JYU DISSERTATIONS 253

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Sobolev Functions and Mappings on Cuspidal Domains

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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List of included articles

This dissertation consists of an introductory part and the following six publications:

- **[A]** P. Koskela and Z. Zhu, *Product of extension domains is still an extension domain,* Indiana Univ. Math. J. 69 No. 1 (2020), 137-150.
- **[B]** T. Iwaniec, J. Onninen and Z. Zhu, *Creating and Flattening Cusp Singularities by Deformations of Bi-conformal Energy*, J. Geom. Anal. (2020). https://doi.org/10.1007/s12220-019-00351-8.
- **[C]** T. Iwaniec, J. Onninen and Z. Zhu, *Deformations of Bi-conformal Energy and a new Characterization of Quasiconformality*, Arch. Rational Mech. Anal. 236 (2020) 1709-1737.
- **[D]** S. Eriksson-Bique, P. Koskela, J. Malý and Z. Zhu, *Point-wise inequalities for Sobolev functions on outward cuspidal domains*, arXiv:1912.04555
- **[E]** T. Iwaniec, J. Onninen and Z. Zhu, *Singularities in* $\mathscr{L}^p$*- quasidisks*, arXiv:1909.01573.
- **[F]** P. Koskela and Z. Zhu, *Sobolev extensions via reflections*, arXiv:1812.09037.

The author of this dissertation has actively taken part in the work of the joint papers [A], [B], [C], [D], [E] and [F].

INTRODUCTION

This dissertation concentrates on two topics. The first one deals with Sobolev homeomorphisms between the unit ball and domains with exemplary singular boundaries, the cuspidal domains in $\mathbb{R}^n$, $n \geq 2$. The second one is about point-wise inequalities for Sobolev functions on cuspidal domains and Sobolev extendability for such domains.

1. Classes of domains

In this introduction, $\mathbb{B}^n := B^n(0,1)$ is the unit ball in $\mathbb{R}^n$, and $\mathbb{X} \subset \mathbb{R}^n$, $n \geq 2$, is always a domain. For every $0 < r < \infty$, the r-neighborhood of a domain $\mathbb{X}$ is defined by setting

$$B(\mathbb{X}, r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : d(y, \mathbb{X}) < r\}$$

where

$$d(y, \mathbb{X}) \stackrel{\text{def}}{=} \inf_{x \in \mathbb{X}} d(x, y).$$

A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Lipschitz continuous, if there exists a constant $C > 1$ such that, for all $x, y \in \mathbb{R}^n$, we have

$$|f(x) - f(y)| \leq C|x - y|.$$

We say that a bounded domain $\mathbb{X} \subset \mathbb{R}^n$ is a Lipschitz domain if, for each $x \in \partial\mathbb{X}$, there exist $r > 0$ and a Lipschitz continuous function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axes if necessary, we have

$$\mathbb{X} \cap Q(x, r) = \{y : \ f(y_1, \cdots, y_{n-1}) < y_n\} \cap Q(x, r),$$

where $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and

$$Q(x, r) \stackrel{\text{def}}{=} \{y : \ |y_i - x_i| < r, i = 1, 2, \cdots, n\}.$$

Lipschitz domains share many nice properties. For a rectifiable curve $\gamma \subset \mathbb{X}$, we define $l(\gamma)$ to be its length. In [26], Jones defined the so-called (ϵ, δ)-domains, which form a much wider class than the class of Lipschitz domains. Fix positive constants ϵ and δ. We say that $\mathbb{X} \subset \mathbb{R}^n$ is an (ϵ, δ)-domain if, for all $x, y \in \mathbb{X}$ with $|x - y| < \delta$, there is a rectifiable curve $\gamma \subset \mathbb{X}$ joining x to y and satisfying

$$l(\gamma) \leq \tfrac{1}{\epsilon}|x - y|$$

and

$$d(z, \partial\mathbb{X}) \geq \tfrac{\epsilon|x-z||y-z|}{|x-y|} \text{ for all } z \in \gamma.$$

A typical example of such a domain which is not a Lipschitz domain is an inward cuspidal domain in $\mathbb{R}^n$ for $n \geq 3$.

We are, however, mostly interested in cuspidal domains which are not (ϵ, δ)-domains and hence neither Lipschitz domains. Towards a precise definition, we distinguish a horizontal coordinate axis in $\mathbb{R}^n$. Accordingly, we write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \left\{(t,x) \colon t \in \mathbb{R} \text{ and } x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\right\},$$

and introduce the notation

$$|x|^2 \overset{\text{def}}{=\!=} x_1^2 + x_2^2 + \cdots + x_{n-1}^2.$$

We write $\widehat{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ for the one-point compactification of $\mathbb{R}^n$. A strictly increasing function $u \colon [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ is said to be a cuspidal function if $u \in \mathscr{C}^1(0,\infty) \cap \mathscr{C}[0,\infty)$, u' is increasing in $(0,\infty)$ and

$$\lim_{\rho \searrow 0} u'(\rho) = 0\,.$$

We normalize the function u by requiring $u(1) = 1$. The model inward cuspidal domain is defined by

$$\mathbb{B}^{\prec}_{u,n} \overset{\text{def}}{=\!=} \mathbb{B}^n(0,1) \setminus \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^{n-1} \colon |x| \leqslant u(t)\,\}\,. \tag{1.1}$$

The model outward cuspidal domain is defined by

$$\mathbb{B}^{\succ}_{u,n} \overset{\text{def}}{=\!=} \mathbb{B}^n((2,0),\sqrt{2}) \cup \{(t,x) \in (0,1] \times \mathbb{R}^{n-1} \colon |x| \leqslant u(t)\ \}\,. \tag{1.2}$$

For $u(t) = t^\beta$, $\beta > 1$, we obtain a power-type cusp with vertex at the origin. Note that

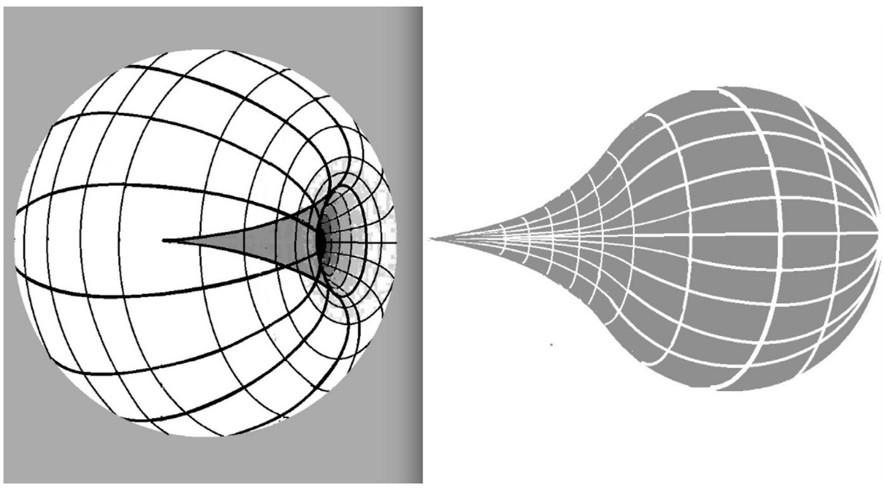

FIGURE 1. Inward and outward cuspidal domains.

the larger the value of β, the sharper the vertex is.

2. Sobolev homeomorphisms

Recall that, for a domain $\mathbb{X} \subset \mathbb{R}^n$, the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X})$, $1 \leqslant p \leqslant \infty$, consists of the functions $u \in \mathscr{L}^p(\mathbb{X})$ whose all first order weak derivatives $D_j u$ belong to $\mathscr{L}^p(\mathbb{X})$. Its norm is given by

$$\|u\|_{\mathscr{W}^{1,p}(\mathbb{X})} = \|u\|_{\mathscr{L}^p(\mathbb{X})} + \sum_{j=1}^{n} \|D_j u\|_{\mathscr{L}^p(\mathbb{X})}.$$

Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{Y} \subset \mathbb{R}^m$ be domains. A mapping $h : \mathbb{X} \to \mathbb{Y}$ $(h = (h_1, h_2, \cdots, h_m))$ is said to be in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, if every component function of h lies in the class $\mathscr{W}^{1,p}(\mathbb{X})$. Its norm is given by

$$\|h\|_{\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})} = \sum_{i=1}^{m} \|h_i\|_{\mathscr{W}^{1,p}(\mathbb{X})}.$$

The local classes are defined accordingly. We say that $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a Sobolev homeomorphism if h maps $\mathbb{X}$ homeomorphically onto $\mathbb{Y}$ and $h \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$. The Riemann Mapping Theorem tells us that every planar simply-connected domain, which is not the whole plane, is conformally equivalent to the unit disk. However, it is rare in higher dimensional spaces that two topological equivalent domains are conformally equivalent because of Liouville's rigidity theorem. Hence, the class of conformal mappings is too restrictive. The class of quasiconformal mappings is a natural generalization of conformal mappings. Let $f \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a Sobolev homeomorphism. Hereafter the symbol $|Df(x)|$ stands for the operator norm of the differential matrix $Df(x) \in \mathbb{R}^{n \times n}$, which is called the deformation gradient ,and $J_f(x)$ for its determinant.

Definition 2.1. *Let $1 \leq K < \infty$. We say that a homeomorphism $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y} \subset \mathbb{R}^n$ on a domain $\mathbb{X} \subset \mathbb{R}^n$ is K-quasiconformal if $f \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ and*

$$|Df(x)|^n \leq K J_f(x) \quad \text{for almost all} \quad x \in \mathbb{X}.$$

A fundamental property of quasiconformal mappings is that the inverse of a quasiconformal mapping is still quasiconformal. In particular, both the mapping and its inverse have finite conformal (or n-harmonic) energy between bounded domains, that is, they belong to the Sobolev class $\mathscr{W}^{1,n}$. Hence, the class of mappings of bi-conformal energy is a generalization of the class of quasiconformal mappings.

Definition 2.2. *A homeomorphism $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$, whose inverse $h^{-1} \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ also belongs to $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$ is called a mapping of bi-conformal energy. If such a homeomorphism exists, $\mathbb{X}$ and $\mathbb{Y}$ are said to be bi-conformally equivalent. The corresponding bi-conformal energy is given by*

$$\mathsf{E}_{\mathbb{X},\mathbb{Y}}[h] \stackrel{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x + \int_{\mathbb{Y}} |Dh^{-1}(y)|^n \, \mathrm{d}y \; < \infty \,. \tag{2.1}$$

Mappings of bi-conformal energy form the widest class of homeomorphisms for which one can hope to build a viable extension of Geometric Function Theory with connections to mathematical models of Nonlinear Elasticity. Such homeomorphisms are exactly the ones with finite conformal energy and integrable inner distortion, as seen in [B, Theorem 1.5]. It is in this way that the studies extend the theory of quasiconformal mappings.

The class of homeomorphisms of finite distortion is another generalization of the class of quasiconformal mappings.

Definition 2.3. *A homeomorphism* $f \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ *is said to have finite distortion if there is a measurable function* $K : \mathbb{X} \to [1, \infty)$ *such that*

$$|Df(x)|^n \leqslant K(x) J_f(x) \ , \quad \text{for almost every } x \in \mathbb{X}. \tag{2.2}$$

The smallest function $K(x) \geqslant 1$ for which (2.2) holds is called the *distortion* of f, denoted by $K_f = K_f(x)$.

Definition 2.4. *Let* $f : \mathbb{X} \to \mathbb{Y}$ *be a homeomorphism in the class* $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$. *We say that* f *has finite inner distortion, if there is a measurable function* $\mathcal{K} : \mathbb{X} \to [1, \infty)$, *such that*

$$|adjDf(x)|^n \leq \mathcal{K}(x) J_f(x)$$

for almost every $x \in \mathbb{X}$. *Here* $\text{adj}Df(x)$ *denotes the adjugate matrix of* $Df(x)$, *i.e. the matrix of the* $(n-1) \times (n-1)$*-subdeterminants of* $Df(x)$.

We define the optimal inner distortion function K_I by setting

$$K_I(x) \overset{\text{def}}{=\!=} \begin{cases} \frac{|\text{adj}Df(x)|^n}{J_f^{n-1}(x)}, & \text{for} \quad J_f(x) > 0, \\ 1, & \text{for} \quad J_f(x) = 0, \end{cases}$$

It is easy to see that a homeomorphism of finite distortion has finite inner distortion. In the Euclidean plane $\mathbb{R}^2$, these two notions coincide. When $n \geq 3$, there are homeomorphisms of finite inner distortion which do not have finite distortion. However, a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ with integrable inner distortion has finite distortion, [5].

3. Sobolev extension domains

The topic about extending functions has a long history. At least, it goes back to the fundamental results of Whitney. In [44, 45], he proved that every $\mathcal{C}^m$-continuous function defined on a closed subset of $\mathbb{R}^n$ can be extended to become a $\mathcal{C}^m$-function on $\mathbb{R}^n$. This extension can be even chosen to be real analytic outside the original closed subset. The class of Sobolev functions is a natural generalization of smooth functions. Sobolev functions are neither necessarily smooth nor differentiable. Instead of this, they have representatives that are absolutely continuous on almost all line segments parallel to the coordinate axes.

Definition 3.1. *A domain* $\mathbb{X} \subset \mathbb{R}^n$ *is said to be a Sobolev* (p, q)*-extension domain for* $1 \leq q \leq p \leq \infty$, *if, for every function* $u \in \mathscr{W}^{1,p}(\mathbb{X})$, *there exists a function* $Eu \in \mathscr{W}^{1,q}_{\text{loc}}(\mathbb{R}^n)$ *with* $Eu\big|_{\mathbb{X}} \equiv u$ *and* $\|Eu\|_{\mathscr{W}^{1,q}(\mathbb{R}^n \setminus \overline{\mathbb{X}})} \leq C\|u\|_{\mathscr{W}^{1,p}(\mathbb{X})}$ *for a positive constant* C *independent of* u.

Thanks to results due to Calderón and Stein [41], Koskela [27], Shvartsman [40], Hajłasz, Koskela and Tuominen [17], Koskela, Rajala and Zhang [29, 30] and so on, the theory of Sobolev (p, p)-extension domains is well understood today, for every $1 \leq p \leq \infty$. Note that even in the planar case there are domains which are not (p,p)-extension domains. The planar unit disk with a removed radial line segment serves as a stardard example of such a domain. It does not allow for (p,p)-extendability, whenever $1 \leq p \leq \infty$.

Hajłasz [16] defined a new class of function spaces on metric measure spaces, the so-called Hajłasz-Sobolev spaces $\mathscr{M}^{1,p}(\mathbb{X})$.

Definition 3.2. *For $u \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{X})$, a non-negative function g is called a p-Hajłasz gradient of u, if $g \in \mathscr{L}^p(\mathbb{X})$, $1 \leq p \leq \infty$ and*

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)), \text{ for a.e. } x, y \in \mathbb{X}.$$

The class of p-Hajłasz gradients of u is denoted by $\mathcal{D}^p(u)$.

Definition 3.3. *The Hajłasz-Sobolev space $\mathscr{M}^{1,p}(\mathbb{X})$, $1 \leq p \leq \infty$, is defined by setting*

$$\mathscr{M}^{1,p}(\mathbb{X}) \stackrel{\text{def}}{=\!=} \{u \in \mathscr{L}^p(\mathbb{X}) : \mathcal{D}^p(u) \neq \emptyset\}.$$

The norm is given by

$$\|u\|_{\mathscr{M}^{1,p}(\mathbb{X})} \stackrel{\text{def}}{=\!=} \|u\|_{\mathscr{L}^p(\mathbb{X})} + \inf_{g \in \mathcal{D}^p(u)} \|g\|_{\mathscr{L}^p(\mathbb{X})}.$$

In the same paper, Hajłasz also proved that the Hajłasz-Sobolev space $\mathscr{M}^{1,p}(\mathbb{R}^n)$ coincides with the classical Sobolev space $\mathscr{W}^{1,p}(\mathbb{R}^n)$, for $1 < p \leq \infty$. In particular, we always have $\mathscr{M}^{1,p}(\mathbb{X}) \subset \mathscr{W}^{1,p}(\mathbb{X})$, $1 \leq p \leq \infty$, and the inclusion is strict for $p = 1$ for any domain $\mathbb{X}$, due to a result of Koskela and Saksman [31]. However, due to the equality $\mathscr{M}^{1,p}(\mathbb{R}^n) = \mathscr{W}^{1,p}(\mathbb{R}^n)$, $1 < p \leq \infty$, we have $\mathscr{M}^{1,p}(\mathbb{X}) = \mathscr{W}^{1,p}(\mathbb{X})$, provided $\mathbb{X}$ is a Sobolev (p,p)-extension domains. This holds especially for (ϵ, δ)-domains by a result of Jones [26]. However, outward cuspidal domains are not (p,p)-extension domains.

4. Mappings of bi-conformal energy from the unit ball onto cuspidal domains

There is broad literature dealing with the question as to when a pair of domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ are quasiconformally equivalent or even bi-Lipschitz equivalent. Domains $\mathbb{X} \subset \mathbb{R}^n$ quasiconformally equivalent with the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ are called quasiballs. It is a highly nontrivial problem to characterize the domains $\mathbb{X} \subset \mathbb{R}^n$ that are quasiballs, when $n \geq 3$. Among geometric obstructions are inward cusps. Outward cuspidal domains, however, are always quasiballs. These fundamental results are due to Gehring and Väisälä [13].

Theorem 4.1. *Let $n \geq 3$. For an arbitrary Lipschitz cuspidal function u, there exists a quasiconformal mapping from the unit ball $\mathbb{B}^n$ onto the outward cuspidal domain $\mathbb{B}^{\succ}_{u,n}$; but there is no quasiconformal mapping from the unit ball $\mathbb{B}^n$ onto the inward cuspidal domain $\mathbb{B}^{\prec}_{u,n}$.*

In particular, every outward cuspidal domain is bi-conformally equivalent with the unit ball. In [B], we established the sharp description of inward cuspidal boundary singularities that can be created and flattened by a mapping of bi-conformal energy. This is in accordance with Hooke's Law in the theory of Nonlinear Elasticity (NE).

Theorem 4.2 (B). *Let $n \geqslant 3$ and*

$$u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}} \quad \text{for } 0 \leqslant t \leqslant 1\,, \quad \text{where } \alpha > 0\,.$$

Then the domains $\mathbb{B}^{\prec}_{u,n}$ and $\mathbb{B}^n$ are bi-conformally equivalent if and only if $\alpha < n$.

In particular, any power-type cuspidal domain, $u(t) = t^{\beta}$, $\beta > 1$, is bi-conformally equivalent with the unit ball. On the other hand, a homeomorphism $h \colon \mathbb{B}^n \xrightarrow{\text{onto}} \mathbb{B}^{\prec}_{u,n}$ of finite bi-conformal energy extends as a homeomorphism up to the closure of $\mathbb{B}^n$, see [B, Theorem 3.1]. Therefore, by the geometry of the inward cuspidal domain $\mathbb{B}^{\prec}_{u,n}$, both the boundary homeomorphism $h \colon \partial\mathbb{B}^n \xrightarrow{\text{onto}} \partial\mathbb{B}^{\prec}_{u,n}$ and its inverse $f := h^{-1}$ enjoys a $\log^{\frac{1}{n}}$-type modulus of continuity estimate, see e.g. [24] and [B]. Such a boundary homeomorphism with given modulus of continuity estimates does not exist between the $(n-1)$-dimensional surface $\partial\mathbb{B}^n$ (smooth) and $\partial\mathbb{B}^{\prec}_{u,n}$ (non-smooth) if

$$u(t) = \exp^{-1}\left(\exp^{\alpha}\left(\frac{1}{t}\right)\right),$$

when $\alpha > n$. Note that, this seemingly natural approach does not lead to a sharp result due to the geometric constrains, see Theorem 4.2. Indeed, if $y_o \in \partial\mathbb{B}^{\prec}_{u,n}$ is the vertex, then $f = h^{-1}$ and h cannot obtain the $\log^{\frac{1}{n}}$-modulus of continuity at y_o and at $f(y_o)$ respectively, at the same time. Without any geometric assumption on the domains, this is however possible, see Theorem 4.6. The nonexistence part of our proof relies on the modulus of continuity of $h \colon \mathbb{B}^n \xrightarrow{\text{onto}} \mathbb{B}^{\prec}_{u,n}$ and the Sobolev embedding on spheres for the inverse mapping. This argument also allows us to substantially relax the regularity assumption of the inverse deformation.

Actually, Theorem 4.2 is a special case of the following theorem.

Theorem 4.3 (B). *Let $n \geqslant 3$ and*

$$u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}} \quad \text{for } 0 \leqslant t \leqslant 1\,, \text{ where } \alpha > 0\,.$$

If $\alpha \geqslant n$ then there is no homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}^{\prec}_{u,n}$ with finite conformal energy whose inverse $h^{-1} = f$ belongs to $\mathscr{W}^{1,p}(\mathbb{B}^{\prec}_{u,n}, \mathbb{R}^n)$, $p > n-1$. If $\alpha < n$, then there exists a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}^{\prec}_{u,n}$ with finite conformal energy such that f is Lipschitz regular.

Symmetry of extremal mappings is a typical speculation in the Calculus of Variations (CV). Several papers, in the intersection of Nonlinear Elasticity (NE) and Geometric Function Theory (GFT), are devoted to understand the expected radial symmetry properties. See [22, 25, 33]. To contribute to such studies in [C], we searched for differences and similarities between mappings of bi-conformal energy and quasiconformal mappings. We examined the modulus of continuity of the mappings.

Definition 4.4. *[Optimal Modulus of Continuity] Every uniformly continuous function $h : \mathbb{X} \to \mathbb{Y}$ admits the optimal modulus of continuity at a given point $x_o \in \mathbb{X}$, given by the* *rule:*

$$\omega_h(x_o;t) \stackrel{\text{def}}{=\!=} \sup\{|h(x)-h(x_o)| : x\in\mathbb{X}, |x-x_o|\le t\}. \tag{4.1}$$

These led us to a new characterization of quasiconformality. Specifically, we observed that quasiconformal mappings behave locally at every point like radial strechings. If a quasiconformal mapping h admits ω as its optimal modulus of continuity at the point x_o, then h^{-1} admits the inverse function ω^{-1} as its modulus of continuity at the point $y_o := h(x_o)$. Second, such a gain/loss rule about moduli of continuity for a homeomorphism and its inverse is typical for radial stretching/ squeezing. It turned out that the gain/loss rule gives a new characterization for the class of quasiconformal mappings.

Theorem 4.5 (C)**.** *Let $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a homeomorphism between domains $\mathbb{X},\mathbb{Y}\subset\mathbb{R}^n$ and let $f\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ denote its inverse. Then h is quasiconformal if and only if, for every pair $(x_\circ,y_\circ)\in\mathbb{X}\times\mathbb{Y}$, $y_\circ = h(x_\circ)$, the optimal modulus of continuity functions $\omega_h=\omega_h(x_\circ;t)$ and $\omega_f=\omega_f(y_\circ;s)$ are quasi-inverse to each other; that is, there is a constant $\mathcal{K}\geqslant 1$ (independent of $(x_\circ,y_\circ)$) such that*

$$\mathcal{K}^{-1}s \leqslant (\omega_h\circ\omega_f)(s) \leqslant \mathcal{K}s$$

for all sufficiently small $s>0$.

The elastic deformations of bi-conformal energy are very different in this respect. We proved unexpectedly that such a mapping may have the same optimal modulus of continuity as its inverse. In line with Hooke's Law, when trying to restore the original shape of the body, the modulus of continuity may neither be improved nor become worse.

Theorem 4.6 (A Representative Example)**.** *Consider a modulus of continuity function $\phi\colon[0,\infty)\xrightarrow{\text{onto}}[0,\infty)$ defined by the rule*

$$\phi(s)=\begin{cases} 0 & \text{if } s=0\\ \left[\log\left(\frac{e}{s}\right)\right]^{-\frac{1}{n}}\ \left[\log\log\left(\frac{e^e}{s}\right)\right]^{-1} & \text{if } 0<s\leqslant 1\\ s & \text{if } s\geqslant 1\end{cases} \tag{4.2}$$

Then there exists a deformation of bi-conformal energy $H\colon\mathbb{R}^n\xrightarrow{\text{onto}}\mathbb{R}^n$ such that

- $H(0)=0$, $H(x)\equiv x$, for $|x|\geqslant 1$
- $|H(x_1)-H(x_2)| \le C\ \phi(|x_1-x_2|)$, *for all* $x_1,x_2\in\mathbb{R}^n$

Its inverse $F \stackrel{\text{def}}{=\!=} H^{-1}\colon\mathbb{R}^n\xrightarrow{\text{onto}}\mathbb{R}^n$ also admits ϕ as a modulus of continuity,

- $|F(y_1)-F(y_2)| \le C\ \phi(|y_1-y_2|)$, *for all* $y_1,y_2\in\mathbb{R}^n$ [1]

Furthermore, ϕ represents the optimal modulus of continuity at the origin for both H and F; that is, for every $0\leqslant s<\infty$ we have

$$\omega_H(0,s) = \phi(s) = \omega_F(0,s)\,. \tag{4.3}$$

[1] In the above estimates the implied constants depend only on n.

5. $\mathscr{L}^p$-QUASIDISKS

Quasidisks have been intensively studied for many years because of their exceptional function-theoretic properties, relationships with Teichmüller theory and Kleinian groups and interesting applications in complex dynamics, see [10] for an elegant survey. Let us start with a definition of quasidisks.

Definition 5.1. *A domain* $\mathbb{X} \subset \mathbb{C}$ *is called a quasidisk if it admits a quasiconformal mapping* $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ *which takes* $\mathbb{X}$ *onto* $\mathbb{D}$. *In symbols, we have* $\mathbb{X} \overset{\text{quasi}}{=\!=} \mathbb{D}$.

Quasidisks can be very complex. There are quasidisks whose boundaries contain no segments with finite length. For every $t \in (1,2)$, one can construct a quasidisk whose boundary has Hausdorff dimension t, see Figure 2. There are many characterizations for quasidisks, see e.g. [12]. Perhaps the best known geometric characterization for a quasidisk is the *Ahlfors' condition* [2].

Theorem 5.2 (Ahlfors)**.** *Let* $\mathbb{X}$ *be a simply-connected Jordan domain in the plane. Then* $\mathbb{X}$ *is a quasidisk if and only if there is a constant* $1 \leqslant \gamma < \infty$, *such that for each pair of distinct points* $a, b \in \partial\mathbb{X}$ *we have*

$$\operatorname{diam} \Gamma \leqslant \gamma\, |a-b| \tag{5.1}$$

where Γ *is the component of* $\partial\mathbb{X} \setminus \{a,b\}$ *with smallest diameter.*

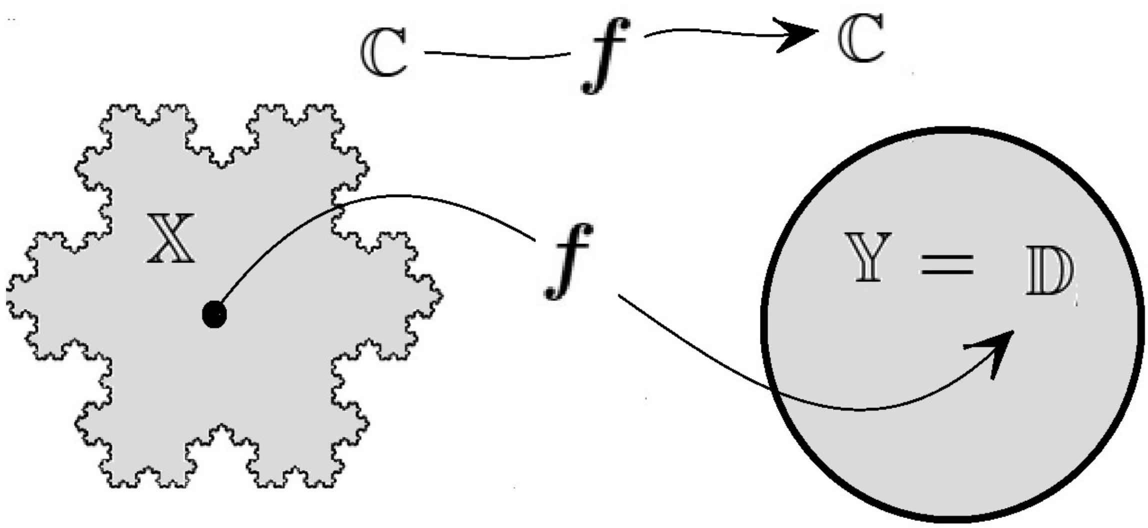

FIGURE 2. Koch snowflake reveals complexity of a quasidisk.

One should infer from the Ahlfors' condition (5.1) that:

Quasidisks do not allow for cusps in the boundary.

This is to say, unfortunately, the point-wise inequality $K(z) \leqslant K < \infty$ in (2.2), precludes f from smoothing even basic singularities. It is therefore of interest to look for more general deformations $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$. We shall see, and it will become intuitively clear, that the act of deviating from conformality should be measured by integral-mean distortions rather than point-wise distortions. A more general class of mappings, for which one might hope to build a viable theory, consists of homeomorphisms with locally $\mathscr{L}^p$-integrable distortion, $1 \leqslant p < \infty$.

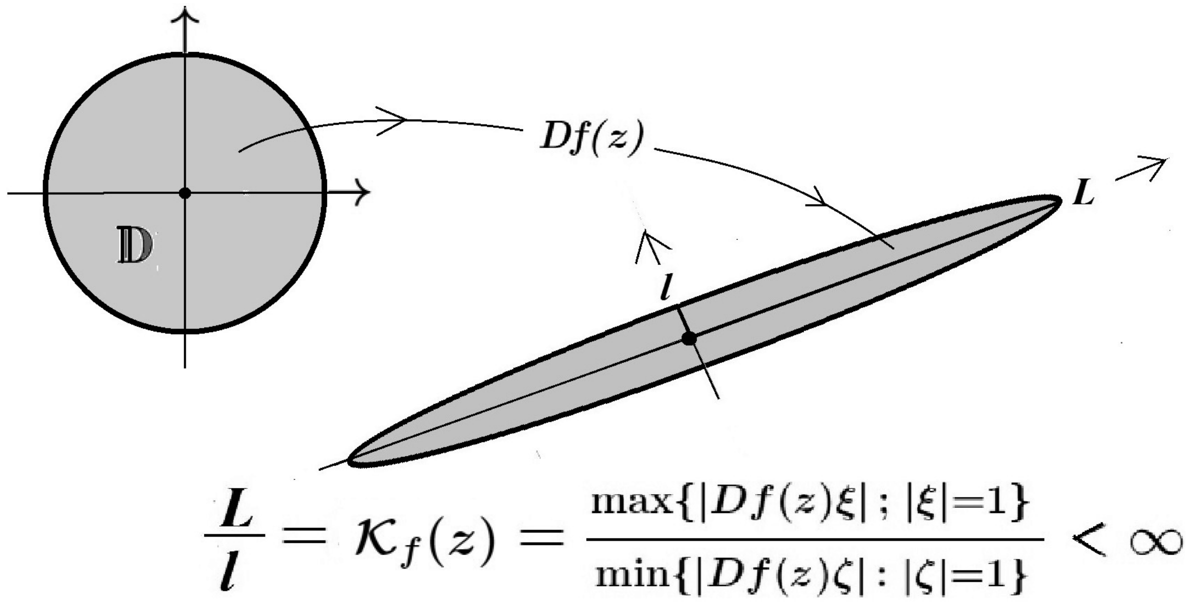

FIGURE 3. The ratio L/l, which measures the infinitesimal distortion of the material structure at the point z, is allowed to be arbitrarily large. Nevertheless, L/l has to be finite almost everywhere.

Definition 5.3. *The term mapping of $\mathscr{L}^p$-distortion, $1 \leqslant p < \infty$, refers to a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ of finite distortion with $K_f \in \mathscr{L}^p_{\text{loc}}(\mathbb{C})$.*

Now, we generalize the notion of quasidisks; simply, replacing the assumption $K_f \in \mathscr{L}^\infty(\mathbb{C})$ by $K_f \in \mathscr{L}^p_{\text{loc}}(\mathbb{C})$ in Definition 5.1.

Definition 5.4. *A domain $\mathbb{X} \subset \mathbb{C}$ is called an $\mathscr{L}^p$-quasidisk if it admits a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ of $\mathscr{L}^p$-distortion such that $f(\mathbb{X}) = \mathbb{D}$.*

Clearly, $\mathscr{L}^p$-quasidisks are Jordan domains. Surprisingly, the $\mathscr{L}^1_{\text{loc}}$-integrability of the distortion seems not to cause any geometric constraint on $\mathbb{X}$. We confirmed this observation for domains with rectifiable boundary.

Theorem 5.5 (E)**.** *All simply-connected Jordan domains with rectifiable boundary are $\mathscr{L}^1$-quasidisks.*

We gave in [E] a full characterization as to when power-type cuspidal domains are $\mathscr{L}^p$-quasidisks.

Theorem 5.6 (E)**.** *Let $\mathbb{X}$ be either $\mathbb{B}^{\prec}_{t^\beta,2}$ or $\mathbb{B}^{\succ}_{t^\beta,2}$ and $1<p<\infty$. Then $\mathbb{X}$ is an $\mathscr{L}^p$-quasidisk if and only if $\beta < \frac{p+3}{p-1}$; equivalently, $p < \frac{\beta+3}{\beta-1}$.*

This is a special case of following theorem.

Theorem 5.7 (E)**.** *Let $u(t)=t^\beta$, $\beta>1$. Consider power-type inward or outward cuspidal domains $\mathbb{X} = \mathbb{B}^{\prec}_{u,2}$ or $\mathbb{B}^{\succ}_{u,2}$ with $\beta > 1$. Given a pair (q,p) of exponents $1 \leqslant q \leqslant \infty$ (for $\mathbb{X}$) and $1 < p \leqslant \infty$ (for the complement of $\mathbb{X}$), define the so-called critical power of the cusp by setting*

$$\beta_{\rm cr} \stackrel{\rm def}{=\!=} \begin{cases} \frac{pq+2p+q}{pq-q}\,, & \text{if } 1<p<\infty \text{ and } q<\infty \\ \frac{2}{q}+1\,, & \text{if } p=\infty \text{ and } q<\infty \\ \frac{p+1}{p-1}\,, & \text{if } 1<p<\infty \text{ and } q=\infty \end{cases} \tag{5.2}$$

Then there exists a Sobolev homeomorphism $f\colon \mathbb{C}\to\mathbb{C}$ which takes $\mathbb{X}$ onto $\mathbb{D}$ such that

- $K_f \in \mathscr{L}^q(\mathbb{X})$

and

- $K_f \in \mathscr{L}^p(\mathbb{B}_R \setminus \overline{\mathbb{X}})$ *for every $R>2$,*

if and only if $\beta < \beta_{\rm cr}$.

The cuspidal domains $\mathbb{B}^{\succ}_{t^\beta,2}$ and $\mathbb{B}^{\prec}_{t^\beta,2}$ satisfy a $\frac{1}{\beta}$-Ahlfors condition, in the sense that we simply replace $|a-b|$ in (5.1) by $|a-b|^{\frac{1}{\beta}}$. Theorem 5.6 tells us how much distortion for a homeomorphism $f\colon\mathbb{C}\to\mathbb{C}$ is needed to flatten (or smoothen) the power-type cusp t^β. Combining this result to the work of Koskela and Takkinen [32], it turns out that a lot more distortion is needed to create a cusp than to smooth it back.

6. Sobolev extensions via reflections

In this section, we introduce the results about Sobolev extendability for the outward and inward power-type cuspidal domains $\mathbb{B}^{\prec}_{t^\beta,n}$ and $\mathbb{B}^{\succ}_{t^\beta,n}$ in $\mathbb{R}^n$. The interesting point of our work in [F] is that we construct the optimal extension operators via reflections. Recall the definition of Sobolev extension domains from Section 3.

Among Sobolev extension domains, the most interesting ones are the (p,p)-extension domains. By results of Calderón and Stein [41], Lipschitz domains are (p,p)-extension domains, for $1\le p\le\infty$. In [26], Jones generalized this result to the class of (ϵ,δ)-domains.

One can easily show that neither the arbitrary n-dimensional outward cuspidal domains $\mathbb{B}^{\succ}_{t^\beta,n}\subset\mathbb{R}^n$ nor the two-dimensional inward cuspidal domains $\mathbb{B}^{\prec}_{t^\beta,2}\subset\mathbb{R}^2$ are (ϵ,δ)-domains, for any $\epsilon,\delta>0$. The inward cuspidal domains $\mathbb{B}^{\prec}_{t^\beta,n}\subset\mathbb{R}^n$, $n\ge 3$, however, are (ϵ,δ)-domains, for some ϵ and δ. The optimal Sobolev extendability results for cuspidal domains are known due to the results of Maz'ya and Poborchi, [35, 36, 37, 38]. The results are rather different between $\mathbb{R}^2$ and $\mathbb{R}^n, n \geq 3$. Let us first give the result in $\mathbb{R}^2$.

Theorem 6.1. *There is a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,2})$ to $\mathscr{W}^{1,q}(\mathbb{R}^2)$ whenever $\frac{1+\beta}{2} < p < \infty$ and $1 \leq q < \frac{2p}{1+\beta}$. Also there exists a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{B}^{\prec}_{t^\beta,2})$ to $\mathscr{W}^{1,q}(\mathbb{R}^2)$, whenever $1 < p < \infty$ and $1 \leq q < \frac{(1+\beta)p}{2+(s-1)\beta}$ or $p = q = 1$.*

As we already know, inward cuspidal domains $\mathbb{B}^{\prec}_{t^\beta,n} \subset \mathbb{R}^n, n \geq 3$, are (ϵ, δ)-domains, and hence they are (p,p)-extension domains, $1 \leq p \leq \infty$. The following theorem gives us the optimal Sobolev extendability for outward cuspidal domains $\mathbb{B}^{\succ}_{t^\beta,n} \subset \mathbb{R}^n, n \geq 3$.

Theorem 6.2. *Let $\mathbb{B}^{\succ}_{t^\beta,n} \subset \mathbb{R}^n (n \geq 3)$, be an outward cuspidal domain with the degree $\beta \in (1, \infty)$. Then*
(1): There exists a bounded linear extension operator E_1 from $\mathscr{W}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,n})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)\beta}{n} < p < \infty$ and $1 \leq q < \frac{np}{1+(n-1)\beta}$.
(2): There exists a bounded linear extension operator E_2 from $\mathscr{W}^{1,\frac{(n-1)+(n-1)^2\beta}{n}}(\mathbb{B}^{\succ}_{t^\beta,n})$ to $\mathscr{W}^{1,n-1}(\mathbb{R}^n)$.
(3): There exists a bounded linear extension operator E_3 from $\mathscr{W}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,n})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)\beta}{2+(n-2)\beta} < p < \infty$ and $1 \leq q < \frac{(1+(n-1)\beta)p}{1+(n-1)\beta+(\beta-1)p}$.

All the above extension results are sharp; the interested reader is refered to [37] and references therein for details. What we are interested in is, whether or not there exists a bounded linear extension operator induced by a reflection. First, let us give the definition of a reflection and explain how does a reflection potentially induce an extension operator.

Definition 6.3. *Let $\mathbb{X} \subset \mathbb{R}^n$ be a domain. A self homeomorphism $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ is said to be a reflection over $\partial\mathbb{X}$, if $\mathcal{R}(\widehat{\mathbb{R}^n} \setminus \overline{\mathbb{X}}) = \mathbb{X}$, $\mathcal{R}(\mathbb{X}) = \widehat{\mathbb{R}^n} \setminus \overline{\mathbb{X}}$ and $\mathcal{R}(x) = x$ for every $x \in \partial\mathbb{X}$.*

Definition 6.4. *Let $\mathbb{X} \subset \mathbb{R}^n$ be a domain with compact boundary. We say that a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\mathbb{X}$ induces a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,q}_{\text{loc}}(\mathbb{R}^n)$, for some $1 \leqslant q \leqslant p \leqslant \infty$, if for every function $u \in \mathscr{W}^{1,p}(\mathbb{X})$ the function*

$$E_{\mathcal{R}}(u)(x) \overset{\text{def}}{=\!=} \begin{cases} u(\mathcal{R}(x)), & \text{for } x \in B(\mathbb{X},1) \setminus \overline{\mathbb{X}}, \\ 0, & \text{for } x \in \partial\mathbb{X}, \\ u(x), & \text{for } x \in \mathbb{X}, \end{cases} \tag{6.1}$$

belongs to $\mathscr{W}^{1,q}_{\text{loc}}(B(\mathbb{X},1))$ and

$$\|E_{\mathcal{R}}(u)\|_{\mathscr{W}^{1,q}(B(\mathbb{X},1)\setminus\overline{\mathbb{X}})} \leq C\|u\|_{\mathscr{W}^{1,p}(\mathbb{X})}.$$

Here C is a positive constant independent of u.

Recall that $B(\mathbb{X}, 1)$ denotes the 1-neighborhood of $\mathbb{X}$. The classical cut-off technique implies that an extension operator from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,q}(B(\mathbb{X}, 1))$ can be upgraded to an extension operator from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$.

In [28], Koskela, Pankka and Zhang proved that, for every planar Jordan (p,p)-extension domain, $1<p<\infty$, there is a reflection which induces a bounded linear extension operator.

Theorem 6.5. *Let $\mathbb{X}\subset\mathbb{R}^2$ be a Jordan (p,p)-extension domain. Then $\mathbb{R}^2\setminus\overline{\mathbb{X}}$ is a $(p^\star,p^\star)$-extension domain with $\frac{1}{p}+\frac{1}{p^\star}=1$. Moreover, there is a reflection over $\partial\mathbb{X}$ which induces a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,p}(\mathbb{R}^2)$ and also a bounded linear extension operator from $\mathscr{W}^{1,p^\star}(\mathbb{R}^2\setminus\overline{\mathbb{X}})$ to $\mathscr{W}^{1,p^\star}(\mathbb{R}^2)$.*

Actually, for the planar outward cuspidal domains $\mathbb{B}^{\succ}_{t^\beta,2}$ and inward cuspidal domains $\mathbb{B}^{\prec}_{t^\beta,2}$, the problem about Sobolev extension via reflection was already studied by Gol'dshtein and Sitnikov [15], around 30 years ago. Their result shows that the corresponding extension result in Theorem 6.1 can be achieved by a bounded linear extension operator induced via a reflection.

Theorem 6.6. [15] *Fix $\beta>1$. There is a reflection $\mathcal{R}:\widehat{\mathbb{R}^2}\to\widehat{\mathbb{R}^2}$ over $\partial\mathbb{B}^{\succ}_{t^\beta,2}$ which induces a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,2})$ to $\mathscr{W}^{1,q}(\mathbb{R}^2)$ whenever $\frac{1+\beta}{2}<p<\infty$ and $1\le q<\frac{2p}{1+\beta}$. Moreover, $\mathcal{R}$ also induces a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{R}^2\setminus\overline{\mathbb{B}^{\succ}_{t^\beta,2}})$ to $\mathscr{W}^{1,q}(\mathbb{R}^2)$ whenever $1<p<\infty, 1\le q<\frac{(1+\beta)p}{2+(\beta-1)p}$ or $p=q=1$.*

Since the domain $\mathbb{R}^2\setminus\overline{\mathbb{B}^{\succ}_{t^\beta,2}}$ has the same singularity on the boundary as the inward cuspidal domain $\mathbb{B}^{\prec}_{t^\beta,2}$, it is easy to see that the Sobolev extendability for $\mathbb{R}^2\setminus\overline{\mathbb{B}^{\succ}_{t^\beta,2}}$ implies the same Sobolev extendability for $\mathbb{B}^{\prec}_{t^\beta,2}$.

After understanding the theory in the Euclidean plane $\mathbb{R}^2$, the similar question arises in $\mathbb{R}^n, n\ge 3$. We obtained in [F] the following result about bounded linear extension operators induced by reflections on outward cuspidal domains $\mathbb{B}^{\succ}_{t^\beta,n}\subset\mathbb{R}^n$.

Theorem 6.7 (F)**.** *Let $n\ge 3$ and $\mathbb{B}^{\succ}_{t^\beta,n}\subset\mathbb{R}^n$ be an outward cuspidal domain with the degree $1<\beta<\infty$. Then*
(1)*: There exists a reflection $\mathcal{R}_1:\widehat{\mathbb{R}^n}\to\widehat{\mathbb{R}^n}$ over $\partial\mathbb{B}^{\succ}_{t^\beta,n}$ which induces a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,n})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)\beta}{n}<p<\infty$ and $1\leqslant q<\frac{np}{1+(n-1)\beta}$.*
(2)*: There exists a reflection $\mathcal{R}_2:\widehat{\mathbb{R}^n}\to\widehat{\mathbb{R}^n}$ over $\partial\mathbb{B}^{\succ}_{t^\beta,n}$ which induces a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,n})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)\beta}{2+(n-2)\beta}<p<\infty$ and $1\leqslant q<\frac{(1+(n-1)\beta)p}{1+(n-1)\beta+(\beta-1)p}$.*

Let $n\ge 3$. It is easy to check that the complement $\mathbb{R}^n\setminus\overline{\mathbb{B}^{\succ}_{t^\beta,n}}$, is an (ϵ,δ)-domain, for some positive ϵ,δ. Hence, $\mathbb{R}^n\setminus\overline{\mathbb{B}^{\succ}_{t^\beta,n}}$ is a (p,p)-extension domain, for every $1\le p\le\infty$, due to a result of Jones [26]. Our following theorem gives the values of $p\in[1,\infty)$, for which the (p,p)-extension of $\mathbb{R}^n\setminus\overline{\mathbb{B}^{\succ}_{t^\beta,n}}$ can be achieved via a bounded linear extension operator induced by a reflection.

Theorem 6.8 (F)**.** *Let $n \geq 3$. For every $1 < \beta < \infty$, $\mathbb{R}^n \setminus \overline{\mathbb{B}^{\succ}_{t^\beta,n}}$ is a (p,p)-extension domain, for every $1 \leq p < \infty$. The reflection $\mathcal{R}_1$ in Theorem 6.7 induces a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{R}^n \setminus \overline{\mathbb{B}^{\succ}_{t^\beta,n}})$ to $\mathscr{W}^{1,p}(\mathbb{R}^n)$, whenever $1 \leq p \leq n-1$. Moreover, for any given $n-1 < p < \infty$, there does not exist a reflection over $\partial\mathbb{B}^{\succ}_{t^\beta,n}$ which can induce a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{R}^n \setminus \overline{\mathbb{B}^{\succ}_{t^\beta,n}})$ to $\mathscr{W}^{1,p}(\mathbb{R}^n)$.*

What about the case $p = \infty$? We say that a domain $\mathbb{X} \subset \mathbb{R}^n$ is uniformly locally quasiconvex if there exist constants $C > 0$ and $R > 0$ such that, for all $x, y \in \mathbb{X}$ with $d(x,y) < R$, there is a rectifiable curve γ connecting x and y in $\mathbb{X}$ such that the length of γ is bounded from above by $Cd(x,y)$. Recall that $\mathbb{X}$ is an (∞,∞)-extension domain if and only if it is uniformly locally quasiconvex, see [17]. One can easily check that both $\mathbb{B}^{\succ}_{t^\beta,n}$ and $\mathbb{R}^n \setminus \overline{\mathbb{B}^{\succ}_{t^\beta,n}}$ are uniformly locally quasiconvex, equivalently, they are (∞,∞)-extension domains. The following theorem is an analog of Theorem 6.8.

Theorem 6.9 (F)**.** *Let $n \geq 3$. For every $1 < \beta < \infty$, both $\mathbb{B}^{\succ}_{t^\beta,n}$ and $\mathbb{R}^n \setminus \overline{\mathbb{B}^{\succ}_{t^\beta,n}}$ are (∞,∞)-extension domains. The reflection $\mathcal{R}_1$ over $\partial\mathbb{B}^{\succ}_{t^\beta,n}$ in Theorem 6.7 induces a bounded linear extension operator from $\mathscr{W}^{1,\infty}(\mathbb{B}^{\succ}_{t^\beta,n})$ to $\mathscr{W}^{1,\infty}(\mathbb{R}^n)$. On the other hand, there is no reflection over $\partial\mathbb{B}^{\succ}_{t^\beta,n}$ which can induce a bounded linear extension operator from $\mathscr{W}^{1,\infty}(\mathbb{R}^n \setminus \overline{\mathbb{B}^{\succ}_{t^\beta,n}})$ to $\mathscr{W}^{1,\infty}(\mathbb{R}^n)$.*

7. The product of Sobolev extension domains

In this section, we introduce our result in [A]. It says that the Sobolev extendability property is stable under products.

First, let us explain why we got interested in this problem. By [15] the Sobolev extendability for planar outward cuspidal domains $\mathbb{B}^{\succ}_{t^\beta,2}$ and inward cuspidal domains $\mathbb{B}^{\prec}_{t^\beta,2}$ can be achieved via a bounded linear extension operator induced by reflections, see Theorem 6.6. By making use of this result, we can easily prove that the domain $\mathbb{B}^{\succ}_{t^\beta,2} \times I \subset \mathbb{R}^3$ has the same Sobolev extendability as $\mathbb{B}^{\succ}_{t^\beta,2}$. Here $I = (0,1) \subset \mathbb{R}$ is the unit interval. This follows as a special case of our next result. The idea of the proof is copied from the proof of [Theorem 1.1, A]. Hence, we only give a rough proof here.

Theorem 7.1. *Let $\mathbb{X} \subset \mathbb{R}^n$ be a bounded simply-connected (p,q)-extension domain, for $1 \leq q \leq p < \infty$. Assume that there exists a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\mathbb{X}$, such that the induced extension operator defined in (6.1) is bounded both from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$ and from $\mathscr{L}^p(\mathbb{X})$ to $\mathscr{L}^q(\mathbb{R}^n)$. Then $\mathbb{X} \times I \subset \mathbb{R}^{n+1}$ is also a (p,q)-extension domain.*

Sketch of proof. Let $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ be the reflection over $\partial\mathbb{X}$, which induces a bounded linear extension operator both from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$ and from $\mathscr{L}^p(\mathbb{X})$ to $\mathscr{L}^q(\mathbb{R}^n)$. For every function $u \in \mathscr{W}^{1,p}(\mathbb{X})$, we define the extension $E_{\mathcal{R}}(u)$ as in (6.1).

We write

$$\mathbb{R}^{n+1} \stackrel{\text{def}}{=\!=} \mathbb{R} \times \mathbb{R}^n = \{(t,x) : t \in \mathbb{R} \text{ and } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n\}.$$

Let $u \in \mathcal{C}^\infty(\mathbb{X}\times I)\cap \mathscr{W}^{1,p}(\mathbb{X}\times I)$ be arbitrary. By the Fubini theorem, for almost every $t \in I \overset{\text{def}}{=\!=} (0,1)$,

$$u_t(\cdot) = u(t,\cdot) \in \mathcal{C}^1(\mathbb{X})\cap \mathscr{L}^\infty(\mathbb{X})\cap \mathscr{W}^{1,p}(\mathbb{X}).$$

For such $t \in I$, $E_{\mathcal{R}}(u_t) \in \mathscr{W}^{1,q}(\mathbb{R}^n)$ and $\|E_{\mathcal{R}}(u_t)\|_{\mathscr{W}^{1,q}(\mathbb{R}^n)} \le C\|u_t\|_{\mathscr{W}^{1,p}(\mathbb{X})}$ with a positive constant C independent of t. Then for $t \in (0,1)$, using the extension (6.1), we set

$$Eu(t,x) \overset{\text{def}}{=\!=} E_{\mathcal{R}}(u_t)(x). \tag{7.1}$$

For every $i \in \{1,2,\cdots,n\}$, define a function by setting

$$\frac{\partial}{\partial x_i}Eu(t,x) \overset{\text{def}}{=\!=} \begin{cases} \frac{\partial}{\partial x_i}E_{\mathcal{R}}(u_t)(x), & t\in(0,1) \text{ with } u_t \in \mathscr{W}^{1,p}(\mathbb{X}),\\ 0, & \text{elsewhere.}\end{cases} \tag{7.2}$$

By some simple computations, $\frac{\partial}{\partial x_i}Eu$ is the distributional derivative of Eu with respect to the x_i-coordinate direction with desired norm control.

