, that $V_2 \oplus \cdots \oplus V_m \subseteq \mathfrak{h}$, for $m \geq 2$. Clearly, we have that $V_2 \subseteq \mathfrak{h}$. Assume then that $V_2 \oplus \cdots \oplus V_{k-1} \subseteq \mathfrak{h}$ and take an element $\widetilde{Y} = [Y, Z]$ of the Hall basis of degree $k$. Recall that by (i) we have $Y < Z$. If $2 \leq \deg Y, \deg Z \leq k-2$, then $\widetilde{Y} \in \mathfrak{h}$ by the induction hypothesis. Assume instead that $\deg Y = 1$ and $\deg Z = k-1$. Thus either $Y = X$ or $Y \in \partial W$ by construction of the basis. If $Y \in \partial W$, we have again that $\widetilde{Y} \in \mathfrak{h}$ since $Z \in V_{k-1} \subseteq \mathfrak{h}$ by the induction hypothesis.

Finally, suppose that $\widetilde{Y} = [X, Z]$. Since $\deg Z > 1$, there exist some $U, V$ with degrees less than $k-1$ such that $Z = [U, V]$. By (iii) then $X \geq U$. Since the ordering for the basis is chosen such that $X$ is the minimal element, this implies that $U = X$ and $\widetilde{Y} = [X, [X, V]]$. Similarly, since $V$ is a degree $k-2$ element of the Hall basis, by (iii) we have again that $V = [X, \widetilde{V}]$ for some $\widetilde{V}$. Repeating this argument gives us finally that $\widetilde{Y} = \mathrm{ad}_X^{k-1} Y_i$ for some $Y_i \in \partial W$. Hence $\widetilde{Y} \in \mathfrak{h}$, by assumption. We have shown that $[\mathfrak{g}, \mathfrak{g}] = V_2 \oplus \cdots \oplus V_s \subseteq \mathfrak{h}$ and the proof is complete. ☐

## 3. Sufficient criteria for semigeneration

In Definition ?? we introduce Carnot groups of type ($\diamondsuit$), which are a generalization of Carnot groups of type ($\star$). After that we present a proof for Theorem 1.5, which is formulated as a corollary of Theorem 3.9 (see Corollary 3.12). We conclude the section with Lemma 3.13, which is useful in the construction of examples in Section 4.

### 3.1. **Carnot groups of type ($\diamondsuit$).**

**Lemma 3.1** (Equivalent conditions for the definition of type ($\diamondsuit$))**.** *For each subalgebra $\mathfrak{h}$ of a stratified Lie algebra $\mathfrak{g}$ for which $\mathfrak{h} \cap V_1$ has codimension one in $V_1$, the following are equivalent:*

(a) *there exists a basis* $\{X_1, \dots, X_m\}$ *of* $V_1$ *such that*

$$\operatorname{ad}^2_{X_i} X_j \in \mathfrak{h} \quad \textit{and} \quad \operatorname{ad}^2_{\operatorname{ad}^k_{X_i} X_j}(X_i) \in \mathfrak{h}, \tag{3.2}$$

*for all* $i, j = 1, \dots, m$ *and* $k \geq 2$*;*

(b) *there exists a basis* $\{Y_1, \dots, Y_{m-1}\}$ *of* $\mathfrak{h} \cap V_1$ *and* $X \in V_1 \setminus \mathfrak{h}$ *such that*

$$\operatorname{ad}^2_X Y_i \in \mathfrak{h}, \quad \operatorname{ad}^2_{Y_i} X \in \mathfrak{h} \quad \textit{and} \quad \operatorname{ad}^2_{\operatorname{ad}^k_X Y_i}(X) \in \mathfrak{h}, \tag{3.3}$$

*for all* $i = 1, \dots, m-1$ *and* $k \geq 2$.

*Proof.* To show that (b) implies (a), assume that there exists a basis $\{Y_1, \dots, Y_{m-1}\}$ of $\mathfrak{h} \cap V_1$ and $X \in V_1 \setminus \mathfrak{h}$ satisfying (3.3). Then we shall check that the basis $\{X, Y_1, \dots, Y_{m-1}\}$ satisfies conditions (3.2). Indeed, from (3.3) the only relations there are left to check are

$$\operatorname{ad}^2_{\operatorname{ad}^k_{Y_i} Y_j}(Y_i) \in \mathfrak{h} \quad \text{and} \quad \operatorname{ad}^2_{\operatorname{ad}^k_{Y_i} X}(Y_i) \in \mathfrak{h}$$

for all $i, j \in \{1, \dots, m-1\}$ and $k \geq 2$. The first one follows from the fact that $Y_i, Y_j \in \mathfrak{h}$ and that $\mathfrak{h}$ is a subalgebra. The second relation follows similarly, since $Y_i \in \mathfrak{h}$ and also $\operatorname{ad}^k_{Y_i} X = \operatorname{ad}^{k-2}_{Y_i}(\operatorname{ad}^2_{Y_i} X) \in \mathfrak{h}$.

For the direction (a) implies (b), let $\{X_1, \dots, X_m\}$ be a basis of $V_1$ for which (3.2) holds. Observe that $X_l \in V_1 \setminus \mathfrak{h}$ for some $l \in \{1, \dots, m\}$ since $\mathfrak{h} \cap V_1$ is $(m-1)$-dimensional. Assume, by possibly changing indexing, that $l = m$. Since now $V_1 = \mathbb{R}X_m \oplus (\mathfrak{h} \cap V_1)$, then for each $i \in \{1, \dots, m-1\}$ there exist $a_i \in \mathbb{R}$ and $Y_i \in \mathfrak{h} \cap V_1$ such that

$$X_i = a_i X_m + Y_i.$$

Therefore, for each $i \in \{1, \dots, m-1\}$ we have

$$Y_i = X_i - a_i X_m$$

and, consequently, $\{Y_1, \dots, Y_{m-1}\}$ is a basis of $\mathfrak{h} \cap V_1$. We claim that this basis together with $X_m \in V_1 \setminus \mathfrak{h}$ satisfies condition (3.3). Indeed, for all $k \geq 2$ we have that

$$\operatorname{ad}^k_{X_m} Y_i = \operatorname{ad}^{k-1}_{X_m}([X_m, Y_i]) = \operatorname{ad}^{k-1}_{X_m}([X_m, X_i - a_i X_m]) = \operatorname{ad}^k_{X_m} X_i \in \mathfrak{h},$$

which proves that $\operatorname{ad}^2_{X_m} Y_i \in \mathfrak{h}$ and $\operatorname{ad}^2_{\operatorname{ad}^k_{X_m} Y_i}(X_m) \in \mathfrak{h}$ for all $i = 1, \dots, m$. Furthermore,

$$\operatorname{ad}^2_{Y_i} X_m = [X_i - a_i X_m, [X_i - a_i X_m, X_m]] = \operatorname{ad}^2_{X_i} X_m + a_i \operatorname{ad}^2_{X_m} X_i \in \mathfrak{h}$$

for every $i = 1, \dots, m$, verifying the last missing condition of (3.3). □

Next definition is a restating of Definition 1.4.

**Definition 3.4** (Diamond type)**.** Let $\mathfrak{g}$ be a stratified Lie algebra. If each subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ for which $\mathfrak{h} \cap V_1$ has codimension 1 in $V_1$ satisfies the equivalent conditions of Lemma 3.1, then we say that $\mathfrak{g}$ is *of type* $(\Diamond)$.

**Remark 3.5.** Every type ($\star$) algebra, as introduced by Marchi, is of type ($\Diamond$). Indeed, we recall that a stratified Lie algebra is of type ($\star$) according to [Mar14] if there exists a basis $\{X_1, \dots, X_m\}$ of $V_1$ such that

$$\operatorname{ad}^2_{X_i} X_j = 0 \quad \forall i, j = 1, \dots, m. \tag{3.6}$$

Therefore, every such an algebra trivially satisfies (3.2) for every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

**Remark 3.7.** If $\mathfrak{g}$ is of type ($\Diamond$) and admits subalgebra $\mathfrak{h}$ of step $\leq 2$ such that $\mathfrak{h} \cap V_1$ has codimension 1 in $V_1$, then $\mathfrak{g}$ is of type ($\star$). Indeed, by (3.2) there exists a basis $\{X_1, \dots, X_m\}$ of $V_1$ such that

$$\operatorname{ad}^2_{X_i} X_j \in \mathfrak{h} \cap V_3 = \{0\},$$

for all $i, j = 1, \dots, m$.

Despite its simplicity, the following remark will prove useful when finding out if a given Lie algebra is of type ($\star$).

**Remark 3.8.** A Lie algebra $\mathfrak{g}$ is of type ($\star$) if and only if there exists a basis $\{X_1, \dots, X_m\}$ of $V_1$ such that

$$\operatorname{ad}^2_{X_i}(V_1) = 0 \quad \forall\, i = 1, \dots, m.$$

Indeed, this follows from the fact that the map $Y \mapsto \operatorname{ad}^2_X Y$ is linear for every $X \in \mathfrak{g}$.

Next we prove a result that is finer than Theorem 1.5. The latter is then obtained in Corollary 3.12 as an immediate consequence of Theorem 3.9.

**Theorem 3.9.** *Let $\mathfrak{g}$ be a stratified Lie algebra. A horizontal half-space $W \subseteq \mathfrak{g}$ is semigenerating if and only if there exists a basis $\{X_1, \dots, X_m\}$ of $V_1$ such that, for $\mathfrak{s} := \operatorname{Cl}(\mathfrak{s}_W)$,*

$$\operatorname{ad}^2_{X_i} X_j \in \mathfrak{e}_{\mathfrak{s}} \quad \textit{and} \quad \operatorname{ad}^2_{\operatorname{ad}^k_{X_i} X_j}(X_i) \in \mathfrak{e}_{\mathfrak{s}}, \tag{3.10}$$

*for all $i, j = 1, \dots, m$ and $k \geq 2$.*

*Proof.* If $W$ is semigenerating, then $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{e}_{\mathfrak{s}}$ and hence (3.10) is satisfied by any basis. Vice versa, we assume a basis satisfying (3.10) exists and plan to show that $W$ is semigenerating.

