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Title: Malliavin smoothness on the Lévy space with Hölder continuous or BV functionals

Year: 2020

Version: Accepted version (Final draft)

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## Please cite the original version:

Laukkarinen, E. (2020). Malliavin smoothness on the Lévy space with Hölder continuous or BV functionals. Stochastic Processes and their Applications, 130(8), 4766-4792.
https://doi.org/10.1016/j.spa.2020.01.016

## Journal Pre-proof

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PII: $\quad$ S0304-4149(18)30334-X
DOI: https://doi.org/10.1016/j.spa.2020.01.016
Reference: SPA 3629

To appear in: Stochastic Processes and their Applications
Received date: 24 July 2018
Revised date: 22 October 2019
Accepted date: 27 January 2020

Please cite this article as: E. Laukkarinen, Malliavin smoothness on the Lévy space with Hölder continuous or BV functionals, Stochastic Processes and their Applications (2020), doi: https://doi.org/10.1016/j.spa.2020.01.016.

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# Malliavin smoothness on the Lévy space with Hölder continuous or $B V$ functionals 

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#### Abstract

We consider Malliavin smoothness of random variables $f\left(X_{1}\right)$, where $X$ is a pure jump Lévy process and the function $f$ is either bounded and Hölder continuous or of bounded variation. We show that Malliavin differentiability and fractional differentiability of $f\left(X_{1}\right)$ depend both on the regularity of $f$ and the BlumenthalGetoor index of the Lévy measure.


Keywords: Lévy process, Malliavin calculus, interpolation 2010 MSC: 60G51, 60H07

## 1. Introduction

Consider a Lévy process $Y$ and the according Malliavin Sobolev space $\mathbb{D}_{1,2}$ based on the Itô chaos decomposition on the Lévy space of square integrable random variables. We recall the space $\mathbb{D}_{1,2}$ in Section 2.1. We are interested in the ways that Malliavin differentiability of $f\left(Y_{1}\right)$ depends on the properties of $f$ and the properties of $Y$.

The process $Y$ consists of three components

$$
Y_{t}=\gamma t+\sigma B_{t}+X_{t}
$$

where $\gamma, \sigma \in \mathbb{R}, B$ is a standard Brownian motion and $X$ is a pure jump process. For the Brownian motion we have that $f\left(B_{1}\right) \in \mathbb{D}_{1,2}$ if and only if $f \in W^{1,2}\left(\mathbb{R} ; \mathbb{P}_{B_{1}}\right)$ (see, for instance, Nualart [23, Exercise 1.2.8]). We also examine fractional differentiability which is determined by the real interpolation spaces $\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}$ between $L_{2}(\mathbb{P})$ and $\mathbb{D}_{1,2}$ (see Section 2.2 ). The fractional smoothness of $f\left(B_{1}\right)$ means that $f$ is in a weighted Besov space (see S . Geiss and Hujo [15], for example). In this paper we focus on the pure jump Lévy process with $\gamma=0$ and $\sigma=0$. We search for properties of the function $f$ and the Lévy measure $\nu$ of $X$, which are related to the smoothness of $f\left(X_{1}\right)$. It

[^0]turns out that Malliavin smoothness is in connection to the Blumenthal-Getoor index
$$
\beta=\inf \left\{\xi \geq 0: m_{\xi}<\infty\right\}, \quad \text { where } \quad m_{\xi}:=\int_{\mathbb{R}}\left(|x|^{\xi} \wedge 1\right) \nu(\mathrm{d} x)
$$

We show that the smaller the index $\beta$ is, the higher smoothness of $f\left(X_{1}\right)$ we have for a given $f$ which is Hölder continuous or of bounded variation.

So far little is known about the question for which $f$ and for which $\nu$ one has $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$ or $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}$. The note [22] enlightens the case where $\nu(\mathbb{R})<\infty$ : Then

$$
f\left(X_{1}\right) \in \mathbb{D}_{1,2} \quad \text { if and only if } \quad \mathbb{E}\left[f^{2}\left(X_{1}\right)(N((0,1] \times \mathbb{R})+1)\right]<\infty
$$

and
$f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, 2} \quad$ if and only if $\mathbb{E}\left[f^{2}\left(X_{1}\right)\left(N((0,1] \times \mathbb{R})^{\theta}+1\right)\right]<\infty$, where $N$ is the Poisson random measure associated with $X$ (see Section 2).

A Lévy measure $\nu$ always satisfies the property $m_{2}<\infty$, and from Solé, Utzet and Vives [26] we know that

$$
\left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}}^{2}=\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\int_{\mathbb{R}} \mathbb{E}\left[\left(f\left(X_{1}+x\right)-f\left(X_{1}\right)\right)^{2}\right] \nu(\mathrm{d} x)
$$

Since $m_{2}<\infty$, it follows that $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$ for any $f$ that is Lipschitz continuous and bounded. On the other hand, if the Lévy measure $\nu$ is finite, then it is sufficient that $f$ is bounded to have $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$. In Section 3 we shall examine intermediate cases, namely that $f$ is bounded and Hölder continuous, that is, in $C_{b}^{\alpha}$. In Theorem 3 we prove that

$$
f\left(X_{1}\right) \in \mathbb{D}_{1,2} \text { for all } f \in C_{b}^{\alpha} \quad \text { if and only if } \quad m_{2 \alpha}<\infty,
$$

where the necessity of the condition $m_{2 \alpha}<\infty$ holds under assumption (A2) given in Section 2.3. For fractional smoothness we obtain in Theorem 5 for $0<\alpha \leq \theta<1$, that

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \text { for all } f \in C_{b}^{\alpha} \quad \text { if } \quad m_{2 \alpha / \theta}<\infty,
$$

and under assumption (A3), that

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \text { for all } f \in C_{b}^{\alpha} \quad \text { only if } \quad m_{2 \alpha / \theta+\varepsilon}<\infty
$$

for all $\varepsilon>0$. In Section 5.1 we see that if the process $X$ is strictly stable and symmetric and $2 \alpha / \theta$ is equal to the Blumenthal-Getoor index $\beta$, then $f\left(X_{1}\right) \in$ $\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$ for all $f \in C_{b}^{\alpha}$ eventhough $m_{2 \alpha / \theta}=m_{\beta}=\infty$.

We also consider normalized functions of bounded variation ( $N B V$, see Section 4). In Theorem 6 we prove that under assumptions (A1) and (A2) it holds that

$$
f\left(X_{1}\right) \in \mathbb{D}_{1,2} \text { for all } f \in N B V \quad \text { if and only if } \quad m_{1}<\infty
$$

In [11, Section 4.2] it was shown that $\mathbb{1}_{(K, \infty)}\left(Y_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{1 / 2, \infty}$, when $Y_{1}$ has a bounded density. We obtain a sharper smoothness index for the pure jump process: Theorem 7 states that under assumption (A1) it holds that

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \text { for all } f \in N B V \quad \text { if } \quad m_{1 / \theta}<\infty
$$

and under assumption (A3) it holds that

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \text { for all } f \in N B V \quad \text { only if } \quad m_{1 / \theta+\varepsilon}<\infty
$$

for all $\varepsilon>0$. In Section 5.1 we see that if the process $X$ is strictly stable and symmetric and $1 / \theta=\beta$, then $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$ for all $f \in N B V$ eventhough $m_{1 / \theta}=m_{\beta}=\infty$.

The method in Section 5 is based on a characterization of fractional smoothness which was introduced for the Brownian motion by S. Geiss and Hujo [15], and which we translate for jump processes in Lemma 9.

### 1.1. Motivation

Malliavin smoothness and fractional smoothness play a role for example in discrete approximation of stochastic integrals and in the investigation of properties of backward stochastic differential equations (BSDEs): Consider the orthogonal Galtchouk-Kunita-Watanabe decomposition of $f\left(Y_{1}\right)$, that is,

$$
f\left(Y_{1}\right)=c+\int_{0}^{1} \varphi_{t} \mathrm{~d} Y_{t}+\mathcal{E}
$$

Then the convergence rate of the equidistant Riemann-approximation of the integral depends on the smoothness parameter of $f\left(Y_{1}\right)$. On the other hand, if $f\left(Y_{1}\right)$ admits fractional smoothness, then it is possible to adjust the discretization points to obtain the best possible convergence rate. (See Geiss et al. [11].) The $L_{p}$-variation of the solution of certain BSDEs depends on the Malliavin fractional smoothness of the terminal condition $f\left(Y_{1}\right)$. This was shown with more general terminal conditions for the Brownian motion by C. Geiss, S. Geiss and Gobet [10] and S. Geiss and Ylinen [17] and for $p=2$ for general $L_{2}$-Lévy processes by C. Geiss and Steinicke [13].

## 2. Preliminaries

Consider a pure jump Lévy process $X=\left(X_{t}\right)_{t \geq 0}$ with càdlàg paths on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is the completion of the sigmaalgebra generated by $X$. The Lévy-Itô decomposition of a pure jump Lévy process is

$$
X_{t}=\iint_{(0, t] \times\{|x|>1\}} x N(\mathrm{~d} s, \mathrm{~d} x)+\iint_{(0, t] \times\{0<|x| \leq 1\}} x \tilde{N}(\mathrm{~d} s, \mathrm{~d} x),
$$

where $N$ is a Poisson random measure on $\mathcal{B}([0, \infty) \times \mathbb{R})$ and $\tilde{N}(\mathrm{~d} s, \mathrm{~d} x)=$ $N(\mathrm{~d} s, \mathrm{~d} x)-\mathrm{d} s \nu(\mathrm{~d} x)$ is the compensated Poisson random measure. The measure $\nu: \mathcal{B}(\mathbb{R}) \rightarrow[0, \infty]$ is the Lévy measure of $X$ satisfying $\nu(\{0\})=0$, $\int_{\mathbb{R}}\left(x^{2} \wedge 1\right) \nu(\mathrm{d} x)<\infty$ and $\nu(B)=\mathbb{E}[N((0,1] \times B)]$.

### 2.1. Itô chaos decomposition and the Malliavin Sobolev space

Denote $\mathbb{R}_{+}:=[0, \infty)$. We consider the following measure $\mathrm{m}: \mathcal{B}\left(\mathbb{R}_{+} \times \mathbb{R}\right) \rightarrow$ $[0, \infty]$ defined as

$$
\mathrm{m}(A):=\int_{A} x^{2} \mathrm{~d} t \nu(\mathrm{~d} x)=\mathbb{E}\left[\left(\int_{A} x \tilde{N}(\mathrm{~d} t, \mathrm{~d} x)\right)^{2}\right]
$$

For $n=1,2, \ldots$ we write $L_{2}\left(\mathbb{m}^{\otimes n}\right):=L_{2}\left(\left(\mathbb{R}_{+} \times \mathbb{R}\right)^{n}, \mathcal{B}\left(\mathbb{R}_{+} \times \mathbb{R}\right)^{\otimes n}, \mathbb{m}^{\otimes n}\right)$ and set $L_{2}\left(\mathrm{~m}^{\otimes 0}\right):=\mathbb{R}$. A function $f_{n}:\left(\mathbb{R}_{+} \times \mathbb{R}\right)^{n} \rightarrow \mathbb{R}$ is said to be symmetric, if it coincides with its symmetrization $\tilde{f}_{n}$,

$$
\tilde{f}_{n}\left(\left(s_{1}, x_{1}\right), \ldots,\left(s_{n}, x_{n}\right)\right)=\frac{1}{n!} \sum_{\pi} f_{n}\left(\left(s_{\pi(1)}, x_{\pi(1)}\right), \ldots,\left(s_{\pi(n)}, x_{\pi(n)}\right)\right)
$$

where the sum is taken over all permutations $\pi:\{1, \ldots, n\} \rightarrow\{1, \ldots, n\}$.
We consider Itô's multiple stochastic integral $I_{n}: L_{2}\left(\mathrm{~m}^{\otimes n}\right) \rightarrow L_{2}(\mathbb{P})$ of order $n$ with respect to the measure $x \tilde{N}(\mathrm{~d} t, \mathrm{~d} x)$. According to [19, Theorem 2] it holds that

$$
L_{2}(\mathbb{P})=\mathbb{R} \oplus \bigoplus_{n=1}^{\infty}\left\{I_{n}\left(f_{n}\right): f_{n} \in L_{2}\left(\mathbb{m}^{\otimes n}\right)\right\}
$$

The functions $f_{n}$ in the representation $F=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right)$ in $L_{2}(\mathbb{P})$ are unique when they are chosen to be symmetric, which is always possible since $I_{n}\left(f_{n}\right)=$ $I_{n}\left(\tilde{f}_{n}\right)$. Moreover, we have

$$
\mathbb{E}\left[I_{n}\left(f_{n}\right) I_{k}\left(g_{k}\right)\right]= \begin{cases}0, & \text { if } n \neq k \\ n!\left(\tilde{f}_{n}, \tilde{g_{n}}\right)_{L_{2}\left(\mathrm{~m}^{\otimes n}\right)} & \text { if } n=k\end{cases}
$$

and

$$
\|F\|_{L_{2}(\mathbb{P})}^{2}=\sum_{n=0}^{\infty} n!\left\|\tilde{f}_{n}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}
$$

In this paper we focus on random variables of the form $f\left(X_{1}\right)$, where $f$ : $\mathbb{R} \rightarrow \mathbb{R}$ is a Borel function. We will take advantage of the following lemma in Sections 3 and 5.

