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Stability and Chaotic Attractors of Memristor-Based Circuit with a Line of Equilibria

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Abstract. This report investigates the stability problem of memristive systems with a line of equilibria on the example of SBT memristor-based Wien-bridge circuit. For the considered system, conditions of local and global partial stability are obtained, and chaotic dynamics is studied.

Keywords: Partial stability · Memristor · Chaos · Hidden attractors

1 Introduction

In 1971, Leon Chua suggested the concept of a memristor [1] as an electrical component that regulates the flow of electrical current in a circuit and remembers the amount of charge that has previously flowed through it. Nowadays, various types of memristors are developing for the realization of memory, computations and many other applications (see e.g. [2–4]).

Consider the dynamical model of SBT memristor-based Wien-bridge circuit [5]:

$$\begin{cases} \dot{x} = \frac{1}{C_1}\big(\frac{1}{R_5}(y-x) - (A+B+g(\varphi)+G)x\big), \\ \dot{y} = \frac{1}{C_2}\big(\frac{1}{R_2}\big(\frac{R_4}{R_3}\,y - z\big) - \frac{1}{R_1}y - \frac{1}{R_5}(y-x)\big), \\ \dot{z} = \frac{1}{C_3}\big(\frac{1}{R_2}\big(\frac{R_4}{R_3}\,y - z\big)\big), \\ \dot{\varphi} = x, \end{cases} \tag{1}$$

where $g(\varphi) = |\varphi|$, the parameters $C_{1,2,3}$, $R_{1,2,3,4,5}$, A, B are positive, and G is negative. Using the notation $\alpha_i = \frac{1}{C_i}$, $(i = 1, 2, 3)$, $\beta_1 = \frac{1}{R_1}$, $\beta_2 = \frac{1}{R_2}$, $\beta_3 = \frac{R_4}{R_3}$, $\beta_4 = \frac{1}{R_5}$ we rewrite system (1) as follows:

$$\begin{cases} \dot{x} = f_1(x,y,z,\varphi) = \alpha_1\big(\beta_4(y-x) - (A+B+g(\varphi)+G)x\big), \\ \dot{y} = f_2(x,y,z,\varphi) = \alpha_2\big(\beta_2\big(\beta_3 y - z\big) - \beta_1 y - \beta_4(y-x)\big), \\ \dot{z} = f_3(x,y,z,\varphi) = \alpha_3\big(\beta_2\big(\beta_3 y - z\big)\big), \\ \dot{\varphi} = f_4(x,y,z,\varphi) = x. \end{cases} \tag{2}$$

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Equating the right-hand of system (2) to zero we obtain a line of equilibria:

$$E = \{(x, y, z, \varphi) \,\big|\, x = y = z = 0,\ \varphi \in \mathbb{R})\}. \tag{3}$$

2 Local Stability Analysis

Let us analyze the local stability of the equilibrium points on the line of equilibria E. Here for simplicity, we approximate continuous function $g(\varphi) = |\varphi|$ by a smooth function $g(\varphi) = \varphi \tanh(\rho\,\varphi) \geq 0$, where $\rho \gg 1$. Since for an arbitrary equilibrium $(0, 0, 0, \varphi_0) \in E$ we have

$$\frac{\partial f_1(x,y,z,\varphi)}{\partial \varphi}\bigg|_{(0,0,0,\varphi_0)} = \lim_{h\to 0} \frac{f_1(x,y,z,\varphi+h)|_{(0,0,0,\varphi_0)} - f_1(x,y,z,\varphi)|_{(0,0,0,\varphi_0)}}{h} = \\ = \lim_{h\to 0} \frac{(\alpha_1 x(g(\varphi+h)-g(\varphi)))|_{(0,0,0,\varphi_0)}}{h} = 0,$$

the Jacobi matrix at $(0, 0, 0, \varphi_0)$ can be expressed as:

$$J = \begin{pmatrix} -\alpha_1(A + Bg(\varphi_0) + G + \beta_4) & \alpha_1\beta_4 & 0 & 0 \\ \alpha_2\beta_4 & \alpha_2(\beta_2\beta_3 - \beta_1 - \beta_4) & -\alpha_2\beta_2 & 0 \\ 0 & \alpha_3\beta_2\beta_3 & -\alpha_3\beta_2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

