Existence, Uniqueness and Malliavin Differentiability of Lévy-driven BSDEs with locally Lipschitz Driver

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Abstract

We investigate conditions for solvability and Malliavin differentiability of backward stochastic differential equations driven by a Lévy process. In particular, we are interested in generators which satisfy a local Lipschitz condition in the Z and U variable. This includes settings of linear, quadratic and exponential growths in those variables.

Extending an idea of Cheridito and Nam to the jump setting and applying comparison theorems for Lévy-driven BSDEs, we show existence, uniqueness, boundedness and Malliavin differentiability of a solution. The pivotal assumption to obtain these results is a boundedness condition on the terminal value ξ and its Malliavin derivative $D\xi$.

Furthermore, we extend existence and uniqueness theorems to cases where the generator is not even locally Lipschitz in U. BSDEs of the latter type find use in exponential utility maximization.

Keywords: BSDEs with jumps; locally Lipschitz generator; quadratic BSDEs; existence and uniqueness of solutions to BSDEs; Malliavin differentiability of BSDEs
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1 Introduction

In this paper, we consider existence, uniqueness and Malliavin differentiability of one-dimensional backward stochastic differential equations (BSDEs) of the type

$$Y_t = \xi + \int_t^T \mathbf{f}(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_{]t,T]\times\mathbb{R}\setminus\{0\}} U_s(x)\tilde{N}(ds, dx). \tag{1}$$

Here W is the Brownian motion and $\tilde{N}$ the Poisson random measure associated to a Lévy process X with Lévy measure ν. In order to compute the Malliavin derivative of $\mathbf{f}$, we require a special structure: We assume that $\mathbf{f}$ can be represented by functions f and g, such that

$$\mathbf{f}(\omega, s, y, z, \mathbf{u}) = f\left((X_r(\omega))_{r\leq s}, s, y, z, \int_{\mathbb{R}\setminus\{0\}} g(s, \mathbf{u}(x))\ (1\wedge|x|)\,\nu(dx)\right), \tag{2}$$

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where $g(s,\cdot)$ is a locally Lipschitz continuous function on $\mathbb{R}$ with $g(s,0)=0$. The function f satisfies the following, in (z,u) only *local* Lipschitz condition:
There are nonnegative functions $a \in L_1([0,T]), b \in L_2([0,T])$ and a nondecreasing, continuous function $\rho : [0,\infty) \to [0,\infty)$ such that $\forall t \in [0,T]$, $\forall \mathtt{x} \in D[0,t]$ (the Skorohod space of càdlàg functions on $[0,T]$), $(y,z,u), (\tilde{y},\tilde{z},\tilde{u}) \in \mathbb{R}^3$:

$$|f(\mathtt{x},t,y,z,u) - f(\mathtt{x},t,\tilde{y},\tilde{z},\tilde{u})| \le a(t)|y-\tilde{y}| + \rho(|z| \vee |\tilde{z}| \vee |u| \vee |\tilde{u}|)\, b(t)(|z-\tilde{z}| + |u-\tilde{u}|).$$

Our first main result is Theorem 3.4 about existence of solutions (Y,Z,U) to the BSDE (1): If the terminal condition ξ and its Malliavin derivative $D\xi$ are bounded, and the Malliavin derivative of the generator is bounded by a certain function depending on time and jump size, then there exists a solution (Y,Z,U) which is Malliavin differentiable, and the paths of Y, Z and U are bounded by a constant a.s. Moreover, within a certain class of bounded processes, this solution is unique.

Following Cheridito and Nam [11], where a similar result is shown for BSDEs driven by a Brownian motion, the proof uses a comparison theorem. For BSDEs with jumps, comparison theorems need an additional assumption on the generator (see (A γ) in Theorem 2.4). The comparison theorem provides not only a bound for Y, but also bounds for Z and U: Indeed, since Z and U can be seen as versions of Malliavin derivatives of Y w.r.t. the Brownian component and the jump component, respectively, one can derive bounds by applying the comparison theorem to the Malliavin derivative of the BSDE.

For BSDEs with quadratic or sub-quadratic growth in z, Briand and Hu [10], Bahlali [5], S. Geiss and Ylinen [19] (all in case of BSDEs driven by a Brownian motion) and Antonelli and Mancini [3] (for BSDEs with jumps and finite Lévy measure), investigate the requirements on the terminal condition such that existence and uniqueness of solutions holds. It is well-known that – in the case of quadratic growth in z, – square integrability of ξ is not sufficient but the assumption that ξ is bounded can be relaxed. However, for super-quadratic drivers and a.s. bounded terminal conditions ξ, Delbaen et al. [12] have shown that there are cases of BSDEs without any solution as well as BSDEs with infinitely many solutions.

For quadratic BSDEs with jumps and infinite Lévy measure there seem to be only results for bounded ξ so far (see Morlais [23] and Becherer et al. [8]), and also for the method we apply here, boundedness is needed.

Our second main result is Theorem 4.2, which states existence and uniqueness for a class of BSDEs where the generator is not even locally Lipschitz w.r.t. $\mathbf{u} \in L_2(\nu)$. As an example, consider

$$\mathbf{f}(s,y,z,\mathbf{u}) = \bar{\mathbf{f}}(s,y,z) + \int_{\mathbb{R}\setminus\{0\}} \mathcal{H}_\alpha(\mathbf{u}(x))\nu(dx), \tag{3}$$

where

$$\mathcal{H}_\alpha(u) := \frac{e^{\alpha u} - \alpha u - 1}{\alpha},$$

for a real $\alpha > 0$ and $\bar{\mathbf{f}}$ being quadratic in z. This particular form of $\mathbf{f}$ arises from exponential utility maximisation, see Morlais [23] or Becherer et al. [8]. Notice that compared to the generator given in (2), the integral in (3) does not contain the factor $1 \wedge |x|$. In Section 4 we address the question to what extent the structure of the generator given in (3) can be generalised. We were not able to show that the factor $1 \wedge |x|$ in (2) can be simply dropped under the given assumptions, but one can generalise (3) to the case where

$$\mathbf{f}(t,y,z,\mathbf{u}) := \varphi\left(\bar{f}\left(t,y,z,G(t,\mathbf{u})\right), \int_{\mathbb{R}\setminus\{0\}} \mathcal{H}(\mathbf{u}(x))\nu(dx)\right)$$

with $G(t,\mathbf{u}) := \int_{\mathbb{R}\setminus\{0\}} g(s,\mathbf{u}(x))\ (1\wedge|x|)\,\nu(dx)$ and $\bar{f}$ satisfying the assumption for (2). The function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $|\partial_v\varphi(v,w)| \le 1$ and $v \mapsto \partial_w\varphi(v,w)$ is a bounded function for any fixed $w \in \mathbb{R}$. Moreover, we require φ to satisfy a condition such that the comparison theorem holds (see ($\mathcal{H}3$)). The function $\mathcal{H}$ is a generalisation of $\mathcal{H}_\alpha$. It turns out that the bounds for (Y,Z,U) do not depend on $\mathcal{H}$.

The paper is structured as follows: In Section 2 we introduce the notation and shortly recall the Skorohod space, Malliavin calculus for Lévy processes and results on existence and uniqueness of solutions to BSDEs as well as a comparison theorem for later use. Sections 3 and 4 contain the main results, Theorems 3.4 and 4.2, and their proofs. Section 5 draws a connection between BSDEs with jumps and partial differential-integral equations (PDIEs). In Appendix A we formulate a result of Malliavin differentiability for Lipschitz BSDEs which slightly generalises [17, Theorem A.1]. It is applied in the proof of Theorem 3.4.

2 Setting and preliminaries

2.1 Lévy process and independent random measure

Let $X = (X_t)_{t\in[0,T]}$ be a càdlàg Lévy process with Lévy measure ν on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will denote the augmented natural filtration of X by $(\mathcal{F}_t)_{t\in[0,T]}$ and assume that $\mathcal{F} = \mathcal{F}_T$.
The Lévy-Itô decomposition of a Lévy process X can be written as

$$X_t = \gamma t + \sigma W_t + \int_{]0,t]\times\{|x|\le 1\}} x\tilde{N}(ds,dx) + \int_{]0,t]\times\{|x|>1\}} xN(ds,dx),$$

where $\gamma \in \mathbb{R}, \sigma \ge 0$, W is a Brownian motion and N ($\tilde{N}$) is the (compensated) Poisson random measure corresponding to X. The process

$$\left(\int_{]0,t]\times\{|x|\le 1\}} x\tilde{N}(ds,dx) + \int_{]0,t]\times\{|x|>1\}} xN(ds,dx)\right)_{t\in[0,T]}$$

is the jump part of X and will be denoted by J. Note that the $\mathbb{P}$-augmented filtrations $(\mathcal{F}_t^W)_{t\in[0,T]}$ and $(\mathcal{F}_t^J)_{t\in[0,T]}$ generated by the processes W and J, respectively, satisfy

$$\mathcal{F}_t^W \vee \mathcal{F}_t^J = \mathcal{F}_t,$$

(see [27, Lemma 3.1]) thus spanning the original filtration generated by X again. Throughout the paper we will use the notation $X(\omega) = (X_t(\omega))_{t\in[0,T]}$ for sample paths.
Let

$$\mu(dx) := \sigma^2\delta_0(dx) + \nu(dx)$$

and

$$\mathbb{m}(dt,dx) := (\lambda\otimes\mu)(dt,dx),$$

where λ denotes the Lebesgue measure. We define the independent random measure (in the sense of [20, p. 256]) M by

$$M(dt,dx) := \sigma dW_t\delta_0(dx) + \tilde{N}(dt,dx) \tag{4}$$

on sets $B\in\mathcal{B}([0,T]\times\mathbb{R})$ with $\mathbb{m}(B)<\infty$. It holds $\mathbb{E}M(B)^2 = \mathbb{m}(B)$. In [27], Solé et al. consider the independent random measure $\sigma dW_t\delta_0(dx) + x\tilde{N}(dt,dx)$. Here, in order to match the notation used for BSDEs, we work with the *equivalent* approach where the Poisson random measure is not multiplied with x.

We close this section with notation for càdlàg processes on the path space, and for BSDEs.

2.2 Notation: Skorohod space

- With $D[0,T]$ we denote the Skorohod space of càdlàg functions on the interval $[0,T]$ equipped with the Skorohod topology. The σ-algebra $\mathcal{B}(D[0,T])$ is the Borel σ-algebra i.e. it is generated by the open sets of $D[0,T]$. It coincides with the σ-algebra generated by the family of coordinate projections $(p_t\colon D[0,T]\to\mathbb{R},\ \mathtt{x}\mapsto\mathtt{x}(t),\ t\in[0,T])$ (see [9, Theorem 12.5] for instance).
- For a fixed $t\in[0,T]$ the notation

  $$\mathtt{x}^t(s) := \mathtt{x}(t\wedge s),\ \text{for all } s\in[0,T]$$

  induces the natural identification

  $$D[0,t] = \left\{\mathtt{x}\in D[0,T] : \mathtt{x}^t = \mathtt{x}\right\}.$$

  By this identification we define a filtration on this space by

  $$\mathcal{G}_t = \sigma\left(\mathcal{B}\left(D[0,t]\right)\cup\mathcal{N}_X[0,T]\right),\quad 0\le t\le T, \tag{5}$$

  where $\mathcal{N}_X[0,T]$ denotes the null sets of $\mathcal{B}\left(D[0,T]\right)$ with respect to the image measure $\mathbb{P}_X$ on $\left(D[0,T],\mathcal{B}\left(D[0,T]\right)\right)$ of the Lévy process $X\colon\Omega\to D[0,T],\omega\mapsto \mathrm{X}(\omega)$. For more details on $D[0,T]$, see [9] and [15, Section 4].

2.3 Notation for BSDEs

- Notice that $|\cdot|$ may denote the absolute value of a real number or a norm in $\mathbb{R}^n$.
- $L_p := L_p(\Omega,\mathcal{F},\mathbb{P}),\quad p\ge 0.$
- $L_p([0,T]) := L_p([0,T],\mathcal{B}([0,T]),\lambda),\quad p\ge 0.$
- $L_2(\nu) := L_2(\mathbb{R}_0,\mathcal{B}(\mathbb{R}_0),\nu)$ with $\|\mathbf{u}\| := \|\mathbf{u}\|_{L_2(\nu)}$ and $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$.

- For $1 \le p \le \infty$ let $\mathcal{S}_p$ denote the space of all $(\mathcal{F}_t)$-progressively measurable and càdlàg processes $Y\colon \Omega \times [0,T] \to \mathbb{R}$ such that

$$\|Y\|_{\mathcal{S}_p} := \| \sup_{0\le t\le T} |Y_t| \, \|_{L_p} < \infty.$$

- We define $L_2(W)$ as the space of all $(\mathcal{F}_t)$-progressively measurable processes $Z\colon \Omega\times[0,T] \to \mathbb{R}$ such that

$$\|Z\|^2_{L_2(W)} := \mathbb{E}\int_0^T |Z_s|^2\, ds < \infty,$$

and $L_\infty(W)$ denotes the space of all $(\mathcal{F}_t)$-progressively measurable processes $Z\colon \Omega\times[0,T] \to \mathbb{R}$ such that

$$\|Z\|_{L_\infty(\mathbb{P}\otimes\lambda)} < \infty.$$

- We define $L_2(\tilde{N})$ as the space of all random fields $U\colon \Omega\times[0,T]\times\mathbb{R}_0 \to \mathbb{R}$ which are measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0)$ (where $\mathcal{P}$ denotes the predictable σ-algebra on $\Omega\times[0,T]$ generated by the left-continuous $(\mathcal{F}_t)$-adapted processes) such that

$$\|U\|^2_{L_2(\tilde{N})} := \mathbb{E}\int_{[0,T]\times\mathbb{R}_0} |U_s(x)|^2\, ds\nu(dx) < \infty,$$

- $L_{2\times\infty}(\tilde{N})$ denotes the space of all random fields $U\colon \Omega\times[0,T]\times\mathbb{R}_0 \to \mathbb{R}$ which are measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_0)$ such that

$$\left\| \int_{\mathbb{R}_0} |U_\cdot(x)|^2\, \nu(dx) \right\|_{L_\infty(\mathbb{P}\otimes\lambda)} < \infty.$$

- $L_{2,b}(\tilde{N}) := \{U \in L_2(\tilde{N}) : \exists A \in L_2(\nu)\cap L_\infty(\nu) \text{ such that } |U_s(x,\omega)| \le A(x)\}$.

- We recall the notion of the predictable projection of a stochastic process depending on parameters. According to [29, Proposition 3] (see also [22, Proposition 3] or [2, Lemma 2.2]) for any non-negative or bounded $z \in L_2(\mathbb{P}\otimes\mathbb{m}) := L_2(\Omega\times[0,T]\times\mathbb{R}, \mathcal{F}_T\otimes\mathcal{B}([0,T]\times\mathbb{R}), \mathbb{P}\otimes\mathbb{m})$ there exists a process

$${}^p z \in \mathrm{L}_2\left(\Omega\times[0,T]\times\mathbb{R}, \mathcal{P}\otimes\mathcal{B}(\mathbb{R}), \mathbb{P}\otimes\mathbb{m}\right)$$

such that for any fixed $x\in\mathbb{R}$ the function $({}^p z)_{\cdot,x}$ is a version of the predictable projection (in the classical sense , see e.g. [2, Definition 2.1]) of $z_{\cdot,x}$. In the following we will always use this result to get predictable projections which are measurable w.r.t. a parameter. Again, we call ${}^p z$ the predictable projection of z.

2.4 Malliavin derivatives

We sketch the definition of the Malliavin derivative using chaos expansions. For details we refer to [26]. According to [20] there exists for any $\xi \in L_2(\Omega,\mathcal{F},\mathbb{P})$ a unique chaos expansion

$$\xi = \sum_{n=0}^{\infty} I_n(\tilde{f}_n),$$

where $f_n \in L_2^n := L_2(([0,T] \times \mathbb{R})^n, \mathfrak{m}^{\otimes n})$, and $\tilde{f}_n((t_1,x_1),...,(t_n,x_n))$ is the symmetrisation of $f_n((t_1,x_1),...,(t_n,x_n))$ w.r.t. the n pairs of variables. The multiple integrals I_n are build with the random measure M from (4). Let $\mathbb{D}_{1,2}$ be the space of all random variables $\xi \in L_2$ such that

$$\|\xi\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1)! \left\| \tilde{f}_n \right\|_{L_2^n}^2 < \infty.$$

For $\xi \in \mathbb{D}_{1,2}$, the Malliavin derivative is defined by

$$D_{t,x}\xi := \sum_{n=1}^{\infty} n I_{n-1}\left(\tilde{f}_n\left((t,x),\ \cdot\ \right)\right),$$

for $\mathbb{P}\otimes\mathfrak{m}$-a.a. $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}$. It holds $D\xi \in L_2(\mathbb{P}\otimes\mathfrak{m})$. We will also use

$$\mathbb{D}_{1,2}^0 := \left\{ \xi = \sum_{n=0}^{\infty} I_n(\tilde{f}_n) \in L_2 \colon f_n \in L_2^n, n \in \mathbb{N}, \sum_{n=1}^{\infty} (n+1)! \int_0^T \|\tilde{f}_n((t,0),\cdot)\|_{L_2^{n-1}}^2 dt < \infty \right\}$$

and

$$\begin{aligned} \mathbb{D}_{1,2}^{\mathbb{R}_0} \quad := \quad & \left\{ \xi = \sum_{n=0}^{\infty} I_n(\tilde{f}_n) \in L_2 \colon f_n \in L_2^n, n \in \mathbb{N}, \right. \\ & \left. \sum_{n=1}^{\infty} (n+1)! \int_{[0,T]\times\mathbb{R}_0} \|\tilde{f}_n((t,x),\cdot)\|_{\mathrm{L}_2^{n-1}}^2 \mathfrak{m}(dt,dx) < \infty \right\}. \end{aligned}$$

The Malliavin derivative $D_{t,x}$ for $x \neq 0$ can be easily characterised without chaos expansions: Here we use that for any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ there exists a measurable function $g_\xi \colon D[0,T] \to \mathbb{R}$ such that

$$\xi(\omega) = g_\xi\left((X_t(\omega))_{0\le t\le T}\right) = g_\xi(X(\omega))$$

for a.a. $\omega \in \Omega$ (see, for instance, [7, Section II.11]).