From the argument above, we already know that the distributional derivatives of Eu exist with respect to the x-coordinate directions. Now we construct the distributional derivative of Eu with respect to the t-coordinate direction. By the definition of reflection, for every $x \in B(\mathbb{X},1)\setminus\overline{\mathbb{X}}$, there exists $x' \in \mathbb{X}$ with $\mathcal{R}(x') = x$. Then by the definition of Eu in (7.1), for every $t \in (0,1)$, we have $Eu(t,x) = u(t,x')$. Since $u \in \mathcal{C}^\infty(\mathbb{X}\times I)\cap\mathscr{W}^{1,p}(\mathbb{X}\times I)$ and $\frac{\partial}{\partial t}Eu(t,x) = \frac{\partial}{\partial t}u(t,x')$ for every $t\in(0,1)$. Hence we define our function $\frac{\partial}{\partial t}Eu$ by setting

$$\frac{\partial}{\partial t}Eu(t,x) \overset{\text{def}}{=\!=} \begin{cases} \frac{\partial}{\partial t}u(t,x'), & x \in B(\mathbb{X},1)\setminus\overline{\mathbb{X}},\\ 0, & x\in\partial\mathbb{X},\\ \frac{\partial}{\partial t}u(t,x), & x\in\mathbb{X}.\end{cases} \tag{7.3}$$

By some simple computations, $\frac{\partial}{\partial t}Eu$ defined in (7.3) is the distributional derivative of Eu with respect to the t-coordinate direction. It is also easy to see that

$$\frac{\partial}{\partial t}Eu(t,x) = E_{\mathcal{R}}\left(\frac{\partial}{\partial t}u\right)(t,x)$$

almost everywhere. Since the extension operator $E_{\mathcal{R}}$ induced by the reflection $\mathcal{R}$ is bounded from $\mathscr{L}^p(\mathbb{X})$ to $\mathscr{L}^q(\mathbb{R}^n)$, we can obtain the desired norm control. Hence, for every $u \in \mathcal{C}^\infty(\mathbb{X}\times I)\cap\mathscr{W}^{1,p}(\mathbb{X}\times I)$, we have

$$\|Eu\|_{\mathscr{W}^{1,q}(B(\mathbb{X},1)\times I)} \le C\|u\|_{\mathbb{X}\times I}$$

with a positive constant C independent of u. By the density of $\mathcal{C}^\infty(\mathbb{X}\times I)\cap\mathscr{W}^{1,p}(\mathbb{X}\times I)$, E can be extended to $\mathscr{W}^{1,p}(\mathbb{X}\times I)$. Let $\mathbb{B}\subset\mathbb{R}^n$ be a large enough ball with $B(\mathbb{X},1)\subset\mathbb{B}$. By the classical cut-off technique, there exists a function $\tilde{E}u \in \mathscr{W}^{1,q}(\mathbb{B}\times I)$ with $\tilde{E}u\big|_{B(\mathbb{X},1)\times I} \equiv u$ and

$$\|\tilde{E}u\|_{\mathscr{W}^{1,q}(\mathbb{B}\times I)} \le C\|u\|_{\mathscr{W}^{1,p}(\mathbb{X}\times I)}$$

with a constant C independent of u. $\square$

If one reads the proof above carefully, one can observe that the fact that the extension operator induced by the reflection is bounded both from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,q}(\mathbb{R}^n)$ and from $\mathscr{L}^p(\mathbb{X})$ to $\mathscr{L}^q(\mathbb{R}^n)$ is the essential point. Regarding the classical Sobolev (p,p)-extension theory, Hajłasz, Koskela and Tuominen [17] proved that for $1 \leq p < \infty$, a (p,p)-extension domain must satisfy an Ahlfors-regularity condition according to which, for every $x \in \mathbb{X}$ and $0 < r < \operatorname{diam} \mathbb{X}$, we have

$$|B(x,r) \cap \mathbb{X}| \geq C|B(x,r)|.$$

They also proved the following theorem in [17, 18].

Theorem 7.2. *Let $1 < p < \infty$. A domain $\mathbb{X} \subset \mathbb{R}^n$ is a Sobolev (p,p)-extension domain if and only if there exists a bounded linear extension operator from $\mathscr{W}^{1,p}(\mathbb{X})$ to $\mathscr{W}^{1,p}(\mathbb{R}^n)$.*

By making use of the operator from this theorem and following the main idea of the proof of Theorem 7.1, we proved the following result in [A].

Theorem 7.3. *Let $1 < p \leq \infty$. If $\mathbb{X}_1 \subset \mathbb{R}^n$ and $\mathbb{X}_2 \subset \mathbb{R}^m$ are (p,p)-extension domains, then $\mathbb{X}_1 \times \mathbb{X}_2 \subset \mathbb{R}^{n+m}$ is also a (p,p)-extension domain. Conversely, if $\mathbb{X}_1 \subset \mathbb{R}^n$ and $\mathbb{X}_2 \subset \mathbb{R}^m$ are domains so that $\mathbb{X}_1 \times \mathbb{X}_2 \subset \mathbb{R}^{n+m}$ is a (p,p)-extension domain, then both $\mathbb{X}_1$ and $\mathbb{X}_2$ are necessarily (p,p)-extension domains.*

8. $\mathscr{M}^{1,p} = \mathscr{W}^{1,p}$ ON OUTWARD CUSPIDAL DOMAINS

In this section, we will introduce the results in [D]. In that paper, we showed that the Hajłasz-Sobolev spaces coincide with the classical Sobolev spaces on a large class of domains including all outward cuspidal domains, see Section 1 for definitions.

In [1, 8], Acerbi, Fusco, Bojarski and Hajłasz proved certain point-wise inequalities for Sobolev functions. That is

$$|u(x) - u(y)| \leq C|x-y| \left(\mathcal{M}[\nabla u](x) + \mathcal{M}[\nabla u](y)\right). \tag{8.1}$$

Here C is a positive constant independent of x, y and u and $\mathcal{M}$ is the usual maximal operator. Motivated by this, P. Hajłasz introduced in [16] the function space $\mathscr{M}^{1,p}(\mathbb{X})$.

It is known that $\mathscr{M}^{1,p}(\mathbb{X}) \subset \mathscr{W}^{1,p}(\mathbb{X})$ and that the inclusion is strict for $p = 1$ for any $\mathbb{X}$, see the work of Koskela and Saksman [19]. By inequality (8.1), in $\mathbb{R}^n$, the opposite inclusion holds for $1 < p \leq \infty$. The opposite inclusion also holds if there is a bounded extension operator from $\mathscr{W}^{1,p}(\mathbb{X})$ into $\mathscr{W}^{1,p}(\mathbb{R}^n)$, for a given $1 < p \leqslant \infty$. Actually, if there exists a constant C so that

$$|B(x,r)| \leq C|B(x,r) \cap \mathbb{X}| \tag{8.2}$$

for every $x \in \mathbb{X}$ and every $0 < r < 1$, where $|\cdot|$ refers to n-measure, then the above two function spaces coincide precisely when such an extension operator exists. For this see [17]. Hence it is easy to exhibit domains $\mathbb{X}$ for which $\mathscr{M}^{1,p}(\mathbb{X}) = \mathscr{W}^{1,p}(\mathbb{X})$ fails for all p; e.g. take $\mathbb{X}$ to be the unit disk minus the interval $[0,1)$ on the real line.

In [D], we did not only work on the outward cuspidal domain $\mathbb{B}^{\succ}_u$ defined in (1.1). Let us consider cuspidal domains of the form

$$\mathbb{X}_\psi \overset{\text{def}}{=\!=} \{(t,x) \in (0,1) \times \mathbb{R}^{n-1}; |x| < \psi(t)\} \cup \{(t,x) \in [1,2) \times \mathbb{R}^{n-1}; |x| < \psi(1)\}, \tag{8.3}$$

where $\psi : (0,1] \to (0,\infty)$ is a left-continuous increasing function. See Figure 4. It is easy

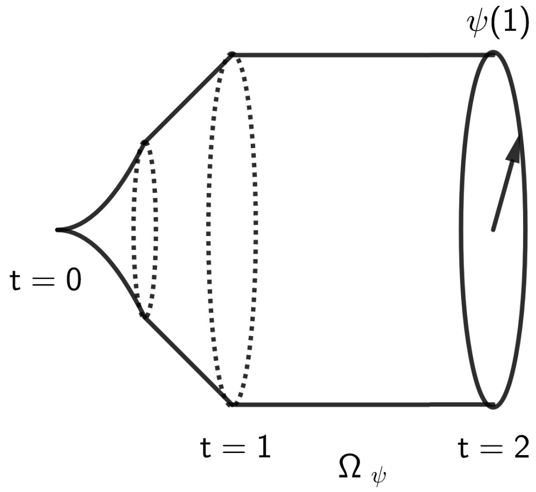

FIGURE 4. Outward cuspidal domain Ω_ψ

to check that $\mathbb{X}_\psi \subset \mathbb{R}^n$ is a domain. If $\lim_{t\to 0^+} \frac{\psi(t)}{t} = 0$, then the measure density condition (8.2) fails, and hence, by [17], there can not exist any bounded extension operator from $\mathscr{W}^{1,p}(\mathbb{X}_\psi)$ to $\mathscr{W}^{1,p}(\mathbb{R}^n)$. According to a somewhat surprising result by A. S. Romanov [39], $\mathscr{W}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,n}) = \mathscr{M}^{1,p}(\mathbb{B}^{\succ}_{t^\beta,n})$ with $\beta > 1$ and $p > \frac{1+(n-1)\beta}{n}$. Once $\psi(t) = t^\beta$ with $\beta > 1$, one can easily check that $\mathbb{B}^{\succ}_{t^\beta,n}$ is bi-Lipschitz equivalent to $\mathbb{X}_\psi$. As we know, bi-Lipschitz transformations preserve both Sobolev and Hajłasz-Sobolev spaces.

Our result from [D] shows that the above restriction on p is superfluous and that u being of the form $u(t) = t^\beta$ can be essentially relaxed to being a nondecreasing left-continuous function.

Theorem 8.1 (D). *Let $\psi : (0,1] \to (0,\infty)$ be a left-continuous increasing function. Define the corresponding cuspidal domain $\mathbb{X}_\psi$ as in (8.3). Then $\mathscr{W}^{1,p}(\mathbb{X}_\psi) = \mathscr{M}^{1,p}(\mathbb{X}_\psi)$, for all $1 < p \leqslant \infty$.*

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Included articles

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Product of extension domains is still an extension domain

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Indiana Univ. Math. J. 69 No. 1 (2020), 137-150.

Product of Extension Domains is still an Extension Domain

PEKKA KOSKELA & ZHENG ZHU

ABSTRACT. Our main result gives a functional property of the class of $W^{1,p}$-extension domains. Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ both be $W^{1,p}$-extension domains for some $1 < p \leq \infty$. We prove that $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{n+m}$ is also a $W^{1,p}$-extension domain. We also establish the converse statement.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open set. For $1 \leq p \leq \infty$, we let $W^{1,p}(\Omega)$ denote the Sobolev space consisting of all functions $u \in L^p(\Omega)$ whose first-order distributional partial derivatives on Ω belong to $L^p(\Omega)$. This space is normed by

$$\|u\|_{W^{1,p}(\Omega)} := \sum_{0 \leq |\alpha| \leq 1} \|D^\alpha u\|_{L^p(\Omega)}.$$

We say that $u \in L^p(\Omega)$ is *ACL* (*absolutely continuous on lines*), if u has a representative $\tilde{u}$ that is absolutely continuous on almost all line segments in Ω, parallel to the coordinate axes. Then, $u \in W^{1,p}(\Omega)$ if and only if u belongs to $L^p(\Omega)$ and has a representative $\tilde{u}$ which is ACL and whose (classical) partial derivatives belong to $L^p(\Omega)$ (see, e.g., Theorem A.15 in [11] and Theorem 2.1.4 in [20]).

We say that $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain if there exists a constant $C \geq 1$ which only depends on Ω, n, p such that for every $u \in W^{1,p}(\Omega)$ there exists a function $Eu \in W^{1,p}(\mathbb{R}^n)$ with $Eu|_\Omega \equiv u$ and so that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

For example, note that every Lipschitz domain is a $W^{1,p}$-extension domain for all $1 \leq p \leq \infty$ by the results of Calderón and Stein [17]. It is easy to give

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examples of domains that fail to be extension domains: for example, the slit disk $\Omega := B^2(0,1) \setminus \{(x_1, 0) : 0 \le x_1 < 1\}$. In general, the extension property for a fixed Ω may depend on the value of p (see [13], [16], and [12]).

In [9], it was shown that any bi-Lipschitz image of a $W^{1,p}$-extension domain, $1 < p \le \infty$, is also a $W^{1,p}$-extension domain: if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain and $f : \Omega \to \Omega' \subset \mathbb{R}^n$ is bi-Lipschitz, then Ω' is also a $W^{1,p}$-extension domain. Our main result gives a second functional property of Sobolev extension domains.

Theorem 1.1. *Let $1 < p \le \infty$. If $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ are $W^{1,p}$-extension domains, then $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{n+m}$ is also a $W^{1,p}$-extension domain. Conversely, if $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ are domains so that $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{n+m}$ is a $W^{1,p}$-extension domain, then both Ω_1 and Ω_2 are necessarily $W^{1,p}$-extension domains.*

According to Theorem 7 in [9] (see [21] for related results), a domain Ω is a $W^{1,\infty}$-extension domain if and only if it is uniformly locally quasiconvex, that is, there exist positive constants C and R, such that for all $x, y \in \Omega$ with $|x - y| < R$, there exists a curve $\gamma_{x,y} \subset \Omega$ from x to y with

$$\ell(\gamma_{x,y}) \le C|x - y|.$$

Here, $\ell(\gamma_{x,y})$ is the length of the curve $\gamma_{x,y}$. It is easy to check that the product of uniformly locally quasiconvex domains is still uniformly locally quasiconvex, and hence we only need to prove the first part of Theorem 1.1 for $1 < p < \infty$.

Our proof of the first part of Theorem 1.1 is based on the existence of an explicit extension operator constructed by Shvartsman in [14]. A result from [9] allows us to employ this operator. This procedure could in principle also be tried for the case of the higher-order Sobolev spaces $W^{k,p}$, $k \ge 2$, but one does not seem to obtain suitable norm estimates. We would like to know whether the first part of Theorem 1.1 extends to the case of higher-order Sobolev spaces or not; the second part does extend, as can be seen from our proof below.

2. PRELIMINARIES

2.1. Definitions and preliminary results. Throughout the paper, C, C_1, C_2, ... or $\gamma, \gamma_1, \gamma_2, \ldots$ will be generic positive constants which depend only on the dimension n, the domain Ω, and indices of the function spaces in question (p, q, etc.). These constants may change even in a single string of estimates. The dependence of a constant on certain parameters in expressed, for example, by the notation $\gamma = \gamma(n,p)$. We write $A \approx B$ if there is a constant $C \ge 1$ such that $A/C \le B \le CA$.

It will be convenient for us to measure distance via the uniform norm

$$\|x\|_\infty := \max\{|x_i| : i = 1, \ldots, n\}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Thus, every Euclidean cube

$$Q = Q(x,r) = \{y \in \mathbb{R}^n : \|y - x\|_\infty \le r\}$$

is a ball in the $\|\cdot\|_\infty$-norm.

Definition 2.1. A measurable set $A \subset \mathbb{R}^n$ is said to be uniformly locally Ahlfors regular (shortly, regular) if there are constants $C_A \geq 1$ and $\delta_A > 0$ such that, for every cube Q with center in A and with diameter $\operatorname{diam} Q \leq \delta_A$, we have

$$|Q| \leq C_A |Q \cap A|.$$

Given $u \in L^p_{\text{loc}}(\mathbb{R}^n)$, $1 < p \leq \infty$, and a cube Q, we set

$$\Lambda(u;Q)_{L^p} := |Q|^{-1/p} \inf_{C\in\mathbb{R}} \|u - C\|_{L^p(Q)} = \inf_{C\in\mathbb{R}} \left(\frac{1}{|Q|}\int_Q |u - C|^p \,\mathrm{d}x\right)^{1/p}$$

(see Brudnyi [3] for related definitions). Sometimes (e.g., in [18]), $\Lambda(u;Q)_{L^p}$ is also called the local oscillation of u. This quantity is the main object in the theory of local polynomial approximation which provides a unified framework for the description of a large family of spaces of smooth functions. We refer the readers to Brudnyi [1]–[6] for the main ideas and results in local approximation theory.

Given a locally integrable function u on $\mathbb{R}^n$, we define its sharp maximal function $u_1^\#$ by setting

$$u_1^\#(x) := \sup_{r>0} r^{-1}\Lambda(u;Q(x,r))_{L^1}.$$

In [8], Calderón proved that, for $1 < p \leq \infty$, a function u is in $W^{1,p}(\mathbb{R}^n)$ if and only if u and $u_1^\#$ are both in $L^p(\mathbb{R}^n)$. Moreover, up to constants depending only on n and p, we have that

$$\|u\|_{W^{1,p}(\mathbb{R}^n)} \approx \|u\|_{L^p(\mathbb{R}^n)} + \|u_1^\#\|_{L^p(\mathbb{R}^n)}.$$

This characterization produces the following definition. Given $1 < p \leq \infty$, a function $u \in L^p_{\text{loc}}(A)$, and a cube Q whose center is in A, we let $\Lambda(u;Q)_{L^p(A)}$ denote the normalized best approximation of f on Q in L^p-norm:

$$\text{(2.1)} \qquad \begin{aligned}\Lambda(u;Q)_{L^p(A)} &:= |Q|^{-1/p} \inf_{C\in\mathbb{R}} \|u - C\|_{L^p(Q\cap A)} \\ &= \inf_{C\in\mathbb{R}} \left(\frac{1}{|Q|}\int_{Q\cap A} |u - C|^p \,\mathrm{d}x\right)^{1/p}.\end{aligned}$$

By $u_{1,A}^\#$, we denote the sharp maximal function of u on A,

$$u_{1,A}^\#(x) := \sup_{r>0} r^{-1}\Lambda(u;Q(x,r))_{L^1(A)}, \quad x \in A.$$

Notice that $u_1^\# = u_{1,\mathbb{R}^n}^\#$.

The following trace theorem by Shvartsman from [14] relates local polynomial approximation to extendability.

Theorem 2.2. *Let A be a regular subset of $\mathbb{R}^n$. Then, a function $u \in L^p(A)$, $1 < p \leq \infty$, can be extended to a function $Eu \in W^{1,p}(\mathbb{R}^n)$ if and only if*

$$u_{1,A}^{\#} := \sup_{r>0} r^{-1}\Lambda(u; Q(\cdot, r))_{L^1(A)} \in L^p(A).$$

In addition,

$$\|u\|_{W^{1,p}(\mathbb{R}^n)\big|_A} \approx \|u\|_{L^p(A)} + \|u_{1,A}^{\#}\|_{L^p(A)}$$

with constants of equivalence depending only on n, p, C_A, and δ_A. Here,

$$\|u\|_{W^{1,p}(\mathbb{R}^n)|_A} := \inf\Big\{\|Eu\|_{W^{1,p}(\mathbb{R}^n)} : Eu \in W^{1,p}(\mathbb{R}^n), Eu\big|_A \equiv u \text{ almost everywhere}\Big\}.$$

For a set $A \subset \mathbb{R}^n$ of positive Lebesgue measure, we set

$$C^{1,p}(A) = \{u \in L^p(A) : u_{1,A}^{\#} \in L^p(A)\},$$
$$\|u\|_{C^{1,p}(A)} = \|u\|_{L^p(A)} + \|u_{1,A}^{\#}\|_{L^p(A)}.$$

A result due to Hajłasz, Koskela, and Tuominen (Theorem 5 in [9]) that partially relies on Theorem 2.2 states the following.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and fix $1 < p < \infty$. Then, the following conditions are equivalent:*

(a) *For every $u \in W^{1,p}(\Omega)$ there exists a function $Eu \in W^{1,p}(\mathbb{R}^n)$ such that $Eu|_\Omega = u$ almost everywhere.*

(b) *Ω is regular, $C^{1,p}(\Omega) = W^{1,p}(\Omega)$ as sets, and the corresponding norms are equivalent.*

(c) *Ω is a $W^{1,p}$-extension domain.*

In [14], Shvartsman constructed an extension operator for Theorem 2.2 explicitly as a variant of the Whitney-Jones extension. We describe this procedure in the next section. In particular, based on Theorem 2.3, for an arbitrary $W^{1,p}$-extension domain Ω with $1 < p < \infty$, there is a Whitney-type extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^n)$. (For an alternate Whitney-type extension operator, see [10].)

2.2. Whitney-type extension. Given a constant $\lambda > 0$ and $Q = Q(x, r)$, we let λQ denote the cube $Q(x, \lambda r)$. By Q^* we denote the cube $Q^* := \frac{9}{8}Q$.

Next, given subsets $A, B \subset \mathbb{R}^n$, we set $\operatorname{diam} A := \sup\{\|a - a'\|_\infty : a,\ a' \in A\}$ and

$$\operatorname{dist}(A, B) := \inf\{\|a - b\|_\infty : a \in A,\ b \in B\}.$$

Furthermore, we set $\operatorname{dist}(x, A) := \operatorname{dist}(\{x\}, A)$ for $x \in \mathbb{R}^n$. The closure of A in $\mathbb{R}^n$ is denoted $\bar{A}$, and we denote the boundary of A by $\partial A := \bar{A} \setminus A$. The characteristic function of A is referred to by χ_A.

The following property is well known (see, e.g., [15]).

Lemma 2.4. *If A is a regular subset of $\mathbb{R}^n$, then $|\partial A| = 0$.*

In what follows, we will assume that S is a closed regular subset of $\mathbb{R}^n$. Since $\mathbb{R}^n \setminus S$ is an open set, it admits a Whitney decomposition W_S (e.g., see Stein [17]). Let us recall the main properties of W_S.

Theorem 2.5. *$W_S = \{Q\}$ is a countable family of closed cubes such that the following hold:*

(i) $\mathbb{R}^n \setminus S = \bigcup\{Q : Q \in W_S\}$.

(ii) *For every cube $Q \in W_S$,*

$$\operatorname{diam} Q \le \operatorname{dist}(Q, S) \le 4 \operatorname{diam} Q.$$

(iii) *No point of $\mathbb{R}^n \setminus S$ is contained in more than $N = N(n)$ distinct cubes from W_S.*

The following properties easily follow from (i)–(iii).

Lemma 2.6.

(1) *If $Q, K \in W_S$ and $Q^* \cap K^* \neq \emptyset$, then*

$$\frac{1}{4} \operatorname{diam} Q \le \operatorname{diam} K \le 4 \operatorname{diam} Q.$$

(2) *For every cube $K \in W_S$ there are at most $N = N(n)$ cubes from the family $W_S^* := \{Q^* : Q \in W_S\}$ which intersect K^*.*

Let $\Phi_S := \{\varphi_Q : Q \in W_S\}$ be a smooth partition of unity subordinated to the Whitney decomposition W_S (see [17]).

Proposition 2.7. *There exists a family Φ_S of functions defined on $\mathbb{R}^n$ with the following properties:*

(a) $0 \le \varphi_Q(x) \le 1$ *for every $Q \in W_S$.*

(b) $\operatorname{supp} \varphi_Q \subset Q^*$ $(:= \frac{9}{8}Q)$, $Q \in W_S$.

(c) $\sum\{\varphi_Q(x) : Q \in W_S\} = 1$ *for every $x \in \mathbb{R}^n \setminus S$.*

(d) *For every multi-index β with $|\beta| = 1$, and every cube $Q \in W_S$,*

$$|D^\beta \varphi_Q(x)| \le C(\operatorname{diam} Q)^{-1}, \quad x \in \mathbb{R}^n,$$

where C is a constant depending only on n.

Actually, the family of cubes W_S constructed in [17] satisfies the conditions of Theorem 2.5 and Lemma 2.6 with respect to the Euclidean norm instead of the uniform one. A simple modification to that construction gives a family of Whitney cubes which have the analogous properties with respect to our uniform norm.

Let $K = Q(x_K, r_K) \in W_S$, and let $a_K \in S$ be a point nearest to x_K on S. Then, by property (ii) of Theorem 2.5,

$$Q(a_K, r_K) \subset 10K.$$

Fix a small $0 < \varepsilon \le 1$ and set $K_\varepsilon := Q(a_K, \varepsilon r_K)$. Let $Q = Q(x_Q, r_Q)$ be a cube from W_S with $\operatorname{diam} Q \le \delta_S$, where δ_S is as in Definition 2.1 for our regular sets. Set

$$\mathcal{A}_Q := \{K = Q(x_K, r_K) \in W_S : K_\varepsilon \cap Q_\varepsilon \ne \varnothing,\ r_K \le \varepsilon r_Q\},$$

where $Q_\varepsilon := Q(a_Q, \varepsilon r_Q)$. We define a "quasi-cube" H_Q by setting

$$H_Q := (Q_\varepsilon \cap S) \setminus \Big(\bigcup\{K_\varepsilon : K \in \mathcal{A}_Q\}\Big).$$

If $\operatorname{diam} Q > \delta_S$, we simply set $H_Q := \varnothing$.

The following result is Theorem 2.4 in [14].

Theorem 2.8. *Let S be a closed regular subset of $\mathbb{R}^n$. Then, there is a family of "quasi-cubes" $\mathcal{H}_\Omega = \{H_Q : Q \in W_S\}$ as discussed above with the following:*

(i) *$H_Q \subset (10Q) \cap S$ whenever $Q \in W_S$.*
(ii) *$|Q| \le \gamma_1 |H_Q|$ whenever $Q \in W_S$ satisfies* $\operatorname{diam} Q \le \delta_S$.
(iii) *$\sum_{Q \in W_S} \chi_{H_Q} \le \gamma_2$.*

Here, γ_1 and γ_2 are positive constants depending only on n and C_A.

Next, we present estimates on local polynomial approximations of the extension Ef, via the corresponding local approximation of a function f defined on a closed regular subset $S \subset \mathbb{R}^n$.

Given a measurable subset $A \subset \mathbb{R}^n$ and a function $u \in L^p(A)$, $1 \le p \le \infty$, we let $\hat{E}_1(u; A)_{L^p}$ denote the local best constant approximation in L^p-norm (see Brudnyi [3]):

$$\hat{E}_1(u; A)_{L^p} := \inf_{C \in \mathbb{R}} \|u - C\|_{L^p(A)}.$$

Thus,

$$\Lambda(u; Q)_{L^p(A)} = |Q|^{-1/p} \hat{E}_1(u; Q \cap A)_{L^p}$$

(see (2.1)). We note a simple property of $\Lambda(u; \cdot)_{L^p(A)}$ as a function of cubes: for every pair of cubes $Q_1 \subset Q_2$,

$$\Lambda(u; Q_1)_{L^p(A)} \le \left(\frac{|Q_2|}{|Q_1|}\right)^{1/p} \Lambda(u; Q_2)_{L^p(A)}.$$

Let A be a subset of $\mathbb{R}^n$ with $|A| > 0$. We set

$$P_A(u) := \fint_A u(x)\,\mathrm{d}x = \frac{1}{|A|}\int_A u(x)\,\mathrm{d}x. \tag{2.2}$$

Then, from a result of Brudnyi in [5] (see also Proposition 3.4 in [14]), we have the following estimate.

Proposition 2.9. *Let A be a subset of a cube Q with $|A| > 0$. Then, the linear operator $P_A : L^1(A) \to \mathbb{R}$ has the property that, for every $1 \le p \le \infty$ and every $u \in L^p(A)$,*

$$\|u - P_A(u)\|_{L^p(A)} \le C\hat{E}_1(u; A)_{L^p}.$$

Here, $C = C(n, |Q|/|A|)$.

According to Lemma 2.4, the boundary of a regular set is of measure zero, and so Proposition 2.9 together with Theorem 2.8 immediately implies the following corollary.

Corollary 2.10. *Let S be a closed regular set and let $Q \in W_S$ be a cube with* $\operatorname{diam} Q \le \delta_S$*. There is a continuous linear operator $P_{H_Q} : L^1(H_Q) \to \mathbb{R}$ such that for every function $u \in L^p(S)$, $1 \le p \le \infty$,*

$$\|u - P_{H_Q}(u)\|_{L^p(H_Q)} \le \gamma\hat{E}_1(u; H_Q)_{L^p}.$$

Here, $\gamma = \gamma(n, k, \theta_S)$.

We set

$$P_{H_Q}u = 0, \quad \text{if } \operatorname{diam} Q > \delta_S.$$

Then, the map $Q \to P_{H_Q}(f)$ is defined on all of the cubes in the family W_S. This map gives rise to a bounded linear extension operator from $L^p(S)$ to $L^p(\mathbb{R}^n)$, defined by the formula

$$Eu(x) := \begin{cases} u(x), & x \in S, \\ \sum_{Q \in W_S} \varphi_Q(x)(P_{H_Q}u)(x), & x \in \mathbb{R}^n \setminus S. \end{cases} \tag{2.3}$$

Given a regular domain $\Omega \subset \mathbb{R}^n$, $\bar{\Omega}$ is a closed regular set with $|\bar{\Omega} \setminus \Omega| = 0$. Given a function $u \in L^p(\Omega)$, the zero extension of u to $S := \bar{\Omega}$ (still denoted u) belongs to $L^p(S)$, and we define the extension Eu of u to $\mathbb{R}^n$ by the formula (2.3). When $u \in C^{1,p}(S)$, Eu here is precisely the Eu from Theorem 2.2. By combining Theorem 2.2 and Theorem 2.3, we obtain the following result.

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^n$ be a $W^{1,p}$-extension domain for a fixed $1 < p < \infty$. Then, for every $u \in W^{1,p}(\Omega)$ and Eu defined as in (2.3) for the zero extension of u to the closure of Ω, we have $Eu \in W^{1,p}(\mathbb{R}^n)$ and*

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)},$$

for a positive constant C independent of u.

3. Proof of Theorem 1.1

The first part of our main theorem (for $1 < p < \infty$) will be obtained as a consequence of the following extension result that we believe to be of independent interest.

Theorem 3.1. *Let* $\Omega_1 \subset \mathbb{R}^n$ *be a* $W^{1,p}$*-extension domain for a given* $1 < p < \infty$, *and* $\Omega_2 \subset \mathbb{R}^m$ *be a domain. Then, for every function* $u \in W^{1,p}(\Omega_1 \times \Omega_2)$, *there exists a function* $E_1 u \in W^{1,p}(\mathbb{R}^n \times \Omega_2)$ *such that* $E_1 u|_{\Omega_1 \times \Omega_2} \equiv u$ *and*

$$\|E_1 u\|_{W^{1,p}(\mathbb{R}^n \times \Omega_2)} \leq C \|u\|_{W^{1,p}(\Omega_1 \times \Omega_2)}$$

with a positive constant C *independent of* u.

Proof. Theorem 2.3.2 in Ziemer's book [20] tells us that

$$C^\infty(\Omega_1 \times \Omega_2) \cap W^{1,p}(\Omega_1 \times \Omega_2)$$

is dense in $W^{1,p}(\Omega_1 \times \Omega_2)$. With a small modification to the proof of this result it is easy to see that $C^1(\Omega_1 \times \Omega_2) \cap L^\infty(\Omega_1 \times \Omega_2) \cap W^{1,p}(\Omega_1 \times \Omega_2)$ is dense in $W^{1,p}(\Omega_1 \times \Omega_2)$. We shall begin by showing that we can extend functions in $C^1(\Omega_1 \times \Omega_2) \cap L^\infty(\Omega_1 \times \Omega_2) \cap W^{1,p}(\Omega_1 \times \Omega_2)$.

According to Theorem 2.3, Ω_1 is regular. Let

$$u \in C^1(\Omega_1 \times \Omega_2) \cap L^\infty(\Omega_1 \times \Omega_2) \cap W^{1,p}(\Omega_1 \times \Omega_2).$$

Then, for $y \in \Omega_2$, using the extension (2.3), we set

$$E_1 u(x,y) = E u_y(x) := \begin{cases} u_y(x), & x \in \overline{\Omega_1}, \\ \sum_{Q \in W_{\overline{\Omega_1}}} \varphi_Q(x)(P_{H_Q} u_y)(x), & x \in \mathbb{R}^n \setminus \overline{\Omega_1}. \end{cases} \tag{3.1}$$

Here, u_y in (3.1) is the zero extension of u_y to the closure $\overline{\Omega_1}$. To show that $E_1 u \in W^{1,p}(\mathbb{R}^n \times \Omega_2)$, we need to show that $E_1 u \in L^p(\mathbb{R}^n \times \Omega_2)$, and for every β with $|\beta| = 1$, we need to find a function $v_\beta \in L^p(\mathbb{R}^n \times \Omega_2)$, such that for every $\psi \in C_0^\infty(\mathbb{R}^n \times \Omega_2)$ we have

$$\int_{\mathbb{R}^n \times \Omega_2} E_1 u(x,y) \mathrm{D}^\beta \psi(x,y)\, \mathrm{d}x\, \mathrm{d}y = -\int_{\mathbb{R}^n \times \Omega_2} v_\beta(x,y)\psi(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

For the convenience of discussion, we divide the rest of the proof into three steps.

STEP 1: In this step, we show that $E_1 u \in L^p(\mathbb{R}^n \times \Omega_2)$ and that the L^p-norm of $E_1 u$ is controlled by the $W^{1,p}$-norm of u.

By Fubini's theorem, $u_y \in W^{1,p}(\Omega_1)$ for almost every $y \in \Omega_2$. As Ω_1 is a $W^{1,p}$-extension domain, by Theorem 2.11, $E_1 u(x,y) = E u_y(x) \in W^{1,p}(\mathbb{R}^n)$ and

$$\|E u_y\|_{L^p(\mathbb{R}^n)} \leq \|E u_y\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u_y\|_{W^{1,p}(\Omega_1)},$$

for every $y \in \Omega_2$ with $u_y \in W^{1,p}(\Omega_1)$. Then, by integrating with respect to $y \in \Omega_2$, we obtain the desired result.

STEP 2: In this step, we show that, for every $\psi \in C_c^\infty(\mathbb{R}^n \times \Omega_2)$, there exist functions $(\partial/\partial x_i)E_1 u \in L^p(\mathbb{R}^n \times \Omega_2)$ (for $i = 1, \dots, n$) such that

$$\int_{\mathbb{R}^n \times \Omega_2} \frac{\partial}{\partial x_i} E_1 u(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = -\int_{\mathbb{R}^n \times \Omega_2} E_1 u(x,y) \frac{\partial}{\partial x_i}\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

For simplicity of notation, we assume that $i = 1$.

Fubini's theorem tells us that $u_y \in W^{1,p}(\Omega_1)$ for almost every $y \in \Omega_2$. Then, by Theorem 2.11, (3.1) gives an extension $Eu_y \in W^{1,p}(\mathbb{R}^n)$ for every $y \in \Omega_2$ with $u_y \in W^{1,p}(\Omega_1)$. Then, we set

$$(3.2) \qquad \frac{\partial}{\partial x_1} E_1 u(x,y) := \begin{cases} \dfrac{\partial}{\partial x_1} Eu_y(x), & \text{if } y \in \Omega_2 \text{ with } u_y \in W^{1,p}(\Omega_1), \\ 0, & \text{otherwise.} \end{cases}$$

Since $Eu_y \in W^{1,p}(\mathbb{R}^n)$ for almost every $y \in \Omega_2$, using Fubini's theorem, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n \times \Omega_2} \frac{\partial}{\partial x_1} E_1 u(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_{\Omega_2}\int_{\mathbb{R}^n} \frac{\partial}{\partial x_1} Eu_y(x)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = -\int_{\Omega_2}\int_{\mathbb{R}^n} Eu_y(x)\frac{\partial}{\partial x_1}\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = -\int_{\mathbb{R}^n \times \Omega_2} E_1 u(x,y)\frac{\partial}{\partial x_1}\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

which means that (3.2) gives a first-order distributional derivative of $E_1 u$ with respect to x_1. Then, using the Fubini theorem twice and the fact that the linear operator E from $W^{1,p}(\Omega_1)$ to $W^{1,p}(\mathbb{R}^n)$ is bounded, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n \times \Omega_2} \left|\frac{\partial}{\partial x_1} E_1 u(x,y)\right|^p \mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_{\Omega_2}\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_1} Eu_y(x)\right|^p \mathrm{d}x\,\mathrm{d}y \\ &\quad \le C\int_{\Omega_2}\int_{\Omega_1} \left(|u_y(x)|^p + \left|\frac{\partial}{\partial x_1} u_y(x)\right|^p\right) \mathrm{d}x\,\mathrm{d}y \\ &\quad \le C\int_{\Omega_1 \times \Omega_2} \left(|u(x,y)|^p + \left|\frac{\partial}{\partial x_1} u(x,y)\right|^p\right) \mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

We have obtained the desired norm estimate.

STEP 3: In this step, we show that, for every $\psi \in C_c^\infty(\mathbb{R}^n \times \Omega_2)$, there exist functions $(\partial/\partial y_j)E_1u \in L^p(\mathbb{R}^n \times \Omega_2)$ (for $j = 1, \dots, m$) such that

$$\int_{\mathbb{R}^n\times\Omega_2} \frac{\partial}{\partial y_j} E_1u(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = -\int_{\mathbb{R}^n\times\Omega_2} E_1u(x,y)\frac{\partial}{\partial y_j}\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

For simplicity of notation, we assume that $j = 1$.

Consider the projection

$$\Pi_1 : \Omega_2 \to \mathbb{R}^{m-1},$$

defined by setting

$$\Pi_1(y) = (y_2, y_3, \dots, y_m) =: \check{y}_1 \quad \text{for } y = (y_1, \dots, y_m) \in \Omega_2.$$

Set $S_1^{\check{y}_1} := \Pi_1^{-1}(\check{y}_1) \subset \Omega_2$, the preimage of $\check{y}_1 \in \Pi_1(\Omega_2)$. Then, $S_1^{\check{y}_1}$ is the union of at most countably many pairwise disjoint segments.

Fix $x \in \mathbb{R}^n \setminus \overline{\Omega_1}$ and $\check{y}_1 \in \Pi_1(\Omega_2)$. To begin, we assume that $S_1^{\check{y}_1}$ is a single segment. Now, for $(y_1^1, \check{y}_1), (y_1^2, \check{y}_1) \in S_1^{\check{y}_1}$, according to (3.1), we have

$$\begin{aligned} E_1u(x,y_1^2,\check{y}_1) &- E_1u(x,y_1^1,\check{y}_1) \\ &= \sum_{Q\in W_{\overline{\Omega_1}}} \varphi_Q(x)((P_{H_Q}u)(x,y_1^2,\check{y}_1) - (P_{H_Q}u)(x,y_1^1,\check{y}_1)). \end{aligned} \tag{3.3}$$

By the definition (2.2) of $P_{H_Q}u$ and the facts that u is C^1 and $H_Q \times S_1^{\check{y}_1} \subset \Omega_1 \times \Omega_2$, we have

$$\begin{aligned} (P_{H_Q}u)(x,y_1^2,\check{y}_1) &- (P_{H_Q}u)(x,y_1^1,\check{y}_1) \\ &= \fint_{H_Q} (u(w,y_1^2,\check{y}_1) - u(w,y_1^1,\check{y}_1))\,\mathrm{d}w \\ &= \fint_{H_Q} \left(\int_{y_1^1}^{y_1^2} \frac{\partial u(w,s,\check{y}_1)}{\partial y_1}\,\mathrm{d}s\right)\mathrm{d}w. \end{aligned} \tag{3.4}$$

By combining (3.3) and (3.4) we obtain

$$\begin{aligned} E_1u(x,y_1^1,\check{y}_1) &- E_1u(x,y_1^2,\check{y}_1) \\ &= \sum_{Q\in W_{\overline{\Omega_1}}} \varphi_Q(x) \fint_{H_Q} \int_{y_1^1}^{y_1^2} \frac{\partial u(w,s,\check{y}_1)}{\partial y_1}\,\mathrm{d}s\,\mathrm{d}w. \end{aligned} \tag{3.5}$$

Since x is contained in the support of only finitely many φ_Q, it follows from the identity (3.5) that $E_1u(x,s,\check{y}_1)$ is absolutely continuous as a function of s on $S_1^{\check{y}_1}$.

By repeating this for each component of $S_1^{\check{y}_1}$, we conclude that $E_1u(x, s, \check{y}_1)$ is absolutely continuous as a function of s on every component of $S_1^{\check{y}_1}$. Furthermore, (3.5) and the Lebesgue differentiation theorem yield that

$$\begin{aligned}
\text{(3.6)} \qquad \frac{\partial E_1u(x, s, \check{y}_1)}{\partial y_1} &:= \lim_{s' \to s} \frac{E_1u(x, s', \check{y}_1) - E_1u(x, s, \check{y}_1)}{s' - s} \\
&= \sum_{Q \in W_{\overline{\Omega_1}}} \varphi_Q(x) \fint_{H_Q} \frac{\partial u(w, s, \check{y}_1)}{\partial y_1} \, \mathrm{d}w \\
&= E_1 \frac{\partial u(x, s, \check{y}_1)}{\partial y_1}
\end{aligned}$$

exists for $\mathcal{H}^1$-almost every s with $(s, \check{y}_1) \in S_1^{\check{y}_1}$. Fix $\psi \in C_c^\infty(\mathbb{R}^n \times \Omega_2)$. Since $E_1u(x, s, \check{y}_1)$ is absolutely continuous as a function of s on each segment of $S_1^{\check{y}_1}$, we conclude that

$$\int_{S_1^{\check{y}_1}} E_1u(x, s, \check{y}_1) \frac{\partial \psi(x, s, \check{y}_1)}{\partial y_1} \, \mathrm{d}s = - \int_{S_1^{\check{y}_1}} \frac{\partial E_1u(x, s, \check{y}_1)}{\partial y_1} \psi(x, s, \check{y}_1) \, \mathrm{d}s.$$

To complete the definition of $\partial E_1u/\partial y_1$, we define $\partial E_1u/\partial y_1 = \partial u/\partial y_1$ when $(x, y) \in \Omega_1 \times \Omega_2$, and set $\partial E_1u/\partial y_1 = 0$ when $(x, y) \in \partial\Omega_1 \times \Omega_2$. Let us show that $\partial E_1u/\partial y_1$ is a first-order distributional derivative of E_1u with respect to the variable y_1. By the Fubini theorem, (3.6) and the fact that $|\partial\Omega_1| = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n \times \Omega_2} E_1u(x, y) \frac{\partial \psi(x, y)}{\partial y_1} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad = \int_{\mathbb{R}^n} \int_{\Pi_1(\Omega_2)} \int_{S_1^{\check{y}_1}} E_1u(x, y) \frac{\partial \psi(x, y)}{\partial y_1} \, \mathrm{d}y_1 \, \mathrm{d}\check{y}_1 \, \mathrm{d}x \\
&\quad = - \int_{\mathbb{R}^n} \int_{\Pi_1(\Omega_2)} \int_{S_1^{\check{y}_1}} \frac{\partial E_1u(x, y)}{\partial y_1} \psi(x, y) \, \mathrm{d}y_1 \, \mathrm{d}\check{y}_1 \, \mathrm{d}x \\
&\quad = - \int_{\mathbb{R}^n \times \Omega_2} \frac{\partial E_1u(x, y)}{\partial y_1} \psi(x, y) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

We continue by showing that $\partial E_1u/\partial y_1 \in L^p(\mathbb{R}^n \times \Omega_2)$ and that its norm is controlled by the Sobolev norm of u. Since $|\partial\Omega_1| = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n \times \Omega_2} \left| \frac{\partial E_1u(x, y)}{\partial y_1} \right|^p \mathrm{d}x \, \mathrm{d}y \\
&\quad = \int_{\Omega_1 \times \Omega_2} \left| \frac{\partial u(x, y)}{\partial y_1} \right|^p \mathrm{d}x \, \mathrm{d}y + \int_{(\mathbb{R}^n \setminus \overline{\Omega_1}) \times \Omega_2} \left| E_1 \frac{\partial u(x, y)}{\partial y_1} \right|^p \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

As we know, for almost every $y \in \Omega_2$, $(\partial u/\partial y_1)|_y \in L^p(\Omega_1)$. Using the fact that $E : L^p(\Omega_1) \to L^p(\mathbb{R}^n)$ is a bounded linear operator, we obtain

$$\int_{\mathbb{R}^n} \left| E_1 \frac{\partial u(x,y)}{\partial y_1} \right|^p \mathrm{d}x \le C \int_{\Omega_1} \left| \frac{\partial u(x,y)}{\partial y_1} \right|^p \mathrm{d}x,$$

for almost every $y \in \Omega_2$. Then, by integration with respect to $y \in \Omega_2$ on the two sides of the inequality above, we obtain the desired inequality

$$\int_{\mathbb{R}^n \times \Omega_2} \left| \frac{\partial E_1 u(x,y)}{\partial y_1} \right|^p \mathrm{d}x\,\mathrm{d}y \le C \int_{\Omega_1 \times \Omega_2} \left| \frac{\partial u(x,y)}{\partial y_1} \right|^p \mathrm{d}x\,\mathrm{d}y.$$

In conclusion, we have shown that the linear extension operator E_1 is bounded from $C^1(\Omega_1 \times \Omega_2) \cap L^\infty(\Omega_1 \times \Omega_2) \cap W^{1,p}(\Omega_1 \times \Omega_2)$ to $W^{1,p}(\mathbb{R}^n \times \Omega_2)$ for our fixed $1 < p < \infty$. Since then $C^1(\Omega_1 \times \Omega_2) \cap L^\infty(\Omega_1 \times \Omega_2) \cap W^{1,p}(\Omega_1 \times \Omega_2)$ is dense in $W^{1,p}(\Omega_1 \times \Omega_2)$, E_1 extends to a bounded linear extension operator from $W^{1,p}(\Omega_1 \times \Omega_2)$ to $W^{1,p}(\mathbb{R}^n \times \Omega_2)$. ◻

Proof of Theorem 1.1. Regarding the first part of the claim, by Theorem 3.1 we have a bounded extension operator $E_1 : W^{1,p}(\Omega_1 \times \Omega_2) \to W^{1,p}(\mathbb{R}^n \times \Omega_2)$, and it thus suffices to extend functions in $W^{1,p}(\mathbb{R}^n \times \Omega_2)$ to $W^{1,p}(\mathbb{R}^n \times \mathbb{R}^m)$. Given $u \in W^{1,p}(\mathbb{R}^n \times \Omega_2)$, define $\hat{u}(x,y) = u(y,x)$. Then, $\hat{u} \in W^{1,p}(\Omega_2 \times \mathbb{R}^n)$, and the desired extension is obtained via Theorem 3.1 as $\Omega_2 \subset \mathbb{R}^m$ is a $W^{1,p}$-extension domain.

Towards the second part, by symmetry, it suffices to prove that $\Omega_1 \subset \mathbb{R}^n$ must be a $W^{1,p}$-extension domain whenever $\Omega_1 \times \Omega_2$ is such a domain.

Suppose first that Ω_2 has finite measure. Given $u \in W^{1,p}(\Omega_1)$, we define $v(x,y) = u(x)$. Then, $v \in W^{1,p}(\Omega_1 \times \Omega_2)$. Let $Ev \in W^{1,p}(\mathbb{R}^n \times \mathbb{R}^m)$ be an extension of v. Then, $Ev \in W^{1,p}(\mathbb{R}^n \times \{y\})$ for almost every $y \in \Omega_2$. This follows via the Fubini theorem from the ACL-characterization of functions in $W^{1,p}$, given in our introduction. Since $v(x,y) = u(x)$, we conclude that u must be the restriction of some function $w \in W^{1,p}(\mathbb{R}^n)$. This allows us to infer from Theorem 2.3 that Ω_1 must be a $W^{1,p}$-extension domain.

In case Ω_2 has infinite measure, we fix a ball $B \subset \Omega_2$ and pick a smooth function ψ with compact support so that ψ is identically 1 on B. We still define v as above and set $w = \psi v$. Then, $w \in W^{1,p}(\Omega_1 \times \Omega_2)$, and we may repeat the above argument as $w(x,y) = u(x)$ for almost every $y \in B \subset \Omega_2$. ◻

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Creating and Flattening Cusp Singularities by Deformations of Bi-conformal Energy

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Abstract

Mappings of bi-conformal energy form the widest class of homeomorphisms that one can hope to build a viable extension of Geometric Function Theory with connections to mathematical models of Nonlinear Elasticity. Such mappings are exactly the ones with finite conformal energy and integrable inner distortion. It is in this way that our studies extend the applications of quasiconformal homeomorphisms to the degenerate elliptic systems of PDEs. The present paper searches a bi-conformal variant of the Riemann Mapping Theorem, focusing on domains with exemplary singular boundaries that are not quasiballs. We establish the sharp description of boundary singularities that can be created and flattened by mappings of bi-conformal energy.

Keywords Cusp · Bi-conformal energy · Mappings of integrable distortion · quasiball

Mathematics Subject Classification Primary 30C65

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1 Introduction

There is a broad literature dealing with a question when a pair of domains $(\mathbb{X}, \mathbb{Y})$ is quasiconformally or even bi-Lipschitz equivalent. Gehring and Väisälä [11] raised the question: *Which domains* $D \subset \mathbb{R}^n$ *are quasiconformally equivalent with the unit ball* $\mathbb{B} \subset \mathbb{R}^n$? Such domains D are called *quasiballs*. The interested reader is referred to the recent book by Gehring et al. [10]. The Riemann Mapping Theorem gives a complete answer to this question when $n = 2$. If $D \subsetneq \mathbb{C}$ is a simply connected domain, then there exists a conformal mapping $h\colon \mathbb{B} \xrightarrow{\text{onto}} D$. It is, however, a highly nontrivial question when a domain $D \subset \mathbb{R}^n$ is a quasiball when $n \geqslant 3$. Among geometric obstructions are the inward cusps. Indeed, Gehring and Väisälä [11] proved that a ball with inward cusp is not a quasiball. A ball with outward cusp, however, is always a quasiball. We denote an n-dimensional unit balls with exemplary boundary singularities of the form of cusps by $\mathbb{B}_u$ where $u\colon [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ is a strictly increasing function and which characterizes the singularity at the origin, see Fig. 1 and Sect. 1.6 for the precise definition.

In this article, we describe boundary singularities that can be created by finite n-*harmonic energy* and return to the original shape by the inverse deformations whose n-harmonic energy is finite as well. This is in accordance with the Hooke's Law, see Sect. 1.4. We remind the reader that there exists a Lipschitz homeomorphism to both directions between $\mathbb{B}$ and $\mathbb{B}_u$. However, it is not always possible to have $\mathscr{W}^{1,n}$-Sobolev bounds to both directions for a single map. We state this as follows.

Theorem 1.1 *Let* $n \geqslant 3$ *and*

$$u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}} \quad \textit{for } 0 \leqslant t \leqslant 1\,, \quad \textit{where } \alpha > 0. \tag{1.1}$$

Then there exists a homeomorphism $h\colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ *in* $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$ *whose inverse* $f = h^{-1}\colon \mathbb{B}_u \xrightarrow{\text{onto}} \mathbb{B}$ *lies in* $\mathscr{W}^{1,n}(\mathbb{B}_u, \mathbb{R}^n)$ *if and only if* $\alpha < n$.

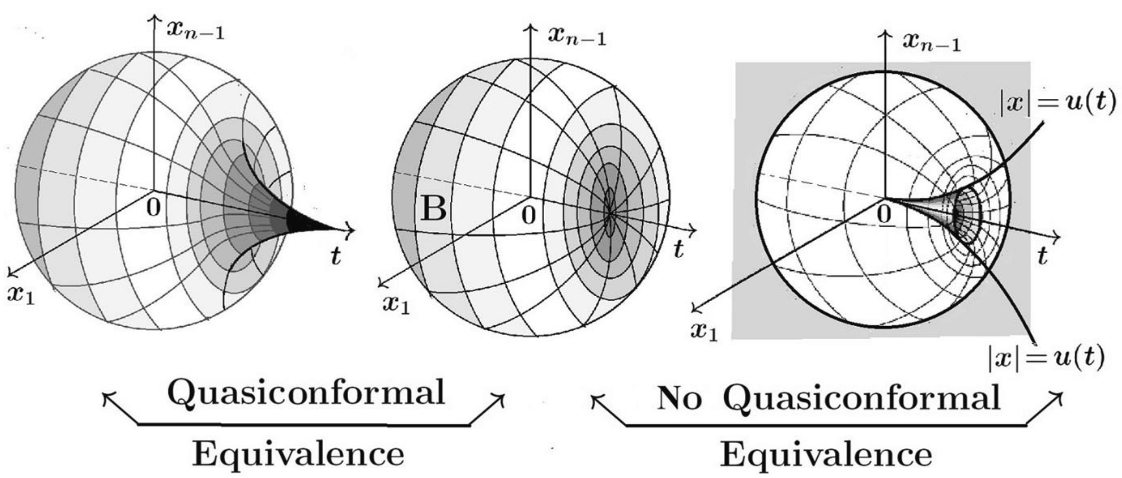


Fig. 1 Quasiconformal mapping can flatten the outward cusp but not the inward cusp

Theorem 1.1 is a special case of our main result (Theorem 1.11). Before formulating Theorem 1.11 we discuss the studied mapping problem in more details.