Lemma 1. Let $f\left(X_{1}\right)=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right) \in L_{2}(\mathbb{P})$ and let $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ be the augmented natural filtration of $X$. Then
(a) there are functions $g_{n} \in L_{2}\left(\left(x^{2} \nu(\mathrm{~d} x)\right)^{\otimes n}\right)$ such that

$$
\tilde{f}_{n}\left(\left(t_{1}, x_{1}\right), \ldots,\left(t_{n}, x_{n}\right)\right)=g_{n}\left(x_{1}, \ldots, x_{n}\right) \mathbb{1}_{[0,1] \times n}\left(t_{1}, \ldots, t_{n}\right)
$$

for $\mathbb{m}^{\otimes n}$-a.e. $\left(\left(t_{1}, x_{1}\right), \ldots,\left(t_{n}, x_{n}\right)\right) \in\left(\mathbb{R}_{+} \times \mathbb{R}\right)^{\times n}$ and
(b) $\mathbb{E}\left[\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]^{2}\right]=\sum_{n=0}^{\infty} t^{n} n!\left\|\tilde{f}_{n}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}$.

Proof. (a) Follows from [3, Remark 6.7]. (b) By analogous argumentation to [23, Lemma 1.2.4] we see that $\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]=\sum_{n=0}^{\infty} I_{n}\left(g_{n} \mathbb{1}_{[0, t] \times n}\right)$. The claim follows from $\left\|\tilde{f}_{n}\right\|_{L_{2}\left(\mathrm{~m}^{\otimes n}\right)}=\left\|g_{n}\right\|_{L_{2}\left(\left(x^{2} \nu(\mathrm{~d} x)\right)^{\otimes n}\right)}$.

We define the Malliavin Sobolev space using Itô's chaos decomposition (as $[24,8,26,27,1,12]$ and many others). We denote by $\mathbb{D}_{1,2}$ the space of all $F=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right) \in L_{2}(\mathbb{P})$ such that

$$
\|F\|_{\mathbb{D}_{1,2}}^{2}:=\|F\|_{L_{2}(\mathbb{P})}^{2}+\sum_{n=1}^{\infty} n n!\left\|\tilde{f}_{n}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}=\sum_{n=0}^{\infty}(n+1)!\left\|\tilde{f}_{n}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}<\infty .
$$

Let us write $L_{2}(\mathbb{m} \otimes \mathbb{P}):=L_{2}\left(\mathbb{R}_{+} \times \mathbb{R} \times \Omega, \mathcal{B}\left(\mathbb{R}_{+} \times \mathbb{R}\right) \otimes \mathcal{F}, \mathbb{m} \otimes \mathbb{P}\right)$. The Malliavin derivative $D: \mathbb{D}_{1,2} \rightarrow L_{2}(\mathrm{~m} \otimes \mathbb{P})$ is defined for $F \in \mathbb{D}_{1,2}$ by

$$
D_{t, x} F=\sum_{n=1}^{\infty} n I_{n-1}\left(\tilde{f}_{n}(\cdot,(t, x))\right) \quad \text { in } L_{2}(\mathrm{~m} \otimes \mathbb{P})
$$

From [26, Proposition 5.4] we have in the canonical probability space that

$$
\begin{align*}
& \left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}}^{2} \\
& =\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\int_{[0,1] \times \mathbb{R} \backslash\{0\}} \mathbb{E}\left[\left(\frac{f\left(X_{1}+x\right)-f\left(X_{1}\right)}{x}\right)^{2}\right] \mathrm{m}(\mathrm{~d} t, \mathrm{~d} x) \\
& =\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\int_{\mathbb{R}} \mathbb{E}\left[\left(f\left(X_{1}+x\right)-f\left(X_{1}\right)\right)^{2}\right] \nu(\mathrm{d} x), \tag{1}
\end{align*}
$$

and when $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$, then

$$
\begin{equation*}
D_{t, x} f\left(X_{1}\right)=\frac{f\left(X_{1}+x\right)-f\left(X_{1}\right)}{x} \mathbb{1}_{[0,1] \times \mathbb{R} \backslash\{0\}}(t, x) \quad \mathrm{m} \otimes \mathbb{P} \text {-a.e. } \tag{2}
\end{equation*}
$$

The result was converted to the general probability space in [14, Lemma 3.2].
For the Brownian motion $B$, the space $\mathbb{D}_{1,2}$ is defined in an analogous way by a chaos decomposition, but the property (1) can not be formulated (see [23]).

### 2.2. Interpolation and Malliavin fractional smoothness

The interpolation space $\left(A_{0}, A_{1}\right)_{\theta, q}$ is a Banach space, intermediate between two Banach spaces $A_{0}$ and $A_{1}$ which are a compatible couple, that is, they are continuously embedded into a Hausdorff topological vector space.

When $\left(A_{0}, A_{1}\right)$ is a compatible couple, the $K$-functional of $a \in A_{0}+A_{1}$ is the mapping $K\left(a, \cdot ; A_{0}, A_{1}\right):(0, \infty) \rightarrow[0, \infty)$ defined by

$$
K\left(a, t ; A_{0}, A_{1}\right):=\inf \left\{\left\|a_{0}\right\|_{A_{0}}+t\left\|a_{1}\right\|_{A_{1}}: a=a_{0}+a_{1}, a_{0} \in A_{0}, a_{1} \in A_{1}\right\} .
$$

Let $\theta \in(0,1)$ and $q \in[1, \infty]$. The real interpolation space $\left(A_{0}, A_{1}\right)_{\theta, q}$ consists of all $a \in A_{0}+A_{1}:=\left\{a_{0}+a_{1}: a_{0} \in A_{0}, a_{1} \in A_{1}\right\}$ such that the norm

$$
\|a\|_{\left(A_{0}, A_{1}\right)_{\theta, q}}= \begin{cases}{\left[\int_{0}^{\infty}\left(t^{-\theta} K\left(a, t ; A_{0}, A_{1}\right)\right)^{q} \frac{\mathrm{~d} t}{t}\right]^{\frac{1}{q}},} & q \in[1, \infty) \\ \sup _{t>0} t^{-\theta} K\left(a, t ; A_{0}, A_{1}\right), & q=\infty\end{cases}
$$

is finite. If $A_{1} \subseteq A_{0}$ with continuous embedding, then

$$
\begin{equation*}
A_{1} \subseteq\left(A_{0}, A_{1}\right)_{\theta, q} \subseteq\left(A_{0}, A_{1}\right)_{\eta, p} \subseteq\left(A_{0}, A_{1}\right)_{\eta, q} \subseteq A_{0} \tag{3}
\end{equation*}
$$

for $0<\eta<\theta<1$ and $1 \leq p \leq q \leq \infty$.
From the Reiteration Theorem we know that for $\eta, \theta \in(0,1)$ and $q \in[1, \infty]$ one has

$$
\begin{equation*}
\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, q}=\left(A_{0}, A_{1}\right)_{\eta \theta, q} \tag{4}
\end{equation*}
$$

with

$$
\begin{equation*}
\|a\|_{\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}} \leq\|a\|_{\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, \infty}} \leq 3\|a\|_{\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}} \tag{5}
\end{equation*}
$$

for all $a \in\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}=\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, \infty}$. In the literature the Reiteration Theorem is usually given in a more general context and the constants 1 and 3 in the norm equivalence (5) are not computed explicitely. Therefore we verify (5) in Lemma 16. For further properties of interpolation spaces, see for instance [4], [5] or [30].

We say that a random variable admits fractional smoothness of order $(\theta, q)$ if it belongs to the interpolation space

$$
\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}
$$

where $\theta \in(0,1)$ and $q \in[1, \infty]$.

### 2.3. Assumptions about a density

Some of the assertions in this paper rest on the following assumptions:
(A1) $X_{1}$ has a bounded density $p_{1}$.
(A2) $X_{1}$ has a density $p_{1}$ and there exist $a, b, c \in \mathbb{R}$ with $c>0$ and $b-a>0$ such that $p_{1}(x) \geq c$ for all $x \in[a, b]$.
(A3) There exist $t_{0} \in(0,1)$ and $a, b, c \in \mathbb{R}$ with $c>0$ and $b-a>0$ such that for all $t \in\left[t_{0}, 1\right]$, the random variable $X_{t}$ has a density $p_{t}$ such that $p_{t}(x) \geq c$ for all $x \in[a, b]$.

Note that the conditions (A1), (A2) and (A3) are satisfied, for example, when the condition

$$
\ell:=\liminf _{|u| \rightarrow \infty} \frac{\int_{\mathbb{R}} \sin ^{2}(u x) \nu(\mathrm{d} x)}{\log |u|}>\frac{1}{2}
$$

of Hartman and Wintner [18] holds. We formulate the argumentation in a lemma as it will be used later.

Lemma 2. Assume that $\ell>1 / 2$. Then (A1), (A2) and (A3) are satisfied.
Proof. By [18, Section 13, statement II], $X_{t}$ has a bounded and continuous density for all $t>\frac{1}{2 \ell}$. The conditions (A1) and (A2) follow immediately. Let
us prove (A3). Let $r>0$. Due to stochastic continuity of Lévy processes, there is $t_{0} \in\left(\frac{1}{2 \ell}, 1\right)$ such that

$$
\mathbb{P}\left(\left|X_{t-t_{0}}\right| \leq r\right) \geq 1 / 2 \quad \text { for all } t \in\left[t_{0}, 1\right] .
$$

Since $\ell>1 / 2$, [25, Theorem 24.10] implies that either the support of $X_{t_{0}}$ is a half line $[\kappa, \infty)$ (or $(-\infty, \kappa])$ for some $\kappa \in \mathbb{R}$, or the support of $X_{s}$ is $\mathbb{R}$ for all $s>0$. The continuous density $p_{t_{0}}$, if supported on a half line, is strictly positive on the open half line $(\kappa, \infty)$ (or $(-\infty, \kappa)$ ) by [28, Chapter IV, Theorem 8.6]. If $X_{s}$ has a bounded and continuous density supported on the whole real line for $\frac{1}{2 \ell}<s<t_{0}$, then [28, Chapter IV, Theorem 8.6] implies that $p_{t_{0}}$ is strictly positive. In any case $p_{t_{0}}$ is continuous and strictly positive on at least a half line, so that we find $K \in \mathbb{R}$ and $c>0$ such that $p_{t_{0}}(x) \geq c$ for all $x \in[K-2 r, K+2 r]$. For any $x \in[K-r, K+r]$ and $t \in\left[t_{0}, 1\right]$ it holds that

$$
\begin{aligned}
p_{t}(x) & =\int_{\mathbb{R}} p_{t_{0}}(x-y) \mathbb{P}_{X_{t-t_{0}}}(\mathrm{~d} y) \geq \int_{[-r, r]} p_{t_{0}}(x-y) \mathbb{P}_{X_{t-t_{0}}}(\mathrm{~d} y) \\
& \geq c \mathbb{P}\left(\left|X_{t-t_{0}}\right| \leq r\right) \geq c / 2
\end{aligned}
$$

## 3. Hölder continuous functions and Malliavin smoothness

For $\alpha \in(0,1]$, the spaces $B(\mathbb{R}), C^{\alpha}$ and $C_{b}^{\alpha}$ are spaces of Borel measurable functions $f$ such that

$$
\|f\|_{\infty}=\sup _{x \in \mathbb{R}}|f(x)|, \quad\|f\|_{C^{\alpha}}=\sup _{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \quad \text { or } \quad\|f\|_{C_{b}^{\alpha}}=\|f\|_{\infty}+\|f\|_{C^{\alpha}},
$$

respectively, is finite. We frequently use the notation Lip $:=C_{b}^{1}$. Note that $\left(B(\mathbb{R}),\|\cdot\|_{\infty}\right)$ and $\left(C_{b}^{\alpha},\|\cdot\|_{C_{b}^{\alpha}}\right)$ are Banach spaces and $\|\cdot\|_{C^{\alpha}}$ is a seminorm. Recall the notation

$$
m_{2 \alpha}=\int_{\mathbb{R}}\left(|x|^{2 \alpha} \wedge 1\right) \nu(\mathrm{d} x)
$$

3.1. Smoothness of first order

Theorem 3. Let $\alpha \in(0,1)$ and $A:=[0,1] \times\{x:|x|>1\}$ and assume that $f\left(X_{1}\right) \in L_{2}(\mathbb{P})$.
(a) If $f \in C^{\alpha}$ and $\int_{\mathbb{R}}|x|^{2 \alpha} \nu(\mathrm{~d} x)<\infty$, then $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$ and

$$
\left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}}^{2} \leq\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\|f\|_{C^{\alpha}}^{2} \int_{\mathbb{R}}|x|^{2 \alpha} \nu(\mathrm{~d} x)
$$

(b) If $f \in C^{\alpha}, m_{2 \alpha}<\infty$ and $\mathbb{E}\left[f^{2}\left(X_{1}\right) N(A)\right]<\infty$, then $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$ and

$$
\left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}}^{2}
$$

$$
\leq\|f\|_{C^{\alpha}}^{2} m_{2 \alpha}+\mathbb{E}\left[f^{2}\left(X_{1}\right) N(A)\right]+\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}(1+\nu(\{|x|>1\}))
$$

(c) If $f \in C_{b}^{\alpha}$ and $m_{2 \alpha}<\infty$, then $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$ and

$$
\begin{equation*}
\left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}}^{2} \leq\left(1+4 m_{2 \alpha}\right)\|f\|_{C_{b}^{\alpha}}^{2} . \tag{6}
\end{equation*}
$$

(d) Assume that (A2) holds and choose $\ell \in\{0,1,2, \ldots\}$ such that there exist $k \in \mathbb{Z}$ and $c>0$ with $p_{1}(x) \geq c$ for all $x \in\left[k 2^{-\ell},(k+1) 2^{-\ell}\right]$. Then for the function $g^{\alpha, \ell}(x)=\sum_{n=\ell}^{\infty} 2^{-\alpha n} d\left(2^{n} x, \mathbb{Z}\right)$ from Lemma 4 it holds that $g^{\alpha, \ell} \in C_{b}^{\alpha}$, and

$$
g^{\alpha, \ell}\left(X_{1}\right) \in \mathbb{D}_{1,2} \quad \text { only if } \quad m_{2 \alpha}<\infty .
$$