Lemma 2.1 ([28], [17, Lemma 3.2]). *If $g_\xi(X) \in L_2$ then*

$$g_\xi(X) \in \mathbb{D}_{1,2}^{\mathbb{R}_0} \iff g_\xi(X + x\mathbb{1}_{[t,T]}) - g_\xi(X) \in L_2(\mathbb{P}\otimes\mathfrak{m}),$$

and it holds then for $x \neq 0$ $\mathbb{P}\otimes\mathfrak{m}$-a.e.

$$D_{t,x}\xi = g_\xi(X + x\mathbb{1}_{[t,T]}) - g_\xi(X). \tag{6}$$

For the canonical Lévy space, this result can be found in [1]. Notice that [1] uses the random measure $\sigma dW_t\delta_0(dx) + x\tilde{N}(dt,dx)$, so that the according Malliavin derivative for $x \neq 0$ and M from (4) is a *difference quotient* while we have just a difference. However, both approaches are equivalent.

Assume for example, that the generator $f\left((X_r(\omega))_{r\le s}, s, y, z, u\right)$ is a.s. Lipschitz in (y,z,u). Then also the Malliavin derivative $D_{t,x}f\left((X_r(\omega))_{r\le s}, s, y, z, u\right)$ for $x \neq 0$ has this property for $\mathbb{P}\otimes\mathfrak{m}$-a.a. $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}_0$. This is an immediate consequence of the next Lemma.

Lemma 2.2 ([17, Lemma 3.3]). *Let $\Lambda \in \mathcal{G}_T$ be a set with $\mathbb{P}(\{X \in \Lambda\}) = 0$. Then*

$$\mathbb{P}\otimes\mathfrak{m}\left(\left\{(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}_0 : X(\omega) + x\mathbb{1}_{[t,T]} \in \Lambda\right\}\right) = 0.$$

2.5 Existence and comparison results for monotonic generators

We consider the BSDE

$$Y_t = \xi + \int_t^T \mathbf{f}(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_{]t,T]\times\mathbb{R}_0} U_s(x)\tilde{N}(ds, dx), \ 0 \le t \le T, \qquad (7)$$

where

$$\mathbf{f} : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times L_2(\nu) \to \mathbb{R}.$$

If a triple $(Y, Z, U) \in \mathcal{S}_2 \times L_2(W) \times L_2(\tilde{N})$ satisfies (7) it is called a solution to the BSDE (7). We will recall first the existence and uniqueness result [18, Theorem 3.1].

Theorem 2.3. *There exists a unique solution to the BSDE* $(\xi, \mathbf{f})$ *with* $\xi \in L_2$ *and generator* $\mathbf{f}$: $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times L_2(\nu) \to \mathbb{R}$ *satisfying the properties*

(H 1) *For all* $(y, z, \mathbf{u}) : (\omega, s) \mapsto \mathbf{f}(\omega, s, y, z, \mathbf{u})$ *is progressively measurable.*

(H 2) *There are nonnegative, progressively measurable processes* K_1, K_2 *and* F *with*

$$\left\| \int_0^T \left(K_1(\cdot, s) + K_2(\cdot, s)^2\right) ds \right\|_\infty < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T |F(t)|dt\right]^2 < \infty$$

such that for all $(y, z, \mathbf{u})$,

$$|\mathbf{f}(s, y, z, \mathbf{u})| \le F(s) + K_1(s)|y| + K_2(s)(|z| + \|\mathbf{u}\|), \quad \mathbb{P}\otimes\lambda\text{-a.e.}$$

(H 3) *For* λ*-almost all* s*, the mapping* $(y, z, \mathbf{u}) \mapsto \mathbf{f}(s, y, z, \mathbf{u})$ *is* $\mathbb{P}$*-a.s. continuous. Moreover, there is a nonnegative function* $\alpha \in L^1([0, T])$*,* $c > 0$ *and a progressively measurable process* β *with* $\int_0^T \beta(\omega, s)^2 ds < c$*,* $\mathbb{P}$*-a.s. such that for all* $(y, z, \mathbf{u}), (y', z', \mathbf{u}')$,

$$\begin{aligned}&(y - y')(\mathbf{f}(s, y, z, \mathbf{u}) - \mathbf{f}(s, y', z', \mathbf{u}')) \\ &\le \alpha(s)\theta(|y - y'|^2) + \beta(s)|y - y'|(|z - z'| + \|\mathbf{u} - \mathbf{u}'\|), \quad \mathbb{P}\otimes\lambda\text{-a.e.}\end{aligned}$$

where θ *is a nondecreasing, continuous and concave function from* $[0, \infty[$ *to itself,* $\theta(0) = 0$, $\limsup_{x\searrow 0} \frac{\theta(x^2)}{x} = 0$ *and* $\int_{0^+} \frac{1}{\theta(x)} dx = \infty$.

We cite also the comparison theorem [18, Theorem 3.5].

Theorem 2.4. *Let* $\mathbf{f}, \mathbf{f}'$ *be two generators satisfying the conditions (H 1)-(H 3) of Theorem 2.3 (*$\mathbf{f}$ *and* $\mathbf{f}'$ *may have different coefficients). We assume* $\xi \le \xi'$*,* $\mathbb{P}$*-a.s. and for all* $(y, z, \mathbf{u})$,

$$\mathbf{f}(s, y, z, \mathbf{u}) \le \mathbf{f}'(s, y, z, \mathbf{u}),$$

for $\mathbb{P}\otimes\lambda$*-a.a.* $(\omega, s) \in \Omega \times [0, T]$*. Moreover, assume that* $\mathbf{f}$ *or* $\mathbf{f}'$ *satisfy the condition (here formulated for* $\mathbf{f}$*)*

(A γ) $\quad \mathbf{f}(s, y, z, \mathbf{u}) - \mathbf{f}(s, y, z, \mathbf{u}') \le \int_{\mathbb{R}_0} (\mathbf{u}'(x) - \mathbf{u}(x))\nu(dx), \quad \mathbb{P}\otimes\lambda\text{-a.e.}$
for all $\mathbf{u}, \mathbf{u}' \in L_2(\nu)$ *with* $\mathbf{u} \le \mathbf{u}'$.

Let (Y, Z, U) *and* (Y', Z', U') *be the solutions to* $(\xi, \mathbf{f})$ *and* $(\xi', \mathbf{f}')$*, respectively. Then,*

$$Y_t \le Y'_t, \quad \mathbb{P}\text{-a.s.}$$

3 Existence result, bounds and Malliavin differentiability for locally Lipschitz generators

To prove Malliavin differentiability we restrict ourselves to the following BSDE

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f\left((X_r)_{r\le s}, s, Y_s, Z_s, \int_{\mathbb{R}_0} g(s, U_s(x))\kappa(x)\nu(dx)\right) ds - \int_t^T Z_s dW_s \\
&\quad - \int_{]t,T]\times\mathbb{R}_0} U_s(x)\tilde{N}(ds, dx),
\end{aligned} \tag{8}
$$

where we use in the future the notation

$$
G(t, \mathbf{u}) := \int_{\mathbb{R}_0} g(t, \mathbf{u}(x))\kappa(x)\nu(dx), \quad \mathbf{u} \in L_2(\nu).
$$

We assume

$$
\kappa(x) := 1 \wedge |x|.
$$

Remark 3.1. *To apply Malliavin calculus in the Lévy setting, one may assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical space in the sense of Solé et al. [27]. On this space, since roughly speaking each $\omega \in \Omega$ represents a path of the Lévy process $X = (X_t)_{t\in[0,T]}$, the Malliavin derivative $D_{t,x}\xi$ has a meaningful definition for every $\omega \in \Omega$ if $\xi \in \mathbb{D}_{1,2}$.*
Here we use a slightly different approach. We keep $(\Omega, \mathcal{F}, \mathbb{P})$ as introduced in Subsection 2.1 but assume any random object to be a functional of X so that for $D_{t,x}$ ($x \neq 0$) one can use Lemma 2.1, and for $D_{t,0}$ we have the chain rule.

- *For the terminal condition ξ the existence of such a functional is guaranteed by Doob's factorisation Lemma: for any $\mathcal{F}_T$-measurable ξ there exists a $g_\xi : D[0,T] \to \mathbb{R}$ such that $\xi = g_\xi(X)$ $\mathbb{P}$-a.s.*

- *For a jointly measurable and adapted generator $\mathbf{f} : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times L_2(\nu) \to \mathbb{R}$ we have by [28, Theorem 3.4] that there exists a jointly measurable $g_{\mathbf{f}} : D[0,T] \times [0,T] \times \mathbb{R} \times \mathbb{R} \times L_2(\nu) \to \mathbb{R}$ such that*

  $$
  \mathbf{f}(\cdot, t, y, z, \mathbf{u}) = g_{\mathbf{f}}((X_s)_{s\in[0,T]}, t, y, z, \mathbf{u})
  $$

  up to indistinguishability for the parameters $(t, y, z, \mathbf{u})$. Moreover, since $\mathbf{f}$ is adapted, for all t, the functional $g_{\mathbf{f}}((X_s)_{s\in[0,T]}, t, \cdot, \cdot, \cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(L_2(\nu))$-measurable. Therefore, using [28, Lemma 3.2], we may find a functional $g^t_{\mathbf{f}} : D[0,T] \times \mathbb{R} \times \mathbb{R} \times L_2(\nu) \to \mathbb{R}$ such that

  $$
  g_{\mathbf{f}}((X_s)_{s\in[0,T]}, t, \cdot, \cdot, \cdot) = g^t_{\mathbf{f}}((X_s)_{s\in[0,t]}, t, \cdot, \cdot, \cdot), \quad \mathbb{P}\text{-a.s.}
  $$

  In other words, $g_{\mathbf{f}}$ is adapted to the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ from (5). As $g_{\mathbf{f}}$ is adapted and measurable, there is a progressively measurable version of $g_{\mathbf{f}}$, denoted by $\bar{g}_{\mathbf{f}}$. Hence we found a progressively measurable functional to represent $\mathbf{f}$ in the way that

  $$
  \mathbf{f}(\cdot, t, y, z, \mathbf{u}) = \bar{g}_{\mathbf{f}}((X_s)_{s\in[0,t]}, t, y, z, \mathbf{u}), \quad \mathbb{P}\text{-a.s.}
  $$

  for all $(t, y, z, \mathbf{u})$.

The previous remark gives us the right to describe the dependency on ω through $(X_t(\omega))_{t\in[0,T]}$ in (8). For shortness of representation we sometimes drop the dependence on $(X_t(\omega))_{t\in[0,T]}$ as it is usually done with ω.
We agree on the following assumptions on ξ, f and g:

Assumption 3.2.

(𝒜1) $A_\xi = \|\xi\|_{L^\infty(\mathbb{P})} < \infty$,
$\xi \in \mathbb{D}_{1,2}$,
$A_{D\xi}(x) := \|(t,\omega) \mapsto D_{t,x}\xi\|_{L^\infty(\lambda\otimes\mathbb{P})} < \infty$,
$\|A_{D\xi}\| < \infty$.

(𝒜2) for all $(y,z,u) \in \mathbb{R}^3$ the map $(\mathtt{x}, t) \mapsto f(\mathtt{x}, t, y, z, u)$ is $(\mathcal{G}_t)_{t\in[0,T]}$- progressively measurable, for all $(\mathtt{x}, t) \in D[0,T] \times [0,T]$, the functions $f, \partial_y f, \partial_z f, \partial_u f$ are continuous in (y,z,u).

(𝒜3) integrability condition: there exists a function $k_f \in L_1([0,T])$ such that $\forall y \in \mathbb{R}$ and $\forall t \in [0,T]$ and $\forall \mathtt{x} \in D[0,T]$ it holds

$$|f(\mathtt{x}, t, 0, 0, 0)| \leq k_f(t).$$

(𝒜4) local Lipschitz condition: there exist nonnegative functions $a \in L_1([0,T]), b \in L_2([0,T])$ and a non-decreasing continuous function $\rho : [0,\infty) \to [0,\infty)$ such that $\forall t \in [0,T]$, (y,z,u), $(\tilde{y}, \tilde{z}, \tilde{u}) \in \mathbb{R}^3$ and $\forall \mathtt{x} \in D[0,T]$ it holds

$$|f(\mathtt{x}, t, y, z, u) - f(\mathtt{x}, t, \tilde{y}, \tilde{z}, \tilde{u})| \leq a(t)|y - \tilde{y}| + \rho(|z| \vee |\tilde{z}| \vee |u| \vee |\tilde{u}|) b(t)(|z - \tilde{z}| + |u - \tilde{u}|).$$

(𝒜5) Malliavin differentiability: Assume that there exists a function $p \in L_1([0,T], \lambda; L_2(\mathbb{R}, \delta_0 + \nu))$ such that if

$$\begin{aligned} R &:= A_\xi e^{\int_0^T a(s)ds} + \int_0^T k_f(s) e^{\int_0^s a(r)dr} ds + 1 \\ Q &:= A_{D\xi}(0) e^{\int_0^T a(s)ds} + \int_0^T p(s,0) e^{\int_0^s a(r)dr} ds + 1 \\ P &:= \rho(2R)\|\kappa\| \left(\|A_{D\xi}\| e^{\int_0^T a(s)ds} + \int_0^T \|p(s,\cdot)\| \, e^{\int_0^s a(r)dr} ds \right) + 1 \end{aligned}$$

and if $rqp := \{(y,z,u) \in \mathbb{R}^3 : |y| \leq R, |z| \leq Q, |u| \leq P\}$, then

(a) $\forall t \in [0,T], (y,z,u) \in rqp : f(X, t, y, z, u) \in \mathbb{D}_{1,2}$,

(b) for a.e. (t,x)

$$A_{Df}(t,x) := \sup_{(y,z,u)\in rqp} \|(\omega, s) \mapsto (D_{s,x} f\,(X, t, y, z, u))(\omega)\|_{L_\infty(\mathbb{P}\otimes\lambda)} \leq p(t,x).$$

(𝒜6) Malliavin regularity: $\forall t \in [0,T]$, $\exists\, K^t \in \bigcup_{p>1} \mathrm{L}_p$ such that for a.a. ω and for all (y,z,u), $(y', z', u') \in rqp$

$$\begin{aligned} &\| \left(D_{\cdot,0} f(X, t, y, z, u)\right)(\omega) - \left(D_{\cdot,0} f(X, t, y', z', u')\right)(\omega)\|_{L_2([0,T])} \\ &\leq K^t(\omega)(|y - y'| + |z - z'| + |u - u'|), \end{aligned}$$

(𝒜7) $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is jointly measurable $u \mapsto \partial_u g(t,u)$ is continuous for all $t \in [0,T]$, and it holds

$$g(t,0) = 0 \quad \text{and} \quad |\partial_u g(t,u)| \leq \rho(|u|).$$

($\mathcal{A}8$) for all $t \in [0,T], \mathtt{x} \in D[0,T]$ and $y, z, u, u' \in \mathbb{R}$ it holds

$$-1 \leq \partial_u f(\mathtt{x}, t, y, z, u)\partial_u g(t, u').$$

We continue with some observations and comments about these assumptions:

Remark 3.3. (i) Notice that $\|A_{D\xi}\| \leq 2A_\xi$ follows immediately from (6).

(ii) Assumption ($\mathcal{A}5$) is trivially satisfied if there exists a $p \in L_1([0,T], \lambda; L_2(\mathbb{R}, \delta_0 + \nu))$ such that

$$\|(\omega, s) \mapsto (D_{s,x} f(X, t, y, z, u))(\omega)\|_{L_\infty(\mathbb{P}\otimes\lambda)} \leq p(t, x)$$

holds uniformly in (y, z, u).

(iii) We need ($\mathcal{A}6$) for the following reason: It is not clear whether the assumption $f(X, t, y, z, u) \in \mathbb{D}_{1,2}$ in ($\mathcal{A}5$) (a) together with the continuity of the partial derivatives required in ($\mathcal{A}2$) implies that $f(X, t, Y_t, Z_t, G(t, \mathbf{u})) \in \mathbb{D}_{1,2}$. Therefore, in [17, Theorem 3.12] it was shown that $G_1, G_2, G_3 \in \mathbb{D}_{1,2} \implies f(X, t, G_1, G_2, G_3) \in \mathbb{D}_{1,2}$ if additionally the Malliavin regularity assumption ($\mathcal{A}6$) holds.

(iv) The mean value theorem implies that condition ($\mathcal{A}8$) is sufficient for (A γ).

The next theorem is a generalisation of Corollary 2.8 [11] to the jump case. For the proof of Corollary 2.8 [11] a comparison theorem is used to show the boundedness of the process Y and its Malliavin derivative. We will follow this idea, but for jump processes stronger conditions are needed for comparison theorems to hold (see the counter example given in [6, Remark 2.7]). In fact, the condition we need is (A γ).
Malliavin differentiability of solutions to BSDEs was considered for example in [25], [14], [16] and [21]. The usual procedure - which we follow here also - is to impose conditions on the generator and show via Picard iteration that the solution is Malliavin differentiable. The approach in [21] is different since there conditions on the Malliavin differentiability of the generator with the solution processes already plugged in were considered.

