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Self-improvement of weighted pointwise inequalities on open sets

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Abstract

We prove a general self-improvement property for a family of weighted pointwise inequalities on open sets, including pointwise Hardy inequalities with distance weights. For this purpose we introduce and study the classes of p-Poincaré and p-Hardy weights for an open set $\Omega \subset X$, where X is a metric measure space. We also apply the self-improvement of weighted pointwise Hardy inequalities in connection with usual integral versions of Hardy inequalities.

Keywords: self-improvement, pointwise Hardy inequality, metric space, weight, maximal operator
2010 MSC: 35A23, 31E05, 42B25

1. Introduction

This paper is continuation of a general program related to various self-improving phenomena, including Poincaré and Hardy inequalities and uniform fatness; see e.g. [3, 11, 15, 19] for earlier results and [5, 6, 18] for recent work by the authors. In this paper we introduce a class of p-Hardy weights and consider for such weights w the pointwise (p, w)-Hardy inequality

$$|u(x)| \leq C\, d(x, \Omega^c) \sup_{0<r<\kappa d(x,\Omega^c)} \left(\frac{1}{w(B(x,r))} \int_{B(x,r)} g(y)^p w(y)\, d\mu(y) \right)^{\frac{1}{p}}. \tag{1}$$

Here Ω is an open subset of a metric space X, $d(x, \Omega^c)$ denotes the distance from $x \in \Omega$ to the complement $\Omega^c = X \setminus \Omega$, and g is a (bounded) upper gradient of $u \in \mathrm{Lip}_0(\Omega)$; see Sections 2 and 3 for definitions. Our main result, Theorem 7.4, shows that these inequalities are self-improving with respect to the exponent p: if a pointwise (p, w)-Hardy inequality holds in Ω with an exponent $1 < p < \infty$, then, under suitable assumptions, there exists $1 < q < p$ such that also a pointwise (q, w)-Hardy inequality holds in Ω. The unweighted case $w = 1$ corresponds to the pointwise p-Hardy inequality, for which the self-improvement was proved in [6]. Our approach relies on the basic ideas and techniques developed in [5, 6]. However, unlike the self-improvement of pointwise p-Hardy inequalities, which was known already before the work in [6] indirectly via the self-improvement of uniform p-fatness (see [3, 19]) and the equivalence between these two concepts (see [13]), the present self-improvement for the weighted pointwise p-Hardy inequalities

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is previously unknown. In particular, our main result is new even for $X = \mathbb{R}^n$, equipped with the Euclidean distance and the Lebesgue measure.

The self-improvement of the pointwise (p, w)-Hardy inequality and a weighted maximal function theorem show that inequality (1), for every $x \in \Omega$, implies the integral version of the (p, w)-Hardy inequality, that is,

$$\int_\Omega \frac{|u(x)|^p}{d(x,\Omega^c)^p}\, w(x)\, d\mu(x) \le C \int_\Omega g(x)^p w(x)\, d\mu(x); \tag{2}$$

see Section 8 for details. This implication is not immediate from inequality (1), since the maximal operator is not typically bounded on $L^1(X)$. In some sense the inbuilt self-improvement of pointwise Hardy inequalities provides a mechanism to bypass the lack of the L^1-boundedness for the maximal operator.

An important model case of (2) is the weighted (p, β)-Hardy inequality in $\mathbb{R}^n$, with $w(x) = d(x, \Omega^c)^\beta$, for $\beta \in \mathbb{R}$; see [16, 20]. Corresponding pointwise theory was developed in [14], but in order to be able to apply the maximal function theorem, it was necessary to assume *a priori* the validity of a stronger variant of (1) in terms of an exponent $1 < q < p$. With the self-improvement results of the present work, the starting point in the weighted pointwise Hardy inequalities as in [14] can now be taken to be the natural candidate involving only the exponent p, at least for $\beta \ge 0$. More motivation and explanation related to (weighted) pointwise Hardy inequalities in Euclidean spaces will be given in Section 8.

Often the theory of weighted inequalities is concerned with doubling weights. In the present setting the natural assumption is a weaker *semilocal* doubling condition with respect the open set $\Omega \subsetneq X$. This class of weights is introduced in Section 3, where we also prove some technical lemmas for such weights. As a tool in pointwise (p, w)-Hardy inequalities we also use a related class of p-Poincaré weights for Ω, see Section 4. In Section 5 we define the p-Hardy weights, which will be crucial for the pointwise (p, w)-Hardy inequalities, and in Section 6 we establish a self-improvement result for p-Hardy weights. This plays a key role also in the self-improvement of pointwise (p, w)-Hardy inequalities, since in Section 7 we show that w being a p-Hardy weight is equivalent to the validity of the pointwise (p, w)-Hardy inequality. Finally, Section 8 contains the applications related to integral versions of weighted Hardy inequalities.

2. Notation and auxiliary results

We make the standing assumption that $X = (X, d, \mu)$, with $\#X \ge 2$, is a metric measure space equipped with a metric d and a positive complete D-doubling Borel regular measure μ such that $0 < \mu(B) < \infty$ and

$$\mu(2B) \le D\,\mu(B) \tag{3}$$

for some $D > 1$ and for all balls $B = B(x, r) = \{y \in X \mid d(y, x) < r\}$. Here we use for $0 < \lambda < \infty$ the notation $\lambda B = B(x, \lambda r)$. It follows that the space X is separable (see e.g. [1, Proposition 1.6]) and $\mu(\{x\}) = 0$ for every $x \in X$ by [1, Corollary 3.9].

For us, a *curve* is a rectifiable and continuous mapping $\gamma\colon [a, b] \to X$. By $\Gamma(X)$ we denote the set of all curves in X. The length of a curve $\gamma \in \Gamma(X)$ is written as $\operatorname{len}(\gamma)$. A curve $\gamma\colon [a, b] \to X$ *connects* $x \in X$ to $y \in X$ (or a point $x \in X$ to a set $E \subset X$), if $\gamma(a) = x$ and $\gamma(b) = y$ ($\gamma(b) \in E$, respectively). We assume throughout that the space X is C_{QC}-quasiconvex for some $C_{\mathrm{QC}} \ge 1$, that is, for every $x, y \in X$ there exists a curve γ connecting x to y such that $\operatorname{len}(\gamma) \le C_{\mathrm{QC}} d(x, y)$.

Fix $x, y \in X$, $E \subset X$ and $\nu \ge 1$. The collection $\Gamma(X)^\nu_{x,y}$ is the set of all curves that connect x to y and whose lengths are at most $\nu d(x, y)$. The set of all curves that connect x to E and whose lengths are at most $\nu d(x, E)$ is denoted by $\Gamma(X)^\nu_{x,E}$.

A Borel function $g \ge 0$ on X is an *upper gradient* of function $u\colon X \to \mathbb{R}$, if for all curves $\gamma\colon [a, b] \to X$, we have

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g\, ds. \tag{4}$$

The space of Lipschitz functions on X is denoted by $\operatorname{Lip}(X)$. By definition $u \in \operatorname{Lip}(X)$ if there exists a constant $\lambda > 0$ such that

$$|u(x) - u(y)| \le \lambda d(x, y), \qquad \text{for all } \; x, y \in X.$$

When $\Omega \subset X$ is an open set, we denote by $\mathrm{Lip}_0(\Omega)$ the space of all Lipschitz functions on X that vanish on $\Omega^c = X \setminus \Omega$. The set of lower semicontinuous functions on X is denoted by $LC(X)$.

Recall that

$$u_E = \fint_E u\, d\mu = \frac{1}{\mu(E)} \int_E u(y)\, d\mu(y)$$

is the integral average of a function $u \in L^1(E)$ over a measurable set $E \subset X$ with $0 < \mu(E) < \infty$. If $1 \le p < \infty$ and $u\colon X \to \mathbb{R}$ is a μ-measurable function, then $u \in L^p_{\mathrm{loc}}(X)$ means that for each $x_0 \in X$ there exists $r > 0$ such that $u \in L^p(B(x_0, r))$, that is, $\int_{B(x_0,r)} |u|^p\, d\mu < \infty$. The characteristic function of a set $E \subset X$ is denoted by $\mathbf{1}_E$; that is, $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \in X \setminus E$.

3. Weights and restricted maximal functions for open sets

We need several classes of weights for open sets. To avoid pathological situations, we assume throughout the paper that the open sets $\Omega \subset X$ under consideration are nonempty.

Definition 3.1. Let $\Omega \subset X$ be an open set. A non-negative Borel function w in X is a weight for Ω, if $\int_B w(x)\, dx < \infty$ for all balls $B \subset X$ and $w(x) > 0$ for almost every $x \in \Omega$. If $E \subset X$ is a measurable set, then we write $w(E) = \int_E w\, d\mu$.

We impose the following localized doubling condition on the weight w. We remark that there are also other uses for the term *semilocally doubling* in the literature, see e.g. [2]. In our definition "local" refers to the fact that the condition is required only for points $x \in \Omega$, but "semi" is added since the balls need not be contained in Ω.

Definition 3.2. Let $\Omega \subsetneq X$ be an open set and let w be a weight for Ω. We say that w is semilocally doubling for Ω if for every $\kappa > 0$ there exists a constant $D(w, \kappa) \ge 1$ such that

$$0 < w(B(x, r)) \le D(w, \kappa) w(B(x, r/2)) < \infty$$

for all $x \in \Omega$ and $0 < r \le \kappa d(x, \Omega^c)$.

In some of our results we will need the following regularity property of w. See [10, Theorem 14.1] for a corresponding statement under slightly different assumptions. We provide a short proof for the reader's convenience.

Lemma 3.3. *Let $\Omega \subset X$ be an open set and let w be a weight for Ω. Then w is outer regular, that is, for every Borel set $E \subset X$ and every $\varepsilon > 0$, there exists an open set $V \supset E$ such that $w(V) \le w(E) + \varepsilon$.*

Proof. Let $\overline{X}$ be a completion of X. We remark that X could fail to be a Borel subset of its completion. We denote by $\mathscr{B}(X)$ and $\mathscr{B}(\overline{X})$ the Borel sets of X and $\overline{X}$, respectively. The measures μ and $w\mu$ extend to Borel regular measures $\overline{\mu}$ and $\overline{w\mu}$ on $\overline{X}$, and $\overline{\mu}$ is doubling, by [21, Lemma 1]. More precisely

$$\{F \in \mathscr{B}(\overline{X}) \mid F \cap X \in \mathscr{B}(X)\} = \mathscr{B}(\overline{X}), \tag{5}$$

and therefore one can define $\overline{\mu}(F) = \mu(F \cap X)$ and $\overline{w\mu}(F) = w(F \cap X)$ for each $F \in \mathscr{B}(\overline{X})$; see the proof of [21, Lemma 1]. This defines the extended measures as Borel measures that are finite on balls, and the Borel regular (complete) extended measures are obtained by completion. The space $\overline{X}$ is complete and the measure $\overline{\mu}$ doubling; thus $\overline{X}$ is proper by [1, Proposition 3.1]. Hence, the measure $\overline{w\mu}$ is outer regular on $\overline{X}$ by [7, Theorem 7.8].