Proof. (a) The claim follows from [26, Proposition 5.4] (see (1)) and the $\alpha$ Hölder continuity.
(c) The claim follows from $\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2} \leq\|f\|_{C_{b}^{\alpha}}^{2}$ and (1), since

$$
\begin{aligned}
& \int_{\mathbb{R}} \mathbb{E}\left[\left|f\left(X_{1}+x\right)-f\left(X_{1}\right)\right|^{2}\right] \nu(\mathrm{d} x) \\
& \leq \int_{\{|x| \leq 1\}}\|f\|_{C^{\alpha}}^{2}|x|^{2 \alpha} \nu(\mathrm{~d} x)+\int_{\{|x|>1\}} 4\|f\|_{\infty}^{2} \nu(\mathrm{~d} x) \\
& \leq\|f\|_{C_{b}^{\alpha}}^{2} \cdot 4 \int_{\mathbb{R}}\left(|x|^{2 \alpha} \wedge 1\right) \nu(\mathrm{d} x) .
\end{aligned}
$$

(b) Consider the chaos expansion $f\left(X_{1}\right)=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right)$ and recall that

$$
\left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}}^{2}=\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\sum_{n=1}^{\infty} n n!\left\|\tilde{f}_{n}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}
$$

We show first that

$$
\begin{align*}
\sum_{n=1}^{\infty} n n!\left\|\tilde{f}_{n}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}= & \int_{[-1,1]} \mathbb{E}\left[\left|f\left(X_{1}+x\right)-f\left(X_{1}\right)\right|^{2}\right] \nu(\mathrm{d} x) \\
& +\sum_{n=1}^{\infty} n n!\left\|\tilde{f}_{n} \mathbb{1}_{\left(\mathbb{R}_{+} \times \mathbb{R}\right)^{\times(n-1)} \times A}\right\|_{L_{2}\left(\mathrm{~m}^{\otimes n}\right)}^{2} \tag{7}
\end{align*}
$$

In fact, it holds that

$$
\begin{align*}
& \int_{\mathbb{R}_{+} \times \mathbb{R} \backslash\{0\}} \mathbb{E}\left[\left|\frac{f\left(X_{1}+x\right)-f\left(X_{1}\right)}{x} \mathbb{1}_{[0,1] \times\{0<|x| \leq 1\}}(t, x)\right|^{2}\right] \mathrm{m}(\mathrm{~d} t, \mathrm{~d} x) \\
& =\int_{[-1,1]} \mathbb{E}\left[\left|f\left(X_{1}+x\right)-f\left(X_{1}\right)\right|^{2}\right] \nu(\mathrm{d} x) \leq\|f\|_{C^{\alpha}}^{2} \int_{[-1,1]}|x|^{2 \alpha} \nu(\mathrm{~d} x)<\infty \tag{8}
\end{align*}
$$

so that there is a chaos representation
$\frac{f\left(X_{1}+x\right)-f\left(X_{1}\right)}{x} \mathbb{1}_{[0,1] \times\{0<|x| \leq 1\}}(t, x)=\sum_{n=0}^{\infty} I_{n}\left(h_{n+1}(\cdot,(t, x))\right) \quad$ in $L_{2}(\mathrm{~m} \otimes \mathbb{P})$
where $h_{n+1} \in L_{2}\left(\mathrm{~m}^{\otimes(n+1)}\right)$ is symmetric in the first $n$ pairs of variables (see [23, Lemma 1.3.1] or [24, Section 4]). Let $\varphi_{k}=-k \vee(f \wedge k)$ so that $\varphi_{k} \in C_{b}^{\alpha}$ and $\varphi_{k}\left(X_{1}\right) \in \mathbb{D}_{1,2}$ by (c). Consider the chaos expansion $\varphi_{k}\left(X_{1}\right)=\sum_{n=0}^{\infty} I_{n}\left(f_{n}^{(k)}\right)$. Then $\tilde{f}_{n}^{(k)} \rightarrow \tilde{f}_{n}$ in $L_{2}\left(\mathrm{~m}^{\otimes n}\right)$, since $\varphi_{k}\left(X_{1}\right) \rightarrow f\left(X_{1}\right)$ in $L_{2}(\mathbb{P})$. It also holds that

$$
\int_{[0,1] \times\{0<|x| \leq 1\}} \mathbb{E}\left[\left|\frac{\varphi_{k}\left(X_{1}+x\right)-\varphi_{k}\left(X_{1}\right)}{x}-\frac{f\left(X_{1}+x\right)-f\left(X_{1}\right)}{x}\right|^{2}\right] \mathrm{m}(\mathrm{~d} t, \mathrm{~d} x)
$$

converges to 0 as $k \rightarrow \infty$ by dominated convergence, since $\mid \varphi_{k}\left(X_{1}+x\right)-$ $\varphi_{k}\left(X_{1}\right)\left|\leq\left|f\left(X_{1}+x\right)-f\left(X_{1}\right)\right|\right.$. From (2) we have that
$\frac{\varphi_{k}\left(X_{1}+x\right)-\varphi_{k}\left(X_{1}\right)}{x} \mathbb{1}_{[0,1] \times \mathbb{R} \backslash\{0\}}(t, x)=D_{t, x} \varphi_{k}\left(X_{1}\right)=\sum_{n=1}^{\infty} n I_{n-1}\left(\tilde{f}_{n}^{(k)}(\cdot,(t, x))\right.$,
in $L_{2}(\mathrm{~m} \otimes \mathbb{P})$, which gives

$$
\begin{aligned}
h_{n} & =\lim _{k \rightarrow \infty} n \tilde{f}_{n}^{(k)} \mathbb{1}_{(\mathbb{R}+\times \mathbb{R}) \times(n-1)} \times([0,1] \times\{0<|x| \leq 1\}) \\
& =n \tilde{f}_{n} \mathbb{1}_{(\mathbb{R}+\times \mathbb{R}) \times(n-1)} \times([0,1] \times\{0<|x| \leq 1\})
\end{aligned}
$$

in $L_{2}\left(\mathrm{~m}^{\otimes n}\right)$ for $n=1,2, \ldots$ Therefore

$$
\begin{aligned}
& \frac{f\left(X_{1}+x\right)-f\left(X_{1}\right)}{x} \mathbb{1}_{[0,1] \times\{0<|x| \leq 1\}}(t, x) \\
& =\sum_{n=1}^{\infty} n I_{n-1}\left(\tilde{f}_{n}(\cdot,(t, x)) \mathbb{1}_{[0,1] \times\{0<|x| \leq 1\}}(t, x)\right)
\end{aligned}
$$

in $L_{2}(\mathbb{m} \otimes \mathbb{P})$. This together with Lemma 1 (a) proves equation (7). For the second term on the right hand side of (7) we have by [22, Proposition 3.4] that
$\sum_{n=1}^{\infty} n n!\left\|\tilde{f}_{n} \mathbb{1}_{\left(\mathbb{R}_{+} \times \mathbb{R}\right) \times(n-1) \times A}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2} \leq \mathbb{E}\left[f^{2}\left(X_{1}\right) N(A)\right]+\mathbb{E}\left[f^{2}\left(X_{1}\right)\right] \mathbb{E}[N(A)]$.
Thus, from (7), (8) and the above inequality we get that
$\sum_{n=1}^{\infty} n n!\left\|\tilde{f}_{n}\right\|_{L_{2}\left(\mathrm{~m} \otimes^{n}\right)}^{2} \leq\|f\|_{C^{\alpha}}^{2} m_{2 \alpha}+\mathbb{E}\left[f^{2}\left(X_{1}\right) N(A)\right]+\mathbb{E}\left[f^{2}\left(X_{1}\right)\right] \mathbb{E}[N(A)]$.
Noting that $\mathbb{E}[N(A)]=\nu(\{|x|>1\})$, we obtain the claim.
(d) We have $g^{\alpha, \ell} \in C_{b}^{\alpha}$ by Lemma 4 below. If $g^{\alpha, \ell}\left(X_{1}\right) \in \mathbb{D}_{1,2}$, then by (1) and Lemma 4 it holds that

$$
\begin{aligned}
\infty & >\int_{\mathbb{R}} \mathbb{E}\left[\left(g^{\alpha, \ell}\left(X_{1}+x\right)-g^{\alpha, \ell}\left(X_{1}\right)\right)^{2}\right] \nu(\mathrm{d} x) \\
& \geq \int_{|x| \leq 2^{-\ell-3}}\left[c \int_{k 2^{-\ell}}^{(k+1) 2^{-\ell}}(g(y+x)-g(y))^{2} \mathrm{~d} y\right] \nu(\mathrm{d} x) \\
& \geq c 2^{-\ell} 2^{8 \alpha-10} \int_{|x| \leq 2^{-\ell-3}}|x|^{2 \alpha} \nu(\mathrm{~d} x) .
\end{aligned}
$$

Hence it must be $m_{2 \alpha}<\infty$.
The idea for the construction of the function $g^{\alpha, \ell}$ below is based on the decomposition of Ciesielski [7].

Lemma 4. Let $\ell \in\{0,1,2 \ldots\}$ and $g^{\alpha, \ell}(x)=\sum_{n=\ell}^{\infty} 2^{-\alpha n} g_{n}(x)$, where

$$
g_{n}(x)=d\left(2^{n} x, \mathbb{Z}\right)=\inf \left\{\left|2^{n} x-z\right|: z \in \mathbb{Z}\right\} .
$$

Then $g^{\alpha, \ell} \in C_{b}^{\alpha}$, and for all $k \in \mathbb{Z}$ and $|x| \leq 2^{-\ell-3}$ it holds that

$$
\int_{k 2^{-\ell}}^{(k+1) 2^{-\ell}}\left[g^{\alpha, \ell}(y+x)-g^{\alpha, \ell}(y)\right]^{2} \mathrm{~d} y \geq 2^{-\ell} 2^{8 \alpha-10}|x|^{2 \alpha}
$$

Proof. Since $\left|g_{n}(x)\right| \leq 1 / 2$ for all $x \in \mathbb{R}$, it is clear that $\left\|g^{\alpha, \ell}\right\|_{\infty}<\infty$. Since we also have that $\left|g_{n}(x)-g_{n}(y)\right| \leq 2^{n}|x-y|$ for all $x, y \in \mathbb{R}$, we get for any $m \geq \ell$ and $2^{-m-1} \leq|x-y| \leq 2^{-m}$, that

$$
\begin{aligned}
\left|g^{\alpha, \ell}(x)-g^{\alpha, \ell}(y)\right| & \leq \sum_{n=\ell}^{\infty} 2^{-\alpha n}\left|g_{n}(x)-g_{n}(y)\right| \\
& \leq \sum_{n=0}^{m} 2^{-\alpha n} 2^{n} 2^{-m}+\sum_{n=m+1}^{\infty} 2^{-\alpha n} \\
& \leq \frac{2\left(2^{-m-1}\right)^{\alpha}}{2^{1-\alpha}-1}+\frac{\left(2^{-m-1}\right)^{\alpha}}{1-2^{-\alpha}} \\
& \leq\left(\frac{1}{\left(2^{1-\alpha}-1\right)\left(1-2^{-\alpha}\right)}\right)|x-y|^{\alpha} .
\end{aligned}
$$

Thus $g^{\alpha, \ell} \in C_{b}^{\alpha}$
The function $g_{m}$ is periodic with period length $2^{-n}$ for all $m \geq n$, so that
via dominated convergence we get that

$$
\begin{aligned}
& \int_{k 2^{-\ell}}^{(k+1) 2^{-\ell}}\left[g^{\alpha, \ell}(y+x)-g^{\alpha, \ell}(y)\right]^{2} \mathrm{~d} y \\
& =\sum_{n=\ell}^{\infty} 2^{n-\ell-2 \alpha n} \int_{0}^{2^{-n}}\left[g_{n}(y+x)-g_{n}(y)\right]^{2} \mathrm{~d} y \\
& \quad+2 \sum_{m>n \geq \ell} 2^{n-\ell-\alpha(n+m)} \int_{0}^{2^{-n}}\left[g_{n}(y+x)-g_{n}(y)\right]\left[g_{m}(y+x)-g_{m}(y)\right] \mathrm{d} y .
\end{aligned}
$$

Let $m>n \geq \ell$. Since $g_{m}$ is periodic with period length $2^{-n-1}$ and

$$
g_{n}(y+x)-g_{n}(y)=-\left(g_{n}\left(y+2^{-n-1}+x\right)-g_{n}\left(y+2^{-n-1}\right)\right)
$$

for all $x, y \in \mathbb{R}$, we have that

$$
\begin{aligned}
& \int_{0}^{2^{-n}}\left[g_{n}(y+x)-g_{n}(y)\right]\left[g_{m}(y+x)-g_{m}(y)\right] \mathrm{d} y \\
& =\int_{0}^{2^{-n-1}}\left[g_{n}(y+x)-g_{n}(y)\right]\left[g_{m}(y+x)-g_{m}(y)\right] \mathrm{d} y \\
& \quad+\int_{2^{-n-1}}^{2^{-n}}\left[g_{n}(y+x)-g_{n}(y)\right]\left[g_{m}(y+x)-g_{m}(y)\right] \mathrm{d} y \\
& =0 .
\end{aligned}
$$

Let $0<|x| \leq 2^{-\ell-3}$ and $m \geq \ell$ such that $2^{-m-4}<|x| \leq 2^{-m-3}$. Since $\left|g_{m}(y+x)-g_{m}(y)\right|=2^{m}|x|$ when both $y+x \in\left(0,2^{-m-1}\right)$ and $y \in\left(0,2^{-m-1}\right)$, we obtain that

$$
\int_{0}^{2^{-m}}\left[g_{m}(y+x)-g_{m}(y)\right]^{2} \mathrm{~d} y \geq \int_{2^{-m-3}}^{3 \cdot 2^{-m-3}}\left[2^{m}|x|\right]^{2} \mathrm{~d} y=2^{m-2} x^{2}
$$

