The parameter identification in the Stokes system with threshold slip boundary conditions

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Summary

The paper is devoted to an identification of the slip bound function g in the Stokes system with threshold slip boundary conditions assuming that g depends on the tangential velocity u_τ. To this end the optimal control approach is used. To remove its nonsmoothness we use a regularized form of the slip conditions in the state problem. The mutual relation between solutions to the original optimization problem and the problems with regularized states is analyzed. The paper is completed by numerical experiments.

KEYWORDS:
threshold slip boundary conditions, Stokes system with slip conditions, parameter identification in flow models

1 | INTRODUCTION

The vanishing velocity $\boldsymbol{u}$ on the boundary of the computational domain is the standard boundary condition in fluid flow models. In practice, depending also on properties of solid surfaces in a fluid's path, a slip along such surfaces can be observed. The slip affects the fluid flow ([1], [2]), therefore it has to be involved into the mathematical model. The Navier slip law is the simplest one. It says that the shear stress σ_τ is a linear function of the tangential velocity u_τ. Clearly, $u_\tau \neq 0$ if and only if $\sigma_\tau \neq 0$, i.e. the slip response is instantaneous. On the other hand, there are situations, for example a water flow along hydrophobic surfaces, when the slip behavior has a threshold character: a slip along surfaces may occur only if the shear stress σ_τ attains certain bound g while below this bound is $\boldsymbol{u} = \boldsymbol{0}$. The constitutive law between σ_τ and u_τ in this case is expressed by inclusions for appropriate multivalued mappings. Consequently, the resulting mathematical models lead to inequality type problems, complexity of which depends on a particular threshold bound g. In the simplest case, g is represented by a function which is given a-priori. This choice of g corresponding to the Tresca model of friction in solid mechanics, has been studied by Fujita in the frame of the Stokes and Navier-Stokes system [3], and in time dependent problems [4]. Later on, the analysis has been extended to g:s depending on the solution itself, e.g. on the tangential velocity u_τ ([5], [6]), or Coulomb's type slip condition in which g depends on the normal stress σ_ν ([7], [8], [9]).

It often happens that not all data we need in computations are available. If it is so then a possible way how to get the missing ones is to use an identification procedure when on the basis of measurements of appropriate physical quantities we are able to reveal remaining data. The present paper is devoted to an identification of the threshold slip function g which depends on u_τ. To this end we use an optimal control approach with the integral least squares cost functional and the target represented by measurements of u_τ in a vicinity of the slip part of the boundary. The Stokes system with this threshold slip condition serves as the state problem and the slip bound g plays the role of the control variable. Owing to the state relation, the resulting optimization problem is nonsmooth, i.e. the minimized functional (or function after a discretization) might be nondifferentiable at some points. This fact restricts the choice of classical gradient type methods. Formally, they can be used together with the standard sensitivity analysis, however obtained results are not satisfactory since minimization methods usually fail due to false gradient information. For this reason a special differential calculus which uses deep tools of the multivalued analysis and special methods for minimization of nonsmooth functions have been developed. It is not surprising that knowledge of such techniques is a domain of a limited community of specialists. To overcome this inconvenience we propose and theoretically justify another way which is based on a regularization approach. The original state relation $(\mathcal{P})$ will be replaced by a sequence of smooth problems $(\mathcal{P}_\varepsilon)$ depending on a parameter $\varepsilon \to 0+$. Under appropriate assumptions, solutions of $(\mathcal{P}_\varepsilon)$ tend to solutions of $(\mathcal{P})$ if $\varepsilon \to 0+$. A natural idea arises, namely to replace the original state $(\mathcal{P})$ in the optimization problem by its regularized form $(\mathcal{P}_\varepsilon)$. The resulting optimization problems then become smooth. Consequently standard optimization techniques can be used.

The rest of the paper is organized as follows. In Section 2 we present the weak velocity-pressure formulation $(\mathcal{P})$ of the Stokes system with the slip bound function g depending on the tangential velocity u_τ, further specify the set $\mathcal{U}_{ad}$ of admissible g:s and define the identification problem $(\mathbb{P})$. We show that solutions to $(\mathcal{P})$ are stable with respect to $g \in \mathcal{U}_{ad}$, i.e. they depend in a continuous manner on such g:s. This property guarantees the existence of a solution to $(\mathbb{P})$. Section 3 starts with the definition of the regularized state problems $(\mathcal{P}_\varepsilon)$. We prove that $(\mathcal{P}_\varepsilon)$ has a solution for any $\varepsilon > 0$ and any $g \in \mathcal{U}_{ad}$. If, in addition g is one sided Lipschitz continuous, the solution is unique. Then keeping $\varepsilon > 0$ fixed, we prove that solutions to $(\mathcal{P}_\varepsilon)$ are stable with respect to $g \in \mathcal{U}_{ad}$. Consequently, the identification problem $(\mathbb{P}_\varepsilon)$ using $(\mathcal{P}_\varepsilon)$ as the state relation has a solution. In Section 4 the mutual relation between solutions to $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ as $\varepsilon \to 0+$ is studied. The rest of the paper is devoted to computational aspects. For the approximation of the regularized state problem, the P2/P1 elements are used together with the Bezier type parametrization of the slip bound function g. We present results of a model example. It deals with the identification of g using measurements of the tangential velocity in a vicinity of the slip part of the boundary S. The stability of numerical results with respect to the regularization parameter, the degree of the Bezier functions and the distance of measurements from S is illustrated.

In the paper we use following notation. For a bounded domain Ω in $\mathbb{R}^2$ we denote by $H^k(\Omega)$, $k \geq 0$ integer, the space of all Lebesgue integrable functions which are together with their generalized derivatives up to order k square integrable in Ω, i.e. belong to $L^2(\Omega) := H^0(\Omega)$. The norm in $H^k(\Omega)$, $L^2(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively. Finally, the cones of all non-negative functions from $L^2(\Omega)$, the space $C(\mathbb{R}_+)$ of all continuous functions in $\mathbb{R}_+$, will be denoted by $L^2_+(\Omega)$, and $C_+(\mathbb{R}_+)$, respectively.

2 | SETTING OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega$ which is decomposed into two non-empty parts Γ and S: $\partial\Omega = \overline{\Gamma} \cup \overline{S}$, $\Gamma \cap S = \emptyset$. In Ω we consider a viscous incompressible Newtonian fluid obeying the Stokes system with the no-slip condition on Γ and the impermeability and slip conditions on S. The classical formulation of this problem reads as follows: find the velocity $\boldsymbol{u} = (u_1, u_2)$ and the pressure p satisfying

$$\begin{cases} -\operatorname{div}(2\eta\mathbb{D}\boldsymbol{u}) + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma \\ u_\nu = 0 & \text{on } S \\ |\sigma_\tau| \leq g(|u_\tau|) & \text{on } S \\ g(|u_\tau|)u_\tau = -|u_\tau|\sigma_\tau & \text{on } S, \end{cases} \tag{2.1}$$

where $\boldsymbol{f}$ is a force acting on the fluid, $\eta > 0$ is the dynamic viscosity, $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ are the unit outward normal, and the tangential vector to $\partial\Omega$, $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $v_\tau = \boldsymbol{v} \cdot \boldsymbol{\tau}$ denote the normal, and tangential component of a vector $\boldsymbol{v}$, respectively. Finally, $\sigma_\tau = (2\eta(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu})_\tau$ is the shear stress, where $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}})$ is the symmetric part of the gradient of $\boldsymbol{u}$. The last two boundary conditions in (2.1) express the threshold slip model. The non-negative function g represents the slip bound which depends on the tangential velocity u_τ. Slip at a point $x \in S$ may occur only if $|\sigma_\tau(x)| = g(|u_\tau(x)|)$ and σ_τ, u_τ have opposite signs.

To give the weak formulation of (2.1) we introduce the velocity, pressure space:

$$\mathbb{V}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma,\ v_\nu = 0 \text{ on } S\}$$

and

$$L^2_0(\Omega) = \{q \in L^2(\Omega) \mid \textstyle\int_\Omega q\,dx = 0\},$$

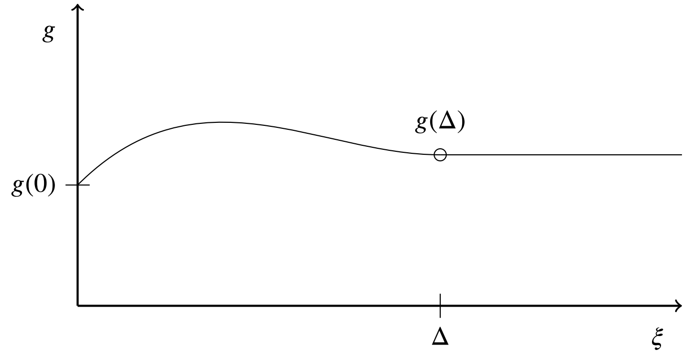

FIGURE 1 Graph of g.

respectively.

The weak formulation of (2.1) leads to the following implicit inequality type problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \mathbb{V}(\Omega) \times L_0^2(\Omega) \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + j_g(|u_\tau|, |v_\tau|) - j_g(|u_\tau|, |u_\tau|) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega), \\ b(\boldsymbol{u}, q) = 0 \quad \forall q \in L_0^2(\Omega), \end{cases} \tag{2.2}$$

where[1] $\boldsymbol{f} \in (L^2(\Omega))^2$ and

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v}\, dx, \qquad b(\boldsymbol{v}, q) = \int_\Omega q \operatorname{div} \boldsymbol{v}\, dx, \qquad j_g(|u_\tau|, |v_\tau|) = \int_S g(|u_\tau|)|v_\tau|\, ds.$$

The mathematical analysis of (2.2) based on a fixed-point approach has been done in [6] and in [10] which contains also a finite element discretization of (2.2) and the respective convergence analysis. It has been shown that under appropriate growth conditions on g, problem (2.2) has a solution. In addition, if g is one-sided Lipschitz continuous in $\mathbb{R}_+$, i.e. there exists a constant $L_{\mathrm{OS}} \geq 0$ such that

$$(g(x_1) - g(x_2))(x_2 - x_1) \leq L_{\mathrm{OS}}(x_1 - x_2)^2 \quad \forall x_1, x_2 \in \mathbb{R}_+ \tag{2.3}$$

then (2.2) has a unique solution provided that L_{OS} is small enough, more precicely it satisfies (3.16), see below. Let us observe that any non-decreasing function g satisfies (2.3) automatically with $L_{\mathrm{OS}} = 0$. Finally, if g is Lipschitz continuous in $\mathbb{R}_+$ with sufficiently small modulus of Lipschitz continuity L then the respective fixed-point mapping is contractive and the method of successive approximations converges to its unique fixed-point. Each iterative step is represented by the Stokes system with the slip conditions of Tresca type, meaning that the unknown slip bound $g(|u_\tau|)$ in (2.1) is replaced by a function $\overline{g}$ given à-priori.