We start by noticing that $\mathfrak{e}(\mathfrak{s}) =: \mathfrak{h}$, as defined in (2.6), is a Lie algebra (see Lemma 2.8.2) for which $\mathfrak{h} \cap V_1 = \partial W$ has codimension 1 in $V_1$ and which satisfies conditions (3.2). Then, by Lemma 3.1, there exists a basis $\{Y_1, \dots Y_{m-1}\}$ for $\mathfrak{e}_{\mathfrak{s}} \cap V_1$ and $X \in V_1 \setminus \mathfrak{e}_{\mathfrak{s}}$ that satisfy equations (3.3). We claim that, to prove that $W$ is semigenerating, it suffices to show that

$$\operatorname{ad}^k_X Y_i \in \mathfrak{e}_{\mathfrak{s}} \quad \forall i = 1, \dots, m-1 \text{ and } k \geq 1. \tag{3.11}$$

Indeed, this is a consequence of Lemma 2.36: being $\mathrm{ad}_X$ linear, conditions (3.11) would imply that $\mathrm{ad}_X^k Y \in \mathfrak{e}(\mathfrak{s})$ for every $Y \in \partial W$. Then, by Lemma 2.36 we have $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{e}_\mathfrak{s} \subseteq \mathfrak{s}$, which would prove that $W$ is semigenerating.

To show (3.11), we treat first the case $k = 1$. Recall that, up to changing sign we have that $X \in \mathfrak{w}_\mathfrak{s}$. Since by (3.3) we have $\mathrm{ad}_{Y_i}^2 X \in \mathfrak{e}_\mathfrak{s}$ for all $i = 1, \dots, m-1$, then by Lemma 2.13 we have that $[X, Y_i] \in \mathfrak{e}_\mathfrak{s}$ for all $i = 1, \dots, m-1$. Since also $[Y_i, Y_j] \in \mathfrak{e}_\mathfrak{s}$ for all $i, j = 1, \dots, m-1$ due to the fact that $\mathfrak{e}_\mathfrak{s}$ is a Lie algebra, we deduce that

$$V_2 = \mathrm{span}\{[X, Y_i], [Y_i, Y_j] \mid i, j = 1, \dots, m-1\} \subseteq \mathfrak{e}_\mathfrak{s}.$$

The case $k \geq 2$ is proven by induction. The first step $\mathrm{ad}_X^2 Y_i \in \mathfrak{e}_\mathfrak{s}$ is given by (3.3). Let us then assume that $\mathrm{ad}_X^k Y_i \in \mathfrak{e}_\mathfrak{s}$ for some $k \geq 2$. Since also $\mathrm{ad}^2_{\mathrm{ad}_X^k Y_i}(X) \in \mathfrak{e}_\mathfrak{s}$ by (3.3), then by Lemma 2.13 again we obtain

$$[X, \mathrm{ad}_X^k Y_i] = \mathrm{ad}_X^{k+1} Y_i \in \mathfrak{e}_\mathfrak{s},$$

which we needed to show. □

**Corollary 3.12** (Theorem 1.5). *Every Carnot algebra of type (◊) is semigenerated.*

*Proof.* Let $\mathfrak{g}$ be of type (◊) and consider a horizontal half-space $W$ in $\mathfrak{g}$. Denoting $\mathfrak{s} := \mathrm{Cl}(\mathfrak{s}_W)$, from Lemma 2.8 we have that $\mathfrak{e}_\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ for which $\mathfrak{e}_\mathfrak{s} \cap V_1 = \partial W$ has codimension 1 in $V_1(\mathfrak{g})$. Being $\mathfrak{g}$ of type (◊) we apply (3.2) with $\mathfrak{h} = \mathfrak{e}_\mathfrak{s}$ to have that there exists a basis $\{X_1, \dots, X_m\}$ of $V_1(\mathfrak{g})$ satisfying

$$\mathrm{ad}_{X_i}^2 X_j \in \mathfrak{e}_\mathfrak{s} \quad \text{and} \quad \mathrm{ad}^2_{\mathrm{ad}_{X_i}^k X_j}(X_i) \in \mathfrak{e}_\mathfrak{s},$$

for all $i, j = 1, \dots, m$ and $k \geq 2$. Hence $W$ is semigenerating by Theorem 3.9. Since $W$ was arbitrary, we conclude that $\mathfrak{g}$ is semigenerated. □

The following lemma gives a method to construct examples of algebras of type (◊) by taking suitable quotients of product Lie algebras. In Example 4.6, we shall use Lemma 3.13 to give an example of a Lie algebra of type (◊) that is not of type (⋆).

**Lemma 3.13.** *Let $n \in \mathbb{N}$ and let $\mathfrak{g}_l$ be a stratified Lie algebra for each $l \in \{1, \dots, n\}$. Let $\mathfrak{g} := \prod_{l=1}^n \mathfrak{g}_l$ with projections $\pi_l \colon \mathfrak{g} \to \mathfrak{g}_l$ and fix a basis $\{X_1^l, \dots, X_{m_l}^l\}$ for each $\mathfrak{g}_l$. If $\mathfrak{i}$ is a homogeneous ideal of $\mathfrak{g}$ such that*

$$\mathrm{ad}^2_{X_i^l} X_j^l \in \pi_l(\mathfrak{i}) \quad \textit{and} \quad \mathrm{ad}^2_{\mathrm{ad}^k_{X_i^l} X_j^l}(X_i^l) \in \pi_l(\mathfrak{i}), \tag{3.14}$$

*for all $i, j \in \{1, \dots, m_l\}$, $k \geq 2$ and $l \in \{1, \dots, n\}$, then $\mathfrak{g}/\mathfrak{i}$ is of type (◊).*

*Proof.* Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}/\mathfrak{i}$ for which $\mathfrak{h} \cap V_1(\mathfrak{g}/\mathfrak{i})$ has codimension 1 in $V_1(\mathfrak{g}/\mathfrak{i})$ and, denoting by $\pi$ the projection $\pi \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, let $\widetilde{\mathfrak{h}}$ be a subalgebra of $\mathfrak{g}$ for which $\pi(\widetilde{\mathfrak{h}}) = \mathfrak{h}$. Notice first that the set $\{\pi(X_1^l), \dots, \pi(X_{m_l}^l)\}_{l=1}^n$ spans $V_1(\mathfrak{g}/\mathfrak{i})$. Taking (3.2) into account, since $[X_i^l, X_j^k] = 0$ whenever $l \neq k$, to prove semigeneration of $\mathfrak{g}/\mathfrak{i}$ it is enough to check that

(3.15)
$$\mathrm{ad}^2_{\pi(X_i^l)} \pi(X_j^l) = \pi(\mathrm{ad}^2_{X_i^l} X_j^l) \in \mathfrak{h} \quad \text{and} \quad \mathrm{ad}^2_{\mathrm{ad}^k_{\pi(X_i^l)} \pi(X_j^l)} \pi(X_i^l) = \pi(\mathrm{ad}^2_{\mathrm{ad}^k_{X_i^l} X_j^l}(X_i^l)) \in \mathfrak{h},$$

for all $i, j \in \{1, \ldots, m_l\}$, $k \geq 2$ and $l \in \{1, \ldots, n\}$.

To do this, let $\langle \cdot, \cdot \rangle$ be a scalar product on $V_1(\mathfrak{g})$ that makes the basis $\{X_1^l, \ldots, X_{m_l}^l\}_{l=1}^n$ orthonormal. Let $\nu \in V_1(\mathfrak{g}) \setminus \{0\}$ be a vector that is orthogonal to $\widetilde{\mathfrak{h}} \cap V_1(\mathfrak{g})$, i.e., let $\nu \in \widetilde{\mathfrak{h}}^\perp \cap V_1(\mathfrak{g})$. Write $\nu$ as

$$\nu = \sum_{l=1}^{n} \sum_{i=1}^{m_l} a_i^l X_i^l, \quad a_i^l \in \mathbb{R}.$$

Without loss of generality, assume that $a_1^1 = 1$. Then, for every $l = 2, \ldots, n$ and $i = 1, \ldots, m_l$ we have that

$$Y_i^l = X_i^l - a_i^l X_1^1 \in \widetilde{\mathfrak{h}},$$

as now $\langle Y_i^l, \nu \rangle = 0$. Since $X_1^1$ commutes with every $\mathfrak{g}_l$ for which $l \in \{2, \ldots, n\}$, we immediately deduce that

$$\widetilde{\mathfrak{h}} \supset \mathrm{Lie}(\{Y_1^l, \ldots, Y_{m_l}^l\}_{l=2}^n) \cap [\mathfrak{g}, \mathfrak{g}] = \mathrm{Lie}(\{X_1^l, \ldots, X_{m_l}^l\}_{l=2}^n) \cap [\mathfrak{g}, \mathfrak{g}] = \prod_{l=2}^{n} [\mathfrak{g}_l, \mathfrak{g}_l].$$

This proves (3.15) for all $i, j \in \{1, \ldots, m_l\}$, $k \geq 2$ and $l \in \{2, \ldots, n\}$. It is then left to show that each term in (3.14) with $l = 1$ is projected to $\mathfrak{h}$. Let $Z$ be such a term. By (3.14), we have $Z \in \pi_1(\mathfrak{i})$. Since $\mathfrak{i}$ is homogeneous and $Z \in [\mathfrak{g}_1, \mathfrak{g}_1]$, there exists some $\widetilde{Z} \in \prod_{l=2}^n [\mathfrak{g}_l, \mathfrak{g}_l]$ such that $Z + \widetilde{Z} \in \mathfrak{i}$. But since $\prod_{l=2}^n [\mathfrak{g}_l, \mathfrak{g}_l] \subseteq \widetilde{\mathfrak{h}}$, we have that $Z \in \widetilde{\mathfrak{h}} + \mathfrak{i}$ and therefore $\pi(Z) \in \mathfrak{h}$. □

## 4. Some results and examples in low-step algebras

In the following section we collect some lemmata that are valid in Carnot algebras of step at most 4 and which will be used later in Section 5. However, these lemmata can also be useful when proving semigeneration of specific examples in low step. In the end of this section we provide two examples in step 3 that show that, on the one hand, algebras of type ($\Diamond$) form a strictly larger class than algebras of type ($\star$) and, on the other hand, that yet being of type ($\Diamond$) is not a necessary condition for semigeneration.

The following result gives, for step $\leq 4$, equivalent conditions for being a semigenerating horizontal half-space.

**Lemma 4.1.** *Let $\mathfrak{g}$ be a stratified Lie algebra of step at most 4. For each horizontal half-space $W$ in $\mathfrak{g}$, writing $\mathfrak{s} = \mathrm{Cl}(\mathfrak{s}_W)$, the following are equivalent:*

*(i)* $V_2 \subseteq \mathfrak{e}_\mathfrak{s}$*;*
*(ii)* $\mathrm{ad}^2_Y X \in \mathfrak{e}_\mathfrak{s}$ *for every* $X \in \mathfrak{w}_\mathfrak{s} \cap V_1$ *and* $Y \in \mathfrak{e}_\mathfrak{s} \cap V_1$*;*

*(iii)* $\mathrm{ad}_Y^2 X \in \mathfrak{e}_{\mathfrak{s}}$, *for every* $X, Y \in V_1$ *;*
*(iv)* $V_3 \subseteq \mathfrak{e}_{\mathfrak{s}}$*;*
*(v)* $W$ *is semigenerating.*

*Proof.* Implications (v) $\implies$ (iv) $\implies$ (iii) $\implies$ (ii) are immediate. Regarding (ii) $\implies$ (i), recall that $V_2$ is spanned by elements of the form $[Y, Y']$ and $[Y, X]$, where $Y, Y' \in \mathfrak{e}_{\mathfrak{s}} \cap V_1$ and $X \in \mathfrak{w}_{\mathfrak{s}} \cap V_1$. Since, by Lemma 2.8.2, $\mathfrak{e}_{\mathfrak{s}}$ is a Lie algebra, each term $[Y, Y'] \in \mathfrak{e}_{\mathfrak{s}}$ and, by Lemma 2.13, the terms $[X, Y]$ belong to $\mathfrak{e}_{\mathfrak{s}}$.