We are concerned with orientation-preserving homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between bounded domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$, $n \geqslant 2$, of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \leqslant p \leqslant \infty$.

1.1 Quasiconformal Deformations

Of particular interest are homeomorphisms of finite *n-harmonic energy*; that is, with $p = n$.

$$\mathsf{E}_{\mathbb{X}}[h] \overset{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x \; < \infty. \tag{1.2}$$

Hereafter the symbol $|Dh(x)|$ stands for the operator norm of the differential matrix $Dh(x) \in \mathbb{R}^{n\times n}$ called the *deformation gradient*. This integral is invariant under the conformal change of variables in the *reference configuration* $\mathbb{X}$ (not in the *deformed configuration* $\mathbb{Y}$). That is, $\mathsf{E}_{\mathbb{X}'}[h'] = \mathsf{E}_{\mathbb{X}}[h]$, where $h' = h \circ \varphi$ for a conformal transformation $\varphi \colon \mathbb{X}' \xrightarrow{\text{onto}} \mathbb{X}$. This motivates us to call $\mathsf{E}_{\mathbb{X}}[h]$ the **conformal energy** of h. Mappings of conformal energy arise naturally in Geometric Function Theory (GFT) for many reasons [2,11,13,16,26].

Definition 1.2 A *Sobolev homeomorphism* $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, that is, of class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$, is said to be *quasiconformal* if there exists a constant $1 \leqslant \mathcal{K} < \infty$ so that for almost every $x \in \mathbb{X}$ it holds:

$$|Dh(x)|^n \leqslant \mathcal{K}\, J_h(x)\,, \qquad \text{where} \quad J_h(x) = \det Dh(x).$$

Every quasiconformal map $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ has finite conformal energy provided $|\mathbb{Y}| < \infty$. Indeed,

$$\mathsf{E}_{\mathbb{X}}[h] = \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x \; \leqslant \mathcal{K} \int_{\mathbb{X}} J_h(x) \mathrm{d}x \; = \; \mathcal{K}\, |\mathbb{Y}|. \tag{1.3}$$

1.2 Mappings of Bi-conformal Energy

The remarkable feature of a quasiconformal mapping is that its inverse $f \overset{\text{def}}{=\!=} h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ is also quasiconformal. In particular, both h and f have finite conformal energy. Their sum

$$\mathsf{E}_{\mathbb{X}\mathbb{Y}}[h] \overset{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x + \int_{\mathbb{Y}} |Df(y)|^n \, \mathrm{d}y \overset{\text{def}}{=\!=} \mathsf{E}_{\mathbb{Y}\mathbb{X}}[f] \tag{1.4}$$

will be called *bi-conformal energy* of h .

This leads us to a viable extension of GFT with connections to mathematical models of Nonlinear Elasticity (NE) [1,4,6,22].

Definition 1.3 A homeomorphism $h\colon \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$, whose inverse $f = h^{-1}\colon \mathbb{Y} \overset{\text{onto}}{\longrightarrow} \mathbb{X}$ also belongs to $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$ is called a *mapping of bi-conformal energy*.

It is equivalent to saying that the inner distortion function of h is integrable over $\mathbb{X}$ and the inner distortion function of f is integrable over $\mathbb{Y}$. For a precise statement (Theorem 1.5) we need some definitions.

1.3 Inner Distortion

Consider a Sobolev mapping $h \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$ and its *co-differential* $D^\sharp h(x) \in \mathbb{R}^{n\times n}$ - the matrix determined by Cramer's rule $D^\sharp h \circ Dh = J_h(x)\,\mathbf{I}$.

Definition 1.4 The inner distortion of h is the smallest measurable function $K_I(x) = K_I(x, h) \in [1, \infty]$ such that

$$|D^\sharp h(x)|^n \leqslant K_I(x) J_h(x)^{n-1} \qquad \text{for almost every } x \in \mathbb{X}. \tag{1.5}$$

The question of finite inner distortion merely asks for the co-differential $D^\sharp h(x) = 0$ at the points where the Jacobian $J_h(x) = 0$. However, for $n \geqslant 3$, the differential $Dh(x)$ need not vanish if $D^\sharp h(x) = 0$.

A formal algebraic computation reveals that the pullback of the n-form $K_I(x, h)\,\mathrm{d}x \in \wedge^n\mathbb{X}$ via the inverse mapping $f\colon \mathbb{Y} \overset{\text{onto}}{\longrightarrow} \mathbb{X}$ equals $|Df(y)|^n\,\mathrm{d}y \in \wedge^n\mathbb{Y}$.

This observation is the key to the fundamental equality between the $\mathscr{L}^1$-norm of $K_I(x, h)$ and conformal energy of the inverse map f, which is usually derived under various regularity assumptions [3,7,12,14,24]. We shall state in the following form:

Theorem 1.5 *Let $h\colon \mathbb{X} \to \mathbb{Y}$ be an orientation-preserving homeomorphism in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$, $n \geqslant 2$. Then the inner distortion of h is integrable if and only if the inverse mapping $f = h^{-1}\colon \mathbb{Y} \to \mathbb{X}$ has finite conformal energy. Furthermore, we have*

$$\int_{\mathbb{Y}} |Df(y)|^n\,\mathrm{d}y = \int_{\mathbb{X}} K_I(x, h)\,\mathrm{d}x. \tag{1.6}$$

Theorem 1.5 is known among the experts in the field and follows combining a few results in the literature. We will provide a proof for the convenience of the reader in the appendix. The interested reader is referred to [20] for planar mappings with integrable distortion (Stoilow factorization). The following corollary is immediate.

Corollary 1.6 *A homeomorphism $h\colon \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ of class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ is quasiconformal if and only if with $K_I(\cdot, h) \in \mathscr{L}^\infty(\mathbb{X})$.*

1.4 Hooke's Law for Materials of Conformal Stored-Energy

In a different direction, the principle of hyper-elasticity is to minimize the given stored-energy functional subject to deformations $h : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ of domains made of elastic materials, see [1,4,6,22]. Here we take on stage the materials of *conformal stored-energy*. This means that the bodies can endure only deformations $h : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ whose gradient Dh is integrable with power n (the dimension of the deformed body). A deformation of infinite n-harmonic energy would break the internal structure of the material causing permanent damage. There are examples galore in which one can return the deformed body to its original shape by a deformation of finite conformal energy, but not necessarily via the inverse mapping $f \overset{\text{def}}{=\!=} h^{-1} : \mathbb{Y} \overset{\text{onto}}{\longrightarrow} \mathbb{X}$. The inverse map need not even belong to $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$. On the other hand the essence of Hooke's Law is reversibility. Accordingly, we wish that both h and $f = h^{-1}$ have finite conformal energy. We call this model n*-harmonic hyper-elasticity*. It is from this point of view that we arrive at the following n-dimensional variant of the conformal Riemann mapping problem.

1.5 Mapping Problems

Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ be bounded domains of the same topological type. For each of the three problems below find conditions on the pair $(\mathbb{X}, \mathbb{Y})$ to ensure that:

(P1) There exists a bi-Lipschitz deformation $h\colon \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$.
(P2) There exists a quasiconformal deformation $h\colon \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$.
(P3) There exists deformation $h : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ of bi-conformal energy.

The implications (P1) $\Longrightarrow$ (P2) $\Longrightarrow$ $(P3)$ are straightforward.

1.6 Ball with Inward Cusp

We shall distinguish a horizontal coordinate axis in $\mathbb{R}^n$,

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x)\colon t \in \mathbb{R} \text{ and } x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\}$$

and introduce the notation

$$\rho = |x| \overset{\text{def}}{=\!=} \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}.$$

Consider a strictly increasing function $u\colon [0, \infty) \overset{\text{onto}}{\longrightarrow} [0, \infty)$ of class $\mathscr{C}^1(0, \infty) \cap \mathscr{C}[0, \infty)$. We assume that u' is increasing in $(0, \infty)$ and

$$\lim_{\rho \searrow 0} u'(\rho) = 0.$$

To every such function there corresponds an $(n-1)$-dimensional surface of revolution $\mathbf{S}_u \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$

$$\mathbf{S}_u \overset{\text{def}}{=\!=} \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{n-1}\colon\ |x| = u(t)\},\ \text{ where } \mathbb{R}_+ = [0, \infty).$$

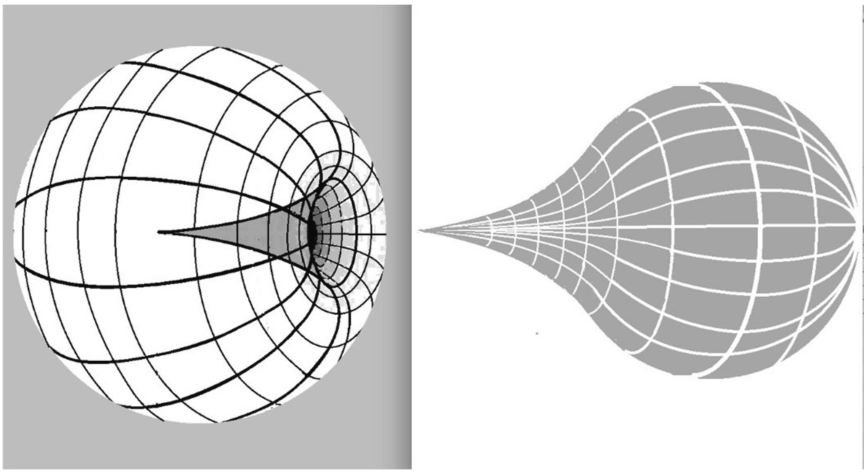


Fig. 2 Inward and outward cusp in a ball

We shall refer to $\mathbf{S}_u$ as *a model cusp* at the origin. Let us emphasize that the case $\limsup_{\rho \searrow 0} u'(\rho) > 0$ is **excluded** from this definition. We may (and do) rescale u so that $u(1) = 1$. The model inward cuspy ball is defined by

$$\mathbb{B}_u \stackrel{\text{def}}{=\!=} \mathbb{B} \setminus \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \colon \; |x| \leqslant u(t)\},$$

see Fig. 1.

1.7 Bi-Lipschitz Deformations

There is no bi-Lipschitz transformation of a cuspy ball (inward or outward as in Fig. 2) onto a ball without cusp. We say that a cusp cannot be flatten via bi-Lipschitz deformation.

However, there always exists a Lipschitz homeomorphism of a cuspy ball onto a round ball and there is a Lipschitz homeomorphism of the round ball onto the cuspy ball, but these two deformations cannot be inverse to each other. The same pertains to a *degenerate cusp* defined by $u \equiv 0$, as in Fig. 3. In this degenerate case, if there would exist a bi-Lipschitz mapping $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$, it would extend as a homeomorphism of $\partial\mathbb{B}$ onto $\partial(\mathbb{B} \setminus \mathbf{I})$, $n \geqslant 3$, see [18] for more details. It is clear that the conflicting topology of the boundaries is an obstruction to the existence of a bi-Lipschitz deformation. This fact is also valid for deformations of bi-conformal energy, but it requires additional arguments.

1.8 Inward Slit in a Ball (the case $u \equiv 0$)

We will discuss the degenerate cups separately ($u \equiv 0$). Let us take a look at the pair $(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$ of a unit ball and the ball with a slit along the line segment

$$\mathbf{I} \stackrel{\text{def}}{=\!=} \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \colon 0 \leqslant t < 1 \text{ and } |x| = 0\},$$

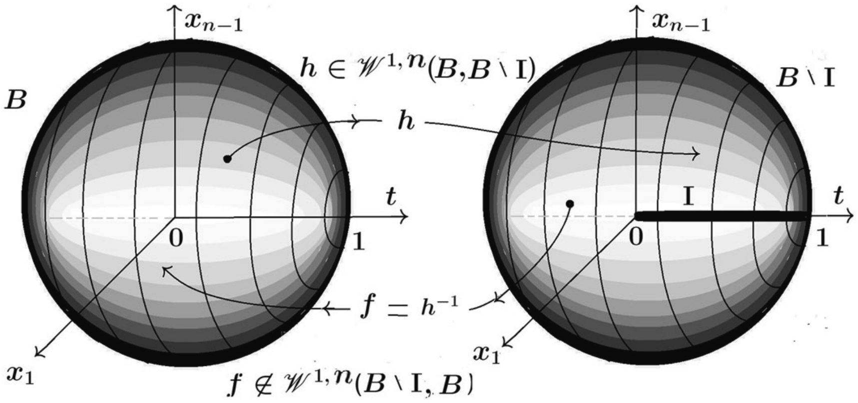


Fig. 3 Two domains which are not of the same bi-conformal energy type

see Fig. 3.

We have already mentioned that there exists a Lipschitz homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$; in particular, $h \in \mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$. The question arises whether there exists a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$ of finite conformal energy whose inverse $f = h^{-1} : \mathbb{B} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{B}$ also has finite conformal energy. Theorem 1.1 answer to this question is in the negative.

Theorem 1.7 *In dimension $n \geqslant 3$ the domains $\mathbb{B}$ and $\mathbb{B} \setminus \mathbf{I}$ are not of the same bi-conformal energy type; that is, there is no homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$ of finite bi-conformal energy.*

On one hand we have:

Example 1.8 There is a homeomorphism $f : \mathbb{B} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{B}$ of finite conformal energy such that $h = f^{-1} \in \mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ for all exponents $p < n$.

On the other hand, Theorem 1.7 is a special case of the following.

Theorem 1.9 *For $p > n - 1 \geqslant 2$ there is no homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$ of finite conformal energy with inverse $h^{-1} = f \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{R}^n)$.*

The lower bound for the Sobolev exponent in this theorem is essentially sharp. More precisely, we have

Theorem 1.10 *For every $p < n - 1$ there is a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$ of finite conformal energy with inverse $f = h^{-1} \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{R}^n)$.*

The borderline case $p = n - 1$ remains open.

1.9 Main Result

Our central question is when the unit ball and the ball with a model inward cusp $\mathbf{S}_u$ are of the same bi-conformal energy type. Let $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ be a deformation of bi-conformal energy. To predict what cusps $\mathbf{S}_u$ can be created, i.e., to predict that u is given by (1.1) it is natural to combine the estimates of the modulus of continuity of h near 0 with those for the inverse deformation $f = h^{-1} : \mathbb{B} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{B}$. From this point of view, deformations of bi-conformal energy are very different from quasiconformal mappings. The latter behave singular-like radial stretchings/squeezing; a poor modulus of continuity is always balanced by a better modulus of continuity of its inverse. Surprisingly, a deformation of bi-conformal energy and its inverse may exhibit the same optimal modulus of continuity [19], locally at a given point. Recall that a homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ satisfies the following estimate of the modulus of continuity:

$$|h(x_1) - h(x_2)|^n \leqslant C_n \left(\int_{2\mathbf{B}} |Dh|^n \right) \log^{-1} \left(e + \frac{\operatorname{diam} \mathbf{B}}{|x_1 - x_2|} \right) , \tag{1.7}$$

where $x_1, x_2 \in \mathbf{B} \stackrel{\text{def}}{=\!=} B(x_\circ, R) \subset B(x_\circ, 2R) \stackrel{\text{def}}{=\!=} 2\mathbf{B} \Subset \mathbb{X}$.

Applying the estimates in (1.7) would give us a nonexistence of a deformation of bi-conformal energy from $\mathbb{B}$ onto $\mathbb{B}_u$ with $u(t) = \exp^{-1}(\exp^\alpha(1/t))$, where $\alpha > n$ (applied to both h and f on the boundaries, see Theorem 3.1). This seemingly natural approach does not lead to a sharp result. Creating and flatting cusp singularities through mappings of bi-conformal energy is in a whole different scale, as stated in (1.1). Even more, Theorem 1.1 is a corollary of the following result.

Theorem 1.11 *(Main Theorem) Let $n \geqslant 3$ and*

$$u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^\alpha} \quad \text{for } 0 \leqslant t \leqslant 1\,, \ \text{where } \alpha > 0.$$

For $\alpha \geqslant n$ there is no homeomorphism $h\colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ with finite conformal energy whose inverse $h^{-1} = f \in \mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$, $p > n-1$. If $\alpha < n$, then there exists a homeomorphism $h\colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ with finite conformal energy such that f is Lipschitz.

2 Prerequisites

Our notation is fairly standard. Throughout the paper $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$. We write $C, C_1, C_2, \ldots$ as generic positive constants. These constants may change even in a single string of estimates. The dependence of constant on a parameter p is expressed by the notation $C = C(p) = C_p$ if needed.

We will appeal to the Sobolev embedding on spheres, see [13, Lemma 2.19].

Lemma 2.1 *Let $h\colon \mathbb{B} \to \mathbb{R}^n$ be a continuous mapping in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$, for some $p > n-1$. Then for almost every $0 < t < 1$ and every $x, y \in \partial\mathbb{B}(0,t) = \mathbb{S}_t$, we have*

$$|h(x) - h(y)| \leqslant C\, t^{1-\frac{n-1}{p}} \left(\int_{\mathbb{S}_t} |Dh(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} .$$

Here the constant C depends only on n and p.

It is relatively easy to conclude from this estimate that a $\mathscr{W}^{1,p}$-homeomorphism when $p > n-1$ is differentiable almost everywhere. It also follows that a homeomorphism $h\colon \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ satisfies Lusin's condition (N). This simply means, by definition, that $|h(E)| = 0$ whenever $|E| = 0$.

Lemma 2.2 *Let $\mathbb{X}$, $\mathbb{Y}$ be domains in $\mathbb{R}^n$ and $h\colon \mathbb{X} \overset{onto}{\longrightarrow} \mathbb{Y}$ be a homeomorphism in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. Then h is differentiable almost everywhere and satisfies Lusin's condition (N).*

Due to Lusin's condition (N) we have the following version of change of variables formula, see, e.g., [16, Theorem 6.3.2] or [13, Corollary A.36].

Lemma 2.3 *Let $h\colon \mathbb{X} \overset{onto}{\longrightarrow} \mathbb{Y}$ be a homeomorphism in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$. If η is a nonnegative Borel measurable function on $\mathbb{R}^n$ and A is a Borel measurable set in $\mathbb{X}$, then we have*

$$\int_A \eta(h(x))|J_h(x)| \, \mathrm{d}x = \int_{h(A)} \eta(y) \, \mathrm{d}y. \tag{2.1}$$

Next, we recall a well-known fact that a function in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R})$, $\mathbb{X} \subset \mathbb{R}^n$, has a representative which is locally Hölder continuous with exponent $1 - p/n$, provided $p > n$. More precisely, we have the following oscillation lemma.

Lemma 2.4 *Let $u \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R})$ where $\mathbb{X} \subset \mathbb{R}^n$ and $p > n$. Then*

$$|u(x) - u(y)| \leqslant C\, r^{1-\frac{n}{p}} \left(\int_{\mathbb{B}_r} |\nabla u|^p \right)^{\frac{1}{p}}$$

for every $x, y \in \mathbb{B}_r = \mathbb{B}(z, r) \subset \mathbb{X}$.

We will employ a higher dimension version of the classical Jordan curve theorem due to Brouwer [5], see also [25, Theorem 6.35].

Lemma 2.5 (Jordan–Brouwer separation theorem) *A topological $(n-1)$-sphere $\mathcal{S}$ disconnects $\mathbb{R}^n$ into a bounded component $\mathcal{S}_\circ$ and an unbounded component $\mathcal{S}_\infty$. Their common boundary is $\overline{\mathcal{S}_\circ} \cap \overline{\mathcal{S}_\infty} = \mathcal{S}$.*

A homeomorphism $h\colon \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_u$ of finite conformal energy extends as a continuous map $h\colon \overline{\mathbb{B}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{B}}_u$. This follows from the following result, see [17, Theorem 1.3].

Lemma 2.6 *Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded domains of finite connectivity. Suppose $\partial\mathbb{X}$ is locally quasiconformally flat and $\partial\mathbb{Y}$ is a neighborhood retract. Then every homeomorphism $h\colon \mathbb{X} \overset{onto}{\longrightarrow} \mathbb{Y}$ in the class $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ extends to a continuous map $h\colon \overline{\mathbb{X}} \overset{onto}{\longrightarrow} \overline{\mathbb{Y}}$.*

The assumed boundary regularities are defined as follows.

Definition 2.7 The boundary $\partial\mathbb{Y}$ is a *neighborhood retract*, if there is a neighborhood $\mathbb{U} \subset \mathbb{R}^n$ of $\partial\mathbb{Y}$ and a continuous map $\chi : \mathbb{U} \to \partial\mathbb{Y}$ which is an identity on $\partial\mathbb{Y}$.

Definition 2.8 The boundary $\partial\mathbb{X}$ is said to be *locally quasiconformally flat* if every point in $\partial\mathbb{X}$ has a neighborhood $\mathbb{U} \subset \mathbb{R}^n$ and a homeomorphism $g : \mathbb{U} \cap \overline{\mathbb{X}} \xrightarrow{\text{onto}} \mathbb{B} \cap (\mathbb{R}^{n-1} \times \mathbb{R}^+)$ which is quasiconformal on $\mathbb{U} \cap \mathbb{X}$; see [27].

Recall that $\mathbb{R}^+ = [0, \infty)$. It is also known that a mapping of bi-conformal energy between domains with locally quasiconformally flat boundaries has a homeomorphic extension up to the boundary, see [17, Corollary 1.1]. Note that $\partial\mathbb{B}_u$ is not locally quasiconformally flat and this result does not apply in our case.

Nevertheless, Lemma 2.6 tells us that h extends as a continuous mapping $h : \overline{\mathbb{B}} \to \overline{\mathbb{B}_u}$. Since $h(\overline{\mathbb{B}})$ is a compact subset of $\overline{\mathbb{B}_u}$, it follows that h takes $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}_u}$. Second, it is a topological fact that such a continuous extension is a monotone mapping $h : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$:

Proposition 2.9 *[8] Suppose that there is a continuous extension $G : \overline{\mathbb{B}} \xrightarrow{onto} \overline{\mathbb{B}}$ of a homeomorphism $g : \mathbb{B} \xrightarrow{onto} \mathbb{B}$. Then $G : \partial\mathbb{B} \xrightarrow{onto} \partial\mathbb{B}$ is monotone.*

By the definition, monotonicity, the concept of Morrey [23], simply means that for a continuous $h : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ the preimage $h^{-1}(y_\circ)$ of a point $y_\circ \in \overline{\mathbb{Y}}$ is a connected set in $\overline{\mathbb{X}}$. It is worth noting that the converse statement of Proposition 2.9 is also valid when $n = 2, 3$. Such an elegant characterization of monotone mappings of a 2-sphere onto itself was obtained by Floyd and Fort [9].

In the next lemmas we will analyze the boundary behavior of continuous extension of homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ with finite conformal energy which we will still denote by $h : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$. The following claim follows from Lemma 2.6 and Proposition 2.9.

Lemma 2.10 *Suppose a homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B}_u$ lies in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$. Then for every $x \in \partial\mathbb{B}_u$ the preimage $h^{-1}(x)$ is a nonempty continuum in $\partial\mathbb{B}$.*

Simplifying writing we set $o' \stackrel{\text{def}}{=\!=} (1, 0, ..., 0) \in \partial\mathbb{B}$ and $o \stackrel{\text{def}}{=\!=} (0, 0, ..., 0) \in \partial\mathbb{B}_u$. Without loss of generality, we may and will assume that $h(o') = o$. For every $0 < t < 1$, we define

$$S_t \stackrel{\text{def}}{=\!=} \{x \in \mathbb{B}_u : |x| = t\} \text{ and } C_t \stackrel{\text{def}}{=\!=} \{x \in \partial\mathbb{B}_u : |x| = t\},$$

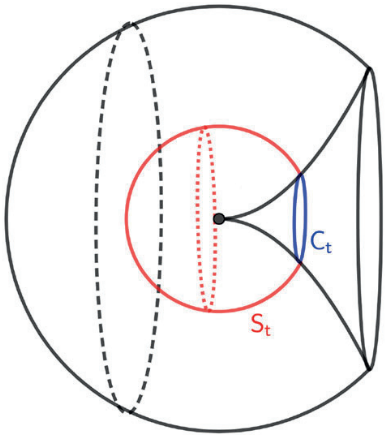
see Fig. 4. Note that here $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$.

Furthermore, let $S'_t \stackrel{\text{def}}{=\!=} h^{-1}(S_t)$ and $C'_t \stackrel{\text{def}}{=\!=} \overline{S'_t} \cap \partial\mathbb{B}$. Since $h : S'_t \xrightarrow{\text{onto}} S_t$ and $\overline{S'_t}$ is compact, the extension of h is also surjective and we have $h : \overline{S'_t} \xrightarrow{\text{onto}} \overline{S_t}$. We state this fact as a lemma.

Lemma 2.11 *Suppose a homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B}_u$ lies in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$. Then we have $h(C'_t) = C_t$.*

The next lemma shows that the Sobolev embedding on spheres, Lemma 2.1, also holds on S_t. In particular, we will need its variant on C_t, see Fig. 4.

Fig. 4 S_t and C_t



Lemma 2.12 *Suppose that a homeomorphism* $h\colon \mathbb{B} \overset{onto}{\longrightarrow} \mathbb{B}_u$ *has finite conformal energy. If the inverse mapping* $f = h^{-1}\colon \mathbb{B}_u \to \mathbb{B}$ *belongs to the Sobolev class* $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$ *for some* $p > n-1$*, then for almost every* $0 < t < 1$ *and every* $x'_t, y'_t \in C'_t$ *we have*

$$|x'_t - y'_t| \leqslant C|x_t - y_t|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df|^p \mathrm{d}x \right)^{\frac{1}{p}} . \tag{2.2}$$

Here $x_t = h(x'_t)$ *and* $y_t = h(y'_t)$ *and* C *is a positive constant independent of* t*,* x_t *and* y_t*.*

Proof Let $x'_t, y'_t \in C'_t$. By Lemma 2.11 there are two sequences $\{x'_{t,i}\}_{i=1}^{\infty}$ and $\{y'_{t,i}\}_{i=1}^{\infty}$ in S'_t such that

$$\lim_{i\to\infty} x'_{t,i} = x'_t, \quad \lim_{i\to\infty} y'_{t,i} = y'_t$$

and

$$\lim_{i\to\infty} x_{t,i} = x_t \in C_t, \quad \lim_{i\to\infty} y_{t,i} = y_t \in C_t.$$

Here,

$$x_{t,i} = h(x'_{t,i}), \quad y_{t,i} = h(y'_{t,i}), \quad x_t = h(x'_t) \text{ and } \quad y_t = h(y'_t).$$

By the classical Sobolev embedding on sphere, Lemma 2.1, we have

$$|x'_{t,i} - y'_{t,i}| \leqslant C|x_{t,i} - y_{t,i}|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df|^p \mathrm{d}x \right)^{\frac{1}{p}} .$$

Passing to the limit, we obtain

$$|x'_t - y'_t| \leqslant C|x_t - y_t|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df|^p \mathrm{d}x \right)^{\frac{1}{p}} .$$

If $f \in \mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$, $p > n-1$, then there is a decreasing sequence $\{t_i\}_{i=1}^{\infty}$ with $0 < t_1 < 1$, which converges to 0, and satisfies (2.2) and □

$$\int_{S_{t_i}} |Df|^p \, \mathrm{d}x < \frac{1}{t_i}.$$

Indeed, if not, then by Fubini's theorem for some $T \in (0, 1)$ we have

$$\int_{\mathbb{B}_u} |Df(x)|^p \, \mathrm{d}x \geqslant \int_0^T \int_{S_t} |Df(x)|^p \, \mathrm{d}x \mathrm{d}t \geqslant \int_0^T \frac{1}{t} \, \mathrm{d}t = \infty.$$

Without loss of generality, we may also assume that $\operatorname{diam} C'_{t_i}$ is decreasing with respect to t_i and $\operatorname{diam} C'_{t_1} < \frac{1}{4}$.

According to Lemmas 2.11 and 2.12 we have that $h: C'_t \overset{\text{onto}}{\longrightarrow} C_t$ is a homeomorphism. Now, Jordan–Brouwer Separation Theorem, Lemma 2.5, yields the following result.

Lemma 2.13 *Suppose that a homeomorphism* $h: \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_u$ *has finite conformal energy and the inverse mapping* $f = h^{-1}: \mathbb{B}_u \to \mathbb{B}$ *belongs to the Sobolev class* $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$ *for some* $p > n-1$. *Then* $\partial\mathbb{B} \setminus C'_t$ *consists of two disjoint connected open sets whose common boundary is* C'_t.

The boundary mapping $h: \partial\mathbb{B} \overset{\text{onto}}{\longrightarrow} \partial\mathbb{B}_u$ is monotone. We can say more about the preimage of the singular point o.

Lemma 2.14 *Suppose that a homeomorphism* $h: \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_u$ *has finite conformal energy and the inverse mapping* $f = h^{-1}: \mathbb{B}_u \to \mathbb{B}$ *belongs to the Sobolev class* $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$ *for some* $p > n-1$. *Then we have* $h^{-1}(o) = o'$.

Proof According to Lemma 2.13, $\partial\mathbb{B} \setminus C'_t$ consists of two disjoint connected open sets in $\partial\mathbb{B}$ whose common boundary is C'_t. We denote the one with smaller diameter by $\mathbb{U}_t$. Now, for $0 < t < \tau < t_1$, we have $U_t \subset U_\tau$ and we denote $U_\circ \overset{\text{def}}{=\!=} \lim_{t \to 0} \overline{U_t}$. Combining this with continuity of $h: \overline{\mathbb{B}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{B}_u}$, we obtain

$$h(U_\circ) = \lim_{t \to 0} h(\overline{U_t}). \tag{2.3}$$

By Lemma 2.11 $h(C'_t) = C_t$. Since further $C'_t \subset \overline{U_t}$ and $\lim_{t\to 0} C_t = o$ we have $o \in h(U_\circ) \subset h(\overline{U_t})$ for every $0 < t < t_1$. By Lemma 2.10 $h^{-1}(o)$ is connected. Thus we obtain that $h^{-1}(o) \subset \overline{U_t}$ for every $0 < t < t_1$. By Lemma 2.12, $\operatorname{diam} C'_t$ will converge to 0 as t goes to 0. Therefore, also the diameter of $\overline{U_t}$ approaches 0. Hence $h^{-1}(o) = o'$. □

Our last lemma in this section gives a precise modulus of continuity estimate for a homeomorphism $h: \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_u$ with finite conformal energy. Recall that such a homeomorphism has a continuous extension up to the boundary. Furthermore, the boundary mapping $h: \partial\mathbb{B} \overset{\text{onto}}{\longrightarrow} \partial\mathbb{B}_u$ is monotone in the sense of Morrey, see Lemma 2.10.

Monotone mappings enjoy a property which is commonly known in literature also as monotonicity. This notion goes back to H. Lebesgue [21] in 1907. To avoid confusion, in the following definition we use the term monotone in the sense of Lebesgue.

Definition 2.15 Let $\mathbb{X}$ be an open subset of $\mathbb{R}^n$. A continuous mapping $h\colon \overline{\mathbb{X}} \to \mathbb{R}^n$ is *monotone in the sense of Lebesgue* if for every compact set $K \subset \overline{\mathbb{X}}$ we have

$$\operatorname{diam} h(K) = \operatorname{diam} h(\partial K). \tag{2.4}$$

Note that for real-valued functions (2.4) can be stated as

$$\min_K h = \min_{\partial K} h \leqslant \max_{\partial K} h = \max_K h.$$

Remark 2.16 A folding map is a characteristic example of continuous nonmonotone mapping which is monotone in the sense of Lebesgue.

Lemma 2.17 *Let $h\colon \mathbb{B} \to \mathbb{B}_u$ be a homeomorphism with finite conformal energy. If $h(o') = o$, then there exists an increasing function $\varepsilon\colon [0,1) \to [0,\infty)$ with $\lim\limits_{t\to 0+} \varepsilon(t) = 0$ such that for $x' \in \overline{\mathbb{B}}$ with $0 < |x'-o'| < 1$ we have*

$$|h(x') - h(o')| \leqslant \frac{\varepsilon(|x'-o'|)}{\log^{\frac{1}{n}}\left(\frac{1}{|x'-o'|}\right)}. \tag{2.5}$$

Proof Set

$$\mathcal{S}_t \overset{\text{def}}{=\!=} \partial\mathbb{B}(o',t) \cap \overline{\mathbb{B}},$$

and

$$\operatorname{osc}(h,\mathcal{S}_t) \overset{\text{def}}{=\!=} \max_{x_t',y_t' \in \mathcal{S}_t} |h(x_t') - h(y_t')|.$$

Since $h\colon \overline{\mathbb{B}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{B}_u}$ is continuous and belongs to the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B},\mathbb{R}^n)$, applying a slightly modified version of the Sobolev embedding on sphere, Lemma 2.1 for almost every $0 < t < 1$ we have

$$(\operatorname{osc}(h,\mathcal{S}_t))^n \leqslant Ct \int_{\mathcal{S}_t} |Dh(x)|^n \, \mathrm{d}x. \tag{2.6}$$

Here C is a positive constant, independent of t. Fix $x' \in \mathbb{B}$ such that $\tau \overset{\text{def}}{=\!=} |x'-o'| < 1$. We write

$$\mathcal{B}(o',t) \overset{\text{def}}{=\!=} \mathbb{B} \cap \mathbb{B}(o',t) \quad \text{for } 0 < t < 1.$$

Choose $t \in [\tau, \sqrt{\tau}]$. Then

$$\operatorname{osc}(h, \overline{\mathcal{B}(o', \tau)}) \leqslant \operatorname{osc}(h, \overline{\mathcal{B}(o', t)}) \leqslant \operatorname{osc}(h, \partial\overline{\mathcal{B}(o', t)}),$$

where the latter inequality follows from the fact that h is monotone in the sense of Lebesgue. Since $\mathcal{S}_t = \partial\overline{\mathcal{B}(o', t)} \cap \mathbb{B}$ and h is monotone in the sense of Lebesgue, we have

$$\operatorname{osc}(h, \partial\overline{\mathcal{B}(o', t)}) = \operatorname{osc}(h, \mathcal{S}_t).$$

Combining this with (2.6) for almost every $t \in [\tau, \sqrt{\tau}]$ we have

$$\frac{\left(\operatorname{osc}(h, \overline{\mathcal{B}(o', \tau)})\right)^n}{t} = C \int_{\mathcal{S}_t} |Dh(x)|^n \, \mathrm{d}x.$$

Integrating this from τ to $\sqrt{\tau}$ with respect to the variable t, the claimed inequality (2.5) follows with

$$\varepsilon(\tau) = C \cdot \left(\int_{\mathcal{B}(o', \sqrt{\tau})} |Dh(x)|^n \mathrm{d}x\right)^{\frac{1}{n}}, \qquad \tau = |x' - o'|. \tag{2.7}$$

□

3 Homeomorphic Boundary Extension

Lemma 2.6 shows that a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ of finite conformal energy can be extended as a continuous mapping from $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}_u}$. In this section we will prove that a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ of bi-conformal energy extends as a homeomorphism up to the boundary.

Theorem 3.1 *Let $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ be a homeomorphism of finite bi-conformal energy. Then h admits a homeomorphic extension to the boundary, again denoted by $h : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$.*

The existence of such an extension is known [17, Corollary 1.1] if the reference and deformed configurations have locally quasiconformally flat boundaries, see Definition 2.8. Obviously, $\partial\mathbb{B}_u$ is not locally quasiconformally flat.

Proof of Theorem 3.1 By Lemma 2.6 a homeomorphism $h : \overline{\mathbb{B}} \to \overline{\mathbb{B}_u}$ with finite conformal energy extends as a continuous mapping $h : \overline{\mathbb{B}} \to \overline{\mathbb{B}_u}$. Since $h(\overline{\mathbb{B}})$ is a compact subset of $\overline{\mathbb{B}_u}$, it follows that $h : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$. Furthermore, by Lemma 2.10 the boundary map $h : \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}_u$ is monotone.

Now, we need to show that the boundary mapping is injective. We again use the notation $o = (0, 0, ..., 0)$ and $o' = (1, 0, ..., 0)$ and assume, without loss of generality, that $h(o') = o$. First, $h^{-1}(o) = o'$ by Lemma 2.14. Second let $y \in \partial\mathbb{B}_u \setminus \{o\}$. Choosing

$0 < r_y < |y - o|$, then $\mathbb{B}(y, r_y) \cap \mathbb{B}_u$ is locally quasiconformally flat. By Lemma 2.6, the homeomorphism $f: \mathbb{B}(y, r_y) \cap \mathbb{B}_u \xrightarrow{\text{onto}} f(\mathbb{B}(y, r_y) \cap \mathbb{B}_u)$ has a continuous extension $f: \overline{\mathbb{B}(y, r_y) \cap \mathbb{B}_u} \xrightarrow{\text{onto}} \overline{f(\mathbb{B}(y, r_y) \cap \mathbb{B}_u)}$. The extension of f is still an inverse of h in the quasiconformally flat part of the boundary; that is, $h^{-1}(y) = f(y)$ is a single point. Now we know that $h: \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$ is a continuous bijection, and therefore it is a homeomorphism. □

4 Construction of Example 1.8

Here we show that there exists a homeomorphism from $\mathbb{B} \setminus \mathbf{I}$ onto $\mathbb{B}$ with finite conformal energy, actually Lipschitz continuous, whose inverse lies in $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ for every $p < n$. To simplify our construction, we may and do replace $\mathbb{B}$ by a bi-Lipschitz equivalent domain; namely,

$$\mathbb{Y} = \{(s, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |y| < 1 \text{ and } -1 < s < |y|\}.$$

As for the reference configuration we replace $\mathbb{B} \setminus \mathbf{I}$ by a cylinder $\mathbf{C} = (-1, 1) \times \mathbb{B}^{n-1}$ with the line segment $\mathbf{I}$ removed from it. Consider the Lipschitz homeomorphism $h: \mathbf{C} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{Y}$ defined by the rule

$$h(t, x) = \begin{cases} (t|x|,\ x) & \text{for } t > 0, \\ (t, x) & \text{for } t < 0. \end{cases} \tag{4.1}$$

Its inverse mapping $f: \mathbb{Y} \xrightarrow{\text{onto}} \mathbf{C} \setminus \mathbf{I}$ takes the form

$$f(s, y) = \begin{cases} \left(\frac{s}{|y|},\ y\right) & \text{for } s \geqslant 0, \\ (s, y) & \text{for } s < 0. \end{cases} \tag{4.2}$$

It is easy to see that

$$|Df(s, y)| \leqslant \frac{C_n}{|y|}.$$

Therefore,

$$\int_{\mathbb{Y}} |Df|^p < \infty \qquad \text{for every } 1 \leqslant p < n$$

as desired.

5 Proof of Theorem 1.9

5.1 The Nonexistence Part of Theorem 1.9

First, we will prove the nonexistence part of Theorem 1.9.

Theorem 5.1 *If $p > n-1$, then there is no homeomorphism $h\colon \mathbb{B} \overset{onto}{\longrightarrow} \mathbb{B} \setminus \mathbf{I}$ with $h \in \mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$ whose inverse $f = h^{-1} \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{B})$.*

Proof Suppose to the contrary that there is a homeomorphism $h\colon \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B} \setminus \mathbf{I}$ in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$ such that $f \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{B})$. Since $\partial(\mathbb{B} \setminus \mathbf{I})$ is a neighborhood retract, Lemma 2.6 tells us that the homeomorphism $h\colon \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B} \setminus \mathbf{I}$ extends as a continuous mapping $h\colon \overline{\mathbb{B}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{B}}$. We denote

$$S_t = \partial B_t \setminus \{x_t\}, \qquad \text{where } x_t \overset{\text{def}}{=\!=} (t, 0, \dots, 0) \quad 0 < t < s < 1.$$

Here $B_t = B(0,t)$. Fubini's theorem implies that for almost every $t \in (0,1)$, $f\big|_{S_t} \in \mathscr{W}^{1,p}(S_t, \mathbb{R}^n)$. Since $p > n-1$ and $n \geqslant 3$, the possible singularity of f at x_t is removable. For such t, applying Lemma 2.4, $f\big|_{S_t}$ extends as a homeomorphism $f\colon \overline{S_t} \overset{\text{onto}}{\longrightarrow} f(\overline{S_t})$. Write $x_t' = f(x_t) \in \partial\mathbb{B}$. Now, Jordan–Brouwer Separation Theorem (Lemma 2.5) tells us that $\mathbb{R}^n \setminus f(\overline{S_t})$ consists of two disjoint connected open sets whose common boundary is $f(\overline{S_t})$. Let us denote the bounded one by U_t. Note that $U_t \subset \mathbb{B}$ and $\overline{U_t} \cap \partial\mathbb{B} = \{x_t'\}$. Since for almost every $t < s \in (0,1)$ we have $B_t \setminus \mathbf{I} \subset B_s \setminus \mathbf{I}$ then $U_t = h^{-1}(B_t \setminus \mathbf{I}) \subset h^{-1}(B_s \setminus \mathbf{I}) = U_s$.

Now comes an elementary topological fact: *given two domains $U \subset V \subset \mathbb{B}$ such that $\overline{U} \cap \partial\mathbb{B} = \{x_\nu\}$ and $\overline{V} \cap \partial\mathbb{B} = \{x_\mu\}$, then $x_\nu = x_\mu$* (Fig. 5).

Now, we have $x_s' = x_t'$. This, however, is impossible since $h(x_s') = (s, 0, \dots, 0)$ and $h(x_t') = (t, 0, \dots, 0)$.

□

5.2 The Existence Part of Theorem 1.9

Here we verify the existence part of Theorem 1.9. Namely,

Theorem 5.2 *There exists a Lipschitz homeomorphism $h\colon \mathbb{B} \to \mathbb{B} \setminus \mathbf{I}$ whose inverse $f \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{B})$ for every $1 \leqslant p < n-1$.*

Proof We shall view $\mathbb{R}^n$ as

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x)\colon t \in \mathbb{R},\ x \in \mathbb{R}^{n-1}\}.$$

To simplify our construction, we may and do replace $\mathbb{B}$ by a bi-Lipschitz equivalent domain; namely $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$, where

$$\mathbb{X}_- = \{(t, x)\colon -1 < t < 0 \text{ and } |x| < 1\}$$

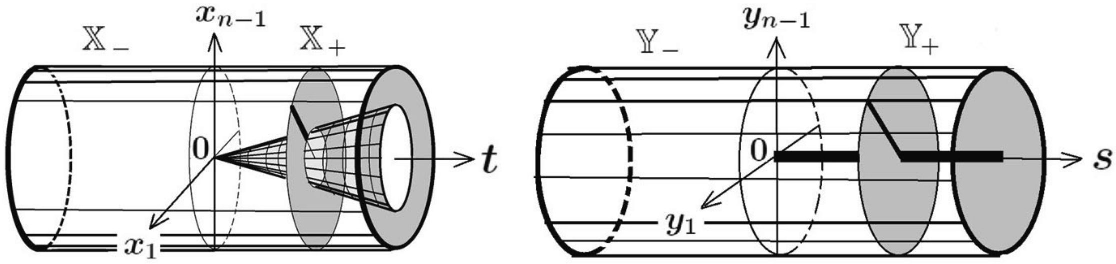


Fig. 5 The domains $\mathbb{X}$ and $\mathbb{Y}$

and

$$\mathbb{X}_+ = \{(t, x) \colon 0 \leqslant t < 1 \text{ and } \frac{t}{2} < |x| < 1\}.$$

As for the reference configuration we consider $\mathbb{Y} = \mathbb{Y}_+ \cup \mathbb{Y}_-$ where $\mathbb{Y}_-$ is the open unit cylinder

$$\mathbb{Y}_- = \{(s, y) \colon -1 < s < 0 \text{ and } |y| < 1\}$$

and

$$\mathbb{Y}_+ = \{(s, y) \colon 0 \leqslant s < 1 \text{ and } 0 < |y| < 1\}.$$

We define a Lipschitz map $h \colon \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ by the rule

$$h(t, x) = \begin{cases} (t, x) & \text{in } \mathbb{X}_-, \\ \left(t, \left[\frac{2|x|}{2-t} - \frac{t}{2-t}\right] \frac{x}{|x|}\right) & \text{in } \mathbb{X}_+. \end{cases}$$

Then the inverse map $f = h^{-1} \colon \mathbb{Y} \overset{\text{onto}}{\longrightarrow} \mathbb{X}$ takes the form

$$f(s, y) = \begin{cases} (s, y) & \text{in } \mathbb{Y}_-, \\ \left(s, \left[\frac{2-s}{2}|y| + \frac{s}{2}\right] \frac{y}{|y|}\right) & \text{in } \mathbb{Y}_+ \end{cases} .$$

It is the identity map on $\mathbb{Y}_-$ while on $\mathbb{Y}_+$ we write it as

$$f(s, y) = \left(s, \frac{2-s}{2} y\right) + \left(0, \frac{sy}{2|y|}\right),$$

where the first term is $\mathscr{C}^\infty$-smooth. It is now easy to verify the estimate

$$|Df(s, y)| \leqslant C \cdot \left(1 + \frac{s}{|y|}\right),$$

where $|s| < 1$ and $y \in \mathbb{R}^{n-1}$, $0 < |y| < 1$. Hence

$$\int_{\mathbb{Y}_+} |Df|^p < \infty \quad \text{for every } 1 \leqslant p < n-1$$

as desired. □

6 Proof of Theorem 1.11

6.1 The Nonexistence Part of Theorem 1.11

Here we give a proof of the nonexistence part of Theorem 1.11. We recall the statement for the convenience of the reader.

Theorem 6.1 *Let $\alpha \geqslant n$ and $p > n-1$ be fixed and $u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}}$. Then there does not exist a homeomorphism $h : \mathbb{B} \to \mathbb{B}_u$ with $h \in \mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B}_u)$ and $h^{-1} \in \mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{B})$.*

Proof Fix $\alpha \geqslant n$ and $p > n-1$. Suppose to the contrary that there exists a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ with finite conformal energy such that its inverse f is in $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$. According to Lemma 2.6, h extends as a continuous mapping $h \colon \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$. Furthermore, by Lemma 2.10 the boundary mapping $h \colon \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}_u$ is monotone.

We follow the notation introduced in Sect. 2 and set $o = (0, 0, \dots, 0)$ and $o' = (1, 0, \dots, 0)$. We may and do assume that $h(o') = o$. Moreover, for every $0 < t < 1$,

$$S_t = \{x \in \mathbb{B}_u \colon |x| = t\} \text{ and } C_t = \{x \in \partial\mathbb{B}_u \colon |x| = t\}$$

and

$$S'_t = h^{-1}(S_t) \quad \text{and} \quad C'_t = \overline{S'_t} \cap \partial\mathbb{B}.$$

Lemma 2.13 tells us that C'_t divides $\partial\mathbb{B}$ into two disjoint components. We denote the component which contains o' by U'_t. Accordingly, we also have

$$\partial U'_t = C'_t. \tag{6.1}$$

Since

$$\int_{\mathbb{B}_u} |Df(x)|^p \mathrm{d}x < \infty,$$

there exists a decreasing sequence $\{t_i\}$, which converges to 0 and satisfies

$$\int_{S_{t_i}} |Df(x)|^p \mathrm{d}x < \frac{1}{t_i}. \tag{6.2}$$

Indeed, by Fubini's theorem we have

$$\int_0^1 \int_{S_t} |Df(x)|^p \, \mathrm{d}x < \infty.$$

Hence,

$$\liminf_{t\to 0} t \int_{S_t} |Df(x)|^p = 0.$$

Now, by Lemma 2.11, we have $h(C'_t) = C_t$. Combining this with Lemma 2.12 we obtain

$$\begin{aligned} \operatorname{diam} C'_{t_i} &\leqslant C \cdot \big(2\, u(t_i)\big)^{1-\frac{n-1}{p}} \left(\int_{S_{t_i}} |Df(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leqslant C \cdot \big(u(t_i)\big)^{1-\frac{n-1}{p}} \left(\frac{1}{t_i} \right)^{\frac{1}{p}} . \end{aligned} \tag{6.3}$$

Here $u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}}$. Especially, this shows that $\operatorname{diam}(C'_{t_i}) \to 0$ as $i \to \infty$ and, therefore, U'_{t_i} lies on the half sphere $\partial\mathbb{B}_+$. We now appeal to the geometric fact if $x, a \in U'_{t_i}$, then $|x - a| \leqslant \operatorname{diam} \partial U'_{t_i}$. Now, for large enough i, by (6.1) we fix $x'_{t_i} \in C'_{t_i}$ and then

$$|x'_{t_i} - o'| \leqslant \operatorname{diam} C'_{t_i}. \tag{6.4}$$

According to Lemma 2.17 and (6.4) we obtain

$$t_i \leqslant |h(x'_{t_i}) - o| \leqslant \varepsilon(t_i) \log^{-\frac{1}{n}} \frac{1}{|x'_{t_i} - o'|} \leqslant \varepsilon(t_i) \log^{-\frac{1}{n}} \frac{1}{\operatorname{diam} C'_{t_i}} , \tag{6.5}$$

where $\varepsilon(t)$ is a positive function defined in (2.7) which converges to 0 as t goes to 0. The estimates (6.3) and (6.5) imply

$$C \cdot u(t_i) \geqslant \left(\frac{t_i^{\frac{1}{p}}}{\exp\left(\frac{\varepsilon(t_i)}{t_i}\right)^n} \right)^{\frac{p}{p+1-n}} . \tag{6.6}$$

Since $\alpha \geqslant n$ we have $\exp(1/t^n) \leqslant \exp(1/t^\alpha)$ for $0 < t \leqslant 1$ and therefore

$$\frac{C \cdot e}{\exp\left(t^{-n}\right)} \geqslant \left(\frac{t_i^{\frac{1}{p}}}{\exp\left(\frac{\varepsilon(t_i)}{t_i}\right)^n} \right)^{\frac{p}{p+1-n}} .$$

This means that there are constants $C_1, C_2 > 0$ satisfying

$$\varepsilon(t_i) \geqslant C_1 \cdot t_i^n \log\left(C_2\, t_i^\beta \exp(t_i^{-n})\right), \quad \beta = \frac{1}{p-n+1}.$$

Letting $i \to \infty$, the right-hand side converses to C_1 and $\varepsilon(t_i) \to 0$. This contradiction completes the proof.

□

6.2 The Existence Part of Theorem 1.11

Theorem 6.2 *Let $u(t) = \frac{e}{\exp(1/t)^\alpha}$ for some $0 < \alpha < n$. Then there exists a homeomorphism $h\colon \mathbb{B} \to \mathbb{B}_u$ with finite conformal energy whose inverse $f = h^{-1}\colon \mathbb{B}_u \to \mathbb{B}$ is Lipschitz regular.*

Proof Fix $0 < \alpha < n$ and the corresponding cusp domain $\mathbb{B}_u$ with $u(t) = \frac{e}{\exp(t^{-1})^\alpha}$. As in the proof of Theorem 5.2 we write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x)\colon t \in \mathbb{R}\,,\ x \in \mathbb{R}^{n-1}\}$$

and replace $\mathbb{B}$ by a bi-Lipschitz equivalent domain, $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$, where

$$\mathbb{X}_- = \{(t, x)\colon\ -1 < t \leqslant 0 \text{ and } |x| < 1\}$$

and

$$\mathbb{X}_+ = \{(t, x)\colon 0 < t < 1 \text{ and } t < |x| < 1\}.$$

We replace the cusp domain $\mathbb{B}_u$ by the following bi-Lipschitz equivalent domain $\mathbb{Y} = \mathbb{Y}_- \cup \mathbb{Y}_+$, where

$$\mathbb{Y}_- = \{(s, y)\colon\ -1 < s \leqslant 0 \text{ and } |y| < 1\}$$

and

$$\mathbb{Y}_+ = \{(s, y)\colon 0 < s < 1 \text{ and } u(s) < |y| < 1\}.$$

We define $h\colon \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ by

$$h(t, x) = \begin{cases} (t, x) & \text{in } \mathbb{X}_-, \\ \left(\frac{u^{-1}(|x|)}{|x|}t,\ x\right) & \text{in } \mathbb{X}_+. \end{cases}$$

Note that the inverse function $u^{-1}(\eta) = \log^{-\frac{1}{\alpha}}\left(\frac{e}{\eta}\right)$. Then the inverse mapping $f = h^{-1} \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f(s, y) = \begin{cases} (s, y) & \text{in } \mathbb{Y}_-, \\ \left(\frac{|y|}{u^{-1}(|y|)} s,\ y\right) & \text{in } \mathbb{Y}_+. \end{cases}$$

Now, f is a Lipschitz regular mapping. Furthermore, we have

$$|Dh(t, x)| \leqslant \frac{C}{|x| \log^{\frac{1}{\alpha}}\left(\frac{e}{|x|}\right)}.$$

Therefore,

$$\int_{\mathbb{X}} |Dh|^n < \infty$$

as claimed. □

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7 Appendix: Proof of Theorem 1.5

Proof First, we assume that $K_I(\cdot, h) \in \mathscr{L}^1(\mathbb{X})$. Then, Theorem 9.1 in [3] states that a homeomorphism $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ satisfies the claimed identity (1.6) if h has a finite (outer) distortion; that is, there is a function $1 \leqslant K_O(x) < \infty$ such that

$$|Dh(x)|^n \leqslant K_O(x)\, J_h(x) \qquad \text{for almost every } x \in \mathbb{X}. \tag{7.1}$$

The proof, however, only uses a consequence of (7.1) the finite inner inequality (1.5) which is stated in [3, (9.10)].