This paper is aimed to the identification of the slip bound g on the basis of the tangential velocity measurements on S. The set of admissible g:s is defined as follows (see Figure 1):

$$\mathcal{U}_{ad} = \{g \in C_+(\mathbb{R}_+) \mid g(0) \leq g_0,\ |g(\xi) - g(\bar{\xi})| \leq L|\xi - \bar{\xi}|\ \forall \xi, \bar{\xi} \in [0, \Delta], g(\xi) = g(\Delta)\ \forall \xi \geq \Delta\}, \tag{2.4}$$

where g_0, L and Δ are given positive constants which do not depend on g, i.e. $\mathcal{U}_{ad}$ is the set of all non-negative functions which are equi-bounded and equi-Lipschitz continuous in $[0, \Delta]$ with modulus $L > 0$. From the definition of $\mathcal{U}_{ad}$ and the Arzelà–Ascoli theorem it follows that $\mathcal{U}_{ad}$ is compact in $C([0, \Delta])$.

In the sequel we shall not impose any other condition on L. Therefore (2.2) with $g \in \mathcal{U}_{ad}$ may have more than one solution.

Let $\mathcal{G}$ be the graph of the generally multivalued control-to-state mapping:

$$\mathcal{G} = \{\ (g, \boldsymbol{u}^g, p^g) \mid g \in \mathcal{U}_{ad},\ (\boldsymbol{u}^p, p^g) \text{ solves } (\mathcal{P}(g))\ \},$$

[1]To simplify presentation we set $\eta = \frac{1}{2}$ here and in what follows.

i.e. $(\boldsymbol{u}^g, p^g) \in \mathbb{V}(\Omega) \times L^2_0(\Omega)$ satisfies

$$\begin{cases} a(\boldsymbol{u}^g, \boldsymbol{v} - \boldsymbol{u}^g) - b(\boldsymbol{v} - \boldsymbol{u}^g, p^g) + j_g(|u^g_\tau|, |v_\tau|) - j_g(|u^g_\tau|, |u^g_\tau|) \\ \qquad\qquad\qquad \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^g)_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega), \\ b(\boldsymbol{u}^g, q) = 0 \quad \forall q \in L^2_0(\Omega). \end{cases} \tag{$\mathcal{P}(g)$}$$

The identification problem that we shall study reads as follows:

$$\text{Find } (g^*, \boldsymbol{u}^{g^*}, p^{g^*}) \in \operatorname{argmin}\{\ I(g, \boldsymbol{u}^g, p^g) \mid (g, \boldsymbol{u}^g, p^g) \in \mathcal{G}\ \}, \tag{$\mathbb{P}$}$$

where I is the least squares functional defined by

$$I(g, \boldsymbol{u}^g, p^g) = \frac{1}{2} \int_S \left(u^g_\tau - \widetilde{u}_\tau\right)^2 \, ds$$

and $\widetilde{u}_\tau \in L^2(S)$ is the measured value of the tangential velocity on S.

Here and in what follows we denote by $\mathcal{U}^\#_{ad} \subset \mathcal{U}_{ad}$ a subset of $\mathcal{U}_{ad}$ for which $(\mathcal{P}(g))$, $g \in \mathcal{U}^\#_{ad}$ has a unique solution. Then $(\mathbb{P})$ can be written in the standard form

$$\begin{cases} \text{Find } g^* \in \mathcal{U}^\#_{ad} \text{ such that} \\ J(g^*) \leq J(g) \quad \forall g \in \mathcal{U}^\#_{ad}, \end{cases} \tag{$\mathbb{P}^\#$}$$

where

$$J(g) := I(g, \boldsymbol{u}^g, p^g)$$

and $(\boldsymbol{u}^g, p^g)$ solves $(\mathcal{P}(g))$. As an example of $\mathcal{U}^\#_{ad}$ which guarantees uniqueness of solutions to $(\mathcal{P}(g))$ and will be used in computations we mention

$$\mathcal{U}^\#_{ad} = \left\{ g \in \mathcal{U}_{ad} \mid g \text{ is non-decreasing in } [0, \Delta] \right\}. \tag{2.5}$$

Our aim is to show that the identification problem $(\mathbb{P})$ has a solution. To this end we prove the following auxiliary result.

Lemma 1. *Let $g_n \rightrightarrows g$ (uniformly) in $[0, \Delta]$, $g_n, g \in \mathcal{U}_{ad}$ and $\{v_n\}, \{w_n\}$ be sequences in $L^2(S)$ such that $v_n \to v$, $w_n \to w$ in $L^2(S)$. Then*

$$\int_S g_n(|v_n|)|w_n| \, ds \to \int_S g(|v|)|w| \, ds, \quad n \to \infty. \tag{2.6}$$

Proof. It holds:

$$\left| \int_S g_n(|v_n|)|w_n| \, ds - \int_S g(|v|)|w| \, ds \right| \leq \int_S \Big| g_n(|v_n|) - g_n(|v|) \Big| |w_n| \, ds$$
$$+ \int_S \Big| g_n(|v|)|w_n| - g(|v|)|w| \Big| \, ds =: I_{1,n} + I_{2,n}.$$

From the definition of $\mathcal{U}_{ad}$ we have:

$$I_{1,n} \leq L \int_S |v_n - v||w_n| \, ds \leq L \|v_n - v\|_{0,S} \|w_n\|_{0,S} \to 0, \quad n \to \infty.$$

Further,

$$I_{2,n} \leq \int_S |g_n(|v|)||w_n - w| \, ds + \int_S |g_n(|v|) - g(|v|)||w| \, ds$$
$$\leq \|g_n(|v|)\|_{0,S} \|w_n - w\|_{0,S} + \|g_n(|v|) - g(|v|)\|_{0,S} \|w\|_{0,S} \to 0, \quad n \to \infty.$$

□

In the next theorem we prove the fundamental property of the graph $\mathcal{G}$.

Theorem 1. *The graph $\mathcal{G}$ is compact in the following sense: for any sequence*[2] $\{(g_n, \boldsymbol{u}^{g_n}, p^{g_n})\}$, $(g_n, \boldsymbol{u}^{g_n}, p^{g_n}) \in \mathcal{G}$ *there exist: its subsequence, which will be denoted by the same symbol and an element* $(g, \boldsymbol{u}^g, p^g) \in \mathcal{G}$ *such that*

$$g_n \rightrightarrows g \text{ in } [0, \Delta], \tag{2.7}$$

$$\boldsymbol{u}^{g_n} \to \boldsymbol{u}^g \text{ in } (H^1(\Omega))^2, \tag{2.8}$$

$$p^{g_n} \to p^g \text{in } L^2(\Omega), \quad n \to \infty. \tag{2.9}$$

Proof. *Since* $\mathcal{U}_{ad}$ *is compact in* $\boldsymbol{C}([0,\Delta])$ *we may assume that* $\{g_n\}$ *satisfies* (2.7)*. Owing to [6], [10], a solution* $(\boldsymbol{u}^{g_n}, p^{g_n})$ *to* $(\mathcal{P}(g_n))$ *exists for any* $g_n \in \mathcal{U}_{ad}$ *and the sequence* $\{(\boldsymbol{u}^{g_n}, p^{g_n})\}$ *is bounded: there exists a constant* $\boldsymbol{C} > 0$ *such that*

$$\|\boldsymbol{u}^{g_n}\|_{1,\Omega} \le \boldsymbol{C} \tag{2.10}$$

$$\|p^{g_n}\|_{0,\Omega} \le \boldsymbol{C} \tag{2.11}$$

holds for any $n \in \mathbb{N}$*. Indeed, it is readily seen that the inequality in* $(\mathcal{P}(g_n))$ *is equivalent to*

$$a(\boldsymbol{u}^{g_n}, \boldsymbol{u}^{g_n}) + j_{g_n}(|u_\tau^{g_n}|, |u_\tau^{g_n}|) = (\boldsymbol{f}, \boldsymbol{u}^{g_n})_{0,\Omega}, \tag{2.12}$$

and

$$a(\boldsymbol{u}^{g_n}, \boldsymbol{v}) - b(\boldsymbol{v}, p^{g_n}) + j_{g_n}(|u_\tau^{g_n}|, |v_\tau|) \ge (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega). \tag{2.13}$$

The upper bound (2.10) *follows from* (2.12) *and Korn's inequality. To prove* (2.11) *we use the following inf-sup condition ([11]):*

$$\beta := \inf_{q \in L_0^2(\Omega)} \sup_{\substack{v \in (H_0^1(\Omega))^2 \\ \boldsymbol{v} \neq \boldsymbol{0}}} \frac{\int_\Omega q \, div \, \boldsymbol{v} \, dx}{\|q\|_{0,\Omega} \|\boldsymbol{v}\|_{1,\Omega}} > 0. \tag{2.14}$$

This, (2.10) *and* (2.13) *with test functions* $\boldsymbol{v} \in (H_0^1(\Omega))^2$ *yield* (2.11)*. Indeed,*

$$\beta \|p^{g_n}\|_{0,\Omega} \le \sup_{\substack{v \in (H_0^1(\Omega))^2 \\ \boldsymbol{v} \neq \boldsymbol{0}}} \frac{\int_\Omega q \, div \, \boldsymbol{v} \, dx}{\|\boldsymbol{v}\|_{1,\Omega}} \le c \|\boldsymbol{u}^{g_n}\|_{1,\Omega} + \|\boldsymbol{f}\|_{0,\Omega},$$

where $c > 0$ *is an absolute constant. Thus one can pass to a subsequence of* $\{(\boldsymbol{u}^{g_n}, p^{g_n})\}$ *tending weakly to an element* $(\bar{\boldsymbol{u}}, \bar{p})$ *in* $(H^1(\Omega))^2 \times L^2(\Omega)$*. Passing to the limit with* $n \to \infty$ *in* (2.12)*,* (2.13) *and using Lemma 1 for the limit passages in the slip terms we see that* $(\bar{\boldsymbol{u}}, \bar{p}) = (\boldsymbol{u}^g, p^g)$ *solves* $(\mathcal{P}(g))$*, i.e.* $(g, \boldsymbol{u}^g, p^g) \in \mathcal{G}$*.*

From (2.12) *and Lemma 1 it follows that* $a(\boldsymbol{u}^{g_n}, \boldsymbol{u}^{g_n}) \to a(\boldsymbol{u}^g, \boldsymbol{u}^g)$ *implying strong convergence of* $\boldsymbol{u}^{g_n}$ *to* $\boldsymbol{u}^g$ *in* $(H^1(\Omega))^2$*. Observe that* (2.13) *with test functions* $\boldsymbol{v} \in (H_0^1(\Omega))^2$ *takes the form*

$$a(\boldsymbol{u}^{g_n}, \boldsymbol{v}) - b(\boldsymbol{v}, p^{g_n}) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega}$$
$$a(\boldsymbol{u}^g, \boldsymbol{v}) - b(\boldsymbol{v}, p^g) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \boldsymbol{v} \in (H_0^1(\Omega))^2.$$

Substracting both equations, and using (2.14)*,* (2.8) *we arrive at* (2.9)*.* □

Theorem 2. *Problem* $(\mathbb{P})$ *has a solution.*

Proof. *Let* $\{(g_n, \boldsymbol{u}^{g_n}, p^{g_n})\}$, $(g_n, \boldsymbol{u}^{g_n}, p^{g_n}) \in \mathcal{G}$ *be a minimizing sequence of* $\boldsymbol{I}$*. From Theorem 1 it follows that there exists its subsequence (denoted by the same symbol) and an element* $(g^*, \boldsymbol{u}^{g^*}, p^{g^*}) \in \mathcal{G}$ *such that* (2.7)–(2.9) *hold with* $g := g^*$*. Hence*

$$\inf_{\mathcal{G}} \boldsymbol{I}(g, \boldsymbol{u}^g, p^g) = \lim_{n \to \infty} \boldsymbol{I}(g_n, \boldsymbol{u}^{g_n}, p^{g_n}) = \boldsymbol{I}(g^*, \boldsymbol{u}^{g^*}, p^{g^*}),$$

making use of continuity of the functional $\boldsymbol{I}$ *in* $\mathcal{G}$*.* □

3 | REGULARIZED OPTIMIZATION PROBLEMS

Due to the presence of the non-smooth slip term j_g in $(\mathcal{P}(g))$ the minimization problem $(\mathbb{P})$ is generally non-smooth, as well. To overcome this drawback, we will regularize $(\mathcal{P}(g))$, i.e. $(\mathcal{P}(g))$ will be approximated by a sequence of nonlinear but smooth

[2] If $(\mathcal{P}(g))$ has more than one solution, we take one of them.

equations which will be used as state relations in ($\mathbb{P}$). As a result, we obtain a sequence of smooth optimization problems which can be solved by standard gradient type methods.