Let $E \in \mathscr{B}(X)$ and $\varepsilon > 0$. By using σ-algebra arguments, one can show that $E = F \cap X$ for some $F \in \mathscr{B}(\overline{X})$. By the outer regularity of $\overline{w\mu}$, there exists an open set U in $\overline{X}$ such that $U \supset F$ and $\overline{w\mu}(U) \le \overline{w\mu}(F) + \varepsilon$. We define $V = U \cap X$, which is an open subset of X. Then $V \supset E$ and

$$w(V) = w(U \cap X) = \overline{w\mu}(U) \le \overline{w\mu}(F) + \varepsilon = w(F \cap X) + \varepsilon = w(E) + \varepsilon.$$

This shows that w is outer regular. $\square$

Let $\Omega \subsetneq X$ be an open set and fix a weight w for Ω. Let $0 < \kappa < \infty$ and $1 \le p < \infty$, and let f be a measurable function in X. We define restricted weighted maximal functions $\mathcal{M}_{p,w,\kappa} f$ and $\mathcal{M}^R_{p,w,\kappa} f$ at $x \in \Omega$ by

$$\mathcal{M}_{p,w,\kappa} f(x) := \sup_{0<r<\kappa d(x,\Omega^c)} \left(\frac{1}{w(B(x,r))} \int_{B(x,r)} |f|^p w \, d\mu \right)^{\frac{1}{p}}$$

and

$$\mathcal{M}^R_{p,w,\kappa} f(x) := \sup_{0<r<\min\{\kappa d(x,\Omega^c),R\}} \left(\frac{1}{w(B(x,r))} \int_{B(x,r)} |f|^p w \, d\mu \right)^{\frac{1}{p}}.$$

Observe that $0 < w(B(x,r)) < \infty$ for all balls $B(x,r)$ that appear within the supremums. The maximal functions $\mathcal{M}_{p,w,\kappa} f$ and $\mathcal{M}^R_{p,w,\kappa} f$ are lower semicontinuous in Ω. This follows easily using monotone convergence theorem and the fact that $B = \bigcup_{0<\varepsilon<1} (1-\varepsilon) B$ for all balls $B \subset X$.

The following lemmas are adaptations of similar results from our prior work [5, 6]. Although the methods are the same, we provide here the full proofs due to subtle technical differences.

Lemma 3.4. *Suppose that w is a semilocally doubling weight for an open set $\Omega \subsetneq X$. Assume that $1 \le q < \infty$ and $\kappa > 1$, and let $f \in L^q_{\mathrm{loc}}(X)$, $x \in \Omega$ and $\tau > 0$ be such that*

$$\mathcal{M}_{q,w,2\kappa} f(x) \le \tau.$$

Fix $\Lambda > 0$ and define

$$E_\Lambda = \{ y \in \Omega \mid \mathcal{M}^{\kappa d(x,\Omega^c)}_{q,w,2\kappa} f(y) > \Lambda \tau \}.$$

Then

$$\mathcal{M}_{1,w,\kappa} \mathbf{1}_{E_\Lambda}(x) \le \frac{D(w,10\kappa)^4}{\Lambda^q}. \tag{6}$$

Proof. Fix $0 < r < \kappa d(x,\Omega^c)$ and let $B = B(x,r)$. We need to show that

$$\frac{1}{w(B)} \int_B \mathbf{1}_{E_\Lambda} w \, d\mu \le \frac{D(w,10\kappa)^4}{\Lambda^q}. \tag{7}$$

The proof of (7) uses a covering argument. For each $y \in E_\Lambda \cap B$ we fix a ball $B_y = B(y,r_y)$ of radius $0 < r_y < \min\{2\kappa d(y,\Omega^c), \kappa d(x,\Omega^c)\}$ such that

$$\left(\frac{1}{w(B_y)} \int_{B_y} |f|^q w \, d\mu \right)^{\frac{1}{q}} > \Lambda \tau. \tag{8}$$

There are two cases to consider.

Case 1: There exists $y \in E_\Lambda \cap B$ with $r < r_y$. Then $B(y,r_y) \subset B(x,2r_y) \subset B(y,3r_y)$ and $3r_y \le 6\kappa d(y,\Omega^c)$. By semilocal doubling, we have $w(B(y,3r_y)) \le D(w,10\kappa)^2 w(B(y,r_y))$. Observe that $2r_y < 2\kappa d(x,\Omega^c)$. Therefore,

$$\begin{aligned}
\frac{1}{w(B)} \int_B \mathbf{1}_{E_\Lambda} w \, d\mu \le 1 &< \frac{\frac{1}{w(B_y)} \int_{B_y} |f|^q w \, d\mu}{\Lambda^q \tau^q} \\
&\le \frac{D(w,10\kappa)^2 \frac{1}{w(B(x,2r_y))} \int_{B(x,2r_y)} |f|^q w \, d\mu}{\Lambda^q \tau^q} \\
&\le \frac{D(w,10\kappa)^2 (\mathcal{M}_{q,w,2\kappa} f(x))^q}{\Lambda^q \tau^q} \le \frac{D(w,10\kappa)^4}{\Lambda^q},
\end{aligned}$$

proving inequality (7).

Case 2: For each $y \in E_\Lambda \cap B$ we have $r \geq r_y$. The $5r$-covering lemma [1, Lemma 1.7] yields a pairwise disjoint subcollection $\mathcal{B} \subset \{B_y \mid y \in E_\Lambda \cap B\}$ of balls such that $E_\Lambda \cap B \subset \bigcup_{B' \in \mathcal{B}} 5B'$. Hence, by (8) and the fact that $5r_y \leq 10\kappa d(y, \Omega^c)$ for every $y \in E_\Lambda \cap B$,

$$\begin{aligned}
\frac{1}{w(B)} \int_B \mathbf{1}_{E_\Lambda} w \, d\mu &\leq \frac{1}{w(B)} \sum_{B' \in \mathcal{B}} w(5B') \\
&\leq \frac{D(w, 10\kappa)^3}{w(B)} \sum_{B' \in \mathcal{B}} w(B') \\
&\leq \frac{D(w, 10\kappa)^3}{\Lambda^q \tau^q w(B)} \sum_{B' \in \mathcal{B}} \int_{B'} |f|^q w \, d\mu.
\end{aligned}$$

Since $r_{B'} \leq r$, we have $B' \subset 2B = B(x, 2r)$ for every $B' \in \mathcal{B}$. Also, since $2r < 2\kappa d(x, \Omega^c)$, we have $w(2B) \leq D(w, 10\kappa) w(B)$. Consequently, inequality (7) follows from the estimates

$$\begin{aligned}
\frac{1}{w(B)} \int_B \mathbf{1}_{E_\Lambda} w \, d\mu &\leq \frac{D(w, 10\kappa)^4}{\Lambda^q \tau^q w(2B)} \int_{2B} |f|^q w \, d\mu \\
&\leq \frac{D(w, 10\kappa)^4 (\mathcal{M}_{q,w,2\kappa} f(x))^q}{\Lambda^q \tau^q} \leq \frac{D(w, 10\kappa)^4}{\Lambda^q}.
\end{aligned}$$

□

The next approximation lemma is a variant of [5, Lemma 3.7]. The outer regularity of the weight, see Lemma 3.3, is needed in the proof. Recall that a Borel function $g \colon X \to [0, \infty)$ is *simple*, if it can be expressed as $g = \sum_{j=1}^k a_j \mathbf{1}_{E_j}$ for some real numbers $a_j > 0$ and Borel sets $E_j \subset X$, $j = 1, \ldots, k$.

Lemma 3.5. *Suppose that w is a semilocally doubling weight for an open set Ω. Assume that $1 \leq p < \infty$ and $\kappa > 1$, and let $g \colon X \to [0, \infty)$ be a simple Borel function. Then, for each finite set $F \subset \Omega$ and every $\varepsilon > 0$, there exists a non-negative and bounded $g_{F,\varepsilon} \in LC(X)$ such that $g(y) \leq g_{F,\varepsilon}(y)$ for all $y \in X \setminus F$ and $\mathcal{M}_{p,w,\kappa} g_{F,\varepsilon}(x) \leq \mathcal{M}_{p,w,\kappa} g(x) + \varepsilon$ for every $x \in F$.*

Proof. It suffices to prove the claim for singletons $F = \{x\}$, since for $F = \{x_1, \ldots, x_n\}$ the function $g_{F,\varepsilon}$ can be obtained as the minimum of the functions $g_{\{x_i\},\varepsilon}$. Fix $x \in \Omega$ and $\varepsilon > 0$.

Step 1: proving the claim for $g = \mathbf{1}_E$ with a Borel set E. We show that there exists an open set $U \subset X$ such that $\mathbf{1}_E \leq \mathbf{1}_U$ in $X \setminus \{x\}$ and

$$\mathcal{M}_{p,w,\kappa}(\mathbf{1}_U - \mathbf{1}_E)(x) < \varepsilon. \tag{9}$$

For each $m \in \mathbb{Z}$, we set

$$A_m = \{y \in X \mid 2^{m-1} < d(x, y) < 2^{m+1}\}.$$

Observe that each $y \in X$ belongs to at most two annuli A_m. Moreover, if $m \in \mathbb{Z}$ then by outer regularity of the weight w (Lemma 3.3) and the fact that A_m is open, there is an open set $U_m \subset A_m$ such that

$$\begin{aligned}
&A_m \cap E \subset U_m \quad \text{and} \\
&w(U_m \setminus E) = w(U_m \setminus (A_m \cap E)) \leq \frac{\varepsilon^p w(A_m)}{2D(w, 4\kappa)^2}.
\end{aligned} \tag{10}$$

In the case $w(A_m) = 0$ we can choose $U_m = A_m$. Define $U = \bigcup_{m \in \mathbb{Z}} U_m$. Then

$$E \setminus \{x\} \subset \bigcup_{m \in \mathbb{Z}} (A_m \cap E) \subset \bigcup_{m \in \mathbb{Z}} U_m = U. \tag{11}$$

As a consequence, we have $\mathbf{1}_E(y) \leq \mathbf{1}_U(y)$ for every $y \in X \setminus \{x\}$.

To prove (9), we let $B(x,r) \subset X$ be a ball with $0 < r < \kappa d(x,\Omega^c)$. Then $\mathbf{1}_U - \mathbf{1}_E = \mathbf{1}_{U\setminus E}$ μ-almost everywhere, and therefore by (10) we obtain

$$\begin{aligned}\frac{1}{w(B(x,r))}\int_{B(x,r)} |\mathbf{1}_U - \mathbf{1}_E|^p w\,d\mu &= \frac{1}{w(B(x,r))}\int_{B(x,r)} \mathbf{1}_{U\setminus E} w\,d\mu \\ &\le \frac{1}{w(B(x,r))}\int_X \sum_{m=-\infty}^{\lceil \log_2 r\rceil} \mathbf{1}_{U_m\setminus E} w\,d\mu \\ &= \frac{\varepsilon^p}{2D(w,4\kappa)^2 w(B(x,r))}\sum_{m=-\infty}^{\lceil \log_2 r\rceil} w(A_m) \\ &\le \frac{\varepsilon^p}{D(w,4\kappa)^2}\frac{w(B(x,4r))}{w(B(x,r))} \le \varepsilon^p \frac{w(B(x,r))}{w(B(x,r))} = \varepsilon^p.\end{aligned}$$

Inequality (9) follows by raising this estimate to power $1/p$ and then taking supremum over all balls $B(x,r)$ as above.