Since $2^{m-2} x^{2} \geq 2^{m-2}\left(2^{-m-4}\right)^{2-2 \alpha}|x|^{2 \alpha}=2^{-m+2 \alpha m+8 \alpha-10}|x|^{2 \alpha}$, we get

$$
\begin{aligned}
\sum_{n=\ell}^{\infty} 2^{n-\ell-2 \alpha n} \int_{0}^{2^{-n}}\left[g_{n}(y+x)-g_{n}(y)\right]^{2} \mathrm{~d} y & \geq 2^{m-\ell-2 \alpha m} 2^{-m+2 \alpha m+8 \alpha-10} \\
& \geq 2^{-\ell} 2^{8 \alpha-10}|x|^{2 \alpha}
\end{aligned}
$$

Remark 1. The function $g^{\alpha, \ell}$ in Theorem 3(d) and Lemma 4 is irregular on the whole real line. If a $C_{b}^{\alpha}$-function is "more smooth", then Theorem 3(d) does not necessarily give the best condition: Take for example $f(x)=|x|^{\alpha} \wedge 1$, which is $C_{b}^{\alpha}$ but not $C_{b}^{\alpha^{\prime}}$ for any $\alpha^{\prime}>\alpha$, and assume that (A1) holds. Then for
$0<|x| \leq 1$ we have that

$$
\begin{aligned}
& \mathbb{E}\left[\left(\left|X_{1}+x\right|^{\alpha} \wedge 1-\left|X_{1}\right|^{\alpha} \wedge 1\right)^{2}\right] \\
& \leq\left\|p_{1}\right\|_{\infty} \int_{-2}^{2}\left(|y+x|^{\alpha}-|y|^{\alpha}\right)^{2} \mathrm{~d} y \\
& =\left\|p_{1}\right\|_{\infty}|x|^{2 \alpha+1} \int_{-\frac{2}{|x|}}^{\frac{2}{|x|}}\left(\left|z+\frac{x}{|x|}\right|^{\alpha}-|z|^{\alpha}\right)^{2} \mathrm{~d} z \\
& \leq\left\|p_{1}\right\|_{\infty}|x|^{2 \alpha+1}\left[\int_{|z|<2} 1 \mathrm{~d} z+\alpha^{2} \int_{2 \leq|z| \leq \frac{2}{|x|}}(|z|-1)^{2 \alpha-2} \mathrm{~d} z\right] \\
& \leq \begin{cases}\left\|p_{1}\right\|_{\infty}|x|^{2 \alpha+1}\left[4+\frac{2 \alpha^{2}}{1-2 \alpha}\right], & \text { for } \alpha<\frac{1}{2} \\
\left\|p_{1}\right\|_{\infty}|x|^{2}\left[4+2 \log \frac{2}{|x|}\right], & \text { for } \alpha=\frac{1}{2} . \\
\left\|p_{1}\right\|_{\infty}|x|^{2 \alpha+1}\left[4+\frac{2^{2 \alpha} \alpha^{2}}{2 \alpha-1}|x|^{1-2 \alpha}\right], & \text { for } \alpha>\frac{1}{2}\end{cases}
\end{aligned}
$$

Since $\mathbb{E}\left[\left(\left|X_{1}+x\right|^{\alpha} \wedge 1-\left|X_{1}\right|^{\alpha} \wedge 1\right)^{2}\right] \leq 1$, we get from (1) that $\left|X_{1}\right|^{\alpha} \wedge 1 \in$ $\mathbb{D}_{1,2}$, if one of the following three conditions holds: $1.0<\alpha<1 / 2$ and $m_{2 \alpha+1}<\infty, 2 . \alpha=1 / 2$ and $\int_{\{0<|x| \leq 1\}} x^{2} \log (1 /|x|) \nu(\mathrm{d} x)<\infty$ or $3 . \alpha>1 / 2$. Note that for the Brownian motion $B$ we have $\left|B_{1}\right|^{\alpha} \wedge 1 \in \mathbb{D}_{1,2}$ if and only if $\alpha>1 / 2$. This can be easily seen using [23, Example 1.2.8].

### 3.2. Fractional smoothness

To find fractional smoothness for $f\left(X_{1}\right)$ with $f \in C_{b}^{\alpha}$ in Corollary 5 below, we take advantage of the fact that $C_{b}^{\alpha}=(B(\mathbb{R}), \text { Lip })_{\alpha, \infty}$ with

$$
\begin{equation*}
\|\cdot\|_{C_{b}^{\alpha}} \leq 3\|\cdot\|_{(B(\mathbb{R}), L i p)_{\alpha, \infty}} \leq 6\|\cdot\|_{C_{b}^{\alpha}} \tag{9}
\end{equation*}
$$

(see Lemma 17 and also [30, Theorem 2.7.2/1] in a slightly different setting).
Theorem 5. Let $0<\alpha \leq \theta<1$.
(a) If $f \in C_{b}^{\alpha}$ and $m_{2 \alpha / \theta}<\infty$, then

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}
$$

and

$$
\left\|f\left(X_{1}\right)\right\|_{\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}} \leq 18 \sqrt{1+4 m_{2 \alpha / \theta}}\|f\|_{C_{b}^{\alpha}}
$$

(b) Assume that (A3) holds and choose $t_{0} \in(0,1)$ and $\ell \in\{0,1,2, \ldots\}$ such that there exist $k \in \mathbb{Z}$ and $c>0$ with $p_{t}(x) \geq c$ for all $t \in\left[t_{0}, 1\right]$ and all $x \in\left[(k-1) 2^{-\ell},(k+2) 2^{-\ell}\right]$. For the function $g^{\alpha, \ell} \in C_{b}^{\alpha}$ of Lemma 4 it holds that

$$
g^{\alpha, \ell}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \quad \text { only if } \quad m_{2 \alpha / \theta+\varepsilon}<\infty \text { for all } \varepsilon>0
$$

Proof. (a) One finds for every $t>0$ and $\varepsilon>0$ a function $f_{t} \in C_{b}^{\alpha / \theta}$ such that

$$
\left(\left\|f-f_{t}\right\|_{\infty}+t\left\|f_{t}\right\|_{C_{b}^{\alpha / \theta}}\right) \leq K\left(f, t ; B(\mathbb{R}), C_{b}^{\alpha / \theta}\right)+\varepsilon
$$

Using inequality (6) for $f_{t}\left(X_{1}\right)$ we get

$$
\begin{aligned}
K\left(f\left(X_{1}\right), t ; L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right) & \leq\left\|\left(f-f_{t}\right)\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}+t\left\|f_{t}\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}} \\
& \leq\left\|f-f_{t}\right\|_{\infty}+t\left\|f_{t}\right\|_{C_{b}^{\alpha / \theta}} \sqrt{1+4 m_{2 \alpha / \theta}} \\
& \leq \sqrt{1+4 m_{2 \alpha / \theta}}\left(K\left(f, t ; B(\mathbb{R}), C_{b}^{\alpha / \theta}\right)+\varepsilon\right)
\end{aligned}
$$

so that

$$
\left\|f\left(X_{1}\right)\right\|_{\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}} \leq \sqrt{1+4 m_{2 \alpha / \theta}}\|f\|_{\left(B(\mathbb{R}), C_{b}^{\alpha / \theta}\right)_{\theta, \infty}}
$$

Using the first inequality of (9), (5), and the second inequality of (9), we obtain that

$$
\begin{aligned}
\|f\|_{\left(B(\mathbb{R}), C_{b}^{\alpha / \theta}\right)_{\theta, \infty}} & \leq 3\|f\|_{\left(B(\mathbb{R}),(B(\mathbb{R}), L i p)_{\alpha / \theta, \infty}\right)_{\theta, \infty}} \\
& \leq 9\|f\|_{(B(\mathbb{R}), L i p)_{\alpha, \infty}} \\
& \leq 18\|f\|_{C_{b}^{\alpha}}
\end{aligned}
$$

and this finishes the proof of (a). The proof of assertion (b) is given in Section 5.

Remark 2. Assertion (a) of Theorem 5 implies that $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\alpha, \infty}$ for all $f \in C_{b}^{\alpha}$ for any pure jump Lévy process $X$. Also for the Brownian motion $B$ we obtain the smoothness of level $(\alpha, \infty)$ for $f\left(B_{1}\right)$ for any $f \in C_{b}^{\alpha}$ : choose $f_{t} \in C_{b}^{1}=\operatorname{Lip}$ like in the proof of Theorem 5 and use the fact that

$$
\left\|f_{t}\left(B_{1}\right)\right\|_{\mathbb{D}_{1,2}} \leq c\left\|f_{t}\right\|_{L i p}
$$

from [29, Lemma A.5], where $c>0$ is a constant not depending on $f_{t}$.

## 4. Functions of bounded variation and smoothness

Let us first recall the space of normalized functions of bounded variation, the space $N B V$. The variation function of $f$ is given by

$$
T_{f}(x)=\sup \left\{\sum_{i=1}^{n}\left|f\left(x_{i}\right)-f\left(x_{i-1}\right)\right|:-\infty<x_{0}<x_{1}<\cdots<x_{n}=x, n \geq 1\right\}
$$

and the total variation of $f$ is $V(f)=\lim _{x \rightarrow \infty} T_{f}(x)$. The space of functions of bounded variation is

$$
B V=\left\{f: \mathbb{R} \rightarrow \mathbb{R}:\|f\|_{B V}=\limsup _{x \rightarrow-\infty}|f(x)|+V(f)<\infty\right\}
$$

Note that when $V(f)<\infty$, then the limit $f(-\infty):=\lim _{x \rightarrow-\infty} f(x)$ exists ([9, Theorem $3.27(\mathrm{c})]$ ) and for $f \in B V$ we may write $\|f\|_{B V}=|f(-\infty)|+V(f)$. Furthermore,

$$
\|f\|_{\infty} \leq\|f\|_{B V}
$$

We denote by $N B V$ the space of normalized functions of bounded variation, that is, the space of all $f \in B V$ such that $f$ is right continuous and $f(-\infty)=0$. When $f \in N B V$, then by [9, Theorem 3.29] there exists a finite signed measure $\mu_{f}$ such that

$$
\begin{equation*}
f(x)=\int_{\mathbb{R}} \mathbb{1}_{(-\infty, x]}(u) \mu_{f}(\mathrm{~d} u)=\int_{\mathbb{R}} \mathbb{1}_{[u, \infty)}(x) \mu_{f}(\mathrm{~d} u)=\int_{\mathbb{R}} \mathbb{1}_{[0, \infty)}(x-u) \mu_{f}(\mathrm{~d} u) \tag{10}
\end{equation*}
$$

for all $x \in \mathbb{R}$. Furthermore, $\mu_{f}$ admits the Jordan decomposition $\mu_{f}=\mu_{f}^{+}-\mu_{f}^{-}$, where $\mu_{f}^{+}$and $\mu_{f}^{-}$are nonnegative finite measures. We write $\left|\mu_{f}\right|=\mu_{f}^{+}+\mu_{f}^{-}$so that $\left|\mu_{f}\right|(\mathbb{R})=\|f\|_{B V}$.

### 4.1. Smoothness of first order

Theorem 6 ([21, Example 3.1]). For normalized functions of bounded variation we have the following.
(a) Assume that (A1) holds. If $f \in N B V$ and $m_{1}<\infty$, then $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$ and

$$
\left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}} \leq \sqrt{1+\left(1 \vee\left\|p_{1}\right\|_{\infty}\right) m_{1}}\|f\|_{B V}
$$

(b) Suppose that $X_{1}$ satisfies (A2) and let $K \in \mathbb{R}$ be such that there is $r>0$ and $c>0$ such that the density $p_{1}$ of $X_{1}$ satisfies $p_{1}(x) \geq c$ for all $x \in$ $[K-r, K+r]$. Then $\mathbb{1}_{[K, \infty)}\left(X_{1}\right) \in \mathbb{D}_{1,2}$ only if $m_{1}<\infty$.
Proof. (a) Let $f \in N B V$ and $\mu_{f}$ be the according signed measure from (10).
We use Hölder's inequality to get

$$
\begin{aligned}
& \int_{\mathbb{R}} \mathbb{E}\left[\left(f\left(X_{1}+x\right)-f\left(X_{1}\right)\right)^{2}\right] \nu(\mathrm{d} x) \\
& =\int_{\mathbb{R}} \mathbb{E}\left[\left(\int_{\mathbb{R}}\left(\mathbb{1}_{[u, \infty)}\left(X_{1}+x\right)-\mathbb{1}_{[u, \infty)}\left(X_{1}\right)\right) \mu_{f}(\mathrm{~d} u)\right)^{2}\right] \nu(\mathrm{d} x) \\
& \leq\left|\mu_{f}\right|(\mathbb{R}) \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}\left[\left(\mathbb{1}_{[u, \infty)}\left(X_{1}+x\right)-\mathbb{1}_{[u, \infty)}\left(X_{1}\right)\right)^{2}\right]\left|\mu_{f}\right|(\mathrm{d} u) \nu(\mathrm{d} x) \\
& \leq\left|\mu_{f}\right|(\mathbb{R}) \int_{\mathbb{R}} \int_{\mathbb{R}}\left(\left\|p_{1}\right\|_{\infty}|x| \wedge 1\right)\left|\mu_{f}\right|(\mathrm{d} u) \nu(\mathrm{d} x) \\
& \leq\|f\|_{B V}^{2}\left(1 \vee\left\|p_{1}\right\|_{\infty}\right) \int_{\mathbb{R}}(|x| \wedge 1) \nu(\mathrm{d} x) .
\end{aligned}
$$

Hence from (1) we obtain that

$$
\begin{aligned}
\left\|f\left(X_{1}\right)\right\|_{\mathbb{D}_{1,2}}^{2} & =\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\int_{\mathbb{R}} \mathbb{E}\left[\left(f\left(X_{1}+x\right)-f\left(X_{1}\right)\right)^{2}\right] \nu(\mathrm{d} x) \\
& \leq\|f\|_{B V}^{2}+\|f\|_{B V}^{2}\left(1 \vee\left\|p_{1}\right\|_{\infty}\right) m_{1}
\end{aligned}
$$

(b) Let $r>0$ and $c>0$ be such that $p_{1}(x) \geq c$ for all $x \in[K-r, K+r]$. Let $f=\mathbb{1}_{[K, \infty)}$. Then $f \in N B V$ and

$$
\begin{aligned}
& \int_{\mathbb{R}} \mathbb{E}\left[\left|f\left(X_{1}+x\right)-f\left(X_{1}\right)\right|^{2}\right] \nu(\mathrm{d} x) \\
& =\int_{(-\infty, 0)} \mathbb{E}\left[\mathbb{1}_{[K, K-x)}\left(X_{1}\right)\right] \nu(\mathrm{d} x)+\int_{(0, \infty)} \mathbb{E}\left[\mathbb{1}_{[K-x, K)}\left(X_{1}\right)\right] \nu(\mathrm{d} x) \\
& \geq c \int_{0<|x| \leq r}|x| \nu(\mathrm{d} x) .
\end{aligned}
$$

By (1) it holds that $m_{1}<\infty$, if $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$.