Second, we assume that $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ and $f = h^{-1} \in \mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$. Then

$$K_I(x, h) = |Df\big(h(x)\big)|^n J_h(x) \quad \text{a.e. } x \in \mathbb{X}. \tag{7.2}$$

Indeed, by Lemma 2.2 both h and f are differentiable almost everywhere. Now, the identity $(f \circ h)(x) = x$, after differentiation, implies that

$$Df(h(x))Dh(x) = \mathbf{I} \quad \text{a.e. in } \mathbb{X}. \tag{7.3}$$

Since both h and f satisfy Lusin's condition (N); that is, preserve sets of zero measure, see Lemma 2.2. This shows that $J_h(x) > 0$ and $J_f(y) > 0$ almost everywhere again we used the fact that h satisfies Lusin's condition (N). Now, the formula (7.2) is a direct consequence of the definition of the inner distortion, Cramer's rule $Dh(x)D^\sharp h(x) = J_h(x)\mathbf{I}$ and (7.3). Indeed,

$$K_I(x,h) = \frac{|D^\sharp h(x)|^n}{|J_h(x)|^{n-1}} = |(Dh(x))^{-1}|^n J_h(x) = |Df\big(h(x)\big)|^n J_h(x).$$

Now the change of variables formula (2.1) gives

$$\int_{\mathbb{X}} K_I(x,h)\,\mathrm{d}x = \int_{\mathbb{Y}} |Df(y)|^n\,\mathrm{d}y.$$

□

Proof of Corollary 1.6 By [16, §6.4] for every $x \in \mathbb{X}$ with $J_h(x) > 0$, we have

$$K_I^{\frac{1}{n-1}}(x,h) \leqslant K_O(x,h) \leqslant K_I^{n-1}(x,h)\,. \tag{7.4}$$

Here $K_O(x,h)$ stands for the smallest function satisfying (7.1). Now, Corollary 1.6 follows immediately from (7.4). □

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Deformations of Bi-conformal Energy and a New Characterization of Quasiconformality

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Abstract

The concept of hyperelastic deformations of bi-conformal energy is developed as an extension of quasiconformality. These deformations are homeomorphisms $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ of the Sobolev class $\mathscr{W}^{1,n}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ whose inverse $f \overset{\text{def}}{=\!=} h^{-1}\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ also belongs to $\mathscr{W}^{1,n}_{\text{loc}}(\mathbb{Y}, \mathbb{X})$. Thus the paper opens new topics in Geometric Function Theory (GFT) with connections to mathematical models of Nonlinear Elasticity (NE). In seeking differences and similarities with quasiconformal mappings we examine closely the modulus of continuity of deformations of bi-conformal energy. This leads us to a new characterization of quasiconformality. Specifically, it is observed that quasiconformal mappings behave locally at every point like radial stretchings; if a quasiconformal map h admits a function ϕ as its *optimal modulus of continuity* at a point $x_\circ$, then $f = h^{-1}$ admits the inverse function $\psi = \phi^{-1}$ as its modulus of continuity at $y_\circ = h(x_\circ)$. That is to say, a poor (possibly harmful) continuity of h at a given point $x_\circ$ is always compensated by a better continuity of f at $y_\circ$, and vice versa. Such a gain/loss property, seemingly overlooked by many authors, is actually characteristic of quasiconformal mappings. It turns out that the elastic deformations of bi-conformal energy are very different in this respect. Unexpectedly, such a map may have the same optimal modulus of continuity as its inverse deformation. In line with Hooke's Law, when trying to restore the original shape of the body (by the inverse transformation), the modulus of continuity may neither be improved nor become worse. However, examples to confirm this phenomenon are far from being obvious; indeed, elaborate computations are on the way. We eventually hope that our examples will gain an interest in the materials science, particularly in mathematical models of hyperelasticity.

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1. Introduction

We study Sobolev homeomorphisms $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$, together with their inverse mappings denoted by $f \overset{\text{def}}{=\!=} h^{-1}\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$. We impose two standing conditions on these mappings:

- The *conformal energy* of h (stored in $\mathbb{X}$) is finite; that is,

$$\mathbf{E}_{\mathbb{X}}[h] \overset{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x \; < \infty; \tag{1.1}$$

- The *conformal energy* of f (stored in $\mathbb{Y}$) is also finite;

$$\mathbf{E}_{\mathbb{Y}}[f] \overset{\text{def}}{=\!=} \int_{\mathbb{Y}} |Df(y)|^n \, \mathrm{d}y \; < \infty. \tag{1.2}$$

Hereafter, $|A|$ stands for the *Hilbert-Schmidt norm* of a linear map A, defined by the rule $|A|^2 = \operatorname{Tr}(A^t A)$. It should be noted that the above energy integrals are invariant under conformal change of variables in their domains of definition ($\mathbb{X}$ and $\mathbb{Y}$, respectively). This motivates us to call such homeomorphisms

Deformations of Bi-conformal Energy

Clearly, such deformations include *quasiconformal* mappings. A Sobolev homeomorphisms $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is said to be a quasiconformal mapping if there exists a constant $\mathbf{K}$ such that

$$|Dh(x)|^n \leqslant \mathbf{K} J(x,h)\,, \qquad J(x,h) = \det Dh(x)\,. \tag{1.3}$$

The conformal energy integral (1.1), an n-dimensional alternative to the classical Dirichlet integral, has drawn the attention of researchers in the multidimensional GFT [7,19,20,27,46,47,51]. In Geometric Analysis, the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ plays a special role for several reasons. First, this space is on the edge of the continuity properties of Sobolev's mappings. Second, just the fact that h is a homeomorphism allows us to establish uniform bounds of its modulus of continuity. Precisely, given a compact subset $\mathbf{X} \Subset \mathbb{X}$, there exists a constant $C(\mathbf{X}, \mathbb{X})$ so that for all distinct points $x_1, x_2 \in \mathbf{X}$, we have

$$|\, h(x_1) - h(x_2)\,| \;\leqslant\; \frac{C(\mathbf{X}, \mathbb{X})\; \sqrt[n]{\mathbf{E}_{\mathbb{X}}[h]}}{\log^{\frac{1}{n}} \left(1 \,+\, \frac{\operatorname{diam} \mathbf{X}}{|x_1 - x_2|}\right)}. \tag{1.4}$$

For a historical account and more details concerning this estimate we refer the reader to Section 7.4, Section 7.5 and Corollary 7.5.1 in the monograph [27].

For the same reasons, to every compact $\mathbf{Y} \Subset \mathbb{Y}$ there corresponds a constant $C(\mathbf{Y}, \mathbb{Y})$ such that, for all distinct points $y_1, y_2 \in \mathbf{Y}$, we have

$$|\, f(y_1) - f(y_2)\,| \;\leqslant\; \frac{C(\mathbf{Y}, \mathbb{Y})\; \sqrt[n]{\mathbf{E}_{\mathbb{Y}}[f]}}{\log^{\frac{1}{n}} \left(1 \,+\, \frac{\operatorname{diam} \mathbf{Y}}{|y_1 - y_2|}\right)}. \tag{1.5}$$

In other words, h and f admit the same function $\omega = \omega(t) \approx \log^{-\frac{1}{n}}(1 + 1/t)$ as a modulus of continuity. Shortly, h and f are ω-continuous. There is still a slight improvement to these estimates; namely,

$$\lim_{|x_1 - x_2| \to 0} |\, h(x_1) - h(x_2) \,| \log^{\frac{1}{n}} \left(1 + \frac{\operatorname{diam} \mathbf{X}}{|x_1 - x_2|} \right) = 0. \tag{1.6}$$

The question whether the modulus of continuity $\omega = \omega(t) \approx \log^{-\frac{1}{n}}(1 + 1/t)$ is the best and universal for all bi-conformal energy mappings remains unclear. We shall not enter this issue here. The *optimal modulus of continuity* of $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ at a given point $x_\circ \in \mathbb{X}$ is defined by

$$\omega_h(x_\circ; t) \stackrel{\text{def}}{=\!=} \max_{|x - x_\circ| = t} |h(x) - h(x_\circ)| \qquad \text{for } 0 \leqslant t < \operatorname{dist}(x_\circ, \partial\mathbb{X}) . \tag{1.7}$$

Nevertheless, it is easy to see, via examples of radial stretchings, that in the class of functions that are powers of logarithms the exponent $\alpha = \frac{1}{n}$ is sharp; meaning that for $\alpha > \frac{1}{n}$ it is not generally true that[1]

$$|\, h(x_1) - h(x_2) \,| \preccurlyeq \log^{-\alpha} \left(1 + \frac{\operatorname{diam} \mathbf{X}}{|x_1 - x_2|} \right). \tag{1.8}$$

To this end, we take a quick look at the radial homeomorphism $h \colon \mathbb{B}^n \xrightarrow{\text{onto}} \mathbb{B}^n$ of the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ onto itself,

$$h(x) = \frac{x}{|x| \big(1 - \log |x|\big)^{\frac{1}{n}} \big[\log(e - \log |x|) \big]^{\beta}} \;, \quad \text{where} \quad \beta > \frac{1}{n}. \tag{1.9}$$

It is often seen that the inverse map $f \stackrel{\text{def}}{=\!=} h^{-1} \colon \mathbb{Y} \to \mathbb{X}$ admits better modulus of continuity than h, or vice versa. Just for h defined in (1.9), its inverse is even $\mathscr{C}^\infty$-smooth. Such a gain/loss rule about the moduli of continuity for a map and its inverse is typical of the radial stretching/squeezing. It turns out that the gain/loss rule gives a new characterization for a widely studied class of quasiconformal mappings.

Theorem 1.1. *Let $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be a homeomorphism between domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ and let $f \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ denote its inverse. Then h is quasiconformal if and only if for every pair $(x_\circ, y_\circ) \in \mathbb{X} \times \mathbb{Y}$, $y_\circ = h(x_\circ)$, the optimal modulus of continuity functions $\omega_h = \omega_h(x_\circ; t)$ and $\omega_f = \omega_f(y_\circ; s)$ are quasi-inverse to each other; that is, there is a constant $\mathcal{K} \geqslant 1$ (independent of $(x_\circ, y_\circ)$) such that*

$$\mathcal{K}^{-1} s \leqslant (\omega_h \circ \omega_f)(s) \leqslant \mathcal{K} s$$

[1] Hereafter the notation $\mathbf{A} \preccurlyeq \mathbf{B}$ stands for the inequality $\mathbf{A} \leqslant c\, \mathbf{B}$ in which $c > 0$, called implied or hidden constant, plays no role. The implied constant may vary from line to line and is easily identified from the context, or explicitely specified if necessary.

for sufficiently small $s > 0$; see Section 3 for fuller discussion. It should be noted that for a radial stretching/squeezing homeomorphism $h(x) = \mathbf{H}(|x|)\frac{x}{|x|}$, $\mathbf{H}(0) = 0$, we always have

$$(\omega_h \circ \omega_f)(s) \equiv s.$$

Thus it amounts to saying that

> *Quasiconformal mappings are characterized by being comparatively radial strectching/squeezing at every point.*

At the first glance, the gain/loss rule seems to generalize to deformations of bi-conformal energy. Here we refute this view, by constructing examples in which both h and f admit the same modulus of continuity. These examples work well regardless of whether or not the modulus of continuity (given upfront) is close to the borderline case $\omega = \omega(t) \approx \log^{-\frac{1}{n}}(1 + 1/t)$. Without additional preliminaries, we now can illustrate this instance with a representative case of Theorem 14.1.

Theorem 1.2. *(A Representative Example). Consider a modulus of continuity function $\phi\colon [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ defined by the rule*

$$\phi(s) = \begin{cases} 0 & \text{if } s = 0 \\ \left[\log\left(\frac{e}{s}\right)\right]^{-\frac{1}{n}} \left[\log\log\left(\frac{e^e}{s}\right)\right]^{-1} & \text{if } 0 < s \leqslant 1 \\ s & \text{if } s \geqslant 1 \end{cases}. \tag{1.10}$$

Then there exists a deformation of bi-conformal energy $H\colon \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ such that

- $H(0) = 0$, $H(x) \equiv x$, for $|x| \geqslant 1$
- $|H(x_1) - H(x_2)| \preccurlyeq \phi(|x_1 - x_2|)$, *for all* $x_1, x_2 \in \mathbb{R}^n$.

Its inverse $F \overset{\text{def}}{=\!=} H^{-1}\colon \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ also admits ϕ as a modulus of continuity,

- $|F(y_1) - F(y_2)| \preccurlyeq \phi(|y_1 - y_2|)$, *for all* $y_1, y_2 \in \mathbb{R}^n$.[2]

Furthermore, ϕ represents the optimal modulus of continuity at the origin for both H and F; that is, for every $0 \leqslant s < \infty$ we have

$$\omega_H(0, s) = \phi(s) = \omega_F(0, s). \tag{1.11}$$

Remark 1.3. More specifically, letting $\psi : [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ denote the inverse of ϕ, the maxima in (1.11) are attained on the vertical axes, where we have

$$H(0,\dots,0,\ x_n) = \begin{cases} (0,\dots,0,\ \phi(x_n)) & \text{if } x_n \geqslant 0 \\ (0,\dots,0,\ \psi(x_n)) & \text{if } x_n \leqslant 0 \end{cases} \tag{1.12}$$

$$F(0,\dots,0,\ y_n) = \begin{cases} (0,\dots,0,\ \psi(y_n)) & \text{if } y_n \geqslant 0 \\ (0,\dots,0,\ \phi(y_n)) & \text{if } y_n \leqslant 0 \end{cases} \tag{1.13}$$

It is worth noting here that in our representative examples the inverse function $\psi : [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ will be even $\mathscr{C}^\infty$-smooth near 0.

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[2] In the above estimates the implied constants depend only on n.

There are many more reasons for studying deformations of bi-conformal energy. First, a homeomorphism $h \colon \mathbb{X} \to \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ whose inverse $f \overset{\text{def}}{=\!=} h^{-1} \colon \mathbb{Y} \to \mathbb{X}$ also lies in $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{X})$ include ones with *integrable inner distortion*, see (1.15). From this point of view our study not only expands the theory of quasiconformal mappings but also mappings of finite distortion. The latter can be traced back to the early paper by Goldstein and Vodop'yanov [17] (1976) who established continuity of such mappings. However, a systematic study of mappings of finite distortion has begun in 1993 with planar mappings of integrable distortion [34] (Stoilow factorization), see also the monographs [3,20,27]. The optimal modulus of continuity for mappings of finite distortion and their inverse deformations have been studied in numerous publications [9,11,21,22,24,38,44,45]. In all of these results, except in [44], the sharp modulus of continuity is obtained among the class of radially symmetric mapping.

In a different direction, the essence of elasticity is reversibility. All materials have limits of the admissible distortions. Exceeding such a limit one breaks the internal structure of the material (permanent damage). Here we take on stage the materials of *bi-conformal stored-energy*

$$\mathbf{E}_{\mathbb{X}\mathbb{Y}}[h, f] \overset{\text{def}}{=\!=} \mathbf{E}_{\mathbb{X}}[h] + \mathbf{E}_{\mathbb{Y}}[f] = \int_{\mathbb{X}} |Dh(x)|^n \mathrm{d}x \; + \int_{\mathbb{Y}} |Df(y)|^n \mathrm{d}y. \quad (1.14)$$

The bi-conformal energy reduces to an integral functional defined solely over the domain $\mathbb{X}$ by the rule

$$\mathbf{E}_{\mathbb{X}\mathbb{Y}}[h, f] = \mathscr{E}_{\mathbb{X}}[h] \overset{\text{def}}{=\!=} \int_{\mathbb{X}} \left\{ \left| Dh(x) \right|^n + \frac{\left| D^{\sharp}h(x) \right|^n}{[\,\mathbf{J}_h(x)\,]^{n-1}} \right\} \mathrm{d}x, \qquad (1.15)$$

where the ratio term represents the inner distortion of h. Here $D^{\sharp}h$ denotes the cofactor matrix of Dh. For more details we refer the reader to [4]. Examples abound in which one can return the deformed body to its original shape with conformal energy, but not necessarily via the inverse mapping $f = h^{-1} \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$, because f need not even belong to $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$. This typically occurs when the boundary of the deformed configuration (like a ball with a straight line slit cut) differs topologically from the boundary of the reference configuration (like a ball without a cut) [29–31]. We believe that the geometric/topological obstructions for reversibility of elastic deformations might be of interest in mathematical models of nonlinear elasticity (NE) [1,5,10,41]. In our setting, by virtue of the Hooke's Law, it is naturally to study deformations of bi-conformal energy. One of the important problems in nonlinear elasticity is whether or not a radially symmetric solution of a rotationally invariant minimization problem is indeed the absolute minimizer. In the case of bi-conformal energy this is proven to be the case in low dimension models ($n = 2, 3$) [32]. The radial symmetric solutions, however, may fail to be absolute minimizers if $n \geqslant 4$ [32]. Several more papers, in the intersection of NE and GFT, are devoted to understand the expected radial symmetric properties [2,6,12,18,23,25,26,28,33,35,36,39,42,43,48–50].

2. Quick Review of the Modulus of Continuity

Let us recall the concept of *modulus of continuity*, also known as *modulus of oscillation*; the concept introduced by H. Lebesgue [40] in 1909.
We are dealing with continuous mappings $h : \mathbb{X} \to \mathbb{Y}$ between subsets $\mathbb{X} \subset \mathscr{X}$ and $\mathbb{Y} \subset \mathscr{Y}$ of normed spaces $(\mathscr{X}, |\cdot|)$ and $(\mathscr{Y}, \|\cdot\|)$.

A modulus of continuity is any continuous function $\omega : [0, \infty) \to [0, \infty)$ that is strictly increasing and $\omega(0) = 0$.

Definition 2.1. A continuous mapping $h : \mathbb{X} \to \mathbb{Y}$ is said to admit ω as its (local) modulus of continuity at the point $x_\circ \in \mathbb{X}$ if

$$\| h(x) - h(x_\circ) \| \preccurlyeq \omega(|x - x_\circ|) \ , \quad \text{for all } x \in \mathbb{X}. \tag{2.1}$$

Here the implied constant may depend on $x_\circ$, but not on x. In short, h is ω-continuous at the point $x_\circ$. If this inequality holds for all $x, x_\circ \in \mathbb{X}$ with an implied constant independent of x and $x_\circ$ then h is said to admit ω as its (global) modulus of continuity in $\mathbb{X}$.

Definition 2.2. (Optimal Modulus of Continuity). Every uniformly continuous function $h : \mathbb{X} \to \mathbb{Y}$ admits the optimal modulus of continuity at a given point $x_\circ \in \mathbb{X}$, given by the rule

$$\omega_h(x_\circ; t) \stackrel{\text{def}}{=\!=} \sup\{ \| h(x) - h(x_\circ) \| : x \in \mathbb{X} \ , \ |x - x_\circ| \leqslant t\}. \tag{2.2}$$

No implied constant is involved in this definition. Similarly, the function

$$\Omega_h(t) \stackrel{\text{def}}{=\!=} \sup\{ \| h(x) - h(x_\circ) \| : x, x_\circ \in \mathbb{X} \ , \ |x - x_\circ| \leqslant t\} \tag{2.3}$$

is referred to as *(globally) optimal modulus of continuity* of h in $\mathbb{X}$.

Definition 2.3. (Bi-modulus of Continuity). The term *bi-modulus of continuity* of a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ refers to a pair (ϕ, ψ) of continuously increasing functions $\phi : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ and $\psi : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ in which ϕ is a modulus of continuity of h and ψ is a modulus of continuity of the inverse map $f \stackrel{\text{def}}{=\!=} h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$. Such a pair is said to be the optimal bi-modulus of continuity at the point $(x_\circ, y_\circ) \in \mathbb{X} \times \mathbb{Y}$, $y_\circ = h(x_\circ)$, if $\phi(t) = \omega_h(x_\circ; t)$ and $\psi(s) = \omega_f(y_\circ; s)$.

3. Quasiconformal Mappings

Let us take a quick look at the radial stretching/squeezing homeomorphism $h : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ defined by

$$h(x) = \mathbf{H}(|x|) \frac{x}{|x|} \ , \quad \text{for } x \in \mathbb{R}^n, \tag{3.1}$$

where the function $\mathbf{H} : [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ (interpreted as radial stress function) is continuous and strictly increasing. Its inverse $f \overset{\text{def}}{=\!=} h^{-1} : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ becomes a squeezing/stretching homeomorphism of the form

$$f(y) = \mathbf{F}(|y|)\, \frac{y}{|y|}\ , \quad \text{for } y \in \mathbb{R}^n, \tag{3.2}$$

where $\mathbf{F} : [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ stands for the inverse function of $\mathbf{H}$. These two radial stress functions are exactly the optimal moduli of continuity at $0 \in \mathbb{R}^n$ of h and f, respectively. By the definition,

$$\omega_h(t) \overset{\text{def}}{=\!=} \omega_h(0,t) = \max_{|x|=t} |h(x)| \; = \; \mathbf{H}(t)$$

$$\omega_f(s) \overset{\text{def}}{=\!=} \omega_f(0,s) = \max_{|y|=s} |f(y)| \; = \; \mathbf{F}(s)\,.$$

Therefore

$$\omega_f(\omega_h(t)) \;\equiv t \;\text{ for all }\; t \geqslant 0\;, \qquad \text{and }\; \omega_h(\omega_f(s)) \equiv s \quad \text{for all } \;\; s \geqslant 0. \tag{3.3}$$

The above identities admit of a simple interpretation:

The better is the optimal modulus of continuity of h, the worse is the optimal modulus of continuity of its inverse map f, and vice versa.

Look at the power type stretching $h(x) = |x|^N \frac{x}{|x|}$ and $f(y) = |y|^{\frac{1}{N}} \frac{y}{|y|}$.
To an extent, this interpretation pertains to all quasiconformal homeomorphisms. There are three main equivalent definitions for quasiconformal mappings: metric, geometric, and analytic. The *analytic definition* (1.3) was first considered by Lavrentiev in connection with elliptic systems of partial differential equations. The *geometric definition* states that a homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a quasiconformal if there is a constant $K \geqslant 1$ such that

$$K^{-1} \operatorname{mod}\big(f(\Gamma)\big) \leqslant \operatorname{mod}(\Gamma) \leqslant K \operatorname{mod}\big(f(\Gamma)\big)$$

for every curve family Γ in $\mathbb{X}$. The *conformal modulus* $\operatorname{mod}(\Gamma)$ of family Γ of curves in $\mathbb{R}^n$ is the infimum of the numbers $\int_{\mathbb{R}^n} (\rho(x))^n \,\mathrm{d}x$ over all nonnegative Borel functions $\rho\colon \mathbb{R}^n \to [0,\infty]$ such that $\int_\gamma \rho \,\mathrm{d}s \geqslant 1$ for every $\gamma \in \Gamma$. The geometric definition quickly yields many strong properties of quasiconformal mappings; for example, the inverse of a quasiconformal mapping is automatically quasiconformal, which is not at all obvious from the analytic definition. Here we, however, will relay on the *metric definition*, which says that “infinitesimal balls are transformed to infinitesimal ellipsoids of bounded eccentricity”. The interested reader is referred to [3, Chapter 3.] to find more about the foundations of quasicoformal mappings.

Definition 3.1. [Metric Definition] Let $\mathbb{X}$ and $\mathbb{Y}$ be domains in $\mathbb{R}^n$, $n \geqslant 2$, and $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ a homeomorphism. For every point $x_\circ \in \mathbb{X}$ we define.

$$\mathcal{H}_h(x_\circ, r) \;\overset{\text{def}}{=\!=}\; \frac{\max_{|x-x_\circ|=r}\; |h(x)-h(x_\circ)|}{\min_{|x-x_\circ|=r}\; |h(x)-h(x_\circ)|} \tag{3.4}$$

whenever $0 < r < \operatorname{dist}(x_\circ, \partial\mathbb{X})$. Also define

$$1 \leqslant \mathcal{H}_h(x_\circ) \stackrel{\text{def}}{=} \limsup_{r\to 0} \mathcal{H}_h(x_\circ, r) \leqslant \infty \tag{3.5}$$

and call it the *linear dilatation* of h at $x_\circ$. If, furthermore,

$$\mathcal{K}_h \stackrel{\text{def}}{=} \sup_{x_\circ\in\mathbb{X}} \mathcal{H}_h(x_\circ) < \infty, \tag{3.6}$$

then we call $\mathcal{K}_h$ the *maximal linear dilatation* of h in $\mathbb{X}$ and h a quasiconformal mapping. Finally, h is K -quasiconformal, $1 \leqslant K < \infty$ if

$$\operatorname*{ess\text{-}sup}_{x_\circ\in\mathbb{X}} \mathcal{H}_h(x_\circ) \leqslant K. \tag{3.7}$$

It should be noted that the inverse map $f \stackrel{\text{def}}{=} h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ is also K - quasiconformal.

Next, we invoke the optimal modulus of continuity at a point $x_\circ \in \mathbb{X}$:

$$\omega_h(t) \stackrel{\text{def}}{=} \omega_h(x_\circ; t) = \max_{|x-x_\circ|=t} |h(x) - h(x_\circ)| \,, \text{ for } 0 \leqslant t < t_\circ \stackrel{\text{def}}{=} \operatorname{dist}(x_\circ; \partial\mathbb{X}).$$

This defines a continuous strictly increasing function $\omega_h : [0, t_\circ) \xrightarrow{\text{onto}} [0, s_\circ)$, where $s_\circ \stackrel{\text{def}}{=} \operatorname{dist}(y_\circ; \partial\mathbb{Y})$. Similar definitions apply to the inverse map $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ which is also K -quasiconformal. Its optimal modulus of continuity at the image point $y_\circ = h(x_\circ)$ is given by

$$\omega_f(s) \stackrel{\text{def}}{=} \omega_f(y_\circ; s) = \max_{|y-y_\circ|=s} |f(y) - f(y_\circ)| \,, \quad \text{for } 0 \leqslant s < s_\circ \,.$$

Therefore, both compositions $\omega_f(\omega_h(t))$ and $\omega_h(\omega_f(s))$ are well defined for $0 \leqslant t < t_\circ$ and $0 \leqslant s < s_\circ$, respectively. Unlike the radial stretchings, the function $\omega_f(s)$ is generally not the inverse of $\omega_h(t)$, but very close to it. Namely, the optimal modulus of continuity of h and that of f are *quasi-inverse* to each other. Let us make this statement more precise by the following theorem:

Theorem 3.2. *(Local quasi-inversion). Let a map $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ be K -quasi conformal and $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ denote its inverse. Then there is a constant $\mathscr{K} = \mathscr{K}(n, K) \geqslant 1$ such that for every point $x_\circ \in \mathbb{X}$ and its image $y_\circ = h(x_\circ) \in \mathbb{Y}$ it holds that*

$$\mathscr{K}^{-1}s \leqslant \omega_h(\omega_f(s)) \leqslant \mathscr{K}s \quad \text{and} \quad \mathscr{K}^{-1}t \leqslant \omega_f(\omega_h(t)) \leqslant \mathscr{K}t \tag{3.8}$$

whenever $0 \leqslant t \leqslant t(x_\circ)$ and $0 \leqslant s \leqslant s(y_\circ)$. Here the upper bounds positive numbers $t(x_\circ)$ and $s(y_\circ)$, depend only on $\operatorname{dist}(x_\circ; \partial\mathbb{X})$ *and* $\operatorname{dist}(y_\circ; \partial\mathbb{Y})$*, respectively.*

Before proceeding to the proof, we recall a very useful Extension Theorem by F. W. Gehring [14], see also the book by J. Väisälä [52] (Theorem 41.6). This theorem allows us to reduce a local quasiconformal problem to an analogous problem for mappings defined in the entire space $\mathbb{R}^n$.

Lemma 3.3. *(F. W. Gehring). Every quasiconformal map* $h : \mathbb{B}(x_\circ, 2r) \overset{\text{into}}{\longrightarrow} \mathbb{R}^n$ *defined in a ball* $\mathbb{B}(x_\circ, 2r) \subset \mathbb{R}^n$ *admits a quasiconformal mapping* $h' : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ *which equals* h *on* $\mathbb{B}(x_\circ, r)$. *The dilatation of* h' *depends only that of* h *and the dimension* n.

Accordingly, we may (and do) assume that $\mathbb{X} = \mathbb{Y} = \mathbb{R}^n$. This will give us a more precise information about the constant $\mathscr{K} = \mathscr{K}(n, K)$.

Theorem 3.4. *(Global quasi-inversion). Let a map* $h : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ *be* K *-quasiconformal and* $f : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ *denote its inverse. Then there is a constant* $\mathscr{K} = \mathscr{K}(n, K) \geqslant 1$ *such that for every point* $x_\circ \in \mathbb{R}^n$ *and its image* $y_\circ = h(x_\circ)$ *it holds that*

$$\mathscr{K}^{-1} s \leqslant \omega_h(\omega_f(s)) \leqslant \mathscr{K} s \quad \text{and} \quad \mathscr{K}^{-1} t \leqslant \omega_f(\omega_h(t)) \leqslant \mathscr{K} t \tag{3.9}$$

for all $s \geqslant 0$ *and* $t \geqslant 0$.

Rather than using the original definition we will appeal to Gehring's characterization of quasiconformal mappings, see Inequality (3.3) in [16] and some related articles [13,15,37,51–53]. The interested reader is referred to a book by P. Caraman [8] on various definitions and extensive early literature on the subject.

Proposition 3.5. *(Three points condition). To every* $\lambda \geqslant 1$ *there corresponds a constant* $1 \leqslant \mathscr{K}_\lambda = \mathscr{K}_\lambda(n, K)$ *such that whenever three distinct points* $x_\circ$, x_1, $x_2 \in \mathbb{R}^n$ *satisfy the ratio condition*

$$\frac{|x_1 - x_\circ|}{|x_2 - x_\circ|} \leqslant \lambda, \tag{3.10}$$

the image points under $h : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ *satisfy analogous condition*

$$\frac{|h(x_1) - h(x_\circ)|}{|h(x_2) - h(x_\circ)|} \leqslant \mathscr{K}_\lambda = \mathscr{K}_\lambda(n, K). \tag{3.11}$$

In particular, we have

Proposition 3.6. *Let* $h : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ *be* K *-quasiconformal. Then for every point* $x_\circ \in \mathbb{X}$ *and* $0 < r < \infty$ *we have*

$$\mathcal{H}_h(x_\circ, r) \overset{\text{def}}{=\!=} \frac{\max_{|x - x_\circ| = r} \; |h(x) - h(x_\circ)|}{\min_{|x - x_\circ| = r} \; |h(x) - h(x_\circ)|} \leqslant \mathscr{K} = \mathscr{K}_1(n, K). \tag{3.12}$$

Proof of Theorem 3.4. It is clearly sufficient to make the computation when $x_\circ = 0$ and $y_\circ = 0$. In this case the condition (3.12) takes the form

$$\frac{1}{\mathscr{K}} |h(x_2)| \leqslant |h(x_1)| \leqslant \mathscr{K} |h(x_2)|, \quad \text{whenever } |x_1| = |x_2| \neq 0. \tag{3.13}$$

By the definition of the optimal modulus of continuity at the origin, we have

- $\omega_h(\omega_f(s)) = |h(x)|$ for some $x \in \mathbb{R}^n$ with $|x| = \omega_f(s)$;
- $\omega_f(s) = |f(y)|$ for some $y \in \mathbb{R}^n$ with $|y| = s$;

- Therefore, $\omega_h(\omega_f(s)) = |h(x)|$, for some $|x| = |f(y)|$.

Now, the right hand side of inequality at (3.13) gives the desired upper bound $\omega_h(\omega_f(s)) = |h(x)| \leqslant \mathscr{K}\,|h(f(y))| = \mathscr{K}\,|y| = \mathscr{K}s$, whereas the left hand side gives the lower bound $\omega_h(\omega_f(s)) = |h(x)| \geqslant \mathscr{K}^{-1}\,|h(f(y))| = \mathscr{K}^{-1}\,|y| = \mathscr{K}^{-1}\,s$. The analogous bounds for $\omega_f(\omega_h(t))$ at (3.8) follow by interchanging the roles of h and f ; as they are both K -quasiconformal. This completes the proof of Theorem 3.4. □

The converse statement to Theorem 3.2 is

Theorem 3.7. *Consider a homeomorphism* $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$*, its inverse mapping* $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$*, and their optimal moduli of continuity at a point* $x_\circ \in \mathbb{X}$ *and* $y_\circ = h(x_\circ)$*, respectively:*

$$\omega_h(t) \stackrel{\text{def}}{=\!=} \max_{|x-x_\circ|=t} |h(x) - h(x_\circ)| \quad \text{and} \qquad \omega_f(s) \stackrel{\text{def}}{=\!=} \max_{|y-y_\circ|=s} |f(y) - f(y_\circ)|,$$

for $0 \leqslant t < \operatorname{dist}(x_\circ, \partial\mathbb{X})$ *and* $0 \leqslant s < \operatorname{dist}(y_\circ, \partial\mathbb{Y})$. *Assume the following one-sided quasi-inverse condition at every point* $x_\circ \in \mathbb{X}$*, with a constant* $\mathscr{K} \geqslant 1$:

$$\omega_h(\omega_f(r)) \leqslant \mathscr{K} r \ \text{ for all sufficiently small } r > 0 \quad (\text{depending on } x_\circ). \qquad (3.14)$$

Then h *is* $\mathscr{K}$ *-quasiconformal.*

Here is a simple geometric proof.

Proof. We shall actually show that Condition (3.14) at the given point $x_\circ \in \mathbb{X}$ implies that

$$\mathcal{H}_h(x_\circ, t) = \frac{\max_{|x-x_\circ|=t}\ |h(x) - h(x_\circ)|}{\min_{|x-x_\circ|=t}\ |h(x) - h(x_\circ)|} \leqslant \mathscr{K} \qquad (3.15)$$

for $t > 0$ sufficiently small. In particular, for every $x_\circ \in \mathbb{X}$ it holds that

$$\limsup_{t\to 0}\ \mathcal{H}_h(x_\circ, t) \leqslant \mathscr{K}, \quad \text{as required.} \qquad (3.16)$$

A sufficient upper bound of t at (3.15) depends on $\operatorname{dist}(x_\circ, \partial\mathbb{X})$, but we shall not enter into this issue. It simplifies the writing, and causes no loss of generality, to assume that $x_\circ = y_\circ = 0$. Thus we are reduced to showing that

$$\max_{|x|=t} |h(x)| \leqslant \mathscr{K} \min_{|x|=t} |h(x)|, \quad \text{for all sufficiently small } t > 0. \qquad (3.17)$$

To this end, consider the ball $\mathbb{B}(x_\circ, t) \subset \mathbb{X}$ centered at $x_\circ = 0$ and with small radius $t > 0$. Its image under h , denoted by $\Omega = h(\mathbb{B}(x_\circ, t)) \subset \mathbb{Y}$, contains the origin $y_\circ = 0$. Let $r > 0$ denote the largest radius of a ball, denoted by $\mathbf{B}_r \subset \Omega$, centered at $y_\circ = 0$. Thus

$$\min_{|x|=t} |h(x)| = r.$$

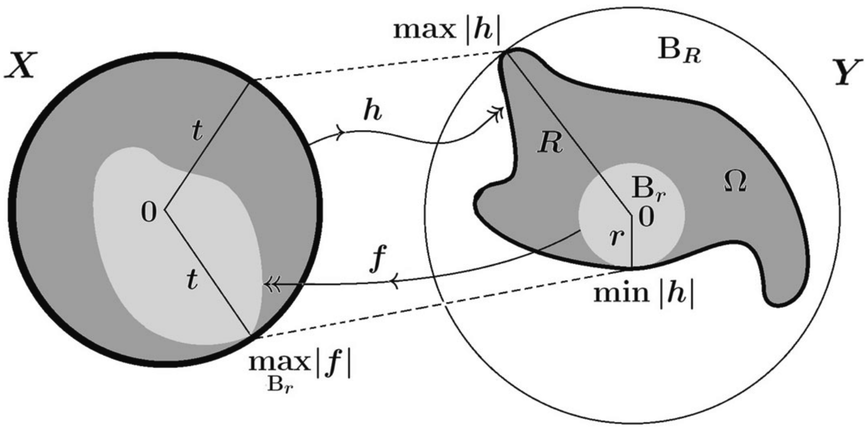

Fig. 1. The ratio $\frac{R}{r} \leqslant \mathscr{K}$

Similarly, denote by R the smallest radius of a ball $\mathbf{B}_R \supset \Omega$ centered at $y_\circ = 0$, see Fig. 1. Thus

$$R = \max_{|x|=t} |h(x)| \stackrel{\text{def}}{=\!=} \omega_h(t).$$

Now the inverse map $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes Ω onto $\mathbb{B}(x_\circ, t)$. In particular, it takes the common point of $\partial \mathbf{B}_r$ and $\partial \Omega$ into a point of $\partial \mathbb{B}(x_\circ, t)$. This means that

$$t = \max_{|y|=r} |f(y)| \stackrel{\text{def}}{=\!=} \omega_f(r).$$

The proof is completed by invoking the quasi-inverse condition at (3.14),

$$R = \omega_h(t) = \omega_h(\omega_f(r)) \leqslant \mathscr{K} r.$$

□

3.1. Doubling Property

It is worth discussing another special property of quasiconformal mappings in relation to their bi-modulus of continuity. To simplify matters we confine ourselves to quasiconformal mappings defined on the entire space, $h : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ and its inverse $f : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$. It turns out that at every point $x_\circ \in \mathbb{R}^n$ the optimal modulus of continuity $\phi(t) \stackrel{\text{def}}{=\!=} \omega_h(x_\circ; t)$, as well as its inverse function $\phi^{-1} : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ have a doubling property. Observe that ϕ^{-1} is not exactly the optimal modulus of continuity of the inverse map $f = h^{-1}$, the latter is only quasi-inverse to ϕ^{-1}. It should be emphasized at this point that doubling property of the modulus of continuity is rather rare, see our representative examples in Section 6.

Proposition 3.8. *Consider all K-quasiconformal mappings $h : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$. To every $\lambda \geqslant 1$ there corresponds a constant $\mathscr{K}_\lambda$ (actually the one specified in*

(3.11)), and there is a constant $C_\lambda = C_\lambda(n, K)$ (independent of h) such that at every point $x_\circ \in \mathbb{R}^n$ we have

$$\omega_h(x_\circ; \lambda t) \leqslant \mathscr{K}_\lambda \; \omega_h(x_\circ; t) \tag{3.18}$$

and

$$\omega_h^{-1}(x_\circ; \lambda s) \leqslant C_\lambda \; \omega_h^{-1}(x_\circ; s) \tag{3.19}$$

for all $0 \leqslant t < \infty$ and $0 \leqslant s < \infty$.

Proof. We may again assume that $x_\circ = 0$ and $h(x_\circ) = 0$. This simplifies the notation $\omega_h(x_\circ; t) \stackrel{\text{def}}{=} \omega_h(t)$. The proof of the first inequality is immediate from the three points ratio condition in Proposition 3.5, which gives us exactly the constant $\mathscr{K}_\lambda$ from this condition. Indeed, we have that

- $\omega_h(\lambda t) = |h(x_1)|$, for some $x_1 \in \mathbb{R}^n$ with $|x_1| = \lambda t$;
- $\omega_h(t) = |h(x_2)|$, for some $x_2 \in \mathbb{R}^n$ with $|x_2| = t$;
- Hence, $\frac{|x_1|}{|x_2|} \leqslant \lambda$;
- Consequently $\frac{|h(x_1)|}{|h(x_2)|} \leqslant \mathscr{K}_\lambda$, which is the desired estimate.

□

Clearly, for every $y_\circ \in \mathbb{R}^n$, we also have

$$\omega_f(y_\circ; \lambda s) \leqslant \mathscr{K}_\lambda \; \omega_f(y_\circ; s) \quad \text{for all} \quad 0 \leqslant s < \infty, \tag{3.20}$$

simply by interchanging the roles of h and f.

We precede the proof of the doubling condition for ω_h^{-1}, with a quick lemma.

3.2. A Quick Lemma on Doubling Condition

Consider an arbitrary continuously increasing function $\phi : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ (in our application, $\phi(t) = \omega_h(t)$). It is commonly said that ϕ satisfies doubling condition if there is a constant $C_\phi \geqslant 1$ such that $\phi(2t) \leqslant C_\phi \, \phi(t)$ for all $t \geqslant 0$. However, it is convenient to work with so-called *generalized doubling condition*, which reads as

$$\phi(\lambda t) \leqslant C_\phi(\lambda) \, \phi(t), \quad \text{for all } t \geqslant 0, \tag{3.21}$$

where the λ - *constant* $C_\phi(\lambda) \geqslant 1$ is obtained by iterating the inequality $\phi(2t) \leqslant C_\phi \, \phi(t)$.

Associated with ϕ is its *quasi-inverse function*. This term pertains to any continuous and strictly increasing function $\psi : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ such that

$$m\, t \leqslant \psi(\phi(t)) \leqslant M t, \quad \text{for all } t \geqslant 0, \tag{3.22}$$

where $0 < m \leqslant 1 \leqslant M < \infty$ are constants. In general, ψ does not satisfy doubling condition, but its inverse $\psi^{-1} : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ does.

Lemma 3.9. *To every factor $\lambda \geqslant 1$ there corresponds a generalized doubling constant for ψ^{-1}. For all $t \geqslant 0$ we have that*

$$\psi^{-1}(\lambda t) \leqslant C_{\psi^{-1}}(\lambda)\, \psi^{-1}(t). \quad \text{Explicitly } C_{\psi^{-1}}(\lambda) \stackrel{\text{def}}{=} C_\phi\,(M\lambda/m)\,. \tag{3.23}$$

Proof. Choose and fix $\lambda \geqslant 1$. Inequality (3.22) is equivalent to

$$\psi^{-1}(m\,t) \leqslant \phi(t) \leqslant \psi^{-1}(Mt)\,, \quad \text{for all } t \geqslant 0. \tag{3.24}$$

Upon substitution $t \rightsquigarrow \frac{\lambda t}{m}$ in the left hand side, we obtain

$$\psi^{-1}(\lambda t) \leqslant \phi\left(\frac{\lambda t}{m}\right) = \phi\left(\frac{M\lambda}{m} \cdot \frac{t}{M}\right) \leqslant C_\phi\left(\frac{M\lambda}{m}\right) \cdot \phi\left(\frac{t}{M}\right).$$

The proof of the lemma is completed by invoking the right hand side of inequality (3.24) which, upon substitution $t \rightsquigarrow \frac{t}{M}$, gives us the desired estimate $\phi(\frac{t}{M}) \leqslant \psi^{-1}(t)$. □

We summarize this section with the following theorem, which is an expanded version of Theorem 1.1:

Theorem 3.10. *Let $h : \mathbb{R}^n \stackrel{\text{onto}}{\longrightarrow} \mathbb{R}^n$ be a K-quasiconformal mapping and $f : \mathbb{R}^n \stackrel{\text{onto}}{\longrightarrow} \mathbb{R}^n$ its inverse. Choose and fix an arbitrary point $x_\circ \in \mathbb{R}^n$ an its image point $y_\circ = h(x_\circ)$. Denote by $\phi(t) = \omega_h(x_\circ; t)$ the optimal modulus of continuity of h at $x_\circ$ and by $\psi(s) = \omega_f(y_\circ; s)$ the optimal modulus of continuity of f at $y_\circ$. Then the following statements hold true:*

(Q1) *The functions ϕ and ψ are quasi-inverse to each other. Precisely, there is a constant $\mathscr{K} = \mathscr{K}(n, K)$ such that*

$$\mathscr{K}^{-1} t \leqslant \psi(\phi(t)) \leqslant \mathscr{K}\, t \quad \text{and} \quad \mathscr{K}^{-1} s \leqslant \phi(\psi(s)) \leqslant \mathscr{K}\, s \tag{3.25}$$

for all $t, s \in [0, \infty)$.

(Q2) *Both ϕ and ψ satisfy the general doubling condition; that is, for every $\lambda \geqslant 1$ there is a constant $\mathscr{K}_\lambda$ such that*

$$\phi(\lambda t) \leqslant \mathscr{K}_\lambda\, \phi(t) \quad \text{and} \quad \psi(\lambda s) \leqslant \mathscr{K}_\lambda\, \psi(s) \tag{3.26}$$

for all $t, s \in [0, \infty)$.

(Q3) *As a consequence of Conditions $(Q1)$ and $(Q2)$, the inverse functions ψ^{-1} and ϕ^{-1} also satisfy a general doubling conditions; namely,*

$$\phi^{-1}(\lambda s) \leqslant C_\lambda\, \phi^{-1}(s) \quad \text{and} \quad \psi^{-1}(\lambda t) \leqslant C_\lambda\, \psi^{-1}(s) \tag{3.27}$$

for all $t, s \in [0, \infty)$, where the constant $C_\lambda = \mathscr{K}_\lambda(\lambda\, \mathscr{K}^2)$.

Let us now proceed to more general mappings of bi-conformal energy.

4. A Handy Metric in $\mathbb{R}^n \simeq \mathbb{R}^{n-1} \times \mathbb{R}$

It will be convenient to consider the space $\mathbb{R}^n$ as Cartesian product $\mathbb{R}^{n-1} \times \mathbb{R}$, with the purpose of using cylindrical coordinates. Accordingly,

$$\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} = \left\{X = (x, t);\ x = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1} \text{ and } t \in \mathbb{R}\right\}.$$

Hereafter, we change the notation of the variables; the lowercase letter x designates a point $(x_1, ...x_{n-1}) \in \mathbb{R}^{n-1}$ while the uppercase letter $X = (x, t)$ is reserved for points in $\mathbb{R}^n$. The Euclidean norm of $x \in \mathbb{R}^{n-1}$ is denoted by $|x| \stackrel{\text{def}}{=} \sqrt{x_1^2 + \cdots + x_{n-1}^2}$. The space $\mathbb{R}^{n-1} \times \mathbb{R}$ is furnished with the norm

$$\| X \| \stackrel{\text{def}}{=} |x| + |t|, \quad \text{for } X = (x, t) = (x_1, ..., x_{n-1}, t) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

In this metric the closed unit ball in $\mathbb{R}^{n-1} \times \mathbb{R}$ becomes the Euclidean double cone

$$\mathcal{C} = \{(x, t) \in \mathbb{R}^n ;\ |x| + |t| \leqslant 1 \} = \mathcal{C}_+ \cup \mathcal{C}_- ,$$

where we split $\mathcal{C}$ into the upper and lower cones:

$$\mathcal{C}_+ = \{(x, t);\ |x| + t \leqslant 1 ,\ t \geqslant 0 \} ,\quad \mathcal{C}_- = \{(x, t);\ |x| - t \leqslant 1 ,\ t \leqslant 0 \}$$

5. The Idea of the Construction of $H : \mathcal{C} \xrightarrow{\text{onto}} \mathcal{C}$

Our construction of a bi-conformal energy map $H : \mathcal{C} \xrightarrow{\text{onto}} \mathcal{C}$, whose optimal modulus of continuity at the origin coincides with that of the inverse map, will be carried out in two steps. First we construct a homeomorphism $H : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ of finite conformal-energy which equals the identity on $\partial \mathcal{C}_+$. Its inverse map $F \stackrel{\text{def}}{=} H^{-1} : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ will also have finite conformal-energy. The substance of the matter is that their optimal moduli of continuity (ω_H and ω_F, respectively) are inverse to each other; thus generally not equal. In fact ω_H will be stronger that ω_F. In the second step we adopt the modulus of continuity of $F : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ to an extension of H to $\mathcal{C}_-$, simply by reflecting F twice about $\mathbb{R}^{n-1}$. Let the reflection $\mathfrak{r} : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ be defined by $\mathfrak{r}(x, t) = (x, -t)$. This gives rise to a map $\mathfrak{r} \circ F \circ \mathfrak{r} : \mathcal{C}_- \xrightarrow{\text{onto}} \mathcal{C}_-$, which we glue to $H : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ along the common base $\partial\mathcal{C}_+ \cap \partial\mathcal{C}_- \subset \mathbb{R}^{n-1}$. Precisely, the desired homeomorphism $H : \mathcal{C} \xrightarrow{\text{onto}} \mathcal{C}$, still denoted by H, will be defined by the rule

$$H \stackrel{\text{def}}{=} \begin{cases} H : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+ \\ \mathfrak{r} \circ F \circ \mathfrak{r} : \mathcal{C}_- \xrightarrow{\text{onto}} \mathcal{C}_- \end{cases} . \tag{5.1}$$

Its inverse, also denoted by $F : \mathcal{C} \xrightarrow{\text{onto}} \mathcal{C}$, is defined analogously by interchanging the roles of F and H:

$$F \stackrel{\text{def}}{=} \begin{cases} F : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+ \\ \mathfrak{r} \circ H \circ \mathfrak{r} : \mathcal{C}_- \xrightarrow{\text{onto}} \mathcal{C}_- \end{cases} . \tag{5.2}$$

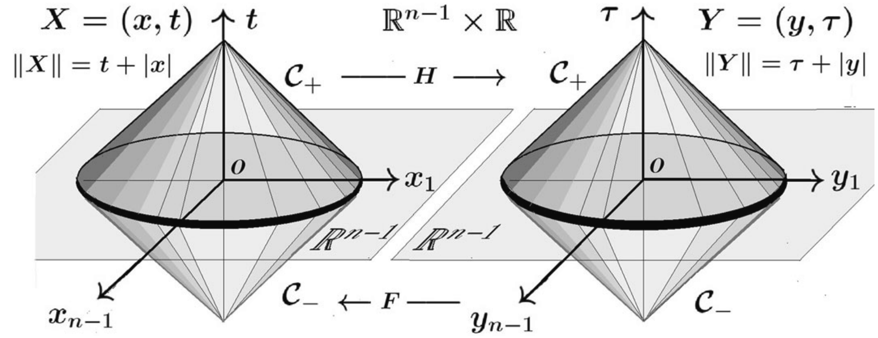

Fig. 2. A mapping $H : \mathcal{C} \overset{\text{onto}}{\longrightarrow} \mathcal{C}$ and its inverse $F : \mathcal{C} \overset{\text{onto}}{\longrightarrow} \mathcal{C}$, will have the same optimal modulus of continuity at the center of $\mathcal{C}$

As a result, the optimal modulus of continuity of H will be attained in the upper cone $\mathcal{C}_+$, whereas the optimal modulus of continuity of F will be attained in the lower cone $\mathcal{C}_-$. Clearly, they are the same for the double cone $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$, and this is the essence of our construction.

The explicit formula for H can easily be stated, see Definition 7.1 in Section 7.

Since $H : \mathcal{C} \overset{\text{onto}}{\longrightarrow} \mathcal{C}$ and its inverse $F \overset{\text{def}}{=\!=} H^{-1} : \mathcal{C} \overset{\text{onto}}{\longrightarrow} \mathcal{C}$ are both equal to the identity on $\partial\mathcal{C}$ we can extend them to $\mathbb{R}^n$ as the identity outside $\mathcal{C}$. Whenever it is convenient, we shall speak of $H : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ and its inverse $F : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ as homeomorphisms of the entire space $\mathbb{R}^n$ onto itself (Fig. 2).