Step 2: proving the claim for a simple Borel function $g = \sum_{j=1}^k a_j \mathbf{1}_{E_j}$. By Step 1, for each $j = 1,\dots,k$, there exists a non-negative and bounded $g_{\{x\},\varepsilon,j} \in LC(X)$ such that $\mathbf{1}_{E_j} \le g_{\{x\},\varepsilon,j}$ in $X \setminus \{x\}$ and

$$\mathcal{M}_{p,w,\kappa}(g_{\{x\},\varepsilon,j} - \mathbf{1}_{E_j})(x) \le \frac{\varepsilon}{k \max_j a_j}. \tag{12}$$

Define $g_{\{x\},\varepsilon} = \sum_{j=1}^k a_j g_{\{x\},\varepsilon,j}$. Then $g \le g_{\{x\},\varepsilon}$ in $X \setminus \{x\}$, and by using the subadditivity and positive homogeneity of the maximal function and inequality (12), we conclude that

$$\begin{aligned}\mathcal{M}_{p,w,\kappa} g_{\{x\},\varepsilon}(x) &= \mathcal{M}_{p,w,\kappa}(g + g_{\{x\},\varepsilon} - g)(x) \\ &\le \mathcal{M}_{p,w,\kappa} g(x) + \mathcal{M}_{p,w,\kappa}(g_{\{x\},\varepsilon} - g)(x) \\ &\le \mathcal{M}_{p,w,\kappa} g(x) + \sum_{j=1}^k a_j \mathcal{M}_{p,w,\kappa}(g_{\{x\},\varepsilon,j} - \mathbf{1}_{E_j})(x) \\ &\le \mathcal{M}_{p,w,\kappa} g(x) + \varepsilon.\end{aligned}$$

□

4. Local Poincaré inequalities in open sets

In the sequel, we will need to assume that a suitable pointwise Poincaré inequality holds with respect to the weight w.

Definition 4.1. Let $1 \le p < \infty$, let $\Omega \subsetneq X$ be an open set and let w be a weight for Ω. We say that w is a p-Poincaré weight for Ω, if there are constants $C_{\mathrm{A}} > 0$, $\nu > C_{\mathrm{QC}}$ and $\kappa > 1$ such that for each non-negative and bounded $g \in LC(X)$ and every $x, y \in \Omega$ with

$$d(x,y) < d(x,\Omega^c)/(3\kappa),$$

it holds that

$$\inf_{\gamma\in\Gamma(X)^{\nu}_{x,y}} \int_\gamma g\,ds \le C_{\mathrm{A}}\, d(x,y)\big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y)\big). \tag{13}$$

Definition 4.1 for a p-Poincaré weight is slightly technical, since it is adjusted to our later purposes. The following lemma provides a more familiar variant of a p-Poincaré inequality that is sufficient for (13). We emphasize the local nature of these Poincaré inequalities with respect to Ω; for instance, we only require inequality (14) for balls B satisfying $2\lambda B \subset \Omega$. Compare also to [9] and [10, Section 3], and the references therein, concerning Poincaré inequalities and pointwise inequalities related to (13).

We write $u_{B;w} = \frac{1}{w(B)}\int_B u(x)w(x)\,d\mu(x)$ whenever $uw \in L^1(B)$ and B is a ball in X.

Lemma 4.2. *Let $1 \le p < \infty$ and $1 \le \lambda < \infty$, let $\Omega \subsetneq X$ be an open set, and let w be a semilocally doubling weight for Ω. Suppose there exists a constant C_1 such that for each $u \in \mathrm{Lip}(X)$ and for every bounded upper gradient g of u we have*

$$\frac{1}{w(B)} \int_B |u - u_{B;w}| w \, d\mu \le C_1 r \left(\frac{1}{w(\lambda B)} \int_{\lambda B} g^p w \, d\mu \right)^{\frac{1}{p}}, \tag{14}$$

whenever $B = B(x, r)$ is a ball with $2\lambda B \subset \Omega$. Then w is a p-Poincaré weight for Ω.

Proof. The proof has two steps.

Step 1: We show that there exist constants $C_2 = 6C_1 D(w, 2^{-1})^2$ and $\kappa = 3\lambda$ such that

$$|u(x) - u(y)| \le C_2 d(x, y) \big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y) \big) \tag{15}$$

for every $x, y \in \Omega$ with $d(x, y) < d(x, \Omega^c)/(3\kappa)$. Here u and g are as in the assumptions of the lemma.

Fix $x, y \in \Omega$, with $x \ne y$ and $r = d(x, y) < d(x, \Omega^c)/(9\lambda)$. Write $B_i = B(x, 2^{-i} r)$, for every $i \in \mathbb{N}_0$. A telescoping argument yields

$$\begin{aligned}
|u(x) - u_{B(x,r);w}| &\le \sum_{i=0}^{\infty} |u_{B_{i+1};w} - u_{B_i;w}| \\
&\le \sum_{i=0}^{\infty} \frac{w(B_i)}{w(B_{i+1})} \frac{1}{w(B_i)} \int_{B_i} |u - u_{B_i;w}| w \, d\mu \\
&\le C_1 D(w, 2^{-1}) \sum_{i=0}^{\infty} (2^{-i} r) \left(\frac{1}{w(\lambda B_i)} \int_{\lambda B_i} g^p w \, d\mu \right)^{\frac{1}{p}} \\
&\le 2C_1 D(w, 2^{-1}) d(x, y) \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x).
\end{aligned}$$

Observe that $B(x, r) \subset B(y, 2r)$ and $2r = 2d(x, y) < d(x, \Omega^c)/(4\lambda) \le d(y, \Omega^c)/(2\lambda)$. Thus, a similar telescoping argument gives

$$|u(y) - u_{B(y,2r);w}| \le 4C_1 D(w, 2^{-1}) d(x, y) \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y).$$

Since $B(x, r) \subset B(y, 2r) \subset B(x, 4r)$, we also have

$$\begin{aligned}
|u_{B(x,r);w} - u_{B(y,2r);w}| &\le \frac{1}{w(B(x, r))} \int_{B(x,r)} |u - u_{B(y,2r);w}| w \, d\mu \\
&\le \frac{w(B(x, 4r))}{w(B(x, r))} \frac{1}{w(B(y, 2r))} \int_{B(y,2r)} |u - u_{B(y,2r);w}| w \, d\mu \\
&\le 2C_1 D(w, 2^{-1})^2 d(x, y) \left(\frac{1}{w(B(y, 2\lambda r))} \int_{B(y,2\lambda r)} g^p w \, d\mu \right)^{\frac{1}{p}} \\
&\le 2C_1 D(w, 2^{-1})^2 d(x, y) \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y).
\end{aligned}$$

By combining the estimates above we obtain

$$\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u_{B(x,r);w}| + |u_{B(x,r);w} - u_{B(y,2r);w}| + |u(y) - u_{B(y,2r);w}| \\
&\le 6C_1 D(w, 2^{-1})^2 d(x, y) \big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y) \big),
\end{aligned}$$

and this completes the proof of inequality (15).

Step 2: With the aid of inequality (15)**, we show that w is a p-Poincaré weight for Ω.** Let $g \in LC(X)$ be a non-negative and bounded function. Fix $x, y \in \Omega$ such that $0 < d(x, y) < d(x, \Omega^c)/(3\kappa)$ and let $\delta > 0$; here $\kappa = 3\lambda$ by Step 1. Define $u \colon X \to [0, \infty)$ by setting

$$u(z) = \inf_{\gamma} \int_{\gamma} h \, ds, \qquad z \in X, \tag{16}$$

where

$$h = g + \big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y) + \delta\big)$$

and the infimum is taken over all curves γ in X connecting z to y. Note that h is a non-negative bounded Borel function, and clearly $u(y) = 0$. Fix $z_1, z_2 \in X$ and consider any curve σ connecting z_1 to z_2. We claim that

$$|u(z_1) - u(z_2)| \le \int_\sigma h\, ds. \tag{17}$$

From this it follows, in particular, that h is an upper gradient of u. Moreover, since X is quasiconvex and h is bounded, estimate (17) implies that $u \in \mathrm{Lip}(X)$.

In order to prove (17), we may assume that $u(z_1) > u(z_2)$. Fix $\varepsilon > 0$ and let γ be a curve in X that connects z_2 to y and satisfies inequality

$$u(z_2) \ge \int_\gamma h\, ds - \varepsilon.$$

Let $\sigma\gamma$ be the concatenation of σ and γ. Then

$$|u(z_1) - u(z_2)| = u(z_1) - u(z_2) \le \int_{\sigma\gamma} h\, ds - \int_\gamma h\, ds + \varepsilon = \int_\sigma h\, ds + \varepsilon.$$

The desired inequality (17) follows by taking $\varepsilon \to 0_+$.

Application of inequality (15) to $u \in \mathrm{Lip}(X)$ and its bounded upper gradient h gives

$$|u(x) - u(y)| \le C_2 d(x,y)\big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} h(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} h(y)\big) < \infty.$$

Since $u(x) \ge \delta d(x,y) > 0$ and $u(y) = 0$, by (16) there is a curve γ in X connecting x to y such that

$$\begin{aligned}
\int_\gamma g\, ds + \Big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) &+ \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y) + \delta\Big) \operatorname{len}(\gamma) \\
&= \int_\gamma h\, ds \le 2u(x) = 2|u(x) - u(y)| \\
&\le 2C_2 d(x,y)\big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} h(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} h(y)\big) \\
&\le 2C_2\, d(x,y)\big(3\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) + 3\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y) + 2\delta\big) \\
&\le 6C_2\, d(x,y)\big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y) + \delta\big).
\end{aligned} \tag{18}$$

The second last inequality follows from the sublinearity of maximal function and definition of h. From (18) we see that $\operatorname{len}(\gamma) \le 6C_2\, d(x,y)$. By taking $\delta \to 0_+$, we also obtain from (18) that inequality (13) holds, that is,

$$\inf_{\gamma \in \Gamma(X)^\nu_{x,y}} \int_\gamma g\, ds \le C_{\mathrm{A}}\, d(x,y)\big(\mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(x) + \mathcal{M}^{\kappa d(x,y)}_{p,w,\kappa} g(y)\big),$$

with $C_{\mathrm{A}} = 6C_2$, $\kappa = 3\lambda$ and $\nu > \max\{C_{\mathrm{QC}}, 6C_2\}$. □

5. The class of p-Hardy weights

The following class of weights turns out to be natural in connection with pointwise Hardy inequalities; see Lemma 7.2, and compare also to the definition of p-Poincaré weights in Definition 4.1.