### 4.2. Fractional smoothness

If $m_{1}<\infty$ does not hold, it is still possible to attain fractional smoothness with functions in $N B V$. In [11, Example 4.2(a)] it is verified that $\mathbb{1}_{(K, \infty)}\left(X_{1}\right) \in$ $\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{1 / 2, \infty}$. Note that in $[11$, Example $4.2(\mathrm{a})]$ it is assumed a small ball estimate for the distribution and this assumption is equivalent with (A1) (one can easily see this by using the steps of the proof of [2, Theorem 2.4(iii)]). In the following theorem we show that the smoothness level increases as the Blumenthal-Getoor index decreases.

Theorem 7. Let $1 / 2 \leq \theta<1$.
(a) Assume that (A1) holds. If $f \in N B V$ and $m_{1 / \theta}<\infty$, then

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}
$$

and

$$
\left\|f\left(X_{1}\right)\right\|_{\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}} \leq\left(\sqrt{\|p\|_{\infty}}+\sqrt{1+2\left(\|p\|_{\infty} \vee 1\right) m_{1 / \theta}}\right)\|f\|_{B V}
$$

Especially, $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\frac{1}{2}, \infty}$ for any Lévy measure $\nu$.
(b) Assume that (A3) holds and let $t_{0} \in(0,1)$ and $K \in \mathbb{R}$ be such that there exist $r>0$ and $c>0$ with $p_{t}(x) \geq c$ for all $x \in[K-2 r, K+2 r]$ and all $t \in\left[t_{0}, 1\right]$. Then

$$
\mathbb{1}_{[K, \infty)}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \quad \text { only if } \quad m_{1 / \theta+\varepsilon}<\infty \text { for all } \varepsilon>0
$$

Proof. (a) Let $f \in N B V$ and $\mu_{f}$ be the according signed measure from (10). For $t \in(0,1)$ we define

$$
g_{t}(x)=\left\{\begin{array}{ll}
0, & x \leq 0 \\
\frac{1}{t} x^{\frac{1}{2 \theta}}, & 0<x<t^{2 \theta} \\
1, & x \geq t^{2 \theta}
\end{array} \quad \text { and } \quad f_{t}(x)=\int_{\mathbb{R}} g_{t}(x-u) \mu_{f}(\mathrm{~d} u)\right.
$$

Then

$$
\begin{aligned}
& \mathbb{E}\left[\left(f_{t}\left(X_{1}+x\right)-f_{t}\left(X_{1}\right)\right)^{2}\right] \\
& =\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left[g_{t}(y+x-u)-g_{t}(y-u)\right] \mu_{f}(\mathrm{~d} u)\right)^{2} p(y) \mathrm{d} y \\
& \leq\left|\mu_{f}\right|(\mathbb{R})\|p\|_{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}}\left(g_{t}(y+x-u)-g_{t}(y-u)\right)^{2}\left|\mu_{f}\right|(\mathrm{d} u) \mathrm{d} y \\
& =\left|\mu_{f}\right|(\mathbb{R})^{2}\|p\|_{\infty} \int_{\mathbb{R}}\left(g_{t}(z+x)-g_{t}(z)\right)^{2} \mathrm{~d} z
\end{aligned}
$$

Note that $g_{t}(\cdot+x)-g_{t}$ is nonzero only on an interval of length $t^{2 \theta}+|x|$ and

$$
\begin{aligned}
\left|g_{t}(z+x)-g_{t}(z)\right| & =\left|\int_{z}^{z+x} \frac{1}{2 \theta t} u^{\frac{1}{2 \theta}-1} \mathbb{1}_{\left(0, t^{2 \theta}\right)}(u) \mathrm{d} u\right| \\
& \leq \int_{0}^{|x|} \frac{1}{2 \theta t} u^{\frac{1}{2 \theta}-1} \mathbb{1}_{\left(0, t^{2 \theta}\right)}(u) \mathrm{d} u \\
& =g_{t}(|x|) \leq 1
\end{aligned}
$$

for all $x, z \in \mathbb{R}$, since $\frac{1}{2 \theta}-1 \leq 0$. When $|x| \geq t^{2 \theta}$, then

$$
\int_{\mathbb{R}}\left(g_{t}(z+x)-g_{t}(z)\right)^{2} \mathrm{~d} z \leq 2|x|=2 t^{2(\theta-1)}|x| t^{2(1-\theta)} \leq 2 t^{2(\theta-1)}|x|^{1 / \theta}
$$

When $|x|<t^{2 \theta}$, then

$$
\int_{\mathbb{R}}\left(g_{t}(z+x)-g_{t}(z)\right)^{2} \mathrm{~d} z \leq 2 t^{2 \theta} g_{t}^{2}(|x|)=2 t^{2(\theta-1)}|x|^{1 / \theta}
$$

On the other hand,

$$
\begin{aligned}
& \mathbb{E}\left[\left(f_{t}\left(X_{1}+x\right)-f_{t}\left(X_{1}\right)\right)^{2}\right] \\
& =\mathbb{E}\left[\left(\int_{\mathbb{R}}\left(g_{t}\left(X_{1}+x-u\right)-g_{t}\left(X_{1}-u\right)\right) \mu_{f}(\mathrm{~d} u)\right)^{2}\right] \\
& \leq\left|\mu_{f}\right|^{2}(\mathbb{R})
\end{aligned}
$$

so that

$$
\begin{aligned}
& \int_{\mathbb{R}} \mathbb{E}\left[\left(f_{t}\left(X_{1}+x\right)-f_{t}\left(X_{1}\right)\right)^{2}\right] \nu(\mathrm{d} x) \\
& \leq \int_{\mathbb{R}}\left|\mu_{f}\right|(\mathbb{R})^{2}\left(\|p\|_{\infty} \vee 1\right)\left(2 t^{2(\theta-1)}|x|^{1 / \theta} \wedge 1\right) \nu(\mathrm{d} x) \\
& \leq\left|\mu_{f}\right|(\mathbb{R})^{2}\left(\|p\|_{\infty} \vee 1\right) 2 t^{2(\theta-1)} m_{1 / \theta}
\end{aligned}
$$

since $0<t<1$, and therefore $f_{t}\left(X_{1}\right) \in \mathbb{D}_{1,2}$. It also holds, by (10), that

$$
\begin{aligned}
\left\|\left(f-f_{t}\right)\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2} & =\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left[\mathbb{1}_{[0, \infty)}(y-u)-g_{t}(y-u)\right] \mu_{f}(\mathrm{~d} u)\right)^{2} \mathbb{P}_{X_{1}}(\mathrm{~d} y) \\
& \leq\left|\mu_{f}\right|(\mathbb{R})^{2}\|p\|_{\infty} \int_{\mathbb{R}}\left(\mathbb{1}_{[0, \infty)}(y)-g_{t}(y)\right)^{2} \mathrm{~d} y \\
& \leq\left|\mu_{f}\right|(\mathbb{R})^{2}\|p\|_{\infty} t^{2 \theta}
\end{aligned}
$$

and

$$
\left\|f_{t}\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})} \leq\left|\mu_{f}\right|(\mathbb{R})
$$

We obtain for $t \in(0,1)$ that

$$
\begin{aligned}
& t^{-\theta}\left(\left\|\left(f-f_{t}\right)\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}+t \sqrt{\left\|f_{t}\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\left\|D f_{t}\left(X_{1}\right)\right\|_{L_{2}(\mathrm{~m} \otimes \mathbb{P})}^{2}}\right) \\
& \leq t^{-\theta}\left(\sqrt{\|p\|_{\infty}}\left|\mu_{f}\right|(\mathbb{R}) t^{\theta}+t \sqrt{\left|\mu_{f}\right|(\mathbb{R})^{2}+\left|\mu_{f}\right|(\mathbb{R})^{2}\left(\|p\|_{\infty} \vee 1\right) 2 t^{2(\theta-1)} m_{1 / \theta}}\right) \\
& \leq\left(\sqrt{\|p\|_{\infty}}+\sqrt{1+2\left(\|p\|_{\infty} \vee 1\right) m_{1 / \theta}}\right)\left|\mu_{f}\right|(\mathbb{R}) .
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \left\|f\left(X_{1}\right)\right\|_{\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}} \\
& =\sup _{t>0} t^{-\theta} \inf \left\{\left\|Y_{0}\right\|_{L_{2}(\mathbb{P})}+t\left\|Y_{1}\right\|_{\mathbb{D}_{1,2}}: Y_{0}+Y_{1}=f\left(X_{1}\right)\right\} \\
& \leq \sup _{t \in(0,1)} t^{-\theta}\left(\left\|\left(f-f_{t}\right)\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}+t \sqrt{\left\|f_{t}\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}^{2}+\left\|D f_{t}\left(X_{1}\right)\right\|_{L_{2}(\mathrm{~m} \otimes \mathbb{P})}^{2}}\right) \\
& \quad \vee\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})} \\
& \leq\left(\sqrt{\|p\|_{\infty}}+\sqrt{1+2\left(\|p\|_{\infty} \vee 1\right) m_{1 / \theta}}\right)\|f\|_{B V} .
\end{aligned}
$$

The proof of assertion (b) is given in Section 5.
5. Sharpness of the connection between the smoothness index and the Blumenthal-Getoor index

In Lemma 9 below, we adapt the characterisation for fractional smoothness from [15, Corollary 2.3], where it is written for the Brownian motion.

Definition 1. For a sequence of Banach spaces $E=\left(E_{n}\right)_{n=0}^{\infty}$ with $E_{n} \neq\{0\}$ we let $\ell_{2}(E)$ and $d_{1,2}(E)$ be the Banach spaces of all $a=\left(a_{n}\right)_{n=0}^{\infty} \in E$ such that

$$
\|a\|_{\ell_{2}(E)}:=\left(\sum_{n=0}^{\infty}\left\|a_{n}\right\|_{E_{n}}^{2}\right)^{\frac{1}{2}} \quad \text { and } \quad\|a\|_{d_{1,2}(E)}:=\left(\sum_{n=0}^{\infty}(n+1)\left\|a_{n}\right\|_{E_{n}}^{2}\right)^{\frac{1}{2}}
$$

respectively, are finite. For $a \in E$ we let $T a:[0,1] \rightarrow \mathbb{R}$ be defined by

$$
(T a)(t):=\sum_{n=0}^{\infty}\left\|a_{n}\right\|_{E_{n}}^{2} t^{n}
$$

We use the notation $A \sim_{c} B$ for $\frac{1}{c} B \leq A \leq c B$, where $A, B \in[0, \infty]$ and $c \geq 1$.
Lemma 8 ([15, Theorem 2.2]). For $\theta \in(0,1), q \in[1, \infty]$ and $a \in \ell_{2}(E)$ one has

$$
\begin{aligned}
& \|a\|_{\left(\ell_{2}(E), d_{1,2}(E)\right)_{\theta, q}} \\
& \sim_{c}\|a\|_{\ell_{2}(E)}+\left\|(1-t)^{\frac{1-\theta}{2}} \sqrt{(T a)^{\prime}(t)}\right\|_{L_{q}\left((0,1), \mathcal{B}(0,1), \frac{\mathrm{d} t}{1-t}\right)} \\
& \sim_{c}\|a\|_{\ell_{2}(E)}+\left\|(1-t)^{-\frac{\theta}{2}} \sqrt{(T a)(1)-(T a)(t)}\right\|_{L_{q}\left((0,1), \mathcal{B}(0,1), \frac{\mathrm{d} t}{1-t}\right)},
\end{aligned}
$$

where $c \geq 1$ depends only on $(\theta, q)$, and the expressions may be infinite.
We will apply this theorem to the Itô chaos decomposition. Let $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ be the augmented natural filtration of $X$. Throughout this section we let $\bar{X}$ be an independent copy of $X$ on $(\bar{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. We will use the notation $\overline{\mathbb{E}}$ for the expectation with respect to the measure $\overline{\mathbb{P}}$.