Let us finally prove (i) $\implies$ (v). We claim that it is enough to show that $\mathrm{ad}_X^k Y \in \mathfrak{e}_{\mathfrak{s}}$ for every $X \in V_1$, $Y \in \mathfrak{e}_{\mathfrak{s}}$ and $k = 1, 2, 3$. Indeed, then by Lemma 2.36 we have $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{e}(\mathfrak{s}) \subseteq \mathfrak{s}$ and $W$ is semigenerating. Now $\mathrm{ad}_X^1 Y = [X, Y] \in \mathfrak{e}_{\mathfrak{s}}$ by (i). Then, as $X \in V_1 \subseteq \mathfrak{w}_{\mathfrak{s}} \cup (-\mathfrak{w}_{\mathfrak{s}})$ and $\mathrm{ad}_{[X,Y]}^2 X \in V_5 = \{0\} \subseteq \mathfrak{e}(\mathfrak{s})$, by Lemma 2.13 we have $[[X, Y], X] = -\mathrm{ad}_X^2 Y \in \mathfrak{e}_{\mathfrak{s}}$. Similarly, $\mathrm{ad}_{\mathrm{ad}_X^2 Y}^2 X = 0$ and therefore $[\mathrm{ad}_X^2 Y, X] = \mathrm{ad}_X^3 Y \in \mathfrak{e}_{\mathfrak{s}}$. So (v) follows. □

**Remark 4.2.** Let us observe what happens to condition (3.10) in low step. Given $k \geq 2$, the vector $\mathrm{ad}_{\mathrm{ad}_X^k Y}^2(X)$ is in $V_{2k+3}$. Hence, if $\mathfrak{g}$ is of step $s$, it is enough to require the conditions (3.2) or (3.3) for all $k \leq (s-3)/2$. In particular, if $s \leq 6$, then a horizontal half-space $W$ of $\mathfrak{g}$ is semigenerated if there exists a basis $\{X_1, \ldots, X_m\}$ of $V_1$ such that

$$\mathrm{ad}_{X_i}^2 X_j \in \mathrm{Lie}(\partial W)$$

for all $i, j = 1, \ldots, m$. Here, we denote by $\mathrm{Lie}(\partial W)$ the Lie subalgebra of $\mathfrak{g}$ generated by the subset $\partial W$.

Similarly to Lemma 2.13, the following lemma gives (in step at most 4) a method to deduce new directions that are contained in the edge of a semigroup generated by a horizontal half-space. Lemma 4.3 below will be used in Example 4.7 and again in the proof of Proposition 5.17.

**Lemma 4.3.** *Let* $\mathfrak{g}$ *be a stratified Lie algebra of step at most 4 and let* $W$ *be a horizontal half-space in* $\mathfrak{g}$*. Let* $\mathfrak{s} := \mathrm{Cl}(\mathfrak{s}_W)$*. If* $Z \in V_2 \cap \mathfrak{e}_{\mathfrak{s}}$*, then* $\mathfrak{I}_{\mathfrak{g}}(Z) \subseteq \mathfrak{e}_{\mathfrak{s}}$*.*

*Proof.* Observe that $V_1 = \mathbb{R}X \oplus \partial W$ for some $X \in W \subseteq \mathfrak{w}_{\mathfrak{s}}$. The ideal $\mathfrak{i} := \mathfrak{I}_{\mathfrak{g}}(Z)$ is graded and, recalling that $Z \in V_2$, we have that its layers are

$$V_1(\mathfrak{i}) = \{0\}, \qquad V_2(\mathfrak{i}) = \mathbb{R}Z, \qquad V_3(\mathfrak{i}) = \mathrm{span}\{[X, Z], [\partial W, Z]\}.$$

$$V_4(\mathfrak{i}) = \mathrm{span}\{[X, [X, Z]], [X, [\partial W, Z]], [\partial W, [X, Z]], [\partial W, [\partial W, Z]]\}.$$

We plan to show that $\mathfrak{I}_{\mathfrak{g}}(Z) \subseteq \mathfrak{e}_{\mathfrak{s}}$, where we recall that $\mathfrak{e}_{\mathfrak{s}}$ is a Lie algebra by Lemma 2.8. On the one hand, by assumption, we have that $Z \in \mathfrak{e}_{\mathfrak{s}}$, so from $\partial W \subseteq \mathfrak{e}_{\mathfrak{s}}$ we get that $[\partial W, Z] \in \mathfrak{e}_{\mathfrak{s}}$. On the other hand, with the aim of applying Lemma 2.13, we observe that, since $Z \in V_2$, we have $\mathrm{ad}_Z^2 X \in V_5 = \{0\} \in \mathfrak{e}_{\mathfrak{s}}$, and hence we also have $[X, Z] \in \mathfrak{e}_{\mathfrak{s}}$. Hence $V_3(\mathfrak{i}) \subseteq \mathfrak{e}_{\mathfrak{s}}$.

We also check that $V_4(\mathfrak{i}) \subseteq \mathfrak{e}_{\mathfrak{s}}$. Since $\mathfrak{e}_{\mathfrak{s}}$ is closed under bracket, we immediately have that $[\partial W, [X, Z]], [\partial W, [\partial W, Z] \subseteq \mathfrak{e}_{\mathfrak{s}}$. Regarding $[X, [X, Z]], [X, [\partial W, Z]]$, we repeat the previous part of the argument of this proof with $Z' \in \{[X, Z]\} \cup [\partial W, Z]$. Indeed, we have that $\mathrm{ad}^2_{Z'} X = 0$ and $Z' \in \mathfrak{e}_{\mathfrak{s}}$. Hence, by Lemma 2.13 we also have $[X, Z'] \in \mathfrak{e}_{\mathfrak{s}}$. □

Next we prove that having a sufficiently large semigenerated subalgebra implies semigeneration. We shall exploit this fact later in the proof of Proposition 5.17.

**Lemma 4.4.** *Let $\mathfrak{g}$ be a stratified Lie algebra of step at most 4. If $\mathfrak{g}$ has a semigenerated proper subalgebra $\mathfrak{h}$ such that $V_3(\mathfrak{g}) \subseteq \mathfrak{h}$, then $\mathfrak{g}$ is semigenerated.*

*Proof.* Let $W$ be a horizontal half-space and let us show that it is semigenerating. Set $H := V_1(\mathfrak{h})$. If $H \subset \partial W$, then

$$V_3(\mathfrak{g}) \subseteq \mathfrak{h} \subseteq \mathrm{Lie}(\partial W) \subseteq \mathfrak{e}(\mathfrak{s}_W).$$

Consequently, by Lemma 4.1 we deduce that $W$ is semigenerating. We then assume that $H \not\subseteq \partial W$. Observe that $\widetilde{W} := H \cap W$ is a horizontal half-space in $H$. Hence, since by assumption $\mathfrak{h}$ is semigenerated, $\widetilde{W}$ is semigenerating within $\mathfrak{h}$. In particular, denoting by $\bar{\mathfrak{s}}^{\mathfrak{h}}_{\widetilde{W}}$ the closure of the (log of the) semigroup generated by $\widetilde{W}$ within $\mathfrak{h}$, we have that $V_3(\mathfrak{h}) \subseteq \bar{\mathfrak{s}}^{\mathfrak{h}}_{\widetilde{W}}$. Since $\mathfrak{h}$ is assumed to contain the third layer of $\mathfrak{g}$, we get the inclusions

$$V_3(\mathfrak{g}) \subseteq V_3(\mathfrak{h}) \subset \bar{\mathfrak{s}}^{\mathfrak{h}}_{\widetilde{W}} \subseteq \bar{\mathfrak{s}}_W,$$

where the last containment is a consequence of the inclusions $\mathfrak{h} \subset \mathfrak{g}$ and $\widetilde{W} \subset W$. Since $V_3(\mathfrak{g})$ is a vector subspace of $\mathfrak{g}$, then by definition of $\mathfrak{e}(\bar{\mathfrak{s}}_W)$ we have $V_3(\mathfrak{g}) \subset \mathfrak{e}(\bar{\mathfrak{s}}_W)$. Hence $W$ is semigenerating again by Lemma 4.1. □

We remark that, actually, the above Lemma 4.4 has the following analogue in algebras of arbitrary step: if $\mathfrak{h}$ is a semigenerated subalgebra of $\mathfrak{g}$ and there exists a basis $\{X_1, \ldots, X_m\}$ of $V_1(\mathfrak{g})$ such that the Diamond-terms (3.2) are in $\mathfrak{h}$, then $\mathfrak{g}$ is semigenerated. The proof is the same, but in the final step one needs to use Theorem 3.9 instead of Lemma 4.1.

**Corollary 4.5** (of Proposition 2.33)**.** *Let $\mathfrak{g}$ be a stratified Lie algebra of step 3. If $\mathfrak{g}$ is not semigenerated, then there exists a quotient algebra of $\mathfrak{g}$ that is trimmed and not semigenerated.*

*Proof.* By Proposition 2.33, there exists a quotient algebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ for which $\mathcal{Z}(\hat{\mathfrak{g}}) \cap V_1(\hat{\mathfrak{g}}) = \mathcal{Z}(\hat{\mathfrak{g}}) \cap V_2(\hat{\mathfrak{g}}) = \{0\}$ and $\dim(\mathcal{Z}(\hat{\mathfrak{g}}) \cap V_3(\hat{\mathfrak{g}})) \leq 1$. Since the center of a stratified Lie algebra is non-trivial and homogeneous (see (2.3)), we deduce that $\dim \mathcal{Z}(\hat{\mathfrak{g}}) = \dim \mathcal{Z}(\hat{\mathfrak{g}}) \cap V_3(\hat{\mathfrak{g}}) = 1$, proving that $\hat{\mathfrak{g}}$ is trimmed. □

In the rest of this section we provide some examples. We first show a 7-dimensional Lie algebra of step 3 that is of type ($\Diamond$) but that is not of type ($\star$), see Example 4.6. Then we provide a 6-dimensional Lie algebra that is semigenerated but not of type ($\Diamond$), see Example 4.7.