Definition 5.1. Let $1 \le p < \infty$, let $\Omega \subsetneq X$ be an open set, and let w be a weight for Ω. We say that w is a p-Hardy weight for Ω if there are constants $C_\Gamma > 0$, $\nu > C_{\mathrm{QC}}$ and $\kappa > 1$ such that for each non-negative and bounded $g \in LC(X)$ and every $x \in \Omega$, we have

$$\inf_{\gamma \in \Gamma(X)^\nu_{x,\Omega^c}} \int_\gamma g\, ds \le C_\Gamma\, d(x,\Omega^c) \mathcal{M}_{p,w,\kappa} g(x). \tag{19}$$

Next we define a convenient albeit slightly abstract α-function that condenses the p-Hardy weight property, specifically inequality (19), in a single function. Indeed, despite the complex appearance this function is a very useful tool in the proof of the self-improvement for p-Hardy weight property.

Definition 5.2. Let $\Omega \subsetneq X$ be an open set and let w be a weight for Ω. If $\tau \geq 0$, $\kappa > 1$, $1 \leq p < \infty$ and $x \in \Omega$, we write

$$\mathcal{E}^{\kappa,\tau}_{p,w,x,\Omega} = \{g \in LC(X) \mid \mathcal{M}_{p,w,\kappa} g(x) \leq \tau \text{ and } g(y) \in [0,1] \text{ for all } y \in X\}.$$

If also $\nu > C_{\mathrm{QC}}$, then we write

$$\alpha_{p,w,\Omega}(\nu,\kappa,\tau) := \sup_{x\in\Omega} \sup_{g\in\mathcal{E}^{\kappa,\tau}_{p,w,x,\Omega}} \frac{\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}} \int_\gamma g\,ds}{d(x,\Omega^c)}. \tag{20}$$

The parameter ν is related to the maximum length of the curves γ, since $\mathrm{len}(\gamma) \leq \nu d(x,\Omega^c)$. The remaining parameters κ and τ are used to control the non-locality and size, or "level", of the maximal function $\mathcal{M}_{p,w,\kappa} g(x)$.

The following lemma codifies the relationship between inequality (19) and the α-function.

Lemma 5.3. *Let $\Omega \subsetneq X$ be an open set and let w be a weight for Ω. Assume that $\kappa > 1$, $1 \leq p < \infty$ and $\nu > C_{\mathrm{QC}}$, and let $g \in LC(X)$ be such that $g(y) \in [0,1]$ for every $y \in X$. Then, for every $x \in \Omega$, we have*

$$\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}} \int_\gamma g\,ds \leq d(x,\Omega^c)\,\alpha_{p,w,\Omega}\big(\nu,\kappa,(\mathcal{M}_{p,w,\kappa} g(x))\big). \tag{21}$$

Proof. Take any $g \in LC(X)$ with $g(y) \in [0,1]$ for all $y \in X$. Fix $x \in \Omega$ and write

$$\tau = \mathcal{M}_{p,w,\kappa} g(x) \geq 0.$$

Then $g \in \mathcal{E}^{\kappa,\tau}_{p,w,x,\Omega}$, and by the definition of $\alpha_{p,w,\Omega}$

$$\frac{\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}} \int_\gamma g\,ds}{d(x,\Omega^c)} \leq \sup_{h\in\mathcal{E}^{\kappa,\tau}_{p,w,x,\Omega}} \frac{\inf_{\gamma\in\Gamma(X)^{\nu}_{x,\Omega^c}} \int_\gamma h\,ds}{d(x,\Omega^c)} \leq \alpha_{p,w,\Omega}(\nu,\kappa,\tau).$$

The last step holds, since $x \in \Omega$. □

In particular, from Lemma 5.3 we obtain the following sufficient condition for p-Hardy weights in terms of a τ-linear upper bound for the α-function.

Lemma 5.4. *Let $1 \leq p < \infty$, let $\Omega \subsetneq X$ be an open set and let w be a weight for Ω. Suppose that there are constants $\nu > C_{\mathrm{QC}}$, $\kappa > 1$ and $C_\alpha > 0$ such that, for any $\tau \geq 0$, we have*

$$\alpha_{p,w,\Omega}(\nu,\kappa,\tau) \leq C_\alpha \tau.$$

Then w is a p-Hardy weight for Ω.

Proof. By Definition 5.1, it suffices to find a constant $C_\Gamma > 0$ such that inequality (19) holds for every non-negative bounded $g \in LC(X)$ and every $x \in \Omega$ — the remaining constants ν and κ are given in the assumptions of the present lemma. Fix such a function g and a point $x \in \Omega$. Since g is bounded and inequality (19) is invariant under multiplication of g with a strictly positive constant, we may further assume that $g(y) \in [0,1]$ for all $y \in X$. Then the desired estimate (19), with $C_\Gamma = C_\alpha$, follows immediately from Lemma 5.3 and the assumptions. □

The converse of Lemma 5.4 is also true, as we will see in Section 6. Therein the following inequalities for the α-function become useful.

Lemma 5.5. *Let $\Omega \subsetneq X$ be an open set. Let $0 \le \tau < \tau'$, $\kappa > 1$, $1 \le p < \infty$ and $\nu > C_{\mathrm{QC}}$. Then*

$$\alpha_{p,w,\Omega}(\nu,\kappa,\tau) \le \alpha_{p,w,\Omega}(\nu,\kappa,\tau'), \qquad \alpha_{p,w,\Omega}(\nu,\kappa,\tau) \le \nu,$$

and, for every $M \ge 1$,

$$\alpha_{p,w,\Omega}(\nu,\kappa,M\tau) \le M\alpha_{p,w,\Omega}(\nu,\kappa,\tau).$$

Proof. These inequalities are clear from the definition of $\alpha_{p,w,\Omega}(\nu,\kappa,\tau)$ in (20). The second inequality also uses the fact that g is bounded by 1 and quasiconvexity, that is, existence of a curve with $\mathrm{len}(\gamma) \le \nu d(x,\Omega^c)$. □

6. Self-improvement property for p-Hardy weights

In this section we examine self-improvement properties of p-Hardy weights for $1 < p < \infty$. We assume that w is a p_0-Poincaré weight for some $p_0 < p$. This assumption allows us to focus on the new phenomena that arise especially in connection with the improvement of pointwise p-Hardy inequalities. Recall that if the metric space X is complete and X supports a $(1,p)$-Poincaré inequality, that is, (14) with $w = 1$ holds for all balls $B \subset X$ whenever $u \in \mathrm{Lip}(X)$ and g is an upper gradient of u, then there exists $p_0 < p$ such that X supports a $(1,p_0)$-Poincaré inequality; see [11] and see also Lemma 8.3 concerning this assumption for distance weights in $\mathbb{R}^n$. It is plausible that also p-Poincaré weights enjoy self-improvement properties, but in the present work we will not focus on this aspect.

The following Theorem 6.1 implies a self-improvement property for p-Hardy weights. This result also provides a converse of Lemma 5.4 for $p > 1$.

Theorem 6.1. *Let $1 < p_0 < p < \infty$, let $\Omega \subsetneq X$ be an open set and let w be a semilocally doubling weight for Ω. Assume that w is a p_0-Poincaré weight for Ω and a p-Hardy weight for Ω. Then there exist an exponent $q \in (p_0,p)$ and constants $N > C_{\mathrm{QC}}$, $K > 1$ and $C_\alpha > 0$ such that*

$$\alpha_{q,w,\Omega}(N,K,\tau) \le C_\alpha \tau \tag{22}$$

whenever $\tau \ge 0$.

Proof. First, we fix some constants to give accurate bounds. In Definition 5.1, inequality (19) holds with constants $C_\Gamma > 0$, $\nu_\Gamma > C_{\mathrm{QC}}$ and $\kappa_\Gamma > 1$. Also, denote by $C_{\mathrm{A}} > 0$, $\nu_{\mathrm{A}} > C_{\mathrm{QC}}$ and $\kappa_{\mathrm{A}} > 1$ the constants from inequality (13) in Definition 4.1, for the exponent $p_0 < p$. By Hölder's inequality we may assume $p/2 < p_0$. Without loss of generality, we also assume that $\kappa_\Gamma = \kappa_{\mathrm{A}} =: \kappa$ and $\nu_\Gamma = \nu_{\mathrm{A}} =: \nu$.

Step 1: Estimate to prove, strategy and parameters. Assume that we have found parameters $k \in \mathbb{N}$, $K, S \in (1,\infty)$, $N \in (C_{\mathrm{QC}},\infty)$, $M > 1$ and $\delta \in (0,1)$ such that, for each $q \in (p_0,p)$ and every $\tau > 0$, we have

$$\alpha_{q,w,\Omega}(N,K,\tau) \le S\tau + \delta \max_{i=1,\dots,k} \big(M^{-iq/p}\alpha_{q,w,\Omega}(N,K,M^i\tau)\big). \tag{23}$$

From this inequality and Lemma 5.5, we obtain

$$\alpha_{q,w,\Omega}(N,K,\tau) \le S\tau + \delta M^{k\frac{p-q}{p}}\alpha_{q,w,\Omega}(N,K,\tau) \qquad \text{for all } q \in (p_0,p) \text{ and } \tau > 0.$$

Observe that the last term on the right is finite by Lemma 5.5. In order to absorb this term to the left-hand side, we need $\delta M^{k\frac{p-q}{p}} < 1$. This can be ensured by choosing $q \in (p_0,p)$ so close to p that

$$0 < p - q < \frac{p\ln(\frac{1}{\delta})}{k\ln(M)}.$$

With this choice of q we find for all $\tau > 0$ that

$$\alpha_{q,w,\Omega}(N,K,\tau) \le \left(\frac{S}{1-\delta M^{k\frac{p-q}{p}}}\right)\tau =: C_\alpha \tau.$$

This inequality holds also for $\tau = 0$, which is seen by using monotonicity property of the α-function, see Lemma 5.5. Thus, the desired inequality (22) follows from (23). Hence, it suffices to find parameters, as above, for which inequality (23) holds for every $q \in (p_0, p)$ and $\tau > 0$.

We begin by fixing the auxiliary parameters

$$K = 2\kappa, \quad N = 3\nu, \quad M = 4, \quad \delta = \frac{1}{6}.$$

We also choose $k \in \mathbb{N}$ so large that $C_\Gamma^p 2^p D(w, 10\kappa)^4 k^{1-p} < (\delta/(3\kappa))^p$. The last parameter is given by $S = 1 + M^k\nu + 3C_{\mathrm{A}}M^k$. For what follows $q \in (p_0, p)$ and $\tau > 0$ are arbitrary.

Now, the overall strategy is to construct, for any $x \in \Omega$ and any $g \in \mathcal{E}^{K,\tau}_{q,w,x,\Omega}$, a curve $\gamma \in \Gamma(X)^N_{x,\Omega^c}$ such that, for some $i_0 = 1, \ldots, k$

$$\int_\gamma g\, ds \le S\tau d(x, \Omega^c) + \delta M^{-i_0 q/p} \alpha_{q,w,\Omega}(N, K, M^{i_0}\tau) d(x, \Omega^c). \tag{24}$$

Estimating the right-hand side by the maximum over possible i_0, then dividing both sides by $d(x, \Omega^c)$, and finally taking the supremum over x and g as above, proves inequality (23). This strategy first involves choosing a good level i_0 along with some proto-curve γ_0 having a small integral, and then adjusting the curve at the level i_0 by filling in certain gaps.