6. Preconditions on the Modulus of Continuity and the Representative Examples

Let us introduce a fairly general class of moduli of continuity to be considered. These classes are intended to unify the proofs. It will also give us an aesthetic appearance of the inequalities. On that account, our moduli of continuity, will be made of functions $\phi : [0, 1] \overset{\text{onto}}{\longrightarrow} [0, 1]$ in $\mathscr{C}[0, 1] \cap \mathscr{C}^1(0, 1]$ such that

($\mathbf{C}_1$) $\phi(0) = 0$, $\phi(1) = 1$ (can be extended by $\phi(s) = s$ for $s \geqslant 1$);
($\mathbf{C}_2$)

$$\phi'(s) \leqslant \frac{\phi(s)}{s} \leqslant M[\phi'(s)]^2 \text{ , for some constant } 1 \leqslant M < \infty \text{;} \tag{6.1}$$

($\mathbf{C}_3$) *Finite Energy Condition* :

$$E[\phi] \overset{\text{def}}{=\!=} \int_0^1 |\phi(s)|^n \frac{\mathrm{d}s}{s} < \infty. \tag{6.2}$$

As a consequence of Conditions ($\mathbf{C}_1$) and ($\mathbf{C}_2$) we have

- $$\lambda(s) \overset{\text{def}}{=\!=} \frac{\phi(s)}{s} \geqslant \phi'(s) \geqslant \frac{1}{M} \quad \text{for all} \quad 0 < s \leqslant 1. \tag{6.3}$$

In the forthcoming representative examples (except for $\phi(s) \equiv s$) we have even stronger property; namely, $\lim_{s\to 0} \phi'(s) = \infty$.

- The function $\lambda(s)$ is non-increasing. This follows from

$$\lambda'(s) = \frac{\phi'(s)}{s} - \frac{\phi(s)}{s^2} \leqslant 0. \tag{6.4}$$

- Thus, in fact,

$$\lambda(s) = \frac{\phi(s)}{s} \geqslant \frac{\phi(1)}{1} = 1 \ , \text{ for all } 0 < s \leqslant 1. \tag{6.5}$$

6.1. Representative Examples

($\mathbf{E}_0$) For $0 < \varepsilon \leqslant 1$, we set

$$\phi_0(s) = s^{\varepsilon}. \quad \text{In the borderline case,} \quad \phi(s) = s.$$

($\mathbf{E}_1$) For $\frac{1}{n} < \alpha \leqslant 1$, we set

$$\phi_1(s) = \log^{-\alpha}\left(\frac{e}{s}\right) = \left(1 + a_1 \log \frac{1}{s}\right)^{-\alpha}.$$

($\mathbf{E}_2$) For $\frac{1}{n} < \alpha \leqslant 1$, we set

$$\phi_2(s) = \left(1 + a_1 \log \frac{1}{s}\right)^{-\frac{1}{n}} \left(1 + a_2 \log\log \frac{e}{s}\right)^{-\alpha}.$$

($\mathbf{E}_3$) For $\frac{1}{n} < \alpha \leqslant 1$, we set

$$\phi_3(s) = \left(1 + a_1 \log \frac{1}{s}\right)^{-\frac{1}{n}} \left(1 + a_2 \log\log \frac{e}{s}\right)^{-\frac{1}{n}} \left(1 + a_3 \log\log\log \frac{e^e}{s}\right)^{-\alpha}.$$

Continuing in this fashion, we define a sequence of functions ϕ_k , $k = 0, 1, 2, ...$ in which the last product-term in the round parantheses involves k-times iterated logarithm and $(k-1)$-times iterated power of e. All the above functions can be extended by setting $\phi_k(s) \equiv s$, for $s \geqslant 1$.

Remark 6.1. The coefficients a_k in the above formulas are adjusted to ensure the inequality $\phi'(s) \leqslant \frac{\phi(s)}{s}$, which is required by Condition ($\mathbf{C}_2$). This works well with $a_k \overset{\text{def}}{=\!=} (1 - \frac{1}{n})^{k-1}$. Indeed, the reader may wish to verify that the expression $\frac{s\phi_k'(s)}{\phi_k(s)}$ is increasing, thus assumes its maximum value at $s = 1$. It is then readily seen that its maximum value is not exceeding $\frac{1}{n}(a_1 + a_2 + ... + a_{k-2}) + \alpha\, a_k = 1 - \big(1 - \alpha\big)\big(1 - \frac{1}{n}\big)^{k-1} \leqslant 1$.

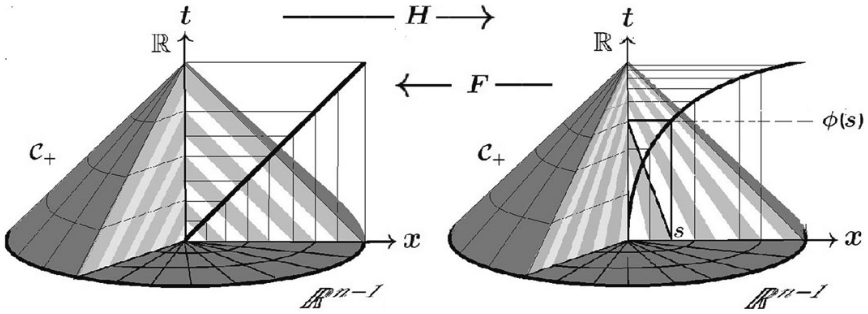

Fig. 3. Diagonals of rectangles built on the curve $t = \phi(s)$. The map F is linear on each such diagonal, as well as on their rotations

7. The Definition of $H : \mathcal{C}_+ \overset{\text{onto}}{\longrightarrow} \mathcal{C}_+$

First we set H on the vertical axis of the upper cone by the rule:

$$H(\mathbf{0}, t) = (\mathbf{0}, \phi(t)) .$$

Here and below $(\mathbf{0}, t) \overset{\text{def}}{=\!=} (0, ..., 0\,,\ t) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We wish H to be the identity map on the base of the cone, which consists of points $(x, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with $|x| \leqslant 1$. The idea is to connect $(x, 0)$ with the point $(\mathbf{0}, |x|)$ by a straight line segment and map it linearly onto the straightline segment with endpoints at $(x, 0)$ and $(\mathbf{0}, \phi(|x|))$. Explicitly, are have

Definition 7.1. The map $H : \mathcal{C}_+ \overset{\text{onto}}{\longrightarrow} \mathcal{C}_+ \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is given by the formula

$$H(x, t) \overset{\text{def}}{=\!=} (x,\ t\,\lambda(t + |x|))\ ,\ \text{for } 0 \leqslant t \leqslant 1 \ \text{ and } |x| + t \leqslant 1, \tag{7.1}$$

where we recall that $\lambda(s) \overset{\text{def}}{=\!=} \frac{\phi(s)}{s}$ for $0 < s \leqslant 1$.

Indeed, for $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$, we have $H\big[\alpha\,(x, 0)\ +\ \beta\,(\mathbf{0}, |x|)\,\big] = \alpha\,(x, 0)\ +\ \beta\,(\mathbf{0}, \phi(|x|)$, which means that H is a linear transformation between the above-mentioned segments. A formula for the inverse map $F : \mathcal{C}_+ \overset{\text{onto}}{\longrightarrow} \mathcal{C}_+$ is not that explicit (Fig. 3).

8. The Jacobian Matrix of H and Its Inverse

A straightforward computation of the Jacobian matrix of H, at the point $X \overset{\text{def}}{=\!=} (x, t) = (x_1, ..., x_{n-1}, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ shows that

$$DH(x,\ t) = \begin{pmatrix} 1 & 0 & 0\dots & 0 & 0 \\ \\ 0 & 1 & 0\dots & 0 & 0 \\ \vdots & \vdots & \vdots\ \ \vdots & \vdots & \vdots \\ 0 & 0 & 0\dots & 1 & 0 \\ \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \dots & \mathfrak{D}_{n-1} & \mathfrak{D} \end{pmatrix}, \tag{8.1}$$

where $\mathfrak{D}_i = \left[\, t\,\lambda'(t+|x|)\,\right] \frac{x_i}{|x|}$ and $\mathfrak{D} = \lambda(t+|x|) \,+\, t\,\lambda'(t+|x|)$ is the Jacobian determinant, later also denoted by $\mathbf{J}_H(X)$. Then the inverse matrix $(DH)^{-1}$ takes the form

$$(DH)^{-1} = \frac{1}{\mathfrak{D}} \begin{pmatrix} \mathfrak{D} & 0 & 0\ldots & 0 & 0 \\ 0 & \mathfrak{D} & 0\ldots & 0 & 0 \\ \vdots & \vdots & \vdots \;\; \vdots & \vdots & \vdots \\ 0 & 0 & 0\ldots & \mathfrak{D} & 0 \\ -\mathfrak{D}_1 & -\mathfrak{D}_2 & -\mathfrak{D}_3\ldots & -\mathfrak{D}_{n-1} & 1 \end{pmatrix}. \tag{8.2}$$

The square of the Hilbert Schmidt norm of a matrix is the sum of squares of its entries. Accordingly,

$$|DH|^2 \;=\; n-1+\left[\, t\,\lambda'(t+|x|)\,\right]^2 + \left[\, \lambda(t+|x|) \,+\, t\,\lambda'(t+|x|)\,\right]^2 \tag{8.3}$$

and

$$|\,(DH)^{-1}|^2 \;=\; \frac{1}{\left[\, \lambda(t+|x|) \,+\, t\,\lambda'(t+|x|)\,\right]^2} + \frac{\left[\, t\,\lambda'(t+|x|)\,\right]^2}{\left[\, \lambda(t+|x|) \,+\, t\,\lambda'(t+|x|)\,\right]^2} \;+\; n-1. \tag{8.4}$$

9. The Jacobian Determinant $\mathfrak{D} \;=\; \mathbf{J}_H(X)$

We have the following bounds of the Jacobian determinant, including a uniform lower bound for all $X=(x,t)\;\in \mathcal{C}_+$:

$$\lambda(\,\|\,X\,\|\,) \geqslant \mathbf{J}_H(X) \;\geqslant \phi'(\,\|\,X\,\|\,) \geqslant \frac{1}{M} \tag{9.1}$$

Proof. Using the notation $\|\,X\,\| = s = t+|x| \leqslant 1$, we write

$$\begin{aligned} \mathfrak{D} \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(t\,\lambda(t+|x|)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(t\;\frac{\phi(t+|x|)}{t+|x|}\Big) \\ &= \frac{\phi(s)}{s} \;+t\left(\frac{\phi'(s)}{s} \;-\; \frac{\phi(s)}{s^2}\right) = \frac{\phi(s)}{s}\left(1-\frac{t}{s}\right) \;+t\,\frac{\phi'(s)}{s}. \end{aligned} \tag{9.2}$$

Now, since $\phi'(s) \leqslant \frac{\phi(s)}{s} = \lambda(s)$, it follows that $\mathfrak{D} \leqslant \lambda(s)$. On the other hand $\phi'(s) \geqslant \frac{1}{M}$ and $\frac{\phi(s)}{s} \geqslant 1 \geqslant \frac{1}{M}$, whence $\mathfrak{D} \geqslant \frac{1}{M}$. □

10. Conformal-energy of $H : \mathcal{C}_+ \overset{\text{onto}}{\longrightarrow} \mathcal{C}_+$

In the forthcoming computation the "implied constants" depend only on the dimension $n \geqslant 2$.

Lemma 10.1. *We have*

$$\int_{\mathcal{C}_+} |DH(x,t)|^n \,\mathrm{d}x\,\mathrm{d}t \;\preccurlyeq\; E[\phi]. \tag{10.1}$$

Proof. Formula (8.3) yields the inequality

$$\begin{aligned}\int_{\mathcal{C}_+} |DH(x,t)|^n \,\mathrm{d}x\,\mathrm{d}t \;\preccurlyeq\; 1 &+ \int_{t+|x|\leqslant 1} |\,\lambda(t+|x|)\,|^n \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{t+|x|\leqslant 1} t^n |\,\lambda'(t+|x|)\,|^n \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{10.2}$$

Obviously, for the constant term we have $1 \preccurlyeq E[\phi]$. For the first integral in the right hand side we make the substitution $t = s - |x|$ and use Fubini's formula to obtain

$$\begin{aligned}\int_{t+|x|} |\lambda(t+|x|)\,|^n \,\mathrm{d}x\,\mathrm{d}t &= \int_{|x|\leqslant s\leqslant 1} |\,\lambda(s)\,|^n \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^1 |\lambda(s)|^n \left(\int_{|x|\leqslant s} \mathrm{d}x\right)\mathrm{d}s = \frac{\omega_{n-2}}{n-1}\int_0^1 s^{n-1}|\lambda(s)|^n \mathrm{d}s \\ &= \frac{\omega_{n-2}}{n-1}\int_0^1 |\phi(s)|^n \,\frac{\mathrm{d}s}{s} \;\preccurlyeq\; E[\phi].\end{aligned}$$

For the second integral in (10.2), we make the same substitution $t = s - |x|$ and proceed as follows:

$$\begin{aligned}\int_{t+|x|\leqslant 1} t^n |\,\lambda'(t+|x|)\,|^n \,\mathrm{d}x\,\mathrm{d}t &= \int_{|x|\leqslant s\leqslant 1} (s-|x|)^n \,|\,\lambda'(s)\,|^n \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^1 |\lambda'(s)|^n \left(\int_{|x|\leqslant s} (s-|x|)^n \,\mathrm{d}x\right)\mathrm{d}s \\ &= c_n \int_0^1 s^{2n-1}|\lambda'(s)|^n \mathrm{d}s \preccurlyeq \int_0^1 |\phi(s)|^n \,\frac{\mathrm{d}s}{s} \;=\; E[\phi].\end{aligned}$$

Here, we used the inequality $|\lambda'(s)\,| = \frac{\phi(s)}{s^2} - \frac{\phi'(s)}{s} \leqslant \frac{\phi(s)}{s^2}$.
The proof is complete. □

11. Conformal-energy of the Inverse Map

This brings us back to the seminal work [4] on extremal mappings of finite distortion. Going into this in detail would take us too far afield, so we confine ourselves to a simplified variant.

Consider a homeomorphism $H : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between bounded domains of Sobolev class $\mathscr{W}^{1,n}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ and assume (just to make it easier) that the Jacobian $\mathbf{J}_H \overset{\text{def}}{=\!=} \det\,[\mathrm{DH}]$ is positive almost everywhere, as in (9.1).

Definition 11.1. The differential expression

$$\mathbf{K}_H(X) \overset{\text{def}}{=\!=} \big|\,[DH(X)]^{-1}\,\big|^n\, \mathbf{J}_H(X) \;=\; \frac{\big|\,D^\sharp H(X)\,\big|^n}{[\,\mathbf{J}_H(X)\,]^{n-1}} \tag{11.1}$$

is called the *inner distortion* function of H. Here the symbol $D^\sharp H$ stands for the cofactor matrix of DH, defined by Cramer's rule.

The following identity was first observed with a complete proof of it in [4] (see Theorem 9.1 therein):

Proposition 11.2. *Under the assumptions above, if* $\mathbf{K}_H \in \mathscr{L}^1(\mathbb{X})$ *then the inverse map* $F : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ *belongs to* $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{X})$ *and*

$$\int_{\mathbb{Y}} \big|\,DF(Y)\,\big|^n\,\mathrm{d}Y \;=\; \int_{\mathbb{X}} \mathbf{K}_H(X)\,\mathrm{d}X. \tag{11.2}$$

In our case, since H is locally Lipschitz on $\mathcal{C}_+$, the derivation of this identity is straightforward. Simply, the differential matrix $DF(Y)$ at the point $Y = H(X)$ equals $[DH(X)]^{-1}$. We may change variables $Y = H(X)$ in the energy integral for F, to obtain

$$\int_{\mathcal{C}_+} \big|\,DF(Y)\big|^n\,\mathrm{d}Y \;= \int_{\mathcal{C}_+} \big|\,[DH(X)]^{-1}\,\big|^n\,\mathbf{J}_H(X)\,\mathrm{d}X \;\overset{\text{def}}{=\!=}\; \int_{\mathcal{C}_+} \mathbf{K}_H(X)\,\mathrm{d}X.$$

Now, by (8.1) and (8.2), we have a point wise inequality $\mathbf{J}_H(X)\,\big|\,[DH(X)]^{-1}\,\big| \leqslant \sqrt{n-1}\,\big|\,DH(X)\,\big|$, which yields

$$\mathbf{K}_H \leqslant \frac{(n-1)^{\frac{n}{2}}\,|\,DH\,|^n}{(\mathbf{J}_H)^{2n-1}} \leqslant (n-1)^{\frac{n}{2}}\,M^{2n-1}\,|\,DH\,|^n \;\in \mathscr{L}^1(\mathcal{C}_+), \tag{11.3}$$

because $\mathbf{J}_H(X) \geqslant \frac{1}{M}$, by (9.1).

12. Modulus of Continuity of $H : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$

We start with the straightforward estimates of the modulus of continuity at $(\mathbf{0}, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$. In consequence of $\lambda(\| X \|) \geqslant 1$, we have

$$\| X \| = |x| + t \leqslant |x| + t\,\lambda(t + |x|) \leqslant |x|\,\lambda(\| X \|) + t\,\lambda(\| X \|) = \phi(\| X \|).$$

Here the middle term $|x| + t\lambda(t + |x|) = \| H(X) \|$. Therefore,

$$\| X \| \leqslant \| H(X) \| \leqslant \phi(\| X \|)\,. \tag{12.1}$$

Corollary 12.1. *The function* ϕ *is the optimal modulus of continuity of the map* $H : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ *at* $(\mathbf{0}, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$*; that is,*

$$\sup_{\| X \| = s} \| H(X) \| = \phi(s)\,, \quad \text{whenever } X \in \mathcal{C}_+ \text{ and } 0 \leqslant s \leqslant 1\,. \tag{12.2}$$

Indeed, the supremum is attained at the point $X = (\mathbf{0}, s)$ on the vertical axis of the cone $\mathcal{C}_+$, because $H(\mathbf{0}, s) = (\mathbf{0}, \phi(s))$.

Remark 12.2. It is perhaps worth remarking in advance that both inequalities at (12.1) remain valid in terms of the Euclidean norm of $\mathbb{R}^n$ as well, where $|X| = |(x, t)| = \sqrt{|x|^2 + t^2} \leqslant \| X \|$. To this end, since λ is decreasing to its minimum value $\lambda(1) = 1$, for $X \in \mathcal{C}_+$ we can write

$$|X|^2 \leqslant |x|^2 + t^2\lambda^2(\| X \|) = |H(X)|^2 \leqslant |x|^2\lambda^2(|X|) + t^2\lambda^2(|X|) = \phi^2(|X|).$$

Let us record this fact as

$$|X| \leqslant |H(X)| \leqslant \phi(|X|)\,. \tag{12.3}$$

For the inverse map $F = F(Y)$, these inequalities take the form

$$\psi(|Y|) \leqslant |F(Y)| \leqslant |Y| \leqslant \phi(|Y|) \ \text{ for all } Y \in \mathcal{C}_+ \text{ because } s \leqslant \phi(s), \tag{12.4}$$

where $\psi : [0, 1] \xrightarrow{\text{onto}} [0, 1]$ denotes the inverse function of ϕ. This, however, does not necessarily imply that F is Lipschitz continuous, as shown by our representative examples.

We shall now prove that H admits ϕ as global modulus of continuity; that is, everywhere in $\mathcal{C}_+$. Precisely, we have

Proposition 12.3. *For* $X = (x, t) \in \mathcal{C}_+$ *and* $X' = (x', t') \in \mathcal{C}_+$ *it holds that*

$$\| H(X) - H(X') \| \leqslant 4\,\phi(\| X - X' \|)\,. \tag{12.5}$$

Thus, according to (2.3)*,*

$$\Omega_H(t) \leqslant 4\phi(t)\,.$$

Proof. Recall that $\| X \| \stackrel{\text{def}}{=} |x| + |t|$ and $H(x,t) \stackrel{\text{def}}{=} (x,\ t\lambda(\| X \|))$. Thus

$$\| H(X) - H(X') \| \ \leqslant |x - x'| \ + \ |t\lambda(\| X \|) \ - \ t'\lambda(\| X' \|)|. \tag{12.6}$$

The first term is easily estimated as $|x - x'| \leqslant \phi(|x - x'|) \leqslant \phi(\| X - X' \|)$, because $s \leqslant \phi(s)$ and ϕ is increasing in $s \in [0, 1]$. The second term needs more work. First observe that for $0 < A \leqslant B \leqslant 1$ it holds that

$$0 < \lambda(A) - \lambda(B) \ \leqslant A^{-1}\phi(B - A). \tag{12.7}$$

Indeed,

$$\begin{aligned} \lambda(A) \ - \ \lambda(B) &= n\frac{\phi(A) - \phi(B)}{A} \ + \ \frac{B - A}{A}\lambda(B) \\ &\leqslant \frac{B - A}{A}\lambda(B - A) = A^{-1}\phi(B - A). \end{aligned}$$

In the above formula, the first term is negative because ϕ is increasing. In the second term we have used the inequality $\lambda(B) \leqslant \lambda(B - A)$, because λ is nonincreasing.
In Inequality (12.5) we may (and do) assume that $\| X' \| \ \leqslant \ \| X \|$, for otherwise we can interchange X with X'. This yields $\frac{1}{2} \ \| X - X' \| \ \leqslant \ \| X \|$ and, consequently, $\lambda(\| X \|) \ \leqslant \ \lambda(\frac{1}{2} \| X - X' \|) \ = \ \phi(\frac{1}{2} \| X - X' \|)/\frac{1}{2} \| X - X' \| \ \leqslant \ 2\phi(\| X - X' \|)/\| X - X' \|$. Having this and (12.7) to hand, we conclude with the desired estimate:

$$\begin{aligned} |\, t\lambda(\| X \|) \ - \ t'\lambda(\| X' \|)\,| &\leqslant |\,t - t'|\,\lambda(\| X \|) + t'\,|\,\lambda(\| X \|) \ - \ \lambda(\| X' \|)\,| \\ &\leqslant \frac{2\,|\,t - t'|}{\| X - X' \|}\ \phi(\| X - X' \|) \\ &\quad + \frac{t'}{\| X' \|}\ \phi(\| X \| \ - \ \| X' \|) \\ &\leqslant 2\,\phi(\| X - X' \|) \ \ + \ \phi(\| X - X' \|) \\ &= 3\,\phi(\| X - X' \|). \end{aligned}$$

□

13. Modulus of Continuity of $F : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$

All representative functions $\phi = \phi_k$, $k = 0, 1, \ldots$ that are listed in $(\mathbf{E}_0) \ldots (\mathbf{E}_k) \ldots$ are concave ($\phi_k'' \ \leqslant \ 0$) near the origin, but not necessarily in the entire interval $[0, 1]$. Actually, upon minor modifications away from the origin all the above functions can be made concave in the entire interval $[0, 1]$, but their aesthetic appearance will be lost. Thus, rather than modifying those examples, in the first step we restrict our attention to a neighborhood of the origin. Outside such a neighborhood the mapping $F : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ is Lipschitz continuous. This will take care of the global estimate.

The additional condition imposed on ϕ reads as follows:

($\mathbf{C}_4$) There is an interval $(0, r] \subset (0, 1]$ in which ϕ is $\mathcal{C}^2$-smooth and concave; that is,

$$\phi''(s) \leqslant 0\,, \quad \text{for } 0 < s \leqslant r. \tag{13.1}$$

We shall now prove that F admits ϕ as global modulus of continuity in $\mathcal{C}_+$.

Proposition 13.1. *For arbitrary two points* $Y = (y, \tau) \in \mathcal{C}_+$ *and* $Y' = (y', \tau') \in \mathcal{C}_+$ *it holds that*

$$\| F(Y) - F(Y') \| \preccurlyeq \phi(\| Y - Y' \|). \tag{13.2}$$

The implied constant depends on the conditions imposed through $(\mathbf{C}_1) - (\mathbf{C}_4)$.

Proof. A seemingly routine proof below, actually took an effort to accomplish all details. Let us begin with the definition of the map $H : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ and some new related notation. For $X = (x, t) \in \mathcal{C}_+ \subset \mathbb{R}^{n-1} \times \mathbb{R}$, we recall that

$$\| X \| = |x| + t \ \text{ and } \ H(X) = \big(x, t\,\lambda(t + |x|)\big) \stackrel{\text{def}}{=\!=} \big(y, \tau\big) = Y \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

For the inverse map $F = H^{-1}$ we write

$$\| Y \| = |y| + \tau \ \text{ and } \ F(Y) = \big(y, T\big) \in \mathcal{C}_+ \subset \mathbb{R}^{n-1} \times \mathbb{R},$$

where the vertical coordinate $T = T(\tau, |y|)$ is determined uniquely from the equation

$$T\,\lambda(T + |y|) = \tau. \tag{13.3}$$

In much the same way as in (9.2) we find that the function $T \rightsquigarrow T\,\lambda(T + |y|)$ is strictly increasing. We actually have

$$\frac{\mathrm{d}\,T\,\lambda(T + |y|)}{\mathrm{d}T} = \lambda(T + |y|) + T\lambda'(T + |y|) \geqslant \frac{1}{M}.$$

Even more can be said about the above expression. Indeed, denoting by $s \stackrel{\text{def}}{=\!=} T + |y| \leqslant 1$, we have the identity

$$\begin{aligned} \lambda(s) + T\lambda'(s) &= \frac{\phi(s)}{s} + T \cdot \left(\frac{\phi'(s)}{s} - \frac{\phi(s)}{s^2} \right) \\ &= \left(1 - \frac{T}{s}\right) \frac{\phi(s)}{s} + \frac{T}{s} \cdot \phi'(s), \end{aligned}$$

which, in view of Condition ($\mathbf{C}_2$) at (6.1), also yields a useful upper bound:

$$\frac{\phi(s)}{s} \geqslant \lambda(s) + T\lambda'(s) \geqslant \phi'(s) \geqslant \frac{1}{M} \tag{13.4}$$

The latter follows from the Condition ($\mathbf{C}_2$) at (6.1) as well.
Now, implicit differentiation in (13.3) with respect to τ-variable gives

$$0 \leqslant \frac{\partial T(\tau, |y|)}{\partial \tau} = \frac{1}{\lambda(T + |y|) + T\lambda'(T + |y|)} \leqslant M. \tag{13.5}$$

On the other hand, differentiation with respect to the $|y|$-variable gives

$$0 \leqslant \frac{\partial T(\tau, |y|)}{\partial |y|} = \frac{-T\lambda'(T+|y|)}{\lambda(T+|y|) + T\lambda'(T+|y|)} = \frac{-T\lambda'(s)}{\lambda(s) + T\lambda'(s)}. \tag{13.6}$$

It should be noted that $\lambda'(s) \leqslant 0$ whenever $s \overset{\text{def}}{=\!=} T + |y| \leqslant 1$. Precisely,

$$0 \leqslant -\lambda'(s) = -\frac{\phi'(s)}{s} + \frac{\phi(s)}{s^2} \leqslant \frac{\phi(s)}{s^2}. \tag{13.7}$$

From this and the lower bound in (13.4) we infer that

$$0 \leqslant \frac{\partial T(\tau, |y|)}{\partial |y|} \leqslant \frac{T\phi(s)}{s^2\phi'(s)} \leqslant \frac{\phi(s)}{s\,\phi'(s)} \leqslant M\phi'(s) = M\phi'(T+|y|),$$

the latter being guaranteed by the right hand side of inequality (6.1).

It is at this point that we are going to use the additional assumption that ϕ is concave near the origin; namely, ϕ' is non-increasing in $(0, r] \subset (0, 1]$. Examine an arbitrary point $Y = (y, \tau) \in \mathcal{C}_+$ of lengths $\| Y \| \overset{\text{def}}{=\!=} \tau + |y| \leqslant \frac{r}{M}$ to show that $T + |y| \leqslant r$. Recall that T is determined by the equation $T\lambda(T + |y|) = \tau$. Thus, we have $\frac{T}{M} \leqslant \phi'(T + |y|)\, T \leqslant \frac{\phi(T+|y|)}{T+|y|}\, T = \tau$, by Condition (6.1). Hence $T + |y| \leqslant M\tau + |y| \leqslant M(\tau + |y|) \leqslant r$. Since $s = T + |y| \geqslant |y|$ and ϕ' is non-increasing in $(0, r]$, we infer that

$$0 \leqslant \frac{\partial T(\tau, |y|)}{\partial |y|} \leqslant M\phi'(|y|)\,, \quad \text{whenever } \tau + |y| \overset{\text{def}}{=\!=} \| Y \| \leqslant \frac{r}{M}. \tag{13.8}$$

We are now ready to formulate an estimate of the modulus of continuity of F within the neighborhood of the origin that is determined by $\| Y \| \leqslant \frac{r}{M}$.

Proposition 13.2. *Let $Y = (y, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $Y' = (y', \tau') \in \mathbb{R}^{n-1} \times \mathbb{R}$ be points in $\mathcal{C}_+$ such that $\| Y \| \leqslant \frac{r}{M}$ and $\| Y' \| \leqslant \frac{r}{M}$. Then*

$$\| F(Y) - F(Y') \| \leqslant 3M\phi(\| Y - Y' \|). \tag{13.9}$$

Proof. With the notation for $F(Y) = (y\,,\ T(\tau, |y|))$ and $F(Y') = (y'\,,\ T(\tau', |y'|))$ we begin with the computation

$$\begin{aligned}
\| F(Y) - F(Y') \| &= |y - y'| + |\, T(\tau, |y|) - T(\tau', |y'|)\, | \\
&\leqslant |y - y'| + |\, T(\tau, |y|) - T(\tau, |y'|)\, | + |\, T(\tau, |y'|) - T(\tau', |y'|)\, | \\
&\leqslant \text{(in view of (13.5))} \\
&\leqslant |y - y'| + |\, T(\tau, |y|) - T(\tau, |y'|)\, | + M\,|\tau - \tau'| \\
&\leqslant |\, T(\tau, |y|) - T(\tau, |y'|)\, | + M\, \| Y - Y' \| \\
&\leqslant |\, T(\tau, |y|) - T(\tau, |y'|)\, | + M\,\phi(\| Y - Y' \|).
\end{aligned}$$

The latter is obtained by the inequality $s \leqslant \phi(s)$, see (6.5). It remains to establish the following estimates, say when $0 < |y'| \leqslant |y| \leqslant r$:

$$|\, T(\tau, |y|) - T(\tau, |y'|)\, | \leqslant 2M\phi(|y - y'|) \leqslant 2M\,\phi(\| Y - Y' \|). \tag{13.10}$$

To that end, we begin with the following expression:

$$T(\tau,|y|)-T(\tau,|y'|) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\gamma}\Big[T(\tau,|\gamma\, y+(1-\gamma)y'|)\Big]\,\mathrm{d}\gamma$$
$$= \int_0^1 T_\xi(\tau,|\gamma\, y+(1-\gamma)y'|)\left\langle \frac{\gamma\, y+(1-\gamma)y'}{|\gamma\, y+(1-\gamma)y'|}\;\Big|\; y-y'\right\rangle \mathrm{d}\gamma,$$

where $T_\xi(\tau,\xi) \stackrel{\text{def}}{=\!=} \frac{\partial T(\tau,\xi)}{\partial \xi}$. In view of (13.8), we obtain

$$|\; T(\tau,|y|)-T(\tau,|y'|)\;| \leqslant M|y-y'| \int_0^1 \phi'(|\gamma\, y+(1-\gamma)y'|)\,\mathrm{d}\gamma. \quad (13.11)$$

It is important to notice that $|\gamma\, y+(1-\gamma)y'| \leqslant r$, which enables us to invoke Condition $(\mathbf{C}_4)$ at (13.1); that is, ϕ' is non-increasing in the interval $(0,r]$. The following interesting lemma comes into play:

Lemma 13.3. *Let* $\Phi : (0,r] \to (0,\infty)$ *be continuous non-increasing and integrable:*

$$\int_0^r \Phi(s)\,\mathrm{d}s \;<\; \infty.$$

Then for every vectors $\mathfrak{a},\mathfrak{b}$ *in a normed space* $(\mathfrak{N}\,;\;|.|)$ *, such that* $0<|\mathfrak{a}|\leqslant r$ *and* $0<|\mathfrak{b}|\leqslant r$ *, it holds that*

$$\int_0^1 \Phi(|\gamma\mathfrak{a} \;+\; (1-\gamma)\,\mathfrak{b}\,|)\,\mathrm{d}\gamma \;\leqslant\; \frac{1}{|\mathfrak{a}|+|\mathfrak{b}|}\left(\int_0^{|\mathfrak{a}|}\Phi(s)\,\mathrm{d}s \;+\; \int_0^{|\mathfrak{b}|}\Phi(s)\,\mathrm{d}s\right). \quad (13.12)$$

Equality occurs if $\mathfrak{a}$ *is a negative multiple of* $\mathfrak{b}$ *.*

Proof. Since ϕ is non-increasing, by triangle inequality it follows that

$$\int_0^1 \Phi(|\gamma\mathfrak{a} \;+\; (1-\gamma)\,\mathfrak{b}\,|)\,\mathrm{d}\gamma \;\leqslant\; \int_0^1 \Phi\left(\big|\,(1-\gamma)\,|\mathfrak{b}| \;-\; \gamma|\mathfrak{a}|\,\big|\right)\,\mathrm{d}\gamma$$
$$= \int_0^{\frac{|\mathfrak{b}|}{|\mathfrak{a}|+|\mathfrak{b}|}} \Phi\big(\,(1-\gamma)\,|\mathfrak{b}| \;-\; \gamma|\mathfrak{a}|\,\big)\,\mathrm{d}\gamma \;+\; \int_{\frac{|\mathfrak{b}|}{|\mathfrak{a}|+|\mathfrak{b}|}}^1 \Phi\big(\,\gamma|\mathfrak{a}| \;-\; (1-\gamma)\,|\mathfrak{b}|\,\big)\,\mathrm{d}\gamma.$$

In the first integral we make a substitution $s=(1-\gamma)\,|\mathfrak{b}| \;-\; \gamma|\mathfrak{a}|$, which places s in the interval $(0,|\mathfrak{b}|\,)$ and $|\mathrm{d}s| \;=\; (|\mathfrak{a}|\;+\;|\mathfrak{b}|)\,\mathrm{d}\lambda$. This gives us the second integral-term of the right hand side of (13.12), and similarly for the first integral-term. □

Since ϕ' is non-increasing in the interval $(0,r]$ (by inequality (13.1) at Condition $(\mathbf{C}_4)$), we may apply Estimate (13.12) to $\Phi=\phi'$. Now, returning to (13.11), the inequality (13.10) is readily inferred as follows:

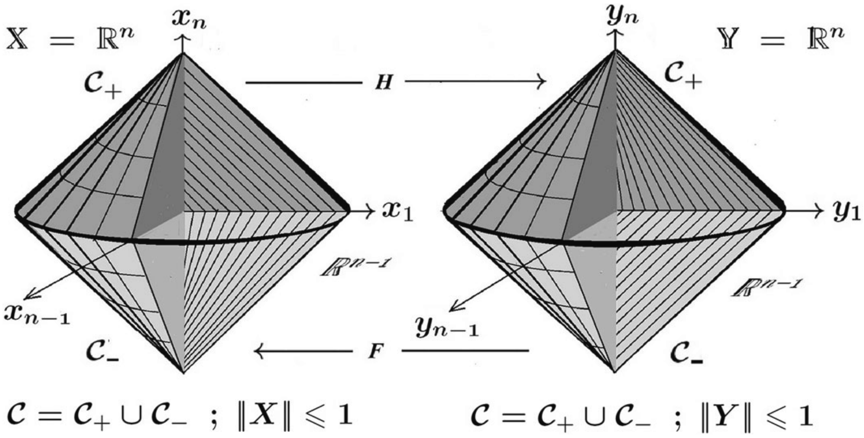

Fig. 4. Bi-conformal energy mapping H and its inverse F exhibit the same optimal modulus of continuity at the origin of the double cone $\mathcal{C}$

$$
\begin{aligned}
|\, T(\tau, |y|) - T(\tau, |y'|)\, | &\leqslant M|y-y'|\, \frac{\phi(|y|)+\phi(|y'|)}{|y|+|y'|} \\
&= M\,\phi(|y-y'|)\, \frac{|y-y'|}{\phi(|y-y'|)}\, \frac{\phi(|y|)+\phi(|y'|)}{|y|+|y'|} \\
&\leqslant M\,\phi(|y-y'|)\, \frac{|y|+|y'|}{\phi(|y|+|y'|)}\, \frac{\phi(|y|)+\phi(|y'|)}{|y|+|y'|} \;\leqslant 2\,M\,\phi(|y-y'|)\,,
\end{aligned}
$$

because $\frac{s}{\phi(s)} = \frac{1}{\lambda(s)}$ is non-decreasing (see (6.4)) and $\phi(s)$ is increasing. The proof of Proposition 13.2 is complete. □

Finally, the global estimate (13.2) in Proposition 13.1 follows from Proposition 13.2, whenever $\| Y \| \leqslant \frac{r}{M}$ and $\| Y' \| \leqslant \frac{r}{M}$, whereas its extension to all points Y and Y' is fairly straightforward by invoking Lipschitz continuity of F away from the origin (Fig. 4). □

14. Conclusion

Choose an arbitrary modulus of continuity function $\phi : [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ that satisfies conditions $(\mathbf{C}_1)\,(\mathbf{C}_2)\,(\mathbf{C}_3)$ and $(\mathbf{C}_4)$. Then consider a bi-conformal energy map $H : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$ defined in (7.1) together with its inverse map $F : \mathcal{C}_+ \xrightarrow{\text{onto}} \mathcal{C}_+$. Extend H and F to the double cone $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ by the reflection rule at (5.1). Afterwards, extend H and F to the entire space $\mathbb{R}^n$ by setting $H = \mathrm{Id} : \mathbb{R}^n \setminus \mathcal{C} \xrightarrow{\text{onto}} \mathbb{R}^n \setminus \mathcal{C}$ and $F = \mathrm{Id} : \mathbb{R}^n \setminus \mathcal{C} \xrightarrow{\text{onto}} \mathbb{R}^n \setminus \mathcal{C}$. Then we obtain

Theorem 14.1. *For every modulus of continuity function* $\phi : [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ *satisfying conditions* $(\mathbf{C}_1)\,(\mathbf{C}_2)\,(\mathbf{C}_3)$ *and* $(\mathbf{C}_4)$, *there exists a homeomorphism* $H : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ *of Sobolev class* $\mathscr{W}^{1,n}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, *whose inverse* $F = H^{-1} : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ *also lies in the Sobolev space* $\mathscr{W}^{1,n}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. *Moreover,*

- $H(0) = 0$, $H(X) \equiv X$, for $\| X \| \geqslant 1$ and for $X = (x_1, ..., x_{n-1}, 0)$;

- *$H : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ admits ϕ as its global modulus of continuity; that is,*

$$\| H(X_1) - H(X_2) \| \preccurlyeq \phi(\| X_1 - X_2 \|) , \quad \text{for all } X_1, X_2 \in \mathbb{R}^n ; \tag{14.1}$$

- *The inverse map $F : \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \mathbb{R}^n$ satisfies the same condition*

$$\| F(Y_1) - F(Y_2) \| \preccurlyeq \phi(\| Y_1 - Y_2 \|) , \quad \text{for all } Y_1, Y_2 \in \mathbb{R}^n ; \tag{14.2}$$

- *H and F share the same optimal moduli of continuity at the origin; namely,*

$$\omega_H(0, r) = \max_{\| X \| = r} |H(X)| = \phi(r) = \max_{\| Y \| = r} |F(Y)| = \omega_F(0, r) \tag{14.3}$$

for all $0 \leqslant r < \infty$.

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Pointwise inequalities for Sobolev functions on outward cuspidal domains

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POINTWISE INEQUALITIES FOR SOBOLEV FUNCTIONS ON OUTWARD CUSPIDAL DOMAINS

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Abstract. We show that the first order Sobolev spaces $W^{1,p}(\Omega_\psi)$, $1 < p \leq \infty$, on cuspidal symmetric domains Ω_ψ can be characterized via pointwise inequalities. In particular, they coincide with the Hajłasz-Sobolev spaces $M^{1,p}(\Omega_\psi)$.

1. Introduction

Optimal definitions for Sobolev spaces are crucial in analysis. It was a remarkable discovery of Hajłasz [4] that distributionally defined Sobolev functions can be characterized using pointwise estimates in the context of Sobolev extension domains. This, in part, has played a crucial role in defining Sobolev spaces for general metric measure spaces. Here, we show that for certain cuspidal domains the pointwise characterization holds without any additional assumptions. These domains do not admit extensions for Sobolev functions. Given a domain $\Omega \subset \mathbb{R}^n$, we denote by $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, the usual first order Sobolev space consisting of all functions $u \in L^p(\Omega)$ whose first order distributional partial derivatives also belong to $L^p(\Omega)$. If $\Omega = \mathbb{R}^n$, then any Sobolev function u satisfies the pointwise inequality

$$|u(x) - u(y)| \leq C|x - y| \left(M[|\nabla u|](x) + M[|\nabla u|](y)\right) \tag{1.1}$$

at Lebesgue points of u, where $M[|\nabla u|]$ is the Hardy-Littlewood maximal function of $|\nabla u|$, see [1, 2, 4, 8]. Motivated by this, P. Hajłasz introduced in [4] the space $M^{1,p}(\Omega)$ consisting of all those $u \in L^p(\Omega)$ for which there exists a set $E \subset \Omega$ of n-measure zero and a function $0 \leq g \in L^p(\Omega)$ so that

$$|u(x) - u(y)| \leq |x - y| \left(g(x) + g(y)\right) \tag{1.2}$$

whenever $x, y \in \Omega \setminus E$.

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One has $M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$ as sets for $1 < p \leq \infty$, and the norms are comparable once $M^{1,p}(\mathbb{R}^n)$ is equipped with the natural norm. Also, for $1 \leq p \leq \infty$, one always has $M^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ and the inclusion is strict for $p = 1$ for any domain Ω, see [7].

A natural question to ask is:

For which domains $\Omega \subset \mathbb{R}^n$ do we have $M^{1,p}(\Omega) = W^{1,p}(\Omega)$?

Indeed, these two spaces coincide if there is a bounded extension operator from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^n)$, for a given $1 < p \leq \infty$. When $p = \infty$ and Ω is bounded, this is the case if Ω is quasiconvex and actually the equality is equivalent to quasiconvexity under these assumptions. This follows from [5, Theorem 7]. Moreover, for $1 < p < \infty$, under the assumption that

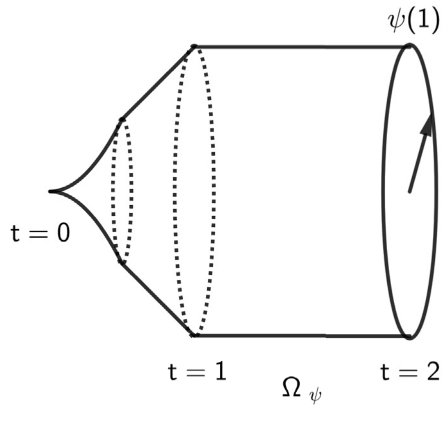
$$|B(x,r)| \leq C|B(x,r) \cap \Omega| \tag{1.3}$$

for every $x \in \Omega$ and every $0 < r < 1$, where $|\cdot|$ refers to n-measure, $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ implies the existence of such an extension operator. Indeed, in this case the spaces coincide precisely when such an extension operator exists. For this see [5]. Using this fact, it is easy to exhibit domains Ω for which $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ fails for all p; e.g. take $\Omega \subset \mathbb{R}^2$ to be the unit disk minus the interval $[0,1)$ on the real axis.

In this paper, we consider this question for cuspidal domains of the form

$$\Omega_\psi := \big\{(t,x) \in (0,1) \times \mathbb{R}^{n-1}; |x| < \psi(t)\big\} \cup \{(t,x) \in [1,2) \times \mathbb{R}^{n-1}; |x| < \psi(1)\}, \tag{1.4}$$

where $\psi\colon (0,1] \to (0,\infty)$ is a left continuous increasing function. (Left continuity is required just to get Ω_ψ open. The term "increasing" is used in the non-strict sense.) The seemingly strange cylindrical annexes are included only to exclude other singularities than the cuspidal one. It is crucial to note that these domains will not, except for limited special cases, be Sobolev extension domains, and thus the methods from [5] do not apply.



It is easy to check that $\Omega_\psi \subset \mathbb{R}^n$ is a domain. If $\lim_{t\to 0} \frac{\psi(t)}{t} = 0$, then the measure density condition (1.3) fails, and hence, by [5], there can not exist any bounded extension operator from $W^{1,p}(\Omega_\psi)$ to $W^{1,p}(\mathbb{R}^n)$. However, according to a somewhat surprising result by A.S. Romanov [9], one still has $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ if $\psi(t) = t^s$ with $s > 1$ and $p > \frac{1+(n-1)s}{n}$. Actually, Romanov proved this statement for a domain which is bi-Lipschitz equivalent to Ω_ψ when $\psi(t) = t^s$, but bi-Lipschitz transforms preserve both Sobolev and Hajłasz-Sobolev spaces.

We show that the above restriction on p is superfluous and that ψ being of the form $\psi(t) = t^s$ can be relaxed to being any left continuous increasing function.

Theorem 1.5. *Let $\psi : (0, 1] \to (0, \infty)$ be a left continuous increasing function. Define the corresponding cuspidal domain Ω_ψ as in* (1.4)*. Then $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ for all $1 < p \le \infty$ with equivalence of norms.*

As a consequence of the bi-Lipschitz invariance stated above, the conclusion $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ then holds for all bi-Lipschitz images of Ω_ψ. Thus, our result covers the result obtained by Romanov.

2. Definitions and Preliminaries

In what follows, $\Omega \subset \mathbb{R}^n$ is always a domain. We write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} := \{z := (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}\}.$$

Throughout the paper, we consider a left continuous increasing function $\psi\colon (0, 1] \to (0, \infty)$, extend the definition of ψ to the interval $(0, 2)$ by setting

$$\psi(t) = \psi(1), \quad \text{for every} \quad t \in (1, 2)$$

and write

$$\Omega_\psi = \{(t, x) \in (0, 2) \times \mathbb{R}^{n-1}; |x| < \psi(t)\}.$$

Typically, c or C will be constants that depend on various parameters and may differ even on the same line of inequalities. The Euclidean distance between points x, y in the Euclidean space $\mathbb{R}^n$ is denoted by $|x - y|$. The open m-dimensional ball of radius r centered at the point x is denoted by $B^m(x, r)$.

The space of locally integrable functions is denoted by $L^1_{\text{loc}}(\Omega)$. For every measurable set $Q \subset \mathbb{R}^n$ with $0 < |Q| < \infty$, and every non-negative measurable or integrable function f on Q we define the integral average of f over Q by

$$⨍_Q f(w)\, dw := \frac{1}{|Q|} \int_Q f(w)\, dw.$$

Let us give the definitions of Sobolev space $W^{1,p}(\Omega)$ and Hajłasz-Sobolev space $M^{1,p}(\Omega)$.

Definition 2.1. We define the first order Sobolev space $W^{1,p}(\Omega)$, $1 \le p \le \infty$, as the set

$$\{\, u \in L^p(\Omega); \nabla u \in L^p(\Omega; \mathbb{R}^n)\,\}.$$

Here $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ is the weak (or distributional) gradient of a locally integrable function u.

We equip $W^{1,p}(\Omega)$ with the non-homogeneous norm:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \||\nabla u|\|_{L^p(\Omega)}$$

for $1 \le p < \infty$, and

$$\|u\|_{W^{1,\infty}(\Omega)} = \|u(z)\|_{L^\infty(\Omega)} + \||\nabla u(z)|\|_{L^\infty(\Omega)}.,$$

where $\|f\|_{L^p(\Omega)}$ denotes the usual L^p-norm for $p \in [1, \infty]$.

For $u \in L^p(\Omega)$, we denote by $\mathcal{D}_p(u)$ the class of functions $0 \le g \in L^p(\Omega)$ for which there exists $E \subset \Omega$ with $|E| = 0$, so that

$$|u(z_1) - u(z_2)| \le |z_1 - z_2| \left(g(z_1) + g(z_2)\right), \quad \text{for} \quad z_1, z_2 \in \Omega \setminus E\,.$$

Definition 2.2. We define the Hajłasz-Sobolev space $M^{1,p}(\Omega)$, $1 \le p \le \infty$, as the set

$$\left\{u \in L^p(\Omega), \mathcal{D}_p(u) \neq \emptyset\right\}\,.$$

We equip $M^{1,p}(\Omega)$ with the non-homogeneous norm:

$$\|u\|_{M^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \inf_{g \in \mathcal{D}_p(u)} \|g\|_{L^p(\Omega)}\,.$$

for $1 \le p < \infty$, and

$$\|u\|_{M^{1,\infty}(\Omega)} = \|u(z)\|_{L^\infty(\Omega)} + \inf_{g \in \mathcal{D}_p(u)} \|g(z)\|_{L^\infty(\Omega)}\,.$$

3. Maximal functions

We will define two maximal functions. The first, $M^\tau[f]$, will vary only the first component t, and the second $M^\chi[f]$ will vary the x-component. For every $x \in B^{n-1}(0, \psi(1))$ set

$$S_x := \{t \in \mathbb{R}; (t, x) \in \Omega_\psi\}.$$

Let $f\colon \Omega_\psi \to \mathbb{R}$ be measurable and let $(t, x) \in \Omega_\psi$. We define the one-dimensional maximal function in the direction of the first variable by setting

$$M^\tau[f](t,x) := \sup_{[a,b] \ni t} \fint_{[a,b] \cap S_x} |f(s,x)|\, ds\,. \tag{3.1}$$

The supremum is taken over all intervals $[a, b]$ containing t.

On the other hand, the second maximal function will be defined for functions $f\colon (0, 2) \times \mathbb{R}^{n-1} \to \mathbb{R}$. For every point $(t, x) \in (0, 2) \times \mathbb{R}^{n-1}$, we define the $(n-1)$-dimensional maximal function $M^\chi[f]$ by setting

$$M^\chi[f](t,x) := \sup_{B^{n-1}(x',r) \ni x} \fint_{B^{n-1}(x',r)} |f(t,y)|\, dy\,, \tag{3.2}$$

where we take the supremum over the $(n-1)$-dimensional balls for which $x \in B^{n-1}(x', r)$. The next lemmas tell us that both M^τ and M^χ enjoy the usual L^p-boundedness property.

Lemma 3.3. *Let $1 < p < \infty$. Then for every $f \in L^p(\Omega_\psi)$, $M^\tau[f]$ is measurable and we have*

$$\int_{\Omega_\psi} |M^\tau[f](z)|^p\, dz \le C \int_{\Omega_\psi} |f(z)|^p\, dz\,, \tag{3.4}$$

where the constant C is independent of f.

Proof. Since the maximal function comes out the same if we consider only segments with rational endpoints, it preserves measurability. Fubini's theorem implies that $f(\cdot, x) \in L^p(S_x)$ for almost every $x \in B^{n-1}(0, \psi(1))$. By the L^p-boundedness of the classical Hardy-Littlewood maximal function on the interval S_x, for such x we have

$$\int_{S_x} |M^\tau[f](t,x)|^p \, dt \leq C \int_{S_x} |f(t,x)|^p \, dt, \tag{3.5}$$

where the constant C is independent of f and x. By combining the inequality (3.5) and Fubini's theorem together, we obtain

$$\begin{aligned} \int_{\Omega_\psi} |M^\tau[f](t,x)|^p \, dx\, dt &= \int_{B^{n-1}(0,\psi(1))} \int_{S_x} |M^\tau[f](t,x)|^p \, dt\, dx \\ &\leq C \int_{B^{n-1}(0,\psi(1))} \int_{S_x} |f(t,x)|^p \, dt\, dx \\ &= C \int_{\Omega_\psi} |f(t,x)|^p \, dx\, dt\,. \end{aligned}$$

□

Lemma 3.6. *Let $1 < p < \infty$. Then for every $f \in L^p((0,2) \times \mathbb{R}^{n-1})$, $M^\chi[f]$ is measurable and we have*

$$\int_{(0,2)\times\mathbb{R}^{n-1}} |M^\chi[f](z)|^p \, dz \leq C \int_{(0,2)\times\mathbb{R}^{n-1}} |f(z)|^p \, dz\,, \tag{3.7}$$

where the constant C is independent of f.