Theorem 3.4. *Assume that ($\mathcal{A}1$) - ($\mathcal{A}8$) hold. Then there exists a solution (Y, Z, U) to (8) which is unique in the class $\mathcal{S}_\infty \times L_\infty(W) \times (L_2(\tilde{N}) \cap L_{2\times\infty}(\tilde{N}))$, and it holds a.s.*

$$|Y_t| \leq A_\xi e^{\int_t^T a(s)ds} + \int_t^T k_f(s) e^{\int_t^s a(r)dr} ds \leq R - 1, \tag{9}$$

and for a.e. $t \in [0,T]$:

$$|Z_t| \leq A_{D\xi}(0) e^{\int_t^T a(s)ds} + \int_t^T p(s,0) e^{\int_t^s a(r)dr} ds \leq Q - 1, \tag{10}$$

and for $\lambda \otimes \nu$-a.e. $(t,x) \in [0,T] \times \mathbb{R}_0$:

$$|U_t(x)| \leq \left(A_{D\xi}(x) e^{\int_t^T a(s)ds} + \int_t^T p(s,x)\, e^{\int_t^s a(r)dr} ds \right) \wedge\ (2R - 2), \tag{11}$$

which means that $U \in L_{2,b}(\tilde{N})$.
Moreover, it holds that (Y, Z, U) is Malliavin differentiable, i.e.

$$Y, Z \in L_2([0,T]; \mathbb{D}_{1,2}), \quad U \in L_2([0,T] \times \mathbb{R}_0; \mathbb{D}_{1,2}),$$

and for $\mathbb{m}$- a.e. (r, x) the triple $(D_{r,x}Y, D_{r,x}Z, D_{r,x}U)$ solves

$$\begin{aligned} D_{r,x}Y_t &= D_{r,x}\xi + \int_t^T F_{r,x}\left(s, D_{r,x}Y_s, D_{r,x}Z_s, D_{r,x}U_s\right) ds - \int_t^T D_{r,x}Z_s dW_s \\ &\quad - \int_{]t,T]\times\mathbb{R}_0} D_{r,x}U_s(v)\tilde{N}(ds,dv), \quad 0 \le r \le t \le T, \end{aligned}$$

where $\Theta_s := (Y_s, Z_s, G(s, U_s))$ and

$$\begin{aligned} F_{r,0}(s, y, z, \mathbf{u}) &:= (D_{r,0}f)(s, \Theta_s) + \partial_y f(s, \Theta_s) y + \partial_z f(s, \Theta_s) z \\ &\quad + \partial_u f(s, \Theta_s) \int_{\mathbb{R}_0} \partial_u g(s, U_s(v)) \mathbf{u}(v) \kappa(v) \nu(dv), \end{aligned}$$

and for $x \neq 0$,

$$\begin{aligned} F_{r,x}(s, y, z, \mathbf{u}) &:= (D_{r,x}f)(X, s, \Theta_s) \\ &\quad + f(X + x\mathbb{1}_{[r,T]}, s, \Theta_s + (y, z, G(s, U_s + \mathbf{u}))) - f(X + x\mathbb{1}_{[r,T]}, s, \Theta_s). \end{aligned}$$

Setting $D_{r,v}Y_r(\omega) := \lim_{t \searrow r} D_{r,v}Y_t(\omega)$ for all (r, v, ω) for which $D_{r,v}Y$ is càdlàg and $D_{r,v}Y_r(\omega) :=$ 0 otherwise, we have

$$\begin{aligned} {}^p\left((D_{r,0}Y_r)_{r\in[0,T]}\right) &\text{ is a version of } (Z_r)_{r\in[0,T]}, \\ {}^p\left((D_{r,v}Y_r)_{r\in[0,T],v\in\mathbb{R}_0}\right) &\text{ is a version of } (U_r(v))_{r\in[0,T],v\in\mathbb{R}_0}. \end{aligned}$$

(The definition of the objects $(D_{r,x}f)(s, \Theta_s)$, where we first apply the Malliavin derivative to f and afterwards insert the expressions in Θ_s, indeed constitute well defined measurable objects because of the continuity assumptions on $\mathbf{f}$, as is explained in [17, Lemma 3.5 ff] or [28, Remark 5.3 (ii)].)

Proof. **Step 1** For $M \in \mathbb{R}_+$ let $b_M : \mathbb{R} \to [-M, M]$ be a smooth monotone function such that $0 \le b'_M(x) \le 1$ and

$$b_M(x) := \begin{cases} M, & x > M+1, \\ x, & |x| \le M-1, \\ -M, & x < -M-1. \end{cases}$$

Notice that $|b_M(x)| \le |x|$. We set (using R, Q and P from ($\mathcal{A}5$))

$$\widehat{f}(s, y, z, G(s, \mathbf{u})) := f(s, b_R(y), b_Q(z), \widehat{G}(s, \mathbf{u})),$$

where

$$\widehat{G}(s, \mathbf{u}) := b_P(G(s, b_{2R}(\mathbf{u}))) \quad \text{and} \quad b_{2R}(\mathbf{u})(x) := b_{2R}(\mathbf{u}(x)).$$

We will show first that $\widehat{\mathbf{f}}(s,y,z,\mathbf{u}) := \widehat{f}(s,y,z,G(s,\mathbf{u}))$ satisfies the assumptions of Theorem 2.3. We have (H 1) because of $(\mathcal{A}2)$ and $(\mathcal{A}7)$; while (H 3) follows from $(\mathcal{A}4)$ and $(\mathcal{A}7)$. Indeed, since $(\mathcal{A}7)$ implies

$$
\begin{aligned}
|\widehat{G}(s,\mathbf{u}) - \widehat{G}(s,\mathbf{u}')| &\le \sup_{u\in[-2R,2R]} |\partial_u g(s,u)| \int_{\mathbb{R}_0} |\mathbf{u}(x) - \mathbf{u}'(x)|\kappa(x)\nu(dx) \\
&\le \rho(2R)\|\kappa\|\|\mathbf{u}-\mathbf{u}'\|,
\end{aligned}
$$

it holds

$$
\begin{aligned}
&|f(\mathrm{x},s,b_R(y),b_Q(z),\widehat{G}(s,\mathbf{u})) - f(\mathrm{x},s,b_R(y'),b_Q(z'),\widehat{G}(s,\mathbf{u}'))| \\
&\quad\le a(s)|y-y'| + \rho(Q\vee P)b(s)(|z-z'| + |\widehat{G}(s,\mathbf{u}) - \widehat{G}(s,\mathbf{u}')|) \\
&\quad\le a(s)|y-y'| + \rho(Q\vee P)(1+\rho(2R)\|\kappa\|)b(s)(|z-z'| + \|\mathbf{u}-\mathbf{u}'\|). \qquad (12)
\end{aligned}
$$

Now we combine the last inequality for $y'=0$, $z'=0$, and $\mathbf{u}'=0$ with $(\mathcal{A}3)$ to get (H 2):

$$
|f(\mathrm{x},s,b_R(y),b_Q(z),\widehat{G}(s,\mathbf{u}))| \le k_f(s) + a(s)|y| + \rho(Q\vee P)(1+\rho(2R)\|\kappa\|)b(s)(|z| + \|\mathbf{u}\|).
$$

Hence by Theorem 2.3 there exists for any $\xi \in L_2$ a unique solution $(\widehat{Y},\widehat{Z},\widehat{U})$ to (8) with data $(\widehat{\mathbf{f}},\xi)$.

Assumption $(\mathcal{A}8)$ implies that $\widehat{\mathbf{f}}$ satisfies (A γ) from Theorem 2.4.

Step 2 From $(\mathcal{A}3)$ and (12) we conclude that $\forall s\in[0,T]$ and $\forall(y,z,\mathbf{u})$ it holds

$$
|f(\mathrm{x},s,b_R(y),b_Q(z),\widehat{G}(s,\mathbf{u}))| \le k_f(s) + a(s)|y| + b(s)\rho(Q\vee P)(|z| + |\widehat{G}(s,\mathbf{u})|).
$$

We want to apply the comparison theorem to the BSDEs:

$$
\begin{aligned}
\overline{Y}_t &= A_\xi + \int_t^T k_f(s) + a(s)|\overline{Y}_s| + b(s)\rho(Q\vee P)(|\overline{Z}_s| + |\widehat{G}(s,\overline{U}_s)|)ds \\
&\quad - \int_t^T \overline{Z}_s dW_s - \int_{]t,T]\times\mathbb{R}_0} \overline{U}_s(x)\tilde{N}(ds,dx), \qquad (13) \\
\widehat{Y}_t &= \xi + \int_t^T \widehat{f}(s,\widehat{Y}_s,\widehat{Z}_s,G(s,\widehat{U}_s))ds \\
&\quad - \int_t^T \widehat{Z}_s dW_s - \int_{]t,T]\times\mathbb{R}_0} \widehat{U}_s(x)\tilde{N}(ds,dx), \\
\underline{Y}_t &= -A_\xi - \int_t^T k_f(s) + a(s)|\underline{Y}_s| + b(s)\rho(Q\vee P)(|\underline{Z}_s| + |\widehat{G}(s,\underline{U}_s)|)ds \\
&\quad - \int_t^T \underline{Z}_s dW_s - \int_{]t,T]\times\mathbb{R}_0} \underline{U}_s(x)\tilde{N}(ds,dx). \qquad (14)
\end{aligned}
$$

By Step 1 the generator $\widehat{f}$ satisfies the conditions (H 1)-(H 3) and (A γ). Since also the generators of $\overline{Y}$ and $\underline{Y}$ satisfy the conditions (H 1)-(H 3), Theorem 2.4 implies that

$$
\underline{Y}_t \le \widehat{Y}_t \le \overline{Y}_t, \quad \forall t\in[0,T] \ \mathbb{P}\text{-a.s.}
$$

By Theorem 2.3 we have that $(\overline{Y},0,0)$ and $(\underline{Y},0,0)$ are the unique solutions to (13) and (14), respectively, and

$$\begin{aligned}\overline{Y}_t=-\underline{Y}_t &= A_\xi e^{\int_t^T a(s)ds}+\int_t^T k_f(s)e^{\int_t^s a(r)dr}ds\\ &\leq A_\xi e^{\int_0^T a(s)ds}+\int_0^T k_f(s)e^{\int_0^s a(r)dr}ds=R-1,\end{aligned}$$

where R was defined in $(\mathcal{A}5)$. This gives (9) for $\widehat{Y}$.

Step 3 To consider Malliavin derivatives we check the conditions of Theorem A.1 for the BSDE with data $(\widehat{f},\xi)$. Condition $(\mathcal{A}1)$ implies that $(\mathbf{A}_\xi)$ is satisfied. Condition $(\mathbf{A}_f)$ a) follows from $(\mathcal{A}2)$. Condition $(\mathcal{A}3)$ implies $(\mathbf{A}_f)$ b).

The Lipschitz continuity required in $(\mathbf{A}_f)$ c) is fulfilled because of $(\mathcal{A}4)$. Furthermore, we have the implications $(\mathcal{A}5)\implies(\mathbf{A}_f)$ d), $(\mathcal{A}6)\implies(\mathbf{A}_f)$ e) and $(\mathcal{A}7)\implies(\mathbf{A}_f)$ f).

Consequently, we may consider the Malliavin derivative of the BSDE (8) with data $(\widehat{f},\xi)$.

Let $\Theta_s=(\widehat{Y}_s,\widehat{Z}_s,\widehat{G}(s,\widehat{U}_s))$. Then

$$\begin{aligned}D_{r,0}\widehat{Y}_t &= D_{r,0}\xi+\int_t^T\Big[(D_{r,0}\widehat{f})(s,\Theta_s)+\partial_y\widehat{f}(s,\Theta_s)D_{r,0}\widehat{Y}_s+\partial_z\widehat{f}(\Theta_s)D_{r,0}\widehat{Z}_s\\ &\quad+\partial_u\widehat{f}(s,\Theta_s)b_P'(G(s,b_{2R}(\widehat{U}_s)))\\ &\qquad\times\int_{\mathbb{R}_0}\partial_u g(s,b_{2R}(\widehat{U}_s(v)))b_{2R}'(\widehat{U}_s(v))D_{r,0}\widehat{U}_s(v)\kappa(v)\nu(dv)\Big]ds\\ &\quad-\int_t^T D_{r,0}\widehat{Z}_s dW_s-\int_{]t,T]\times\mathbb{R}_0}D_{r,0}\widehat{U}_s(v)\tilde{N}(ds,dv).\end{aligned}\tag{15}$$

By $(\mathcal{A}5)$ we have $|(D_{r,0}\widehat{f})(s,\Theta_s)|\leq p(s,0)$, and the Lipschitz coefficients from (12) are bounds for the partial derivatives, so that by Theorem 2.4, the solutions of

$$\begin{aligned}\overline{\mathcal{Y}_t^{r,0}} = &\ A_{D\xi}(0)+\int_t^T\Big[p(s,0)+a(s)\,|\overline{\mathcal{Y}_s^{r,0}}|+\rho(Q\vee P)b(s)\,|\overline{\mathcal{Z}_s^{r,0}}|\\ &+\rho(Q\vee P)b(s)\rho(2R)\int_{\mathbb{R}_0}|\overline{\mathcal{U}_s^{r,0}(v)}|\,\kappa(v)\nu(dv)\Big]ds\\ &-\int_t^T\overline{\mathcal{Z}_s^{r,0}}dW_s-\int_{]t,T]\times\mathbb{R}_0}\overline{\mathcal{U}_s^{r,0}(v)}\tilde{N}(ds,dv),\end{aligned}$$

and

$$\begin{aligned}\underline{\mathcal{Y}_t^{r,0}} = &\ -A_{D\xi}(0)-\int_t^T\Big[p(s,0)+a(s)\,|\underline{\mathcal{Y}_s^{r,0}}|+\rho(Q\vee P)b(s)\,|\underline{\mathcal{Z}_s^{r,0}}|\\ &+\rho(Q\vee P)b(s)\rho(2R)\int_{\mathbb{R}_0}|\underline{\mathcal{U}_s^{r,0}(v)}|\,\kappa(v)\nu(dv)\Big]ds\\ &-\int_t^T\underline{\mathcal{Z}_s^{r,0}}dW_s-\int_{]t,T]\times\mathbb{R}_0}\underline{\mathcal{U}_s^{r,0}(v)}\tilde{N}(ds,dv),\end{aligned}$$

satisfy $\underline{\mathcal{Y}_t^{r,0}}\leq D_{r,0}\widehat{Y}_t\leq\overline{\mathcal{Y}_t^{r,0}}$. Note that condition (A γ) required in Theorem 2.4 is satisfied by the linear generator of equation (15) using assumption $(\mathcal{A}8)$. The above equations do have unique solutions where $\overline{\mathcal{Y}}$ and $\underline{\mathcal{Y}}$ are given by

$$\overline{\mathcal{Y}_t^{r,0}} = -\underline{\mathcal{Y}_t^{r,0}} = A_{D\xi}(0)e^{\int_t^T a(s)ds} + \int_t^T p(s,0)e^{\int_t^s a(r)dr}ds \le Q-1.$$

According to Theorem A.1 (iv) we have that $D_{r,0}\widehat{Y}_r(\omega) := \lim_{t\searrow r} D_{r,0}\widehat{Y}_t(\omega)$. Since $D_{r,0}\widehat{Y}_r(\omega)$ is measurable and bounded, its predictable projection exists and is a version of $(Z_r)_{r\in[0,T]}$ which proves (10) for the BSDE with $(\widehat{f},\xi)$. For $x\neq 0$ we get a similar result:

$$\begin{aligned}
D_{r,x}\widehat{Y}_t &= D_{r,x}\xi + \int_t^T \Big[(D_{r,x}\widehat{f})(s,\Theta_s) + \widehat{f}(X + x\mathbb{1}_{[r,T]}, s, \Theta_s + D_{r,x}\Theta_s) \\
&\quad - \widehat{f}(X + x\mathbb{1}_{[r,T]}, s, \Theta_s)\Big]ds - \int_t^T D_{r,x}\widehat{Z}_s dW_s - \int_{]t,T]\times\mathbb{R}_0} D_{r,x}\widehat{U}_s(v)\tilde{N}(ds,dv), \\
\overline{\mathcal{Y}_t^{r,x}} &= A_{D\xi}(x) + \int_t^T \Big[p(s,x) + a(s)\,|\overline{\mathcal{Y}_s^{r,x}}| + \rho(Q\vee P)b(s)\,|\overline{\mathcal{Z}_s^{r,x}}| \\
&\quad + \rho(Q\vee P)b(s)\rho(2R)\int_{\mathbb{R}_0} |\overline{\mathcal{U}_s^{r,x}(v)}|\,\kappa(v)\nu(dv)\Big]ds \\
&\quad - \int_t^T \overline{\mathcal{Z}_s^{r,x}}dW_s - \int_{]t,T]\times\mathbb{R}_0} \overline{\mathcal{U}_s^{r,x}(v)}\tilde{N}(ds,dv),
\end{aligned}$$

and

$$\begin{aligned}
\underline{\mathcal{Y}_t^{r,x}} &= -A_{D\xi}(x) - \int_t^T \Big[p(s,x) + a(s)\,|\underline{\mathcal{Y}_s^{r,x}}| + \rho(Q\vee P)b(s)\,|\underline{\mathcal{Z}_s^{r,x}}| \\
&\quad + \rho(Q\vee P)b(s)\rho(2R)\int_{\mathbb{R}_0} |\underline{\mathcal{U}_s^{r,x}(v)}|\,\kappa(v)\nu(dv)\Big]ds \\
&\quad - \int_t^T \underline{\mathcal{Z}_s^{r,x}}dW_s - \int_{]t,T]\times\mathbb{R}_0} \underline{\mathcal{U}_s^{r,x}(v)}\tilde{N}(ds,dv),
\end{aligned}$$

We get $\underline{\mathcal{Y}}_t^{r,x} \le D_{r,x}\widehat{Y}_t \le \overline{\mathcal{Y}}_t^{r,x}$, where

$$\overline{\mathcal{Y}_t^{r,x}} = -\underline{\mathcal{Y}_t^{r,x}} = A_{D\xi}(x)e^{\int_t^T a(s)ds} + \int_t^T p(s,x)e^{\int_t^s a(r)dr}ds.$$