**Example 4.6.** Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two copies of the four-dimensional Engel algebra $\mathbb{E}\mathrm{n}^1$ and consider their product Lie algebra $\mathfrak{h}_1 \times \mathfrak{h}_2$. Denoting by $Z_1$ and $Z_2$ the generators of $V_3(\mathfrak{h}_1)$ and $V_3(\mathfrak{h}_2)$, respectively, and identifying $\mathfrak{h}_1$ and $\mathfrak{h}_2$ with the respective subalgebras of $\mathfrak{h}_1 \times \mathfrak{h}_2$, we have that $V_3(\mathfrak{h}_1 \times \mathfrak{h}_2) = \mathrm{span}\{Z_1, Z_2\}$. Then $\mathbb{R}(Z_1 - Z_2)$ is an ideal of $\mathfrak{h}_1 \times \mathfrak{h}_2$ and the quotient algebra

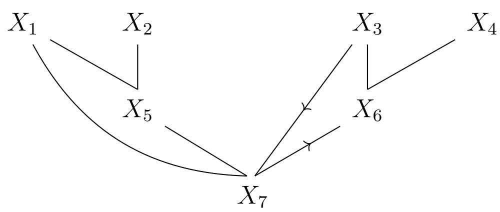
$$\mathfrak{g} \coloneqq (\mathfrak{h}_1 \times \mathfrak{h}_2)/\mathbb{R}(Z_1 - Z_2)$$

is a 7-dimensional (trimmed) stratified Lie algebra of step 3 (which in the Gong's classification [Gon98, p. 57] is denoted by (137A)). We claim that $\mathfrak{g}$ is of type ($\Diamond$) but it is not of type ($\star$). Indeed, the fact that $\mathfrak{g}$ is of type ($\Diamond$) follows immediately from Lemma 3.13 with $\mathfrak{i} = \mathbb{R}(Z_1 - Z_2)$, as now $\pi_l(\mathfrak{i}) = V_3(\mathfrak{h}_l)$ for both $l = 1$ and $l = 2$.

We argue next that $\mathfrak{g}$ is not of type ($\star$). Let $\{X_1, \ldots, X_7\}$ be a basis of $\mathfrak{g}$ for which $\{X_1, \ldots, X_4\}$ is a basis of $V_1(\mathfrak{g})$ and the only nonzero brackets are $[X_1, X_2] = X_5, [X_3, X_4] = X_6$ and $[X_1, [X_1, X_2]] = [X_3, [X_3, X_4]] = X_7$, as presented in the diagram below. Then for a vector $Y \coloneqq \sum_{i=1}^4 a_i X_i \in V_1$ we have, for instance, that

$$\mathrm{ad}_Y^2(X_2) = a_1^2 X_7.$$

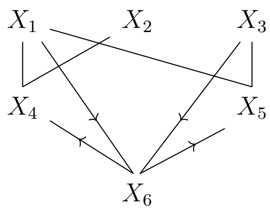
In particular, if $Y$ is such that $\mathrm{ad}_Y^2(V_1) = 0$, then $a_1 = 0$. Consequently, the set of vectors $Y \in V_1$ satisfying $\mathrm{ad}_Y^2(V_1) = 0$ is contained in a 3-dimensional subspace of $V_1(\mathfrak{g})$. We conclude by Remark 3.8 that $\mathfrak{g}$ is not of type ($\star$).




**Example 4.7.** Let $\mathfrak{g}$ be the 6-dimensional step-3 Lie algebra ($N_{6,2,6}$ in [Gon98, p. 33]), where the only non-trivial brackets are given by

$$[X_1, X_2] = X_4,\ [X_1, X_3] = X_5,\ [X_1, X_4] = [X_3, X_5] = X_6.$$

The Lie brackets can be described by the following diagram:




This Carnot algebra $\mathfrak{g}$ is semigenerated but it is not of type ($\Diamond$). Indeed, to prove that it is semigenerated, let $W \subseteq \mathfrak{g}$ be a horizontal half-space and let us show that $V_3 = \mathbb{R}X_6 \subseteq \mathfrak{e}_{\mathfrak{s}}$, where $\mathfrak{s} := \mathrm{Cl}(\mathfrak{s}_W)$. This would show that $W$ is semigenerating by Lemma 4.1.

Suppose first that $X_1 \notin \partial W$. The rank of $\mathfrak{g}$ is 3, so $\partial W$ has dimension 2. Then there exist some $a, b \in \mathbb{R}$ such that $Y_2 := X_2 - aX_1$ and $Y_3 := X_3 - bX_1$ form a basis for $\partial W$. Since $\partial W \subseteq \mathfrak{e}_{\mathfrak{s}}$ and $\mathfrak{e}_{\mathfrak{s}}$ is a Lie algebra by Lemma 2.8, we obtain

$$[Y_2, Y_3] = [X_2 - aX_1, X_3 - bX_1] = bX_4 - aX_5 \in \mathfrak{e}_{\mathfrak{s}}.$$

If $a \neq 0$ or $b \neq 0$, we get

$$V_3 \subseteq \mathfrak{I}([Y_2, Y_3]) \subseteq \mathfrak{e}_{\mathfrak{s}},$$

where the last inclusion comes from Lemma 4.3. If instead $a = b = 0$, then $X_2 = Y_2 \in \mathfrak{e}_{\mathfrak{s}}$. Since $\mathrm{ad}^2_{X_2} X_1 = 0$, by Lemma 2.13 we get that $[X_1, X_2] = X_4 \in \mathfrak{e}_{\mathfrak{s}}$. Again, since $V_3 \subseteq \mathfrak{I}(X_4)$, by Lemma 4.3 we get that $V_3 \subseteq \mathfrak{e}_{\mathfrak{s}}$.

The cases $X_2 \notin \partial W$ and $X_3 \notin \partial W$ are easier: if $X_2 \notin \partial W$ we find, like above, some $a, b \in \mathbb{R}$ such that $\partial W = \mathrm{span}\{X_1 - aX_2, X_3 - bX_2\}$. It then suffices to notice that $X_6 \in \mathrm{Lie}(X_1 - aX_2, X_3 - bX_2) \subseteq \mathfrak{e}_{\mathfrak{s}}$, for all choices of $a, b \in \mathbb{R}$. Similarly, the case $X_3 \notin \partial W$ follows from the fact that $X_6 \in \mathrm{Lie}(X_1 - aX_3, X_2 - bX_3)$ for all $a, b \in \mathbb{R}$. We conclude that $\mathfrak{g}$ is semigenerated.

Finally, to justify that $\mathfrak{g}$ is not of type ($\Diamond$), observe that the span of $X_2$ and $X_3$ is an abelian stratified subalgebra of $\mathfrak{g}$. If $\mathfrak{g}$ were type of ($\Diamond$), then by Remark 3.7 it would be of type ($\star$). However, similarly to Example 4.6, we have for an arbitrary element $Y = a_1X_1 + a_2X_2 + a_3X_3 \in V_1$ that

$$\mathrm{ad}^2_Y(X_2) = a_1^2 X_6.$$

Hence vectors $Y \in V_1$ for which $\mathrm{ad}^2_Y(V_1) = 0$ must satisfy $a_1 = 0$, which proves the non-existence of type ($\star$)-basis by Remark 3.8.

## 5. Engel-type algebras

In the rest of the paper we concentrate on a family of Carnot algebras that we call of Engel type. These algebras can be constructed through an iterative process from the classical 4-dimensional Engel algebra. Similarly to the Engel algebra, every Engel-type algebra is trimmed and non-semigenerated, as we shall show in Propositions 5.11

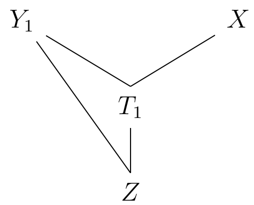
and 5.13. A more subtle result is that, at least in step 3, the Engel-type algebras are the only Carnot algebras with these properties. For this last part, see Proposition 5.17. The proof of Theorem 1.2 will then be straightforward.

### 5.1. **Definition and properties.**

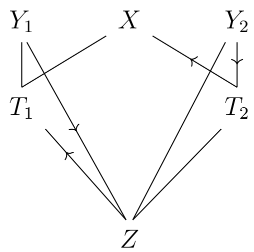
**Definition 5.1** (Engel-type algebra $\mathbb{E}\mathrm{n}^n$)**.** For each $n \in \mathbb{N}$, we denote by $\mathbb{E}\mathrm{n}^n$ and call it the *n-th Engel-type algebra* the $2(n+1)$-dimensional Lie algebra (of step 3 and rank $n+1$) with basis $\{X, Y_i, T_i, Z\}_{i=1}^n$, where the only non-trivial brackets are given by

$$[Y_i, X] = T_i \quad \text{and} \quad [Y_i, T_i] = Z \quad \forall i \in \{1, \dots, n\}. \tag{5.2}$$

The first two Engel-type algebras are the following. The first is $\mathbb{E}\mathrm{n}^1$, and it is commonly known as Engel algebra, see [BL13], and we represent it by the diagram below.




The second Engel-type algebra $\mathbb{E}\mathrm{n}^2$ is the six-dimensional algebra $N_{6,3,1a}$ in [Gon98, p. 135] and its diagram is presented below.




Each Engel-type algebra is a Lie algebra (see the simple verification in Remark 5.3) and admits a step-3 stratification:

$$\begin{aligned} V_1(\mathbb{E}\mathrm{n}^n) &:= \operatorname{span}\{X, Y_1, \dots, Y_n\}, \\ V_2(\mathbb{E}\mathrm{n}^n) &:= \operatorname{span}\{T_1, \dots, T_n\}, \\ V_3(\mathbb{E}\mathrm{n}^n) &:= \operatorname{span}\{Z\}. \end{aligned}$$

We shall also give another equivalent definition for $\mathbb{E}\mathrm{n}^n$ in Proposition 5.9. We first list some properties of such Lie algebras.