Step 2: Choosing a good level i_0 and the proto-curve γ_0. Fix $x \in \Omega$ and $g \in \mathcal{E}^{K,\tau}_{q,w,x,\Omega}$. For each $i \ge 1$, we write

$$E_i := \{y \in \Omega \mid \mathcal{M}^{\kappa d(x,\Omega^c)}_{q,w,K} g(y) > M^i\tau\},$$

and define a bounded function $h\colon X \to [0, \infty)$ by setting

$$h = \frac{1}{k}\sum_{i=1}^{k} \mathbf{1}_{E_i} M^{iq/p}.$$

Since $E_j \supset E_i$ if $j \le i$ and $p/2 < p_0 < q < p$, it follows that

$$h^p \le \frac{1}{k^p}\sum_{j=1}^{k}\left(\sum_{i=1}^{j} M^{iq/p}\right)^p \mathbf{1}_{E_j} \le \frac{2^p}{k^p}\sum_{j=1}^{k}\mathbf{1}_{E_j} M^{jq}.$$

In the final estimate, we also use the choice $M = 4$ to obtain the factor 2^p. Observe that $\mathbf{1}_{E_i} \in LC(X)$ since E_i is open, for each $i = 1, \ldots, k$, by the lower semicontinuity of $\mathcal{M}^{\kappa d(x,\Omega^c)}_{q,w,K} g$. Hence, we have $h \in LC(X)$. By sublinearity and monotonicity of the maximal function, Lemma 3.4, and the assumption that $g \in \mathcal{E}^{K,\tau}_{q,w,x,\Omega}$, where $K = 2\kappa$, we obtain

$$\begin{aligned}(\mathcal{M}_{p,w,\kappa} h(x))^p &\le \frac{2^p}{k^p}\sum_{j=1}^{k}(\mathcal{M}_{1,w,\kappa}\mathbf{1}_{E_j}(x))M^{jq} \\ &\le \frac{2^p}{k^p}\sum_{j=1}^{k}\frac{D(w,10\kappa)^4}{M^{jq}}M^{jq} \le \frac{2^p D(w,10\kappa)^4}{k^{p-1}}.\end{aligned} \tag{25}$$

Then, by the choice of k and estimate (25), we have

$$C_\Gamma d(x, \Omega^c)\mathcal{M}_{p,w,\kappa} h(x) < \frac{\delta}{3\kappa} d(x, \Omega^c).$$

Therefore by Definition 5.1, with exponent p, there is a curve $\gamma_0 \in \Gamma(X)^\nu_{x,\Omega^c}$, which is parameterized by the arc length and defined on the interval $[0, \mathrm{len}(\gamma_0)]$, such that

$$\frac{1}{k}\sum_{i=1}^{k} M^{iq/p}\int_{\gamma_0}\mathbf{1}_{E_i}\, ds = \int_{\gamma_0} h\, ds \le \frac{\delta}{3\kappa} d(x, \Omega^c) \tag{26}$$

and

$$\operatorname{len}(\gamma_0) \le \nu d(x, \Omega^c). \tag{27}$$

Without loss of generality, we may assume that $\gamma_0([0, \operatorname{len}(\gamma_0)) \subset \Omega$. By inequality (26), there exists $i_0 \in \{1, \dots, k\}$ such that

$$\int_{\gamma_0} \mathbf{1}_{E_{i_0}}\, ds \le \frac{\delta}{3\kappa} M^{-i_0 q/p} d(x, \Omega^c). \tag{28}$$

Step 3: Adjusting the curve at level i_0 by filling in gaps. Recall that the proto-curve γ_0 is parameterized by arc length. Let $O = \gamma_0^{-1}(E_{i_0})$ and write $T = [0, \operatorname{len}(\gamma_0)] \setminus O$. By the lower semicontinuity of g and the definition of E_{i_0} we have, for all $t \in T \setminus \{\operatorname{len}(\gamma_0)\}$,

$$g(\gamma_0(t)) \le \mathcal{M}^{\kappa d(x,\Omega^c)}_{q,w,K} g(\gamma_0(t)) \le M^{i_0}\tau. \tag{29}$$

Since E_{i_0} is open in X, the set O is relatively open in $[0, \operatorname{len}(\gamma_0)]$. Observe that $0 \notin O$ since $g \in \mathcal{E}^{K,\tau}_{q,w,x,\Omega}$. Likewise $\operatorname{len}(\gamma_0) \notin O$ since $\gamma_0(\operatorname{len}(\gamma_0)) \in \Omega^c$. Hence, we can write O as a union of so-called gaps:

$$O = \bigcup_{i \in I} (a_i, b_i), \tag{30}$$

where $I \subset \mathbb{N}$ is a finite or infinite indexing set. We also write $x_i := \gamma_0(a_i)$, $y_i := \gamma_0(b_i)$ and $d_i := d(x_i, y_i)$ for each $i \in I$. There are two cases to consider: either $d_i < d(x_i, \Omega^c)/(3\kappa)$ for all $i \in I$ or there exists $i \in I$ such that $d_i \ge d(x_i, \Omega^c)/(3\kappa)$. The latter also includes the case when $y_i \in \Omega^c$ for some $i \in I$. In both cases the gaps (a_i, b_i) are pairwise disjoint and $0 \le a_i < b_i \le \operatorname{len}(\gamma_0)$ for each $i \in I$. By inequality (28), we have

$$\sum_{i \in I} d_i \le \sum_{i \in I} \operatorname{len}(\gamma_0|_{[a_i,b_i]}) = \sum_{i \in I} \int_{\gamma_0|_{[a_i,b_i]}} \mathbf{1}_{E_{i_0}}\, ds \le \int_{\gamma_0} \mathbf{1}_{E_{i_0}}\, ds \le \frac{\delta}{3\kappa} M^{-i_0 q/p} d(x, \Omega^c). \tag{31}$$

For each i we next define a filling curve $\gamma_i \colon [a_i, b_i] \to X$ connecting $\gamma_0(a_i)$ and $\gamma_0(b_i)$.

Case 1: We have $d(x_i, y_i) = d_i < d(x_i, \Omega^c)/(3\kappa)$ for all $i \in I$. Fix $i \in I$. If $d_i = 0$, we define $\gamma_i(t) = \gamma_0(a_i) = \gamma_0(b_i)$ for each $t \in [a_i, b_i]$. From now on we assume that $d_i > 0$ and proceed as follows. Observe that $\kappa < K$ and $x_i, y_i \in \Omega \setminus E_{i_0}$. This gives

$$\mathcal{M}^{\kappa d(x,\Omega^c)}_{q,w,\kappa} g(x_i) \le M^{i_0}\tau \quad \text{and} \quad \mathcal{M}^{\kappa d(x,\Omega^c)}_{q,w,\kappa} g(y_i) \le M^{i_0}\tau. \tag{32}$$

We apply Definition 4.1 to the points x_i and y_i. After a reparameterization, this yields a curve $\gamma_i \colon [a_i, b_i] \to X$ such that $\gamma_i(a_i) = x_i$, $\gamma_i(b_i) = y_i$,

$$\operatorname{len}(\gamma_i) \le \nu d(x_i, y_i) = \nu d_i, \tag{33}$$

and, by using also Hölder's inequality and the fact that $p_0 < q$,

$$\int_{\gamma_i} g\, ds \le C_{\mathrm{A}} d(x_i, y_i) \left(\mathcal{M}^{\kappa d(x_i,y_i)}_{q,w,\kappa} g(x_i) + \mathcal{M}^{\kappa d(x_i,y_i)}_{q,w,\kappa} g(y_i) \right) + \underbrace{C_{\mathrm{A}} M^{i_0} \tau d(x_i, y_i)}_{>0}. \tag{34}$$

Here $\kappa d(x_i, y_i) \le \kappa d(x, \Omega^c)$, since by (31) we have

$$d(x_i, y_i) = d_i \le \sum_{i \in I} d_i \le d(x, \Omega^c).$$

This estimate together with (32) and (34) gives

$$\int_{\gamma_i} g\, ds \le 3 C_{\mathrm{A}} M^{i_0} \tau d_i. \tag{35}$$

We define a curve $\gamma\colon [0,\mathrm{len}(\gamma_0)] \to X$ by setting $\gamma(t) = \gamma_0(t)$ if $t \in T$ and $\gamma(t) = \gamma_i(t)$ if $t \in (a_i, b_i)$ for some $i \in I$ that is uniquely determined by t. Then, by the length estimates (27) and (33), followed by inequality (31), we obtain

$$\mathrm{len}(\gamma) \le \mathrm{len}(\gamma_0) + \sum_{i\in I} \mathrm{len}(\gamma_i) \le \nu d(x,\Omega^c) + \nu \sum_{i\in I} d_i \le 2\nu d(x,\Omega^c) \le N d(x,\Omega^c).$$

From this it follows that $\gamma \in \Gamma(X)^N_{x,\Omega^c}$; we remark that the required continuity and connecting properties of γ are straightforward to establish, and we omit the details. Also, by inequalities (27), (29), (31) and (35), we have

$$\begin{aligned}\int_\gamma g\,ds &= \int_T g(\gamma_0(t))\,dt + \sum_{i\in I}\int_{\gamma_i} g\,ds \\ &\le M^{i_0}\tau\nu d(x,\Omega^c) + 3C_{\mathrm{A}} M^{i_0}\tau d(x,\Omega^c) \\ &\le (M^{i_0}\nu + 3C_{\mathrm{A}} M^{i_0})\tau d(x,\Omega^c) \le S\tau d(x,\Omega^c).\end{aligned}$$

Thus curve γ satisfies inequality (24) and this concludes the proof of the first case.

Case 2: There exists $i \in I$ such that $d(x_i,y_i) = d_i \ge d(x_i,\Omega^c)/(3\kappa)$. This includes the case when $b_i = \mathrm{len}(\gamma_0)$ for some $i \in I$. Write

$$t = \inf\{a_i \mid i \in I \text{ and } d(x_i,y_i) \ge d(x_i,\Omega^c)/(3\kappa)\} \in [0,\mathrm{len}(\gamma_0)).$$

The infimum is reached, that is, there exists an index $i_0 \in I$ such that $t = a_{i_0}$ and $d(x_{i_0},y_{i_0}) \ge d(x_{i_0},\Omega^c)/(3\kappa)$. Indeed, otherwise there would exist a strictly decreasing sequence $(a_{i_k})_{k\in\mathbb{N}}$ such that $i_k \in I$ and $d(x_{i_k},y_{i_k}) \ge d(x_{i_k},\Omega^c)/(3\kappa)$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} a_{i_k} = t$. Clearly $a_{i_{k-1}} - a_{i_k} \to 0$ as $k \to \infty$. Since γ_0 is parameterized by arc length, we obtain for all $k > 1$

$$\begin{aligned}d(\gamma_0(a_{i_k}),\Omega^c)/(3\kappa) &= d(x_{i_k},\Omega^c)/(3\kappa) \le d(x_{i_k},y_{i_k}) \\ &= d(\gamma_0(a_{i_k}),\gamma_0(b_{i_k})) \le b_{i_k} - a_{i_k} \le a_{i_{k-1}} - a_{i_k} \xrightarrow{k\to\infty} 0.\end{aligned}$$

Hence, by continuity, we have $d(\gamma_0(t),\Omega^c) = \lim_{k\to\infty} d(\gamma_0(a_{i_k}),\Omega^c) = 0$. Since Ω^c is closed, this implies $\gamma_0(t) \in \Omega^c$. This is a contradiction, since $t < \mathrm{len}(\gamma_0)$ and, on the other hand, we have assumed that $\gamma_0([0,\mathrm{len}(\gamma_0)) \subset \Omega$.