Lemma 9. For $\theta \in(0,1), q \in[1, \infty]$ and $f\left(X_{1}\right) \in L_{2}(\mathbb{P})$ one has

$$
\begin{aligned}
& \left\|f\left(X_{1}\right)\right\|_{\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}} \\
& \sim_{c}\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}+\left\|(1-t)^{-\frac{\theta}{2}}\right\| f\left(X_{1}\right)-\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]\left\|_{L_{2}(\mathbb{P})}\right\|_{L_{q}\left((0,1), \mathcal{B}(0,1), \frac{\mathrm{d} t}{1-t}\right)} \\
& =\left\|f\left(X_{1}\right)\right\|_{L_{2}(\mathbb{P})}+\frac{1}{\sqrt{2}}\left\|(1-t)^{-\frac{\theta}{2}}\right\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\left\|_{L_{2}(\overline{\mathbb{P}})}\right\|_{L_{q}\left(\frac{\mathrm{~d} t}{1-t}\right)},
\end{aligned}
$$

where $c \geq 1$ depends only on $(\theta, q)$ and the expressions may be infinite.
Proof. Let $f\left(X_{1}\right)=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right) \in L_{2}(\mathbb{P}), E=\left(L_{2}\left(\mathrm{~m}^{\otimes n}\right)\right)_{n=0}^{\infty}$ and $a=$ $\left(\sqrt{n!} \tilde{f}_{n}\right)_{n=0}^{\infty}$. By orthogonality the equality

$$
\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right)=\sum_{n=0}^{\infty} I_{n}\left(g_{n}\right)+\sum_{n=0}^{\infty} I_{n}\left(h_{n}\right)
$$

holds in $L_{2}(\mathbb{P})$ if and only if $\tilde{f}_{n}=\tilde{g_{n}}+\tilde{h_{n}}$ holds $\mathrm{m}^{\otimes n}$-a.e. Therefore

$$
\begin{aligned}
& K\left(f\left(X_{1}\right), t ; L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right) \\
& =\inf _{\tilde{f_{n}}=\tilde{g_{n}}+\tilde{h_{n}}}\left(\sqrt{\sum_{n=0}^{\infty} n!\left\|\tilde{g_{n}}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}}+t \sqrt{\sum_{n=0}^{\infty}(n+1)!\left\|\tilde{h_{n}}\right\|_{L_{2}(\mathrm{~m} \otimes n)}^{2}}\right) \\
& =K\left(a, t ; \ell_{2}(E), d_{1,2}(E)\right)
\end{aligned}
$$

and Lemma 1(b) gives

$$
\begin{aligned}
\left\|f\left(X_{1}\right)-\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]\right\|_{L_{2}(\mathbb{P})}^{2} & =\mathbb{E}\left[f\left(X_{1}\right)^{2}\right]-\mathbb{E}\left[\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]^{2}\right] \\
& =(T a)(1)-(T a)(t)
\end{aligned}
$$

The equivalence follows now from Lemma 8. To conclude with the equality below, we use the facts that $\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]=\overline{\mathbb{E}}\left[f\left(X_{t}+\bar{X}_{1-t}\right)\right]$ a.s. and $X_{t}+$ $\bar{X}_{1-t} \stackrel{d}{=} X_{1}$ to get that

$$
\begin{aligned}
& \left\|f\left(X_{1}\right)-\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]\right\|_{L_{2}(\mathbb{P})}^{2} \\
& =\mathbb{E}\left[f\left(X_{1}\right)\left(f\left(X_{1}\right)-\mathbb{E}\left[f\left(X_{1}\right) \mid \mathcal{F}_{t}\right]\right)\right] \\
& =\overline{\mathbb{E}} \mathbb{E}\left[f\left(X_{1}\right)\left(f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right)\right] \\
& =-\overline{\mathbb{E}} \mathbb{E}\left[f\left(X_{t}+\bar{X}_{1-t}\right)\left(f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right)\right] \\
& =\frac{1}{2} \overline{\mathbb{E}} \mathbb{E}\left[\left(f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right)^{2}\right]
\end{aligned}
$$

where the last line is obtained as the average of the two previous lines.
Lemma 10. Let $\tilde{X}$ be a pure jump Lévy process with càdlàg paths on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Let $\tilde{\nu}$ be its Lévy measure and $\beta$ be its BlumenthalGetoor index. Let $t_{0}>0$ and define a constant $\kappa$ by letting

$$
\kappa= \begin{cases}\int_{\{|x| \leq 1\}} x \tilde{\nu}(\mathrm{~d} x), & \text { if } \int_{\{|x| \leq 1\}}|x| \tilde{\nu}(\mathrm{d} x)<\infty \\ 0, & \text { if } \int_{\{|x| \leq 1\}}|x| \tilde{\nu}(\mathrm{d} x)=\infty .\end{cases}
$$

(a) For all $\beta^{\prime}>\beta$ it holds that

$$
\int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\frac{\left|\tilde{X}_{t}+\kappa t\right|}{t^{1 / \beta^{\prime}}}>c\right) \frac{\mathrm{d} t}{t}<\infty \quad \text { for all } c>0
$$

(b) For any $\beta^{\prime \prime}<\beta$ there exists $c^{\prime}>0$ such that

$$
\int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\frac{\left|\tilde{X}_{t}+\kappa t\right|}{t^{1 / \beta^{\prime \prime}}}>c^{\prime}\right) \frac{\mathrm{d} t}{t}=\infty
$$

(c) It holds that

$$
\int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\left|\tilde{X}_{t}\right|>c\right) \frac{\mathrm{d} t}{t}<\infty \quad \text { for all } c>0
$$

Proof. By [6, Theorems 3.1 and 3.3] it holds for all $\beta^{\prime \prime}<\beta<\beta^{\prime}$ that

$$
\lim _{t \rightarrow 0} \frac{\tilde{X}_{t}+\kappa t}{t^{1 / \beta^{\prime}}}=0 \text { a.s. } \quad \text { and } \quad \limsup _{t \rightarrow 0} \frac{\left|\tilde{X}_{t}+\kappa t\right|}{t^{1 / \beta^{\prime \prime}}}=\infty \text { a.s. }
$$

By a result of Khintchine [20, Section 2], we have that if $u:\left(0, t_{0}\right) \rightarrow(0, \infty)$ is non-decreasing and $\lim _{t \rightarrow 0} u(t)=0$, then for any Lévy process $Y$ it holds that

$$
\lim _{t \rightarrow 0} \frac{Y_{t}}{u(t)}=0 \text { a.s. } \quad \text { if and only if } \quad \int_{0}^{t_{0}} \mathbb{P}\left(\frac{\left|Y_{t}\right|}{u(t)}>c\right) \frac{\mathrm{d} t}{t}<\infty \text { for all } c>0
$$

The claims (a) and (b) follow by choosing $Y_{t}=\tilde{X}_{t}+\kappa t$ and $u(t)=t^{1 / \beta^{\prime}}$ in (a) and $u(t)=t^{1 / \beta^{\prime \prime}}$ in (b).
(c) Let $u(t)=t^{1 / 3} \wedge 1$. Then

$$
\lim _{t \rightarrow 0} \frac{\tilde{X}_{t}}{u(t)} \leq \lim _{t \rightarrow 0} \frac{\left|\tilde{X}_{t}+\kappa t\right|}{t^{1 / 3}}+\frac{|\kappa t|}{t^{1 / 3}}=0
$$

by (a) so that [20, Section 2] implies that

$$
\int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\left|\tilde{X}_{t}\right|>c\right) \frac{\mathrm{d} t}{t} \leq \int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\frac{\left|\tilde{X}_{t}\right|}{u(t)}>c\right) \frac{\mathrm{d} t}{t}<\infty \quad \text { for all } c>0
$$

Lemma 11. Assume that (A3) holds and let $R>0, a<b, t_{0} \in(0,1)$ and $c_{1}>0$ be such that $p_{t}(x) \geq c_{1}$ for all $x \in[a-R, b+R]$ and $t \in\left[t_{0}, 1\right]$. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable and there exist $r>0, c_{2}>0$ and $\eta>0$ such that

$$
\begin{equation*}
\int_{a}^{b}[f(y+x)-f(y)]^{2} \mathrm{~d} y \geq c_{2}|x|^{\eta} \quad \text { for all }|x| \leq r \tag{11}
\end{equation*}
$$

then

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \text { only if } m_{\eta / \theta+\varepsilon}<\infty
$$

for all $\varepsilon>0$.
Proof. The assumptions (A3) and (11) yield for $t \in\left[t_{0}, 1\right]$ that

$$
\begin{align*}
& \left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \\
& =\overline{\mathbb{E}} \mathbb{E}\left[\int_{\mathbb{R}}\left(f\left(y+X_{1}-X_{t}\right)-f\left(y+\bar{X}_{1-t}\right)\right)^{2} p_{t}(y) \mathrm{d} y\right] \\
& =\overline{\mathbb{E}} \mathbb{E}\left[\int_{\mathbb{R}}\left(f\left(y+X_{1-t}-\bar{X}_{1-t}\right)-f(y)\right)^{2} p_{t}\left(y-\bar{X}_{1-t}\right) \mathrm{d} y\right] \\
& \geq \overline{\mathbb{E}} \mathbb{E}\left[\int_{a}^{b}\left(f\left(y+X_{1-t}-\bar{X}_{1-t}\right)-f(y)\right)^{2} c_{1} \mathrm{~d} y \mathbb{1}_{\left\{\left|\bar{X}_{1-t}\right| \leq R\right\}}\right] \\
& \geq c_{1} c_{2} \overline{\mathbb{E}} \mathbb{E}\left[\left|X_{1-t}-\bar{X}_{1-t}\right|^{\eta} \mathbb{1}_{\left\{\left|X_{1-t}-\bar{X}_{1-t}\right| \leq r,\left|\bar{X}_{1-t}\right| \leq R\right\}}\right] . \tag{12}
\end{align*}
$$

Since $X$ and $\bar{X}$ are independent, the process $\tilde{X}=X-\bar{X}$ with $\tilde{X}_{t}(\omega, \bar{\omega})=$ $X_{t}(\omega)-\bar{X}_{t}(\bar{\omega})$ is a Lévy process on $(\Omega \times \bar{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \otimes \overline{\mathbb{P}})$ with Lévy measure $\tilde{\nu}(B)=\nu(B)+\nu(-B)$, and its Blumenthal-Getoor index is the same $\beta$ as for $X$. Let $0<\theta^{\prime}<\theta$ and $c>0$ and set $c_{3}=c_{1} c_{2} c^{\eta}$. Then (12) has the lower bound

$$
\begin{aligned}
& c_{3}(1-t)^{\theta^{\prime}}(\mathbb{P} \otimes \overline{\mathbb{P}})\left(\frac{\left|X_{1-t}-\bar{X}_{1-t}\right|^{\eta}}{(1-t)^{\theta^{\prime}} c^{\eta}}>1,\left|X_{1-t}-\bar{X}_{1-t}\right| \leq r,\left|\bar{X}_{1-t}\right| \leq R\right) \\
& \geq c_{3}(1-t)^{\theta^{\prime}}\left[\tilde{\mathbb{P}}\left(\frac{\left|\tilde{X}_{1-t}\right|}{(1-t)^{\theta^{\prime} / \eta}}>c\right)-\tilde{\mathbb{P}}\left(\left|\tilde{X}_{1-t}\right|>r\right)-\overline{\mathbb{P}}\left(\left|\bar{X}_{1-t}\right|>R\right)\right]
\end{aligned}
$$

If $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$, then $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta^{\prime}, 2}$ by (3). Using Lemma 8 we get that

$$
\begin{aligned}
\infty & >\int_{0}^{1}(1-t)^{-\theta^{\prime}}\| \| f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\left\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \frac{\mathrm{~d} t}{1-t} \\
& \geq c_{3} \int_{1-t_{0}}^{1}\left[\tilde { \mathbb { P } } \left(\frac{\left|\tilde{X}_{1-t}\right|}{\left.\left.(1-t)^{\theta^{\prime} / \eta}>c\right)-\tilde{\mathbb{P}}\left(\left|\tilde{X}_{1-t}\right|>r\right)-\overline{\mathbb{P}}\left(\left|\bar{X}_{1-t}\right|>R\right)\right] \frac{\mathrm{d} t}{1-t}}\right.\right. \\
& =c_{3}\left[\int _ { 0 } ^ { t _ { 0 } } \tilde { \mathbb { P } } \left(\frac{\left|\tilde{X}_{t}\right|}{\left.\left.t^{\theta^{\prime} / \eta}>c\right) \frac{\mathrm{~d} t}{t}-\int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\left|\tilde{X}_{t}\right|>r\right) \frac{\mathrm{d} t}{t}-\int_{0}^{t_{0}} \overline{\mathbb{P}}\left(\left|\bar{X}_{t}\right|>R\right) \frac{\mathrm{d} t}{t}\right],}\right.\right.
\end{aligned}
$$

where

$$
\int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\left|\tilde{X}_{t}\right|>r\right) \frac{\mathrm{d} t}{t}+\int_{0}^{t_{0}} \overline{\mathbb{P}}\left(\left|\bar{X}_{t}\right|>R\right) \frac{\mathrm{d} t}{t}<
$$

by Lemma 10(c). Hence

$$
\int_{0}^{t_{0}} \tilde{\mathbb{P}}\left(\frac{\left|\tilde{X}_{t}\right|}{t^{\theta^{\prime} / \eta}}>c\right) \frac{\mathrm{d} t}{t}<\infty \quad \text { for all } c>0 \text { and for all } 0<\theta^{\prime}<\theta
$$

Since $\tilde{\nu}$ is symmetric, the constant $\kappa$ of Lemma 10 is zero and Lemma 10 (b) implies $\beta \leq \eta / \theta^{\prime}$ for all $0<\theta^{\prime}<\theta$, so that $\beta \leq \eta / \theta$.

Proof of Theorem 5(b). By Lemma 4, the function $g^{\alpha, \ell}$ satisfies (11) with $[a, b]=$ $\left[k 2^{-\ell},(k+1) 2^{-\ell}\right], r=2^{-\ell-3}, c_{2}=2^{-\ell} 2^{8 \alpha-10}$ and $\eta=2 \alpha$. If $g^{\alpha, \ell}\left(X_{1}\right) \in$ $\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$, then by Lemma 11 it holds that $\beta \leq 2 \alpha / \theta$.