**Remark 5.3.** Each Engel-type algebra given by the brackets (5.2) is indeed a Lie algebra. Namely, let us verify that the Jacobi identity is satisfied. Since the basis has a natural stratification, it is enough to check the identity for triples in the set

$$\{X, Y_1, \dots, Y_n\}.$$

Hence, we just consider the case $X, Y_i, Y_j$ or $Y_i, Y_j, Y_k$. In the first case, we have

$$\begin{aligned}[X,[Y_i,Y_j]] + [Y_i,[Y_j,X]] + [Y_j,[X,Y_i]] &= 0 + [Y_i, T_j,] + [Y_j, -T_i] \\ &= \delta_{ij} Z - \delta_{ij} Z = 0.\end{aligned}$$

In the second case, we have

$$[Y_i,[Y_j,Y_k]] + [Y_j,[Y_k,Y_i]] + [Y_k,[Y_i,Y_j]] = 0 + 0 + 0 = 0.$$

**Lemma 5.4** (Properties of Engel-type algebras)**.** *The $n$-th Engel-type algebra $\mathbb{E}\mathrm{n}^n$ with a basis satisfying* (5.2) *has the following properties.*

*(i) If $n \geq 2$, then $\operatorname{span}\{Y_1, \dots, Y_n\}$ is the unique abelian $n$-dimensional subspace of $V_1(\mathbb{E}\mathrm{n}^n)$;*
*(ii) the line $\mathbb{R}X$ is the unique horizontal line satisfying $[\mathbb{R}X, V_2] = \{0\}$;*
*(iii) for every nonzero $Y \in \operatorname{span}\{Y_1, \dots, Y_n\}$, we have*

$$\operatorname{ad}_Y^2(V_1) = \mathbb{R}Z.$$

*Proof.* (i) Obviously, the space $\operatorname{span}(Y_1, \dots, Y_n)$ is an abelian $n$-space. Vice versa, let $H$ be an $n$-dimensional subspace of $V_1(\mathbb{E}\mathrm{n}^n)$ such that $H \neq \operatorname{span}(Y_1, \dots, Y_n)$. Then there exists $\nu \in H$ of the form

$$\nu := X + \sum_{i=1}^{n} a_i Y_i, \qquad \text{with } a_i \in \mathbb{R} \text{ for } i = 1, \dots, n,$$

and a nonzero $Y \in H \cap \operatorname{span}(Y_1, \dots, Y_n)$. Writing $Y = \sum_{i=1}^{n} b_i Y_i$ for some $b_i \in \mathbb{R}$, we obtain

$$[Y, \nu] = \sum_i b_i T_i \neq 0,$$

proving that $H$ is nonabelian.

(ii) Let now

$$\nu := aX + \sum_{i=1}^{n} a_i Y_i, \quad a, a_i \in \mathbb{R}\ \forall\, i = 1, \dots, n,$$

where $a_k \neq 0$ for some $k \in \{1, \dots, n\}$. Then

$$[\nu, T_k] = a_k Z \neq 0,$$

which shows that $[\nu, V_2] \neq 0$ if $\nu \notin \mathbb{R}X$.

(iii) Let again $Y = \sum_{i=1}^n b_i Y_i$ for some real numbers $b_i$ not all identically zero. Since

$$\mathrm{ad}_Y^2(\mathbb{R}X) = \mathbb{R}\ \mathrm{ad}_Y^2 X = \mathbb{R}\sum_{i=1}^n b_i^2 Z = \mathbb{R}Z$$

and also $V_3(\mathbb{E}\mathfrak{n}^n) = \mathbb{R}Z$, we get

$$\mathbb{R}Z = \mathrm{ad}_Y^2(\mathbb{R}X) \subseteq \mathrm{ad}_Y^2(V_1) \subseteq \mathbb{R}Z. \qquad \square$$

We provide next the automorphism group of the Engel-type algebras, which will be used later in the proof of Lemma 5.7 where we characterize all stratified subalgebras of the Engel-type algebras. For the automorphism group of the Engel algebra, we refer to [BL13, Lemma 2.3].

**Lemma 5.5.** *Consider the basis* $\{X, Y_1, \ldots, Y_n\}$ *of* $V_1(\mathbb{E}\mathfrak{n}^n)$, $n \geq 2$, *defined in* (5.2). *Fix a scalar product* $\langle\cdot,\cdot\rangle$ *on* $V_1(\mathbb{E}\mathfrak{n}^n)$ *that makes* $\{Y_1, \ldots, Y_n\}$ *orthonormal. Then every linear transformation on* $V_1(\mathbb{E}\mathfrak{n}^n)$ *that in the basis* $\{X, Y_1, \ldots, Y_n\}$ *is given by the block matrix*

$$\begin{pmatrix} a & 0 \\ 0 & bA \end{pmatrix}, \quad a, b \in \mathbb{R}\setminus\{0\} \quad \text{and} \quad A \in O(n),$$

*induces a Lie algebra automorphism of* $\mathbb{E}\mathfrak{n}^n$. *Moreover, every automorphism of* $\mathbb{E}\mathfrak{n}^n$ *is induced by such a transformation on* $V_1(\mathbb{E}\mathfrak{n}^n)$.

*Proof.* By Lemma 5.4.(i) and (ii), every $\Phi \in \mathrm{Aut}(\mathbb{E}\mathfrak{n}^n)$ must fix the subspaces $\mathrm{span}\{Y_1, \ldots, Y_n\}$ and $\mathbb{R}X$. Moreover, notice that any linear map $\Phi$ on $V_1(\mathbb{E}\mathfrak{n}^n)$ fixing these subspaces satisfies, for some $a \neq 0$, the two equalities:

$$[\Phi(Y_i), \Phi(X)] = [\Phi(Y_i), aX] = \sum_{k=1}^n a\langle\Phi(Y_i), Y_k\rangle T_k \quad \text{and}$$

$$[\Phi(Y_i), T_k] = [\sum_{\ell=1}^n \langle\Phi(Y_i), Y_\ell\rangle Y_\ell, T_k] = \langle\Phi(Y_i), Y_k\rangle Z.$$

Therefore, using again that $\{Y_j\}_j$ are orthonormal, we deduce that

$$(5.6) \qquad [\Phi(Y_i), [\Phi(Y_j), \Phi(X)]] = \sum_{k=1}^n a\langle\Phi(Y_j), Y_k\rangle\langle\Phi(Y_i), Y_k\rangle Z = a\langle\Phi(Y_i), \Phi(Y_j)\rangle Z.$$

Recall that the basis vectors of $\mathbb{E}\mathfrak{n}^n$ satisfy $[Y_i, [Y_j, X]] = \delta_{ij} Z$. Therefore, the map $\Phi$ induces a Lie algebra automorphism of $\mathbb{E}\mathfrak{n}^n$ if and only if there exists some $b \neq 0$ such that

$$[\Phi(Y_i), [\Phi(Y_j), \Phi(X)]] = b\,\delta_{ij} Z, \quad \forall i, j \in \{1, \ldots, n\}$$

According to (5.6), this is equivalent to saying that the map $\Phi$ is, up to scaling, an orthogonal transformation on $\mathrm{span}\{Y_1, \ldots, Y_n\}$ . $\square$

For a good understanding of the rest of this section, we stress that we say that a subalgebra of a stratified Lie algebra is stratified if it is homogeneous and stratified with respect to the induced grading.

**Lemma 5.7.** *Let $1 \leq k \leq n$. If $\mathfrak{h}$ is a stratified rank-$k$ subalgebra of the $n$-th Engel-type algebra $\mathbb{E}\mathfrak{n}^n$, then either $\mathfrak{h}$ is abelian or it is isomorphic to $\mathbb{E}\mathfrak{n}^{k-1}$.*

*Proof.* If $n = 1$, the claim is trivially true. Assume then that $n \geq 2$ and let $\{X, Y_1, \ldots, Y_n\}$ be a basis of $V_1(\mathbb{E}\mathfrak{n}^n)$ as in (5.2). We start by proving the case $k = n$. Assume first that $\mathfrak{h}$ is the subalgebra generated by $\{X + aY_n, Y_1, \ldots, Y_{n-1}\}$ for some $a \in \mathbb{R}$. Observe that then $\mathfrak{h}$ is isomorphic to $\mathbb{E}\mathfrak{n}^{n-1}$ for every value of $a \in \mathbb{R}$ since, for each $i = 1, \ldots, n-1$, we have that

$$[Y_i, X + aY_n] = [Y_i, X] = T_i \quad \text{and} \quad [X + aY_n, T_i] = [X, T_i] = 0.$$

Recall that, by Lemma 5.4.i, the subspace $\text{span}\{Y_1, \ldots, Y_n\}$ is the unique abelian stratified subalgebra of $\mathbb{E}\mathfrak{n}^n$. Since every $n$-dimensional subspace of $V_1(\mathbb{E}\mathfrak{n}^n)$ which is not equal to $\text{span}\{Y_1, \ldots, Y_n\}$ can be realized from some subspace $\text{span}\{X + aY_n, Y_1, \ldots, Y_{n-1}\}$, $a \in \mathbb{R}$, by a rotation of $V_1(\mathbb{E}\mathfrak{n}^n)$ around the $X$-axis, we infer by Lemma 5.5 that any subalgebra generated by such subspace is isomorphic to $\mathbb{E}\mathfrak{n}^{n-1}$.

Regarding the case $k < n$, fix a non-abelian stratified rank-$k$ subalgebra $\mathfrak{h}_k$ of $\mathbb{E}\mathfrak{n}^n$. Then we find a filtration $\mathfrak{h}_k \subset \mathfrak{h}_{k+1} \subset \cdots \subset \mathfrak{h}_n \subset \mathbb{E}\mathfrak{n}^n$ of non-abelian stratified rank-$l$ subalgebras $\mathfrak{h}_l$, $l = k+1, \ldots, n$. The claim follows now by the first part of this proof. □

There are plenty of Lie algebras of arbitrarily large step whose first three layers coincide with the first Engel-type algebra, for example, the filiform algebras. The same phenomenon does not happen for the other Engel-type algebras. Since we need this latter fact in the proof of Proposition 5.9, we clarify such a phenomenon in the next remark.

**Remark 5.8.** For each $n \geq 2$, the Lie algebra $\mathbb{E}\mathfrak{n}^n$ cannot be 'prolonged' in the following sense: if $\mathfrak{g}$ is a stratified Lie algebra for which $\mathfrak{g}/\mathfrak{g}^{(4)}$ is isomorphic to $\mathbb{E}\mathfrak{n}^n$ for some $n \geq 2$, where $\mathfrak{g}^{(4)} = V_4 \oplus \cdots \oplus V_s$, then $\mathfrak{g} = \mathbb{E}\mathfrak{n}^n$.