Additionally, notice that according to Theorem A.1 (iv), $\widehat{U}_t(x)$ can be expressed $\mathbb{P}\otimes\mathbb{m}$-a.e. as $\lim_{r\searrow t} D_{t,x}\widehat{Y}_r$ for $x\neq 0$. Thus, by the representation (6) of D as difference operator in this case, we end this step by stating

$$|\widehat{U}_t(x)| \le 2\sup_{0\le t\le T}|\widehat{Y}_t| \le 2(R-1), \quad \mathbb{P}\otimes\mathbb{m}\text{-a.e.},$$

which implies the estimate (11) for $\widehat{U}$. Again, since $\widehat{U}_t(x) = \lim_{r\searrow t} D_{t,x}\widehat{Y}_r$ is measurable and bounded, we can take its predictable projection. Moreover, by the definition of P in $(\mathcal{A}5)$,

$$\|\widehat{U}_t\| \le \|A_{D\xi}\|e^{\int_0^T a(s)ds} + \int_0^T \|p(s,\cdot)\|e^{\int_0^s a(r)dr}ds = \frac{P-1}{\rho(2R)\|\kappa\|}, \quad \text{so that } \widehat{U}\in L_{2,b}(\tilde{N}).$$

Step 4 The assertion is now shown for the generator $\widehat{\mathbf{f}}$, thus the goal of this step is to obtain the results also for $\mathbf{f}$ without any cut-off restraints. By Step 3 we get $|\widehat{Y}_t| \leq R-1, |\widehat{Z}_t| \leq Q-1$, a.s., $|\widehat{U}_t(x)| \leq 2R-2$ $\mathbb{P}\otimes\mathbb{m}$-a.e. and

$$\begin{aligned}|G(s,\widehat{U}_s)| &= \left|\int_{\mathbb{R}_0} g(s,\widehat{U}_s(x))\kappa(x)\nu(dx)\right| \\ &\leq \rho(2R)\int_{\mathbb{R}_0}|\widehat{U}_s(x)|\kappa(x)\nu(dx) \\ &\leq \rho(2R)\|\kappa\|\,\|\widehat{U}_s\| \leq P-1.\end{aligned}$$

In that case, the equality

$$\widehat{f}(t,\widehat{Y}_t,\widehat{Z}_t,G(t,\widehat{U}_t)) = f(t,\widehat{Y}_t,\widehat{Z}_t,G(t,\widehat{U}_t))$$

holds almost surely for every $t \in [0,T]$. Therefore, the solution $(\widehat{Y},\widehat{Z},\widehat{U})$ to the BSDE with data $(\widehat{\mathbf{f}},\xi)$ also serves as solution of the equation given by $(\mathbf{f},\xi)$, which is (8).
Step 5 Finally we show the uniqueness of solutions in the space $\mathcal{S}_\infty \times L_\infty(W)\times(L_2(\tilde{N})\cap L_{2\times\infty}(\tilde{N}))$. Assume that we have two solutions $(Y^{(j)},Z^{(j)},U^{(j)})$ $(j=1,2)$ with $\sup|Y_t^{(j)}| \leq R_j-1$ and $\sup|Z_t^{(j)}| \leq Q_j-1$ and $\operatorname{ess\,sup}_{s,\omega}\|U_s^{(j)}\| \leq C_j$. Then $\sup|U_t^{(j)}(x)| \leq 2R_j-2$, and by $(\mathcal{A}7)$

$$\begin{aligned}|G(s,U_s^{(j)})| &= \left|\int_{\mathbb{R}_0} g(t,U_s^{(j)}(x))\kappa(x)\nu(dx)\right| \\ &\leq \rho(2R_j-2)\int_{\mathbb{R}_0}|U_s^{(j)}(x)|\kappa(x)\nu(dx) \\ &\leq \rho(2R_j-2)\|\kappa\|\operatorname{ess\,sup}_{s,\omega}\|U_s^{(j)}\| \\ &\leq \rho(2R_j-2)\|\kappa\|C_j =: M_j.\end{aligned}$$

Hence from $(\mathcal{A}4)$ and it follows that

$$\begin{aligned}&|\mathbf{f}(\omega,t,Y_t^{(1)},Z_t^{(1)},U_t^{(1)}) - \mathbf{f}(\omega,t,Y_t^{(2)},Z_t^{(2)},U_t^{(2)})| \\ &\leq \; a(t)|Y_t^1-Y_t^2| + b(t)\rho(Q_1\vee Q_2\vee M_1\vee M_2)\left(|Z_t^1-Z_t^2| + |G(t,U_t^{(1)})-G(t,U_t^{(2)})|\right)\end{aligned}$$

And by $(\mathcal{A}7)$

$$|G(t,U_t^{(1)})-G(t,U_t^{(2)})| \leq \rho(|2R_1|\vee|2R_2|)\,\|\kappa\|\,\|U_t^{(1)}-U_t^{(2)}\|.$$

Because the processes $(Y^{(j)},Z^{(j)},U^{(j)})$ are bounded and f is locally Lipschitz, f restricted to a bounded set is Lipschitz, and uniqueness follows from Theorem 2.3. □

Remark 3.5. *By a mollifying argument, to weaken differentiability conditions on the generator, assumptions $(\mathcal{A}2)$ and $(\mathcal{A}7)$ may be relaxed to*

(A2') for all $(y,z,u)\in\mathbb{R}^3$ the map $(\mathtt{x},t)\mapsto f(\mathtt{x},t,y,z,u)$ is $(\mathcal{G}_t)_{t\in[0,T]}$- progressively measurable, for all $(\mathtt{x},t)\in D[0,T]\times[0,T]$ and the function f is continuous in (y,z,u).

(A7') $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ is jointly measurable and for all $t\in[0,T]$ it holds $g(t,0)=0$ and

$$\forall u,u'\in\mathbb{R}: |g(t,u)-g(t,u')| \leq \rho(|u|\vee|u'|)|u-u'|.$$

However, instead of (A5) we have to impose the slightly stronger condition

(A5') Assume that there exists $\varepsilon > 0$ and a function $p \in L_1([0,T], \lambda; L_2(\mathbb{R}, \delta_0 + \nu))$ such that if

$$rqp_\varepsilon := \{(y,z,u) \in \mathbb{R}^3 : |y| \le R + \varepsilon, |z| \le Q, |u| \le P + \varepsilon\},$$

with R, Q, P from (A5), then

(a) $\forall t \in [0,T], (y,z,u) \in rqp_\varepsilon : f(X,t,y,z,u) \in \mathbb{D}_{1,2}$,

(b) for a.e. (t,x)

$$\begin{aligned} A_{Df}(t,x) &:= \sup_{(y,z,u)\in rqp_\varepsilon} \|(\omega,s) \mapsto D_{s,x} f\left(X(\omega),t,y,z,u\right)\|_{L_\infty(\mathbb{P}\otimes\lambda)} \\ &\le p(t,x). \end{aligned}$$

As was the case for (A5), if there exists a $p \in L_1([0,T], \lambda; L_2(\mathbb{R}, \delta_0 + \nu))$ such that

$$\|(\omega,s) \mapsto D_{s,x} f\left(X(\omega),t,y,z,u\right)\|_{L_\infty(\mathbb{P}\otimes\lambda)} \le p(t,x)$$

holds uniformly in (y,z,u), then (A5') is trivially satisfied. Condition (A8) becomes

(A8') for all $t \in [0,T], \mathbf{x} \in D[0,T]$ and $y,z \in \mathbb{R}$, the generalised function (in the sense of distributions on $\mathbb{R}^2$ and using weak derivatives)

$$(\partial_u f(\mathbf{x},t,y,z,\cdot) \otimes \partial_{u_1} g(t,\cdot)) + 1$$

is nonnegative.

4 A generalisation of the local Lipschitz condition

In this section we address the question whether one may remove the factor $\kappa(x) = 1 \wedge |x|$ in

$$G(t,\mathbf{u}) = \int_{\mathbb{R}_0} g(t,\mathbf{u}(x))\,(1 \wedge |x|)\,\nu(dx)$$

of the generator $f(t,y,z,G(t,\mathbf{u}))$. For this, one could replace $\kappa(x)$ by $\kappa_n(x) := 1 \wedge |nx|$ and let $n \to \infty$. Notice that $\kappa_n(x) \to 1$ for all $x \in \mathbb{R}_0$. If we consider for example for some $\alpha > 0$ the expression

$$G_{\alpha,n}(t,\mathbf{u}) := \int_{\mathbb{R}\setminus\{0\}} \mathcal{H}_\alpha(\mathbf{u}(x))\kappa_n(x)\nu(dx) \quad \text{with} \quad \mathcal{H}_\alpha(u) = \frac{e^{\alpha u} - \alpha u - 1}{\alpha},$$

then $|G_{\alpha,n}(t,\mathbf{u})| \le \frac{e^{\alpha\|u\|_\infty}}{\alpha} \int_{\mathbb{R}\setminus\{0\}} |\mathbf{u}(x)|^2 \nu(dx)$ for all $n \in \mathbb{N}$, so that it seems possible to consider the limit $n \to \infty$ for $\mathbf{u} \in L^\infty(\nu) \cap L^2(\nu)$.
However, in condition (A5) the factor $\|\kappa\|$ appears for the constant P, and since $\|\kappa_n\| \to \infty$ if $\nu(\mathbb{R}_0) = \infty$, this would lead to $P = \infty$.

Nevertheless, generators including the case $f(t,y,z) + \int_{\mathbb{R}\setminus\{0\}} \mathcal{H}_\alpha(\mathbf{u}(x))\nu(dx)$ have been treated in [8] (see also [23]).

We will consider the following situation:
Let $\bar{f}$ be a generator satisfying assumptions $(\mathcal{A}1)$-$(\mathcal{A}8)$ (so that Theorem 3.4 applies) and define

$$\mathbf{f}(t,y,z,\mathbf{u}) := \varphi\left(\bar{f}\left(t,y,z,G(t,\mathbf{u})\right), \int_{\mathbb{R}_0} \mathcal{H}(\mathbf{u}(x))\nu(dx)\right) \tag{16}$$

with

$$G(t,\mathbf{u}) := \int_{\mathbb{R}_0} g(t,\mathbf{u}(x))\kappa(x)\nu(dx), \qquad \text{for } \ \mathbf{u} \in L_2(\nu). \tag{17}$$

For the functions $\bar{f}, \mathcal{H} : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ we require the following conditions:

Assumption 4.1 (A$\mathcal{H}$).
Suppose that $\bar{f}$ satisfies $(\mathcal{A}1)$-$(\mathcal{A}7)$ and assume that $\mathbf{f}$ is given by (16) and (17).

($\mathcal{H}1$) *Let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ be such that $\mathcal{H}(0) = 0$.*

($\mathcal{H}2$) *We assume that $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable, and the following conditions hold:*

$$\forall v, w \in \mathbb{R} : |\partial_v \varphi(v,w)| \leq 1,$$
$$\forall w \in \mathbb{R} : v \mapsto \partial_w \varphi(v,w) \quad \text{is a bounded function.}$$

($\mathcal{H}3$) *Instead of $(\mathcal{A}8)$, we impose that for all $t \in [0,T]$, $\mathbf{x} \in D[0,T]$ and $w, y, z, u, u' \in \mathbb{R}$ it holds*

$$-1 \leq \partial_v \varphi\left(\bar{f}\left(\mathbf{x},t,y,z,u\right), w\right) \partial_u \bar{f}(\mathbf{x},t,y,z,u)\partial_u g(t,u')$$

and

$$-1 \leq \partial_v \varphi\left(\bar{f}\left(\mathbf{x},t,y,z,u\right), w\right) \partial_u \bar{f}(\mathbf{x},t,y,z,u)\partial_u g(t,u') + \partial_w \varphi\left(\bar{f}\left(\mathbf{x},t,y,z,u\right), w\right) \mathcal{H}'(u').$$

($\mathcal{H}4$) *For any $R' > 0$, there is a constant $c_{R'}$ such that $|\mathcal{H}'(u)| \leq c_{R'}|u|$ for all $|u| \leq R'$.*

Note that the generator $\mathbf{f}$ satisfying (A$\mathcal{H}$) is not locally Lipschitz in $\mathbf{u} \in L_2(\nu)$.

Theorem 4.2. *Under Assumption 4.1 (A$\mathcal{H}$) (with notation taken from Assumption 3.2) there exists a solution to (8) if f is replaced by (16). The solution processes (Y,Z,U) of this equation have the same bounds which Theorem 3.4 states for the solution of the BSDE given by $(\xi, \bar{f})$. The solution (Y,Z,U) is unique in the class $\mathcal{S}_\infty \times L_\infty(W) \times L_{2,b}(\tilde{N})$.*

Proof. We define for $n \in \mathbb{N}$,

$$\mathcal{H}^n(u,x) := \mathcal{H}(u)\min\{1, n|x|\}$$

and

$$\mathbf{f}^n(s,y,z,\mathbf{u}) := \varphi\left(\bar{f}\left(s,y,z,G(s,\mathbf{u})\right), \int_{\mathbb{R}_0} \mathcal{H}^n(\mathbf{u}(x),x)\nu(dx)\right).$$

Step 1
For $n \in \mathbb{N}$ let (Y^n, Z^n, U^n) be the unique solution to $(\xi, \mathbf{f}^n)$ which exists, since by the conditions in Assumption 4.1 (A$\mathcal{H}$) also Assumption 3.2 is met. Like in Step 4 of Theorem 3.4 we see that for $\bar{f}$ and G there are Lipschitz functions $\hat{\bar{f}}, \hat{G}$ (in the sense of $(\mathcal{A}4)$ and $(\mathcal{A}7)$ with constant ρ) which, if the solution processes are inserted, give the same values as $\bar{f}$ and G, $\mathbb{P} \otimes \lambda \otimes \nu$-a.e.

Moreover, Theorem 3.4 implies that for all $n \in \mathbb{N}$

$$|U_t^n(x)| \quad \leq \quad A_{D\xi}(x) e^{\int_t^T a(s)ds} + \int_t^T p(s,x) e^{\int_t^s a(r)dr} ds.$$

We also know that $|U_t^n(x)| \leq R'$ for $\mathbb{P} \otimes \mathfrak{m}$-a.a. (ω, t, x), where $R' = 2R - 2$ is the constant bound for U^n appearing in (11). The $\mathcal{H}^n$ are a deterministic functions, and therefore do not contribute to the integral term $\int_t^T p(s,x) e^{\int_t^s a(r)dr} ds$ which bounds the size of $U_t^n(x)$. By ($\mathcal{H}1$) and ($\mathcal{H}4$), on $\{u : |u| \leq R'\}$ there is $c_{R'} > 0$ such that $|\mathcal{H}(u)| \leq c_{R'} u^2$. Therefore, we observe by the use of ($\mathcal{H}2$) that

$$\begin{aligned}
&|\mathbf{f}^n(s, Y_s^n, Z_s^n, U_s^n) - \mathbf{f}^n(s, Y_s^m, Z_s^m, U_s^m)| \\
&= \Bigg| \varphi\left(\hat{\bar{f}}(s, Y_s^n, Z_s^n, \hat{G}(s, U_s^n)), \int_{\mathbb{R}_0} \mathcal{H}^n(U_s^n(x), x)\nu(dx)\right) \\
&\qquad - \varphi\left(\hat{\bar{f}}(s, Y_s^m, Z_s^m, \hat{G}(s, U_s^m)), \int_{\mathbb{R}_0} \mathcal{H}^n(U_s^m(x), x)\nu(dx)\right)\Bigg| \\
&\leq \left|\hat{\bar{f}}(s, Y_s^n, Z_s^n, \hat{G}(s, U_s^n)) - \hat{\bar{f}}(s, Y_s^m, Z_s^m, \hat{G}(s, U_s^m))\right| \\
&\quad + \left|\partial_w \varphi\left(\hat{\bar{f}}(s, Y_s^m, Z_s^m, \hat{G}(s, U_s^m)), \vartheta\right)\right| \\
&\qquad \times \left(\int_{\mathbb{R}_0} |\mathcal{H}(U_s^n(x)) \min\{1, n|x|\} - \mathcal{H}(U_s^m(x)) \min\{1, n|x|\}| \,\nu(dx)\right),
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\min&\left\{\int_{\mathbb{R}_0} \mathcal{H}^n(U_s^n(x), x)\nu(dx), \int_{\mathbb{R}_0} \mathcal{H}^n(U_s^m(x), x)\nu(dx)\right\} \leq \vartheta \\
&\leq \max\left\{\int_{\mathbb{R}_0} \mathcal{H}^n(U_s^n(x), x)\nu(dx), \int_{\mathbb{R}_0} \mathcal{H}^n(U_s^m(x), x)\nu(dx)\right\}.
\end{aligned}$$