Let $\varepsilon > 0$ be a regularization parameter destined to tend to zero and define the following C^1-approximation of the absolute value:

$$T_\varepsilon(q) = \begin{cases} |q| & \text{if } |q| \geq \varepsilon \\ \frac{q^2+\varepsilon^2}{2\varepsilon} & \text{if } |q| < \varepsilon. \end{cases} \tag{3.1}$$

The regularization (3.1) enjoys the following important properties.

Lemma 2. *Let $v_n \to v$ in $L^2(S)$. Then*

$$T_{\varepsilon_n}(v_n) \to |v| \quad \text{in } L^2(S), \tag{3.2}$$

$$T'_{\varepsilon_n}(v_n)v_n \to |v| \quad \text{in } L^2(S), \quad n \to \infty, \tag{3.3}$$

for any $\varepsilon_n \to 0+$.

Proof. *It holds:*

$$\left\|T_{\varepsilon_n}(v_n) - |v|\right\|_{0,S} \leq \left\|T_{\varepsilon_n}(v_n) - |v_n|\right\|_{0,S} + \left\||v_n| - |v|\right\|_{0,S} =: I_{1,n} + I_{2,n}.$$

By the assumption, $I_{2,n} \to 0$ as $n \to \infty$. To estimate $I_{1,n}$ we use the definition of T_ε:

$$\left\|T_{\varepsilon_n}(v_n) - |v_n|\right\|_{0,S}^2 = \int\limits_{|v_n|<\varepsilon_n} \left(\frac{v_n^2 + \varepsilon_n^2}{2\varepsilon_n} - |v_n|\right)^2 ds \leq \int\limits_{|v_n|<\varepsilon_n} \left(\frac{v_n^2 + \varepsilon_n^2}{2\varepsilon_n}\right)^2 ds = \mathcal{O}(\varepsilon_n^2),$$

since $0 \leq |v_n| \leq \frac{v_n^2+\varepsilon_n^2}{2\varepsilon_n}$ for $|v_n| < \varepsilon_n$. To prove (3.3) it is sufficient to estimate the term $\|T'_{\varepsilon_n}(v_n)v_n - |v_n|\|_{0,S}$:

$$\|T'_{\varepsilon_n}(v_n)v_n - |v_n|\|_{0,S}^2 = \int\limits_{|v_n|\leq\varepsilon_n} \left(\frac{v_n^2}{\varepsilon_n} - |v_n|\right)^2 ds \leq 2 \int\limits_{|v_n|\leq\varepsilon_n} \left(\frac{v_n^4}{\varepsilon_n^2} + |v_n|^2\right) ds = \mathcal{O}(\varepsilon_n^2).$$

□

Using T_ε instead of the absolute values in j_g we obtain the following problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \in \mathbb{V}(\Omega) \times L_0^2(\Omega) \text{ such that} \\ a(\boldsymbol{u}_\varepsilon^g, \boldsymbol{v} - \boldsymbol{u}_\varepsilon^g) - b(\boldsymbol{v} - \boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \\ \qquad + j_g(T_\varepsilon(u_{\varepsilon,\tau}^g), T_\varepsilon(v_\tau)) - j_g(T_\varepsilon(u_{\varepsilon,\tau}^g), T_\varepsilon(u_{\varepsilon,\tau}^g)) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon^g)_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega), \\ b(\boldsymbol{u}_\varepsilon^g, q) = 0 \quad \forall q \in L_0^2(\Omega), \end{cases} \tag{3.4}$$

where $u_{\varepsilon,\tau}^g = \boldsymbol{u}_\varepsilon^g \cdot \boldsymbol{\tau}$. Further

$$\begin{aligned} j_g(T_\varepsilon(u_{\varepsilon,\tau}^g), T_\varepsilon(v_\tau)) - j_g(T_\varepsilon(u_{\varepsilon,\tau}^g), T_\varepsilon(u_{\varepsilon,\tau}^g)) &= \int\limits_S g(T_\varepsilon(u_{\varepsilon,\tau}^g))(T_\varepsilon(v_\tau) - T_\varepsilon(u_{\varepsilon,\tau}^g))\, ds \\ &\geq \int\limits_S g(T_\varepsilon(u_{\varepsilon,\tau}^g))T'_\varepsilon(u_{\varepsilon,\tau}^g)(v_\tau - u_{\varepsilon,\tau}^g)\, ds \end{aligned} \tag{3.5}$$

making use of convexity of T_ε.

The final form of the regularized state problem ($\mathcal{P}(g)$) reads as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \in \mathbb{V}(\Omega) \times L_0^2(\Omega) \text{ such that} \\ a(\boldsymbol{u}_\varepsilon^g, \boldsymbol{v}) - b(\boldsymbol{v}, p_\varepsilon^g) + \int\limits_S g(T_\varepsilon(u_{\varepsilon,\tau}^g))T'_\varepsilon(u_{\varepsilon,\tau}^g)v_\tau\, ds = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega), \\ b(\boldsymbol{u}_\varepsilon^g, q) = 0 \quad \forall q \in L_0^2(\Omega). \end{cases} \tag{$\mathcal{P}_\varepsilon(g)$}$$

Let us note that ($\mathcal{P}_\varepsilon(g)$) and (3.4) are equivalent and both will be used in the forthcoming analysis.

Now we prove that $(\mathcal{P}_\varepsilon(g))$ has a solution for any $\varepsilon > 0$ and $g \in \mathcal{U}_{ad}$. Let $g \in \mathcal{U}_{ad}$ and $\varepsilon > 0$ be fixed. For any $\varphi \in L^2_+(S)$ we introduce the following auxiliary problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}^g_\varepsilon(\varphi), p^g_\varepsilon(\varphi)) \in \mathbb{V}(\Omega) \times L^2_0(\Omega) \text{ such that} \\ a(\boldsymbol{u}^g_\varepsilon(\varphi), \boldsymbol{v}) - b(\boldsymbol{v}, p^g_\varepsilon(\varphi)) + \int\limits_S g(\varphi) T'_\varepsilon(u^g_{\varepsilon,\tau}(\varphi)) v_\tau \, ds = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega), \\ b(\boldsymbol{u}^g_\varepsilon(\varphi), q) = 0 \quad \forall q \in L^2_0(\Omega) \end{cases} \tag{$\mathcal{P}^\varphi_\varepsilon(g)$}$$

which is nothing else than the Stokes system with the regularized Tresca slip condition. Using standard arguments one can show that $(\mathcal{P}^\varphi_\varepsilon(g))$ has a unique solution $(\boldsymbol{u}^g_\varepsilon(\varphi), p^g_\varepsilon(\varphi))$ for any $\varphi \in L^2_+(S)$, $g \in \mathcal{U}_{ad}$, and $\varepsilon > 0$.

To prove the existence of a solution to $(\mathcal{P}_\varepsilon(g))$ we use the weak form of the Schauder fixed-point theorem. For the sake of simplicity we shall suppose that the slip part S is represented by a straight line segment which is parallel to one of the axes. By $H^{1/2}(S)$ we denote the trace space on S defined by

$$H^{1/2}(S) = \{\varphi \in L^2(S) \mid \exists v \in H^1(\Omega),\ v = \varphi \text{ on } S\} = \{\varphi \in L^2(S) \mid \|\varphi\|_{1/2,S} < \infty\},$$

where

$$\|\varphi\|_{1/2,S} = \left(\|\varphi\|^2_{0,S} + \int\limits_S \int\limits_S \tfrac{|\varphi(x)-\varphi(y)|^2}{|x-y|^2} \, dx\, dy \right)^{1/2}$$

is the norm in $H^{1/2}(S)$ (for details see [12]). From the definition of $\|\cdot\|_{1/2,S}$ and (3.1) it easily follows that

$$\|T_\varepsilon(\varphi)\|_{1/2,S} \le \|\varphi\|_{1/2,S} + c(\varepsilon) \tag{3.6}$$

holds for every $\varphi \in H^{1/2}(S)$, where $c(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Finally, let $H^{1/2}_+(S)$ stand for the cone of non-negative functions from $H^{1/2}(S)$.

Define the mapping $\Phi : H^{1/2}_+(S) \to H^{1/2}_+(S)$ by

$$\Phi(\varphi) = T_\varepsilon(u^g_{\varepsilon,\tau}(\varphi)). \tag{3.7}$$

Comparing the definitions of $(\mathcal{P}_\varepsilon(g))$ and $(\mathcal{P}^\varphi_\varepsilon(g))$ we see that $(\boldsymbol{u}^g_\varepsilon(\varphi), p^g_\varepsilon(\varphi))$ being the solution to $(\mathcal{P}^\varphi_\varepsilon(g))$ solves $(\mathcal{P}_\varepsilon(g))$ if and only of φ is a fixed-point of the mapping Φ.