*Proof.* Let $\{X, Y_i\}_{i=1}^n$, $\{T_i\}_{i=1}^n$ and $\{Z\}$ be bases of $V_1(\mathfrak{g})$, $V_2(\mathfrak{g})$ and $V_3(\mathfrak{g})$, respectively, satisfying the bracket relations (5.2) modulo $\mathfrak{g}^{(4)}$. Observe that, being $\mathfrak{g}$ stratified, the subspace $V_4$ is spanned by elements of the form $[\nu, Z]$, where $\nu \in \{X, Y_i\}_{i=1}^n$. We need to show that $V_4 = \{0\}$. Indeed, by the Jacobi identity, we have

$$[X, Z] = [X, [Y_i, [Y_i, X]]] = -[Y_i, [[Y_i, X], X]] - [[Y_i, X], [X, Y_i]] = 0,$$

where we deduced that $[Y_i, [[Y_i, X], X]] = 0$, since $[X, V_2] = \{0\}$ by Lemma 5.4.ii. Moreover, for every $j = 1, \ldots, n$ and $i \neq j$ (which exists since $n \geq 2$) one has

$$[Y_j, Z] = [Y_j, [Y_i, [Y_i, X]]] = -[Y_i, [[Y_i, X], Y_j]] - [[Y_i, X], [Y_j, Y_i]] = 0,$$

where we used that $[Y_j,[Y_i,X]] = [Y_j,T_i] = 0$ and that $[Y_j,Y_i] = 0$. This proves that $V_4 = \{0\}$, in which case also $\mathfrak{g}^{(4)} = \{0\}$ and hence $\mathfrak{g} \cong \mathfrak{g}/\mathfrak{g}^{(4)} \cong \mathbb{E}\mathrm{n}^n$. $\square$

The Engel-type algebras have the following equivalent definition using induction.

**Proposition 5.9** (A characterization of Engel-type algebras). *Let $\mathfrak{g}$ be a stratified Lie algebra of rank $n+1 \geq 4$. Then $\mathfrak{g}$ is isomorphic to $\mathbb{E}\mathrm{n}^n$ if and only if $\mathfrak{g}$ has a unique abelian stratified subalgebra of rank $n$ and every other stratified subalgebra of rank $n$ is isomorphic to $\mathbb{E}\mathrm{n}^{n-1}$. Moreover, this characterization holds for rank $n+1=3$ if in addition* $\dim V_3(\mathfrak{g}) = 1$.

*Proof.* One direction is proven in Lemmata 5.4.i and 5.7. Regarding the other direction, let $\mathfrak{g}$ be a rank $n+1$ stratified Lie algebra, with $n+1 \geq 3$, which has a unique abelian stratified subalgebra $\mathfrak{h}_0$ of rank $n$ and every other rank-$n$ stratified subalgebra is isomorphic to $\mathbb{E}\mathrm{n}^{n-1}$.

Our first aim is to show that the condition $\dim V_3(\mathfrak{g}) = 1$ always holds as long as $n+1 \geq 4$. We start by claiming the following property:

(5.10)

$$\text{If } n+1 \geq 4 \text{ and } \mathfrak{l} \subset V_1(\mathfrak{g}) \text{ is an } (n-1)\text{-dimensional abelian subspace of } V_1(\mathfrak{g}), \text{ then } \mathfrak{l} \subset \mathfrak{h}_0.$$

Indeed, let $\nu \in V_1(\mathfrak{g}) \setminus \mathfrak{l}$. On the one hand, if $\mathfrak{l} \oplus \mathbb{R}\nu$ is an abelian subalgebra of $\mathfrak{g}$, then $\mathfrak{l} \oplus \mathbb{R}\nu = \mathfrak{h}_0$ by uniqueness of $\mathfrak{h}_0$ and so $\mathfrak{l} \subset \mathfrak{h}_0$. On the other hand, if $\mathfrak{l} \oplus \mathbb{R}\nu$ generates a nonabelian subalgebra, then $\mathfrak{l} \oplus \mathbb{R}\nu$ is isomorphic to $V_1(\mathbb{E}\mathrm{n}^{n-1})$, where $n-1 \geq 2$. Since $\mathfrak{h}_0 \cap (\mathfrak{l} \oplus \mathbb{R}\nu)$ is an abelian $(n-1)$-dimensional subspace of $V_1(\mathbb{E}\mathrm{n}^{n-1})$, we deduce that $\mathfrak{h}_0 \cap (\mathfrak{l} \oplus \mathbb{R}\nu) = \mathfrak{l}$ by uniqueness of $(n-1)$-dimensional abelian subspaces of $V_1(\mathbb{E}\mathrm{n}^{n-1})$ because $n-1 \geq 2$ (see Lemma 5.4.i). Then again $\mathfrak{l} \subset \mathfrak{h}_0$ and (5.10) is proven.

Recall that

$$V_3(\mathfrak{g}) = \operatorname{span}\{[X_1,[X_2,X_3]] \mid X_i \in V_1,\ i = 1,2,3\}$$

as $\mathfrak{g}$ is stratified. We are going to show that vectors $[X_1,[X_2,X_3]]$ and $[\widetilde{X}_1,[\widetilde{X}_2,\widetilde{X}_3]]$ are linearly dependent, for every choice of vectors $X_i, \widetilde{X}_i \in V_1$, $i = 1,2,3$. So let $X_i, \widetilde{X}_i \in V_1$ for $i = 1,2,3$ be such that $[X_1,[X_2,X_3]]$ and $[\widetilde{X}_1,[\widetilde{X}_2,\widetilde{X}_3]]$ are nonzero and let $\mathfrak{h}_1 \supseteq \{X_1,X_2,X_3\}$ and $\mathfrak{h}_2 \supseteq \{\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3\}$ be rank-$n$ subalgebras of $\mathfrak{g}$. Since $\mathfrak{h}_1$ and $\mathfrak{h}_2$ have nonzero third layers, they are isomorphic to $\mathbb{E}\mathrm{n}^{n-1}$. We may assume that $V_1(\mathfrak{h}_1) \neq V_1(\mathfrak{h}_2)$, since otherwise $[X_1,[X_2,X_3]]$ and $[\widetilde{X}_1,[\widetilde{X}_2,\widetilde{X}_3]]$ are linearly dependent. Hence, being it the intersection of two different hyperplanes, the space $V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2)$ is a codimension 2 subspace of $V_1(\mathfrak{g})$, i.e., it has dimension $n-1$.

To prove that $\dim V_3(\mathfrak{g}) = 1$, we have to show that $V_3(\mathfrak{h}_1) = V_3(\mathfrak{h}_2)$, for all such $\mathfrak{h}_1$ and $\mathfrak{h}_2$ as above. The proof for the latter fact is divided into two cases depending on whether $V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2)$ is closed under brackets (or equivalently, it is an abelian subalgebra) or not. Assume first that $V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2)$ forms an abelian subalgebra. Then by claim (5.10) we have that $V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2) \subseteq \mathfrak{h}_0$. Let us fix a basis $\{Z_1, \ldots, Z_{n-1}\}$ for $V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2)$ and let also $Z_n \in \mathfrak{h}_0$ and $X \in V_1(\mathfrak{g})$ be such that $\{Z_1, \ldots, Z_n, X\}$

is a basis of $V_1(\mathfrak{g})$. Fix next $Y_i \in V_1(\mathfrak{h}_i) \setminus (V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2))$ for $i = 1, 2$ and write it in terms of this basis as

$$Y_i = a_i X + \sum_{j=1}^{n} b_j^i Z_j$$

for some $a_i, b_j^i \in \mathbb{R}$. Notice that, since $\mathfrak{h}_i$ is not abelian, we have $a_i \neq 0$. Since now $\{Y_i, Z_1, \dots, Z_{n-1}\}$ is a basis of $V_1(\mathfrak{h}_i)$ for $i = 1, 2$ and since $\mathfrak{h}_0 = \operatorname{span}\{Z_1, \dots, Z_n\}$ is abelian, we obtain

$$V_3(\mathfrak{h}_1) = \operatorname{ad}^2_{Z_1}(V_1(\mathfrak{h}_1)) = \operatorname{ad}^2_{Z_1}(\mathbb{R}Y_1) = \operatorname{ad}^2_{Z_1}(\mathbb{R}X) = \operatorname{ad}^2_{Z_1}(\mathbb{R}Y_2) = \operatorname{ad}^2_{Z_1}(V_1(\mathfrak{h}_2)) = V_3(\mathfrak{h}_2),$$

where the first and the last equality follow from Lemma 5.4 (iii). Since the third layers of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are one dimensional, we deduce that $[X_1, [X_2, X_3]]$ and $[\widetilde{X}_1, [\widetilde{X}_2, \widetilde{X}_3]]$ are linearly dependent.

Assume instead that $V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2)$ is not a subalgebra. Then, by Lemma 5.7, the Lie algebra generated by $V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2)$ is isomorphic to $\mathbb{E}\mathfrak{n}^{n-2}$. In particular, it has step 3. Exploiting again the fact $\dim V_3(\mathfrak{h}_1) = \dim V_3(\mathfrak{h}_2) = 1$, we get that

$$V_3(\mathfrak{h}_1) = V_3(\operatorname{Lie}(V_1(\mathfrak{h}_1) \cap V_1(\mathfrak{h}_2))) = V_3(\mathfrak{h}_2).$$

Therefore $[X_1, [X_2, X_3]]$ and $[\widetilde{X}_1, [\widetilde{X}_2, \widetilde{X}_3]]$ are linearly dependent as in the previous case. This concludes the proof for the fact that $\dim V_3(\mathfrak{g}) = 1$ when $n + 1 \geq 4$. Hence, in what follows we may assume that $V_3(\mathfrak{g})$ is one dimensional, and that $n + 1 \geq 3$.

In the rest of this proof we are going to construct a basis of $\mathfrak{g}$ that satisfies the defining commutator relations (5.2) of the Engel-type algebra $\mathbb{E}\mathfrak{n}^n$. Let $\mathfrak{h} \cong \mathbb{E}\mathfrak{n}^{n-1}$ be some nonabelian rank-$n$ subalgebra of $\mathfrak{g}$ and let $\{X, Y_i, T_i, Z\}_{i=1}^{n-1}$ be a basis of $\mathfrak{h}$ satisfying relations (5.2). We aim to find a vector $Y_n \in V_1$ which together with $T_n := [Y_n, X]$ completes $\{X, Y_i, T_i, Z\}_{i=1}^{n-1}$ to the defining basis of $\mathbb{E}\mathfrak{n}^n$. Since $\mathbb{E}\mathfrak{n}^n$ cannot be prolonged (see Remark 5.8), this is enough to prove that $\mathfrak{g}$ is isomorphic to $\mathbb{E}\mathfrak{n}^n$. Notice that $\mathfrak{h}_0 \cap \mathfrak{h} = \operatorname{span}\{Y_1, \dots, Y_{n-1}\}$ since $\operatorname{span}\{Y_1, \dots, Y_{n-1}\}$ is the unique abelian subspace of $V_1(\mathfrak{h})$ by Lemma 5.4 (i). Moreover, $V_3(\mathfrak{g}) = \mathbb{R}Z$ as $V_3(\mathfrak{g})$ is one dimensional. Fix next $\hat{Y}_n \in \mathfrak{h}_0 \setminus \mathfrak{h}$ and write

$$Y_n := a\left(\hat{Y}_n + \sum_{i=1}^{n-1} a_i Y_i\right) \in \mathfrak{h}_0,$$

where $a, a_i \in \mathbb{R}$, $i = 1, \dots, n-1$, are values to be determined later. Now whenever $a \neq 0$ we have that $\operatorname{span}\{Y_1, \dots, Y_n\} = \mathfrak{h}_0$ is abelian and $\{Y_1, \dots, Y_n, X\}$ is a basis of $V_1(\mathfrak{g})$. To conclude the proof of the proposition, we claim that it suffices to show that

(i) there exist $a_i \in \mathbb{R}$, $i = 1, \dots, n-1$, such that $[Y_n, T_i] = 0$ for all $i = 1, \dots, n-1$;
(ii) there exists $a \in \mathbb{R}$ such that $[Y_n, [Y_n, X]] = Z$;
(iii) with the above choices of $a_i, a \in \mathbb{R}$, the vector $T_n := [Y_n, X]$ is linearly independent of $T_1, \dots, T_{n-1}$.