Condition ($\mathcal{H}4$) and the bounds for U^n and U^m imply that

$$\begin{aligned}
&\int_{\mathbb{R}_0} \max\{|\mathcal{H}(U_s^n(x))|, |\mathcal{H}(U_s^m(x))|\}\nu(dx) \\
&\leq \int_{\mathbb{R}_0} \left| c_{R'} \left(A_{D\xi}(x) e^{\int_s^T a(v)dv} + \int_s^T p(r,x) e^{\int_s^r a(v)dv} dr\right)^2 \right| \nu(dx) \\
&\leq C_1 \left(\|A_{D\xi}\|^2 + \int_0^T \|p(r,\cdot)\|^2 dr\right) < \infty
\end{aligned}$$

for a constant $C_1 = C_1\left(c_{R'}, \int_0^T a(v)dv\right)$. Therefore, because ϑ is bounded and by ($\mathcal{H}2$), there is a constant K such that $\left|\partial_w \varphi\left(\bar{f}(s, Y_s^m, Z_s^m, G(s, U_s^m)), \vartheta\right)\right| \leq K$.
The estimate (18) can then be continued to

$$\begin{aligned}
&|\mathbf{f}^n(s, Y_s^n, Z_s^n, U_s^n) - \mathbf{f}^n(s, Y_s^m, Z_s^m, U_s^m)| \\
&\leq \left|\hat{\bar{f}}(s, Y_s^n, Z_s^n, \hat{G}(s, U_s^n)) - \hat{\bar{f}}(s, Y_s^m, Z_s^m, \hat{G}(s, U_s^m))\right| + K \int_{\mathbb{R}_0} |\mathcal{H}(U_s^n(x)) - \mathcal{H}(U_s^m(x))| \,\nu(dx) \\
&\leq \left|\hat{\bar{f}}(s, Y_s^n, Z_s^n, \hat{G}(s, U_s^n)) - \hat{\bar{f}}(s, Y_s^m, Z_s^m, \hat{G}(s, U_s^m))\right| + K \int_{\mathbb{R}_0} |\mathcal{H}'(\vartheta')| \, |U_s^n(x) - U_s^m(x)| \,\nu(dx),
\end{aligned} \tag{19}$$

with $\min\{U_s^n(x), U_s^m(x)\} \le \vartheta' \le \max\{U_s^n(x), U_s^m(x)\}$. Using ($\mathcal{H}4$) and the bound for U^n and U^m again, we estimate the integral from (19) by

$$\begin{aligned}
&K\int_{\mathbb{R}_0} |\mathcal{H}(U_s^n(x)) - \mathcal{H}(U_s^m(x))|\,\nu(dx)\\
\le &K\int_{\mathbb{R}_0} |c_{R'}\max\{|U_s^n(x)|, |U_s^m(x)|\}|\,|U_s^n(x) - U_s^m(x)|\,\nu(dx),\\
\le &K\int_{\mathbb{R}_0} c_{R'}\left(A_{D\xi}(x)e^{\int_s^T a(v)dv} + \int_s^T p(r,x)e^{\int_s^T a(v)dv}dr\right)|U_s^n(x) - U_s^m(x)|\,\nu(dx).
\end{aligned}$$

Finally, by the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}
&|\mathbf{f}^n(s, Y_s^n, Z_s^n, U_s^n) - \mathbf{f}^n(s, Y_s^m, Z_s^m, U_s^m)|\\
&\quad\le \left|\hat{\bar{f}}(s, Y_s^n, Z_s^n, \hat{G}(s, U_s^n)) - \hat{\bar{f}}(s, Y_s^m, Z_s^m, \hat{G}(s, U_s^m))\right|\\
&\qquad + KC_2\left(\|A_{D\xi}\|^2 + \int_0^T \|p(r,\cdot)\|^2 dr\right)^{\frac{1}{2}}\|U_s^n - U_s^m\|,
\end{aligned}$$

for $C_2 = C_2\Big(c_{R'}, \int_0^T a(v)dv\Big)$. Since $\hat{\bar{f}}$ and $\hat{G}$ satisfy a Lipschitz condition, the above inequality shows that also all $\mathbf{f}^n$ applied to (Y^n, Z^n, U^n) and (Y^m, Z^m, U^m) behave like Lipschitz functions with Lipschitz coefficients that do not depend on n or m.

Exploiting this property, very similar methods as the standard procedure used in [6, Proposition 2.2] show that there exists a constant $C > 0$ (only dependent on the Lipschitz coefficients of $\hat{\bar{f}}$) such that

$$\begin{aligned}
&\|Y^n - Y^m\|_{\mathcal{S}_2}^2 + \|Z^n - Z^m\|_{L_2(W)}^2 + \|U^n - U^m\|_{L_2(\tilde{N})}^2\\
&\le\ C\mathbb{E}\int_0^T \bigg|\varphi\left(\bar{f}(s, Y_s^n, Z_s^n, G(s, U_s^n)), \int_{\mathbb{R}_0}\mathcal{H}^n(U_s^n(x), x)\nu(dx)\right)\\
&\qquad -\varphi\left(\bar{f}(s, Y_s^n, Z_s^n, G(s, U_s^n)), \int_{\mathbb{R}_0}\mathcal{H}^m(U_s^n(x), x)\nu(dx)\right)\bigg|^2 ds. \qquad (20)
\end{aligned}$$

The mean value theorem applied to the second variable of φ helps to estimate the latter term by

$$\begin{aligned}
&C\mathbb{E}\int_0^T \left|\partial_w\varphi\left(\bar{f}(s, Y_s^n, Z_s^n, G(s, U_s^n)), \vartheta\right)\right|^2\\
&\qquad\times\left(\int_{\mathbb{R}_0}|\mathcal{H}(U_s^n(x))\min\{1, n|x|\} - \mathcal{H}(U_s^n(x))\min\{1, m|x|\}|\,\nu(dx)\right)^2 ds\\
=\ &C\mathbb{E}\int_0^T \left|\partial_w\varphi\left(\bar{f}(s, Y_s^n, Z_s^n, G(s, U_s^n)), \vartheta\right)\right|^2 \qquad (21)\\
&\times\left(\int_{\mathbb{R}_0}|\mathcal{H}(U_s^n(x))||\min\{1, n|x|\} - \min\{1, m|x|\}|\,\nu(dx)\right)^2 ds.
\end{aligned}$$

As in (18)-(19) above, we continue to estimate inequalities (20) and (21) similarly by

$$\|Y^n - Y^m\|_{\mathcal{S}_2}^2 + \|Z^n - Z^m\|_{L_2(W)}^2 + \|U^n - U^m\|_{L_2(\tilde{N})}^2$$

$$\leq \quad CK^2 \int_0^T \Bigg(\int_{\mathbb{R}_0} \left| c_{R'} \left(A_{D\xi}(x) e^{\int_s^T a(v)dv} + \int_s^T p(r,x) e^{\int_s^r a(v)dv} dr \right)^2 \right| \times |\min\{1, n|x|\} - \min\{1, m|x|\}| \nu(dx) \Bigg)^2 ds \tag{22}$$

$$\leq \quad CK^2 T C_1 \left(\|A_{D\xi}\|^2 + \int_0^T \|p(r,\cdot)\|^2 dr \right)^2 < \infty, \tag{23}$$

where $C_1 = C_1\Big(c_{R'}, \int_0^T a(v)dv\Big)$. The last estimate allows us to use dominated convergence. Hence,

$$\|Y^n - Y^m\|_{\mathcal{S}_2}^2 + \|Z^n - Z^m\|_{L_2(W)}^2 + \|U^n - U^m\|_{L_2(\tilde{N})}^2 \to 0, \quad \text{as} \quad m, n \to \infty,$$

because of $\lim_{n,m\to\infty} |\min\{1, n|x|\} - \min\{1, m|x|\}| = 0$ for all x. This proves the existence of the limits

$$Y^n \xrightarrow[\mathcal{S}_2]{} Y, \quad Z^n \xrightarrow[L_2(W)]{} Z, \quad U^n \xrightarrow[L_2(\tilde{N})]{} U.$$

Note that the triplet (Y, Z, U) obeys the same bounds as all (Y^n, Z^n, U^n) do.

Step 2

It remains to show that (Y, Z, U) indeed solves the BSDE given by $(\xi, \mathbf{f})$:
By the convergence of (Y^n, Z^n, U^n) to (Y, Z, U), we know that for all $t \in [0, T]$

$$Y_t^n \xrightarrow[L_2]{} Y_t, \quad \int_t^T Z_s^n dW_s \xrightarrow[L_2]{} \int_t^T Z_s dW_s,$$
$$\int_{]t,T]\times\mathbb{R}_0} U_s^n(x) \tilde{N}(ds, dx) \xrightarrow[L_2]{} \int_{]t,T]\times\mathbb{R}_0} U_s(x) \tilde{N}(ds, dx),$$

so three terms of the BSDE of $(\xi, \mathbf{f}^n)$ already converge to the respective terms of the one given by $(\xi, \mathbf{f})$.
The last term which needs to converge to the right limit is $\int_t^T \mathbf{f}(s, Y^n, Z^n, U^n)ds$. Therefore consider

$$\left| \int_t^T \mathbf{f}^n(s, Y_s^n, Z_s^n, U_{s,\cdot}^n) ds - \int_t^T \mathbf{f}(s, Y_s, Z_s, U_{s,\cdot}) ds \right|$$
$$\leq \int_0^T \left| \bar{f}(s, Y_s^n, Z_s^n, G(s, U_s^n)) - \bar{f}(s, Y_s, Z_s, G(s, U_s)) \right| ds$$
$$+ K \int_0^T \int_{\mathbb{R}_0} |\mathcal{H}^n(U_s^n(x), x) - \mathcal{H}(U_s(x))| \, \nu(dx) ds,$$

where the constant K is chosen in the same way as in the previous step, replacing $\mathcal{H}^m, U^m$ by $\mathcal{H}, U$. Having (Y^n, Z^n, U^n) and (Y, Z, U) estimated by the same bounds for all $n \in \mathbb{N}$, like in Step 4 of the proof of Theorem 3.4, $\bar{f}$ acts as Lipschitz function and yields a constant C with

$$\int_0^T \left| \bar{f}(s, Y_s^n, Z_s^n, G(s, U_s^n)) - \bar{f}(s, Y_s, Z_s, G(s, U_s)) \right| ds$$
$$\leq C \int_0^T (|Y_s^n - Y_s| + |Z_s^n - Z_s| + \|U_s^n - U_s\|) \, ds \xrightarrow[L_2]{n\to\infty} 0.$$

The only term left is now

$$\int_0^T \int_{\mathbb{R}_0} |\mathcal{H}^n(U_s^n(x), x) - \mathcal{H}(U_s(x))| \, \nu(dx)ds =$$
$$\int_0^T \int_{\mathbb{R}_0} |\mathcal{H}(U_s^n(x)) \min\{1, n|x|\} - \mathcal{H}(U_s(x))| \, \nu(dx)ds,$$

which approaches zero by a similar dominated convergence argument as in the inequalities (20) and (22) replacing $\min\{1, m|x|\}$ by 1. Thus (Y, Z, U) solves the BSDE $(\xi, \mathbf{f})$.

Step 3

This final step shows uniqueness of solutions to this equation in the class $\mathcal{S}_\infty \times L_\infty(W) \times L_{2,b}(\tilde{N})$. Let $(Y^j, Z^j, U^j), j = 1, 2$ be two solution to the BSDE $(\xi, \mathbf{f})$ with bounds $(R^j, Q^j, 2R^j)$ as in (9), (10) and (11), and assume that $A^j \in L_2(\nu)$ such that $|U_s^j(x)| \leq A^j(x)$ for the respective solution processes.

We start, similarly to the last step (and to Step 4 from the proof of Theorem 3.4), to consider $\bar{f}$ as a Lipschitz function. We look at the difference

$$\left|\varphi\left(\bar{f}(s, Y_s^1, Z_s^1, U_s^1), \int_{\mathbb{R}_0} \mathcal{H}(U_s^1(x))\nu(dx)\right) - \varphi\left(\bar{f}(s, Y_s^2, Z_s^2, U_s^2), \int_{\mathbb{R}_0} \mathcal{H}(U_s^2(x))\nu(dx)\right)\right|$$

and estimate it by

$$\left|\bar{f}(s, Y_s^1, Z_s^1, U_s^1) - \bar{f}(s, Y_s^2, Z_s^2, U_s^2)\right| + K \int_{\mathbb{R}_0} \left|\mathcal{H}(U_s^1(x)) - \mathcal{H}(U_s^2(x))\right| \nu(dx).$$

The constant K is chosen similarly as in Step 1, here using the bounds for U^1 and U^2. Assumption $(\mathcal{H}4)$ and the mean value theorem now imply that the last term is smaller than

$$\begin{aligned} &\left|\bar{f}(s, Y_s^1, Z_s^1, U_s^1) - \bar{f}(s, Y_s^2, Z_s^2, U_s^2)\right| \\ &+ K \int_{\mathbb{R}_0} c_{2R^1 \vee 2R^2} \left(|U_s^1(x)| + |U_s^2(x)|\right) \left|U_s^1(x) - U_s^2(x)\right| \nu(dx). \end{aligned}$$

By the bounds A^1, A^2, and the Cauchy-Schwarz inequality, we arrive at the inequality

$$\begin{aligned} &\left|\bar{f}(s, Y_s^1, Z_s^1, U_s^1) - \bar{f}(s, Y_s^2, Z_s^2, U_s^2)\right| \\ &+ K \int_{\mathbb{R}_0} c_{2R^1 \vee 2R^2} \left(A^1(x) + A^2(x)\right) \left|U_s^1(x) - U_s^2(x)\right| \nu(dx) \\ &\leq \left|\bar{f}(s, Y_s^1, Z_s^1, U_s^1) - \bar{f}(s, Y_s^2, Z_s^2, U_s^2)\right| + K c_{2R^1 \vee 2R^2} \|A^1 + A^2\| \left\|U_s^1 - U_s^2\right\|, \end{aligned}$$

which shows that the standard procedure for Lipschitz generators (e.g. the one from [18, Proposition 4.2]) is applicable. The uniqueness of the solution then follows. □

Remark 4.3. *The setting of Theorem 4.2 contains the example*

$$\mathcal{H}_\alpha(u) := \frac{e^{\alpha u} - \alpha u - 1}{\alpha}$$

for some fixed $\alpha > 0$. This type of generators appears in BSDEs related to utility optimisation, see the work of Morlais [23] and Becherer et al. [8].

5 Locally Lipschitz BSDEs and PDIEs

In this section we apply our results to the correspondence between Lévy-driven forward-backward SDEs with locally Lipschitz generators and partial differential-integral equations (PDIEs). In the case of Lipschitz generators this has been investigated before e.g. in [6] and [24], and in the Brownian setting for locally Lipschitz generators in [11]. We recall the setting of [6] (in a one-dimensional version) but relax the Lipschitz condition from there to a local Lipschitz condition:

Assume a generator function of the type

$$\mathbf{f}(\omega, t, y, z, u) = f(\Psi_t(\omega), t, y, z, u),$$

with

(i) f is Lipschitz in y and locally Lipschitz in (z, u),

(ii) f is non-decreasing in u,

(iii) f has a continuous partial derivative in the first variable bounded by $K > 0$, which is also locally Lipschitz in (y, z, u).

Here, $(\Psi^t(v))_{(t,v)\in[0,T]\times\mathbb{R}}$ denotes a family of forward processes given by the SDEs

$$d\Psi^t_s(v) = b(\Psi^t_s(v))ds + \sigma(\Psi^t_s(v))dW_s + \beta(\Psi^t_{s-}(v), x)\tilde{N}(ds, dx), \quad s \in [t, T],$$

with $\Psi^t_t = v$ and the following requirements:

(iv) The functions $b\colon \mathbb{R} \to \mathbb{R}$ and $\sigma\colon \mathbb{R} \to \mathbb{R}$ are continuously differentiable with bounded derivative.

(v) $\beta\colon \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$ is measurable, satisfies

$$|\beta(\psi, x)| \le C_\beta(1 \wedge |x|), \quad (\psi, x) \in \mathbb{R} \times \mathbb{R}_0$$

and is continuously differentiable in ψ with bounded derivative for fixed $x \in \mathbb{R}_0$.

Define the partial differential-integral operator

$$\begin{aligned} -\partial_t u(t, v) - \mathfrak{L}u(t, v) - \tilde{f}(v, t, u(t, v), \partial_v u(t, v), \mathfrak{B}u(t, v)) &= 0, \quad (t, v) \in [0, T] \times \mathbb{R}, \\ u(T, v) &= \mathfrak{g}(v), \quad v \in \mathbb{R}, \end{aligned} \tag{24}$$

for a bounded, Lipschitz function $\mathfrak{g}$ and the operator $\mathfrak{L} = \mathfrak{A} + \mathfrak{K}$ given by

$$\begin{aligned} \mathfrak{A}\varphi(v) &= \frac{\sigma^2(v)}{2}\partial_v^2\varphi(v) + b(v)\partial_v\varphi(v), \\ \mathfrak{K}\varphi(v) &= \int_{\mathbb{R}_0} \left(\varphi(v + \beta(v, x)) - \varphi(v) - \beta(v, x)\partial_v\varphi(v)\right)\nu(dx), \end{aligned}$$

and the integral operator $\mathfrak{B}$ given by

$$\mathfrak{B}\varphi(v) = \int_{\mathbb{R}_0} \left(\varphi(v + \beta(v, x)) - \varphi((v)\right)\kappa(x)\nu(dx).$$

Denote by $(Y^t(v), Z^t(v), U^t(v,x))$ the solution to the BSDE

$$Y_s^t(v) = \mathfrak{g}(\Psi_T^t(v)) + \int_s^T f\left(\Psi_r^t(v), Y_r^t(v), Z_r^t(v), \int_{\mathbb{R}_0} U_r^t(v,x)\kappa(x)\nu(dx)\right) dr$$
$$- \int_s^T Z_r^t(v)dr - \int_{]r,T]\times\mathbb{R}_0} U_r^t(v,x)\tilde{N}(ds,dx), \quad s \in [t,T], \tag{25}$$

for $(t,v) \in [0,T] \times \mathbb{R}$.
For the notion of a viscosity solution we refer to [6, Section 3]. The following theorem is a one-dimensional version of [6, Theorem 3.4], where the generator may be locally Lipschitz in (z,u).

Theorem 5.1. *With the assumptions from this section, and the additional conditions (*$\mathcal{A}2$*), (*$\mathcal{A}3$*), (*$\mathcal{A}6$*) and*

$$\int_0^T \left(\int_{\mathbb{R}_0} \|(r,x) \mapsto D_{r,x}\Psi_s\|^2_{L^\infty(\lambda\otimes\mathbb{P})}\,\nu(dx)\right)^{1/2} ds < \infty \text{ and } \|(r,x) \mapsto D_{r,x}\Psi_T\|^2_{L^\infty(\lambda\otimes\mathbb{P})} \in L_2(\nu), \tag{26}$$

the function

$$u(t,v) = Y_t^t(v), \quad (t,v) \in [0,T] \times \mathbb{R},$$

with $Y_t^t(v)$ *given by* (25) *is a viscosity solution to the PDIE* (24).