Lemma 3. *It holds:*

- *there exists $R > 0$ which does not depend on $g \in \mathcal{U}_{ad}$, $\varphi \in H^{1/2}_+(S)$, and $\varepsilon > 0$ such that*

$$\Phi(B_R(0) \cap H^{1/2}_+(S)) \subset B_R(0) \cap H^{1/2}_+(S), \tag{3.8}$$

  where $B_R(0)$ is the ball of radius R and center at zero;

- $$\Phi \text{ is weakly continuous in } H^{1/2}_+(S). \tag{3.9}$$

Proof. *The first equation in $(\mathcal{P}^\varphi_\varepsilon(g))$ yields:*

$$\alpha \|\boldsymbol{u}^g_\varepsilon(\varphi)\|^2_{1,\Omega} \le a(\boldsymbol{u}^g_\varepsilon(\varphi), \boldsymbol{u}^g_\varepsilon(\varphi)) + \int\limits_S g(\varphi) T'_\varepsilon(u^g_{\varepsilon,\tau}(\varphi)) u^g_{\varepsilon,\tau}(\varphi)\, ds = (\boldsymbol{f}, \boldsymbol{u}^g_\varepsilon(\varphi))_{0,\Omega} \tag{3.10}$$

making use of the Korn inequality with the constant $\alpha > 0$ and the fact that $T'_\varepsilon(u^g_{\varepsilon,\tau}(\varphi)) u^g_{\varepsilon,\tau}(\varphi) \ge 0$ on S holds for any $g \in \mathcal{U}_{ad}$, $\varphi \in H^{1/2}_+(S)$, and $\varepsilon > 0$. The existence of R such that (3.8) *is satisfied follows from the trace theorem and* (3.6). *Clearly, R does not depend on the parameters mentioned in the lemma.*

Let $\varphi_n \rightharpoonup \varphi$ in $H^{1/2}(S)$, $\varphi_n, \varphi \in H^{1/2}_+(S)$ and $(\boldsymbol{u}^g_\varepsilon(\varphi_n), p^g_\varepsilon(\varphi_n))$ be the unique solution to $(\mathcal{P}^{\varphi_n}_\varepsilon(g))$. Then

$$\|\boldsymbol{u}^g_\varepsilon(\varphi_n)\|_{1,\Omega} \le R \quad \forall n \in \mathbb{N} \tag{3.11}$$

and also

$$\|p^g_\varepsilon(\varphi_n)\|_{0,\Omega} \le C \quad \forall n \in \mathbb{N} \tag{3.12}$$

using (3.10) *and the inf-sup condition* (2.14) *in* $(\mathcal{P}_\varepsilon^{\varphi_n}(g))$. *The constant* C *in* (3.12) *does not depend on* $g \in \mathcal{U}_{ad}, \varphi_n \in H_+^{1/2}(S)$, *and* $\varepsilon > 0$ *just as* R. *Thus one can pass to subsequences denoted by the same symbol such that*

$$\boldsymbol{u}_\varepsilon^g(\varphi_n) \rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } (H^1(\Omega))^2, \quad n \to \infty, \tag{3.13}$$

$$p_\varepsilon^g(\varphi_n) \rightharpoonup \overline{p} \quad \text{in } L^2(S), \quad n \to \infty, \tag{3.14}$$

where $(\overline{\boldsymbol{u}}, \overline{p}) \in \mathbb{V}(\Omega) \times L_0^2(S)$. *It is easy to show that* $(\overline{\boldsymbol{u}}, \overline{p}) = (\boldsymbol{u}_\varepsilon^g(\varphi), p_\varepsilon^g(\varphi))$ *is a solution to* $(\mathcal{P}_\varepsilon^{\varphi}(g))$. *Only what we have to verify is the limit passage with the slip term (recall that* $\varepsilon > 0$ *is fixed):*

$$\begin{aligned}
&\left| \int_S g(\varphi_n) T_\varepsilon'(\boldsymbol{u}_{\varepsilon,\tau}^g(\varphi_n)) v_\tau \, ds - \int_S g(\varphi) T_\varepsilon'(\overline{u}_\tau) v_\tau \, ds \right| \\
&\qquad \le \int_S |(g(\varphi_n) - g(\varphi))| \left| T_\varepsilon'(\boldsymbol{u}_{\varepsilon,\tau}^g(\varphi_n)) \right| |v_\tau| \, ds + \int_S g(\varphi) \left| T_\varepsilon'(\boldsymbol{u}_{\varepsilon,\tau}^g(\varphi_n)) - T_\varepsilon'(\overline{u}_\tau) \right| |v_\tau| \, ds \\
&\qquad \le L \|\varphi_n - \varphi\|_{0,S} \|v_\tau\|_{0,S} + \int_S g(\varphi) \left| T_\varepsilon'(\boldsymbol{u}_{\varepsilon,\tau}^g(\varphi_n)) - T_\varepsilon'(\overline{u}_\tau) \right| |v_\tau| \, ds \to 0, \quad n \to \infty
\end{aligned} \tag{3.15}$$

using that $g \in \mathcal{U}_{ad}$ *is Lipschitz continuous,* $|T_\varepsilon'(q)| \le 1 \ \forall q \in \mathbb{R}_+, \forall \varepsilon > 0$ *and the fact that* $\varphi_n \to \varphi$ *in* $L^2(S)$ *owing to compact embedding of* $H^{1/2}(S)$ *into* $L^2(S)$. *The last term in* (3.15) *tends to zero because of* (3.13), *continuity and boundedness of* T_ε' *together with the Lebesgue dominated convergence theorem. Thus* $(\overline{\boldsymbol{u}}, \overline{p}) = (\boldsymbol{u}_\varepsilon^g(\varphi), p_\varepsilon^g(\varphi))$ *solves* $(\mathcal{P}_\varepsilon^{\varphi}(g))$ *and the whole sequence* $\{(\boldsymbol{u}_\varepsilon^g(\varphi_n), p_\varepsilon^g(\varphi_n))\}$ *tends weakly to* $(\boldsymbol{u}_\varepsilon^g(\varphi), p_\varepsilon^g(\varphi))$. *From* (3.13) *we obtain:*

$$u_{\varepsilon,\tau}^g(\varphi_n) \to u_{\varepsilon,\tau}^g(\varphi) \quad \text{in } L^2(S), \quad n \to \infty,$$

and consequently

$$T_\varepsilon(u_{\varepsilon,\tau}^g(\varphi_n)) \to T_\varepsilon(u_{\varepsilon,\tau}^g(\varphi)) \quad \text{in } L^2(S), \quad n \to \infty.$$

Since $\{\|T_\varepsilon(u_{\varepsilon,\tau}^g(\varphi_n))\|_{1/2,S}\}$ *is bounded there exists* $\chi \in H_+^{1/2}(S)$ *such that* $T_\varepsilon(u_{\varepsilon,\tau}^g(\varphi_n)) \rightharpoonup \chi$ *in* $H_+^{1/2}(S)$. *Hence* $\chi = T_\varepsilon(u_{\varepsilon,\tau}^g(\varphi))$ *proving* (3.9). □

Corollary 1. *From* (3.8), (3.9), *and the weak version of the Schauder fixed-point theorem it follows that* Φ *has a fixed point in* $H_+^{1/2}(S)$ *or equivalently* $(\mathcal{P}_\varepsilon(g))$ *has at least one solution for any* $g \in \mathcal{U}_{ad}$ *and* $\varepsilon > 0$.

Theorem 3. *Let* $g \in \mathcal{U}_{ad}$ *be one-sided Lipschitz continuous in* $\mathbb{R}_+$ *with modulus* $L_{\text{OS}} \ge 0$ *satisfying*

$$L_{\text{OS}} < \frac{1}{c_{\text{tr}}^2}, \tag{3.16}$$

where $c_{\text{tr}} > 0$ *is the norm of the trace mapping* $\text{tr} : \mathbb{V}(\Omega) \to L^2(S)$, $\text{tr}\, \boldsymbol{v} = v_\tau$ *assuming that* $\mathbb{V}(\Omega)$ *is equipped with the norm* $\sqrt{a(\boldsymbol{v}, \boldsymbol{v})}$. *Then the mapping* Φ *has a unique fixed-point. Consequently* $(\mathcal{P}_\varepsilon(g))$ *has a unique solution for any* $g \in \mathcal{U}_{ad}$ *and* $\varepsilon > 0$.

Proof. *Let* $(\boldsymbol{u}_\varepsilon^g, p_\varepsilon^g)$, $(\overline{\boldsymbol{u}}_\varepsilon^g, \overline{p}_\varepsilon^g)$ *be two solutions of* $(\mathcal{P}_\varepsilon(g))$. *Then*

$$\begin{aligned}
a(\boldsymbol{u}_\varepsilon^g, \boldsymbol{v} - \boldsymbol{u}_\varepsilon^g) - b(\boldsymbol{v} - \boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) + j_g(T_\varepsilon(u_{\varepsilon,\tau}^g), T_\varepsilon(v_\tau)) - j_g(T_\varepsilon(u_{\varepsilon,\tau}^g), T_\varepsilon(u_{\varepsilon,\tau}^g)) \\
\ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon^g) \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
a(\overline{\boldsymbol{u}}_\varepsilon^g, \boldsymbol{v} - \overline{\boldsymbol{u}}_\varepsilon^g) - b(\boldsymbol{v} - \overline{\boldsymbol{u}}_\varepsilon^g, \overline{p}_\varepsilon^g) + j_g(T_\varepsilon(\overline{u}_{\varepsilon,\tau}^g), T_\varepsilon(v_\tau)) - j_g(T_\varepsilon(\overline{u}_{\varepsilon,\tau}^g), T_\varepsilon(\overline{u}_{\varepsilon,\tau}^g)) \\
\ge (\boldsymbol{f}, \boldsymbol{v} - \overline{\boldsymbol{u}}_\varepsilon^g) \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega).
\end{aligned} \tag{3.18}$$

Inserting $\boldsymbol{v} := \overline{\boldsymbol{u}}_\varepsilon^g$, $\boldsymbol{v} := \boldsymbol{u}_\varepsilon^g$ *into* (3.17), *and* (3.18), *respectively and adding both inequalities we obtain:*

$$\begin{aligned}
a(\boldsymbol{u}_\varepsilon^g - \overline{\boldsymbol{u}}_\varepsilon^g, \boldsymbol{u}_\varepsilon^g - \overline{\boldsymbol{u}}_\varepsilon^g) &\le \int_S \left(g(T_\varepsilon(\overline{u}_{\varepsilon,\tau}^g)) - g(T_\varepsilon(u_{\varepsilon,\tau}^g)) \right) (T_\varepsilon(u_{\varepsilon,\tau}^g) - T_\varepsilon(\overline{u}_{\varepsilon,\tau}^g)) \, ds \\
&\overset{(2.3)}{\le} L_{\text{OS}} \int_S |T_\varepsilon(u_{\varepsilon,\tau}^g) - T_\varepsilon(\overline{u}_{\varepsilon,\tau}^g)|^2 \, ds \le L_{\text{OS}} \|u_{\varepsilon,\tau}^g - \overline{u}_{\varepsilon,\tau}^g\|_{0,S}^2.
\end{aligned} \tag{3.19}$$

Here we used the fact that $b(\boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) = b(\overline{\boldsymbol{u}}_\varepsilon^g, \overline{p}_\varepsilon^g) = 0$ *and* $|T_\varepsilon'(q)| \leq 1\ \forall q \in \mathbb{R}_+,\ \forall \varepsilon > 0$. *From* (3.19) *and the definition of the constant* $c_{\mathrm{tr}} > 0$ *it follows:*

$$\frac{1}{c_{\mathrm{tr}}^2}\|u_{\varepsilon,\tau}^g - \overline{u}_{\varepsilon,\tau}^g\|_{0,S}^2 \leq a(\boldsymbol{u}_\varepsilon^g - \overline{\boldsymbol{u}}_\varepsilon^g, \boldsymbol{u}_\varepsilon^g - \overline{\boldsymbol{u}}_\varepsilon^g) \leq L_{\mathrm{OS}}\|u_{\varepsilon,\tau}^g - \overline{u}_{\varepsilon,\tau}^g\|_{0,S}^2.$$