Proof of Theorem 7(b). We have that

$$
\begin{align*}
& \int_{K-r}^{K+r}\left(\mathbb{1}_{[K, \infty)}(y+x)-\mathbb{1}_{[K, \infty)}(y)\right)^{2} \mathrm{~d} y \\
& =\int_{K-r}^{K+r}\left(\mathbb{1}_{[K-x, K)}(y) \mathbb{1}_{(0, \infty)}(x)+\mathbb{1}_{[K, K-x)}(y) \mathbb{1}_{(-\infty, 0)}(x)\right) \mathrm{d} y \\
& =|x| \tag{13}
\end{align*}
$$

for all $|x| \leq r$, so that $\mathbb{1}_{[K, \infty)}$ satisfies (11) with $[a, b]=[K-r, K+r]$. Choosing $R=r$ it now follows from Lemma 11 , that if $\mathbb{1}_{[K, \infty)}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$ then $\beta \leq 1 / \theta$.

Remark 3. (a) If $m_{\beta}<\infty$ and (A3) holds, then we get for $0<\alpha \leq \theta<1$ from Theorem 5 the "if and only if"-condition

$$
m_{2 \alpha / \theta}<\infty \Longleftrightarrow f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \forall f \in C_{b}^{\alpha}
$$

and if also (A1) holds, then Theorem 7 implies for $1 / 2 \leq \theta<1$ that

$$
m_{1 / \theta}<\infty \Longleftrightarrow f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \forall f \in N B V
$$

Note that $m_{\beta}<\infty$ is indeed possible: choose for example

$$
\nu(\mathrm{d} x)=\frac{b}{|x|^{1+\beta}\left(\log ^{2} x+1\right)} \mathrm{d} x \quad \text { for some } b>0
$$

for $\beta \in(0,2]$. Using Lemma 2 we see that this process satisfies (A1)-(A3).
(b) If $m_{\beta}=\infty$, then Theorems 5 and 7 do not give an "if and only if"-result in general: In Theorems 13-15 in Section 5.1 we consider the symmetric strictly stable process with

$$
\nu(\mathrm{d} x)=\frac{b}{|x|^{1+\beta}} \mathrm{d} x \quad \text { for some } b>0 \text { and } \beta \in(0,1)
$$

and the process satisfies (A1)-(A3) by Lemma 2. Theorems 13 and 14 show that when $0<\alpha<\theta<1$, then

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \forall f \in C_{b}^{\alpha} \quad \text { for } 2 \alpha / \theta=\beta
$$

and that for $\frac{1}{2} \leq \theta<1$ it holds that

$$
f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \forall f \in N B V \quad \text { for } 1 / \theta=\beta,
$$

eventhough $m_{\beta}=\infty$. However, we obtain for $0<\alpha<\theta<1$ from Theorem 13, that

$$
\begin{aligned}
m_{2 \alpha / \theta}<\infty & \Longleftrightarrow f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q} \forall f \in C_{b}^{\alpha} \text { for some } q \in[1, \infty) \\
& \Longleftrightarrow f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q} \forall f \in C_{b}^{\alpha} \text { for all } q \in[1, \infty) .
\end{aligned}
$$

Theorems 14 and 15 imply for $0<\theta<1$ that

$$
\begin{aligned}
m_{1 / \theta}<\infty & \Longleftrightarrow f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q} \forall f \in N B V \text { for some } q \in[1, \infty) \\
& \Longleftrightarrow f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q} \forall f \in N B V \text { for all } q \in[1, \infty)
\end{aligned}
$$

### 5.1. Symmetric strictly stable process

We consider the symmetric strictly stable process which has the characteristic function $\varphi(u)=e^{-c|u|^{\beta}}$ for some $c>0$ and $\beta \in(0,2]$ ([25, Theorem 14.14]). If $\beta=2$, then the process is the Brownian motion $\sqrt{2 c} B$, and otherwise it is a pure jump Lévy process $X$ with Lévy measure

$$
\nu(\mathrm{d} x)=b|x|^{-\beta-1} \mathrm{~d} x \quad \text { for some } b>0,
$$

where $\beta$ is the Blumenthal-Getoor index of the process. We will later take advantage of the property that $X_{t} \stackrel{d}{=} t^{1 / \beta} X_{1}$, which follows from

$$
\mathbb{E}\left[e^{i u X_{t}}\right]=e^{-t c|u|^{\beta}}=e^{-c\left|u t^{1 / \beta}\right|^{\beta}}=\mathbb{E}\left[e^{i u t^{1 / \beta} X_{1}}\right]
$$

Using Lemma 2 one can easily check that assumptions (A1), (A2) and (A3) are satisfied. For the rest of this section we assume that $X$ is the symmetric and strictly stable process of index $\beta \in(0,2)$.

Lemma 12. Let $a<b$ and $t_{0} \in(0,1)$. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable and there exist $r>0, c_{2}>0$ and $\eta>0$ such that (11) holds, then there exists $c>0$ such that

$$
\left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \geq c(1-t)^{\eta / \beta} \quad \text { for all } t \in\left[t_{0}, 1\right] .
$$

Proof. Let $R>0$. Since $X_{1}$ has the support $\mathbb{R}$ by [25, Theorem 24.10(ii)], then $p_{1}$ is strictly positive and continuous on $\mathbb{R}$ by the proof of Lemma 2. Hence we find $c_{1}>0$ such that $p_{1}(x) \geq c_{1}$ for all $-|a-R| t_{0}^{-1 / \beta} \leq x \leq|b+R| t_{0}^{-1 \beta}$. Using the fact that $X_{t} \stackrel{d}{=} t^{1 / \beta} X_{1}$, we obtain for any $x \in[a-R, b+R]$ that

$$
p_{t}(x)=t^{-1 / \beta} p_{1}\left(t^{-1 / \beta} x\right) \geq p_{1}\left(t^{-1 / \beta} x\right) \geq c_{1}
$$

for all $t \in\left[t_{0}, 1\right]$. Using (12) we get that

$$
\begin{aligned}
& \left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \\
& \geq c_{1} c_{2} \overline{\mathbb{E}} \mathbb{E}\left[\left|X_{1-t}-\bar{X}_{1-t}\right|^{\eta} \mathbb{1}_{\left\{\left|X_{1-t}-\bar{X}_{1-t}\right| \leq r,\left|\bar{X}_{1-t}\right| \leq R\right\}}\right] \\
& =c_{1} c_{2}(1-t)^{\eta / \beta} \overline{\mathbb{E}} \mathbb{E}\left[\left|X_{1}-\bar{X}_{1}\right|^{\eta} \mathbb{1}_{\left\{\left|X_{1}-\bar{X}_{1}\right| \leq r(1-t)^{-1 / \beta},\left|\bar{X}_{1}\right| \leq R(1-t)^{-1 / \beta}\right\}}\right] \\
& \geq c_{1} c_{2}(1-t)^{\eta / \beta} \overline{\mathbb{E}} \mathbb{E}\left[\left|X_{1}-\bar{X}_{1}\right|^{\eta} \mathbb{1}_{\left\{\left|X_{1}-\bar{X}_{1}\right| \leq r,\left|\bar{X}_{1}\right| \leq R\right\}}\right] \\
& \geq c(1-t)^{\eta / \beta}
\end{aligned}
$$

for some $c>0$, where we used the fact that since $X_{1}-\bar{X}_{1}$ is strictly stable with Lévy measure $2 \nu$, it must be that $\overline{\mathbb{E}} \mathbb{E}\left[\left|X_{1}-\bar{X}_{1}\right| \mathbb{1}_{\left\{\left|X_{1}-\bar{X}_{1}\right| \leq r,\left|\bar{X}_{1}\right| \leq R\right\}}\right]$ is strictly positive.

Theorem 13. Let $0<\alpha<\theta<1$ and assume that $f \in C_{b}^{\alpha}$.
(a) It holds that $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$, if $\beta \leq 2 \alpha / \theta$.
(b) Let $q \in[1, \infty)$ and $\ell \in\{0,1,2, \ldots\}$. For the function $g^{\alpha, \ell} \in C_{b}^{\alpha}$ from Lemma 4 we have that
(i)

$$
g^{\alpha, \ell}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q} \quad \text { if and only if } \quad \beta<2 \alpha / \theta
$$

(ii)

$$
g^{\alpha, \ell}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty} \quad \text { if and only if } \quad \beta \leq 2 \alpha / \theta
$$

Proof. (a) If $\beta \leq 2 \alpha$, then $m_{2 \alpha / \theta}<\infty$ and the claim follows from Theorem 5(a). Assume now that $\beta>2 \alpha$. We have

$$
\begin{aligned}
\left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} & \leq 2 \overline{\mathbb{E}} \mathbb{E}\left[\left|X_{1-t}-\bar{X}_{1-t}\right|^{2 \alpha}\|f\|_{C_{b}^{\alpha}}^{2}\right] \\
& \leq 2 \overline{\mathbb{E}} \mathbb{E}\left[\left|(1-t)^{1 / \beta}\left(X_{1}-\bar{X}_{1}\right)\right|^{2 \alpha}\|f\|_{C_{b}^{\alpha}}^{2}\right] \\
& \leq 2(1-t)^{2 \alpha / \beta}\|f\|_{C_{b}^{\alpha}}^{2} \overline{\mathbb{E}} \mathbb{E}\left[\left|X_{1}-\bar{X}_{1}\right|^{2 \alpha}\right]
\end{aligned}
$$

Since the process $X-\bar{X}$ on $\Omega \times \bar{\Omega}$ has the Lévy measure $2 \nu$ and $\beta>2 \alpha$, we get that

$$
\int_{\{|x|>1\}}|x|^{2 \alpha} 2 \nu(\mathrm{~d} x)=2 \int_{\{|x|>1\}}|x|^{2 \alpha-\beta-1} \mathrm{~d} x<\infty
$$

which implies $\overline{\mathbb{E} E}\left[\left|X_{1}-\bar{X}_{1}\right|^{2 \alpha}\right]<\infty$ by [25, Theorem 25.3]. Thus

$$
\left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \leq C(1-t)^{2 \alpha / \beta}
$$

for all $t \in(0,1)$ for some $C \in(0, \infty)$ and the claim (a) follows from Lemma 9. The "if"-parts of (b) follow from (a) and (3). By Lemma 4, the function $g^{\alpha, \ell}$ satisfies (11) with $[a, b]=\left[k 2^{-\ell},(k+1) 2^{-\ell}\right], r=2^{-\ell-3}, c_{2}=2^{-\ell} 2^{8 \alpha-10}$ and $\eta=2 \alpha$. Thus, Lemma 12 implies that

$$
\left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \geq c(1-t)^{2 \alpha / \beta}
$$

for some $c>0$, and with the use of Lemma 9 this proves the "only if"-parts of (b).

Theorem 14. Let $f \in N B V$.
(a) If $\beta<1$, then $f\left(X_{1}\right) \in \mathbb{D}_{1,2}$
(b) If $\beta=1$, then $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}$ for all $\theta \in(0,1)$ and $q \in[1, \infty]$.
(c) Let $\theta \in(0,1)$. If $\beta \leq 1 / \theta$, then $f\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$.

Proof. (a) The claim follows from Theorem 6(a).
(b) The claim follows from Theorem 7 and (3).
(c) If $\beta \leq 1$, then the claim follows from (b). Assume that $\beta>1$. We have

$$
\begin{aligned}
& \left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \\
& =\overline{\mathbb{E}} \mathbb{E}\left[\left(\int_{\mathbb{R}} \mathbb{1}_{[u, \infty)}\left(X_{1}\right)-\mathbb{1}_{[u, \infty)}\left(X_{t}+\bar{X}_{1-t}\right) \mu_{f}(\mathrm{~d} u)\right)^{2}\right] \\
& \leq\left|\mu_{f}\right|(\mathbb{R}) \int_{\mathbb{R}} \overline{\mathbb{E}} \mathbb{E}\left[\mathbb{1}_{\left[u-X_{1-t}, u-\bar{X}_{t-1}\right) \cup\left[u-\bar{X}_{1-t}, u-X_{t-1}\right)}\left(X_{t}\right)\right] \mu_{f}(\mathrm{~d} u) \\
& \leq\left|\mu_{f}\right|^{2}(\mathbb{R})\left\|p_{t}\right\|_{\infty} \overline{\mathbb{E}} \mathbb{E}\left[\left|\bar{X}_{1-t}-X_{1-t}\right|\right] \\
& =\left|\mu_{f}\right|^{2}(\mathbb{R}) t^{-1 / \beta}\left\|p_{1}\right\|_{\infty} \overline{\mathbb{E}} \mathbb{E}\left[(1-t)^{1 / \beta}\left|\bar{X}_{1}-X_{1}\right|\right] \\
& \leq(1-t)^{1 / \beta}\left|\mu_{f}\right|^{2}(\mathbb{R}) t^{-1 / \beta}\left\|p_{1}\right\|_{\infty} \overline{\mathbb{E}} \mathbb{E}\left[\left|\bar{X}_{1}-X_{1}\right|\right]
\end{aligned}
$$

Since the process $X-\bar{X}$ has the Lévy measure $2 \nu$ and

$$
\int_{\{|x|>1\}}|x| 2 \nu(\mathrm{~d} x)=2 \int_{\{|x|>1\}}|x|^{-\beta} \mathrm{d} x<\infty
$$

for $\beta>1$, we get $\overline{\mathbb{E}} \mathbb{E}\left[\left|\bar{X}_{1}-\hat{X}_{1}\right|\right]<\infty$ from [25, Theorem 25.3]. Thus

$$
\left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \leq C(1-t)^{1 / \beta}
$$

for all $t \in(1 / 2,1)$ for some $C \in(0, \infty)$. When $t \in(0,1 / 2]$, then

$$
\left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \leq\|f\|_{B V}^{2} \leq\|f\|_{B V}^{2} 2^{1 / \beta}(1-t)^{1 / \beta}
$$

and the claim follows from Lemma 9.
Theorem 15. Let $K \in \mathbb{R}$.
(a) It holds that $\mathbb{1}_{[K, \infty)}\left(X_{1}\right) \in \mathbb{D}_{1,2}$ if and only if $\beta<1$.
(b) It holds that $\mathbb{1}_{[K, \infty)}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}$ for all $\theta \in(0,1)$ and $q \in[1, \infty]$ if and only if $\beta \leq 1$.
(c) Let $\theta \in(0,1)$ and $q \in[1, \infty)$. Then
(i) $\mathbb{1}_{[K, \infty)}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}$ if and only if $\beta<1 / \theta$ and
(ii) $\mathbb{1}_{[K, \infty)}\left(X_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$ if and only if $\beta \leq 1 / \theta$.
(d) Let $\theta \in(0,1)$ and $q \in[1, \infty)$. For the Brownian motion $B$ we have that
(i) $\mathbb{1}_{[K, \infty)}\left(B_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, q}$ if and only if $2<1 / \theta$ and
(ii) $\mathbb{1}_{[K, \infty)}\left(B_{1}\right) \in\left(L_{2}(\mathbb{P}), \mathbb{D}_{1,2}\right)_{\theta, \infty}$ if and only if $2 \leq 1 / \theta$.