Let $J := \{i \in I \mid a_i < a_{i_0}\}$. Then $d_i < d(x_i,\Omega^c)/(3\kappa)$ for all $i \in J$. As in the previous case, for each $i \in J$, we can first construct curves $\gamma_i\colon [a_i,b_i] \to X$ such that

$$\mathrm{len}(\gamma_i) \le \nu d(x_i,y_i) = \nu d_i, \tag{36}$$

and

$$\int_{\gamma_i} g\,ds \le 3C_{\mathrm{A}} M^{i_0}\tau d_i. \tag{37}$$

For $i = i_0$ we are too close to the boundary and must proceed more carefully. By using (31) and the equality $3K\delta = \kappa$, we first observe that

$$Kd(x_{i_0},\Omega^c) \le 3\kappa K d(x_{i_0},y_{i_0}) = 3\kappa K d_{i_0} \le 3K\delta d(x,\Omega^c) \le \kappa d(x,\Omega^c).$$

We still have that $x_{i_0} \in \Omega \setminus E_{i_0}$, and thus

$$\mathcal{M}_{q,w,K} g(x_{i_0}) \le \mathcal{M}^{\kappa d(x,\Omega^c)}_{q,w,K} g(x_{i_0}) \le M^{i_0}\tau.$$

From this it follows that $g \in \mathcal{E}^{K,M^{i_0}\tau}_{q,w,x_{i_0},\Omega}$. By definition (20) of the function $\alpha_{q,w,\Omega}(N,K,M^{i_0}\tau)$, we obtain a curve $\gamma_{i_0}\colon [a_{i_0},b_{i_0}] \to X$ connecting $x_{i_0} \in \Omega$ to Ω^c such that

$$\mathrm{len}(\gamma_{i_0}) \le N d(x_{i_0},\Omega^c) \le 3\kappa N d(x_{i_0},y_{i_0}) = 3\kappa N d_{i_0} \tag{38}$$

and

$$\begin{aligned}\int_{\gamma_{i_0}} g\,ds &\le d(x_{i_0},\Omega^c)\alpha_{q,w,\Omega}(N,K,M^{i_0}\tau) + \underbrace{\tau d(x,\Omega^c)}_{>0}\\ &\le 3\kappa d_{i_0}\alpha_{q,w,\Omega}(N,K,M^{i_0}\tau)+\tau d(x,\Omega^c).\end{aligned} \tag{39}$$

We now define a curve $\gamma\colon [0,b_{i_0}]\to X$ by setting $\gamma(t)=\gamma_0(t)$ if $t\in T\cap[0,a_{i_0}]$, $\gamma(t)=\gamma_i(t)$ if $t\in(a_i,b_i)$ for some $i\in J$, which is uniquely determined by t, and $\gamma(t)=\gamma_{i_0}(t)$ for every $t\in(a_{i_0},b_{i_0}]$. Then by (27), (31), (36), (38), and our choices of N and δ, we obtain

$$\begin{aligned}\operatorname{len}(\gamma) &\le \operatorname{len}(\gamma_0)+\sum_{i\in J}\operatorname{len}(\gamma_i)+\operatorname{len}(\gamma_{i_0})\\ &\le (\nu+\nu+\delta N)d(x,\Omega^c)\le N d(x,\Omega^c),\end{aligned}$$

and thus $\gamma\in\Gamma(X)^N_{x,\Omega^c}$. Finally, by inequalities (27), (29), (31), (37), and (39) we have

$$\begin{aligned}\int_\gamma g\,ds &= \int_{T\cap[0,a_{i_0}]} g(\gamma_0(t))\,dt+\sum_{i\in J}\int_{\gamma_i} g\,ds+\int_{\gamma_{i_0}} g\,ds\\ &\le M^{i_0}\tau\nu d(x,\Omega^c)+3C_{\mathrm{A}}M^{i_0}\tau d(x,\Omega^c)+3\kappa d_{i_0}\alpha_{q,w,\Omega}(N,K,M^{i_0}\tau)+\tau d(x,\Omega^c)\\ &\le S\tau d(x,\Omega^c)+\delta M^{-i_0 q/p}\alpha_{q,w,\Omega}(N,K,M^{i_0}\tau)d(x,\Omega^c).\end{aligned}$$

This shows that (24) holds also in Case 2, and the proof is complete. □

7. Pointwise (p,w)-Hardy inequalities

The definition of a pointwise (p,w)-Hardy inequality is as follows; recall that $\Omega^c = X\setminus\Omega$.

Definition 7.1. Let $1\le p<\infty$, let $\Omega\subsetneq X$ be an open set, and let w be a weight for Ω. We say that a pointwise (p,w)-Hardy inequality holds in Ω if there exist constants $C_{\mathrm{H}}>0$ and $\kappa>1$ such that for every Lipschitz function $u\in\operatorname{Lip}_0(\Omega)$, every bounded upper gradient g of u and every $x\in\Omega$, we have

$$|u(x)|\le C_{\mathrm{H}}\,d(x,\Omega^c)\mathcal{M}_{p,w,\kappa}g(x). \tag{40}$$

These pointwise (p,w)-Hardy inequalities in fact characterize the class of p-Hardy weights for Ω, thus explaining the terminology.

Lemma 7.2. *Let $1\le p<\infty$, let $\Omega\subsetneq X$ be an open set and let w be a semilocally doubling weight for Ω. Then a pointwise (p,w)-Hardy inequality holds in Ω if, and only if, w is a p-Hardy weight for Ω.*

Proof. Throughout this proof, we tacitly assume that curves are parameterized by arc length. First assume that a pointwise (p,w)-Hardy inequality (40) holds in Ω with constants $C_{\mathrm{H}}>0$ and $\kappa_\Gamma>1$. Let $g\in LC(X)$ be a non-negative and bounded function, and fix $x\in\Omega$ and $\delta>0$. We define a function $u\colon X\to[0,\infty)$ by setting

$$u(y)=\inf_\gamma\int_\gamma h\,ds,\qquad y\in X, \tag{41}$$

where $h=g+\mathcal{M}_{p,w,\kappa_\Gamma}g(x)+\delta$ and the infimum is taken over all curves γ in X connecting y to Ω^c; note that h is a non-negative bounded Borel function. Clearly, we have $u=0$ in Ω^c. Fix $y,z\in X$ and consider any curve σ connecting y to z. As in Step 2 of the proof of Lemma 4.2, we assume that $u(y)>u(z)$ and fix $\varepsilon>0$. We let γ be a curve in X that connects z to Ω^c and satisfies inequality

$$u(z)\ge\int_\gamma h\,ds-\varepsilon,$$

and define $\sigma\gamma$ to be the concatenation of σ and γ. Then, as in the proof of Lemma 4.2,

$$|u(y) - u(z)| \le \int_\sigma h\, ds + \varepsilon,$$

and by taking $\varepsilon \to 0_+$ we obtain

$$|u(y) - u(z)| \le \int_\sigma h\, ds. \tag{42}$$

This shows that h is an upper gradient of u. Moreover, since X is quasiconvex and h is bounded, it follows from (42) that $u \in \mathrm{Lip}_0(\Omega)$.

Now, applying the assumed pointwise (p, w)-Hardy inequality (40) to $u \in \mathrm{Lip}_0(\Omega)$ and its bounded upper gradient h yields

$$u(x) \le C_{\mathrm{H}}\, d(x, \Omega^c)\mathcal{M}_{p,w,\kappa_\Gamma} h(x) < \infty.$$

Since $u(x) \ge \delta d(x, \Omega^c) > 0$, by (41) there is a curve γ in X connecting x to Ω^c such that

$$\begin{aligned}\int_\gamma g\, ds + (\mathcal{M}_{p,w,\kappa_\Gamma} g(x) + \delta)\,\mathrm{len}(\gamma) = \int_\gamma h\, ds &\le 2u(x)\\ &\le 2C_{\mathrm{H}}\, d(x, \Omega^c)(\mathcal{M}_{p,w,\kappa_\Gamma} h(x))\\ &\le 2C_{\mathrm{H}}\, d(x, \Omega^c)(2\mathcal{M}_{p,w,\kappa_\Gamma} g(x) + \delta)\\ &\le 4C_{\mathrm{H}}\, d(x, \Omega^c)(\mathcal{M}_{p,w,\kappa_\Gamma} g(x) + \delta).\end{aligned} \tag{43}$$

Here the penultimate inequality follows from the sublinearity of maximal function. We can now conclude from (43) that $\mathrm{len}(\gamma) \le 4C_{\mathrm{H}}\, d(x, \Omega^c)$. By taking $\delta \to 0_+$, we also obtain from (43) that (19) holds, that is,

$$\inf_{\gamma \in \Gamma(X)^\nu_{x,\Omega^c}} \int_\gamma g\, ds \le C_\Gamma\, d(x, \Omega^c)\mathcal{M}_{p,w,\kappa} g(x)$$

with

$$C_\Gamma = 4C_{\mathrm{H}}, \qquad \kappa = \kappa_\Gamma, \qquad \nu > \max\{C_{\mathrm{QC}}, 4C_{\mathrm{H}}\}.$$

For the converse implication, we assume that inequality (19) holds for all non-negative and bounded $g \in LC(X)$ and for all $x \in \Omega$. We need to prove that a pointwise (p, w)-Hardy inequality holds in Ω. To this end, we fix $x \in \Omega$ and $u \in \mathrm{Lip}_0(\Omega)$, and let g be a bounded upper gradient of u. Since g is not necessarily lower semicontinuous, some approximation is first needed so that we get to apply (19) and thereby establish inequality (40).