If $L_{\mathrm{OS}} < 1/c_{\mathrm{tr}}^2$ *then* $u_{\varepsilon,\tau}^g = \overline{u}_{\varepsilon,\tau}^g$ *on* S. *Consequently,* $\boldsymbol{u}_\varepsilon^g = \overline{\boldsymbol{u}}_\varepsilon^g$ *in* Ω. *Uniqueness of* p_ε^g *is due to the inf-sup condition* (2.14). □

Now we proceed to the definition of the identification problem which uses the penalized state problem $(\mathcal{P}_\varepsilon(g))$, $\varepsilon > 0$ fixed. As before, we introduce the graph $\mathcal{G}_\varepsilon$ of the respective generally multivalued control-to-state mapping:

$$\mathcal{G}_\varepsilon = \left\{(g, \boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \mid g \in \mathcal{U}_{ad},\ (\boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \text{ solves } (\mathcal{P}_\varepsilon(g))\right\}.$$

The identification problem reads as follows:

$$\text{Find } (g_\varepsilon^*, \boldsymbol{u}_\varepsilon^{g_\varepsilon^*}, p_\varepsilon^{g_\varepsilon^*}) \in \operatorname{argmin}\left\{I(g, \boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \mid (g, \boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \in \mathcal{G}_\varepsilon\right\}, \qquad (\mathbb{P}_\varepsilon)$$

where

$$I(g, \boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) = \frac{1}{2}\int_S (u_{\varepsilon,\tau}^g - \widetilde{u}_\tau)^2\, ds$$

with a straightforward modification of $(\mathbb{P}_\varepsilon)$ when $(\mathcal{P}_\varepsilon(g))$ has a unique solution for any $g \in \mathcal{U}_{ad}$. This is the case when $g \in \mathcal{U}_{ad}^{\#}$ is defined by (2.5), e.g.

To prove the existence of solutions to $(\mathbb{P}_\varepsilon)$ we shall need the compactness property of the graph $\mathcal{G}_\varepsilon$ for any $\varepsilon > 0$ fixed. We shall prove the following result parallel to Theorem 1.

Theorem 4. *The graph* $\mathcal{G}_\varepsilon$ *is compact for any* $\varepsilon > 0$ *in the following sense: for any sequence* $\{(g_n, \boldsymbol{u}_\varepsilon^{g_n}, p_\varepsilon^{g_n})\}_n$, $(g_n, \boldsymbol{u}_\varepsilon^{g_n}, p_\varepsilon^{g_n}) \in \mathcal{G}_\varepsilon$ *there exist: its subsequence denoted by the same symbol and an element* $(g, \boldsymbol{u}_\varepsilon^g, p_\varepsilon^g) \in \mathcal{G}_\varepsilon$ *such that*

$$g_n \rightrightarrows g \quad \text{in } [0, \Delta], \qquad (3.20)$$

$$\boldsymbol{u}_\varepsilon^{g_n} \to \boldsymbol{u}_\varepsilon^g \quad \text{in } (H^1(\Omega))^2, \quad n \to \infty, \qquad (3.21)$$

$$p_\varepsilon^{g_n} \to p_\varepsilon^g \quad \text{in } L^2(\Omega), \quad n \to \infty. \qquad (3.22)$$

Proof. *We may assume that* $\{g_n\}$ *satisfies* (3.20) *for some* $g \in \mathcal{U}_{ad}$. *From* (3.11) *and* (3.12) *which are valid with the constants* R *and* C *independent of* $g \in \mathcal{U}_{ad}$, $\varphi \in H_+^{1/2}(S)$ *and* $\varepsilon > 0$ *it follows that* $\{(\boldsymbol{u}_\varepsilon^{g_n}, p_\varepsilon^{g_n})\}$ *is bounded in* $(H^1(\Omega))^2 \times L^2(\Omega)$. *Thus one can pass to a subsequence such that*

$$\boldsymbol{u}_\varepsilon^{g_n} \rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } (H^1(\Omega))^2, \quad n \to \infty, \qquad (3.23)$$

$$p_\varepsilon^{g_n} \rightharpoonup \overline{p} \quad \text{in } L^2(\Omega), \quad n \to \infty. \qquad (3.24)$$

To show that $(\overline{\boldsymbol{u}}, \overline{p})$ *solves* $(\mathcal{P}_\varepsilon(g))$ *we pass to the limit with* $n \to \infty$ *in* $(\mathcal{P}_\varepsilon(g_n))$ *written now in the form*

$$a(\boldsymbol{u}_\varepsilon^{g_n}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon^{g_n}) - b(\boldsymbol{v} - \boldsymbol{u}_\varepsilon^{g_n}, p_\varepsilon^{g_n}) + j_{g_n}(T_\varepsilon(u_{\varepsilon,\tau}^{g_n}), T_\varepsilon(v_\tau)) - j_{g_n}(T_\varepsilon(u_{\varepsilon,\tau}^{g_n}), T_\varepsilon(u_{\varepsilon,\tau}^{g_n})) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon^{g_n})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega),$$

$$b(\boldsymbol{u}_\varepsilon^{g_n}, q) = 0 \quad \forall q \in L_0^2(\Omega).$$

Clearly $(\overline{\boldsymbol{u}}, \overline{p}) \in \mathbb{V}(\Omega) \times L_0^2(\Omega)$ *and* $\operatorname{div} \overline{\boldsymbol{u}} = 0$ *in* Ω. *Weak lower semicontinuity of* a *and* (3.23) *entail:*

$$\limsup_{n\to\infty} a(\boldsymbol{u}_\varepsilon^{g_n}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon^{g_n}) \geq a(\overline{\boldsymbol{u}}, \boldsymbol{v} - \overline{\boldsymbol{u}}). \qquad (3.25)$$

Further

$$\lim_{n\to\infty} b(\boldsymbol{v} - \boldsymbol{u}_\varepsilon^{g_n}, p_\varepsilon^{g_n}) = \lim_{n\to\infty} b(\boldsymbol{v}, p_\varepsilon^{g_n}) = b(\boldsymbol{v}, \overline{p}) = b(\boldsymbol{v} - \overline{\boldsymbol{u}}, \overline{p}) \qquad (3.26)$$

using that $\operatorname{div} \boldsymbol{u}_\varepsilon^{g_n} = \operatorname{div} \overline{\boldsymbol{u}} = 0$ *in* Ω *and* (3.24). *Finally*

$$\begin{cases} j_{g_n}(T_\varepsilon(u_{\varepsilon,\tau}^{g_n}), T_\varepsilon(v_\tau)) \to j_g(T_\varepsilon(\overline{u}_\tau), T_\varepsilon(v_\tau)) \\ j_{g_n}(T_\varepsilon(u_{\varepsilon,\tau}^{g_n}), T_\varepsilon(u_{\varepsilon,\tau}^{g_n})) \to j_g(T_\varepsilon(\overline{u}_\tau), T_\varepsilon(\overline{u}_\tau)) \end{cases} \qquad (3.27)$$

making use of Lemma 1 and the fact that $\|T_\varepsilon(u^{g_n}_{\varepsilon,\tau}) - T_\varepsilon(\overline{u}_\tau)\|_{0,S} \to 0$ *as* $n \to \infty$ *which is an easy consequence of* (3.23) *and the definition of* T_ε. *Since* $\operatorname{div} \overline{\boldsymbol{u}} = 0$ *in* Ω *it follows from* (3.25)-(3.26) *that the pair* $(\overline{\boldsymbol{u}}, \overline{p}) = (\boldsymbol{u}^g_\varepsilon, p^g_\varepsilon)$ *i.e. it solves* $(\mathcal{P}_\varepsilon(g))$. *To prove strong convergence in* (3.21) *we use the definition of* $(\mathcal{P}_\varepsilon(g_n))$:

$$\lim_{n\to\infty}\left\{a(\boldsymbol{u}^{g_n}_\varepsilon, \boldsymbol{u}^{g_n}_\varepsilon) + \int_S g_n(T_\varepsilon(u^{g_n}_{\varepsilon,\tau}))T'(u^{g_n}_{\varepsilon,\tau})u^{g_n}_{\varepsilon,\tau}\, ds\right\}$$
$$= \lim_{n\to\infty}(\boldsymbol{f}, \boldsymbol{u}^{g_n}_\varepsilon) = (\boldsymbol{f}, \boldsymbol{u}^g_\varepsilon) = a(\boldsymbol{u}^g_\varepsilon, \boldsymbol{u}^g_\varepsilon) + \int_S g(T_\varepsilon(u^g_{\varepsilon,\tau}))T'_\varepsilon(u^g_{\varepsilon,\tau})u^g_{\varepsilon,\tau}\, ds. \quad (3.28)$$

Further,

$$\lim_{n\to\infty}\int_S g_n(T_\varepsilon(u^{g_n}_{\varepsilon,\tau}))T'_\varepsilon(u^{g_n}_{\varepsilon,\tau})u^{g_n}_{\varepsilon,\tau}\, ds = \lim_{n\to\infty}\int_S g_n(T_\varepsilon(u^{g_n}_{\varepsilon,\tau}))T'_\varepsilon(u^{g_n}_{\varepsilon,\tau})u^{g}_{\varepsilon,\tau}\, ds$$
$$+ \lim_{n\to\infty}\int_S g_n(T_\varepsilon(u^{g_n}_{\varepsilon,\tau}))T'_\varepsilon(u^{g_n}_{\varepsilon,\tau})(u^{g_n}_{\varepsilon,\tau} - u^g_{\varepsilon,\tau})\, ds =: I_{1,n} + I_{2,n}.$$

It easily follows that

$$I_{1,n} \to \int_S g(T_\varepsilon(u^g_{\varepsilon,\tau}))T'_\varepsilon(u^g_{\varepsilon,\tau})u^g_{\varepsilon,\tau}\, ds, \quad n \to \infty,$$

and trivially $I_{2,n} \to 0$ *as* $n \to \infty$. *From this and* (3.28) *we see that* $a(\boldsymbol{u}^{g_n}_\varepsilon, \boldsymbol{u}^{g_n}_\varepsilon) \to a(\boldsymbol{u}^g_\varepsilon, \boldsymbol{u}^g_\varepsilon)$, *as* $n \to \infty$. *Strong convergence* (3.22) *follows from the inf-sup condition* (2.14) *and* (3.21). □

A direct consequence of this theorem is the following existence result.

Theorem 5. *Problem* $(\mathbb{P}_\varepsilon)$ *has a solution for any* $\varepsilon > 0$.

4 | RELATION BETWEEN $(\mathbb{P})$ AND $(\mathbb{P}_\varepsilon)$, $\varepsilon \to 0+$

In this section we shall study the mutual relation between the original identification problem $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ if $\varepsilon \to 0+$. To this end we shall need the following stability result of solutions to $(\mathcal{P}_\varepsilon(g))$ with respect to $g \in \mathcal{U}_{ad}$ and $\varepsilon \to 0+$.