Proof. Again, the maximal function preserves measurability, as it comes out the same if we consider only balls with rational centers and radii (a point is rational if all its coordinates are rational). By Fubini's theorem, $f(t, \cdot) \in L^p(\mathbb{R}^{n-1})$ for almost every $t \in (0, 2)$. By the L^p-boundedness of the Hardy-Littlewood maximal operator we have

$$\int_{\mathbb{R}^{n-1}} |M^\chi[f](t,x)|^p \, dx \leq C \int_{\mathbb{R}^{n-1}} |f(t,x)|^p \, dx\,,$$

where the positive constant C is independent of f and t. Then Fubini's theorem gives

$$\begin{aligned} \int_{(0,2)\times\mathbb{R}^{n-1}} |M^\chi[f](z)|^p \, dz &= \int_0^2 \int_{\mathbb{R}^{n-1}} |M^\chi[f](t,x)|^p \, dx\, dt \\ &\leq C \int_0^2 \int_{\mathbb{R}^{n-1}} |f(t,x)|^p \, dx\, dt \\ &\leq C \int_{(0,2)\times\mathbb{R}^{n-1}} |f(z)|^p \, dz\,. \end{aligned}$$

□

4. Proof of the Main theorem

Let us begin by sketching a simple proof for Theorem 1.5 in the Euclidean plane $\mathbb{R}^2$, for $1 < p < \infty$. In this case the maximal function $M^\chi[f]$, with respect to x-coordinate, can be replaced by

$$\tilde{M}^\chi[f](t,x) := \sup_{[z,w]\ni x} \fint_{\{y\in[z,w];(t,y)\in\Omega_\psi\}} |f(t,y)|\, dy\,, \tag{4.1}$$

for every $(t,x) \in \Omega_\psi$. As in Lemma 3.3 we obtain

$$\int_{\Omega_\psi} |\tilde{M}^\chi[f](z)|^p\, dz \le C \int_{\Omega_\psi} |f(z)|^p\, dz\,. \tag{4.2}$$

By [4], there is a bounded inclusion $\iota\colon\ M^{1,p}(\Omega_\psi) \hookrightarrow W^{1,p}(\Omega_\psi)$. To show that ι is an isomorphism, it suffices to show that its inverse ι^{-1} is both densely defined and bounded on $W^{1,p}(\Omega_\psi)$. Let $C^1(\Omega_\psi)$ be the set of continuously differentiable functions. Since $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ is dense in $W^{1,p}(\Omega_\psi)$, it suffices to show that $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi) \subset M^{1,p}(\Omega_\psi)$ and that for each $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ we have $||u||_{M^{1,p}(\Omega_\psi)} \lesssim ||u||_{W^{1,p}(\Omega_\psi)}$.

Fix $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$. Let $z_1 := (t_1, x_1), z_2 := (t_2, x_2) \in \Omega_\psi$ be arbitrary. Without loss of generality, we assume $0 < t_1 \le t_2 < 2$. From the definition of Ω_ψ, the point $z' := (t_2, x_1)$ is also in Ω_ψ. Using the triangle inequality, we have

$$|u(z_1) - u(z_2)| \le |u(z_1) - u(z')| + |u(z') - u(z_2)|\,. \tag{4.3}$$

Since $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$, the fundamental theorem of calculus implies

$$|u(z_1) - u(z')| \le \int_{t_1}^{t_2} |\nabla u(s, x_1)| ds \le |z_1 - z_2| M^\tau[|\nabla u|](z_1) \tag{4.4}$$

and

$$|u(z') - u(z_2)| \le \int_{x_1}^{x_2} |\nabla u(t_2, y)| dy \le |z_1 - z_2| \tilde{M}^\chi[|\nabla u|](z_2)\,. \tag{4.5}$$

Combining inequalities (4.3), (4.4) and (4.5) together, we have

$$|u(z_1) - u(z_2)| \le |z_1 - z_2| \Big(M^\tau[|\nabla u|](z_1) + \tilde{M}^\chi[|\nabla u|](z_2) \Big) \le |z_1 - z_2| (g(z_1) + g(z_2))\,,$$

where

$$g(z) := M^\tau[|\nabla u|](z) + \tilde{M}^\chi[|\nabla u|](z)\,.$$

By inequalities (3.4) and (4.2), we have

$$\int_{\Omega_\psi} |g(z)|^p dz \le C \int_{\Omega_\psi} |\nabla u(z)|^p\, dz$$

which immediately gives that $g \in \mathcal{D}_p(u)$, and $\|u\|_{M^{1,p}(\Omega_\psi)} \le C \|u\|_{W^{1,p}(\Omega_\psi)}$.

In higher dimensions, we have to work harder. Let us fix some notation.

Let $\eta\colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth cut-off function such that $\eta = 1$ on $B^{n-1}(0,1)$ and $\eta = 0$ on the complement of $B^{n-1}(0,2)$. Consider the standard extension operator $E^R\colon W^{1,p}(B^{n-1}(0,R)) \to W^{1,p}(\mathbb{R}^{n-1})$ given by

$$E^R u(x) = \begin{cases} u(x), & |x| < R, \\ 0, & |x| = R, \\ u\big(\frac{R^2}{|x|^2}\, x\big)\eta\big(\frac{x}{R}\big), & |x| > R. \end{cases}$$

Then

$$\|\nabla E^R u\|_{L^p(\mathbb{R}^{n-1})} \le C\|\nabla u\|_{L^p(B^{n-1}(0,R))} \tag{4.6}$$

with C independent of u and R.

Let $u \in W^{1,p}(\Omega_\psi)$ be arbitrary, $1 < p < \infty$. Extend the function u to $(0,2)\times\mathbb{R}^{n-1}$ by setting

$$\tilde{u}(t,\cdot) = E^{\psi(t)}(u(t,\cdot)), \qquad t \in (0,2). \tag{4.7}$$

Denoting the gradient with respect to the x-variable by ∇^χ, from (1.1) we immediately obtain

$$|\tilde{u}(z_1) - \tilde{u}(z_2)| \le C|z_1 - z_2|(M^\chi[|\nabla^\chi \tilde{u}|](z_1) + M^\chi[|\nabla^\chi \tilde{u}|](z_2)) \tag{4.8}$$

for a.e. $t \in (0,2)$ and a.e. z_1, $z_2 \in \{t\}\times\mathbb{R}^{n-1}$. It is easily seen, when $u \in C^1(\Omega_\psi)$, that the function $\tilde{u}$ and $\nabla^\chi \tilde{u}$ are measurable on $(0,2)\times\mathbb{R}^{n-1}$. In fact, it could be shown that both of these would be measurable even if u were just in $W^{1,p}(\Omega_\psi)$.

Next, we prove the main estimate.

Lemma 4.9. *Let $z_1 = (t_1,x_1), z_2 := (t_2,x_2) \in \Omega_\psi$ be two points with $t_1 < t_2$. Suppose that $u \in W^{1,p}(\Omega_\psi)\cap C^1(\Omega_\psi)$ and that $\tilde{u}$ is its extension given by (4.7). Then we have*

$$\begin{aligned} |u(z_1) - u(z_2)| \;\le\; C|z_1 - z_2|\big(&M^\tau[|\nabla u|](z_1) + M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z_1) + \\ &M^\tau[|\nabla u|](z_2) + M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z_2)\big). \end{aligned} \tag{4.10}$$

Proof. Similarly to the two-dimensional argument, we will compare the change in the function via additional values $\tilde{u}(s,x_i)$ for some $s \in (0,2)$. Without knowing exactly which s yields an optimal estimate, we will instead average over a range of possible s with the hope that, on average, the differences are better controlled. Indeed, let

$$T_2 = \min\Big\{2, t_2 + \frac{t_2 - t_1}{2}\Big\},$$
$$T_1 = T_2 - \frac{t_2 - t_1}{2}.$$

Notice that $t_2 \in [T_1,T_2]$ and $[T_1,T_2]\times\{x_1,x_2\} \subset \Omega_\psi$. When we average over different possible $s \in [T_1,T_2]$ and use the triangle inequality we obtain that

$$\begin{aligned} |u(z_2) - u(z_1)| \le &\underbrace{\left|\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} |u(t_2,x_2) - u(s,x_2)|\, ds\right|}_{I} + \underbrace{\left|\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} |u(s,x_2) - u(s,x_1)|\, ds\right|}_{II} \\ &+ \underbrace{\left|\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} |u(s,x_1) - u(t_1,x_1)|\, ds\right|}_{III}. \end{aligned} \tag{4.11}$$

First, we estimate the terms I and III. Let $i \in \{1,2\}$. If $t_i < s$, by the fundamental theorem of calculus we have

$$|u(t_i,x_i) - u(s,x_i)| \leq \int_{t_i}^{s} |\nabla u(r,x_i)|\, dr \leq |t_i - s| M^\tau[|\nabla u|](z_i) \leq 3(T_2 - T_1) M^\tau[|\nabla u|](z_i). \tag{4.12}$$

Similarly, (4.12) holds also if $t_i \geq s$. Integrating with respect to s we obtain

$$I \leq 3(T_2 - T_1) M^\tau[|\nabla u|](z_2) \leq 2|z_2 - z_1| M^\tau[|\nabla u|](z_2). \tag{4.13}$$

and

$$III \leq 3(T_2 - T_1) M^\tau[|\nabla u|](z_1) \leq 2|z_2 - z_1| M^\tau[|\nabla u|](z_1) \tag{4.14}$$

Next, we apply (4.8) to the second term:

$$\begin{aligned} II &\leq \frac{C|x_1 - x_2|}{T_2 - T_1} \int_{T_1}^{T_2} (M^\chi[|\nabla^\chi \tilde{u}|](s,x_1) + M^\chi[|\nabla^\chi \tilde{u}|](s,x_2))\, ds \\ &\leq C|x_1 - x_2| \left(\frac{1}{T_2 - t_1} \int_{t_1}^{T_2} (M^\chi[|\nabla^\chi \tilde{u}|](s,x_1)\, ds + \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} (M^\chi[|\nabla^\chi \tilde{u}|](s,x_2)\, ds \right) \\ &\leq C|z_1 - z_2| \big(M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z_1) + M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z_2) \big)\,. \end{aligned} \tag{4.15}$$

Finally, by combining inequalities (4.13), (4.14), (4.15) and (4.11), we obtain the desired inequality (4.10). □

Recall that a domain Ω is quasiconvex if there exists a $C \geq 1$ such that, for every pair of points $x, y \in \Omega$, there is a rectifiable curve $\gamma \subset \Omega$ joining x to y so that $\operatorname{len}(\gamma) \leq C|x - y|$.

Proof of Theorem 1.5. Because Ω_ψ is quasiconvex for every ψ, the case of $p = \infty$ is a consequence of [5, Theorem 7]. Thus, fix $1 < p < \infty$. By [4], we know that there is a bounded inclusion $\iota\colon M^{1,p}(\Omega_\psi) \hookrightarrow W^{1,p}(\Omega_\psi)$. To show that ι is an isomorphism it suffices to show that the dense subspace $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ of $W^{1,p}(\Omega_\psi)$ is contained in $M^{1,p}(\Omega_\psi)$, and that the restricted inverse $\iota^{-1}|_{C^1(\Omega_\psi)\cap W^{1,p}(\Omega_\psi)}$ is defined and bounded.

Let $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ be arbitrary, and define $\tilde{u}$ as in (4.7). Set

$$\hat{g}(z) = M^\tau[|\nabla u|](z) + M^\chi[|\nabla^\chi \tilde{u}|](z) + M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z)\,. \tag{4.16}$$

By (4.8) and Lemma 4.9, for every $z_1, z_2 \in \Omega_\psi$, we get the estimate

$$|u(z_1) - u(z_2)| \leq C|z_1 - z_2|(\hat{g}(z_1) + \hat{g}(z_2))\,.$$

Hence (1.2) holds for $g := C\hat{g}$ for a suitable constant $C > 1$. The triangle inequality gives

$$\int_{\Omega_\psi} |g(z)|^p\, dz \leq C \left(\int_{\Omega_\psi} M^\tau[|\nabla u|](z)^p\, dz + \int_{\Omega_\psi} M^\chi[|\nabla^\chi \tilde{u}|](z)^p\, dz + \int_{\Omega_\psi} M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z)^p\, dz \right).$$

Lemmata 3.3 and 3.6 and (4.6) lead to the estimates

$$\int_{\Omega_\psi} |M^\tau[|\nabla u|](z)|^p\, dz \leq C \int_{\Omega_\psi} |\nabla u(z)|^p\, dz$$

and

$$\begin{aligned}\int_{\Omega_\psi} |M^\tau[M^\chi[|\nabla^\chi \tilde{u}|]](z)|^p\,dz &\leq C\int_{\Omega_\psi} M^\chi[|\nabla^\chi \tilde{u}|](z)^p\,dz \leq C\int_{(0,2)\times\mathbb{R}^{n-1}} |\nabla^\chi \tilde{u}(z)|^p\,dz \\ &\leq C\int_0^2\int_{\mathbb{R}^{n-1}} |\nabla^\chi \tilde{u}(t,x)|^p\,dx\,dt \leq C\int_0^2\int_{B(0,\psi(t))} |\nabla^\chi u(t,x)|^p\,dx\,dt \\ &\leq C\int_{\Omega_\psi} |\nabla u(z)|^p\,dz\,,\end{aligned}$$

which imply that $g \in \mathcal{D}_p(u)$ and that $\|u\|_{M^{1,p}(\Omega_\psi)} \leq C\|u\|_{W^{1,p}(\Omega_\psi)}$. That is, $\iota^{-1}|_{C^1(\Omega_\psi)\cap W^{1,p}(\Omega_\psi)}$ is both well-defined and bounded. □

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Singularities In $\mathscr{L}^p$-quasidisks

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SINGULARITIES IN $\mathscr{L}^p$-QUASIDISKS

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Abstract. We study planar domains with exemplary boundary singularities of the form of cusps. A natural question is how much elastic energy is needed to flatten these cusps; that is, to remove singularities. We give, in a connection of quasidisks, a sharp integrability condition for the distortion function to answer this question.

1. Introduction and Overview

The subject matter emerge most clearly when the setting is more general than we actually present it here. Thus we suggest, as a possibility, to consider two planar sets $\mathbb{X}, \mathbb{Y} \subset \mathbb{C}$ of the same global topological configuration, meaning that there is a sense preserving homeomorphism $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ which takes $\mathbb{X}$ onto $\mathbb{Y}$. Clearly $f : \mathbb{C} \setminus \mathbb{X} \xrightarrow{\text{onto}} \mathbb{C} \setminus \mathbb{Y}$. We choose two examples; one from naturally occurring Geometric Function Theory (GFT) and the other from mathematical models of Nonlinear Elasticity (NE). The first one deals with *quasiconformal mappings* $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ and the associated concept of a *quasidisk*, whereas the unexplored perspectives come from NE. From these perspectives we look at the ambient space $\mathbb{C}$ as made of a material whose elastic properties are characterized by a *stored energy function* $E : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{2\times 2} \to \mathbb{R}$, and $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ as a deformation of finite energy,

$$\mathbf{E}[f] \overset{\text{def}}{=\!=} \int_{\mathbb{C}} E(z, f, Df)\, \mathbf{d}z < \infty\,. \tag{1.1}$$

Hereafter the differential matrix $Df(z) \in \mathbb{R}^{2\times 2}$ is referred to as *deformation gradient*. A Sobolev homeomorphism $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ of finite energy is understood as a *hyper-elastic deformation* of $\mathbb{C}$. Our concept of finite energy, suited to the purpose of the present paper, is clearly inspired by *mappings of finite distortion* [3, 10, 12], including *quasiconformal mappings*. Therefore, omitting necessary details, the stored

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energy function will take the form $E(z, f, Df) = E(z, |Df|^2/\det Df)$. We adopt interpretations from NE where a great part of our paper is highly motivated. Let us take a quick look at such mappings.

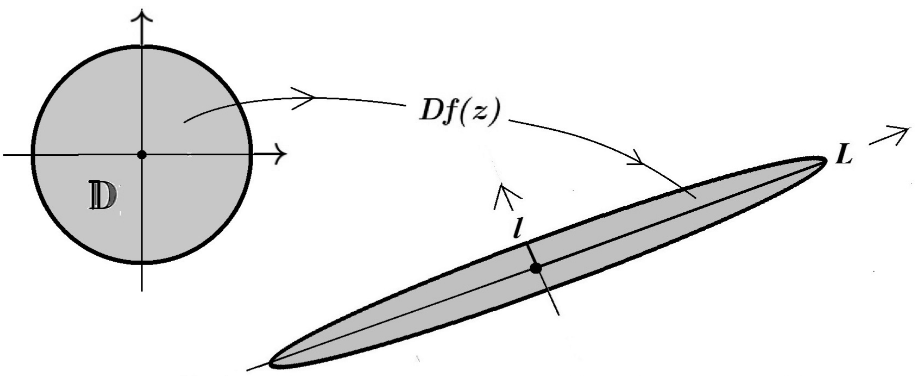
1.1. **Mappings of finite distortion.** Throughout this paper the domain of definition of such mappings consists of sense preserving homeomorphisms $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ of Sobolev class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{C}, \mathbb{C})$.

Definition 1.1. A homeomorphism $f \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{C}, \mathbb{C})$ is said to have *finite distortion* if there is a measurable function $K : \mathbb{C} \to [1, \infty)$ such that

$$|Df(z)|^2 \leqslant K(z) J_f(z) \,, \quad \text{for almost every } z \in \mathbb{C}. \tag{1.2}$$

Hereafter $|Df(z)|$ stands for the operator norm of the differential matrix $Df(z) \in \mathbb{R}^{2\times 2}$, and $J_f(z)$ for its determinant. The smallest function $K(x) \geqslant 1$ for which (1.2) holds is called the *distortion* of f, denoted by $K_f = K_f(x)$. In terms of d'Alembert complex derivatives, we have $|Df(z)| = |f_z| + |f_{\bar z}|$ and $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$. Thus f can be viewed as a *very weak solution* to the *Beltrami equation*:

$$\frac{\partial f}{\partial \bar z} = \mu(z) \frac{\partial f}{\partial z} \,, \quad \text{where } |\mu(z)| = \frac{K_f(z) - 1}{K_f(z) + 1} < 1 \tag{1.3}$$



$$\frac{L}{l} = \mathcal{K}_f(z) = \frac{\max\{|Df(z)\xi| \,;\, |\xi|=1\}}{\min\{|Df(z)\zeta| : |\zeta|=1\}} < \infty$$

FIGURE 1. The ratio L/l, which measures the infinitesimal distortion of the material structure at the point z, is allowed to be arbitrarily large. Nevertheless, L/l has to be finite almost everywhere.

The distortion inequality (1.2) asks that $Df(z) = 0 \in \mathbb{R}^{2\times 2}$ at the points where the Jacobian $J_f(z) = \det Df(z)$ vanishes.

Definition 1.2. A homeomorphism $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ of Sobolev class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{C},\mathbb{C})$ is said to be *quasiconformal* if $K_f \in \mathscr{L}^\infty(\mathbb{C})$. It is K-*quasiconformal* $(1 \leqslant K < \infty)$ if $1 \leqslant K_f(z) \leqslant K$ everywhere.

1.2. **Quasi-equivalence.** It should be pointed out that the inverse map $f^{-1} : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ is also K-quasiconformal and a composition $f \circ g$ of K_1 and K_2-quasiconformal mappings is $K_1 \cdot K_2$-quasiconformal. These special features of quasiconformal mappings furnish an equivalence relation between subsets of $\mathbb{C}$ that is reflexive, symmetric and transitive.

Definition 1.3. We say that $\mathbb{X} \subset \mathbb{C}$ is *quasi-equivalent* to $\mathbb{Y} \subset \mathbb{C}$, and write $\mathbb{X} \overset{\text{quasi}}{=\!=} \mathbb{Y}$, if $Y = f(\mathbb{X})$ for some quasiconformal mapping $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$.

1.3. **Quasidisks.** One exclusive class of quasi-equivalent subsets is represented by the open unit disk $\mathbb{D} \subset \mathbb{C}$. Thus we introduce the following:

Definition 1.4. A domain $\mathbb{X} \subset \mathbb{C}$ is called *quasidisk* if it admits a quasiconformal mapping $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ which takes $\mathbb{X}$ onto $\mathbb{D}$. In symbols, we have $\mathbb{X} \overset{\text{quasi}}{=\!=} \mathbb{D}$.

Quasidisks have been studied intensively for many years because of their exceptional functional theoretical properties, relationships with Teichmüller theory and Kleinian groups and interesting applications in complex dynamics, see [6] for an elegant survey. Perhaps the best know geometric characterization for a quasidisk is the *Ahlfors' condition* [1].

Theorem 1.5 (Ahlfors)**.** *Let $\mathbb{X}$ be a (simply connected) Jordan domain in the plane. Then $\mathbb{X}$ is a quasidisk if and only if there is a constant $1 \leqslant \gamma < \infty$, such that for each pair of distinct points $a, b \in \partial\mathbb{X}$ we have*

$$\operatorname{diam} \Gamma \leqslant \gamma\, |a-b| \tag{1.4}$$

where Γ is the component of $\partial\mathbb{X} \setminus \{a,b\}$ with smallest diameter.

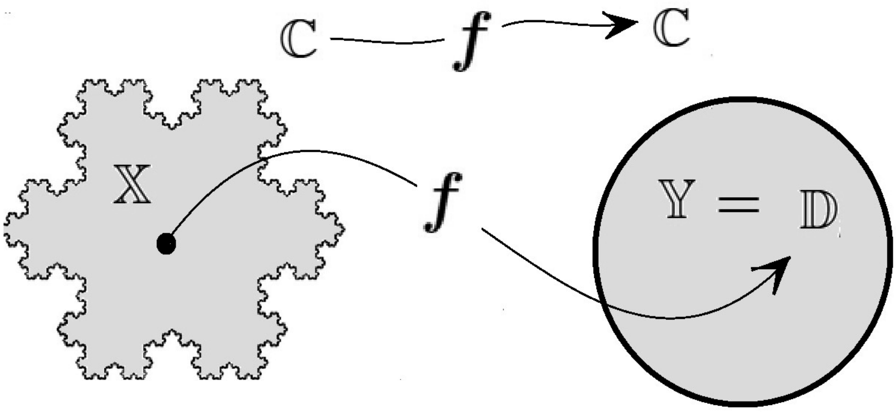


FIGURE 2. Koch snowflake reveals complexity of a quasidisk.

One should infer from the Ahlfors' condition (1.4) that:

Quasidisks do not allow for cusps in the boundary.

That is to say, unfortunately, the point-wise inequality $K_f(z) \leqslant K < \infty$ precludes f from smoothing even basic singularities. It is therefore of interest to look for more general deformations $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$. We shall see, and it will become intuitively clear, that the act of deviating from conformality should be measured by integral-mean distortions rather than point-wise distortions. More general class of mappings, for which one might hope to build a viable theory, consists of homeomorphisms with locally $\mathscr{L}^p$-integrable distortion, $1 \leqslant p < \infty$.

Definition 1.6. The term *mapping of $\mathscr{L}^p$-distortion*, $1 \leqslant p < \infty$, refers to a homeomorphism $f \colon \mathbb{C} \to \mathbb{C}$ of class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{C}, \mathbb{C})$ with $K_f \in \mathscr{L}^p_{\text{loc}}(\mathbb{C})$.

Now, we generalize the notion of quasidisks; simply, replacing the assumption $K_f \in \mathscr{L}^\infty(\mathbb{C})$ by $K_f \in \mathscr{L}^p_{\text{loc}}(\mathbb{C})$.

Definition 1.7. A domain $\mathbb{X} \subset \mathbb{C}$ is called an *$\mathscr{L}^p$-quasidisk* if it admits a homeomorphism $f \colon \mathbb{C} \to \mathbb{C}$ of $\mathscr{L}^p$-distortion such that $f(\mathbb{X}) = \mathbb{D}$.

Clearly, $\mathscr{L}^p$-quasidisks are Jordan domains. Surprisingly, the $\mathscr{L}^1_{\text{loc}}$-integrability of the distortion seems not to cause any geometric constraint on $\mathbb{X}$. We confirm this observation for domains with rectifiable boundary.

Theorem 1.8. *Simply-connected Jordan domains with rectifiable boundary are $\mathscr{L}^1$-quasidisks.*

Nevertheless, the $\mathscr{L}^p$-quasidisks with $p > 1$ can be characterized by model singularities at their boundaries. The most specific singularities, which fail to satisfy the Ahlfors' condition (1.4), are cusps. Let us consider the power-type inward and outward cusp domains, see Figure 3. For $\beta > 1$ we consider a disk with inward cusp defined by

$$\mathbb{D}^{\prec}_{\beta} = \mathbb{B}(1-\beta, r_\beta) \setminus \{z = x+iy \in \mathbb{C}\colon x \geqslant 0, |y| \leqslant x^\beta\}\ , \quad r_\beta = \sqrt{\beta^2+1}\,.$$

Whereas a disk with outer cusp will be defined by

$$\mathbb{D}^{\succ}_{\beta} = \{z = x+iy \in \mathbb{C}\colon 0 < x < 1, |y| < x^\beta\} \cup \mathbb{B}(1+\beta, r_\beta)\,.$$

Here, $r_\beta = \sqrt{\beta^2+1}$.

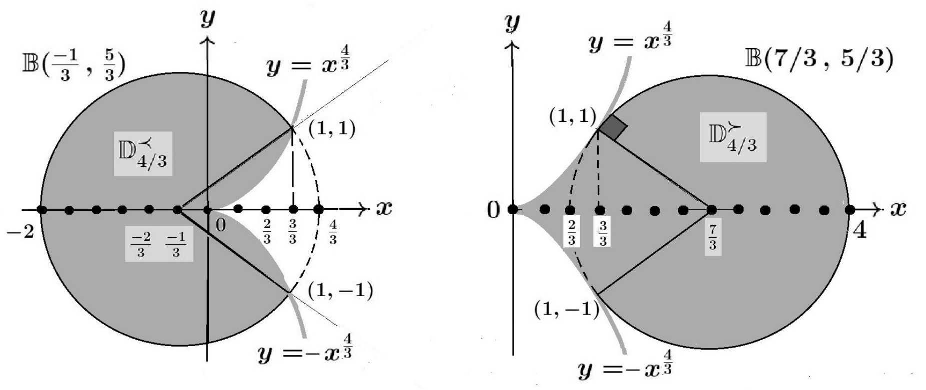


FIGURE 3. The inner and outer *power cusps* in the disks $\mathbb{D}^{\prec}_{\beta}$ and $\mathbb{D}^{\succ}_{\beta}$, with $\beta = \frac{4}{3}$.

Note, all of these domains fail to satisfy the Alhfors' condition (1.4). However, replacing $|a-b|$ in (1.4) by $|a-b|^\alpha$ we obtain:

Definition 1.9. A Jordan domain $\mathbb{X} \subset \mathbb{C}$ is *α-Ahlfors regular*, with $\alpha \in (0,1]$, if there is a constant $1 \leqslant \gamma < \infty$ such that for each pair of distinct points $a, b \in \partial\mathbb{X}$ we have

$$\operatorname{diam}\Gamma \leqslant \gamma\,|a-b|^\alpha \tag{1.5}$$

where Γ is the component of $\partial\mathbb{X} \setminus \{a,b\}$ with smallest diameter.

Theorem 1.10. *Let $\mathbb{X}$ be either $\mathbb{D}^{\prec}_{\beta}$ or $\mathbb{D}^{\succ}_{\beta}$ and $1 < p < \infty$. Then $\mathbb{X}$ is a $\mathscr{L}^p$-quasidisk if and only if $\beta < \frac{p+3}{p-1}$; equivalently, $p < \frac{\beta+3}{\beta-1}$.*

This simply means that $\mathbb{X}$ is $\frac{1}{\beta}$-Ahlfors regular. Theorem 1.10 tells us how much the distortion of a homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is needed to flatten (or smoothen) the power type cusp t^β. It turns out that a lot more distortion is needed to create a cusp than to smooth it back. Indeed, in a series of papers [14, 15, 16], Koskela and Takkinen raised such an inverse question. For which cusps does there exist a homeomorphism $h\colon \mathbb{C} \to \mathbb{C}$ of finite distortion $1 \leqslant K_h < \infty$ which takes $\mathbb{D}$ onto $\mathbb{D}_\beta^{\prec}$? A necessary condition turns out to be that $e^{K_h} \notin \mathscr{L}^p_{\text{loc}}(\mathbb{C})$ with $p > \frac{2}{\beta-1}$. However, if $p < \frac{2}{\beta-1}$ there is such a homeomorphism. Especially, each power-type cusp domain can be obtained as the image of open disk by a homeomorphism $h\colon \mathbb{C} \to \mathbb{C}$ with $K_h \in \mathscr{L}^p_{\text{loc}}(\mathbb{C})$ for all $p < \infty$. Combining this with Theorem 1.10 boils down to the following postulate:

Creating singularities takes almost no efforts (just allow for a little distortion) while tidying them up is a whole new story.

1.4. **The energy for $\mathscr{L}^p$-distortion.** We need to pullback to $\mathbb{C}$ the Euclidean area element $\mathrm{d}\sigma(\xi)$ of $\mathbb{S}^2 \subset \mathbb{R}^3$ by stereographic projection $\Pi : \mathbb{S}^\circ \xrightarrow{\text{onto}} \mathbb{C}$, where

$$\mathbb{S}^\circ \stackrel{\text{def}}{=\!=} \{\, \xi = (w,t)\colon w \in \mathbb{C}\,,\ -1 \leqslant t < 1\,,\ |w|^2 + t^2 = 1 \,\} \subset \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 \,.$$

The image point $z = \Pi\xi$ is defined by the rule $\Pi(w,t) = \frac{w}{1-t}$. For the inverse projection $\Pi^{-1} : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{S}^\circ$ we have:

$$\xi = \Pi^{-1} z = (w,t)\ , \quad \text{where } w = \frac{2\,z}{1+|z|^2} \ \text{ and } \ t = \frac{|z|^2-1}{|z|^2+1}\,.$$

Denote by $\mathbf{d}z = \mathrm{d}x\,\mathrm{d}y$ the area element in $\mathbb{C}$, $z = x + iy$. The general formula of integration by change of variables reads as follows:

$$\mathrm{d}\sigma(\xi) = \frac{4\,\mathbf{d}z}{(|z|^2+1)^2}\ , \ \text{hence} \ \int_{\mathbb{C}} \frac{4\,G(z)\,\mathbf{d}z}{(|z|^2+1)^2} = \int_{\mathbb{S}^\circ} G(\Pi\xi)\,\mathrm{d}\sigma(\xi)$$

Now, one might consider mappings of $\mathscr{L}^p$-distortion which have finite $\mathscr{L}^p$-energy:

$$\mathbf{E}[f] \stackrel{\text{def}}{=\!=} 4 \int_{\mathbb{C}} \frac{[K_f(z)]^p\,\mathbf{d}z}{(|z|^2+1)^2} = \int_{\mathbb{S}^\circ} [K_{\mathcal{F}}(\xi)]^p\,\mathrm{d}\sigma(\xi) < \infty, \tag{1.6}$$

where $K_{\mathcal{F}} : \mathbb{S}^\circ \to [1,\infty)$ stands for the distortion function of the mapping $\mathcal{F} = f \circ \Pi : \mathbb{S}^\circ \xrightarrow{\text{onto}} \mathbb{C}$. For the energy formula (1.6), we invoke the equality $K_{\mathcal{F}}(\xi) = K_f(\Pi\xi)$ which is due to the fact that Π is conformal. This formula makes it clear that K-quasiconformal mappings $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ have finite $\mathscr{L}^p$-energy and $\mathbf{E}[f] \leqslant 4\pi K^p$.

In the spirit of extremal quasiconformal mappings in Teichmüller spaces, one might be interested in studying homeomorphisms $f : \mathbb{C} \xrightarrow{\text{onto}}$

$\mathbb{C}$ of smallest $\mathscr{L}^p$-energy, subject to the condition $f(\mathbb{X}) = \mathbb{Y}$. Here the given pair $\mathbb{X}, \mathbb{Y}$ of subsets in $\mathbb{C}$ is assumed to admit at least one such homeomorphism of finite energy. To look at a more specific situation, take for $\mathbb{X}$ an $\mathscr{L}^p$-quasidisk from Theorem 1.10, and the unit disk $\mathbb{D}$ for $\mathbb{Y}$. What is then the energy-minimal map $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$? Polyconvexity of the integrand will certainly help us find what conditions are needed for the existence of energy-minimal mappings. We shall not enter these topics here, but refer to [2, 13, 19] for related results.

1.5. **The main result.** Since a simply connected Jordan domain is conformally equivalent with the unit disk, it is natural to consider special $\mathscr{L}^p$-quasidisks; namely, the domains $\mathbb{X}$ which can be mapped onto an open disk under a homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ with p-integrable distortion and to be quasiconformal when restricted to $\mathbb{X}$.

The answer to this question can be inferred from our main result which also generalizes Theorem 1.10.

Theorem 1.11 (Main Theorem). *Consider power-type inward cusp domains $\mathbb{X} = \mathbb{D}_\beta^{\prec}$ with $\beta > 1$. Given a pair (q,p) of exponents $1 \leqslant q \leqslant \infty$ (for $\mathbb{X}$) and $1 < p \leqslant \infty$ (for the complement of $\mathbb{X}$), define the so-called critical power of inward cusps*

$$\beta_{\text{cr}} \stackrel{\text{def}}{=\!=} \begin{cases} \frac{pq+2p+q}{pq-q}\,, & \text{if } 1<p<\infty \text{ and } q<\infty \\ \frac{2}{q}+1\,, & \text{if } p=\infty \text{ and } q<\infty \\ \frac{p+1}{p-1}\,, & \text{if } 1<p<\infty \text{ and } q=\infty \end{cases} \tag{1.7}$$

Then there exists a Sobolev homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ which takes $\mathbb{X}$ onto $\mathbb{D}$ such that

- $K_f \in \mathscr{L}^q(\mathbb{X})$

and

- $K_f \in \mathscr{L}^p(\mathbb{B}_R \setminus \overline{\mathbb{X}})$ *for every $R > 2$,*

if and only if $\beta < \beta_{\text{cr}}$.

Here and what follows $\mathbb{B}_R = \{z \in \mathbb{C} \colon |z| < R\}$ for $R > 0$. Applying the standard inversion of unit disk, Theorem 1.11 extends to the power-type outer cusp domains as well. In this case the roles of p and q are interchanged. The reader interested in learning more about the conformal case $f\colon \mathbb{D}_\beta^{\prec} \xrightarrow{\text{onto}} \mathbb{D}$ is refer to [22].

Our proof of Theorem 1.11 is self-contained. The "only if" part of Theorem 1.11 relies on a regularity estimate of a reflection in $\partial\mathbb{D}_\beta^{\prec}$.

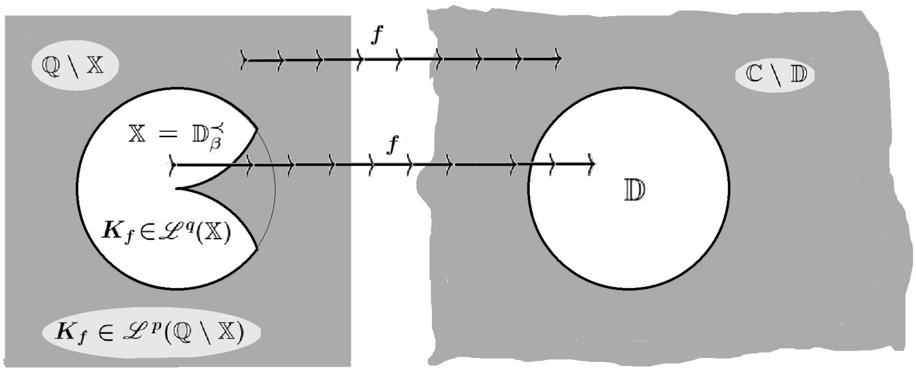


FIGURE 4. An $\mathscr{L}^{q,p}$-quasidisk

Such a reflection is defined and examined in the boundary of an arbitrary $\mathscr{L}^p$-quasidisk. In this connection we recall a classical result of Kühnau [18] which tells us that a Jordan domain is a qusidisk if and only if it admits a qusiconformal reflection in its boundary. Before going into details about the boundary reflection proceeders (Section 3) we need some preliminaries.

2. Preliminaries

First we recall a well-known theorem of Gehring and Lehto [9] which asserts that a planar open mapping with finite partial derivatives at almost every point is differentiable at almost every point. For homeomorphisms the result was earlier established by Menchoff [20].

Lemma 2.1. *Suppose that $f\colon \mathbb{C}\to\mathbb{C}$ is a homeomorphism in the class $\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$. Then f is differentiable almost everywhere.*

It is easy to see, at least formally, applying a change of variables that the integral of distortion function equals the Dirichlet integral of inverse mapping. This observation is the key to the fundamental identity which we state next, see [10, 11, 21].

Lemma 2.2. *Suppose that a homeomorphism $f\colon \mathbb{C}\xrightarrow{\text{onto}}\mathbb{C}$ of Sobolev class $\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$. Then f is a mapping of $\mathscr{L}^1$-distortion if and only if the inverse $h\overset{\text{def}}{=\!=} f^{-1}\in W^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$. Furthermore, then for every bounded domain $\mathbb{U}\subset\mathbb{C}$ we have*

$$\int_{f(\mathbb{U})}|Dh(y)|^2\,\mathrm{d}y=\int_{\mathbb{U}}K_f(x)\,\mathrm{d}x$$

and $J_f(x)>0$ a.e.

At least formally the identity $(h \circ f)(x) = x$, after differentiation, implies that $Dh(f(x))Df(x) = \mathbf{I}$. The validity of such identity under minimal regularity assumptions on the mappings is the essence of the following lemma, see [10, Lemma A.29].

Lemma 2.3. *Let $f\colon \mathbb{X} \to \mathbb{Y}$ be a homeomorphism which is differentiable at $x \in \mathbb{X}$ with $J_f(x) > 0$. Let $h\colon \mathbb{Y} \to \mathbb{X}$ be the inverse of f. Then h is differentiable at $f(x)$ and $Dh(f(x)) = (Df(x))^{-1}$.*

Next we state a crucial version of the area formula for us.

Lemma 2.4. *Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{C}$ be domains and $g\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ a homeomorphism. Suppose that $\mathbb{V} \subset \mathbb{X}$ be a measurable set and g is differentiable at every point of $\mathbb{V}$. If η is a nonnegative Borel measurable function, then*

$$\int_{\mathbb{V}} \eta(g(x))|J_g(x)| \, \mathrm{d}x \leqslant \int_{g(\mathbb{V})} \eta(y) \, \mathrm{d}y \,. \tag{2.1}$$

This follows from [5, Theorem 3.1.8] together with the area formula for Lipschitz mappings.

The circle is uniquely characterized by the property that among all closed Jordan curves of given length L, the circle of circumference L encloses maximum area. This property is expressed in the well-known isoperimetric inequality.

Lemma 2.5. *Suppose $\mathbb{U}$ is a bounded Jordan domain with rectifiable boundary $\partial\mathbb{U}$. Then*

$$|\mathbb{U}| \leqslant \frac{1}{4\pi}[\ell(\partial\mathbb{U})]^2 \tag{2.2}$$

where $|\mathbb{U}|$ is the area of $\mathbb{U}$ and $\ell(\partial\mathbb{U})$ is the length of $\partial\mathbb{U}$.

3. Reflection

We denote the one point compactification of the complex plane by $\widehat{\mathbb{C}} \stackrel{\text{def}}{=\!=} \mathbb{C} \cup \{\infty\}$.

Definition 3.1. A domain $\Omega \subset \widehat{\mathbb{C}}$ admits a *reflection in its boundary* $\partial\Omega$ if there exists a homeomorphism g of $\widehat{\mathbb{C}}$ such that

- $g(\Omega) = \widehat{\mathbb{C}} \setminus \overline{\Omega}$, and
- $g(z) = z$ for $z \in \partial\Omega$.

A domain $\Omega \subset \widehat{\mathbb{C}}$ is a Jordan domain if and only if it admits a reflection in its boundary, see [7]. In this section we raise a question what else can we say about the reflection if the domain is an $\mathscr{L}^p$-quasidisk. A classical result of Kühnau [18] tells us that $\Omega \subset \widehat{\mathbb{C}}$ is a qusidisk if and only if it admits a qusiconformal reflection in $\partial\Omega$. Let $\mathbb{X} \subset \mathbb{C}$ be an $\mathscr{L}^p$-quasidisk. Then there exists a homeomorphism $f \colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ such that $f(\mathbb{X}) = \mathbb{D}$. We extend f by setting $f(\infty) = \infty$ and still denote the extended mapping by f. This way we obtain a homeomorphism $f \colon \widehat{\mathbb{C}} \xrightarrow{\text{onto}} \widehat{\mathbb{C}}$. We also denote its inverse by $h \colon \widehat{\mathbb{C}} \xrightarrow{\text{onto}} \widehat{\mathbb{C}}$.

The circle inversion map $\Psi \colon \widehat{\mathbb{C}} \xrightarrow{\text{onto}} \widehat{\mathbb{C}}$,

$$\Psi(z) \overset{\text{def}}{=\!=} \begin{cases} \frac{z}{|z|^2} & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \end{cases}$$

is anticonformal, which means that at every point it preserves angles and reverses orientation. The circle inversion defines a reflection in $\partial\mathbb{X}$ by the rule

$$g \colon \widehat{\mathbb{C}} \xrightarrow{\text{onto}} \widehat{\mathbb{C}} \qquad g(x) \overset{\text{def}}{=\!=} h \circ \Psi \circ f(x)\,. \tag{3.1}$$

Theorem 3.2. *Let $\mathbb{X}$ be an $\mathscr{L}^p$-quasidisk and g the reflection in $\partial\mathbb{X}$ given by* (3.1). *Then for a bounded domain $\mathbb{U} \subset \mathbb{C}$ such that $h(0) \notin \overline{\mathbb{U}}$ we have $g \in \mathscr{W}^{1,1}(\mathbb{U}, \mathbb{C})$ and*

$$\int_{\mathbb{U}} \frac{|Dg(x)|^p}{|J_g(x)|^{\frac{p-1}{2}}}\, \mathrm{d}x \leqslant \left(\int_{g(\mathbb{U})} K_f^p(x)\, \mathrm{d}x \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{U}} K_f^p(x)\, \mathrm{d}x \right)^{\frac{1}{2}}\,. \tag{3.2}$$

Proof. Let $\mathbb{U}$ be a bounded domain in $\mathbb{C}$ such that $h(0) \notin \overline{\mathbb{U}}$. For $x \in \mathbb{U}$ we denote

$$\tilde{f}(x) \overset{\text{def}}{=\!=} \Psi \circ f(x) \quad \text{and} \quad \tilde{h}(y) \overset{\text{def}}{=\!=} (\tilde{f})^{-1}(y)\,.$$

We write

$$\mathbb{V} \overset{\text{def}}{=\!=} \{x \in \mathbb{U} \colon f \text{ is differentiable at } x \ \text{ and } \ J_f(x) > 0\}\,.$$

Then by Lemma 2.1 and Lemma 2.2 we obtain $|\mathbb{V}| = |\mathbb{U}|$.

Fix $x \in \mathbb{V}$. Then $\tilde{f}$ is differentiable at x. Furthermore, h is differentiable at $f(x)$, see Lemma 2.3. Therefore, for $x \in \mathbb{V}$ the chain rule gives

$$|Dg(x)| \leqslant |Dh(\tilde{f}(x))|\,|D\tilde{f}(x)| \ \text{ and } J_g(x) = J_h(\tilde{f}(x))J_{\tilde{f}}(x)\,. \tag{3.3}$$

Hence, applying Hölder's inequality we have

$$
\begin{aligned}
\int_{\mathbb{U}} \frac{|Dg(x)|^p}{|J_g(x)|^{\frac{p-1}{2}}}\,\mathrm{d}x &= \int_{\mathbb{V}} \frac{|Dg(x)|^p}{|J_g(x)|^{\frac{p-1}{2}}}\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{V}} \frac{|Dh(\tilde f(x))|^p}{|J_h(\tilde f(x))|^{\frac{p-1}{2}}} \frac{|D\tilde f(x)|^p}{|J_{\tilde f}(x)|^{\frac{p-1}{2}}}\,\mathrm{d}x \\
&\leqslant \left(\int_{\mathbb{V}} \frac{|Dh(\tilde f(x))|^{2p}}{|J_h(\tilde f(x))|^{p-1}} |J_{\tilde f}(x)|\,\mathrm{d}x\right)^{\frac12} \\
&\quad \cdot \left(\int_{\mathbb{V}} \frac{|D\tilde f(x)|^{2p}}{|J_{\tilde f}(x)|^{p}}\,\mathrm{d}x\right)^{\frac12}.
\end{aligned} \tag{3.4}
$$

According to Lemma 2.4 we obtain

$$
\int_{\mathbb{V}} \frac{|Dh(\tilde f(x))|^{2p}}{|J_h(\tilde f(x))|^{p-1}} |J_{\tilde f}(x)|\mathrm{d}x \leqslant \int_{\tilde f(\mathbb{V})} \frac{|Dh(y)|^{2p}}{[J_h(y)]^{p-1}}\,\mathrm{d}y\,. \tag{3.5}
$$

Applying Lemma 2.4 again this time for h, we have

$$
\int_{\tilde f(\mathbb{V})} \frac{|Dh(y)|^{2p}}{[J_h(y)]^{p}} J_h(y)\,\mathrm{d}y \leqslant \int_{g(\mathbb{V})} [Dh\big(f(x)\big)]^{2p}[J_f(x)]^p\,\mathrm{d}x\,.
$$

This together with Lemma 2.3 gives

$$
\int_{\tilde f(\mathbb{V})} \frac{|Dh(y)|^{2p}}{[J_h(y)]^{p}} J_h(y)\,\mathrm{d}y \leqslant \int_{g(\mathbb{V})} \big[(Df(x))^{-1}\big]^{2p}\,[J_f(x)]^p\,\mathrm{d}x\,.
$$

The familiar Cramer's rule implies

$$
\int_{g(\mathbb{V})} \big[(Df(x))^{-1}\big]^{2p}\,[J_f(x)]^p\,\mathrm{d}x = \int_{g(\mathbb{V})} \frac{|Df(x)|^{2p}}{[J_f(x)]^p}\,. \tag{3.6}
$$

Combining the estimate (3.5) with (3.6) we have

$$
\int_{\mathbb{V}} \frac{|Dh(\tilde f(x))|^{2p}}{|J_h(\tilde f(x))|^{p-1}} |J_{\tilde f}(x)|\mathrm{d}x \leqslant \int_{g(\mathbb{U})} K_f^p(x)\,\mathrm{d}x\,. \tag{3.7}
$$

Estimating the second term on the right hand side of (3.4) we simply note that $|D\Psi(z)|^2 = J(z,\Psi)$ for $z \in \mathbb{C}\setminus\{0\}$ and so

$$
\int_{\mathbb{V}} \frac{|D\tilde f(x)|^{2p}}{|J_{\tilde f}(x)|^{p}}\,\mathrm{d}x = \int_{\mathbb{V}} K_f^p(x)\,\mathrm{d}x \leqslant \int_{\mathbb{U}} K_f^p(x)\,\mathrm{d}x\,. \tag{3.8}
$$

The claim follows from (3.4), (3.7) and (3.8). □

4. Proof of Theorem 1.8

The proof is based on a Sobolev variant of the Jordan-Schönflies theorem.

Lemma 4.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded simply connected Jordan domains, $\partial\mathbb{Y}$ being rectifiable. A boundary homeomorphism $\phi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfying*

$$\int_{\partial\mathbb{Y}}\int_{\partial\mathbb{Y}} \left|\log|\phi^{-1}(\xi)-\phi^{-1}(\eta)|\right| |\mathrm{d}\xi||\mathrm{d}\eta| < \infty \tag{4.1}$$

admits a homeomorphic extension $h\colon \mathbb{C}\to\mathbb{C}$ of Sobolev class $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$.

This result is from [17, Theorem 1.6]. Note that if one asks the existence of homeomorphic extension $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ (on one side of $\partial\mathbb{X}$) in the Sobolev class $\mathscr{W}^{1,2}(\mathbb{X},\mathbb{C})$. First, applying the Riemann Mapping Theorem we may assume that $\mathbb{X}=\mathbb{D}$. Second, a necessary condition is that the mapping ϕ is the Sobolev trace of some (possibly non-homeomorphic) mapping in $\mathscr{W}^{1,2}(\mathbb{X},\mathbb{C})$. The class of boundary functions which admit a harmonic extension with finite Dirichlet energy was characterized by Douglas [4]. The Douglas condition for a function $\phi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ reads as

$$\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \left|\frac{\phi(\xi)-\phi(\eta)}{\xi-\eta}\right|^2 |\mathrm{d}\xi||\mathrm{d}\eta| < \infty. \tag{4.2}$$

In [2] it was shown that for $\mathscr{C}^1$-smooth $\mathbb{Y}$ the Douglas condition (4.2) can be equivalently given in terms of the inverse mapping $\phi^{-1}\colon \partial\mathbb{Y} \xrightarrow{\text{onto}} \partial\mathbb{D}$ by (4.1). Beyond the $\mathscr{C}^1$-smooth domains, if $\mathbb{Y}$ is a Lipschitz regular, then a boundary homeomorphism $\phi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a homeomorphic extension $h\colon \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,2}(\mathbb{D},\mathbb{C})$ if and only if ϕ satisfies the Douglas condition. There is, however, an inner chordarc domain $\mathbb{Y}$ and a homeomorphism $\phi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfying the Douglas condition which does not admit a homeomorphic extension $h\colon \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ with finite Dirichlet energy. Recall that $\mathbb{Y}$ is an inner chordarc domain if there exists a homeomorphism $\Upsilon\colon \overline{\mathbb{Y}} \xrightarrow{\text{onto}} \overline{\mathbb{D}}$ which is $\mathscr{C}^1$-diffeomorphic in $\mathbb{Y}$ with bounded gradient matrices $D\Upsilon$ and $(D\Upsilon)^{-1}$. These and more about Sobolev homeomorphic extension results we refer to [17].

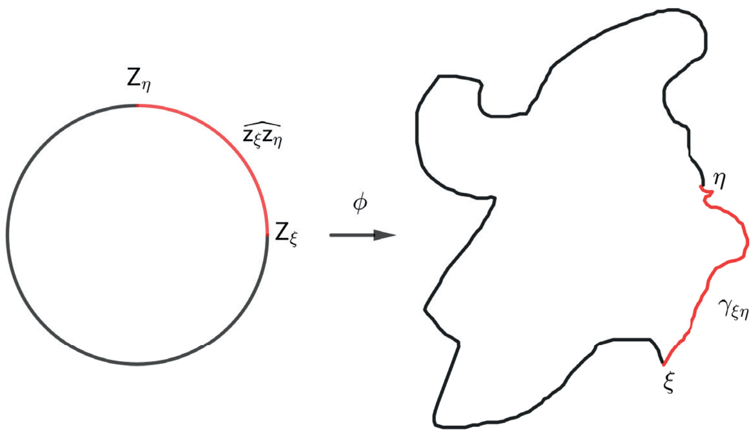
Proof of Theorem 1.8. Let $\mathbb{X}\subset\mathbb{C}$ be a simply connected Jordan domain, $\partial\mathbb{X}$ being rectifiable. According to Lemma 2.2, $\mathbb{X}$ is an $\mathscr{L}^1$-quasidisk if and only if there exists a homeomorphism $h\colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ in $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$ such that $h(\mathbb{D})=\mathbb{X}$. Therefore, by Lemma 4.1 it suffices to construct a boundary homeomorphism $\phi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{X}$ which satisfies

$$\int_{\partial\mathbb{X}}\int_{\partial\mathbb{X}} \left|\log|\phi^{-1}(\xi)-\phi^{-1}(\eta)|\right| |\mathrm{d}\xi||\mathrm{d}\eta| < \infty\,.$$

Let $\xi, \eta \in \partial\mathbb{X}$ be arbitrary. We denote by $\gamma_{\xi\eta}$ the subcurve of $\partial\mathbb{X}$, connecting ξ and η. The curve $\gamma_{\xi\eta}$ is parametrized counterclockwise. Setting $z_\xi = 1$. For arbitrary $z \in \partial\mathbb{D}$ let $\widehat{z_\xi z} \subset \partial\mathbb{D}$ be the circular arc starting from z_ξ ending at z. The arc is parametrized counterclockwise. For $\eta \in \partial\mathbb{X}$, there exists a unique $z_\eta \in \partial\mathbb{D}$ with

$$\frac{\ell(\gamma_{\xi\eta})}{\ell(\partial\mathbb{X})} = \frac{\ell(\widehat{z_\xi z_\eta})}{\ell(\partial\mathbb{D})}.$$

Now, we define the boundary homeomorphism $\phi : \partial\mathbb{D} \to \partial\mathbb{X}$ by setting $\phi(z_\eta) = \eta$.