Let $(g_N)_{N\in\mathbb{N}}$ be a pointwisely increasing sequence of simple Borel functions such that $\lim_{N\to\infty} g_N = g$ uniformly in X. Fix $\varepsilon > 0$. By the uniform convergence, there exists $N \in \mathbb{N}$ such that for all $\gamma \in \Gamma(X)^\nu_{x,\Omega^c}$ we have

$$\begin{aligned}\int_\gamma g\, ds &= \int_\gamma g_N\, ds + \int_\gamma (g - g_N)\, ds\\ &\le \int_\gamma g_N\, ds + \sup_{y\in X}(g(y) - g_N(y))\,\mathrm{len}(\gamma)\\ &\le \int_\gamma g_N\, ds + \sup_{y\in X}(g(y) - g_N(y))\nu d(x, \Omega^c)\\ &\le \int_\gamma g_N\, ds + \varepsilon.\end{aligned} \tag{44}$$

Let $g_{N,x,\varepsilon} \in LC(X)$ be the non-negative bounded approximant of g_N given by Lemma 3.5 with $F = \{x\}$. By inequality (19) and Lemma 3.5, there exists $\gamma_N \in \Gamma(X)^\nu_{x,\Omega^c}$ defined on the interval $[0, \ell(\gamma_N)]$ such that

$$\begin{aligned}\int_{\gamma_N} g_{N,x,\varepsilon}\, ds &\le C_\Gamma d(x, \Omega^c)\mathcal{M}_{p,w,\kappa} g_{N,x,\varepsilon}(x) + \varepsilon\\ &\le C_\Gamma d(x, \Omega^c)\,(\mathcal{M}_{p,w,\kappa} g_N(x) + \varepsilon) + \varepsilon\\ &\le C_\Gamma d(x, \Omega^c)\,(\mathcal{M}_{p,w,\kappa} g(x) + \varepsilon) + \varepsilon.\end{aligned} \tag{45}$$

Without loss of generality, we may assume that $\gamma_N(t) = x$ only if $t = 0$. On the other hand, by Lemma 3.5, we have $g_N \le g_{N,x,\varepsilon}$ in $X \setminus \{x\}$. Inequalities (44) and (45), with $\gamma = \gamma_N$, imply that

$$\begin{aligned}\int_{\gamma_N} g\,ds &\le \int_{\gamma_N} g_N\,ds + \varepsilon \le \int_{\gamma_N} g_{N,x,\varepsilon}\,ds + \varepsilon \\ &\le C_\Gamma d(x,\Omega^c)\,(\mathcal{M}_{p,w,\kappa} g(x) + \varepsilon) + 2\varepsilon.\end{aligned}$$

Since g is an upper gradient of $u \in \mathrm{Lip}_0(\Omega)$ and $\gamma_N(\mathrm{len}(\gamma_N)) \in \Omega^c$, we obtain

$$\begin{aligned}|u(x)| &= |u(\gamma_N(0)) - u(\gamma_N(\mathrm{len}(\gamma_N)))| \\ &\le \int_{\gamma_N} g\,ds \le C_\Gamma d(x,\Omega^c)\,(\mathcal{M}_{p,w,\kappa} g(x) + \varepsilon) + 2\varepsilon,\end{aligned}$$

and letting $\varepsilon \to 0_+$ gives the pointwise (p,w)-Hardy inequality (40) with $C_{\mathrm{H}} = C_\Gamma$ and κ. □

Remark 7.3. Let $\Omega \subsetneq X$ be an open set and let w be a weight for Ω such that a pointwise (p,w)-Hardy inequality holds in Ω, with constants $C_{\mathrm{H}} > 0$ and $\kappa > 1$. Then the proof of Lemma 7.2, with $g = 0$, shows that for every $\varepsilon > 0$ and every $x \in \Omega$ there exists a curve γ that connects x to Ω^c in X such that $\mathrm{len}(\gamma) \le (1+\varepsilon) C_{\mathrm{H}} d(x,\Omega^c)$.

The following is our main result.

Theorem 7.4. *Let $1 \le p_0 < p < \infty$, let $\Omega \subsetneq X$ be an open set, and assume that w is a semilocally doubling p_0-Poincaré weight for Ω. If a pointwise (p,w)-Hardy inequality holds in Ω, then there exists $q \in (p_0,p)$ such that a pointwise (q,w)-Hardy inequality holds in Ω.*

Proof. By Lemma 7.2, we find that w is a p-Hardy weight for Ω. Theorem 6.1 and Lemma 5.4 imply that there exists $q \in (p_0,p)$ such that w is a q-Hardy weight for Ω. Lemma 7.2 implies that a pointwise (q,w)-Hardy inequality holds in Ω. □

Remark 7.5. The proofs of the results in Sections 3, 4, 6 and 7 show that the semilocal doubling property in Definition 3.1 is not really needed to hold for every $\kappa > 0$ but for every $0 < \kappa \le \kappa_0$ with a large enough κ_0 depending on the parameters in the assumed p_0-Poincaré weight property and pointwise (p,w)-Hardy inequality.

8. Applications

In this section we show how the self-improvement of pointwise (p,w)-Hardy inequalities can be applied in the context of integral versions of weighted Hardy inequalities. Here we need to know, for all $1 < q < \infty$ and all $0 < \kappa < \infty$, the boundedness of the restricted weighted maximal operator $\mathcal{M}_{1,w,\kappa} \colon L^q(X; w\,d\mu) \to L^q(\Omega; w\,d\mu)$, where w is a semilocally doubling weight for an open set Ω. If w is a doubling weight in X, then this L^q-boundedness of $\mathcal{M}_{1,w,\kappa}$ follows from the maximal function theorem in X; see, for instance [1, Section 3.2]. In our case the weight w is not necessarily doubling, but the boundedness follows with a suitable adaptation of the proof of the doubling case, given by the following lemma.

Lemma 8.1. *Let $0 < \kappa < \infty$ and $1 < q < \infty$, let $\Omega \subsetneq X$ be an open set, and assume that w is a semilocally doubling weight for Ω. Then the restricted weighted maximal operator $\mathcal{M}_{1,w,\kappa} \colon L^q(X; w\,d\mu) \to L^q(\Omega; w\,d\mu)$ is bounded, that is, there is a constant $C = C_{q,\kappa,w}$ such that*

$$\int_\Omega \big(\mathcal{M}_{1,w,\kappa} f\big)^q w\,d\mu \le C \int_X |f|^q w\,d\mu,$$

for every $f \in L^q(X; w\,d\mu)$.

Proof. Clearly $\mathcal{M}_{1,w,\kappa}\colon L^\infty(X; w\,d\mu) \to L^\infty(\Omega; w\,d\mu)$ is bounded. Hence by interpolation it suffices to prove that $\mathcal{M}_{1,w,\kappa}$ is of corresponding weak type $(1,1)$, compare to the proof of [1, Theorem 3.13]. Let $f \in L^1(X; w\,d\mu)$ and $0 < \tau < \infty$. We estimate the $w\,d\mu$-measure of $E = \{x \in \Omega \mid \mathcal{M}_{1,w,\kappa} f(x) > \tau\}$. The set E has a cover by balls in

$$\mathcal{B} = \left\{ B = B(x,r) \;\Big|\; x \in \Omega,\ \frac{1}{w(B)} \int_B |f| w\,d\mu > \tau,\ 0 < r < \kappa d(x,\Omega^c) \right\}.$$

By the $5r$-covering lemma [1, Lemma 1.7], we obtain a countable subfamily $\mathcal{B}' \subset \mathcal{B}$ of pairwise disjoint balls such that

$$E \subset \bigcup_{B \in \mathcal{B}'} 5B.$$

Then

$$\begin{aligned} w(E) &\le \sum_{B(x,r)\in\mathcal{B}'} w(B(x,5r)) \le D(w,5\kappa)^3 \sum_{B(x,r)\in\mathcal{B}'} w(B(x,r)) \\ &\le D(w,5\kappa)^3 \sum_{B\in\mathcal{B}'} \frac{\int_B |f| w\,d\mu}{\tau} \le \frac{D(w,5\kappa)^3}{\tau} \int_X |f| w\,d\mu. \end{aligned}$$

On the first line we used semilocal doubling with $x \in \Omega$ and $0 < r < \kappa d(x,\Omega^c)$, and on the last line we used the fact that the balls in $\mathcal{B}'$ are pairwise disjoint. This shows the desired weak type $(1,1)$ property and the proof is completed by interpolation. □

Theorem 8.2. *Let $1 \le p_0 < p < \infty$, let $\Omega \subsetneq X$ be an open set, and assume that w is a semilocally doubling p_0-Poincaré weight for Ω. Assume that a pointwise (p,w)-Hardy inequality holds in Ω. Then there exists a constant $C > 0$ such that the (p,w)-Hardy inequality*

$$\int_\Omega \frac{|u(x)|^p}{d(x,\Omega^c)^p} w(x)\,d\mu(x) \le C \int_\Omega g(x)^p w(x)\,d\mu(x)$$

holds for every $u \in \mathrm{Lip}_0(\Omega)$ and for all bounded upper gradients g of u.

Proof. By Theorem 7.4 there exists $q \in (p_0, p)$ such that a pointwise (q,w)-Hardy inequality holds in Ω with $1 < \kappa < \infty$. Let $u \in \mathrm{Lip}_0(\Omega)$ and let g be a bounded upper gradient of u. Without loss of generality, we may assume that $g = 0$ in Ω^c. It is immediate that

$$\big(\mathcal{M}_{q,w,\kappa} g(x)\big)^p = \big(\mathcal{M}_{1,w,\kappa} g^q(x)\big)^{\frac{p}{q}},$$

for every $x \in \Omega$, and on the other hand the pointwise (q,w)-Hardy inequality, raised to power p, implies

$$\frac{|u(x)|^p}{d(x,\Omega^c)^p} \le C\big(\mathcal{M}_{q,w,\kappa} g(x)\big)^p$$

for every $x \in \Omega$. Since $p/q > 1$, by the $L^{p/q}$-boundedness of $\mathcal{M}_{1,w,\kappa}$ from Lemma 8.1 we obtain

$$\begin{aligned} \int_\Omega \frac{|u(x)|^p}{d(x,\Omega^c)^p} w(x)\,d\mu(x) &\le C \int_\Omega \big(\mathcal{M}_{q,w,\kappa} g(x)\big)^p w(x)\,d\mu(x) \\ &= C \int_\Omega \big(\mathcal{M}_{1,w,\kappa} g^q(x)\big)^{\frac{p}{q}} w(x)\,d\mu(x) \\ &\le C \int_X g(x)^p w(x)\,d\mu(x) = C \int_\Omega g(x)^p w(x)\,d\mu(x), \end{aligned}$$

and this proves the claim. □

Next we concentrate on the special case where $X = \mathbb{R}^n$ is equipped with the Euclidean distance and the Lebesgue measure, and weights w are powers of the distance function $x \mapsto d(x, \Omega^c)$. Let $1 \le p < \infty$ and let $\Omega \subsetneq \mathbb{R}^n$ be an open set. We say that a pointwise p-Hardy inequality holds in Ω, if there exists a constant $C > 0$ such that

$$|u(x)| \le C d(x, \Omega^c)\big(\mathcal{M}_{2d(x,\Omega^c)}|\nabla u|^p(x)\big)^{\frac{1}{p}}, \tag{46}$$

for every $x \in \Omega$ and every $u \in \mathrm{Lip}_0(\Omega)$. Here $\mathcal{M}_{2d(x,\Omega^c)}$ is the usual restricted maximal operator, which corresponds to $\mathcal{M}_{1,1,2}$ in the notation introduced in Section 3. These pointwise inequalities were introduced and studied by Hajłasz [8] and Kinnunen and Martio [12], and they can be regarded as pointwise variants of the usual p-Hardy inequality

$$\int_\Omega \frac{|u(x)|^p}{d(x,\Omega^c)^p}\, dx \le C \int_\Omega |\nabla u(x)|^p\, dx. \tag{47}$$

If (46) holds for a function $u \in \mathrm{Lip}_0(\Omega)$ at every $x \in \Omega$, but with an exponent $1 < q < p$, then the maximal function theorem implies that (47) with exponent p holds for u with a constant C independent of u. However, the passage from (46) to (47), with the same exponent $1 < p < \infty$, is not at all obvious. This was established in [13] using an indirect route, first showing the equivalence between the validity of (46) and the uniform p-fatness of Ω^c, and then applying the known self-improvement of the latter, which in $\mathbb{R}^n$ is by Lewis [19] and in metric spaces by Björn, MacManus and Shanmugalingam [3]. A direct proof for the self-improvement of pointwise p-Hardy inequalities, which applies also in metric spaces, was recently given in [6].