Theorem 6. *For any* $\{(g_\varepsilon, \boldsymbol{u}^{g_\varepsilon}_\varepsilon, p^{g_\varepsilon}_\varepsilon)\}_\varepsilon$, $(g_\varepsilon, \boldsymbol{u}^{g_\varepsilon}_\varepsilon, p^{g_\varepsilon}_\varepsilon) \in \mathcal{G}_\varepsilon$, $\varepsilon \to 0+$ *there exist: a sequence* $\{\varepsilon_n\}$, $\varepsilon_n \to 0+$ *as* $n \to \infty$ *and an element* $(g, \boldsymbol{u}^g, p^g) \in \mathcal{G}$ *such that*

$$g_n \rightrightarrows g \quad \text{in } [0, \Delta], \quad (g_n := g_{\varepsilon_n}) \quad (4.1)$$
$$\boldsymbol{u}^{g_n}_{\varepsilon_n} \to \boldsymbol{u}^g \quad \text{in } (H^1(\Omega))^2, \quad (4.2)$$
$$p^{g_n}_{\varepsilon_n} \to p^g \quad \text{in } L^2(\Omega), \quad n \to \infty. \quad (4.3)$$

Proof. *Since* $\mathcal{U}_{ad}$ *is compact in* $\boldsymbol{C}([0, \Delta])$ *and* $\{(g_\varepsilon, \boldsymbol{u}^{g_\varepsilon}_\varepsilon, p^{g_\varepsilon}_\varepsilon)\}_\varepsilon$ *is bounded in* $(H^1(\Omega))^2 \times L^2(\Omega)$ *uniformly with respect to* $g \in \mathcal{U}_{ad}$ *and* $\varepsilon \to 0+$, *we obtain* (4.1) *and the existence of* $\overline{\boldsymbol{u}} \in \mathbb{V}(\Omega)$, $\overline{p} \in L^2_0(\Omega)$ *such that*

$$\boldsymbol{u}^{g_n}_{\varepsilon_n} \rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } (H^1(\Omega))^2$$
$$p^{g_n}_{\varepsilon_n} \rightharpoonup \overline{p} \quad \text{in } L^2(\Omega), \quad n \to \infty.$$

To prove that $(\overline{\boldsymbol{u}}, \overline{p})$ *solves* $(\mathcal{P}(g))$ *we use* (3.4):

$$a(\boldsymbol{u}^{g_n}_{\varepsilon_n}, \boldsymbol{v} - \boldsymbol{u}^{g_n}_{\varepsilon_n}) - b(\boldsymbol{v} - \boldsymbol{u}^{g_n}_{\varepsilon_n}, p^{g_n}_{\varepsilon_n}) + j_{g_n}(T_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau}), T_{\varepsilon_n}(v_\tau)) - j_{g_n}(T_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau}), T_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau}))$$
$$\geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^{g_n}_{\varepsilon_n})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega),$$
$$b(\boldsymbol{u}^{g_n}_{\varepsilon_n}, q) = 0 \quad \forall q \in L^2_0(\Omega).$$

Letting $n \to \infty$ and using that

$$j_{g_n}(T_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau}), T_{\varepsilon_n}(v_\tau)) \to j_g(|\bar{u}_\tau|, |v_\tau|)$$
$$j_{g_n}(T_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau}), T_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau})) \to j_g(|\bar{u}_\tau|, |\bar{u}_\tau|), \quad n \to \infty$$

as follows from Lemma 1, (3.2) *and the fact that* $u^{g_n}_{\varepsilon_n,\tau} \to \bar{u}_\tau$ *in* $L^2(S)$ *as* $n \to \infty$ *we see that* $(\bar{\boldsymbol{u}}, \bar{p}) = (\boldsymbol{u}^g, p^g)$ *solves* $(\mathcal{P}(g))$. *To prove strong convergence in* (4.2) *we use the definition of* $(\mathcal{P}_{\varepsilon_n}(g_n))$ *with the test function* $\boldsymbol{v} = \boldsymbol{u}^{g_n}_{\varepsilon_n}$*:*

$$\lim_{n\to\infty}\left[a(\boldsymbol{u}^{g_n}_{\varepsilon_n}, \boldsymbol{u}^{g_n}_{\varepsilon_n}) + \int_S g_n(T_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau}))T'_{\varepsilon_n}(u^{g_n}_{\varepsilon_n,\tau})\, ds\right] = \lim_{n\to\infty}(\boldsymbol{f}, \boldsymbol{u}^{g_n}_{\varepsilon_n})_{0,\Omega} = (\boldsymbol{f}, \boldsymbol{u}^g)_{0,\Omega} \overset{(2.12)}{=} a(\boldsymbol{u}^g, \boldsymbol{u}^g) + j_g(|u^g_\tau|, |u^g_\tau|).$$

From this, Lemma 1 and (3.3) *it follows that*

$$a(\boldsymbol{u}^{g_n}_{\varepsilon_n}, \boldsymbol{u}^{g_n}_{\varepsilon_n}) \to a(\boldsymbol{u}^g, \boldsymbol{u}^g), \quad n \to \infty.$$

Strong convergence (4.3) *is a consequence of the inf-sup condition* (2.14) *and* (3.21). □

Let $\tilde{\mathcal{G}}$ be a set of all accumulation points of $\{(g_\varepsilon, \boldsymbol{u}^{g_\varepsilon}_\varepsilon, p^{g_\varepsilon}_\varepsilon)\}_\varepsilon$, $(g_\varepsilon, \boldsymbol{u}^{g_\varepsilon}_\varepsilon, p^{g_\varepsilon}_\varepsilon) \in \mathcal{G}_\varepsilon$, $\varepsilon \to 0+$, in the sense of (4.1)–(4.3). From the previous theorem it follows that $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ but it is not clear whether $\tilde{\mathcal{G}} = \mathcal{G}$ or not. In other words, are all solutions to $(\mathcal{P}(g))$, $g \in \mathcal{U}_{ad}$ attainable by solutions to regularized problems? The answer is positive if $g \in \mathcal{U}^\#_{ad}$ since (by the definition of $\mathcal{U}^\#_{ad}$) $(\mathcal{P}(g))$ has unique solution which is the limit of solutions to $(\mathcal{P}_\varepsilon(g))$. Now we are ready to prove the following convergence result.

Theorem 7. *From any sequence* $\{(g^*_\varepsilon, \boldsymbol{u}^{g^*_\varepsilon}_\varepsilon, p^{g^*_\varepsilon}_\varepsilon)\}_\varepsilon$ *of solutions to* $(\mathbb{P}_\varepsilon)$, $\varepsilon \to 0+$ *one can find a subsequence denoted by the same symbol and an element* $(g^*, \boldsymbol{u}^{g^*}, p^{g^*}) \in \tilde{\mathcal{G}}$ *such that*

$$\begin{cases} g^*_\varepsilon \rightrightarrows g^* & \text{in } [0, \Delta], \\ \boldsymbol{u}^{g^*_\varepsilon}_\varepsilon \to \boldsymbol{u}^{g^*} & \text{in } (H^1(\Omega))^2, \\ p^{g^*_\varepsilon}_\varepsilon \to p^{g^*} & \text{in } L^2(\Omega), \quad \varepsilon \to 0+ \end{cases} \tag{4.4}$$

and

$$(g^*, \boldsymbol{u}^{g^*}, p^{g^*}) \in \operatorname{argmin}\{I(g, \boldsymbol{u}^g, p^g) \mid (g, \boldsymbol{u}^g, p^g) \in \tilde{\mathcal{G}}\}.$$

In addition, any accumulation point of $\{(g^*_\varepsilon, \boldsymbol{u}^{g^*_\varepsilon}_\varepsilon, p^{g^*_\varepsilon}_\varepsilon)\}_\varepsilon$ *in the sense of* (4.4) *enjoys this property.*

Proof. *It easily follows from the definition of* $(\mathbb{P}_\varepsilon)$*, Theorem 6 and continuity of* I. □

It is worth noticing that if the state problems $(\mathcal{P}(g))$, $g \in \mathcal{U}_{ad}$ have multiple solutions then the triplet $(g^*, \boldsymbol{u}^{g^*}, p^{g^*})$ might be suboptimal since it minimizes I on $\tilde{\mathcal{G}}$. On the other hand if $g \in \mathcal{U}^\#_{ad}$, i.e. $(\mathcal{P}(g))$ has a unique solution for any $g \in \mathcal{U}^\#_{ad}$ then $(g^*, \boldsymbol{u}^{g^*}, p^{g^*})$ is an optimal solution to $(\mathbb{P})$.

Remark 1. So far we did not specify any particular form of $g \in \mathcal{U}_{ad}$. In this remark we shall present a class of affine g:s, i.e.

$$g(|u_\tau|) = \kappa + k|u_\tau|, \tag{4.5}$$

where κ, k are nonnegative functions from $L^\infty(S)$. The slip conditions $(2.1)_{5,6}$ with g defined by (4.5) read as follows:

$$\begin{cases} u_\tau(x) = 0 \implies |\sigma_\tau(x)| \le \kappa(x) \\ u_\tau(x) \neq 0 \implies \sigma_\tau(x) = -k(x)u_\tau(x) - \kappa(x)\operatorname{sign}(u_\tau(x)), \quad x \in S. \end{cases} \tag{4.6}$$

The conditions in (4.6) involve three important slip models:

a) classical Navier slip if $\kappa \equiv 0$ on S;

b) Tresca slip if $k \equiv 0$ on S;

c) threshold Navier slip if both $\kappa, k \not\equiv 0$ on S.

The weak formulation of the Stokes system with the slip conditions (4.6) is defined as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \mathbb{V}(\Omega) \times L^2_0(\Omega) \text{ such that} \\ a_k(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + j_\kappa(|v_\tau|) - j_\kappa(|u_\tau|) \\ \qquad\qquad \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega), \\ b(\boldsymbol{u}, q) = 0 \quad \forall q \in L^2_0(\Omega), \end{cases} \tag{$\mathcal{P}(k,\kappa)$}$$

where

$$a_k(\boldsymbol{u},\boldsymbol{v}) = \int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v}\,dx + \int\limits_S k u_\tau v_\tau\,ds, \quad b(\boldsymbol{v},q) = \int\limits_\Omega q \operatorname{div}\boldsymbol{v}\,dx, \quad j_\kappa(|v_\tau|) = \int\limits_S \kappa|v_\tau|\,ds.$$

Problem $(\mathcal{P}(k,\kappa))$ has a unique solution for any $(k,\kappa) \in (L^\infty_+(S))^2$. It can be used as the state problem in $(\mathbb{P}^\#)$ with the admissible set

$$\mathcal{U}^\#_{ad} = \left\{(k,\kappa) \in (L^\infty(S))^2 \mid 0 \le k \le k_{\max},\ 0 \le \kappa \le \kappa_{\max}\right\}, \tag{4.7}$$

where $k_{\max}, \kappa_{\max} > 0$ are given (now the parameter Δ appearing in (2.4) is not needed). Let us observe that all the results which have been proven above, remain valid also in this case. In addition, the proofs are much more simpler due to the fact that g is a linear function of $|u_\tau|$ and the set $\mathcal{U}^\#_{ad}$ defined by (4.7) is weak* compact in $L^\infty(S)$.

5 | APPROXIMATION AND NUMERICAL REALIZATION OF $(\mathbb{P}_\varepsilon)$

In this section we describe how to discretize and numerically realize optimal control problems governed by the Stokes system with the regularized threshold slip condition. The slip bound function $g \in \mathcal{U}_{ad}$ will be discretized by a linear combination of suitable functions, while a finite element method will be used to discretize the state equation $(\mathcal{P}_\varepsilon(g))$.