Proof. By the consequences of Theorem 3.4, the solution $(Y^t(v), Z^t(v), U^t(v,x))$ may be regarded as solution to a Lipschitz BSDE provided that f satisfies ($\mathcal{A}1$)-($\mathcal{A}8$). We show that this is indeed the case.
The boundedness condition (26) on $D\Psi$ and the Lipschitz continuity and boundedness property of $\mathfrak{g}$ imply that ($\mathcal{A}1$) is satisfied. Moreover, by the boundedness and differentiability assumption (in the first variable) on f, ($\mathcal{A}5$) is satisfied. The conditions ($\mathcal{A}2$), ($\mathcal{A}3$), ($\mathcal{A}6$) were assumed and since f is Lipschitz in y and locally Lipschitz in (z,u), also ($\mathcal{A}4$) is satisfied.
In the present generator, the function g is given by $g(t,u) = u$, thus f) is readily checked and ($\mathcal{A}8$) follows as f increases in u.
Having verified ($\mathcal{A}1$)-($\mathcal{A}8$), we may now assume that $Y^t(v)$ is a bounded solution to a BSDE with a Lipschitz generator fitting the assumptions of [6, Theorem 3.4], which in turn guarantees that $(t,v) \mapsto Y_t^t(v)$ solves (24). □

Remark 5.2. *The conditions on* f *in this section were taken from [6] and may surely be relaxed for Theorem 5.1 to hold. The proof of [6, Theorem 3.4] relies on the comparison theorem for BSDEs, which is valid in a more general setting than required in [6], see e.g. Theorem 2.4. Note also that the function* $\kappa(x) = 1 \wedge |x|$ *can be generalised like in [6] to a function depending also on the forward process,* $\kappa(\Psi_t(\omega), x)$*, under suitable conditions.*

We finish with an example where (26) is satisfied for $D_{r,x}\Psi$ with $x = 0$. For $x \neq 0$ it follows from Lemma 2.1 and Lemma 2.2 that $D_{r,x}\Psi$ is bounded if Ψ is.

Example 5.3. *We have for $t \leq r \leq s \leq T$ that*

$$\begin{aligned} D_{r,0}\Psi_s &= \sigma(\Psi_r) + \int_r^s b'(\Psi_u) D_{r,0}\Psi_u du + \int_r^s \sigma'(\Psi_u) D_{r,0}\Psi_u dW_u \\ &\quad + \int_{]r,s]\times\mathbb{R}_0} \partial_\psi \beta(\Psi_{u-}^t, x) D_{r,0}\Psi_u \tilde{N}(du,dx) \end{aligned}$$

which implies that

$$\begin{aligned} D_{r,0}\Psi_s &= \sigma(\Psi_r) \exp\left(\int_r^s b'(\Psi_u)du + \int_r^s \sigma'(\Psi_u)dW_u - \frac{1}{2}\int_r^s (\sigma'(\Psi_u))^2 du \right) \\ &\quad \times \prod_{r<u\leq s} \big(1 + \partial_\psi \beta(\Psi_{u-}, \Delta L_u)\big) \exp\left(-\int_r^s \int_{\mathbb{R}_0} \partial_\psi \beta(\Psi_u, x)\nu(dx)du \right). \end{aligned} \tag{27}$$

We first check conditions to have the stochastic integral $\int_r^s \sigma'(\Psi_u)dW_u$ bounded from above. By Itô's formula,

$$\begin{aligned} \log(|\sigma(\Psi_s)|) &= \log(|\sigma(\Psi_r)|) + \int_r^s \sigma'(\Psi_u)dW_u + \int_r^s \frac{\sigma'(\Psi_u)}{\sigma(\Psi_u)} b(\Psi_u)du \\ &\quad + \frac{1}{2}\int_r^s \sigma(\Psi_u)\sigma''(\Psi_u) - (\sigma'(\Psi_u))^2 du \\ &\quad + \int_{]r,s]\times\mathbb{R}_0} [\log(|\sigma(\Psi_{u-}) + \beta(\Psi_{u-},x)|) - \log(|\sigma(\Psi_{u-})|)]\tilde{N}(du,dx) \\ &\quad + \int_r^s \int_{\mathbb{R}_0} \Big[\log(|\sigma(\Psi_{u-}) + \beta(\Psi_{u-},x)|) - \log(|\sigma(\Psi_{u-})|) \\ &\qquad - \frac{\sigma'(\Psi_u)}{\sigma(\Psi_u)} \beta(\Psi_{u-},x) \Big] \nu(dx)du \end{aligned}$$

so that

$$\begin{aligned} \int_r^s \sigma'(\Psi_u)dW_u &= \log\left(\frac{|\sigma(\Psi_s)|}{|\sigma(\Psi_r)|} \right) \\ &\quad + \int_r^s \left[\frac{(\sigma'(\Psi_u))^2}{2} - \frac{\sigma'(\Psi_u)b(\Psi_u)}{\sigma(\Psi_u)} - \frac{1}{2}\sigma(\Psi_u)\sigma''(\Psi_u) \right] du \\ &\quad + \int_r^s \int_{\mathbb{R}_0} \frac{\sigma'(\Psi_u)}{\sigma(\Psi_u)} \beta(\Psi_u,x)\nu(dx)du \\ &\quad - \log\Big(\prod_{r<u\leq s} \Big| 1 + \frac{\beta(\Psi_{u-}^t, \Delta L_u)}{\sigma(\Psi_{u-})} \Big| \Big), \end{aligned}$$

which is bounded from above if

* $c^{-1} \leq \sigma(\Psi_u) \leq c$ *for all* $u \in [t,T]$, *for some* $c > 0$,

* $\frac{(\sigma'(\Psi_u))^2}{2} - \frac{\sigma'(\Psi_u)b(\Psi_u)}{\sigma(\Psi_u)} - \frac{1}{2}\sigma(\Psi_u)\sigma''(\Psi_u) + \int_{\mathbb{R}_0} \frac{\sigma'(\Psi_u)}{\sigma(\Psi_u)} \beta(\Psi_u,x)\nu(dx) \leq c'$ *for all* $u \in [t,T]$, *for some* $c' > 0$,

* $\beta \geq 0$.

Additional conditions that guarantee the boundedness of the right hand side of equation (27) are

* $b'(\Psi_u) - \frac{(\sigma'(\Psi_u))^2}{2} \leq \tilde{c}$ *for all* $u \in [t,T]$, *for some* $\tilde{c} > 0$,

* $-1 < \partial_\psi \beta(\Psi_u, x) \leq 0$ *for all* $u \in [t,T]$,

* $-\int_{\mathbb{R}} \partial_\psi \beta(\Psi_u, x)\nu(dx) \leq \tilde{c}'$, *for all* $u \in [t,T]$, *for some* $\tilde{c}' > 0$.

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A Appendix

Malliavin differentiability for Lipschitz generators

For the terminal value ξ and the function $\mathbf{f}$ with

$$\mathbf{f}(\omega, t, y, z, \mathbf{u}) = f\left(X(\omega), t, y, z, \int_{\mathbb{R}_0} g(s, \mathbf{u}(x))\kappa(x)\nu(dx)\right)$$

we agree upon the following assumptions:

$(\mathbf{A}_\xi)$ $\xi \in \mathbb{D}_{1,2}$.

$(\mathbf{A}_f)$ *a)* $f\colon D[0,T] \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is jointly measurable, adapted to $(\mathcal{G}_t)_{t\in[0,T]}$ defined in (5), and for all $t \in [0,T]$ and $i = 1,2,3$, $\exists\ \partial_{\eta_i} f(\mathtt{x}, t, \eta)$, and the functions

$$\mathbb{R}^3 \ni \eta \mapsto \partial_{\eta_i} f(\mathtt{x}, t, \eta)$$

are continuous.

b) There exist functions $k_f \in L_1([0,T])$, $K_f \in L_2(W)$, such that

$$|f(X, t, 0, 0, 0)| \leq k_f(t) + K_f(t), \quad \mathbb{P}\text{-a.s.}$$

c) f satisfies the following Lipschitz condition: There exist nonnegative functions $a \in L_1([0,T]), b \in L_2([0,T])$ such that for all $t \in [0,T]$, $(y,z,u), (\tilde{y}, \tilde{z}, \tilde{u}) \in \mathbb{R}^3$

$$|f(\mathtt{x}, t, y, z, u) - f(\mathtt{x}, t, \tilde{y}, \tilde{z}, \tilde{u})| \leq a(t)|y - \tilde{y}| + b(t)(|z - \tilde{z}| + |u - \tilde{u}|).$$

d) Assume there is a nonnegative random field $\Gamma \in \mathrm{L}_2(\lambda \otimes \mathbb{P} \otimes \mathbb{m})$, and a nonnegative $\rho^D \in L_2(\mathbb{m})$ such that for all random vectors $G = (G_1, G_2, G_3) \in (\mathrm{L}_2)^3$ and for a.e. t it holds

$$|(D_{s,x}f)\,(t, G)| \leq \Gamma_{s,x}(t) + \rho^D_{s,x}|G|, \quad \mathbb{P} \otimes \mathbb{m}\text{-a.e.}$$

where $(D_{s,x}f)\,(t, G) := D_{s,x}f(X, t, \eta)\,|_{\eta=G}$.

e) $f(X, t, \eta) \in \mathbb{D}_{1,2}$ for all $(t, \eta) \in [0, T] \times \mathbb{R}^3$, and $\forall t \in [0, T]$, $\forall N \in \mathbb{N}\ \exists\, K^t_N \in \bigcup_{p>1} \mathrm{L}_p$ such that for a.a. ω

$$\begin{aligned}&\forall \eta, \tilde{\eta} \in B_N(0) :\\ &\|\,(D_{\cdot,0}f(X, t, \eta))\,(\omega) - (D_{\cdot,0}f(X, t, \tilde{\eta}))\,(\omega)\|_{L_2([0,T])} < K^t_N(\omega)\,|\eta - \tilde{\eta}|\,,\end{aligned}$$

where for $D_{\cdot,0}f(X, t, \eta)$ we always take a progressively measurable version in t.

f) $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is jointly measurable, $g(t, \cdot) \in \mathcal{C}^1(\mathbb{R})$ with $g(t, 0) = 0$, bounded derivative $|g'(t, \cdot)| \leq L_g$, and $\kappa \in \mathrm{L}_2(\nu)$.

For similar results on differentiability of BSDEs with jumps in the Lévy case, see [14], [13] or [17]. The following result generalises [17, Theorem 4.4] and – up to the time delay – [14, Theorem 4.1].

Theorem A.1. *Assume ($\mathbf{A}_\xi$) and ($\mathbf{A}_f$). Then the following assertions hold.*

(i) For $\mathbb{m}$- a.e. $(r, v) \in [0, T] \times \mathbb{R}$ there exists a unique solution $(\mathcal{Y}^{r,v}, \mathcal{Z}^{r,v}, \mathcal{U}^{r,v}) \in \mathcal{S}_2 \times L_2(W) \times L_2(\tilde{N})$ to the BSDE

$$\begin{aligned}\mathcal{Y}^{r,v}_t &= D_{r,v}\xi + \int_t^T F_{r,v}\,(s, \mathcal{Y}^{r,v}_s, \mathcal{Z}^{r,v}_s, \mathcal{U}^{r,v}_s)\,ds\\ &\quad - \int_t^T \mathcal{Z}^{r,v}_s dW_s - \int_{]t,T]\times\mathbb{R}_0} \mathcal{U}^{r,v}_{s,x}\tilde{N}(ds, dx), \quad 0 \leq r \leq t \leq T,\\ \mathcal{Y}^{r,v}_s &= \mathcal{Z}^{r,v}_s = \mathcal{U}^{r,v}_s = 0, \quad 0 \leq s < r \leq T,\end{aligned} \tag{28}$$

where $\Theta_s := (Y_s, Z_s, G(s, U_s))$ and

$$\begin{aligned}F_{r,0}(s, y, z, \mathbf{u}) \quad &:= \quad (D_{r,0}f)(s, \Theta_s) + \partial_y f(s, \Theta_s)y + \partial_z f(s, \Theta_s)z\\ &\qquad + \partial_u f(s, \Theta_s) \int_{\mathbb{R}_0} \partial_u g(s, U_s(v))\mathbf{u}(v)\kappa(v)\nu(dv),\end{aligned}$$

and for $v \neq 0$,

$$\begin{aligned}F_{r,v}(s, y, z, \mathbf{u}) \quad := \quad &(D_{r,v}f)(X, s, \Theta_s)\\ &+ f(X + v\mathbb{1}_{[r,T]}, s, \Theta_s + (y, z, G(s, U_s + \mathbf{u}))) - f(X + v\mathbb{1}_{[r,T]}, s, \Theta_s).\end{aligned}$$

(ii) For the solution (Y, Z, U) of (8) it holds

$$Y, Z \in L_2([0, T]; \mathbb{D}_{1,2}), \quad U \in L_2([0, T] \times \mathbb{R}_0; \mathbb{D}_{1,2}), \tag{29}$$

and $D_{r,y}Y$ admits a càdlàg version for $\mathbb{m}$- a.e. $(r, y) \in [0, T] \times \mathbb{R}$.

(iii) (DY, DZ, DU) *is a version of* $(\mathcal{Y}, \mathcal{Z}, \mathcal{U})$*, i.e. for* $\mathbb{m}$*- a.e.* (r, v) *it solves*

$$D_{r,v}Y_t = D_{r,v}\xi + \int_t^T F_{r,v}\left(s, D_{r,v}Y_s, D_{r,v}Z_s, D_{r,v}U_s\right) ds \\ - \int_t^T D_{r,v}Z_s dW_s - \int_{]t,T]\times\mathbb{R}_0} D_{r,v}U_s(x)\tilde{N}(ds, dx), \quad 0 \le r \le t \le T. \tag{30}$$

(iv) Setting $D_{r,v}Y_r(\omega) := \lim_{t\searrow r} D_{r,v}Y_t(\omega)$ *for all* (r, v, ω) *for which* $D_{r,v}Y$ *is càdlàg and* $D_{r,v}Y_r(\omega) := 0$ *otherwise, we have that*

$$\left((\mathbb{E}[D_{r,0}Y_r|\mathcal{F}_{r-}])_{r\in[0,T]}\right) \text{ is a version of } (Z_r)_{r\in[0,T]},$$
$$\left((\mathbb{E}[D_{r,v}Y_r|\mathcal{F}_{r-}])_{r\in[0,T],v\in\mathbb{R}_0}\right) \text{ is a version of } (U_r(v))_{r\in[0,T],v\in\mathbb{R}_0}.$$

Proof of Theorem A.1

Let us start with a lemma providing estimates for the Malliavin derivative of the generator.

Lemma A.2. *Let* $G = (G_1, G_2, G_3) \in (L_2)^3$ *and* $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in (L_2(\mathbb{P} \otimes \mathbb{m}))^3$*. If* f *satisfies* $(\mathbf{A}_f)$ *it holds for* $\mathbb{P} \otimes \mathbb{m}$*-a.a.* (ω, r, v), $v \neq 0$*, that*

$$|f(X + v\mathbb{1}_{[r,T]}, t, G + \Phi_{r,v}) - f(X, t, G)| \\ \le a(t)|\Phi_{1,r,v}| + b(t)(|\Phi_{2,r,v}| + |\Phi_{3,r,v}|) + \Gamma_{r,v}(t) + \rho^D_{r,v}|G|. \tag{31}$$

Moreover, for $G \in (\mathbb{D}_{1,2})^3$ *it holds* $f(X, t, G) \in \mathbb{D}_{1,2}$ *and*

$$|D_{r,v}f(X, t, G)| \le a(t)|D_{r,v}G_1| + b(t)(|D_{r,v}G_2| + |D_{r,v}G_3|) + \Gamma_{r,v}(t) + \rho^D_{r,v}|G|, \quad \mathbb{P} \otimes \mathbb{m}\text{-a.e.} \tag{32}$$

Proof. According to Lemma 2.2 we may replace X by $X + v\mathbb{1}_{[r,T]}$ and get from the Lipschitz property $(\mathbf{A}_f)$ c) that

$$\left|f(X + v\mathbb{1}_{[r,T]}, t, G + \Phi_{r,v}) - f(X + v\mathbb{1}_{[r,T]}, t, G)\right| \le a(t)|\Phi_{1,r,v}| + b(t)(|\Phi_{2,r,v}| + |\Phi_{3,r,v}|)$$

for $\mathbb{P} \otimes \mathbb{m}$-a.e. (ω, r, v) with $v \neq 0$. From $(\mathbf{A}_f)$ d) one concludes then (31).
For $v \neq 0$ we apply Lemma 2.1 to get

$$D_{r,v}f(X, t, G) = f(X + v\mathbb{1}_{[r,T]}, t, G + D_{r,v}G) - f(X, t, G),$$

and hence (32) follows from (31). In the case $v = 0$, [17, Theorem 3.12] implies that under the assumptions $(\mathbf{A}_f)$ a) and $(\mathbf{A}_f)$ e) the Malliavin derivative $D_{r,0}f(X, t, G)$ exists and it holds that

$$D_{r,0}f(X, t, G) = (D_{r,0}f)(t, G) + \partial_{\eta_1}f(X, t, G)D_{r,0}G_1 + \partial_{\eta_2}f(X, t, G)D_{r,0}G_2 \\ + \partial_{\eta_3}f(X, t, G)D_{r,0}G_3 \tag{33}$$

for $\mathbb{P} \otimes \lambda$-a.a. $(\omega, r) \in \Omega \times [0, T]$. Relation (32) follows from conditions $(\mathbf{A}_f)$ c) and d) using that the partial derivative $\partial_{\eta_1}f(X, t, \eta)$ is bounded by $a(t)$ and the derivatives $\partial_{\eta_i}f(X, t, \eta)$, $i = 2, 3$ are bounded by $b(t)$. □

Proof of Theorem A.1. The core of the proof is to conclude assertion (ii) which is done by an iteration argument. To simplify the notation, in most places we do not mention the dependency of f on X.