The characteristic polynomial for the Jacobi matrix J is as follows:

$$\det(\lambda I - J) = \lambda\big(\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0\big), \tag{5}$$

where

$$\begin{aligned} P_2 &= \alpha_1 (A + Bg(\varphi_0) + G + \beta_4) + \alpha_2 (\beta_1 - \beta_2\,\beta_3 + \beta_4) + \alpha_3\beta_2, \\ P_1 &= \alpha_1\alpha_2 \left((A + Bg(\varphi_0) + G + \beta_4)(\beta_1 - \beta_2\,\beta_3 + \beta_4) - \beta_4^2\right) + \\ &\qquad + \alpha_1\alpha_3 ((A + Bg(\varphi_0) + G + \beta_4)\,\beta_2) + \alpha_2\,\alpha_3\,\beta_2\,(\beta_1 + \beta_4), \\ P_0 &= \alpha_1\alpha_2\alpha_3\,\beta_2 \left((A + Bg(\varphi_0) + G + \beta_4)(\beta_1 + \beta_4) - \beta_4^2\right). \end{aligned} \tag{6}$$

Expression (5) indicates that the characteristic equation of Jacobi matrix J has a zero eigenvalue with corresponding eigenvector $(0, 0, 0, 1)^*$, and three non-zero eigenvalues. Since the central manifold of each equilibrium $p \in E$ is placed on the line of equilibria E, local dynamics of the nonlinear system near p is described by the local dynamics of linearized system (see, e.g. *Shoshitaishvili reduction principle* [6] and related results). Thus, if p has three eigenvalues with negative real parts, then it is locally stable.

According to the Routh-Hurwitz criterion of stability, all the non-zero eigenvalues of (5) have negative real parts, iff $P_2 > 0$, $P_0 > 0$ and $P_2P_1 - P_0 > 0$. Left-hand side of the latter inequality has the form of quadratic equation:

$$P_2\,P_1 - P_0 = Q_2\,\nu^2 + Q_1\,\nu + Q_0 \tag{7}$$

with respect to $\nu = \alpha_1 B g(\varphi_0)$, where

$$\begin{aligned} Q_2 &= \alpha_2\left(\beta_1 - \beta_2\beta_3 + \beta_4\right) + \alpha_3\beta_2, \\ Q_1 &= Q_2^2 + \alpha_1\left(2\left(A + G + \beta_4\right)Q_2 - \alpha_2\beta_4^2\right), \\ Q_0 &= \alpha_1\left(\left(A + G + \beta_4\right)Q_2 - \alpha_2\beta_4^2\right)\left(\left(A + G + \beta_4\right)\alpha_1 + Q_2\right) \\ &\qquad + \alpha_2\alpha_3\beta_2\left(\alpha_1\beta_4^2 + \left(\beta_1 + \beta_4\right)Q_2\right). \end{aligned} \tag{8}$$

The discriminant of (7) has the following form:

$$\mathcal{D} = \left(Q_2^2 + \alpha_1\alpha_2\,\beta_4^2\right)^2 - 4\alpha_2\,\alpha_3\,\beta_2\,Q_2\left(\left(\beta_1 + \beta_4\right)Q_2 + \alpha_1\beta_4^4\right). \tag{9}$$

For stability of all points on the line of equilibria (3), the branches of parabola Eq. (7) has to be directed upwards, i.e. the inequality $Q_2 > 0$ is needed. Since $\nu = \alpha_1 B g(\varphi_0) \geq 0$ for all $\varphi_0 \in \mathbb{R}$, to satisfy the inequality $P_2P_1 - P_0 > 0$ it is necessary and sufficient to have either no real roots (i.e $\mathcal{D} < 0$), or all negative roots of the Eq. (7). The latter condition is satisfied, iff $\mathcal{D} \geq 0$, $Q_1 > 0$, $Q_0 > 0$. Inequalities $P_2 > 0$, $P_0 > 0$ are satisfied for all $\varphi_0 \in \mathbb{R}$, iff