Let us assume that Ω is a polygonal domain such that $\bar{S}$ is a line segment in the x_1-coordinate direction. Let $\{\mathcal{T}_h\}, h \to 0$ be a regular family of triangulations of $\bar{\Omega}$. With any $\mathcal{T}_h$ we associate the following finite dimensional spaces:

$$\mathbb{V}_h = \{\boldsymbol{v}^h = (v_1^h, v_2^h) \in (C(\bar{\Omega}))^2 \mid \boldsymbol{v}^h|_T \in (P_2(T))^2\ \forall T \in \mathcal{T}_h,\ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma,\ v_2^h = 0 \text{ on } S\}$$
$$Q_h = \{q^h \in C(\bar{\Omega}) \mid q^h|_T \in P_1(T)\ \forall T \in \mathcal{T}_h,\ \textstyle\int_\Omega q^h\,dx = 0\}.$$

This choice corresponds to the P2/P1 element satisfying the Ladyzhenskaya–Babuška–Brezzi (LBB) condition [13].

In addition, we define the following Bezier (or polynomial in Bernstein form) type parametrization of g:

$$g(\xi) \approx g_{\mathbf{a}}(\xi) := \sum_{i=0}^m a_i B_i(\xi), \tag{5.1}$$

where $\{B_i\}_{i=0}^m$ are the Bernstein basis polynomials of degree m on the interval $[0,\Delta]$. Thus the discrete control variable is the vector $\mathbf{a} = (a_0, a_1, ..., a_m) \in \mathbb{R}^{m+1}$. The use of the Bernstein basis polynomials instead of the monomials $\{\xi^i\}_{i=0}^m$ in the parametrization allows us later on to include easily the sign and slope constraints imposed on $g_{\mathbf{a}}$.

The finite element approximation of the state problem with the parameterized control $g_{\mathbf{a}}$ then reads (the superscript $g_{\mathbf{a}}$ and subscript ε are omitted for clarity of notation, e.g. $\boldsymbol{u}^h := \boldsymbol{u}_\varepsilon^{g_{\mathbf{a}},h}$):

$$\begin{cases} \text{Find } (\boldsymbol{u}^h, p^h) \in \mathbb{V}_h \times Q_h \text{ such that} \\ a(\boldsymbol{u}^h,\boldsymbol{v}^h) - b(\boldsymbol{v}^h,p^h) + \int\limits_S g_{\mathbf{a}}(T_\varepsilon(u_\tau^h))T'_\varepsilon(u_\tau^h)v_\tau^h\,ds = (\boldsymbol{f},\boldsymbol{v}^h)_{0,\Omega} \quad \forall \boldsymbol{v}^h \in \mathbb{V}_h, \\ b(\boldsymbol{u}^h,q^h) = 0 \quad \forall q^h \in Q_h. \end{cases} \tag{$\mathcal{P}_\varepsilon^h(\mathbf{a})$}$$

In the rest of the paper we shall suppose that $\mathcal{U}^\#_{ad}$ is defined by (2.5) and $g_{\mathbf{a}} \in \mathcal{U}^\#_{ad}$ which together with the LBB condition guarrantee uniqueness of the solution to $(\mathcal{P}_\varepsilon^h(\mathbf{a}))$. The algebraic form of $(\mathcal{P}_\varepsilon^h(\mathbf{a}))$ leads to the system of nonlinear algebraic equations:

$$\mathbf{r}(\mathbf{a};\mathbf{q}) := \begin{bmatrix} \mathbf{A}\mathbf{u} + \mathbf{c}_\varepsilon(\mathbf{a},\mathbf{u}) - \mathbf{B}\mathbf{p} - \mathbf{f} \\ \mathbf{B}^{\mathrm{T}}\mathbf{u} \end{bmatrix} = \mathbf{0}, \tag{5.2}$$

where $\mathbf{q} := [\mathbf{u}^{\mathrm{T}}, \mathbf{p}^{\mathrm{T}}]^{\mathrm{T}}$ is the vector of degrees of freedom containing the nodal values of the discrete velocity $\boldsymbol{u}^h$ and the pressure p^h. The block matrices $\mathbf{A}, \mathbf{B}$ and the vector $\mathbf{f}$ arise from the standard discretized Stokes system. The nonlinear term $\mathbf{c}_\varepsilon(\mathbf{a},\mathbf{u})$ is obtained by integrating numerically the boundary integral term in $(\mathcal{P}_\varepsilon^h(\mathbf{a}))$. The system will be solved efficiently by Newton's method.

Owing to the use of Newton's method we need the C^1-regularity of $g \in \mathcal{U}^\#_{ad}$ in $\mathbb{R}_+$. To this end we will use a slightly smaller set of admissibe slip bounds in the numerical examples, namely

$$\mathcal{U}^\dagger_{ad} = \left\{g \in \mathcal{U}^\#_{ad} \mid g \in C^1([0,\Delta]),\ g'(\Delta) = 0\right\}.$$

Let us recall that $\mathcal{U}_{ad}^{\#}$ is defined by (2.5). From the basic properties of the polynomials in Bernstein form ([14, Ch. 7]) it follows that the constraint $g_{\mathbf{a}} \in \mathcal{U}_{ad}^{\dagger}$ is equivalent to $\mathbf{a} \in \mathcal{U}$, where

$$\mathcal{U} = \Big\{ \mathbf{a} \in \mathbb{R}^{m+1} \mid a_i \geq 0,\ i=0,...,m, \quad a_{i+1} - a_i \geq 0,\ i=0,...,m-1, \quad a_m = a_{m-1}, \\ -\frac{L\Delta}{m} \leq a_{i+1} - a_i \leq \frac{L\Delta}{m},\ i = 0,...,m-1 \Big\}.$$

Let $\mathcal{I}$ be the discretization of the cost functional I and denote $\mathfrak{J} : \mathcal{U} \to \mathbb{R}$, $\mathfrak{J}(\mathbf{a}) := \mathcal{I}(\mathbf{a}, \mathbf{q}(\mathbf{a}))$, where $\mathbf{q}(\mathbf{a})$ solves (5.2). Then the mathematical programming problem to be solved reads:

$$\text{Find } \mathbf{a}^* \in \operatorname{argmin}\{\mathfrak{J}(\mathbf{a}) \mid \mathbf{a} \in \mathcal{U}\}. \tag{5.3}$$

Gradient type optimization algorithms for the numerical solution of (5.3) need the gradient of $\mathfrak{J}$. To compute the partial derivatives of $\mathfrak{J}$, the standard discrete adjoint variable approach (see e.g. [15], [16]) is used:

$$\frac{\partial \mathfrak{J}(\mathbf{a})}{\partial a_k} = \frac{\partial \mathcal{I}(\mathbf{a}, \mathbf{q})}{\partial a_k} + \mathbf{w}^{\mathrm{T}} \left[\frac{\partial \mathbf{r}(\mathbf{a}, \mathbf{q})}{\partial a_k} \right], \quad k = 0, ..., m, \tag{5.4}$$

where $\mathbf{w}$ solves the linear adjoint problem

$$\left[\frac{\partial \mathbf{r}(\mathbf{a}, \mathbf{q})}{\partial \mathbf{q}} \right]^{\mathrm{T}} \mathbf{w} = -\nabla_{\mathbf{q}} \mathcal{I}(\mathbf{a}, \mathbf{q}). \tag{5.5}$$

Here $\frac{\partial \mathbf{r}(\mathbf{a},\mathbf{q})}{\partial \mathbf{q}}$ is the Jacobian matrix of $\boldsymbol{r}$ at $(\mathbf{a}, \mathbf{q})$ that is needed in Newton's method, too.

Remark 2. The evaluation of the partial derivatives of $\mathbf{r}(\mathbf{a}, \mathbf{q})$ in (5.4) and (5.5) requires in fact C^2-continuity of the smooth approximation of the absolute value function. However, this is not the case of T_ε defined by (3.1). The additional smoothness is obtained if we use the piecewise quartic regularization

$$T_\varepsilon(q) = \begin{cases} |q| & \text{if } |q| \geq \varepsilon \\ \dfrac{-q^4 + 6\varepsilon^2 q^2 + 3\varepsilon^4}{8\varepsilon^3} & \text{if } |q| < \varepsilon. \end{cases} \tag{5.6}$$

It is obvious that the regularization (5.6) also enjoys the properties (3.2) and (3.3).

Remark 3. In practice, one usually is not able to measure the target tangential velocity directly on S to obtain the target $\tilde{u}_\tau$. Therefore, the slightly modified cost functional, e.g.

$$I(g, \boldsymbol{u}^g, p^g) = \tfrac{1}{2} \|\boldsymbol{u}^g - \tilde{\boldsymbol{u}}\|_{0,S_H}^2$$

can be used. Here $S_H := S + (0, H)$, $H \geq 0$ is a parallel shift of S inside of Ω and $\tilde{\boldsymbol{u}}$ is the measured velocity field on S_H.

Remark 4. Let us comment the choice of Δ from (2.4). For a given forcing function $\boldsymbol{f}$ there exists a constant $\Delta_f > 0$ such that $\max_S |\tilde{u}_\tau| = \Delta_f$. Ideally, Δ should be chosen such that $\Delta > \Delta_f$ but not too large. Indeed, if $\Delta \gg \Delta_f$, there is not enough information (and therefore too much freedom) to determine g on the interval $]\Delta_f, \Delta]$. In practice one can do several measurements with different forcing terms $\{\boldsymbol{f}^{(k)}\}$ and to set $\Delta = \max\{\Delta_f^k\}$, where $\Delta_f^k = \max_S |\tilde{u}_\tau^{(k)}|$ and $\tilde{u}_\tau^{(k)}$ is the tangential velocity on S corresponding to $\boldsymbol{f}^{(k)}$. Similarly we define Δ when the velocity vector is measured on S_H, $H > 0$ small enough.

6 | NUMERICAL EXAMPLES

In this section we present numerical results of a model identification problem. We will study effects of the choise of the regularization parameter ε, the location of the observed velocity (see Remark 3), and the degree of the Bezier functions on numerical results.

In what follows we assume that $\Omega =]0,1[\times]0,1[$, $S =]0,1[\times\{0\}$, $\Gamma = \partial\Omega \setminus \bar{S}$, and $\eta = 1$. We use a structured but non-uniform P2/P1 triangular mesh whose elements are defined by a grid of $N_x \times N_y$ points (see Figure 2). The state solver was implemented using MATLAB [17]. In minimization, the `fmincon` (with `'interior-point'` option) from MATLAB Optimization Toolbox was used. As expected, using gradient based optimization methods with exact analytic derivatives resulted in very efficient implementation. The elapsed real computing time of the computations needed in each example varied between 1 to 3 minutes on a desktop computer with Intel Core i7-7700 CPU @3.6 GHz and 16GB RAM.

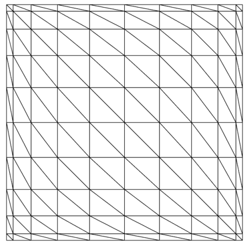

FIGURE 2 A non-uniform structured mesh ($N_x = N_y = 11$).