Indeed, we stress again that, by Remark 5.8, the Lie algebra $\mathbb{E}\mathrm{n}^n$ cannot be prolonged and hence we indeed have that the set $\{X, Y_i, T_i, Z\}_{i=1}^{n-1}$ is a basis of a step-3 Lie algebra isomorphic to $\mathbb{E}\mathrm{n}^n$.

To show (i), observe that for every $i = 1, \ldots, n-1$,

$$[Y_n, T_i] = a([\hat{Y}_n, T_i] + a_i Z).$$

Since $V_3(\mathfrak{g})$ is one dimensional, the two vectors $[\hat{Y}_n, T_i]$ and $Z$ are linearly dependent. Hence for every $i = 1, \ldots, n-1$ there exists $a_i \in \mathbb{R}$ such that $[Y_n, T_i] = 0$, which proves (i).

Regarding (ii), let then $\mathfrak{h}' := \mathrm{Lie}(Y_2, \ldots, Y_n, X) \cong \mathbb{E}\mathrm{n}^{n-1}$. Since $\mathrm{span}\{Y_2, \ldots, Y_n\}$ is again the unique abelian $(n-1)$-dimensional subspace of $V_1(\mathfrak{h}')$, it holds

$$[Y_n, [Y_n, \mathbb{R}X]] = [Y_n, [Y_n, V_1(\mathfrak{h}')]].$$

As $[Y_n, [Y_n, V_1(\mathfrak{h}')]] \neq 0$ by Lemma 5.4 (iii) and since $V_3(\mathfrak{g}) = \mathbb{R}Z$, we may choose $a \in \mathbb{R}$ such that

$$[Y_n, [Y_n, X]] = Z.$$

Thus (ii) is proven.

Regarding (iii), it is enough to notice that from (i) we have

$$[Y_n, \sum_{i=1}^{n-1} b_i T_i] = \sum_{i=1}^{n-1} b_i [Y_n, T_i] = 0,$$

for every choice of $b_i \in \mathbb{R}$, $i = 1, \ldots, n-1$, whereas from (ii) we have

$$[Y_n, T_n] = [Y_n, [Y_n, X]] \neq 0.$$

This finishes the proof of (iii) and of the proposition. □

Next we show that each Lie algebra $\mathbb{E}\mathrm{n}^n$ is trimmed and that a horizontal half-space $W$ is, for $n \geq 2$, not semigenerating if and only if $\partial W$ is the abelian codimension 1 subspace of $V_1$. Recall our definition of trimmed algebra from Proposition 2.31.

**Proposition 5.11.** *Every Engel-type algebra is trimmed.*

*Proof.* Let $n \in \mathbb{N}$ and let $\mathbb{E}\mathrm{n}^n$ be the Engel-type algebra with the defining basis (5.2). We shall use the third definition of trimmed Lie algebra in Proposition 2.31. Recall, see (2.3), that in every step-$s$ stratified Lie algebra $\mathfrak{g}$, we have that the center $\mathcal{Z}(\mathfrak{g})$ is graded and $V_s \subseteq \mathcal{Z}(\mathfrak{g})$. Hence, noticing that $\dim V_3 = 1$, it suffices to show that $\mathcal{Z}(\mathbb{E}\mathrm{n}^n) \cap V_1 = \mathcal{Z}(\mathbb{E}\mathrm{n}^n) \cap V_2 = \{0\}$. By Lemma 5.4.ii, we have $[Y, V_2] \neq 0$ for every $Y \in V_1 \setminus \mathbb{R}X$. Since $X$ is not in the center either, this proves that $\mathcal{Z}(\mathbb{E}\mathrm{n}^n) \cap V_1 = \{0\}$. To show that $\mathcal{Z}(\mathbb{E}\mathrm{n}^n) \cap V_2 = \{0\}$, let $T \in V_2$ be nonzero and write

$$T := \sum_{i=1}^{n} a_i T_i, \quad a_i \in \mathbb{R}\ \forall\, i = 1, \ldots, n.$$

Then $a_i \neq 0$ for some $i = 1, \dots, n$ and hence

$$[Y_i, T] = a_i Z \neq 0. \qquad \square$$

**Remark 5.12.** A horizontal half-space $W \subseteq \mathbb{E}\mathrm{n}^n$ for $n \geq 2$ is semigenerating whenever $\partial W \neq \operatorname{span}\{Y_1, \dots, Y_n\}$.

*Proof.* By Lemma 5.7, we have that $\operatorname{Lie}(\partial W)$ is isomorphic to $\mathbb{E}\mathrm{n}^{n-1}$. In particular, $V_3(\operatorname{Lie}(\partial W)) \neq \{0\}$. Since $V_3(\operatorname{Lie}(\partial W)) \subseteq V_3(\mathbb{E}\mathrm{n}^n)$ and $V_3(\mathbb{E}\mathrm{n}^n)$ has dimension one, we deduce that $V_3(\mathbb{E}\mathrm{n}^n) \subseteq \operatorname{Lie}(\partial W) \subseteq \mathfrak{e}(\mathfrak{s}_W)$. The proof is finished by Lemma 4.1. $\square$

**Proposition 5.13.** *None of the* $\mathbb{E}\mathrm{n}^n$ *is semigenerated. Indeed, using the defining basis* (5.2)*, every (of the two) horizontal half-space* $W$ *such that* $\partial W = \operatorname{span}\{Y_1, \dots, Y_n\}$ *is not semigenerating.*

*Proof.* Let $n \geq 1$ and let us consider the following explicit representation of the basis elements as vectors in $\mathbb{R}^{2(n+1)}$:

$$\begin{aligned} Y_i &= \partial_i; \\ X &= \partial_{n+1} + \sum_{i=1}^{n} x_i \partial_{n+1+i} + \frac{x_i^2}{2} \partial_{2(n+1)}; \\ T_i &= \partial_{n+1+i} + x_i \partial_{2(n+1)}; \\ Z &= \partial_{2(n+1)}, \end{aligned}$$

where $i = 1, \dots, n$. It is readily checked that these vector fields satisfy the commutator relations given in Definition 5.1. We consider the set

$$C := \{x \in \mathbb{R}^{2(n+1)} : x_{2(n+1)} \geq 0\}.$$

We shall show that, for $W := \mathbb{R}_+ X \oplus \operatorname{span}\{Y_1, \dots, Y_n\}$, the set $C$ contains $\operatorname{Cl}(S_W)$ but $\exp(V_3)$ is not contained in $C$.

Regarding $\operatorname{Cl}(S_W) \subset C$, we claim that it is enough to show that

$$p \exp(\mathbb{R}_+ Y) \subseteq C, \quad \forall p \in C, \quad \forall Y \in W. \tag{5.14}$$

Indeed, assume that (5.14) holds. Since $C$ is closed, it suffices to prove that $S_W \subseteq C$. As $0 \in C$, then by (5.14) we have that $\exp(Y) \in C$ for all $Y \in W$. Then, again by (5.14), for every finite collection $Y_1, \dots, Y_k \in W$ it holds

$$\exp(Y_1) \cdots \exp(Y_k) \in C.$$

Therefore, we conclude by (2.2) that

$$S_W \overset{(2.2)}{=} \bigcup_{k=1}^{\infty} (\exp(W))^k \subseteq C,$$

which we needed to show.

To prove claim (5.14), we use the fact that the curve $t \mapsto p\exp(tY)$ is the flow line of the vector field $Y$ starting from $p$. Write $Y = aX + \sum_{i=1}^n b_i Y_i$ with $a \geq 0$ and $b_i \in \mathbb{R}$. The ODE given by this vector field is

$$\partial_t\big(p\exp(tX)\big) = aX_{p\exp(tX)} + \sum_{i=1}^n b_i (Y_i)_{p\exp(tX)}.$$

In particular, from the above expression of the vector fields in coordinates, we have that the $2(n+1)$-th component of $p\exp(tX)$ satisfies

$$\partial_t\big(p\exp(tX)\big)_{2(n+1)} = \sum_{i=1}^n \frac{\big(p\exp(tX)\big)_i^2}{2}, \tag{5.15}$$

which is non-negative. Notice that if $p \in C$, then at time $t = 0$ the $2(n+1)$-th component is non-negative, i.e., $\big(p\exp(tX)\big)_{2(n+1)} = p_{2(n+1)} \geq 0$. Therefore, $p\exp(tX) \in C$ for all $t \in \mathbb{R}_+$ by (5.15), and so claim (5.14) is proven.

To see that $\exp(V_3)$ is not contained in $C$, we observe that $V_3 = \mathbb{R}Z$ and $Z = \partial_{2(n+1)}$. We get the conclusion since $C$ is not $\partial_{2(n+1)}$-invariant. □

For the time being, we do not know if the intrinsic $C^1$-rectifiability result for finite-perimeter sets according to [FSS03] holds in non-semigenerated Carnot groups. However, in [DLMV19, Corollary 5.12], *a priori* weaker intrinsic Lipschitz rectifiability of finite-perimeter sets was shown in every Carnot group that admits a non-abnormal horizontal line. As a final remark before proceeding to the proof of Theorem 1.2, we shall characterize non-abnormal horizontal lines of the $n$-th Engel-type algebras for $n \geq 2$. The case $n = 1$ is treated in [DLMV19, Section 6]. As a consequence, we have that the reduced boundary of a finite-perimeter set in any Engel-type algebra is intrinsically Lipschitz rectifiable.

**Remark 5.16.** Let $n \geq 2$ and consider $\mathbb{E}\mathrm{n}^n$ with the basis $\{X, Y_1, \dots, Y_n\}$ satisfying (5.2). Then, for $\nu \in V_1 \setminus \{0\}$, the line $t \mapsto \exp(t\nu)$ is abnormal if and only if $\nu \in \mathrm{span}\{Y_1, \dots, Y_n\} \cup \mathbb{R}X$.