First, we observe that $|\phi'(z)| = \frac{\ell(\partial\mathbb{X})}{\ell(\partial\mathbb{D})}$ for every $z \in \partial\mathbb{D}$. Furthermore since the length of the shorter circular arc between two points in $\partial\mathbb{D}$ is comparable to their Euclidean distance the change of variables formula gives

$$\begin{aligned}\int_{\partial\mathbb{X}} |\log|\phi^{-1}(\xi) - \phi^{-1}(\eta)|| \, |\mathrm{d}\eta| &\leqslant C \int_{\partial\mathbb{D}} |\log|\phi^{-1}(\xi) - \phi^{-1}(\eta)|| \, |\mathrm{d}\phi^{-1}(\eta)| \\ &\leqslant C \int_0^{2\pi} |\log t| \, \mathrm{d}t < \infty.\end{aligned}$$

□

5. Proof of Theorem 1.11

Before jumping into the proof we fix a few notation and prove two auxiliary results. Fix a power-type inward cusp domain $\mathbb{D}_\beta^{\prec}$. For $0 < t < 1$ we write

$$\mathbb{I}_t \overset{\text{def}}{=\!=} \{t + iy \in \mathbb{C} \colon 0 \leqslant |y| < t^\beta\}$$

and

$$\mathbb{U}_t \overset{\text{def}}{=\!=} \{x + iy \in \mathbb{C} \colon 0 < x < t \text{ and } 0 \leqslant |y| < x^\beta\}.$$

The area of $\mathbb{U}_t$ is given by

$$|\mathbb{U}_t| = \int_0^t \int_{-s^\beta}^{s^\beta} 1 \, \mathrm{d}y \, \mathrm{d}s = \frac{2t^{\beta+1}}{\beta+1} \, .$$

Suppose the cusp domain $\mathbb{D}_\beta^{\prec}$ is an $\mathscr{L}^s$-quasidisk for $1 \leqslant s < \infty$. Note that according to Theorem 1.8 the domain $\mathbb{D}_\beta^{\prec}$ is always an $\mathscr{L}^1$-quasidisk for every β. Therefore, there exists a homeomorphism $f \colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$ of $\mathscr{L}^1$-distortion such that $f(\mathbb{D}_\beta^{\prec}) = \mathbb{D}$. We denote the inverse of f by $h \colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$. After first extending the homeomorphisms f and h by $f(\infty) = \infty = h(\infty)$ we define a homeomorphism $g \colon \widehat{\mathbb{C}} \xrightarrow{\text{onto}} \widehat{\mathbb{C}}$ by the formula (3.1). The mapping g gives a reflection in the boundary of $\mathbb{D}_\beta^{\prec}$; that is,

- $g(\mathbb{D}_\beta^{\prec}) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_\beta^{\prec}}$,
- $g(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_\beta^{\prec}}) = \mathbb{D}_\beta^{\prec}$ and
- $g(x) = x$ for $x \in \partial \mathbb{D}_\beta^{\prec}$.

Lemma 5.1. *Let $\epsilon_n = 2^{-n}$ for $n \in \mathbb{N}$. Then there exists a subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ such that for every $k \in \mathbb{N}$ we have either*

- $|g(\mathbb{U}_{\epsilon_{n_k}})| \leqslant \epsilon_{n_k}^2$ *or*
- $|g(\mathbb{U}_{\epsilon_{n_k}})| \leqslant 5 |g(\mathbb{U}_{\epsilon_{n_k+1}})|$ *and* $|g(\mathbb{U}_{\epsilon_{n_k}})| > \epsilon_{n_k}^2$.

Proof. Assume to the contrary that the claim is not true, then there exists $n_o \in \mathbb{N}$ such that for every $i \geqslant n_o$, we have $|g(U_{\epsilon_i})| > \epsilon_i^2$ and $|g(U_{\epsilon_i})| > 5|g(U_{\epsilon_{i+1}})|$. Hence we have

$$|g(U_{\epsilon_{n_o}})| > 5|g(U_{\epsilon_{n_o+1}})| > \dots > 5^n |g(U_{\epsilon_{n_0+n}})| > \dots.$$

which implies that for every $n \in \mathbb{N}$, we have

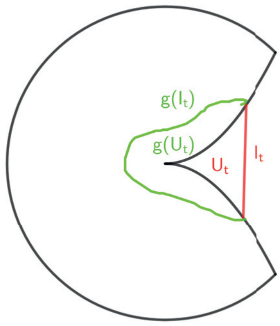
$$|g(U_{\epsilon_{n_o}})| > \left(\frac{5}{4}\right)^n 4^{-n_o}. \tag{5.1}$$

Letting $n \to \infty$ the term on the right hand side of (5.1) converges to ∞ which contradicts with $|g(U_{\epsilon_{n_o}})| < |\mathbb{D}_\beta^{\prec}| < \infty$. □

The key observation to show that $\mathbb{D}_\beta^{\prec}$, $\beta > 1$, is not an $\mathscr{L}^s$-quasidisk for sufficiently large $s > 1$ is to compare the length of curves $g(\mathbb{I}_t)$ and $\mathbb{I}_t$.

Lemma 5.2. *Suppose that $\mathbb{D}_\beta^{\prec}$ is an $\mathscr{L}^s$-quasidisk for $1 < s < \infty$. Then for almost every $0 < t < 1$ we have*

$$\ell(g(\mathbb{I}_t)) \leqslant \left(\int_{\mathbb{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}} \, \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_{\mathbb{I}_t} |J_g(x)|^{\frac{1}{2}} \, \mathrm{d}x \right)^{\frac{s-1}{s}} . \tag{5.2}$$



Proof. The second estimate in (5.2) follows immediately from Hölder's inequality

$$\begin{aligned}\ell(g(\mathbb{I}_t)) &\leqslant \int_{\mathbb{I}_t} |Dg(x)|\, \mathrm{d}x \leqslant \int_{\mathbb{I}_t} \frac{|Dg(x)|}{|J_g(x)|^{\frac{s-1}{2s}}} \cdot |J_g(x)|^{\frac{s-1}{2s}}\, \mathrm{d}x \\ &\leqslant \left(\int_{\mathbb{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\, \mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbb{I}_t} |J_g(x)|^{\frac{1}{2}} dx\right)^{\frac{s-1}{s}} .\end{aligned}$$

□

Now, we are ready to prove our main result Theorem 1.11.

5.1. **The nonexistence part.** Recall that critical power of inward cusps β_{cr} is given by the formula (1.7). Here we prove that if $\beta \geqslant \beta_{\mathrm{cr}}$, then there is no homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ of finite distortion with $f(\mathbb{D}_\beta^{\prec}) = \mathbb{D}$ and $K_f \in \mathscr{L}^p(\mathbb{B}_R \setminus \overline{\mathbb{D}_\beta^{\prec}}) \cap \mathscr{L}^q(\mathbb{D}_\beta^{\prec})$ for every $R > 2$. For that suppose that there exists such a homeomorphism. Write

$$s \stackrel{\text{def}}{=\!=} \min\{p, q\} > 1 .$$

We will split our argument into two parts. According to Lemma 5.1 (we denote $\mathcal{J} = \{n_k \in \mathbb{N} \colon k \in \mathbb{N}\}$) there exists a set $\mathcal{J} \subset \mathbb{N}$ and a decreasing sequence ϵ_j such that $\epsilon_j \to 0$ as $j \to \infty$ and for every $j \in \mathcal{J}$ we have either

(i) $|g(\mathbb{U}_{\epsilon_j})| \leqslant \epsilon_j^2$ or
(ii) $|g(\mathbb{U}_{\epsilon_j})| \leqslant 5|g(\mathbb{U}_{\epsilon_{j+1}})|$, $|g(\mathbb{U}_{\epsilon_j})| > \epsilon_j^2$ and $\epsilon_j = 2\epsilon_{j+1}$.

We simplify the notation a little bit and write $\mathbb{U}_j = \mathbb{U}_{\epsilon_j}$. In both cases we will integrate the inequality (5.2) with respect to the variable t and then bound the right hand side by the following basic estimate.

$$\begin{aligned}&\left(\int_{\mathbb{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{2}{s}} \left(\int_{\mathbb{U}_j} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{2(s-1)}{s}}\\ &\leqslant \begin{cases} C_1(\epsilon_j)\,|\mathbb{U}_j|^{\frac{p-1}{p}}\cdot|g(\mathbb{U}_j)|^{\frac{q-1}{q}} & \text{when } q\,,p<\infty\\ C_2(\epsilon_j)\,|\mathbb{U}_j|\cdot|g(\mathbb{U}_j)|^{\frac{q-1}{q}} & \text{when } p=\infty\\ C_3(\epsilon_j)\,|\mathbb{U}_j|^{\frac{p-1}{p}}\cdot|g(\mathbb{U}_j)| & \text{when } q=\infty\,.\end{cases}\end{aligned} \tag{5.3}$$

Here the functions $C_1(\epsilon_j)$, $C_2(\epsilon_j)$ and $C_3(\epsilon_j)$ converge to 0 as $j\to\infty$.

Proof of (5.3). Since f is a mapping of $\mathscr{L}^s$-distortion and $h(0)=f^{-1}(0)\notin\overline{\mathbb{U}_j}$ applying Theorem 3.2 we have

$$\int_{\mathbb{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x \leqslant \left(\int_{g(\mathbb{U}_j)} K_f^s(x)\,\mathrm{d}x\right)^{\frac12}\cdot\left(\int_{\mathbb{U}_j} K_f^s(x)\,\mathrm{d}x\right)^{\frac12}. \tag{5.4}$$

Especially, Theorem 3.2 tells us that $g\in\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$. Therefore, Lemma 2.1 and Lemma 2.4 give

$$\int_{\mathbb{U}_j} |J_g(x)|\,\mathrm{d}x \leqslant |g(\mathbb{U}_j)|\,. \tag{5.5}$$

This together with Hölder's inequality implies

$$\int_{\mathbb{U}_j} |J_g(x)|^{\frac12}\,\mathrm{d}x \leqslant |\mathbb{U}_j|^{\frac12}|g(\mathbb{U}_j)|^{\frac12}\,. \tag{5.6}$$

Combining (5.4) and (5.6) we conclude that

$$\begin{aligned}&\left(\int_{\mathbb{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{2}{s}} \left(\int_{\mathbb{U}_j} |J_g(x)|^{\frac12}\,\mathrm{d}x\right)^{\frac{2(s-1)}{s}}\\ &\leqslant \left(\int_{g(\mathbb{U}_j)} K_f^s(x)\,\mathrm{d}x\cdot\int_{\mathbb{U}_j} K_f^s(x)\,\mathrm{d}x\right)^{\frac1s}\left(|\mathbb{U}_j|\cdot|g(\mathbb{U}_j)|\right)^{\frac{s-1}{s}}\,.\end{aligned} \tag{5.7}$$

Recall that $1<s=\min\{p,q\}<\infty$. Now the claimed inequality (5.3) follows from the estimate (5.7) after applying Hölder's inequality with

$$\begin{aligned} C_1(\epsilon_j) &\overset{\text{def}}{=\!=} ||K_f||_{\mathscr{L}^p(\mathbb{U}_j)}||K_f||_{\mathscr{L}^q(g(\mathbb{U}_j))}\\ C_2(\epsilon_j) &\overset{\text{def}}{=\!=} ||K_f||_{\mathscr{L}^\infty(\mathbb{U}_j)}||K_f||_{\mathscr{L}^q(g(\mathbb{U}_j))}\\ C_3(\epsilon_j) &\overset{\text{def}}{=\!=} ||K_f||_{\mathscr{L}^p(\mathbb{U}_j)}||K_f||_{\mathscr{L}^\infty(g(\mathbb{U}_j))}\,.\end{aligned} \tag{5.8}$$

□

5.1.1. *Case* (i). Recall that in this case we assume that $|g(\mathbb{U}_j)| \leqslant \epsilon_j^2$. The homeomorphism f is a mapping of $\mathscr{L}^s$-distortion, Lemma 5.2 implies that for almost every $0 < t < 1$ we have

$$\ell(g(\mathbb{I}_t)) \leqslant \left(\int_{\mathbb{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}} \, \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_{\mathbb{I}_t} |J_g(x)|^{\frac{1}{2}} \, \mathrm{d}x \right)^{\frac{s-1}{s}} . \tag{5.9}$$

Since the curve $g(\mathbb{I}_t)$ connects the points (t, t^β) and $(t, -t^\beta)$ staying in $\mathbb{D}_\beta^{\prec}$, the length of $g(\mathbb{I}_t)$ is at least $2t$. Therefore,

$$2t \leqslant \left(\int_{\mathbb{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}} \, \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_{\mathbb{I}_t} |J_g(x)|^{\frac{1}{2}} \, \mathrm{d}x \right)^{\frac{s-1}{s}} . \tag{5.10}$$

Integrating this estimate from 0 to ϵ_j with respect to the variable t and applying Hölder's inequality we obtain

$$\epsilon_j^2 \leqslant \left(\int_{\mathbb{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}} \, \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_{\mathbb{U}_j} |J_g(x)|^{\frac{1}{2}} \, \mathrm{d}x \right)^{\frac{s-1}{s}} . \tag{5.11}$$

After squaring this and applying the basic estimate (5.3) we conclude that

$$\epsilon_j^4 \leqslant \begin{cases} C_1(\epsilon_j) \, |\mathbb{U}_j|^{\frac{p-1}{p}} \cdot |g(\mathbb{U}_j)|^{\frac{q-1}{q}} & \text{when } q \,, p < \infty \\ C_2(\epsilon_j) \, |\mathbb{U}_j| \cdot |g(\mathbb{U}_j)|^{\frac{q-1}{q}} & \text{when } p = \infty \\ C_3(\epsilon_j) \, |\mathbb{U}_j|^{\frac{p-1}{p}} \cdot |g(\mathbb{U}_j)| & \text{when } q = \infty \, . \end{cases}$$

Now, since $|\mathbb{U}_j| = \frac{2\epsilon_j^{\beta+1}}{\beta+1} \leqslant \epsilon_j^{\beta+1}$ and $|g(\mathbb{U}_j)| \leqslant \epsilon_j^2$ we have

$$1 \leqslant \begin{cases} C_1(\epsilon_j) \, \epsilon_j^{\frac{(\beta-\beta_{\mathrm{cr}})(pq-q)}{pq}} & \text{when } q \,, p < \infty \\ C_2(\epsilon_j) \, \epsilon_j^{\beta-\beta_{\mathrm{cr}}} & \text{when } p = \infty \\ C_3(\epsilon_j) \, \epsilon_j^{\frac{(\beta-\beta_{\mathrm{cr}})(p-1)}{p}} & \text{when } q = \infty \, . \end{cases}$$

Note that $C_1(\epsilon_j)$, $C_2(\epsilon_j)$ and $C_3(\epsilon_j)$ converge to 0 as $j \to \infty$. Therefore, $\beta < \beta_{\mathrm{cr}}$, this finishes the proof of Theorem 1.11 in Case (i).

5.1.2. *Case* (ii). As in the previous case applying Lemma 5.2 for almost every $0 < t < 1$ we have

$$\ell(g(\mathbb{I}_t)) \leqslant \left(\int_{\mathbb{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}} \, \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_{\mathbb{I}_t} |J_g(x)|^{\frac{1}{2}} \, \mathrm{d}x \right)^{\frac{s-1}{s}} . \tag{5.12}$$

Now, we first note that $2\,\ell(g(\mathbb{I}_t)) \geqslant \ell\big(\partial g(\mathbb{U}_t)\big)$ and then apply the isoperimetric inequality, Lemma 2.5 we get

$$|g(\mathbb{U}_t)|^{\frac{1}{2}} \leqslant \left(\int_{\mathbb{I}_t} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}} \, \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_{\mathbb{I}_t} |J_g(x)|^{\frac{1}{2}} \, \mathrm{d}x \right)^{\frac{s-1}{s}} . \tag{5.13}$$

Integrating from ϵ_{j+1} to ϵ_j with respect to t we obtain

$$(\epsilon_j - \epsilon_{j+1})|g(\mathbb{U}_{j+1})|^{\frac{1}{2}} \leqslant \left(\int_{\mathbb{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbb{U}_j} |J_g(x)|^{\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{s-1}{s}} .$$

Since by the assumptions of Case (ii), $|g(\mathbb{U}_j)| \leqslant 5|g(\mathbb{U}_{j+1})|$ and $\epsilon_j = 2\epsilon_{j+1}$ we have

$$\epsilon_j|g(\mathbb{U}_j)|^{\frac{1}{2}} \leqslant 10 \left(\int_{\mathbb{U}_j} \frac{|Dg(x)|^s}{|J_g(x)|^{\frac{s-1}{2}}}\,\mathrm{d}x\right)^{\frac{1}{s}} \left(\int_{\mathbb{U}_j} |J_g(x)|^{\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{s-1}{s}} .$$

Combining this with (5.3) we obtain

$$\epsilon_j^2|g(\mathbb{U}_j)| \leqslant 100 \cdot \begin{cases} C_1(\epsilon_j)\,|\mathbb{U}_j|^{\frac{p-1}{p}} \cdot |g(\mathbb{U}_j)|^{\frac{q-1}{q}} & \text{when } q\,, p < \infty \\ C_2(\epsilon_j)\,|\mathbb{U}_j| \cdot |g(\mathbb{U}_j)|^{\frac{q-1}{q}} & \text{when } p = \infty \\ C_3(\epsilon_j)\,|\mathbb{U}_j|^{\frac{p-1}{p}} \cdot |g(\mathbb{U}_j)| & \text{when } q = \infty\,. \end{cases}$$

Therefore,

$$\epsilon_j^2 \leqslant 100 \cdot \begin{cases} C_1(\epsilon_j)\,|\mathbb{U}_j|^{\frac{p-1}{p}} \cdot |g(\mathbb{U}_j)|^{-\frac{1}{q}} & \text{when } q\,, p < \infty \\ C_2(\epsilon_j)\,|\mathbb{U}_j| \cdot |g(\mathbb{U}_j)|^{-\frac{1}{q}} & \text{when } p = \infty \\ C_3(\epsilon_j)\,|\mathbb{U}_j|^{\frac{p-1}{p}} & \text{when } q = \infty\,. \end{cases}$$

This time $|\mathbb{U}_j| = \frac{2\epsilon_j^{\beta+1}}{\beta+1} \leqslant \epsilon_j^{\beta+1}$ and $|g(\mathbb{U}_j)| > \epsilon_j^2$. Therefore,

$$1 \leqslant 100 \cdot \begin{cases} C_1(\epsilon_j)\,\epsilon_j^{\frac{(\beta-\beta_{\mathrm{cr}})(pq-q)}{pq}} & \text{when } q\,, p < \infty \\ C_2(\epsilon_j)\,\epsilon_j^{\beta-\beta_{\mathrm{cr}}} & \text{when } p = \infty \\ C_3(\epsilon_j)\,\epsilon_j^{\frac{(\beta-\beta_{\mathrm{cr}})(p-1)}{p}} & \text{when } q = \infty\,. \end{cases}$$

Therefore $\beta < \beta_{\mathrm{cr}}$. This finishes the proof of nonexistence part of Therorem 1.11.

5.2. **The existence part.** In this section, we construct a homeomorphism of finite distortion $f\colon \mathbb{C} \to \mathbb{C}$ with $f(\mathbb{D}^{\prec}_{\beta}) = \mathbb{D}$ and $K_f \in \mathscr{L}^p(\mathbb{B}_R \setminus \overline{\mathbb{D}^{\prec}_{\beta}}) \cap \mathscr{L}^q(\mathbb{D}^{\prec}_{\beta})$ for every $R > 2$, whenever $1 \leqslant \beta < \beta_{\mathrm{cr}}$. Simplifying the construction we will replace the unit disk $\mathbb{D}$ by $\mathbb{D}^{\prec}_1$. This causes no loss of generality because $\mathbb{D}^{\prec}_1$ is Lipschitz regular. Indeed, for every Lipschitz domain Ω there exists a global bi-Lipschitz change of variables $\Phi\colon \mathbb{C} \to \mathbb{C}$ for which $\Phi(\Omega)$ is the unit disk. Therefore, the domains $\mathbb{D}^{\prec}_1$ and $\mathbb{D}$ are bi-Lipschitz equivalent. Especially, $\mathbb{D}^{\prec}_1$ is a quasidisk. Hence we may also assume the strict inequality $1 < \beta < \beta_{\mathrm{cr}}$ in the construction.

In addition to these we will construct a self-homeomorphism of the unit disk onto itself which coincide with identity on the boundary. Note that this causes no loss of generality since $1 \pm i \in \mathbb{D}_\beta^{\prec}$ and therefore extending the constructed homeomorphism as the identity map to the complement of unit disk. In summary, it suffices to construct a homeomorphism $f \colon \mathbb{D} \xrightarrow{\text{onto}} \mathbb{D}$, $f(z) = z$ on $\partial\mathbb{D}$, $f(\mathbb{D}_\beta^{\prec}) = \mathbb{D}_1^{\prec}$ and $K_f \in \mathscr{L}^p(\mathbb{D} \setminus \overline{\mathbb{D}_\beta^{\prec}}) \cap \mathscr{L}^q(\mathbb{D}_\beta^{\prec})$. We will use the polar coordinates (r, θ) and write $f \colon \mathbb{D} \to \mathbb{D}$ in the form $f(r,\theta) = (\tilde{r}(r), \tilde{\theta}(\theta, r))$. Here $\tilde{r} : [0,1] \xrightarrow{\text{onto}} [0,1]$ is a strictly increasing function defined by

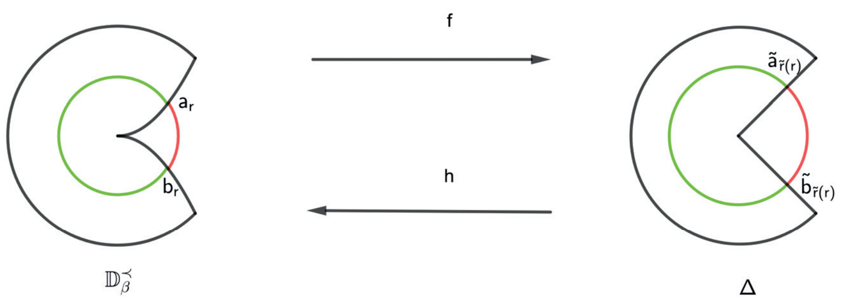
$$\tilde{r}(r) \stackrel{\text{def}}{=\!=} \begin{cases} \frac{e}{\exp\left(\left(\frac{1}{r}\right)^{\gamma_\beta}\right)} & \text{when } q < \infty \\ r & \text{when } q = \infty\,. \end{cases} \tag{5.14}$$

The value γ_β is chosen so that

$$\begin{cases} \max\left\{\frac{\beta(p-1)-(p+1)}{p}, 0\right\} < \gamma_\beta < \frac{2}{q} & \text{when } p < \infty \\ \gamma_\beta = \beta - 1 & \text{when } p = \infty\,. \end{cases} \tag{5.15}$$

For every $0 < r < 1$ we choose $a_r, b_r \in S(0,r) \cap \partial\mathbb{D}_\beta^{\prec}$ such that $\operatorname{Im} a_r > 0$ and $\operatorname{Im} b_r < 0$. Here and what follows we write $S(0,r) = \partial\mathbb{D}(0,r)$. Respectively, we choose $\tilde{a}_{\tilde{r}(r)}, \tilde{b}_{\tilde{r}(r)} \in S(0, \tilde{r}(r)) \cap \partial\mathbb{D}_1^{\prec}$ such that $\operatorname{Im} \tilde{a}_{\tilde{r}(r)} > 0$ and $\operatorname{Im} \tilde{b}_{\tilde{r}(r)} < 0$. We define the argument function $\tilde{\theta}(r, \theta)$ so that it satisfies the following three properties

(1) $f(a_r) = \tilde{a}_{\tilde{r}(r)}$ and $f(b_r) = \tilde{b}_{\tilde{r}(r)}$.
(2) f maps the circular arc $S(0,r) \cap \mathbb{D}_\beta^{\prec}$ onto the circular arc $S(0, \tilde{r}(r)) \cap \mathbb{D}_1^{\prec}$ linearly as a function of θ.
(3) f maps the circular arc $S(0,r) \cap \left(\mathbb{D} \setminus \overline{\mathbb{D}_\beta^{\prec}}\right)$ onto the circular arc $S(0, \tilde{r}(r)) \cap \left(\mathbb{D} \setminus \overline{\mathbb{D}_1^{\prec}}\right)$ linearly as a function of θ.



We have

$$\mathbb{D} \cap \mathbb{D}_\beta^{\prec} = \left\{(r,\theta) \in \mathbb{C} \colon 0 < r < 1 \text{ and } \arctan t^{\beta-1} < \theta < 2\pi - \arctan t^{\beta-1}\right\}$$

and

$\mathbb{D}\setminus\overline{\mathbb{D}^{\prec}_{\beta}} = \left\{(r,\theta)\in\mathbb{C}\colon 0<r<1 \ \text{ and } \ -\arctan t^{\beta-1}<\theta<\arctan t^{\beta-1}\right\}.$

Here $t>0$ and solves the equation $t^2+t^{2\beta}=r^2$. We also have

$$\mathbb{D}\cap\mathbb{D}^{\prec}_1 = \left\{(\tilde r,\tilde\theta)\in\mathbb{C}\colon 0<\tilde r<1 \ \text{ and } \ \frac{\pi}{4}<\tilde\theta<\frac{7\pi}{4}\right\}$$

and

$$\mathbb{D}\setminus\overline{\mathbb{D}^{\prec}_1} = \left\{(\tilde r,\tilde\theta)\in\mathbb{C}\colon 0<\tilde r<1 \ \text{ and } \ \frac{-\pi}{4}<\tilde\theta<\frac{\pi}{4}\right\}.$$

Using the polar coordinates we have

$$\tilde\theta(\theta,r)=\begin{cases}\frac{3\pi\theta}{4\left(\pi-\arctan t^{\beta-1}\right)}+\left(\frac{\pi}{4}-\frac{3\pi\arctan t^{\beta-1}}{4(\pi-\arctan t^{\beta-1})}\right) & \text{when } (r,\theta)\in\mathbb{D}^{\prec}_{\beta}\\ \frac{\pi\theta}{4\arctan t^{\beta-1}} & \text{when } (r,\theta)\in\mathbb{D}\setminus\overline{\mathbb{D}^{\prec}_{\beta}}.\end{cases}$$

For $(r,\theta)\in\mathbb{D}$, the differential matrix of f reads as

$$Df(r,\theta)=\begin{pmatrix}\frac{\partial}{\partial r}\tilde r(r) & 0\\ \tilde r(r)\frac{\partial}{\partial r}\tilde\theta(r,\theta) & \frac{\tilde r(r)}{r}\frac{\partial}{\partial\theta}\tilde\theta(r,\theta)\end{pmatrix}.$$

Computing the derivative of radial part $\tilde r(r)$ we have

$$\frac{\partial}{\partial r}\tilde r(r)=\begin{cases}\gamma_\beta\left(\frac{1}{r}\right)^{\gamma_\beta+1}\tilde r(r) & \text{when } q<\infty\\ 1 & \text{when } q=\infty\,.\end{cases} \tag{5.16}$$

5.2.1. *Proof of $K_f\in\mathscr{L}^q(\mathbb{D}^{\prec}_{\beta})$.* For $(r,\theta)\in\mathbb{D}^{\prec}_{\beta}$, we have

$$\tilde r(r)\frac{\partial}{\partial r}\tilde\theta(r,\theta)=\tilde r(r)\frac{\partial}{\partial r}\left[\frac{3\pi\theta}{4\left(\pi-\arctan t^{\beta-1}\right)}+\left(\pi-\frac{3\pi^2}{4\left(\pi-\arctan t^{\beta-1}\right)}\right)\right]$$

and

$$\frac{\tilde r(r)}{r}\frac{\partial}{\partial\theta}\tilde\theta(r,\theta)=\frac{\tilde r(r)}{r}\frac{3\pi}{4\left(\pi-\arctan t^{\beta-1}\right)}.$$

Since $t>0$ solves the equation $t^2+t^{2\beta}=r^2$, for $0<r<1$, we have $\frac{\partial t}{\partial r}\approx 1$ and $0<\arctan t^{\beta-1}<\frac{\pi}{4}$. Here and what follows the notation $A\approx B$ is a shorter form of two inequalities $A\leqslant cB$ and $B\leqslant cA$ for some positive constant c. Therefore, there exists a constant $C>1$ independent of r and θ, such that

$$|\tilde r(r)\frac{\partial}{\partial r}\tilde\theta(r,\theta)|\leqslant C\cdot\begin{cases}\left(\frac{1}{r}\right)^{\gamma_\beta+1}\tilde r(r) & \text{when } q<\infty\\ 1 & \text{when } q=\infty\,.\end{cases}$$

and

$$\frac{\tilde r(r)}{r}\frac{\partial}{\partial\theta}\tilde\theta(r,\theta)\approx\begin{cases}\frac{\tilde r(r)}{r} & \text{when } q<\infty\\ 1 & \text{when } q=\infty\,.\end{cases}$$

Now, we have

$$K_f(r,\theta) \leqslant C \cdot \begin{cases} r^{-\gamma_\beta} & \text{when } q < \infty \\ 1 & \text{when } q = \infty\,. \end{cases}$$

for some constant $C > 0$.

Since γ_β is chosen so that $0 < \gamma_\beta < \frac{2}{q}$ for $q < \infty$, we have $K_f \in \mathscr{L}^q(\mathbb{D}_\beta^{\prec})$. Also if $q = \infty$, then the distortion function $K_f \in \mathscr{L}^\infty(\mathbb{D}_\beta^{\prec})$, as claimed.

5.2.2. *Proof of $K_f \in \mathscr{L}^p(\mathbb{D} \setminus \overline{\mathbb{D}_\beta^{\prec}})$.* For $(r,\theta) \in \mathbb{D} \setminus \overline{\mathbb{D}_\beta^{\prec}}$, we have

$$\tilde{r}(r)\tfrac{\partial}{\partial r}\tilde{\theta}(r,\theta) = \tilde{r}(r)\tfrac{\partial}{\partial r}\left(\tfrac{\pi\theta}{4\arctan t^{\beta-1}}\right)$$

and

$$\tfrac{\tilde{r}(r)}{r}\tfrac{\partial}{\partial\theta}\tilde{\theta}(r,\theta) = \tfrac{\tilde{r}(r)}{r}\tfrac{\pi}{4\arctan t^{\beta-1}}.$$

Recall that since $t > 0$ solves the equation $t^2 + t^{2\beta} = r^2$, for $0 < r < 1$, we have $\frac{\partial t}{\partial r} \approx 1$. In this case, $-\arctan t^{\beta-1} < \theta < \arctan t^{\beta-1}$, therefore there exists a constant $C > 0$ such that

$$|\tilde{r}(r)\tfrac{\partial}{\partial r}\tilde{\theta}(r,\theta)| \leqslant C\left(\tfrac{1}{r}\right)^{\gamma_\beta+1}\tilde{r}(r).$$

Since

$$\lim_{t\to 0^+}\tfrac{\arctan t^{\beta-1}}{t^{\beta-1}} = 1 \text{ and } t < r < 2t,$$

we have

$$\tfrac{\pi}{4\arctan t^{\beta-1}}\tfrac{\tilde{r}(r)}{r} \approx \tfrac{\tilde{r}(r)}{r^\beta}.$$

Therefore,

$$K_f(r,\theta) \leqslant \frac{C}{r^{|\beta-\gamma_\beta-1|}} \qquad \text{when } (r,\theta) \in \mathbb{D} \setminus \overline{\mathbb{D}_\beta^{\prec}}$$

For $p = \infty$, since $\gamma_\beta = \beta - 1$, we have $K_f \in \mathscr{L}^\infty(\mathbb{D} \setminus \mathbb{D}_\beta^{\prec})$. For $p < \infty$, β is chosen so that $1 < \beta < \beta_{\mathrm{cr}}$. When $q < \infty$, γ_β is chosen so that

$$\max\left\{\frac{\beta(p-1)-(p+1)}{p}, 0\right\} < \gamma_\beta < \frac{2}{q},$$

and when $q = \infty$, γ_β is set to be 0. Since $|\gamma_\beta + 1 - \beta| < \frac{2}{p}$ we have

$$\int_{\mathbb{D}\setminus\overline{\mathbb{D}_\beta^{\prec}}} K_f^p(x)\,\mathrm{d}x \leqslant \int_0^{2\pi}\int_0^1 \frac{1}{r^{p|\beta-\gamma_\beta-1|-1}}\,\mathrm{d}r\,\mathrm{d}\theta < \infty.$$

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Sobolev extension via reflections

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Sobolev extensions via reflections

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Abstract

We show that the extension results by Maz'ya and Poborchi for polynomial cusps can be realized via composition operators generated by reflections. We also study the case of the complementary domains.

1 Introduction

A domain $\Omega \subset \mathbb{R}^n$ is called a (p,q)-extension domain, $1 \le q \le p \le \infty$, if every $u \in W^{1,p}(\Omega)$ has an extension $Eu \in W^{1,q}_{\rm loc}(\mathbb{R}^n)$ with $||Eu||_{W^{1,q}(\mathbb{R}^n\setminus\overline{\Omega})} \le C||u||_{W^{1,p}(\Omega)}$. A Lipschitz domain Ω is a (p,p)-extension domain for all $1 \le p \le \infty$ by results due to Calderón and Stein [30]. Jones generalized this result to a much larger class of domains, so-called (ϵ, δ)-domains, but general domains are not necessarily extension domains for any p, q. For example, in [23, 24, 25], Maz'ya and Poborchi investigated in detail a typical case where the above extension property fails: the case of a domain with an outward peak, also see [22, 28] for related results. Once a polynomial degree of the peak was fixed, they found the optimal p, q for the (p,q)-extendability.

The idea of using reflections to construct extension operators is implicit in the results for Lipschitz domains. Gol'dshtein, Latfullin and Vodop'yanov initiated the systematic use of reflections for constructing extension operators in the Euclidean plane $\mathbb{R}^2$ in [7, 10]. In [8], Gol'dshtein and Sitnikov showed that the Sobolev extendability for planar outward and inward cuspidal domains of polynomial order can be achieved by a bounded linear extension operator induced by reflections. Very recently, Koskela, Pankka and Zhang [21] proved that for every planar Jordan (p,p)-extension domain with $1 < p < \infty$, there exists a reflection over the boundary $\partial\Omega$ which induces a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^2)$.

In this paper, we study the Sobolev extension via reflections on outward cuspidal domains in the Euclidean space $\mathbb{R}^n$ with $n \ge 3$. From now on, we alway assume $n \ge 3$.

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We distinguish a horizontal coordinate axis in $\mathbb{R}^n$,

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{z := (t, x) : t \in \mathbb{R} \text{ and } x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

Let us consider the model case of Ω^s, the outward cuspidal domain with the degree $1 < s < \infty$, defined by setting

$$\Omega^s := \left\{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n : 0 < t \le 1, |x| < t^s\right\} \cup B((2, 0), \sqrt{2}). \tag{1.1}$$

See Figure 1. For the case of this model domain, the results due to Maz'ya and Poborchi state that there exists a bounded linear extension operator E_1 from $W^{1,p}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \le q < \frac{np}{1+(n-1)s}$, and there exists another bounded linear extension operator E_2 from $W^{1,p}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{2+(n-2)s} < p < \infty$ and $1 \le q < \frac{p+(n-1)sp}{1+(n-1)s+(s-1)p}$. For $p = \frac{1+(n-1)s}{n}$, one has

$$\frac{np}{1+(n-1)s} = \frac{p+(n-1)sp}{1+(n-1)s+(s-1)p} = n-1.$$

Hence both E_1 and E_2 extend functions in $W^{1,\frac{1+(n-1)s}{n}}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $1 \le q < n-1$. However, surprisingly, Maz'ya and Poborchi also constructed a bounded linear extension operator E_3 from $W^{1,\frac{(n-1)+(n-1)^2 s}{n}}(\Omega^s)$ to $W^{1,n-1}(\mathbb{R}^n)$. All these results are sharp, see [22] and references therein. For a detailed exposition of these results, see [22]. Interestingly, the given extension operators for the domain Ω^s above are linear and the formulas defining the operators do not depend on p once s and the range of p are fixed. Our main result explains this phenomenon.

Theorem 1.1. *Let $\Omega^s \subset \mathbb{R}^n$ be an outward cuspidal domain with the degree $s > 1$. Then*
(1) : There exists a reflection $\mathcal{R}_1 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$ which induces a bounded linear extension operator from $W^{1,p}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \le q < \frac{np}{1+(n-1)s}$.
(2) : There exists another reflection $\mathcal{R}_2 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$ which induces a bounded linear extension operator from $W^{1,p}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{2+(n-2)s} < p < \infty$ and $1 \le q < \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$.

Theorem 1.1 implies that both reflections $\mathcal{R}_1$ and $\mathcal{R}_2$ induce a bounded linear extension operator from $W^{1,\frac{(n-1)+(n-1)^2 s}{n}}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $1 \le q < n-1$. We would like to know if there exists a further reflection $\mathcal{R}_3$ which induces a bounded linear extension operator from $W^{1,\frac{(n-1)+(n-1)^2 s}{n}}(\Omega^s)$ to $W^{1,n-1}(\mathbb{R}^n)$.

In general, we say that a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega$, for a bounded domain Ω (whose boundary has volume zero) induces a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ if there is an open set U containing $\partial\Omega$ so that, for every $u \in W^{1,p}(\Omega)$, the function v defined by setting $v = u$ on $\Omega \cap U$ and $v = u \circ \mathcal{R}$ on $U \setminus \overline{\Omega}$ has a representative that belongs to $W^{1,q}(U)$ with

$$\|v\|_{W^{1,q}(U)} \leq C\|u\|_{W^{1,p}(U\cap\Omega)}, \tag{1.2}$$

for some positive constant C independent of u. Similarly, we say that the reflection $\mathcal{R}$ induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega})$ to $W^{1,q}(\mathbb{R}^n)$, if for every $u \in W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega})$ the function $\tilde{v}$ defined by setting $\tilde{v} = u$ on $U \setminus \overline{\Omega}$ and $\tilde{v} = u \circ \mathcal{R}$ on $U \cap \Omega$ has a representative that belongs to $W^{1,q}(U)$ with

$$\|\tilde{v}\|_{W^{1,q}(U)} \leq C\|u\|_{W^{1,p}(U\setminus\overline{\Omega})}. \tag{1.3}$$

Here the introduction of the open set U is a convenient way to overcome the non-essential difficulty that functions in $W^{1,p}(G)$ do not necessarily belong to $W^{1,q}(G)$ when $1 \leq q < p < \infty$ and G has infinite volume. It follows from the assumption (1.2) (or (1.3)) via the use of a suitable cut-off function that Ω (or $\mathbb{R}^n \setminus \overline{\Omega}$, respectively) is a $(p, q)-$extension domain with a bounded linear extension operator. For this, see Section 2.

The crucial point behind Theorem 1.1 is that we obtain Sobolev estimates on $u \circ \mathcal{R}$ in terms of the data on u. There is a rather long history of such results, for example see [9, 11, 15, 31] and references therein. In the setting of our problem, the most relevant reference is the paper [31] by Ukhlov. What we find surprising in our situation is that a single $\mathcal{R}_1$ induces the best bounded linear extension operator for all values $\frac{(n-1)+(n-1)^2s}{n} < p < \infty$ and another single $\mathcal{R}_2$ induces the best bounded linear extension operator for all values $\frac{1+(n-1)s}{2+(n-2)s} < p < \frac{(n-1)+(n-1)^2s}{n}$, but neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ can induce a best linear extension operator for $p = \frac{(n-1)+(n-1)^2s}{n}$. In the case of compositions from $W^{1,p}$ to $W^{1,p}$, the relevant estimate is

$$|D\mathcal{R}(z)|^p \leq C|J_{\mathcal{R}}(z)| \tag{1.4}$$

almost everywhere, which for $p = n$ is the pointwise condition of quasiconformality. Mappings satisfying (1.4) with $p \neq n$ apparently appeared for the first time in the works of Gehring [6] and of Maz'ya [26], independently. With some work one can show that (1.4) implies the corresponding inequality with p replaced by q when either $q > p > n$ or $1 \leq q < p < n$, but not in other cases. On the other hand, for $n - 1 < p < \infty$, a result in [8] shows that (1.4) together with $W^{1,p}$-regularity of $\mathcal{R}$ implies the dual estimate

$$|D\mathcal{R}^{-1}(z)|^{\frac{p}{p+1-n}} \leq C'|J_{\mathcal{R}^{-1}}(z)|. \tag{1.5}$$

This kind of duality actually also holds for compositions from $W^{1,p}$ to $W^{1,q}$ with $q < p$, see [31]. Also see [16, 31, 34] for general results on the regularity of $\mathcal{R}^{-1}$.

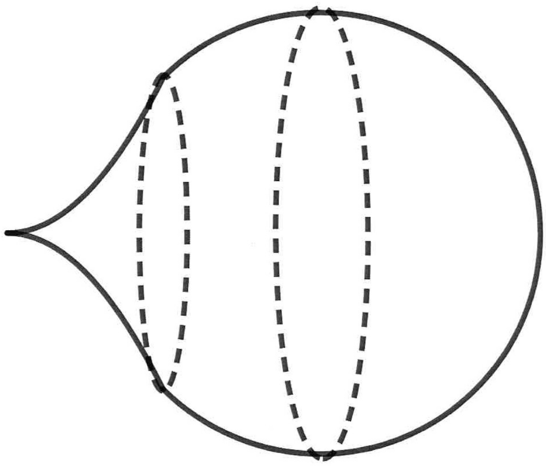

Figure 1: Ω^s

From the argument above, one could also expect that the reflections $\mathcal{R}_1$ and $\mathcal{R}_2$ induce a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega^s})$ to $W^{1,q}(\mathbb{R}^n)$, for some $1 \le q \le p < \infty$. As one can easily check, for every $1 < s < \infty$, $\mathbb{R}^n \setminus \overline{\Omega^s}$ is a so-called (ϵ, δ)-domain and hence a (p,p)-extension domain for every $1 \le p < \infty$ due to Jones [19]. Our next theorem relates this to our reflections.

Theorem 1.2. *For every $1 < s < \infty$, $\mathbb{R}^n \setminus \overline{\Omega^s}$ is a (p,p)-extension domain, for every $1 \le p < \infty$. The reflection $\mathcal{R}_1$ over $\partial\Omega^s$ in Theorem 1.1 induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$, whenever $1 \le p \le n-1$. Moreover, for each $n - 1 < p < \infty$, no reflection over $\partial\Omega^s$ can induce a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$.*

What then about the case $p = \infty$? We say that a domain $\Omega \subset \mathbb{R}^n$ is uniformly locally quasiconvex if there exist constants $C > 0$ and $R > 0$ such that for every pair of points $x, y \in \Omega$ with $d(x,y) < R$, there is a rectifiable curve γ connecting x and y in Ω such that the length of γ is bounded from above by $Cd(x,y)$. If the above holds without the distance restriction, Ω is said to be quasiconvex. Recall that Ω is an (∞,∞)-extension domain if and only if it is uniformly locally quasiconvex, see [12] by Hajłasz, Koskela and Tuominen. One can easily check that both Ω^s and $\mathbb{R}^n \setminus \overline{\Omega^s}$ are uniformly locally quasiconvex, equivalently, they are (∞,∞)-extension domains. We close this introduction with the following analog of Theorem 1.2.

Theorem 1.3. *Given $1 < s < \infty$, both Ω^s and $\mathbb{R}^n \setminus \overline{\Omega^s}$ are (∞,∞)-extension domains. The reflection $\mathcal{R}_1$ over $\partial\Omega^s$ in Theorem 1.1 induces a bounded linear extension operator from $W^{1,\infty}(\Omega^s)$ to $W^{1,\infty}(\mathbb{R}^n)$. On the other hand, no reflection over $\partial\Omega^s$ can induce a bounded linear extension operator from $W^{1,\infty}(\mathbb{R}^n \setminus \overline{\Omega^s})$ to $W^{1,\infty}(\mathbb{R}^n)$.*

2 Preliminaries

In this paper, $\widehat{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^n$. Next, $z = (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$ means a point in the n-dimensional Euclidean space $\mathbb{R}^n$. We write $C = C(a_1, a_2, ..., a_n)$ to indicate a constant C that depends only on the parameters $a_1, a_2, ..., a_n$; the notation $A \lesssim B$ means there exists a finite constant c with $A \leq cB$, and $A \sim_c B$ means $\frac{1}{c}A \leq B \leq cA$ for a constant $c > 1$. Typically $c, C, ...$ will be constants that depend on various parameters and may differ even on the same line of inequalities. The Euclidean distance between given points z_1, z_2 in Euclidean space $\mathbb{R}^n$ is denoted by $d(z_1, z_2)$ or $|z_1 - z_2|$. Then the distance between two sets $A, B \subset \mathbb{R}^n$ is denoted by

$$d(A, B) := \inf\{d(z_1, z_2) : z_1 \in A, z_2 \in B\}.$$

The open ball of radius r centered at the point z is denoted by $B(z, r)$. In what follows, $\Omega \subset \mathbb{R}^n$ is always a domain, and $\partial\Omega$ is the boundary of Ω. The r-neighborhood of Ω is

$$B(\Omega, r) := \{z \in \mathbb{R}^n : d(z, \Omega) < r\}.$$

Given a Lebesgue measurable set $A \subset \mathbb{R}^n$, $|A|$ refers to the n-dimensional Lebesgue measure. The interior of a set $A \subset \mathbb{R}^n$ is denoted by $\mathring{A}$. For a locally integrable function u and a measurable set $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$, we define the integral average of u over A by setting

$$\fint_A u(z)dz := \frac{1}{|A|} \int_A u(z)dz.$$

The Sobolev space $W^{1,p}(\Omega)$ for $p \in [1, \infty]$ is the collection of all functions $u \in L^p(\Omega)$ whose norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \||Du|\|_{L^p(\Omega)}$$

is finite. Here $Du = (g_1, g_2, ..., g_n)$ is the *distributional gradient* of u, where g_i is the weak partial derivative of u with respect to x_i. A mapping $f = (f_1, f_2, \cdots, f_m) : \Omega \to \Omega'$ is said to be in the class $W^{1,p}(\Omega, \Omega')$, if every component f_i is in the Sobolev space $W^{1,p}(\Omega)$.

The outward cuspidal domain Ω^s has a boundary singularity but it is still rather nice. For example, both the outward cuspidal domain Ω^s and its complement $\mathbb{R}^n \setminus \overline{\Omega^s}$ satisfy the segment condition.

Definition 2.1. *We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the segment condition if every $x \in \partial\Omega$ has a neighborhood U_x and a nonzero vector y_x such that if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$.*

For a domain satisfying the segment condition, we have the following lemma. See [1, Theorem 3.22].

Lemma 2.1. *If the domain $\Omega \subset \mathbb{R}^n$ satisfies the segment condition, then the set of restrictions to Ω of functions in $C_o^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$. In short, $C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$.*

Let us give the definition of Sobolev extension domains.

Definition 2.2. *Let $1 \leq q \leq p \leq \infty$. We say that a domain $\Omega \subset \mathbb{R}^n$ is a (p,q)-extension domain, if for every $u \in W^{1,p}(\Omega)$, there exists a function $Eu \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ with $Eu\big|_\Omega \equiv u$ and*

$$\|Eu\|_{W^{1,q}(\mathbb{R}^n \setminus \overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}$$

with a constant C independent of u.

Lipschitz domains are typical examples of Sobolev extension domains. By the results due to Calderón and Stein [30], Lipschitz domains are (p,p)-extension domains for $1 \leq p \leq \infty$. For the definition of Lipschitz domains, please see [4, Definition 4.4]. As a generalization of the extension result for Lipschitz domains, Jones [19] proved that (ϵ, δ)-domains are also (p,p)-extension domains.

Definition 2.3. *We say $\Omega \subset \mathbb{R}^n$ is an (ϵ,δ)-domain for some positive constant $0 < \epsilon < 1$ and $\delta > 0$ if whenever $z_1, z_2 \in \Omega$ with $|z_1 - z_2| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ joining x to y and satisfying*

$$l(\gamma) \leq \frac{1}{\epsilon}|z_1 - z_2|$$

and

$$d(z, \Omega^c) \geq \frac{\epsilon|z_1 - z||z_2 - z|}{|z_1 - z_2|} \text{ for all } z \text{ on} \gamma.$$

Definition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain. A self-homeomorphism $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ is called a reflection over $\partial\Omega$, if $\mathcal{R}(\widehat{\mathbb{R}^n} \setminus \overline{\Omega}) = \Omega$, $\mathcal{R}(\Omega) = \widehat{\mathbb{R}^n} \setminus \overline{\Omega}$ and for every $z \in \partial\Omega$, $\mathcal{R}(z) = z$.*

The following technical lemma justifies our terminology.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $|\partial\Omega| = 0$ and $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ be a reflection over $\partial\Omega$. If $\mathcal{R}$ induces a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ in the sense of (1.2) (from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega})$ to $W^{1,q}(\mathbb{R}^n)$, respectively) for $1 \leq q \leq p < \infty$, then Ω ($\mathbb{R}^n \setminus \overline{\Omega}$, respectively) is a (p,q)-extension domain with a linear extension operator.*

Proof. We only consider the case of Ω, since the case of $\mathbb{R}^n \setminus \overline{\Omega}$ is analogous. Let $U \subset \mathbb{R}^n$ be the corresponding open set which contains $\partial\Omega$. For a given function $u \in W^{1,p}(\Omega)$, we define a function $E_{\mathcal{R}}(u)$ by setting

$$E_{\mathcal{R}}(u)(z) := \begin{cases} u(\mathcal{R}(z)), & \text{for } z \in U \setminus \overline{\Omega}, \\ 0, & \text{for } z \in \partial\Omega, \\ u(z), & \text{for } z \in \Omega. \end{cases} \tag{2.1}$$

Then $E_{\mathcal{R}}(u)$ has a representative that belongs to $W^{1,q}(U)$ with

$$\|E_{\mathcal{R}}(u)\|_{W^{1,q}(U)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function such that $\psi\big|_{\overline{\Omega}} \equiv 1$, $\psi\big|_{\mathbb{R}^n \setminus U} \equiv 0$ and $0 \leq \psi(z) \leq 1$ for every $z \in \mathbb{R}^n$. For every function $u \in W^{1,p}(\Omega)$, we define a function on $\mathbb{R}^n$ by setting

$$\tilde{E}_{\mathcal{R}}(u) := \psi \cdot E_{\mathcal{R}}(u). \tag{2.2}$$

Since ψ is Lipschitz with $0 \leq \psi \leq 1$, $\tilde{E}_{\mathcal{R}}(u)$ has a representative that belongs to $W^{1,p}(\mathbb{R}^n)$. Now

$$\begin{aligned}\int_{\mathbb{R}^n} |\tilde{E}_{\mathcal{R}}(u)(z)|^q dz &\leq \int_{\Omega} |u(z)|^q dz + \int_U |E_{\mathcal{R}}(u)(z)|^q dz \\ &\leq \left(\int_{\Omega} |u(z)|^p dz + \int_{\Omega} |Du(z)|^p dz\right)^{\frac{q}{p}},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} |D\tilde{E}_{\mathcal{R}}(u)|^q dz &\leq C\int_U |E\mathcal{R}(u)D\psi|^q dz + C\int_U |\psi\nabla E_{\mathcal{R}}(u)|^q dz \\ &\quad + C\int_{\Omega} |Du|^q dz \\ &\leq C\left(\int_{\Omega} |u|^p dz + \int_{\Omega} |\nabla u|^p dz\right)^{\frac{q}{p}}.\end{aligned}$$

By combining these two inequalities, we obtain that $\tilde{E}_{\mathcal{R}}(u) \in W^{1,q}(\mathbb{R}^n)$ with $\tilde{E}_{\mathcal{R}}(u)\big|_{\Omega} \equiv u$ and

$$\|\tilde{E}_{\mathcal{R}}(u)\|_{W^{1,q}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Hence, we defined a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ in (2.2). □

By Proposition 2.1, in order to prove that a reflection $\mathcal{R}$ over $\partial\Omega^s$ can induce a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ for some $1 \leq q \leq p \leq \infty$, it suffices to prove that for every $u \in W^{1,p}(\Omega)$, the function $E_{\mathcal{R}}(u)$ defined in (2.1) satisfies the inequality

$$\|E_{\mathcal{R}}(u)\|_{W^{1,q}(U)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

with a constant C independent of u.

Let $f : \Omega \to \Omega'$ be a homeomorphism. If for every $z \in U$ there is an open set containing z and a constant $C > 1$ such that for every $x, y \in U$, we have

$$\frac{1}{C}|x-y| \leq |f(x) - f(y)| \leq C|x-y|,$$

we call it a locally bi-Lipschitz homeomorphism.

By combining results in [31, 32, 33, 35], we obtain following two lemmas.