(i) For those (r, v) such that $D_{r,v}\xi \in L_2$ the existence and uniqueness of a solution $(\mathcal{Y}^{r,v}, \mathcal{Z}^{r,v}, \mathcal{U}^{r,v})$ to (28) follows from Theorem 2.3 since $F_{r,v}$ meets the assumptions of the theorem.
By the a priori estimate shown in [18, Proposition 4.2], a solution to a BSDE satisfying (H 1) - (H 3) depends continuously on the terminal condition, i.e. the mapping

$$L_2 \to L_2(W) \times L_2(W) \times L_2(\tilde{N}) \colon \zeta \mapsto (\mathcal{Y}^\zeta, \mathcal{Z}^\zeta, \mathcal{U}^\zeta)$$

is continuous. The existence of a jointly measurable version of

$$(\mathcal{Y}^{r,v}, \mathcal{Z}^{r,v}, \mathcal{U}^{r,v}), \quad (r, v) \in [0, T] \times \mathbb{R}$$

follows then by approximating $D\xi$ (which is measurable w.r.t. (r, v).) by simple functions in $L_2(\mathbb{P} \otimes \mathbb{m})$. Joint measurability (for example for $\mathcal{Z}$) in all arguments can be gained by identifying the spaces

$$L_2(\lambda, L_2(\mathbb{P} \otimes \mathbb{m})) \cong L_2(\lambda \otimes \mathbb{P} \otimes \mathbb{m}).$$

The quadratic integrability with respect to (r, v) also follows from [18, Proposition 4.2] since $\xi \in \mathbb{D}_{1,2}$.
(ii) We use the iteration scheme introduced in [25]. Starting in our setting with $(Y^0, Z^0, U^0) = (0, 0, 0)$, we get Y^{n+1} by taking the optional projection which implies that $\mathbb{P}$-a.s.

$$Y_t^{n+1} = \mathbb{E}_t \left(\xi + \int_t^T f\left(s, Y_s^n, Z_s^n, G(s, U_s^n)\right) ds \right). \tag{34}$$

The processes Z^{n+1}, U^{n+1} one gets by the martingale representation theorem w.r.t. $dW_s + N(ds, dx)$ (see, for example, [4]):

$$\begin{aligned} &\xi + \int_0^T f\left(s, Y_s^n, Z_s^n, G(s, U_s^n)\right) ds \\ &= \mathbb{E}\left(\xi + \int_0^T f\left(s, Y_s^n, Z_s^n, G(s, U_s^n)\right) ds \right) + \int_0^T Z_s^{n+1} dW_s + \int_{]0,T]\times\mathbb{R}_0} U_{s,x}^{n+1} \tilde{N}(ds, dx). \end{aligned} \tag{35}$$

Step 1.
In this first step we will show convergence of the so defined sequence $(Y^n, Z^n, U^n) \to (Y, Z, U)$ in $L_2(W) \times L_2(W) \times L_2(\tilde{N})$.
Equations (34) and (35) mean that

$$Y_t^{n+1} = \xi + \int_t^T f\left(s, Y_s^n, Z_s^n, G(s, U_s^n)\right) ds - \int_t^T Z_s^{n+1} dW_s - \int_{]t,T]\times\mathbb{R}_0} U_{s,x}^{n+1} \tilde{N}(ds, dx),$$

which can be considered as BSDE with a generator not depending on the y, z and u variables.
With $\Delta Y^{n+1} = Y^{n+2} - Y^{n+1}$ and $\Delta Y^n = Y^{n+1} - Y^n$ and similar notations for the Z and U processes, we get, with the help of Itô's formula ($\gamma \in L_2([0, T])$ will be determined later)

$$e^{\int_0^t \gamma(s) ds} \left|\Delta Y_t^{n+1}\right|^2 + \int_t^T e^{\int_0^s \gamma(\tau) d\tau} \left(\gamma(s) \left|\Delta Y_s^{n+1}\right|^2 + |\Delta Z_s^{n+1}|^2 + \|U_s^{n+1}\|^2 \right) ds$$

$$= \int_t^T e^{\int_0^s \gamma(\tau)d\tau} 2\Delta Y_s^{n+1} \left(f(s, Y^{n+1}, Z_s^{n+1}, G(s, U_s^{n+1})) - f(s, Y_s^n, Z_s^n, G(s, U_s^n)) \right) ds \quad (36)$$
$$- M(t).$$

In this equation, $M(t)$ consists of the stochastic integrals

$$\int_t^T e^{\int_0^s \gamma(\tau)d\tau} 2\Delta Y_s^{n+1} \Delta Z_s^{n+1} dW_s + \int_{]t,T]\times\mathbb{R}_0} \left((\Delta U_s^{n+1} + \Delta Y_s^{n+1})^2 - |\Delta U_s^{n+1}|^2 \right) \tilde{N}(ds, dx).$$

By a standard procedure (see [18, Proposition 4.1] for the present setting), one concludes from $(Y^n, Z^n, U^n) \in L_2(W) \times L_2(W) \times L_2(\tilde{N})$ that $Y^{n+1} \in \mathcal{S}_2$. This fact, together with the Burkholder-Davis-Gundy inequality implies that $\mathbb{E}M(t) = 0$.
We use conditions $(\mathbf{A_f})$ c) and f) and apply Young's inequality to the resulting terms to get

$$\begin{aligned}
&2|\Delta Y_s^{n+1}|\, |f(s, Y^{n+1}, Z_s^{n+1}, G(s, U_s^{n+1})) - f(s, Y_s^n, Z_s^n, G(s, U_s^n))| \\
&\le 2|\Delta Y_s^{n+1}| \left(a(s)|\Delta Y_s^n| + (1 \vee L_g\|\kappa\|) b(s)(|\Delta Z_s^n| + \|\Delta U_s^n\|) \right) \\
&\le \left(2\left(a(s) + 2(1 \vee L_g\|\kappa\|)^2 b(s)^2 \right) |\Delta Y_s^{n+1}|^2 + \frac{a(s)}{2} |\Delta Y_s^n|^2 + \frac{|\Delta Z_s^n|^2 + \|\Delta U_s^n\|^2}{2} \right).
\end{aligned}$$

The right hand side increases if we replace the factor $\frac{a(s)}{2}$ before $|\Delta Y_s^n|^2$ by $\frac{a(s)+1}{2}$. Thus (36), after using this inequality and taking expectations turns into

$$\begin{aligned}
&\mathbb{E} e^{\int_0^t \gamma(s)ds} \left| \Delta Y_t^{n+1} \right|^2 + \mathbb{E} \int_t^T e^{\int_0^s \gamma(\tau)d\tau} \left(\gamma(s) \left| \Delta Y_s^{n+1} \right|^2 + |\Delta Z_s^{n+1}|^2 + \|U_s^{n+1}\|^2 \right) ds \\
&\le \mathbb{E} \int_t^T e^{\int_0^s \gamma(\tau)d\tau} \Big(2(a(s) + (1 \vee L_g\|\kappa\|)^2 b(s)^2) |\Delta Y_s^{n+1}|^2 \\
&\qquad + \frac{a(s)+1}{2} |\Delta Y_s^n|^2 + \frac{|\Delta Z_s^n|^2 + \|\Delta U_s^n\|^2}{2} \Big) ds.
\end{aligned}$$

Setting $\gamma = 1 + 3a + 2(1 \vee L_g\|\kappa\|)^2 b^2$ and omitting the first term of the inequality, we have for $t = 0$ that

$$\begin{aligned}
&\mathbb{E} \int_0^T e^{\int_0^s \gamma(\tau)d\tau} \left((1 + a(s)) \left| \Delta Y_s^{n+1} \right|^2 + |\Delta Z_s^{n+1}|^2 + \|U_s^{n+1}\|^2 \right) ds \\
&\le \frac{1}{2} \mathbb{E} \int_0^T e^{\int_0^s \gamma(\tau)d\tau} \left((1 + a(s)) |\Delta Y_s^n|^2 + |\Delta Z_s^n|^2 + \|\Delta U_s^n\|^2 \right) ds.
\end{aligned}$$

The last inequality states that the sequence $(Y^n, Z^n, U^n)_{n\ge 0}$ is subject to a contraction in the Banach space of all $(\bar{y}, \bar{z}, \bar{u}) \in L_2(W) \times L_2(W) \times L_2(\tilde{N})$, such that

$$\|(\bar{y}, \bar{z}, \bar{u})\|_{1+a,\gamma}^2 := \left\| e^{\int_0^\cdot \gamma(\tau)d\tau} (1 + a)\bar{y} \right\|_{L_2(W)}^2 + \left\| e^{\int_0^\cdot \gamma(\tau)d\tau} \bar{z} \right\|_{L_2(W)}^2 + \left\| e^{\int_0^\cdot \gamma(\tau)d\tau} \bar{u} \right\|_{L_2(\tilde{N})}^2 < \infty.$$

This norm is stronger than $\sqrt{\|\cdot\|_{L_2(W)}^2 + \|\cdot\|_{L_2(W)}^2 + \|\cdot\|_{L_2(\tilde{N})}^2}$ on this space, hence the Picard iteration converges to the unique fixed point (Y, Z, U).

Step 2.

Our aim in this step is to show that Y^n, Z^n and U^n are uniformly bounded in n as elements of

$L_2(\lambda;\mathbb{D}_{1,2})$ and $L_2(\lambda\otimes\nu;\mathbb{D}_{1,2})$, respectively. This will follow from (39) below.
We recall the notation for M and $\mathbb{m}$ from (4) and define for $n\geq 0$,

$$\underline{Z}^n_{t,x}=\begin{cases} Z^n_t, & x=0,\\ U^n_t(x), & x\neq 0.\end{cases}$$

Given that $Y^n, Z^n\in L_2(\lambda;\mathbb{D}_{1,2})$ and $U^n\in L_2(\lambda\otimes\nu;\mathbb{D}_{1,2})$ one can infer that this also holds for $n+1$: Indeed, $(\mathbf{A}_f)$ $f)$ implies that $G(s,U^n_s)\in\mathbb{D}_{1,2}$ for a.e. s and

$$|D_{r,v}G(s,U^n_s)|\leq L_g\|\kappa\|\|D_{r,v}U^n_s\|. \tag{37}$$

From [17, Theorem 3.12] and Lemma A.2 we conclude that $f(X,s,Y^n_s,Z^n_s,G(s,U^n_s))\in\mathbb{D}_{1,2}$. The above estimate and (32) as well as the Malliavin differentiation rules shown by Delong and Imkeller in [14, Lemma 3.1. and Lemma 3.2.] imply that Y^{n+1} as defined in (34) is in $L_2(\lambda;\mathbb{D}_{1,2})$. Then we conclude that both stochastic integrals in (35) are in $\mathbb{D}_{1,2}$ and [14, Lemma 3.3.] implies that for the corresponding integrals one has $Z^{n+1}\in L_2(\lambda;\mathbb{D}_{1,2})$ and $U^{n+1}\in L_2(\lambda\otimes\nu;\mathbb{D}_{1,2})$. Especially, we get for $t\in[0,T]$ that $\mathbb{P}$ -a.e.

$$\begin{aligned} D_{r,v}Y^{n+1}_t &= D_{r,v}\xi+\int_t^T D_{r,v}f\left(X,s,Y^n_s,Z^n_s,G(s,U^n_s)\right)ds\\ &\quad-\int_{]t,T]\times\mathbb{R}} D_{r,v}\underline{Z}^{n+1}_{s,x}M(ds,dx),\ \text{for } \mathbb{m}\text{ - } a.a.\ (r,v)\in[0,t]\times\mathbb{R},\\ D_{r,v}Y^{n+1}_t &= 0 \quad \text{for } \mathbb{m}\text{ - } a.a.\ (r,v)\in(t,T]\times\mathbb{R},\\ D_{r,v}\underline{Z}^{n+1}_{t,x} &= 0 \quad \text{for } \mathbb{m}\otimes\mu\text{ - } a.a.\ (r,v,x)\in(t,T]\times\mathbb{R}^2. \end{aligned} \tag{38}$$

Since by [4, Theorem 4.2.12] the process $\left(\int_{]0,t]\times\mathbb{R}} D_{r,v}\underline{Z}^{n+1}_{s,x}M(ds,dx)\right)_{t\in[0,T]}$, admits a càdlàg version, we may take a càdlàg version of both sides.
By Itô's formula, we conclude that for $0<r<t$ and $\beta\in L_1([0,T])$ it holds

$$\begin{aligned} e^{\int_0^T\beta(s)ds}(D_{r,v}\xi)^2 = &\ e^{\int_0^t\beta(s)ds}(D_{r,v}Y^{n+1}_t)^2+\int_t^T\beta(s)e^{\int_0^s\beta(\tau)d\tau}(D_{r,v}Y^{n+1}_s)^2ds\\ &-2\int_t^T e^{\int_0^s\beta(\tau)d\tau}\left[D_{r,v}f\left(X,s,Y^n_s,Z^n_s,G(s,U^n_s)\right)\right]D_{r,v}Y^{n+1}_s ds\\ &+\int_{]t,T]\times\mathbb{R}} e^{\int_0^s\beta(\tau)d\tau}[2(D_{r,v}Y^{n+1}_{s-})D_{r,v}\underline{Z}^{n+1}_{s,x}\\ &\qquad\qquad+\mathbb{1}_{\mathbb{R}_0}(x)(D_{r,v}\underline{Z}^{n+1}_{s,x})^2]M(ds,dx)\\ &+\int_{]t,T]\times\mathbb{R}} e^{\int_0^s\beta(\tau)d\tau}(D_{r,v}\underline{Z}^{n+1}_{s,x})^2ds\mu(dx), \quad \mathbb{P}\otimes\mathbb{m}\text{ - } a.e. \end{aligned}$$

By (32), the requirements of the a priori estimate [18, Proposition 4.1] are met, which shows that

$$\mathbb{E}\sup_{t\in[0,T]}\left|D_{r,v}Y^{n+1}_t\right|^2<\infty, \quad \mathbb{P}\otimes\mathbb{m}\text{-a.e.}$$

Thus, the integral w.r.t. M is a uniformly integrable martingale and hence has expectation zero. Therefore, using (38), we have for $0<u<t\leq T$ that

$$\mathbb{E}e^{\int_0^t\beta(s)ds}(D_{r,v}Y^{n+1}_t)^2+\mathbb{E}\int_{]r,T]\times\mathbb{R}} e^{\int_0^s\beta(\tau)d\tau}(D_{r,v}\underline{Z}^{n+1}_{s,x})^2ds\mu(dx)$$

$$\leq \quad e^{\int_0^T \beta(s)ds}\mathbb{E}(D_{r,v}\xi)^2 + 2\int_r^T e^{\int_0^s \beta(\tau)d\tau}\mathbb{E}\left|\left[D_{r,v}f\left(X,s,Y_s^n,Z_s^n,G(s,U_s^n)\right)\right]D_{r,v}Y_s^{n+1}\right| ds$$
$$-\mathbb{E}\int_r^T \beta(s)e^{\int_0^s \beta(\tau)d\tau}(D_{r,v}Y_s^{n+1})^2 ds.$$

Similar as in Step 1 we estimate the integrand containing the generator. Here we use Lemma A.2 and (37), and then again Young's inequality:

$$\begin{aligned}
&2\left|\left[D_{r,v}f\left(X,s,Y_s^n,Z_s^n,G(s,U_s^n)\right)\right]D_{r,v}Y_s^{n+1}\right| \\
&\leq \quad 2\left|D_{r,v}Y_s^{n+1}\right|\Big(|\Gamma_{r,v}(s)| + a(s)\,|D_{r,v}Y_s^n| + b(s)\left(|D_{r,v}Z_s^n| + L_g\|\kappa\|\|D_{r,v}U_s^n\|\right) \\
&\qquad\qquad +\rho_{r,v}^D\left(|Y_s^n| + |Z_s^n| + L_g\|\kappa\|\|U_s^n\|\right)\Big) \\
&\leq \quad |\Gamma_{r,v}(s)|^2 + |\rho_{r,v}^D|^2\left(|Y_s^n|^2 + |Z_s^n|^2 + \|U_s^n\|^2\right) + \frac{a(s)}{2}\,|D_{r,v}Y_s^n|^2 + \frac{1}{2}\,|D_{r,v}Z_s^n|^2 \\
&\qquad +\frac{1}{2}\|D_{r,v}U_s^n\|^2 + \left(1 + 2a(s) + (1\vee L_g\|\kappa\|)^2\left(3+2b(s)^2\right)\right)\left|D_{r,v}Y_s^{n+1}\right|^2.
\end{aligned}$$

Choosing $\beta = 2 + 3a + (1\vee L_g\|\kappa\|)^2(3+2b^2)$ leads to

$$\begin{aligned}
&\mathbb{E}\int_r^T e^{\int_0^s \beta(\tau)d\tau}(1+a(s))\left|D_{r,v}Y_s^{n+1}\right|^2 ds + \mathbb{E}\int_{]r,T]\times\mathbb{R}} e^{\int_0^s \beta(\tau)d\tau}\left|D_{r,v}\underline{Z}_{s,x}^{n+1}\right|^2 \mathbb{m}(ds,dx) \\
&\leq \quad e^{\int_0^T \beta(s)ds}\mathbb{E}\,|D_{r,v}\xi|^2 + \mathbb{E}\int_r^T e^{\int_0^s \beta(\tau)d\tau}\,|\Gamma_{r,v}(s)|^2\,ds \\
&\quad +|\rho_{r,v}^D|^2\,\mathbb{E}\int_r^T e^{\int_0^s \beta(\tau)d\tau|}\left(|Y_s^n|^2 + |Z_s^n|^2 + \|U_s^n\|^2\right) ds \\
&\quad +\frac{1}{2}\left(\mathbb{E}\int_r^T e^{\int_0^s \beta(\tau)d\tau}(1+a(s))\,|D_{r,v}Y_s^n|^2\,ds + \mathbb{E}\int_{]r,T]\times\mathbb{R}} e^{\int_0^s \beta(\tau)d\tau}\left|D_{r,v}\underline{Z}_{s,x}^n\right|^2 \mathbb{m}(ds,dx)\right).
\end{aligned}$$