$$\begin{aligned} &(P_2 \geq) \quad \underbrace{\alpha_1\left(A + G + \beta_4\right) + \alpha_2\left(\beta_1 - \beta_2\,\beta_3 + \beta_4\right) + \alpha_3\beta_2}_{=\kappa_1} > 0, \\ &(P_0 \geq) \quad \underbrace{\alpha_1\alpha_2\alpha_3\,\beta_2\left(\left(A + G + \beta_4\right)\left(\beta_1 + \beta_4\right) - \beta_4^2\right)}_{=\kappa_2} > 0. \end{aligned} \tag{10}$$

Thus, it is possible to formulate the following statement

Lemma 1. *If the values of parameters $\alpha_{1,2,3}$, $\beta_{1,2,3,4}$, A, B, G are such that the conditions $\kappa_1 > 0$, $\kappa_2 > 0$, $Q_2 > 0$, and*

$$\mathcal{D} \geq 0, \quad Q_1 > 0, \quad Q_0 > 0, \qquad or \qquad \mathcal{D} < 0 \tag{11}$$

hold, then each point at the line of equilibria E is locally Lyapunov stable[1].

3 Global Stability Analysis

In order to study the global stability of the line (3) let us consider the following Lyapunov function:

$$V = \tfrac{1}{2}\left(\tfrac{x^2}{\alpha_1} + \tfrac{y^2}{\alpha_2} + \tfrac{z^2}{\alpha_3}\right), \tag{12}$$

which has the following derivative along the solutions of system (2):

$$\begin{aligned} \dot{V} &= -(\beta_4 + A + Bg(\varphi) + G)x^2 + 2\beta_4 xy - (\beta_1 - \beta_2\beta_3 + \beta_4)y^2 + (\beta_2(\beta_3 - 1))yz - \beta_2 z^2 \\ &= -\gamma_1\left(x - \tfrac{\beta_4 y}{\gamma_1}\right)^2 - \gamma_2\left(y - \tfrac{1}{2}\tfrac{\beta_2(\beta_3 - 1)z}{\gamma_2}\right)^2 - \gamma_3 z^2, \end{aligned} \tag{13}$$

[1] For any $\varepsilon > 0$ there exists $\delta > 0$, such that, if $|u(0) - u_{eq}| < \delta$, then $|u(t) - u_{eq}| < \varepsilon$ is valid for all $t > 0$. Recall that *local asymptotic stability* of u_{eq} means that u_{eq} is locally Lyapunov stable and also there exists $\delta > 0$, such that if $|u(0) - u_{eq}| < \delta$, then $\lim_{t\to\infty} |u(t) - u_{eq}| = 0$. Thus, due to the noise, the state of the physical model ~~could~~ drift along the line of equilibria.

where

$$\gamma_1 = \beta_4 + A + Bg(\varphi) + G, \qquad \gamma_2 = \beta_1 - \beta_2\beta_3 + \beta_4\big(1 - \tfrac{\beta_4}{\gamma_1}\big), \qquad \gamma_3 = \beta_2\big(1 - \tfrac{\beta_2(\beta_3-1)^2}{4\gamma_2}\big). \tag{14}$$

Since $B > 0$, we have

$$\gamma_1 \geq \underbrace{\beta_4 + A + G}_{=\mu_1}, \qquad \gamma_2 \geq \underbrace{\beta_1 - \beta_2\beta_3 + \beta_4\left(1 - \frac{\beta_4}{\mu_1}\right)}_{=\mu_2}, \qquad \gamma_3 \geq \underbrace{\beta_2\left(1 - \frac{\beta_2(\beta_3-1)^2}{4\mu_2}\right)}_{=\mu_3}. \tag{15}$$