Lemma 2.2. *Suppose that $f:\Omega\to\Omega'$ is a homeomorphism in the class $W^{1,1}_{\mathrm{loc}}(\Omega,\Omega')$. Fix $1\le p<\infty$. Then the following assertions are equivalent:*
(1) : *for every locally Lipschitz function u, the inequality*

$$\int_\Omega |D(u\circ f)(z)|^p dz \le C\int_{\Omega'} |Du(z)|^p dz$$

holds for a positive constant C independent of u;
(2) : *the inequality*

$$|Df(z)|^p \le C(p)|J_f(z)|$$

holds almost everywhere in Ω.

Lemma 2.3. *Let $1\le q<p<\infty$. Suppose that $f:\Omega\to\Omega'$ is a homeomorphism in the class $W^{1,1}_{\mathrm{loc}}(\Omega,\Omega')$. Then the following assertions are equivalent:*
(1) : *for every locally Lipschitz function u, the inequality*

$$\left(\int_\Omega |Du\circ f(z)|^q dz\right)^{\frac{1}{q}} \le C\left(\int_{\Omega'} |Du(z)|^p dz\right)^{\frac{1}{p}}$$

holds for a positive constant C independent of u;
(2) :

$$\int_\Omega \frac{|Df(z)|^{\frac{pq}{p-q}}}{|J_f(z)|^{\frac{q}{p-q}}} dz<\infty.$$

The following lemma is a special case of [34, Theorem 3].

Lemma 2.4. *Let $\Omega,\Omega'\subset\mathbb{R}^n$ be domains, and let $f:\Omega\to\Omega'$ be a homeomorphism in the class $W^{1,p}_{\mathrm{loc}}(\Omega,\Omega')$ for a fixed $n-1<p<\infty$. If*

$$|Df(z)|^p \le C(p)|J_f(z)| \tag{2.3}$$

holds for almost every $z\in\Omega$, then the inverse homeomorphism $f^{-1}:\Omega'\to\Omega$ belongs to the class $W^{1,\frac{p}{p+1-n}}_{\mathrm{loc}}(\Omega',\Omega)$ with

$$|Df^{-1}(z)|^{\frac{p}{p+1-n}} \le C(p)|J_{f^{-1}}(z)| \tag{2.4}$$

for almost every $z\in\Omega'$.

3 Main Results

In this section, we show that the Sobolev extension results for outward cuspidal domain $\Omega^s\subset\mathbb{R}^n$ from [23, 24, 25] can be achieved by bounded linear extension operators induced by reflections, except possibly for the case from $W^{1,\frac{1+(n-1)s}{n}}(\Omega^s)$ to $W^{1,n-1}(\mathbb{R}^n)$. Let us begin by introducing two reflections.

3.1 Reflection $\mathcal{R}_1$ over $\partial\Omega^s$

In order to introduce the reflection $\mathcal{R}_1 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$, we define a domain $\Delta \subset \mathbb{R}^n$ by setting

$$\Delta := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^{n-1}; \frac{-1}{2} < t < \frac{1}{2}, |x| < \frac{1}{2}\right\} \cup \Omega^s. \tag{3.1}$$

See Figure 2. To begin, we divide $\Delta \setminus \overline{\Omega^s}$ into three parts A, B, C by setting

$$A := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^{n-1}; \tfrac{-1}{2} < t \leq 0, |x| \leq |t|\right\},$$

$$B := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^{n-1}; \tfrac{-1}{2} < t < \tfrac{1}{2}, |t| \leq |x| < \tfrac{1}{2}\right\}$$

and

$$C := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^{n-1}; 0 \leq t < \tfrac{1}{2}, t^s \leq |x| \leq t\right\}.$$

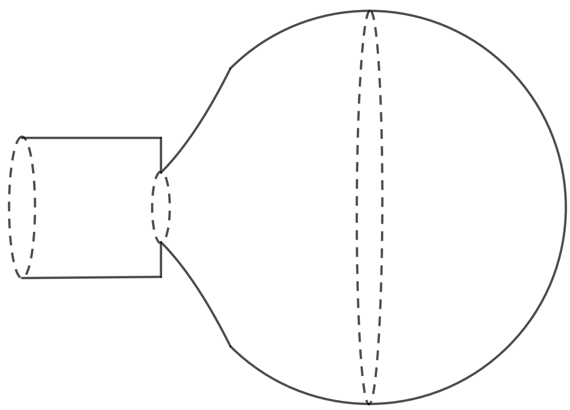

Figure 2: The domain Δ

We define a subdomain $\Omega_1^s \subset \Omega^s$ by setting

$$\Omega_1^s := \left\{(t,x) \in \Omega^s; 0 < t < \frac{1}{2}, |x| < t^s\right\}. \tag{3.2}$$

We will construct a reflection $\mathcal{R}_1$ which maps $\Delta \setminus \overline{\Omega^s}$ onto Ω_1^s. We define $\mathcal{R}_1$ on $\Delta \setminus \overline{\Omega^s}$ by setting

$$\mathcal{R}_1(t,x) := \begin{cases} \left(-t, \frac{1}{6}|t|^{s-1}x\right), & \text{if } (t,x) \in A, \\ \left(|x|, \frac{t}{6}|x|^{s-2}x + \frac{1}{3}|x|^{s-1}x\right), & \text{if } (t,x) \in B, \\ \left(t, \frac{t^{s-1}}{2(t^{s-1}-1)}x + \left(t^s - \frac{t^{2s-1}}{2(t^{s-1}-1)}\right)\frac{x}{|x|}\right), & \text{if } (t,x) \in C. \end{cases} \tag{3.3}$$

We extend $\mathcal{R}_1$ to $\partial\Omega^s$ as the identity. Since both $\partial\Delta$ and $\partial(\Omega^s \setminus \Omega_1^s)$ are bi-Lipschitz equivalent to the unit sphere, it is easy to check that we can construct a reflection $\mathcal{R}_1 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$ such that $\mathcal{R}_1$ is defined as above on $\Delta \setminus \Omega^s$, and $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega^s, 1) \setminus \Delta$.

For $(t,x) \in \mathring{A}$, the resulting differential matrix of $\mathcal{R}_1$ is

$$D\mathcal{R}_1(t,x) = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ \frac{1-s}{6}|t|^{s-2}x_1 & \frac{1}{6}|t|^{s-1} & 0 & \cdots & 0 \\ \frac{1-s}{6}|t|^{s-2}x_2 & 0 & \frac{1}{6}|t|^{s-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1-s}{6}|t|^{s-2}x_{n-1} & 0 & \cdots & 0 & \frac{1}{6}|t|^{s-1} \end{pmatrix}. \tag{3.4}$$

Hence, for every $(t,x) \in \mathring{A}$, we have

$$|D\mathcal{R}_1(t,x)| \lesssim 1 \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c |t|^{(n-1)(s-1)}. \tag{3.5}$$

For $(t,x) \in \mathring{B}$, the resulting differential matrix $\mathcal{R}_1$ is

$$D\mathcal{R}_1(t,x) = \begin{pmatrix} 0 & \frac{x_1}{|x|} & \frac{x_2}{|x|} & \cdots & \frac{x_{n-1}}{|x|} \\ \frac{x_1}{6}|x|^{s-2} & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ \frac{x_2}{6}|x|^{s-2} & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{x_{n-1}}{6}|x|^{s-2} & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix}, \tag{3.6}$$

where, for every $i, j \in \{1, 2, \cdots, n-1\}$, we set

$$A_j^i(t,x) := \begin{cases} \left(\frac{t}{6}|x|^{s-2} + \frac{1}{3}|x|^{s-1}\right) + \left(\frac{t}{6}(s-2)\frac{x_i^2}{|x|^{4-s}} + \frac{s-1}{3}\frac{x_i^2}{|x|^{3-s}}\right), & \text{if } i = j, \\ \frac{t}{6}(s-2)\frac{x_i x_j}{|x|^{4-s}} + \frac{s-1}{3}\frac{x_i x_j}{|x|^{3-s}}, & \text{if } i \neq j. \end{cases}$$

Since $|t| \leq |x| < \frac{1}{2}$, a simple computation gives

$$|D\mathcal{R}_1(t,x)| \lesssim 1 \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c \sum_{k=1}^{n-1} \frac{x_k^2}{6}|x|^{s-3} \prod_{i \neq k} A_1^i \sim_c |x|^{(n-1)(s-1)}. \tag{3.7}$$

For $(t,x) \in \mathring{C}$, the resulting differential matrix of $\mathcal{R}_1$ is

$$
(3.8) \quad D\mathcal{R}_1(t,x) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ A_t^1(t,x) & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ A_t^2(t,x) & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_t^{n-1}(t,x) & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix},
$$

where, for every $i,j \in \{1,2,\cdots,n-1\}$, we have

$$
A_j^i(t,x) := \begin{cases} \frac{t^{s-1}}{2(t^{s-1}-1)} + \left(t^s - \frac{t^{2s-1}}{2(t^{s-1}-1)}\right)\left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right), & \text{if} \quad i=j, \\ \left(\frac{t^{2s-1}}{2(t^{s-1}-1)} - t^s\right)\frac{x_i x_j}{|x|^3}, & \text{if} \quad i \neq j. \end{cases}
$$

and

$$
\begin{aligned} A_t^i(t,x) := & \, x_i \left(\frac{(s-1)t^{s-2}}{2(t^{s-1}-1)} - \frac{(s-1)t^{2s-3}}{2(t^{s-1}-1)^2} \right) \\ & + \frac{x_i}{|x|}\left(st^{s-1} - \frac{(2s-1)t^{2s-2}}{2(t^{s-1}-1)} + \frac{(s-1)t^{3s-3}}{2(t^{s-1}-1)^2} \right). \end{aligned}
$$

Since $t^s < |x| < t$, a simple computation gives

$$
(3.9) \qquad |D\mathcal{R}_1(t,x)| \lesssim 1 \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c t^{(n-1)(s-1)}.
$$

Finally, since $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega^s,1) \setminus \overline{\Delta}$, there exists a positive constant $C > 1$ such that for almost every $(t,x) \in B(\Omega^s,1) \setminus \overline{\Delta}$, we have

$$
\frac{1}{C} \le |D\mathcal{R}_1(t,x)| \le C \text{ and } \frac{1}{C} \le |J_{\mathcal{R}_1}(t,x)| \le C.
$$

It is easy to see that the restriction of $\mathcal{R}_1$ to $B(\Omega^s,1) \setminus (\Omega^s \cup \{0\})$ is locally bi-Lipschitz.

3.2 Reflection $\mathcal{R}_2$ over $\partial\Omega^s$

In order to introduce the reflection $\mathcal{R}_2 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$, we define a domain $\Delta' \subset \mathbb{R}^n$ by setting

$$
(3.10) \qquad \Delta' := \left\{ (t,x) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n : \frac{-1}{2} < t < \frac{1}{2}, |x| < \left(\frac{1}{2}\right)^s \right\} \cup \Omega^s.
$$

See Figure 3. We divide $\Delta' \setminus \overline{\Omega^s}$ into two parts D, E by setting

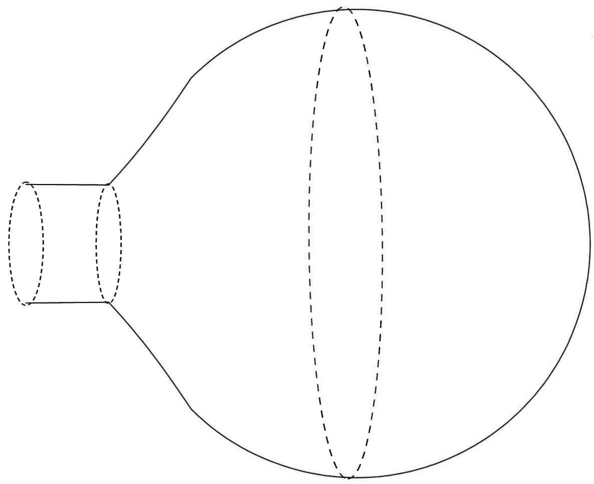

Figure 3: The domain Δ'

$$D := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^{n-1} = \mathbb{R}^n : \frac{-1}{2} < t \leq 0, |x| \leq |t|^s\right\},$$

and

$$E := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^{n-1} = \mathbb{R}^n : \frac{-1}{2} < t < \frac{1}{2}, |t|^s < |x| < \left(\frac{1}{2}\right)^s\right\}.$$

We will construct a reflection $\mathcal{R}_2$ over $\partial\Omega^s$ which maps $\Delta' \setminus \overline{\Omega^s}$ onto Ω^s_1. We define the reflection $\mathcal{R}_2$ on $\Delta' \setminus \overline{\Omega^s}$ by setting

$$\mathcal{R}_2(t,x) := \begin{cases} \left(-t, \frac{1}{2}x\right), & \text{if } (t,x) \in D, \\ \left(|x|^{\frac{1}{s}}, \frac{t}{4}\frac{x}{|x|^{\frac{1}{s}}} + \frac{3}{4}x\right), & \text{if } (t,x) \in E. \end{cases} \tag{3.11}$$

We extend $\mathcal{R}_2$ on $\partial\Omega^s$ as the identity. Since both $\partial\Delta'$ and $\partial(\Omega^s \setminus \Omega^s_1)$ are bi-Lipschitz equivalent to the unit sphere, we can construct a reflection $\mathcal{R}_2$ which is defined on $\Delta' \setminus \overline{\Omega^s}$ as in (3.11) and is bi-Lipschitz on $B(\Omega^s, 1) \setminus \Delta'$.

For $z = (t,x) \in \mathring{D}$, the resulting differential matrix of $\mathcal{R}_2$ is

$$D\mathcal{R}_2(t,x) = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} \end{pmatrix}. \tag{3.12}$$

Hence,

$$|D\mathcal{R}_2|(t,x) = 1 \text{ and } |J_{\mathcal{R}_2}(t,x)| = \frac{1}{2^{n-1}}. \tag{3.13}$$

For $z = (t,x) \in \mathring{E}$, the resulting matrix of $\mathcal{R}_2$ is

$$D\mathcal{R}_2(t,x) = \begin{pmatrix} 0 & \frac{x_1}{s|x|^{2-\frac{1}{s}}} & \frac{x_2}{s|x|^{2-\frac{1}{s}}} & \cdots & \frac{x_{n-1}}{s|x|^{2-\frac{1}{s}}} \\ \frac{x_1}{4|x|^{\frac{1}{s}}} & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ \frac{x_2}{4|x|^{\frac{1}{s}}} & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{x_{n-1}}{4|x|^{\frac{1}{s}}} & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix}, \tag{3.14}$$

where, for every $i,j \in \{1,2,\cdots,n-1\}$, we have

$$A_j^i(t,x) := \begin{cases} \frac{t}{4}\left(\frac{1}{|x|^{\frac{1}{s}}} - \frac{x_i^2}{s|x|^{2+\frac{1}{s}}}\right) + \frac{3}{4}, & \text{if } i = j, \\ \frac{-tx_ix_j}{4s|x|^{2+\frac{1}{s}}}, & \text{if } i \neq j. \end{cases} \tag{3.15}$$

After a simple computation, for every $(t,x) \in \mathring{E}$ we have

$$|D\mathcal{R}_2(t,x)| \lesssim \frac{1}{|x|^{\frac{s-1}{s}}} \text{ and } |J_{\mathcal{R}_2}(t,x)| \sim_c \sum_{k=1}^{n-1} \frac{x_k^2}{4s|x|^2} \prod_{i\neq k} A_i^i = \sim_c 1. \tag{3.16}$$

Finally, since $\mathcal{R}_2$ is bi-Lipschitz on $B(\Omega^s,1) \setminus \overline{\Delta'}$, there exists a positive constant $C > 1$ such that for almost every $(t,x) \in B(\Omega^s,1) \setminus \overline{\Delta'}$, we have

$$\frac{1}{C} \leq |D\mathcal{R}_2(t,x)| \leq C \text{ and } \frac{1}{C} \leq |J_{\mathcal{R}_2}(t,x)| \leq C. \tag{3.17}$$

It is easy to see that the restriction of $\mathcal{R}_2$ to $B(\Omega^s,1) \setminus (\Omega^s \cup \{0\})$ is locally bi-Lipschitz.

3.3 Proof of Theorem 1.1

We prove Theorem 1.1 in two parts, considering the two reflections separately.

Theorem 3.1. *Let $\Omega^s \subset \mathbb{R}^n$ be an outward cuspidal domain with the degree $s > 1$. Then the reflection $\mathcal{R}_1 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$ induces a bounded linear extension operator from $W^{1,p}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \leq q < \frac{np}{1+(n-1)s}$.*

Proof. Since Ω^s satisfies the segment condition, by Lemma 2.1, $C_o^\infty(\mathbb{R}^n)\cap W^{1,p}(\Omega^s)$ is dense in $W^{1,p}(\Omega^s)$. Let $u\in C_o^\infty(\mathbb{R}^n)\cap W^{1,p}(\Omega^s)$ be arbitrary. We define a function $E_{\mathcal{R}_1}(u)$ as in (2.1) and another function w by setting

$$w(z) := \begin{cases} u\circ\mathcal{R}_1(z), & \text{if } z\in B(\Omega^s,1)\setminus\overline{\Omega^s},\\ u(z), & \text{if } z\in\overline{\Omega^s}. \end{cases} \tag{3.18}$$

Since $u\in C_o^\infty(\mathbb{R}^n)\cap W^{1,p}(\Omega^s)$ and $\mathcal{R}_1$ is locally Lipschitz on $B(\Omega^s,1)\setminus(\Omega^s\cup\{0\})$, the function w is locally Lipchitz on $B(\Omega^s,1)\setminus\{0\}$. We claim that $w\in W^{1,q}(B(\Omega^s,1))$ with

$$\|w\|_{W^{1,q}(B(\Omega^s,1))}\le C\|u\|_{W^{1,p}(\Omega^s)}$$

for a constant $C>1$ independent of u. These claims follow if we prove the above norm estimate with $B(\Omega^s,1)$ replaced by $B(\Omega^s,1)\setminus\{0\}$. Next, since w is locally Lipschitz and $|\partial\Omega^s|=0$, it suffices to estimate the norm over the union of Ω^s and $B(\Omega^s,1)\setminus\overline{\Omega^s}$. Since $w=u\in W^{1,p}(\Omega^s)$ on Ω^s, our domain Ω^s has finite measure and $q<p$, we are reduced to estimating the norm over the second set in question. On this set, $w=u\circ\mathcal{R}_1$ almost everywhere and hence it suffices to prove the inequality

$$\left(\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}}|u\circ\mathcal{R}_1(z)|^q dz\right)^{\frac{1}{q}}\le C\left(\int_{\Omega^s}|u(z)|^p dz\right)^{\frac{1}{p}} \tag{3.19}$$

and the inequality

$$\left(\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}}|D(u\circ\mathcal{R}_1)(z)|^q dz\right)^{\frac{1}{q}}\le C\left(\int_{\Omega^s}|Du(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.20}$$

It is easy to see that

$$B(\Omega^s,1)\setminus\overline{\Omega^s}=(B(\Omega^s,1)\setminus\Delta)\cup(\Delta\setminus\overline{\Omega^s})$$

and $\Delta\setminus\overline{\Omega^s}=A\cup B\cup C$. Since

$$|\partial\Delta|=|\partial A|=|\partial B|=|\partial C|=0,$$

we have

$$\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}}|u\circ\mathcal{R}_1(z)|^q dz=\int_{B(\Omega^s,1)\setminus\overline{\Delta}}|u\circ\mathcal{R}_1(z)|^q dz + \left(\int_{\mathring{A}}+\int_{\mathring{B}}+\int_{\mathring{C}}\right)|u\circ\mathcal{R}_1(z)|^q dz. \tag{3.21}$$

Since $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega^s,1)\setminus\overline{\Delta}$ and $|\Omega^s|<\infty$, by the Hölder inequality, we have

$$\int_{B(\Omega^s,1)\setminus\overline{\Delta}}|u\circ\mathcal{R}_1(z)|^q dz\le C\left(\int_{\Omega^s}|u(z)|^p dz\right)^{\frac{q}{p}}. \tag{3.22}$$

By the Hölder inequality and a change of variable, we have

$$\int_{\mathring{A}} |u \circ \mathcal{R}_1(z)|^q dz \le \left(\int_{\mathring{A}} |u \circ \mathcal{R}_1(z)|^p |J_{\mathcal{R}_1}(z)| dz \right)^{\frac{1}{p}} \cdot \left(\int_{\mathring{A}} \frac{1}{|J_{\mathcal{R}_1}^{\frac{q}{p-q}}(z)|} dz \right)^{\frac{p-q}{p}}$$

$$\le \left(\int_{\Omega^s} |u(z)|^p dz \right)^{\frac{q}{p}} \cdot \left(\int_{\mathring{A}} \frac{1}{|J_{\mathcal{R}_1}^{\frac{q}{p-q}}(z)|} dz \right)^{\frac{p-q}{p}}. \tag{3.23}$$

By (3.5), we have

$$\int_{\mathring{A}} \frac{1}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \le C \int_0^{\frac{1}{2}} t^{(n-1) - \frac{(n-1)(s-1)q}{p-q}} dt < \infty,$$

whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \le q < \frac{np}{1+(n-1)s}$. Hence, we have

$$\int_{\mathring{A}} |u \circ \mathcal{R}_1(z)|^q dz \le C \left(\int_{\Omega^s} |u(z)|^p dz \right)^{\frac{q}{p}}. \tag{3.24}$$

Next, via (3.7) and (3.9), we obtain the estimates

$$\int_{\mathring{B}} \frac{1}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \le C \int_0^{\frac{1}{2}} |x|^{(n-1) - \frac{(n-1)(s-1)q}{p-q}} d|x| < \infty$$

and

$$\int_{\mathring{C}} \frac{1}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \le C \int_0^{\frac{1}{2}} x_1^{(n-1) - \frac{(n-1)(s-1)q}{p-q}} dx_1 < \infty,$$

whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \le q < \frac{np}{1+(n-1)s}$. By repeating the argument leading to (3.24), we obtain the following desired analogs of (3.24):

$$\int_{\mathring{B}} |u \circ \mathcal{R}_1(z)|^q dz \le C \left(\int_{\Omega^s} |u(z)|^p dz \right)^{\frac{q}{p}} \tag{3.25}$$

and

$$\int_{\mathring{C}} |u \circ \mathcal{R}_1(z)|^q dz \le C \left(\int_{\Omega^s} |u(z)|^p dz \right)^{\frac{q}{p}}, \tag{3.26}$$

whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \le q < \frac{np}{1+(n-1)s}$. Hence, (3.19) follows.

To prove inequality (3.20), by Lemma 2.2, it suffices to show that

$$\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz < \infty.$$

Clearly

$$\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz = \int_{B(\Omega^s,1)\setminus\overline{\Delta}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz + \int_{\Delta\setminus\overline{\Omega^s}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz.$$

First, by inequality (3.17), we have

$$\int_{B(\Omega^s,1)\setminus\overline{\Delta}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz < \infty.$$

Since $\Delta \setminus \overline{\Omega_s} = A \cup B \cup C$ and $|\partial A| = |\partial B| = |\partial C| = 0$, we have

$$\int_{\Delta\setminus\overline{\Omega_s}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz = \left(\int_{\mathring{A}} + \int_{\mathring{B}} + \int_{\mathring{C}}\right) \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz.$$

By (3.5), (3.7) and (3.9), we obtain

$$\int_{\mathring{A}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz \le C\int_0^{\frac{1}{2}} t^{(n-1)-\frac{(n-1)(s-1)q}{p-q}}dt < \infty,$$

$$\int_{\mathring{B}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dz \le C\int_0^{\frac{1}{2}} |x|^{(n-1)-\frac{(n-1)(s-1)q}{p-q}}d|x| < \infty,$$

and

$$\int_{\mathring{C}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}}dx \le C\int_0^{\frac{1}{2}} x_1^{(n-1)-\frac{(n-1)(s-1)q}{p-q}}dx_1 < \infty,$$

whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \le q < \frac{np}{1+(n-1)s}$. In conclusion, we have proved that $w \in W^{1,q}(B(\Omega^s,1))$ with the bound

$$\|w\|_{W^{1,q}(B(\Omega^s,1))} \le C\|u\|_{W^{1,p}(\Omega^s)}$$

whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \le q < \frac{np}{1+(n-1)s}$. Since $E_{\mathcal{R}_1}(u) = w$ almost everywhere, the above also holds with w replaced by $E_{\mathcal{R}_1}(u)$.

For an arbitrary $u \in W^{1,p}(\Omega^s)$, by the density of $C_o^\infty(\mathbb{R}^n)\cap W^{1,p}(\Omega^s)$, we can find a sequence of functions $\{u_i\}_{i=1}^\infty \subset C_o^\infty(\mathbb{R}^n)\cap W^{1,p}(\Omega^s)$ and a subset $N \subset \Omega^s$ with $|N| = 0$ such that

$$\lim_{i\to\infty} \|u_i - u\|_{W^{1,p}(\Omega^s)} = 0, \tag{3.27}$$

and for every $z \in \Omega^s \setminus N$,

$$\lim_{i\to\infty} |u_i(z) - u(z)| = 0. \tag{3.28}$$

By the argument above, for every $u_i \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega^s)$, we have $E_{\mathcal{R}_1}(u_i) \in W^{1,q}(B(\Omega^s, 1))$ and

$$\|E_{\mathcal{R}_1}(u_i)\|_{W^{1,q}(B(\Omega^s,1))} \le C\|u_i\|_{W^{1,p}(\Omega^s)} \tag{3.29}$$

with a constant C independent of u_i. Since $\mathcal{R}_1$ is locally bi-Lipschitz on $B(\Omega^s, 1)\setminus\overline{\Omega^s}$, we have $\mathcal{R}_1(N) \subset B(\Omega^s, 1) \setminus \overline{\Omega^s}$ with $|\mathcal{R}_1(N)| = 0$. By the definition of $E_{\mathcal{R}_1}(u_i)$ in (2.1), the sequence $\{E_{\mathcal{R}_1}(u_i)\}_{i=1}^\infty$ has a limit at every point $z \in B(\Omega^s, 1) \setminus (N \cup \mathcal{R}_1(N))$. Define

$$v(z) := \begin{cases} \lim_{i\to\infty} E_{\mathcal{R}_1}(u_i)(z) & \text{if } \; z \in B(\Omega^s, 1) \setminus (N \cup \mathcal{R}_1(N)), \\ 0, & \text{if } \; z \in N \cup \mathcal{R}_1(N). \end{cases} \tag{3.30}$$

Since $\{u_i\}_{i=1}^\infty$ is a Cauchy sequence in $W^{1,p}(\Omega^s)$, the inequalities (3.27) and (3.29) yields that $\{E_{\mathcal{R}_1}(u_i)\}_{i=1}^\infty$ is also a Cauchy sequence in $W^{1,q}(B(\Omega^s, 1))$. Hence $v \in W^{1,q}(B(\Omega^s, 1))$ with

$$\|v\|_{W^{1,q}(B(\Omega^s,1))} \le C\|u\|_{W^{1,p}(\Omega^s)}.$$

By definition, we conclude that $E_{\mathcal{R}_1}(u)(z) = v(z)$ for every $z \in B(\Omega^s, 1) \setminus (N \cup \mathcal{R}_1(N))$. Since $|N \cup \mathcal{R}_1(N)| = 0$, we have $E_{\mathcal{R}_1}(u) \in W^{1,q}(B(\Omega^s, 1))$ with

$$\|E_{\mathcal{R}_1}(u)\|_{W^{1,q}(B(\Omega^s,1))} = \|v\|_{W^{1,q}(B(\Omega^s,1))} \le C\|u\|_{W^{1,p}(\Omega^s)}.$$

□

Theorem 3.2. *Let $\Omega^s \subset \mathbb{R}^n$ be an outward cuspidal domain with the degree $s > 1$. Then the reflection $\mathcal{R}_2 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$ induces a bounded linear extension operator from $W^{1,p}(\Omega^s)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{2+(n-2)s} < p < \infty$ and $1 \le q < \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$.*

Proof. Let $u \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega^s)$ be arbitrary. We define a function $E_{\mathcal{R}_2}(u)$ as in (2.1) and another function w by setting

$$w(z) := \begin{cases} u \circ \mathcal{R}_2(z), & \text{if } \; z \in B(\Omega^s, 1) \setminus \overline{\Omega^s}, \\ u(z), & \text{if } \; z \in \overline{\Omega^s}. \end{cases} \tag{3.31}$$

We claim that $w \in W^{1,q}(B(\Omega^s, 1))$ with

$$\|w\|_{W^{1,q}(B(\Omega^s,1))} \le C\|u\|_{W^{1,p}(\Omega^s)}$$

for a constant $C > 1$ independent of u. As in the proof of Theorem 3.1, it suffices to estimate the norm over the union of Ω^s and $B(\Omega^s, 1) \setminus \overline{\Omega^s}$ and we are again reduced to estimating the norm over the second set in question. On this set, $w = u \circ \mathcal{R}_2$ almost everywhere and hence it suffices to prove the inequality

$$\left(\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}} |u \circ \mathcal{R}_2(z)|^q dz\right)^{\frac{1}{q}} \le C \left(\int_{\Omega^s} |u(z)|^p dz\right)^{\frac{1}{p}} \tag{3.32}$$

and the inequality

$$\left(\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}}|D(u\circ\mathcal{R}_2)(z)|^q dz\right)^{\frac{1}{q}}\leq C\left(\int_{\Omega^s}|Du(z)|^p dz\right)^{\frac{1}{p}}. \tag{3.33}$$

Now

$$B(\Omega^s,1)\setminus\overline{\Omega^s}=(B(\Omega^s,1)\setminus\Delta')\cup(\Delta'\setminus\overline{\Omega^s})$$

and $\Delta'\setminus\overline{\Omega^s}=D\cup E$. Since

$$|\partial\Delta'|=|\partial D|=|\partial E|=0,$$

we have

$$\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}}|u\circ\mathcal{R}_2(z)|^q dz=\int_{B(\Omega^s,1)\setminus\overline{\Delta'}}|u\circ\mathcal{R}_2(z)|^q dz+\left(\int_{\mathring{D}}+\int_{\mathring{E}}\right)|u\circ\mathcal{R}_2(z)|^q dz. \tag{3.34}$$

Since $\mathcal{R}_2$ is bi-Lipschitz on $B(\Omega^s,1)\setminus\overline{\Delta'}$ and $|\Omega^s|<\infty$, by the Hölder inequality, we have

$$\int_{B(\Omega^s,1)\setminus\overline{\Delta'}}|u\circ\mathcal{R}_2(z)|^q dz\leq C\left(\int_{\Omega^s}|u(z)|^p dz\right)^{\frac{q}{p}}. \tag{3.35}$$

Since $|J_{\mathcal{R}_2}(t,x)|\sim 1$ on $\mathring{E}\cup\mathring{D}$, by (3.13) and (3.16), we conclude by computing as in (3.23) that

$$\int_{\mathring{E}\cup\mathring{D}}|u\circ\mathcal{R}_2(z)|^q dz\leq C\left(\int_{\Omega^s}|u(z)|^p dz\right)^{\frac{q}{p}}, \tag{3.36}$$

whenever $\frac{1+(n-1)s}{2+(n-2)s}<p<\infty$ and $1\leq q<\frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$. By combining inequalities (3.34)-(3.36), we obtain inequality (3.32).

To prove inequality (3.33), by Lemma 2.2, it suffices to show that

$$\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}}\frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}}dz<\infty.$$

Trivially,

$$\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}}\frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}}dz=\int_{B(\Omega^s,1)\setminus\overline{\Delta'}}\frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}}dz+\int_{\Delta'\setminus\overline{\Omega^s}}\frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}}dz.$$

Since $\mathcal{R}_2$ is bi-Lipschitz on $B(\Omega^s, 1) \setminus \overline{\Delta'}$, we have

$$\int_{B(\Omega^s,1)\setminus\overline{\Delta'}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz < \infty.$$

Since $\Delta' \setminus \overline{\Omega^s} = D \cup E$, $|\partial D| = |\partial E| = 0$, inequalities (3.13), (3.16) give

$$\begin{aligned} \int_{\Delta'\setminus\overline{\Omega^s}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz &\leq \int_{\mathring{D}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz + \int_{\mathring{E}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz \\ &\leq C \int_0^{\frac{1}{2}} \int_{t^s}^{\left(\frac{1}{2}\right)^s} |x|^{(n-2)-\frac{(s-1)pq}{s(p-q)}} d|x| dt + C \\ &\leq C \int_0^{\frac{1}{2}} t^{(n-1)s-\frac{(s-1)pq}{p-q}} dt + C < \infty, \end{aligned} \tag{3.37}$$

whenever $\frac{1+(n-1)s}{2+(n-2)s} < p < \infty$ and $1 \leq q < \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$. In conclusion, we have proved that $w \in W^{1,q}(B(\Omega^s, 1))$ with the bound

$$\|w\|_{W^{1,q}(B(\Omega^s,1))} \leq C\|u\|_{W^{1,p}(\Omega^s)}$$

whenever $\frac{1+(n-1)s}{2+(n-2)s} < p < \infty$ and $1 \leq q < \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$. Since $E_{\mathcal{R}_2}(u) = w$ almost everywhere, the above also holds with w replaced by $E_{\mathcal{R}_2}(u)$. Hence, we may complete the proof by following the argument of the proof of Theorem 3.1.

□

3.4 Proof of Theorem 1.2

We begin with a useful observation.

Lemma 3.1. *Let $1 < s < \infty$ and $1 < p < \infty$. If there is a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$ which induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$, then $\mathcal{R} \in W^{1,p}_{\mathrm{loc}}(G \cap \Omega^s, \mathbb{R}^n)$ and*

$$|D\mathcal{R}(z)|^p \leq C|J_{\mathcal{R}}(z)|$$

for almost every $z \in G \cap \Omega^s$, where G is a bounded open set containing $\partial\Omega^s$.

Proof. Let $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ be a reflection over $\partial\Omega^s$ which induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$. Then there exists a bounded open set U containing $\partial\Omega^s$ so that the function

$$E_{\mathcal{R}}(u)(z) := \begin{cases} u \circ \mathcal{R}(z), & \text{for } z \in U \cap \Omega^s, \\ 0, & \text{for } z \in \partial\Omega^s, \\ u(z), & \text{for } z \in U \setminus \overline{\Omega^s} \end{cases} \tag{3.38}$$

belongs to $W^{1,p}(U)$ and satisfies

$$\|E_{\mathcal{R}}(u)\|_{W^{1,p}(U)} \le C\|u\|_{W^{1,p}(U\setminus\Omega^s)}$$

for a positive constant C independent of u. It follows that $\mathcal{R} \in W^{1,p}_{\mathrm{loc}}(U\cap\Omega^s, \mathbb{R}^n)$. We employ an idea from [18] and pick a Lipschitz domain G so that $\overline{\Omega^s} \subset G$ and $\partial G \subset U$. Since G is Lipschitz and contains the closure of Ω^s, the geometry of Ω^s easily yields that $G\setminus\overline{\Omega^s}$ is an (ϵ,δ)-domain for some positive ϵ,δ. Since $u-u_{G\setminus\overline{\Omega^s}} \in W^{1,p}(G\setminus\overline{\Omega^s})$ and (ϵ,δ)-domains are (p,p)-extension domains, we find a function $v \in W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega^s})$ such that $v = u - u_{G\setminus\overline{\Omega^s}}$ on $G\setminus\overline{\Omega^s}$ and

$$\|v\|_{W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega^s})} \le C\|u-u_{G\setminus\overline{\Omega^s}}\|_{W^{1,p}(G\setminus\overline{\Omega^s})}. \tag{3.39}$$

Next, since $G\setminus\overline{\Omega^s}$ is a bounded (ϵ,δ)-domain, we have

$$\int_{G\setminus\overline{\Omega^s}} |u(z)-u_{G\setminus\overline{\Omega^s}}|^p dz \le C\int_{G\setminus\overline{\Omega^s}} |Du(z)|^p dz, \tag{3.40}$$

see [3, 29]. By our assumption, (3.39) and (3.40), we have

$$\begin{aligned}\|v\circ\mathcal{R}\|_{W^{1,p}(G\cap\Omega^s)} &\le \|v\circ\mathcal{R}\|_{W^{1,p}(U\cap\Omega^s)}\\ &\le C\|u-u_{G\setminus\overline{\Omega^s}}\|_{W^{1,p}(G\setminus\overline{\Omega^s})} \le C\|Du\|_{L^p(G\setminus\overline{\Omega^s})}.\end{aligned}$$

It is easy to check that $v\circ\mathcal{R} = E_{\mathcal{R}}(v)$ on $G\cap\Omega^s$ and that $Du = Dv$ almost everywhere on $G\setminus\overline{\Omega^s}$. Hence, we have

$$\int_{G\cap\Omega^s} |DE_{\mathcal{R}}(v)(z)|^p dz \le C\int_{G\setminus\overline{\Omega^s}} |Du(z)|^p dz.$$

Since $u \in W^{1,p}(\Omega^s)$ is arbitrary, Lemma 2.2 gives the asserted inequality. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix $1 < s < \infty$. It is easy to check that $\mathbb{R}^n\setminus\overline{\Omega^s}$ is an (ϵ,δ)-domain, for some positive constants ϵ and δ. Hence, by [19], $\mathbb{R}^n\setminus\overline{\Omega^s}$ is a (p,p)-extension domain, for every $p\in[1,\infty)$.

We begin by showing that the reflection $\mathcal{R}_1$ induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$, whenever $1\le p\le n-1$. Define the domain Δ as in (3.1) and the domain Ω^s_1 as in (3.2). By (3.3), the formula of the reflection $\mathcal{R}_1$ on Ω^s_1 is

$$\mathcal{R}_1(t,x) = \begin{cases} \left(-t, \frac{6x}{t^{s-1}}\right), & \text{if } 0\le |x| < \frac{1}{6}t^s,\\ \left(\frac{12|x|}{t^{s-1}} - 3t, t\frac{x}{|x|}\right), & \text{if } \frac{1}{6}t^s \le |x| < \frac{1}{3}t^s,\\ \left(t, \frac{3(t^s-t)}{2t^s}x + \left(\frac{3t}{2}-\frac{t^s}{2}\right)\frac{x}{|x|}\right), & \text{if } \frac{1}{3}t^s\le |x| < t^s. \end{cases} \tag{3.41}$$

For every $(t,x)\in\Omega_1^s$ with $0<|x|<\frac{1}{6}t^s$, the resulting differential matrix of $\mathcal{R}_1$ is

$$D_{\mathcal{R}_1}(t,x)=\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ (1-s)\frac{6x_1}{t^s} & \frac{6}{t^{s-1}} & 0 & \cdots & 0 \\ (1-s)\frac{6x_2}{t^s} & 0 & \frac{6}{t^{s-1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-s)\frac{6x_{n-1}}{t^s} & 0 & 0 & \cdots & \frac{6}{t^{s-1}} \end{pmatrix}.$$

After a simple computation, for every $(t,x)\in\Omega_1^s$ with $0<|x|<\frac{1}{6}t^s$, we have

$$|D\mathcal{R}_1(t,x)|=\frac{6}{t^{s-1}} \text{ and } |J_{\mathcal{R}_1}(t,x)|=\left(\frac{6}{t^{s-1}}\right)^{n-1}. \tag{3.42}$$

For every $(t,x)\in\Omega_1^s$ with $\frac{1}{6}t^s<|x|<\frac{1}{3}t^s$, the resulting differential matrix is

$$D\mathcal{R}_1(t,x)=\begin{pmatrix} \frac{12(1-s)|x|}{t^s}-3 & \frac{12x_1}{|x|t^{s-1}} & \frac{12x_2}{|x|t^{s-1}} & \cdots & \frac{12x_{n-1}}{|x|t^{s-1}} \\ \frac{x_1}{|x|} & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ \frac{x_2}{|x|} & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{x_{n-1}}{|x|} & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix},$$

where, for every $i,j\in\{1,2,\cdots,n-1\}$, we have

$$A_j^i(t,x):=\begin{cases} \frac{t}{|x|}-\frac{tx_i^2}{|x|^3}, & \text{if } i=j, \\ \frac{-tx_ix_j}{|x|^3}, & \text{if } i\neq j. \end{cases}$$

After a simple computation, for every $(t,x)\in\Omega_1^s$ with $\frac{1}{6}t^s<|x|<\frac{1}{3}t^s$, we have

$$|D\mathcal{R}_1(t,x)|\leq\frac{C}{t^{s-1}} \text{ and } |J_{\mathcal{R}_1}(t,x)|\sim_c\left(\frac{1}{t^{s-1}}\right)^{n-1}. \tag{3.43}$$

For every $(t,x)\in\Omega_1^s$ with $\frac{1}{3}t^s<|x|<t^s$, the resulting differential matrix is

$$D\mathcal{R}_1(t,x)=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ A_t^1(t,x) & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ A_t^2(t,x) & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_t^{n-1}(t,x) & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix},$$

where, for every $i,j\in\{1,2,\cdots,n-1\}$, we have

$$A_j^i(t,x):=\begin{cases} \left(\frac{3}{2}-\frac{3}{2t^{s-1}}\right)+\left(\frac{3t}{2}-\frac{t^s}{2}\right)\left(\frac{1}{|x|}-\frac{x_i^2}{|x|^3}\right), & \text{if } i=j, \\ -\left(\frac{3t}{2}-\frac{t^s}{2}\right)\frac{x_ix_j}{|x|^3}, & \text{if } i\neq j. \end{cases}$$

and

$$A_t^i(t,x) := (s-1)\frac{3x_i}{2t^s} + \left(\frac{3}{2} - \frac{s}{2}t^{s-1}\right)\frac{x_i}{|x|}.$$

After a simple computation, for every $(t,x) \in \Omega_1^s$ with $\frac{1}{3}t^s < |x| < t^s$, we have

$$|D\mathcal{R}_1(t,x)| \le \frac{C}{t^{s-1}} \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c \left(\frac{1}{t^{s-1}}\right)^{n-1}. \tag{3.44}$$

By combining (3.42), (3.43) and (3.44), we conclude that

$$|D\mathcal{R}_1(z)|^p \le C|J_{\mathcal{R}_1}(z)| \tag{3.45}$$

for almost every $z \in \Delta\cap\Omega^s$. By the same inequalities, since $\mathcal{R}_1$ is locally bi-Lipschitz on $B(\Omega^s,1)\setminus\overline{\Delta}$, for every $u \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega^s})$, we have

$$\begin{aligned}\int_{\mathcal{R}_1(B(\Omega^s,1)\setminus\overline{\Omega^s})} |u\circ\mathcal{R}_1(z)|^p dz &\le C\int_{\mathcal{R}_1(B(\Omega^s,1)\setminus\overline{\Omega^s})} |u\circ\mathcal{R}_1(z)|^p |J_{\mathcal{R}_1}(z)| dz \\ &\le \int_{B(\Omega^s,1)\setminus\overline{\Omega^s}} |u(z)|^p dz.\end{aligned} \tag{3.46}$$

Moreover, by Lemma 2.2 and (3.45), we have

$$\int_{B(\Omega^s,1)\setminus\overline{\Omega^s}} |D(u\circ\mathcal{R}_1)(z)|^p dz \le \int_{\mathbb{R}^n\setminus\overline{\Omega^s}} |Du(z)|^p dz. \tag{3.47}$$

Since $\mathbb{R}^n\setminus\overline{\Omega^s}$ satisfies the segment condition, (3.46) and (3.47) allow us to repeat the argument in the proof of Theorem 3.1 so as to conclude that $\mathcal{R}_1$ induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$, whenever $1 \le p \le n-1$.

Next, we show that there is no reflection over $\partial\Omega^s$ which can induce a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$, for any $n-1 < p < \infty$. Let $n-1 < p < \infty$ be fixed. Suppose that there exists a reflection $\mathcal{R} : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ over $\partial\Omega^s$, which induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega^s})$ to $W^{1,p}(\mathbb{R}^n)$. By Lemma 3.1, there exists an open set G which contains $\partial\Omega^s$ such that for almost every $z \in G\cap\Omega^s$, we have

$$|D\mathcal{R}(z|^p \le C|J_{\mathcal{R}}(z)|.$$

Then, by Lemma 2.4, for almost every $(t,x) \in \mathcal{R}(G\cap\Omega^s)$, we have

$$|D\mathcal{R}(z)|^{\frac{p}{p+1-n}} \le C|J_{\mathcal{R}}(z)|. \tag{3.48}$$

Let $u \in C_o^\infty(\mathbb{R}^n)\cap W^{1,p}(\Omega^s)$ be arbitrary. By definition, $E_{\mathcal{R}}(u)$ is bounded and continuous on G. Pick a Lipschitz domain $\widetilde{G}$ so that $\overline{\Omega^s} \subset G$ and $\partial\widetilde{G} \subset G$. By Lemma 2.2, we have

$$\|DE_{\mathcal{R}}(u)\|_{L^{\frac{p}{p+1-n}}(\widetilde{G})} \le C\|Du\|_{L^{\frac{p}{p+1-n}}(\Omega^s)}. \tag{3.49}$$

We conclude that $E_{\mathcal{R}}(u) \in W^{1,\frac{p}{p+1-n}}(\widetilde{G})$. Since $\widetilde{G}$ is a Lipschitz domain, [18, Lemma 4.1] implies

$$\int_{\widetilde{G}} |E_{\mathcal{R}}(u)(z) - u_{\Omega^s}|^{\frac{p}{p+1-n}} dz \leq C(\widetilde{G}, \Omega^s) \int_{\widetilde{G}} |DE_{\mathcal{R}}(u)(z)|^{\frac{p}{p+1-n}} dz. \tag{3.50}$$

Hence, we have

$$\|E_{\mathcal{R}}(u)\|_{L^{\frac{p}{p+1-n}}(\widetilde{G})} \leq C \left(\|DE_{\mathcal{R}}(u)\|_{L^{\frac{p}{p+1-n}}(\widetilde{G})} + \|u\|_{L^{\frac{p}{p+1-n}}(\Omega^s)} \right). \tag{3.51}$$

By combining inequalities (3.49) and (3.51), we obtain

$$\|E_{\mathcal{R}}(u)\|_{W^{1,\frac{p}{p+1-n}}(\widetilde{G})} \leq \|u\|_{W^{1,\frac{p}{p+1-n}}(\Omega^s)}. \tag{3.52}$$

Since $C_o^\infty(\mathbb{R}^n) \cap W^{1,\frac{p}{p+1-n}}(\Omega^s)$ is dense in $W^{1,\frac{p}{p+1-n}}(\Omega^s)$, for every function $u \in W^{1,\frac{p}{p+1-n}}(\Omega^s)$, there exists a sequence of functions $u_i \in C_o^\infty(\mathbb{R}^n) \cap W^{1,\frac{p}{p+1-n}}(\Omega^s)$ such that

$$\lim_{i \to \infty} \|u_i - u\|_{W^{1,\frac{p}{p+1-n}}(\Omega^s)} = 0, \tag{3.53}$$

and for almost every $z \in \Omega^s$,

$$\lim_{i \to \infty} |u_i(z) - u(z)| = 0.$$

By (3.49) and (3.53), $\{E_{\mathcal{R}}(u_i)\}_{i=1}^\infty$ is a Cauchy sequence in $W^{1,\frac{p}{p+1-n}}(\widetilde{G})$. By the completeness of $W^{1,\frac{p}{p+1-n}}(\widetilde{G})$, there exits a function $\omega \in W^{1,\frac{p}{p+1-n}}(\widetilde{G})$ with

$$\lim_{i \to \infty} \|w - E_{\mathcal{R}}(u_i)\|_{W^{1,\frac{p}{p+1-n}}(\widetilde{G})} = 0$$

and $\omega(z) = u(z)$ for almost every $z \in \Omega^s$. We define $E_{\mathcal{R}}(u)(z) := \omega(z)$ on $\widetilde{G}$. By (3.49) and (3.53) again, we have

$$\|E_{\mathcal{R}}(u)\|_{W^{1,\frac{p}{p+1-n}}(\widetilde{G})} \leq C\|u\|_{W^{1,\frac{p}{p+1-n}}(\Omega^s)}.$$

Hence, Ω^s is a Sobolev $\left(\frac{p}{p+1-n}, \frac{p}{p+1-n}\right)$-extension domain. This contradicts the classical result that Ω^s is not a (q, q)-extension domain, for any $1 \leq q < \infty$, see [22] and references therein. □

3.5 Proof of Theorem 1.3

Proof of Theorem 1.3. Fix $1 < s < \infty$. It is easy to see both Ω^s and $\mathbb{R}^n \setminus \overline{\Omega^s}$ are uniformly locally quasiconvex. By [12], they are (∞, ∞)-extension domains.

To begin, we show that the reflection $\mathcal{R}_1$ induces a bounded linear extension operator from $W^{1,\infty}(\Omega^s)$ to $W^{1,\infty}(\mathbb{R}^n)$. Since Ω^s is uniformly quasiconvex, every function in $W^{1,\infty}(\Omega^s)$ has a Lipschitz representative. Without loss of generality, we assume every function in $W^{1,\infty}(\Omega^s)$ is Lipschitz. Let $u \in W^{1,\infty}(\Omega^s)$ be arbitrary. Define the extension $E_{\mathcal{R}_1}(u)$ on $B(\Omega^s, 1)$ as in (2.1). Since $u \in W^{1,\infty}(\Omega^s)$ is Lipschitz and $\mathcal{R}_1$ is locally Lipschitz on $B(\Omega^s,1)\setminus(\Omega^s\cup\{0\})$, we have $E_{\mathcal{R}_1}(u) \in W^{1,1}_{\rm loc}(B(\Omega^s,1)\setminus\overline{\Omega^s})$. By (3.5), (3.7), (3.9) and the fact that $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega^s,1)\setminus\overline{\Delta}$, for almost every $z \in B(\Omega^s,1)\setminus\overline{\Omega^s}$, we have

$$|DE_{\mathcal{R}_1}(u)(z)| \leq C|Du(\mathcal{R}_1(z))|.$$

This implies that

$$\|E_{\mathcal{R}_1}(u)\|_{W^{1,\infty}(B(\Omega^s,1))} \leq C\|u\|_{W^{1,\infty}(\Omega^s)}$$

as desired.

Next, we show that there does not exist a reflection over $\partial\Omega^s$ which can induce a bounded linear extension operator from $W^{1,\infty}(\mathbb{R}^n\setminus\overline{\Omega^s})$ to $W^{1,\infty}(\mathbb{R}^n)$. Define a function $u \in W^{1,\infty}(\mathbb{R}^n\setminus\overline{\Omega^s})$ by setting

$$u(t,x) = \begin{cases} 1, & \text{if } (t,x) \in \mathbb{R}^n\setminus\overline{\Omega^s} \text{ and } t \geq 1, \\ t, & \text{if } (t,x) \in \mathbb{R}^n\setminus\overline{\Omega^s} \text{ and } 0 < t < 1, \\ 0, & \text{if } (t,x) \in \mathbb{R}^n\setminus\overline{\Omega^s} \text{ and } t \leq 0. \end{cases} \tag{3.54}$$

For every $t \in (0,1)$ fixed, we define a 2-dimensional disk $D_t \subset \Omega^s$ by setting

$$D_t := \{(t,x) \in \mathbb{R}^n; |x| < t^s\}$$

and define

$$S_t := \{(t,x) \in \mathbb{R}^n; |x| = 2t^s\}.$$

Suppose to the contrary that there exists a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega^s$ which induces a bounded linear extension operator from $W^{1,\infty}(\mathbb{R}^n\setminus\overline{\Omega^s})$ to $W^{1,\infty}(\mathbb{R}^n)$. Define the function $E_{\mathcal{R}}(u)$ on $B(\Omega^s,1)$ as in (2.1). By the geometry of Ω^s and the fact that $\mathcal{R}$ is continuous and $\mathcal{R}(z) = z$ whenever $z \in \partial\Omega^s$, there exists a small enough $t_o \in (0,1)$ such that for every $t \in (0,t_o)$, there exists $(t,x_t) \in D_t$ with $E_{\mathcal{R}}(u)((t,x_t)) = 0$ and there exists $(t,x'_t) \in S_t$ with $E_{\mathcal{R}}(u)((t,x'_t)) = t$ and $d((t,x_t),(t,x'_t)) \leq 2t^s$. Hence for every $0 < t < t_o$, we have

$$|E_{\mathcal{R}}(u)((t,x_t)) - E_{\mathcal{R}}(u)((t,x'_t))| \geq t \geq Cd^{\frac{1}{1+s}}((t,x_t),(t,x'_t)).$$

This contradicts the assumption that $E_{\mathcal{R}}(u) \in W^{1,\infty}(B(\Omega_s,1))$: since $B(\Omega^s,1)$ is uniformly locally quasiconvex, a function in $W^{1,\infty}(B(\Omega^s,1))$ must have a Lipschitz representative. □

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