Proof. (a) The claim follows from Theorem 6(a) and the proof of Theorem 6(b), since by [25, Theorem 24.10 (ii)] the continuous density of $X_{1}$ is strictly positive on the whole real line.
(b) The "if" follows from Theorem 14 and the "only if" follows from (c), since $\beta<1 / \theta$ for all $\theta \in(0,1)$.
(c) The "if"-parts of (i) and (ii) follow from Theorem 14(c) and (3).

Fix $r>0$ and $t_{0} \in(0,1)$. By (13), the function $\mathbb{1}_{[K, \infty)}$ satisties (11) with $[a, b]=[K-r, K+r], c_{2}=1$ and $\eta=1$. Thus, Lemma 12 implies that

$$
\left\|\left\|f\left(X_{1}\right)-f\left(X_{t}+\bar{X}_{1-t}\right)\right\|_{L_{2}(\mathbb{P})}\right\|_{L_{2}(\overline{\mathbb{P}})}^{2} \geq c(1-t)^{1 / \beta}
$$

for some $c>0$. The "only if"-parts of (c) follow now from Lemma 9.
(d) We choose $E$ and $a$ like in the proof of Lemma 9 on the corresponding Wiener chaos. The claim follows from Lemma 8 and the proof in [16, Example 4.7], where it is shown that $(T a)^{\prime}(t) \sim(1-t)^{-1 / 2}$.

## 6. Acknowledgments

The author would like to thank Christel Geiss and Stefan Geiss for fruitful discussions and valuable suggestions, and the unknown referees for their remarks and questions, that considerably help improving the article.

## Appendix A.

The reiteration theorem states that $\left(A_{0}, A_{1}\right)_{\eta \theta, q}=\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, q}$ for all $\eta, \theta \in(0,1)$ and $q \in[1, \infty]$ with equivalent norms. In the following lemma we compute the explicit constants for the equivalence of the norms for $q=\infty$.

Lemma 16. Let $\left(A_{0}, A_{1}\right)$ be a compatible couple and $\eta, \theta \in(0,1)$. Then

$$
\|f\|_{\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}} \leq\|f\|_{\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, \infty}} \leq 3\|f\|_{\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}}
$$

for all $f \in\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}=\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, \infty}$.
Proof. First inequality: Let $t>0$ and $\varepsilon>0$. There exist $f_{0}, g_{0} \in A_{0}, g \in$ $\left(A_{0}, A_{1}\right)_{\eta, \infty}$ and $g_{1} \in A_{1}$ such that $f=f_{0}+g=f_{0}+g_{0}+g_{1}$ and

$$
\begin{aligned}
K\left(f, t^{\eta} ; A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right) & \geq\left\|f_{0}\right\|_{A_{0}}+t^{\eta}\|g\|_{\left(A_{0}, A_{1}\right)_{\eta, \infty}}-\frac{\varepsilon}{2} \\
& \geq\left\|f_{0}\right\|_{A_{0}}+t^{\eta} t^{-\eta}\left(\left\|g_{0}\right\|_{A_{0}}+t\left\|g_{1}\right\|_{A_{1}}-\frac{\varepsilon}{2}\right)-\frac{\varepsilon}{2} \\
& \geq\left\|f_{0}+g_{0}\right\|_{A_{0}}+t\left\|g_{1}\right\|_{A_{1}}-\varepsilon \\
& \geq K\left(f, t ; A_{0}, A_{1}\right)-\varepsilon .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|f\|_{\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, \infty}} & =\sup _{t>0}\left(t^{\eta}\right)^{-\theta} K\left(f, t^{\eta} ; A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right) \\
& \geq \sup _{t>0} t^{-\eta \theta} K\left(f, t ; A_{0}, A_{1}\right) \\
& =\|f\|_{\left(A_{0}, A_{1}\right)_{\eta \theta, \infty} .} .
\end{aligned}
$$

Second inequality: Let $f \in\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}$ and $\varepsilon>0$. For all $t>0$ we find $g_{t} \in A_{0}$ and $h_{t} \in A_{1}$ such that $f=g_{t}+h_{t}$ and

$$
\left\|g_{t}\right\|_{A_{0}}+t\left\|h_{t}\right\|_{A_{1}} \leq K\left(f, t ; A_{0}, A_{1}\right)+\frac{\varepsilon}{2} t^{\eta \theta} .
$$

Then

$$
\begin{aligned}
& K\left(g_{t}, s ; A_{0}, A_{1}\right) \leq\left\|g_{t}\right\|_{A_{0}} \leq K\left(f, t ; A_{0}, A_{1}\right)+\frac{\varepsilon}{2} t^{\eta \theta} \text { and } \\
& K\left(h_{t}, s ; A_{0}, A_{1}\right) \leq s\left\|h_{t}\right\|_{A_{1}} \leq \frac{s}{t}\left[K\left(f, t ; A_{0}, A_{1}\right)+\frac{\varepsilon}{2} t^{\eta \theta}\right]
\end{aligned}
$$

for all $s \in(0, \infty)$. These inequalities give, keeping in mind that $h_{t}=f-g_{t}$,
that

$$
\begin{aligned}
t^{\eta}\left\|h_{t}\right\|_{\left(A_{0}, A_{1}\right)_{\eta, \infty} \leq}= & t^{\eta} \sup _{s>0} s^{-\eta} K\left(h_{t}, s ; A_{0}, A_{1}\right) \\
\leq & \left(\sup _{0<s \leq t}\left(\frac{s}{t}\right)^{-\eta} \frac{s}{t}\left[K\left(f, t ; A_{0}, A_{1}\right)+\frac{\varepsilon}{2} t^{\eta \theta}\right]\right) \vee \\
& \left(\sup _{s \geq t}\left(\frac{s}{t}\right)^{-\eta}\left[K\left(f, s ; A_{0}, A_{1}\right)+K\left(g_{t}, s ; A_{0}, A_{1}\right)\right]\right) \\
\leq & \left(K\left(f, t ; A_{0}, A_{1}\right)+\frac{\varepsilon}{2} t^{\eta \theta}\right) \vee \\
& \left(\sup _{s \geq t}\left(\frac{s}{t}\right)^{-\eta}\left[K\left(f, s ; A_{0}, A_{1}\right)+K\left(f, t ; A_{0}, A_{1}\right)+\frac{\varepsilon}{2} t^{\eta \theta}\right]\right) \\
\leq & K\left(f, t ; A_{0}, A_{1}\right)+\frac{\varepsilon}{2} t^{\eta \theta}+\sup _{s \geq t}\left(\frac{s}{t}\right)^{-\eta \theta} K\left(f, s ; A_{0}, A_{1}\right) .
\end{aligned}
$$

We obtain

$$
\begin{aligned}
& \|f\|_{\left(A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)_{\theta, \infty}} \\
& =\sup _{t>0}\left(t^{\eta}\right)^{-\theta} K\left(f, t^{\eta} ; A_{0},\left(A_{0}, A_{1}\right)_{\eta, \infty}\right) \\
& \leq \sup _{t>0} t^{-\eta \theta}\left(\left\|g_{t}\right\|_{A_{0}}+t^{\eta}\left\|h_{t}\right\|_{\left.\left(A_{0}, A_{1}\right)_{\eta, \infty}\right)}\right) \\
& \leq \sup _{t>0} t^{-\eta \theta}\left(2 K\left(f, t ; A_{0}, A_{1}\right)+\varepsilon t^{\eta \theta}+\sup _{s \geq t}\left(\frac{s}{t}\right)^{-\eta \theta} K\left(f, s ; A_{0}, A_{1}\right)\right) \\
& \leq 3 \sup _{s>0} s^{-\eta \theta} K\left(f, s ; A_{0}, A_{1}\right)+\varepsilon \\
& =3\|f\|_{\left(A_{0}, A_{1}\right)_{\eta \theta, \infty}}+\varepsilon
\end{aligned}
$$

Lemma 17. Let $\alpha \in(0,1)$. Then $C_{b}^{\alpha}=(B(\mathbb{R}), \text { Lip })_{\alpha, \infty}$ with

$$
\|\cdot\|_{C_{b}^{\alpha}} \leq 3\|\cdot\|_{(B(\mathbb{R}), L i p)_{\alpha, \infty}} \leq 6\|\cdot\|_{C_{b}^{\alpha}} .
$$

Proof. First inequality: Let $f \in(B(\mathbb{R}) \text {, Lip })_{\alpha, \infty}$ and $\varepsilon>0$. For all $t>0$ we find $f_{t} \in$ Lip such that

$$
t^{-\alpha}\left(\left\|f-f_{t}\right\|_{\infty}+t\left\|f_{t}\right\|_{L i p}\right) \leq\|f\|_{(B(\mathbb{R}), L i p)_{\alpha, \infty}}+\varepsilon
$$

Let $x \neq y \in \mathbb{R}$ and $t=|x-y|>0$. By the triangle inequality we have

$$
\begin{aligned}
\frac{|f(x)-f(y)|}{|x-y|^{\alpha}} & \leq|x-y|^{-\alpha}\left(\left|f(x)-f_{t}(x)\right|+\left|f(y)-f_{t}(y)\right|+\left|f_{t}(x)-f_{t}(y)\right|\right) \\
& \leq t^{-\alpha}\left(2\left\|f-f_{t}\right\|_{\infty}+t\left\|f_{t}\right\|_{\text {Lip }}\right) \\
& \leq 2\left(\|f\|_{(B(\mathbb{R}), \text { Lip })_{\alpha, \infty}}+\varepsilon\right)
\end{aligned}
$$

It also holds that

$$
\|f\|_{\infty} \leq\left\|f-f_{1}\right\|_{\infty}+\left\|f_{1}\right\|_{\infty} \leq\|f\|_{(B(\mathbb{R}), L i p)_{\alpha, \infty}}+\varepsilon
$$

so that

$$
\|f\|_{C_{b}^{\alpha}}=\|f\|_{\infty}+\sup _{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \leq 3\|f\|_{(B(\mathbb{R}), \text { Lip })_{\alpha, \infty}}
$$

Second inequality: Let $f \in C_{b}^{\alpha}$ and $t>0$ and define $f_{t}$ so that $f_{t}(k t)=f(k t)$ for $k \in \mathbb{Z}$ and $f_{t}$ is linear on each interval $[k t,(k+1) t], k \in \mathbb{Z}$. Then for $x \in[k t,(k+1) t]$ there is $s \in[0,1]$ such that $f_{t}(x)=s f(k t)+(1-s) f((k+1) t)$ and we get that

$$
\begin{aligned}
\left\|f-f_{t}\right\|_{\infty} & =\sup _{k \in \mathbb{Z}} \sup _{x \in[k t,(k+1) t]}\left|f(x)-f_{t}(x)\right| \\
& \leq \sup _{k \in \mathbb{Z}} \sup _{x \in[k t,(k+1) t]}(s|f(x)-f(k t)|+(1-s)|f(x)-f((k+1) t)|) \\
& \leq \sup _{|x-y| \leq t}|f(x)-f(y)| \\
& \leq t^{\alpha}\|f\|_{C_{b}^{\alpha}} .
\end{aligned}
$$

For the function $f_{t}$ it holds for $0<t \leq 1$ that

$$
\begin{aligned}
\left\|f_{t}\right\|_{L i p} & =\left\|f_{t}\right\|_{\infty}+\sup _{x \neq y} \frac{\left|f_{t}(x)-f_{t}(y)\right|}{|x-y|} \\
& \leq\|f\|_{\infty}+\sup _{k \in \mathbb{Z}} \frac{|f(k t)-f((k+1) t)|}{t} \\
& \leq\|f\|_{\infty}+t^{\alpha-1} \sup _{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \\
& \leq t^{\alpha-1}\|f\|_{C_{b}^{\alpha}} .
\end{aligned}
$$

Hence we obtain that

$$
\begin{aligned}
\|f\|_{(B(\mathbb{R}), \text { Lip })_{\alpha, \infty}} & \leq\left[\sup _{0<t \leq 1} t^{-\alpha}\left(\left\|f-f_{t}\right\|_{\infty}+t\left\|f_{t}\right\|_{\text {Lip }}\right)\right] \vee \sup _{t \geq 1} t^{-\alpha}\|f\|_{\infty} \\
& \leq 2\|f\|_{C_{b}^{\alpha}}
\end{aligned}
$$

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