The following weighted version of the pointwise p-Hardy inequality was considered in [14]:

$$|u(x)| \le C d(x, \Omega^c)^{1-\frac{\beta}{p}} \big(\mathcal{M}_{2d(x,\Omega^c)}\big(|\nabla u|^q d(\cdot, \Omega^c)^{\frac{\beta q}{p}}\big)(x)\big)^{\frac{1}{q}}, \tag{48}$$

for every $x \in \Omega$ and every $u \in \mathrm{Lip}_0(\Omega)$, where $1 < q < p$ are fixed. As in the unweighted case, with an application of the maximal function theorem for exponent $\frac{p}{q} > 1$, this implies the weighted (p, β)-Hardy inequality

$$\int_\Omega |u(x)|^p d(x, \Omega^c)^{\beta - p}\, dx \le C \int_\Omega |\nabla u(x)|^p d(x, \Omega^c)^\beta\, dx. \tag{49}$$

A more natural formulation for the weighted pointwise Hardy inequality (48) would have been with $q = p$, but then the passage to inequality (49) would not have been possible with a direct use of the maximal function theorem.

Now, using the general technology developed in this paper, we can show that the validity of (48), with $1 < q = p < \infty$, implies (49), at least in the case $\beta \ge 0$. We begin by proving that in this case the weight $w(x) = d(x, \Omega^c)^\beta$, for $x \in \mathbb{R}^n$, is a semilocally doubling p_0-Poincaré weight for Ω, for every $1 \le p_0 < \infty$.

Lemma 8.3. *Let $1 \le p_0 < \infty$ and $\beta \ge 0$, and let $\Omega \subsetneq \mathbb{R}^n$ be an open set. Define $w(x) = d(x, \Omega^c)^\beta$ for all $x \in \mathbb{R}^n$. Then w is a semilocally doubling p_0-Poincaré weight for Ω.*

Proof. Let $\kappa > 0$, $x \in \Omega$ and $0 < r \le \kappa d(x, \Omega^c)$. There exists $C = C(n, \beta, \kappa)$ such that

$$C^{-1} r^n d(x, \Omega^c)^\beta \le w(B(x,r)) = \int_{B(x,r)} d(y, \Omega^c)^\beta\, dy \le C r^n d(x, \Omega^c)^\beta, \tag{50}$$

and this shows that w is a semilocally doubling weight for Ω.

To prove the p_0-Poincaré weight property, we let $u \in \mathrm{Lip}(\mathbb{R}^n)$. There exists a bounded upper gradient g_u of u such that

$$g_u = |\nabla u| \qquad \text{and} \qquad |\nabla u| \le g \tag{51}$$

almost everywhere in $\mathbb{R}^n$ whenever g is a bounded upper gradient of u; we refer to the proof of [1, Corollary 1.47] and [1, Proposition A.3]. Let $x \in \Omega$ and let $B = B(x,r)$ be a ball with $2B = B(x, 2r) \subset \Omega$. We have $0 < r \le d(x,\Omega^c)/2$ and $d(y,\Omega^c) \le 2d(x,\Omega^c) \le 4d(y,\Omega^c)$ for every $y \in B$. By (50), with $\kappa = 1/2$, and the well-known 1-Poincaré inequality in $\mathbb{R}^n$, we have

$$
\begin{aligned}
\frac{1}{w(B)} \int_B |u(y) - u_{B;w}| w(y)\, dy &\le \frac{2}{w(B)} \int_B |u(y) - u_{B;1}| w(y)\, dy \\
&\le C(\beta) \frac{d(x,\Omega^c)^\beta}{w(B)} \int_B |u(y) - u_{B;1}|\, dy \le \frac{C(n,\beta)}{|B|} \int_B |u(y) - u_{B;1}|\, dy \\
&\le C(n,\beta) \frac{r}{|B|} \int_B |\nabla u(y)|\, dy \le C(n,\beta) \frac{r}{w(B)} \int_B |\nabla u(y)| w(y)\, dy \\
&\le C(n,\beta) \frac{r}{w(B)} \int_B g(y) w(y)\, dy
\end{aligned}
$$

whenever g is a bounded upper gradient of u, where the final step follows from the second inequality in (51). This together with Hölder's inequality and Lemma 4.2, with $\lambda = 1$, proves that w is a p_0-Poincaré weight in Ω. □

The claim that weighted pointwise (p,β)-Hardy inequality (52), with $\beta \ge 0$, implies the integral version of the (p,β)-Hardy inequality is now a special case of Theorem 8.2.

Theorem 8.4. *Let $1 < p < \infty$ and $\beta \ge 0$, and let $\Omega \subsetneq \mathbb{R}^n$ be an open set. Assume that there exists a constant $C > 0$ such that*

$$
|u(x)| \le C d(x,\Omega^c)^{1-\frac{\beta}{p}} \big(\mathcal{M}_{2d(x,\Omega^c)} \big(|\nabla u|^p d(\cdot,\Omega^c)^\beta \big)(x) \big)^{\frac{1}{p}}, \tag{52}
$$

for every $x \in \Omega$ and every $u \in \operatorname{Lip}_0(\Omega)$. Then the weighted (p,β)-Hardy inequality (49) *holds for every $u \in \operatorname{Lip}_0(\Omega)$, with a constant independent of u.*

Proof. Define $w(x) = d(x,\Omega^c)^\beta$ for every $x \in \mathbb{R}^n$ and let $\kappa = 2$. By Lemma 8.3, w is a semilocally doubling 1-Poincaré weight for Ω. Let $u \in \operatorname{Lip}_0(\Omega)$. From the estimates in (50) it follows that inequality (52) is comparable to (40), with $\kappa = 2$ and $g = |\nabla u|$, and therefore a pointwise (p,w)-Hardy inequality holds in Ω by (51). Hence all assumptions of Theorem 8.2 are valid and the claim follows from the (p,w)-Hardy inequality in Theorem 8.2, applied with the bounded upper gradient g_u that is given in connection with (51). □

Remark 8.5. It is possible to extend Lemma 8.3 and Theorem 8.4 also to some $-n < \beta < 0$. In this case it is natural to add the condition that $w = 0$ in Ω^c. The obstruction with $\beta < 0$ is that clearly the last inequality in (50) is not valid for every $\beta < 0$ if the ball $B(x,r)$ intersects the boundary of Ω, since for small enough β the integral in (50) becomes infinite. On the other hand, if the last inequality in (50) is valid for some $\beta < 0$, then everything else in Lemma 8.3 and Theorem 8.4 works, and we conclude that for such $\beta < 0$ the weighted pointwise (p,β)-Hardy inequality (52) implies the weighted (p,β)-Hardy inequality (49).

The validity of the last inequality in (50) is closely related to the Assouad dimension of $\partial\Omega$ via the so-called Aikawa condition, but we omit any further discussion related to these concepts and refer to [4, 17] for details.

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References

[1] A. Björn and J. Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.

[2] A. Björn and J. Björn. Local and semilocal Poincaré inequalities on metric spaces. *J. Math. Pures Appl.*, 119:158-192, 2018.

[3] J. Björn, P. MacManus, and N. Shanmugalingam. Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces. *J. Anal. Math.*, 85:339–369, 2001.

[4] B. Dyda, L. Ihnatsyeva, J. Lehrbäck, H. Tuominen, and A. V. Vähäkangas. Muckenhoupt A_p-properties of distance functions and applications to Hardy-Sobolev–type inequalities. *Potential Anal.*, 50(1):83–105, 2019.

[5] S. Eriksson-Bique. Alternative proof of Keith–Zhong self-improvement and connectivity. *Ann. Acad. Sci. Fenn. Math.* 44(1): 1, 407–425, 2019

[6] S. Eriksson-Bique and A. Vähäkangas, Self-improvement of pointwise Hardy inequality. *Trans. Amer. Math. Soc.* 372(3): 2235–2250, 2019.

[7] G. B. Folland. *Real analysis: Modern techniques and their applications.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984.

[8] P. Hajłasz. Pointwise Hardy inequalities. *Proc. Amer. Math. Soc.*, 127(2):417–423, 1999.

[9] P. Hajłasz. Sobolev spaces on metric-measure spaces. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 173–218. Amer. Math. Soc., Providence, RI, 2003.

[10] P. Hajłasz and P. Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.

[11] S. Keith and X. Zhong. The Poincaré inequality is an open ended condition. *Ann. of Math. (2)*, 167(2):575–599, 2008.

[12] J. Kinnunen and O. Martio. Hardy's inequalities for Sobolev functions. *Math. Res. Lett.*, 4(4):489–500, 1997.

[13] R. Korte, J. Lehrbäck, and H. Tuominen. The equivalence between pointwise Hardy inequalities and uniform fatness. *Math. Ann.*, 351(3):711–731, 2011.

[14] P. Koskela and J. Lehrbäck. Weighted pointwise Hardy inequalities. *J. Lond. Math. Soc. (2)*, 79(3):757–779, 2009.

[15] P. Koskela and X. Zhong. Hardy's inequality and the boundary size. *Proc. Amer. Math. Soc.*, 131(4):1151–1158 (electronic), 2003.

[16] A. Kufner. *Weighted Sobolev spaces.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985.

[17] J. Lehrbäck and H. Tuominen. A note on the dimensions of Assouad and Aikawa. *J. Math. Soc. Japan*, 65(2):343–356, 2013.

[18] J. Lehrbäck, H. Tuominen, and A. V. Vähäkangas. Self-improvement of uniform fatness revisited. *Math. Ann.*, 368(3-4):1439–1464, 2017.

[19] J. L. Lewis. Uniformly fat sets. *Trans. Amer. Math. Soc.*, 308(1):177–196, 1988.

[20] J. Nečas. Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. *Ann. Scuola Norm. Sup. Pisa (3)*, 16:305–326, 1962.

[21] E. Saksman, Remarks on the nonexistence of doubling measures. *Ann. Acad. Sci. Fenn. Math.* 24(1):155–163, 1999.