Thus, it is possible to formulate the following statement

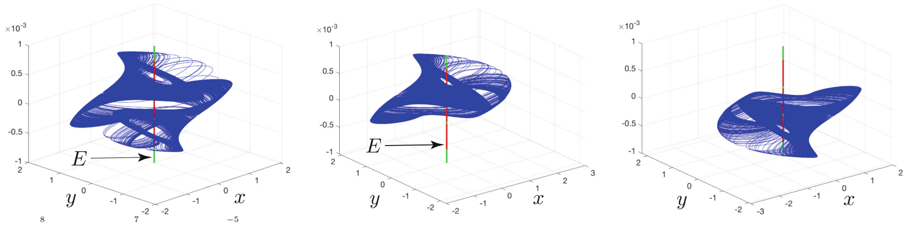
Lemma 2. *If the values of parameters $\alpha_{1,2,3}$, $\beta_{1,2,3,4}$, A, B, G are such that the conditions $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$ hold, then the line of equilibria E is partially globally stable, i.e. for any ellipsoidal cylinder $\varepsilon = V(x, y, z)$ defined by (12) with a sufficiently small radius ε and for any trajectory from outside it there exists a moment of time T after which the trajectory enters the cylinder and remain there (see, e.g. [7]).*

4 Chaotic Attractors

For parameters $\alpha_1 = 10^8$, $\alpha_2 = \alpha_3 = 5 \cdot 10^7$, $\beta_1 = \beta_2 = 4 \cdot 10^{-5}$, $\beta_3 = 2.5$, $\beta_4 = 2.22 \cdot 10^{-5}$, $A = 0.0676$, $B = 0.3682$, $G = -0.0677$ chaotic attractors [5] can be found in system (2) (see Fig. 1). These attractors are self-excited ones with respect to some unstable points on the line of equilibria E according to the definition from [8–13]. However since there is a continuum of unstable equilibria on E, the unstable manifold of which may ~~from~~ attractors, the revealing of all co-existing attractors is a challenging task, and, thus, attractors ~~is~~ such systems sometimes are also called "hidden". For their search one can use, e.g., various evolutionary algorithms [14,15]. The search of all co-existing attractors and determination of their mutual disposition in dynamical systems can be regarded as a generalization [16] of the second part of Hilbert's 16th problem on the number and mutual disposition of limit cycles in two-dimensional polynomial systems. Remark, that since there is an unbounded line of equilibria E in system (1), one has to consider cylindrical absorbing sets and unbounded attractors.

One can see that the region of parameters given by the conditions of Lemma 1 does not coincide with the region of parameters corresponding to the conditions of Lemma 2. When all equilibria are locally stable, the following cases are of interest:

(a) system (1) can be partially globally stable when all trajectories tend to the line of equilibria E;
(b) system (1) may have hidden attractors with respect to E;
(c) system (1) can be dichotomic (some trajectories can tend to infinity in the (x, y, z) subspace).



(a) Initial point $u_0 = (10^{-2}, 0, 0, 0)$.

(b) Initial point $u_0 = (10^{-2}, 0, 0, 5 \cdot 10^{-4})$.

(c) Initial point $u_0 = (10^{-2}, 0, 0, -5 \cdot 10^{-4})$.

Fig. 1. Self-excited chaotic attractors (blue) in system (2) for parameters $\alpha_1 = 10^8$, $\alpha_2 = \alpha_3 = 5 \cdot 10^7$, $\beta_1 = \beta_2 = 4 \cdot 10^{-5}$, $\beta_3 = 2.5$, $\beta_4 = 2.22 \cdot 10^{-5}$, $A = 0.0676$, $B = 0.3682$, $G = -0.0677$, which are visualized by trajectories with initial data in vicinity of the line of equilibria E (stable equilibria are green, unstable – red).

When some of the equilibria on E are unstable, the following cases are of interest:

(a) system (1) can be gradient-like (i.e. when all trajectories except unstable equilibria tend to the stable equilibria on E);
(b) system (1) can be partially dissipative (all trajectories do not leave an absorbing cylinder; in this case system (1) can have self-excited attractors);
(c) system (1) can be dichotomic.

5 Conclusion

In this report we discussed some basic ideas of the stability theory for memristive systems with a line of equilibria. For the SBT memristor-based Wien-bridge chaotic circuit the conditions of local and global partial stability are obtained. Using [17,18], various other memristive circuits can be studied similarly. More detailed studies and results will be included in the forthcoming survey "*Theory of stability for memristive systems with a line of equilibria*" [19].

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