*Proof.* By [DLMV19, Proposition 5.10] and the fact that $\mathbb{E}\mathrm{n}^n$ is stratified, for a given $\nu \in V_1 \setminus \{0\}$, the curve $t \mapsto \exp(t\nu)$ is non-abnormal if and only if

$$\text{(i)}\quad \mathrm{ad}_\nu(V_1) = V_2 \quad \text{and} \quad \text{(ii)}\quad \mathrm{ad}_\nu^2(V_1) = V_3.$$

We prove first that, for every $\nu \in \mathrm{span}\{Y_1, \dots, Y_n\} \cup \mathbb{R}X$, the line $t \mapsto \exp(t\nu)$ is abnormal. Indeed, if $\nu \in \mathbb{R}X$, then $[\nu, V_2] = \{0\}$ by Lemma 5.4.ii. So $\mathrm{ad}_\nu^2(V_1) \subseteq [\nu, V_2] = \{0\}$ and condition (ii) is not satisfied. Let then $\nu \in \mathrm{span}\{Y_1, \dots, Y_n\}$ and consider the linear map $\mathrm{ad}_\nu \colon V_1 \to V_2$. Notice that $\mathrm{span}\{Y_1, \dots, Y_n\} \subseteq \mathrm{Ker}(\mathrm{ad}_\nu)$, since $\mathrm{span}\{Y_1, \dots, Y_n\}$ is abelian by Lemma 5.4.i. Therefore, $\mathrm{ad}_\nu(V_1)$ is 1-dimensional, which violates condition (i) as now $\dim V_2 > 1$.

Assume then $\nu \notin \operatorname{span}\{Y_1, \dots, Y_n\} \cup \mathbb{R}X$ and let us show that the line $t \mapsto \exp(t\nu)$ is non-abnormal by proving that $\nu$ satisfies conditions (i) and (ii). Now $\nu$ can be written as

$$\nu = aX + \sum_{i=1}^{n} a_i Y_i,$$

where $a, a_i \in \mathbb{R}$ are such that $a \neq 0$ and $a_j \neq 0$ for some $j = 1, \dots, n$. Then

$$\begin{aligned}[\nu, \operatorname{span}\{Y_1, \dots, Y_n\}] &= \operatorname{span}\{[\nu, Y_i] \,:\, i = 1, \dots, n\} \\ &= \operatorname{span}\{[aX, Y_i] \,:\, i = 1, \dots, n\} \\ &= \operatorname{span}\{T_i \,:\, i = 1, \dots, n\} \\ &= V_2.\end{aligned}$$

Consequently, condition (i) is satisfied. To prove that also (ii) holds, let $j \in \{1, \dots, n\}$ be such that $a_j \neq 0$ and notice that $[\nu, \mathbb{R}Y_j] = \mathbb{R}T_j$. Then

$$\operatorname{ad}_\nu^2(\mathbb{R}Y_j) = [aX + \sum_{i=1}^{n} a_i Y_i, \mathbb{R}T_j] = [a_j Y_j, \mathbb{R}T_j] = \mathbb{R}Z = V_3.$$

□

5.2. **Proof of Theorem 1.2.** We conclude with the following characterization of trimmed non-semigenerated Carnot algebras in step 3. As a corollary, we shall obtain Theorem 1.2.

**Proposition 5.17.** *Let $\mathfrak{g}$ be a stratified Lie algebra of step 3. If $\mathfrak{g}$ is trimmed and not semigenerated, then it is isomorphic to some $\mathbb{E}\mathrm{n}^n$.*

*Proof.* We start by proving the following fact.

If $\mathfrak{g}$ is a trimmed Lie algebra of step 3 and rank $n+1$ with a non-semigenerating half-space $W \subseteq V_1$, then

$$\partial W \text{ is abelian;} \tag{5.18}$$

and

$$\begin{gathered}\text{for } n+1 \geq 3 \text{ every rank-}n \text{ Carnot subalgebra } \mathfrak{h} \text{ with } V_1(\mathfrak{h}) \neq \partial W, \\ \text{is trimmed and not semigenerated.}\end{gathered} \tag{5.19}$$

Regarding the proof of (5.18), suppose by contradiction that there exist $Y_1, Y_2 \in \partial W$ such that $[Y_1, Y_2] \neq 0$. Set $\mathfrak{s} := \operatorname{Cl}(\mathfrak{s}_W)$. As $\partial W \subseteq \mathfrak{e}_{\mathfrak{s}}$, we have by Lemma 2.8.2 that $[Y_1, Y_2] \subseteq \mathfrak{e}_{\mathfrak{s}}$. Then on the one hand, by Lemma 4.3 we have $\mathfrak{I}_{\mathfrak{g}}([Y_1, Y_2]) \subseteq \mathfrak{e}_{\mathfrak{s}}$. On the other hand, since $\mathfrak{g}$ is trimmed, we have that $V_3 \subseteq \mathfrak{I}_{\mathfrak{g}}([Y_1, Y_2])$, as the latter is a nontrivial ideal. Hence, we infer that $V_3 \subseteq \mathfrak{e}_{\mathfrak{s}}$. Consequently, according to Lemma 4.1 we get a contradiction, since $W$ was assumed not to be semigenerating. Thus (5.18) is proved.

Regarding the proof of (5.19), we show first that

$$\mathscr{Z}(\mathfrak{h}) \cap V_1(\mathfrak{h}) = \{0\}, \tag{5.20}$$

$$\mathscr{Z}(\mathfrak{h}) \cap V_2(\mathfrak{h}) = \{0\}. \tag{5.21}$$

Before proving (5.20), setting $H := V_1(\mathfrak{h})$, we claim that

$$\text{if } X \in H \setminus \partial W \text{ and } Y \in \partial W \cap H \text{ satisfy } [X,Y] = 0, \text{ then } Y = 0. \tag{5.22}$$

Indeed, since we have the decomposition $V_1(\mathfrak{g}) = \partial W \oplus \mathbb{R}X$ and $\partial W$ is abelian by (5.18), we get that $Y$ commutes with $V_1(\mathfrak{g})$. Since $V_1(\mathfrak{g})$ generates $\mathfrak{g}$, we get that $Y \in \mathscr{Z}(\mathfrak{g}) \cap V_1(\mathfrak{g})$, but the latter is trivial since $\mathfrak{g}$ is trimmed. Hence, we get (5.22).

Regarding the proof of (5.20), we assume, aiming for contradiction, that there exists a nonzero element in $\mathscr{Z}(\mathfrak{h}) \cap V_1(\mathfrak{h}) = \mathscr{Z}(\mathfrak{h}) \cap H$. We have two possibilities: either the element is of the form $X \in H \setminus \partial W$ or it is of the form $Y \in \partial W \cap H$. In the first case, since $\partial W \cap H$ is not trivial being the intersection of two hyperplanes, we can find a nonzero $Y$ in it contradicting (5.22). In the second case, since $H \setminus \partial W$ is not empty being the two sets different hyperplanes, we can find $X$ in it contradicting (5.22). Together these two cases prove (5.20).

Regarding (5.21), assume the contrary and let $Z \in \mathscr{Z}(\mathfrak{h}) \cap V_2(\mathfrak{h})$ be nonzero. Applying (2.21) for $\mathfrak{h}$ gives $Z \in \mathfrak{e}(\mathrm{Cl}(\mathfrak{s}_{W\cap H}))$ , where the latter is a subset of $\mathfrak{e}_{\mathfrak{s}}$. Similarly to the proof of (5.18), then by the fact that $\mathfrak{g}$ is trimmed and by Lemma 4.3 we have

$$V_3 \subseteq \mathfrak{I}_{\mathfrak{g}}(Z) \subseteq \mathfrak{e}_{\mathfrak{s}}.$$

Since $W$ is not semigenerating. by Lemma 4.1 we get a contradiction. So (5.21) is proved.

Properties (5.20) and (5.21), together with the fact that $\mathscr{Z}(\mathfrak{h})$ is a non-trivial homogeneous subspace of $\mathfrak{h}$, imply that

$$\{0\} \neq \mathscr{Z}(\mathfrak{h}) \subseteq V_3(\mathfrak{h}) \subseteq V_3(\mathfrak{g}).$$

Since $V_3(\mathfrak{g})$ has dimension 1 as $\mathfrak{g}$ is trimmed, we get that $\mathscr{Z}(\mathfrak{h})$ has dimension 1, i.e., we obtained that $\mathfrak{h}$ is a trimmed step-3 Lie algebra.

The fact that $\mathfrak{h}$ is not semigenerated follows then from Lemma 4.4: since $\dim V_3(\mathfrak{g}) = 1$ and $V_3(\mathfrak{h}) \neq \{0\}$, then $V_3(\mathfrak{h}) = V_3(\mathfrak{g})$. Thus (5.19) is proved.

We complete the proof by an induction argument. The initial step is given for rank $n+1 = 2$ by the classical Engel algebra $\mathbb{E}\mathrm{n}^1$, since every trimmed Carnot algebra of rank 2 and step 3 is isomorphic to $\mathbb{E}\mathrm{n}^1$ (see Remark 2.30). Let then $n+1 \geq 3$ and suppose, by induction assumption, that every trimmed step-3 non-semigenerated Lie algebra of rank $n$ is isomorphic to $\mathbb{E}\mathrm{n}^{n-1}$. Let $\mathfrak{g}$ be a trimmed step-3 non-semigenerated Lie algebra of rank $n+1$. By (5.18) and (5.19), the Lie algebra $\mathfrak{g}$ has a unique abelian stratified subalgebra of rank $n$ and, by induction assumption, every other rank-$n$ stratified subalgebra is isomorphic to $\mathbb{E}\mathrm{n}^{n-1}$. As $\dim V_3(\mathfrak{g}) = 1$ since $\mathfrak{g}$ is trimmed, we conclude that $\mathfrak{g}$ is isomorphic to $\mathbb{E}\mathrm{n}^n$ by Proposition 5.9. □

*Proof of Theorem 1.2.* To prove the easy direction of Theorem 1.2, we recall that every Engel-type algebra is not semigenerated, see Proposition 5.13. Hence, if a Carnot algebra $\mathfrak{g}$ has an Engel-type algebra as a quotient, then $\mathfrak{g}$ is not semigenerated, see Proposition 2.29. Regarding the other implication of Theorem 1.2, let $\mathfrak{g}$ be a step-3 Carnot algebra that is not semigenerated. By Proposition 2.33, there exists a quotient algebra of $\mathfrak{g}$ that is trimmed and not semigenerated. Notice that, being not semigenerated, still such a quotient has step 3. Proposition 5.17 makes us conclude that such a quotient is isomorphic to some Engel-type algebra. □

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