Since $\|(Y^n, Z^n, U^n)\|_{L_2(W)\times L_2(W)\times L_2(\tilde{N})}$ converges, we have that

$$C_{\text{sup}} := \sup_{l\geq 0}\mathbb{E}\int_0^T \left(|Y_s^l|^2 + |Z_s^l|^2 + \|U_s^l\|^2\right) ds < \infty.$$

Finally, we use (38) to extend the integrals w.r.t. ds onto $[0,T]$, and conclude by an elementary elementary recursion inequality (see Lemma A.1 in [17]) that

$$\begin{aligned}
&\int_0^T e^{\int_0^s \beta(\tau)d\tau}\|(1+a(s))DY_s^n\|^2_{L_2(\mathbb{m}\otimes\mathbb{P})}ds + \int_{[0,T]\times\mathbb{R}} e^{\int_0^T \beta(\tau)d\tau}\|D\underline{Z}_{s,x}^n\|^2_{L_2(\mathbb{m}\otimes\mathbb{P})}\mathbb{m}(ds,dx) \\
&\quad \leq c_\beta\left(\|D\xi\|^2_{L_2(\mathbb{P}\otimes\mathbb{m})} + \|\Gamma\|^2_{L_2(\lambda\otimes\mathbb{P}\otimes\mathbb{m})}\right) + c_\beta\|\rho^D\|^2_{L_2(\mathbb{m})}C_{\text{sup}} \quad \text{for all } n\in\mathbb{N}.
\end{aligned} \tag{39}$$

Step 3.
We now prove that

$$\left\|\mathcal{Y} - DY^{n+1}\right\|^2_{\mathrm{L}^2(\mathbb{P}\otimes\lambda\otimes\mathbb{m})} + \left\|\underline{\mathcal{Z}} - D\underline{Z}^{n+1}\right\|^2_{\mathrm{L}^2(\mathbb{P}\otimes(\mathbb{m})^{\otimes 2})} \to 0, \quad n\to\infty. \tag{40}$$

To show (40), one can repeat the above computations, now for the difference $\mathcal{Y}_t^{r,v} - D_{r,v}Y_t^{n+1}$, to get

$$\begin{aligned} &\mathbb{E}\int_r^T e^{\int_0^s \beta(\tau)d\tau}\beta(s)(\mathcal{Y}_s^{r,v} - D_{r,v}Y_s^{n+1})^2 ds + \mathbb{E}\int_{]r,T]\times\mathbb{R}} e^{\int_0^s \beta(\tau)d\tau}(\underline{\mathcal{Z}}_{s,x}^{r,v} - D_{r,v}\underline{Z}_{s,x}^{n+1})^2 ds\mu(dx) \\ &\leq \mathbb{E}\int_r^T e^{\int_0^s \beta(\tau)d\tau} 2\left|\mathcal{Y}_s^{r,v} - D_{r,v}Y_s^{n+1}\right| \\ &\qquad \times \left|F_{r,v}(s,\mathcal{Y}_s^{r,v},\mathcal{Z}_s^{r,v},\mathcal{U}_s^{r,v}) - D_{r,v}f(s,Y_s^n,Z_s^n,G(s,U_s^n))\right| ds. \end{aligned} \tag{41}$$

We first consider the case $v=0$. By using Lipschitz properties of f (which also imply the boundedness of the partial derivatives) and (33) it follows that

$$\begin{aligned} &\left|F_{r,0}(s,\mathcal{Y}_s^{r,0},\mathcal{Z}_s^{r,0},\mathcal{U}_s^{r,0}) - D_{r,0}f(s,Y_s^n,Z_s^n,G(s,U_s^n))\right| \\ &\leq a(s)\left|\mathcal{Y}_s^{r,0} - D_{r,0}Y_s^n\right| + b(s)\left(\left|\mathcal{Z}_s^{r,0} - D_{r,0}Z_s^n\right| + L_g\|\kappa\|\left\|\mathcal{U}_s^{r,0} - D_{r,0}U_s^n\right\|\right) \\ &\quad + \left|\mathcal{Y}_s^{r,0}\right|\left|\partial_y f(s,Y_s,Z_s,G(s,U_s)) - \partial_y f(s,Y_s^n,Z_s^n,G(s,U_s^n))\right| + \lambda_n(r,s), \end{aligned}$$

where

$$\begin{aligned} \lambda_n(r,s) &:= \Big(\left|(D_{r,0}f)(s,Y_s,Z_s,G(s,U_s)) - (D_{r,0}f)(s,Y_s^n,Z_s^n,G(s,U_s^n))\right| \\ &\quad \wedge\Big(2\Gamma_{r,0}(s) + \rho_{r,0}^D\Big(|Y_s^n| + |Y_s| + |Z_s^n| + |Z_s| + L_g\|\kappa\|\Big(\|U_s^n\| + \|U_s\|\Big)\Big) \\ &\quad + \left|\mathcal{Z}_s^{r,0}\right|\left|\partial_z f_g(s,Y_s,Z_s,U_s) - \partial_z f_g(s,Y_s^n,Z_s^n,U_s^n)\right| \\ &\quad + \left\|\mathcal{U}_s^{r,0}\right\|\Big(\left|\partial_u f(s,Y_s,Z_s,G(s,U_s)) - \partial_u f(s,Y_s^n,Z_s^n,G(s,U_s^n))\right| L_g\|\kappa\| \\ &\qquad + b(s)\left\|\left|g'(s,U_s) - g'(s,U_s^n)\right|\kappa\right\|\Big). \end{aligned}$$

Thus, using Young's inequality again, and the fact that $|\partial_y f(X,s,\eta)| \leq a(s)$, we estimate

$$2\left|\mathcal{Y}_s^{r,0} - D_{r,0}Y_s^{n+1}\right|\ \left|F_{r,0}(s,\mathcal{Y}_s^{r,0},\mathcal{Z}_s^{r,0},\mathcal{U}_s^{r,0}) - D_{r,0}f(s,Y_s^n,Z_s^n,G(s,U_s^n))\right|$$

by

$$\begin{aligned} &\left(1 + 4a(s) + 2(1\vee L_g^2\|\kappa\|)^2 b(s)^2\right)\left|\mathcal{Y}_s^{r,0} - D_{r,0}Y_s^{n+1}\right|^2 \\ &+ \lambda_n(r,s)^2 + \left|\mathcal{Y}_s^{r,0}\right|^2\left|\partial_y f(s,Y_s,Z_s,G(s,U_s)) - \partial_y f(s,Y_s^n,Z_s^n,G(s,U_s^n))\right| \\ &+ \frac{1}{2}\left(a(s)\left|\mathcal{Y}_s^{r,0} - D_{r,0}Y_s^n\right|^2 + \left|\mathcal{Z}_s^{r,0} - D_{r,0}Z_s^n\right|^2 + \left\|\mathcal{U}_s^{r,0} - D_{r,0}U_s^n\right\|^2\right). \end{aligned}$$

We notice that $C(s) := 1 + 4a(s) + 2(1\vee L_g^2\|\kappa\|)^2 b(s)^2$ from the expression above is in $L_1([0,T])$. For the case $v=0$, we set

$$\begin{aligned} \delta_n := \mathbb{E}\int_0^T\int_0^T e^{\int_0^s \beta(\tau)d\tau}\Big(&\left|\mathcal{Y}_s^{r,0}\right|^2\left|\partial_y f(s,Y_s,Z_s,G(s,U_s)) - \partial_y f(s,Y_s^n,Z_s^n,G(s,U_s^n))\right| \\ &+ \lambda_n(r,s)^2\Big) dr ds \end{aligned}$$

and are now in the position to infer relation (43) below for the r.h.s. of (41) by using the above inequalities. The fact that

$$\delta_n \to 0 \ \text{ for } \ n\to\infty$$

can be seen by Vitali's convergence theorem taking into consideration that $\beta \in L_1([0,T])$ and $\mathbb{E}\sup_{t\in[0,T]} \left|\mathcal{Y}_t^{r,0}\right|^2 < \infty$; that (Y^n, Z^n, U^n) converges to (Y, Z, U) in $L_2(W) \times L_2(W) \times L_2(\tilde{N})$; that $\partial_y f$ is continuous and bounded by the function a. For the convergence of the integral part containing $\lambda_n(r,s)^2$ we need additionally that $\eta \mapsto (D_{r,0}f)(s,\eta)$ is continuous (which follows from $(\mathbf{A}_f)$ $e)$), that $\partial_z f, \partial_u f$ are continuous and bounded by b, and that g' is continuous and bounded by L_g. Thanks to the minimum in its first term, $\lambda_n(r,s)^2$ is uniformly integrable in n.

Now we continue with the case $v \neq 0$. We first realise that for a given $\varepsilon > 0$ we may choose $\alpha > 0$ small enough such that for all $n \geq 1$

$$
\begin{aligned}
&\mathbb{E}\int_0^T \int_0^T \int_{\{|v|<\alpha\}} e^{\int_0^T \beta(\tau)d\tau} \left|\mathcal{Y}_s^{r,v} - D_{r,v}Y_s^{n+1}\right| \\
&\quad \times \left|F_{r,v}(s, \mathcal{Y}_s^{r,v}, \mathcal{Z}_s^{r,v}, \mathcal{U}_s^{r,v}) - D_{r,v}f(s, Y_s^n, Z_s^n, G(s, U_s^n))\right| \nu(dv)drds < \varepsilon. \qquad (42)
\end{aligned}
$$

This is because from (31), $(\mathbf{A}_f)$ $d)$ and (32) we have that

$$
\begin{aligned}
|F_{r,v}(s, \mathcal{Y}_s^{r,v}, \mathcal{Z}_s^{r,v}, \mathcal{U}_s^{r,v})| \leq& \Gamma_{r,v}(s) + a(s)|\mathcal{Y}_s^{r,v}| + b(s)(|\mathcal{Z}_s^{r,v}| + L_g\|\kappa\|\|\mathcal{U}_s^{r,v}\|) \\
&+ \rho_{r,v}^D(|Y_s| + |Z_s| + L_g\|\kappa\|\|U_s\||)
\end{aligned}
$$

and

$$
\begin{aligned}
|D_{r,v}f(s, Y_s^n, Z_s^n, G(s, U_s^n))| \leq& \Gamma_{r,v}(s) + a(s)|D_{r,v}Y_s^n| + b(s)(|D_{r,v}Z_s^n| + L_g\|\kappa\|_{L_2(\nu)}\|D_{r,v}U^n{}_s\|) \\
&+ \rho_{r,v}^D(|Y_s^n| + |Z_s^n| + L_g\|\kappa\|\|U_s^n\||),
\end{aligned}
$$

holds. Then, Young's inequality and inequality (39) imply the boundedness of the integral in (42). On the set $\{|v| \geq \alpha\}$ we use the Lipschitz properties $(\mathbf{A}_f)$ $c)$ and $(\mathbf{A}_f)$ $f)$ to get the estimate

$$
\begin{aligned}
&|F_{r,v}(s, \mathcal{Y}_s^{r,v}, \mathcal{Z}_s^{r,v}, \mathcal{U}_s^{r,v}) - D_{r,v}f(s, Y_s^n, Z_s^n, G(s, U_s^n))| \\
\leq\ & \big| f\left((X + v\mathbb{1}_{[r,T]}), s, Y_s + \mathcal{Y}_s^{r,v}, Z_s + \mathcal{Z}_s^{r,v}, G(s, U_s + \mathcal{U}_s^{r,v})\right) \\
&\quad - f\left((X + v\mathbb{1}_{[r,T]}), s, Y_s^n + D_{r,v}Y_s^n, Z_s^n + D_{r,v}Z_s^n, G(s, U_s^n + D_{r,v}U_s^n)\right) \big| \\
&\quad + \big| f\left(X, s, Y_s, Z_s, G(s, U_s)\right) - f\left(X, s, Y_s^n, Z_s^n, G(s, U_s^n)\right) \big| \\
\leq\ & a(s)|\mathcal{Y}_s^{r,v} - D_{r,v}Y_s^n| + b(s)\left[|\mathcal{Z}_s^{r,v} - D_{r,v}Z_s^n| + L_g\|\kappa\|\|\mathcal{U}_s^{r,v} - D_{r,v}U_s^n\|\right] \\
&\quad + 2a(s)|Y_s - Y_s^n| + b(s)\left[|Z_s - Z_s^n| + L_g\|\kappa\|\|U_s - U_s^n\|)\right].
\end{aligned}
$$

This helps us to estimate an integrated version of the r.h.s. of (41) for any $n \in \mathbb{N}$: Using Young's inequality once again, we arrive for $v = 0$ and $v \neq 0$ at

$$
\begin{aligned}
&\mathbb{E}\int_r^T \int_{[0,T]\times\mathbb{R}} e^{\int_0^T \beta(\tau)d\tau} \left|\mathcal{Y}_s^{r,v} - D_{r,v}Y_s^{n+1}\right| \\
&\quad \times \left|F_{r,v}(s, \mathcal{Y}_s^{r,v}, \mathcal{Z}_s^{r,v}, \mathcal{U}_s^{r,v}) - D_{r,v}f(s, Y_s^n, Z_s^n, G(s, U_s^n))\right| \mathbb{m}(dr, dv)ds \\
&\quad \leq \frac{1}{2}\mathbb{E}\int_r^T e^{\int_0^T \beta(\tau)d\tau} \left(a(s)\|\mathcal{Y}_s - DY_s^n\|_{L_2(\mathbb{m})}^2 + \|\underline{\mathcal{Z}}_{s,.} - D\underline{Z}_{s,.}^n\|_{L_2(\mathbb{m}\otimes\mu)}^2\right)ds \\
&\quad + \nu(\{|v| \geq \alpha\})\mathbb{E}\int_r^T e^{\int_0^T \beta(\tau)d\tau} \left(a(s)|Y_s - Y_s^n|^2 + \|\underline{Z}_{s,.} - \underline{Z}_{s,.}^n\|_{L_2(\mathbb{m})}^2\right)ds \\
&\quad + \delta_n + \varepsilon + \mathbb{E}\int_r^T e^{\int_0^T \beta(\tau)d\tau} C(s)\|\mathcal{Y}_s - DY_s^{n+1}\|_{L_2(\mathbb{m})}^2 ds. \qquad (43)
\end{aligned}
$$

Choosing $\beta = 1 + a(s) + C(s)$ in (41) and applying (43) leads to

$$\left\| \sqrt{(1+a)}(\mathcal{Y} - DY^{n+1}) \right\|^2_{L^2(\mathbb{P}\otimes\lambda\otimes\mathbb{m})} + \|\underline{\mathcal{Z}} - D\underline{Z}^{n+1}\|^2_{L^2(\mathbb{P}\otimes(\mathbb{m})^{\otimes 2})}$$
$$\le \varepsilon + C_n + \frac{1}{2}\left(\left\| \sqrt{(1+a)}(\mathcal{Y} - DY^{n}) \right\|^2_{L^2(\mathbb{P}\otimes\lambda\otimes\mathbb{m})} + \|\underline{\mathcal{Z}} - D\underline{Z}^{n}\|^2_{L^2(\mathbb{P}\otimes(\mathbb{m})^{\otimes 2})} \right)$$

with

$$C_n = C_n(\alpha)$$
$$= \delta_n + \nu(\{|v| \ge \alpha\})\mathbb{E}\int_r^T e^{\int_0^T \beta(\tau)d\tau}\left(a(s)|Y_s - Y_s^n|^2 + \|\underline{Z}_{s,\cdot} - \underline{Z}^n_{s,\cdot}\|^2_{L_2(\mathbb{m})}\right)ds$$

tending to zero if $n \to \infty$ for any fixed $\alpha > 0$. We now use the recursion inequality from ([17, Lemma A.1]) and end up with

$$\limsup_{n\to\infty}\left(\left\| \sqrt{(1+a)}(\mathcal{Y} - DY^{n}) \right\|^2_{L^2(\mathbb{P}\otimes\lambda\otimes\mathbb{m})} + \|\underline{\mathcal{Z}} - D\underline{Z}^{n}\|^2_{L^2(\mathbb{P}\otimes(\mathbb{m})^{\otimes 2})} \right) \le 2\varepsilon.$$

This implies (29) since $\left\| \sqrt{(1+a)}\mathcal{Y} \right\|_{L^2(\mathbb{P}\otimes\lambda\otimes\mathbb{m})} < \infty$ and $\left\| \sqrt{(1+a)}\cdot \right\|_{L^2(\mathbb{P}\otimes\lambda\otimes\mathbb{m})} \ge \|\cdot\|_{L^2(\mathbb{P}\otimes\lambda\otimes\mathbb{m})}$. Hence we can take the Malliavin derivative $D_{r,v}$ of (8) and get (30) for $0 \le r \le t \le T$ as well as

$$0 = D_{r,v}\xi + \int_r^T F_{r,v}\left(s, D_{r,v}Y_s, D_{r,v}Z_s, D_{r,v}U_s\right)ds - \underline{Z}_{r,v} - \int_{]r,T]\times\mathbb{R}} D_{r,v}\underline{Z}_{s,x}M(ds,dx), \quad (44)$$

for $0 \le t < r \le T$. By the same reasoning as for $D_{r,v}Y^n$ we may conclude that the RHS of (30) has a càdlàg version which we take for $D_{r,v}Y$.

(iii) This assertion we get by comparing (28) and (30) because of the uniqueness of $(\mathcal{Y}, \mathcal{Z}, \mathcal{U})$.

(iv) For the discussion on the measurability of $\lim_{t\searrow r} D_{r,v}Y_t$ w.r.t. (r, v, ω) which is needed to take the conditional expectation $\mathbb{E}[\,\cdot\,|\mathcal{F}_{r-}]$ we refer the reader to the proof of [17, Theorem 4.4]. The assertion follows then from comparing (30) with (44) and the uniqueness of solutions. If the predictable projections ${}^p\left((D_{r,0}Y_r)_{r\in[0,T]}\right)$ and ${}^p\left((D_{r,v}Y_r)_{r\in[0,T],v\in\mathbb{R}_0}\right)$ exist, this has the benefit that one has jointly measurable processes which are unique up to indistinguishability. □

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