6.1 Identification problem

We assume that the velocity field on S (or S_H) is known. It will be represented by the numerical solution of $(\mathcal{P}_\varepsilon(g))$, with

$$g(\xi) = 3 + \frac{1}{2}\arctan(3\xi), \quad \xi \in \mathbb{R}_+, \tag{6.1}$$

and the force $\boldsymbol{f}$ acting on the fluid given by

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} (0,120), & x_1 > 0.7 \\ (0,0), & \text{otherwise.} \end{cases}$$

The functions g and $\boldsymbol{f}$ are chosen in such a way that both slip and no-slip occur on S. We use the finite element mesh with $N_x = N_y = 31$ (8403 DOFs) and the regularization parameter $\varepsilon = 10^{-8}$. The same mesh size is used in all numerical examples in this section. The identification problems considered are not very sensitive to mesh size and results obtained using finer meshes would be visually indistinguishable. Computed results serve as the reference solution which will be denoted as $(\boldsymbol{u}^{\text{ref}}, p^{\text{ref}})$. The tangential velocity u_τ^{ref} and tangential stress σ_τ^{ref} on S are shown in Figure 3 . Pressure contours and streamlines of the reference solution are plotted in Figure 4 .

In the next two examples we assume $\Delta = 1$. Moreover, it turned out that there is no need to impose the Lipchitz constraints on $g_{\mathbf{a}}$ in computations.

Example 1. Our aim is to reconstruct g from (6.1) by minimizing the cost functional

$$I(g, \boldsymbol{u}^g, p^g) = \frac{1}{2}\int_S (u_1^g - \widetilde{u}_1)^2\, ds,$$

with $\widetilde{u}_1 = u_1^{\text{ref}}$ taking into account that $\boldsymbol{\tau} = (1,0)$ on S. We use the Bernstein polynomials of degree $m = 4$ and $a_i^0 = 1.25$, i=0, ..., m as the initial guess of $\mathbf{a}$. The minimization problem used three different values of the penalty parameter, namely $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}\}$.

The reconstructed slip bounds g_ε^* are depicted in Figure 5 . One can observe that they are not very sensitive to the choice of the regularization parameter and practically coincide with the target g for ε small enough. In Figure 6 we compare u_τ^{ref} with $u_\tau^{g_\varepsilon^*}$ and σ_τ^{ref} with $\sigma_\tau^{g_\varepsilon^*}$ for $\varepsilon = 10^{-2}$.

Finally, the dependence of the identified g_ε^* on the degree m of the Bezier polynomials is visualized in Figure 7 . One can observe the stable behavior of g_ε^* with respect to m.

Example 2. As mentioned in Remark 3, the measurements of $\widetilde{u}_\tau$ on S are not usually possible in practice. If the measurements of both velocity components are available on the shifted line segment S_H, we minimize the modified cost functional

$$I(g, \boldsymbol{u}^g, p^g) = \frac{1}{2}\|\boldsymbol{u}^g - \widetilde{\boldsymbol{u}}\|_{0,S_H}^2,$$

where $\widetilde{\boldsymbol{u}} = \boldsymbol{u}^{\text{ref}}$ in our case. We study the sensitivity of the reconstructed slip bound g_ε^*, $\varepsilon = 10^{-5}$ on the choice of H. The results are shown in Figure 8 . The sensitivity of g_ε^* on the choice of H is surprisingly small.

7 | CONCLUSIONS

The paper deals with an identification problem for the Stokes system with threshold slip boundary conditions. Our aim is to identify the slip bound function g depending on the solution having the use of measurements of the velocity vector or its components. To this end we use an optimal control approach with an appropriate choice of a least squares cost functional. Due to the slip conditions, the state and hence also optimization problems are nonsmooth. To handle the problem in its original non-smooth form, we need deep knowledge of the generalized differential calculus [18] to get subgradient information in the theoretical part and to use non-smooth bundle trust methods in the computational part [19]. For the application of the nonsmooth approach in solid mechanics, in particular in shape optimization, we refer to [20]. To make the implementation comprehensible for a wide community of potential users, we use a regularized, i.e. smoothed form of the slip conditions in the Stokes system. As a result, the whole optimization problem becomes smooth, so classical gradient type minimization methods can be used in computations. The theoretical justification of this approach is done in the first part of the paper. The second part is concerned with computational aspects. The state problem is discretized using the P2/P1 finite elements in the velocity-pressure formulation of the problem. The slip bound functions $g \in \mathcal{U}_{ad}$ which play the role of the control variables are modelled by the Bezier functions. The paper is completed by numerical results of a model example. For the numerical minimization of the nonlinear objective function we used the default interior point algorithm of the MATLAB Optimization Toolbox with hand coded analytic gradient computations. These results confirm the efficiency of this approach, in particular its stability with respect to the choice of the regularization parameter, the degree of the Bezier functions and the place of measurements.

ACKNOWLEDGEMENT

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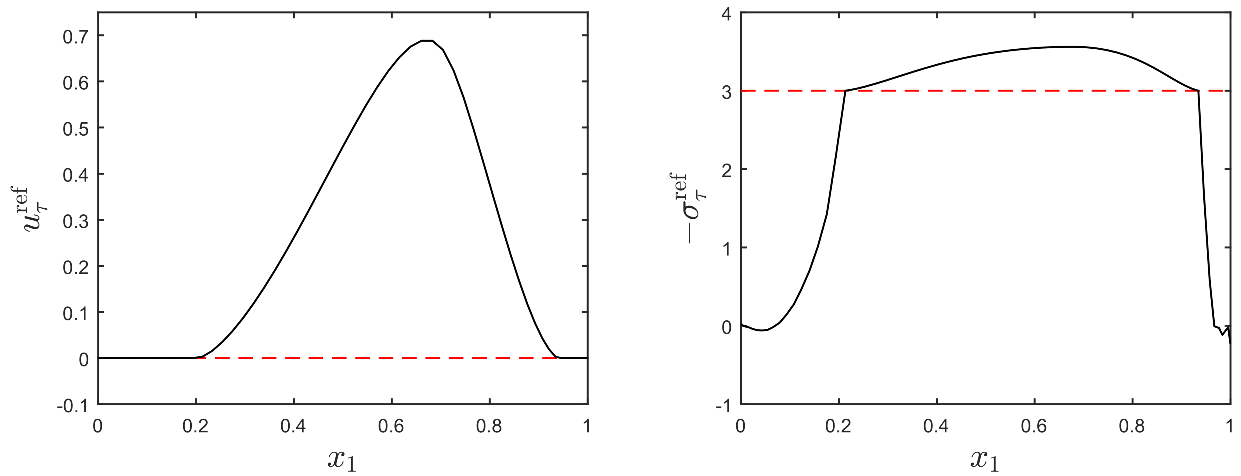

FIGURE 3 Tangential velocity (left) and tangential stress (right) on S of the reference solution.

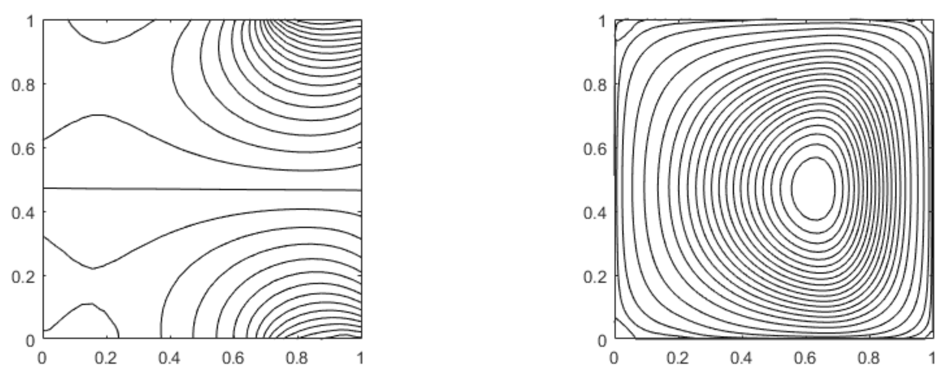

FIGURE 4 Pressure contours (left) and streamlines (right) of the reference solution.

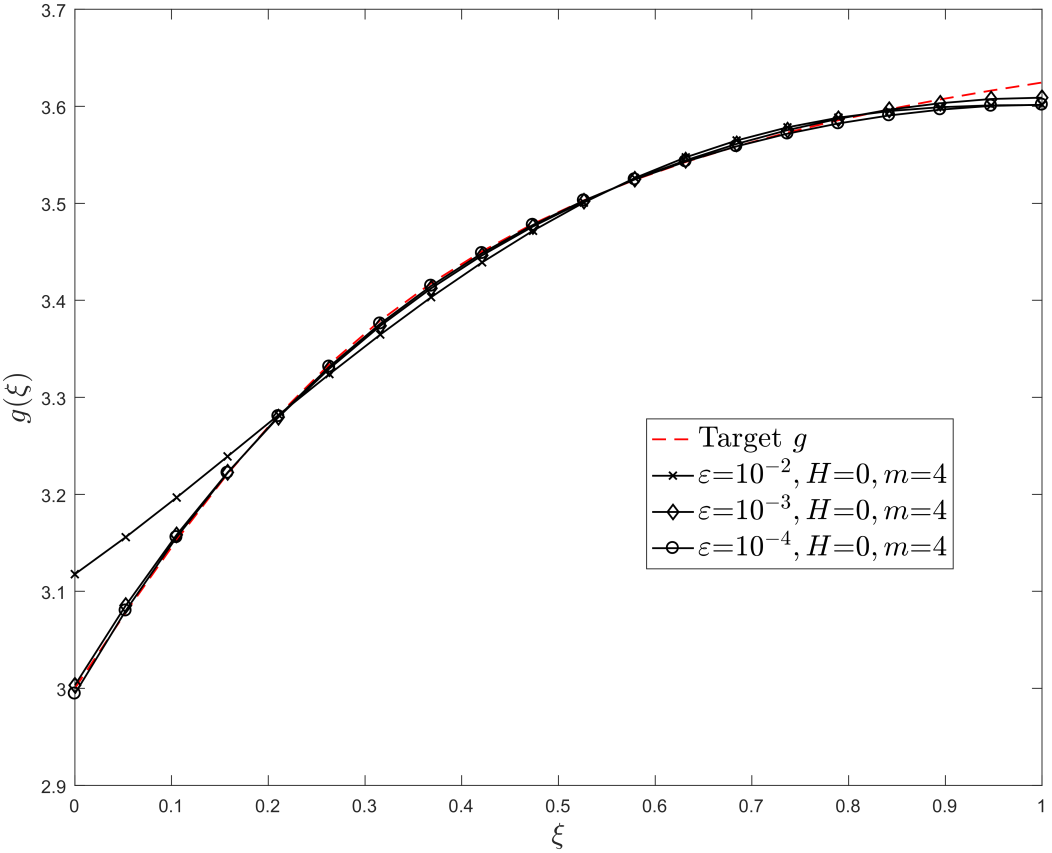

FIGURE 5 Target slip bound g (dashed line) and reconstructed slip bounds g^*_ε for different values of ε.

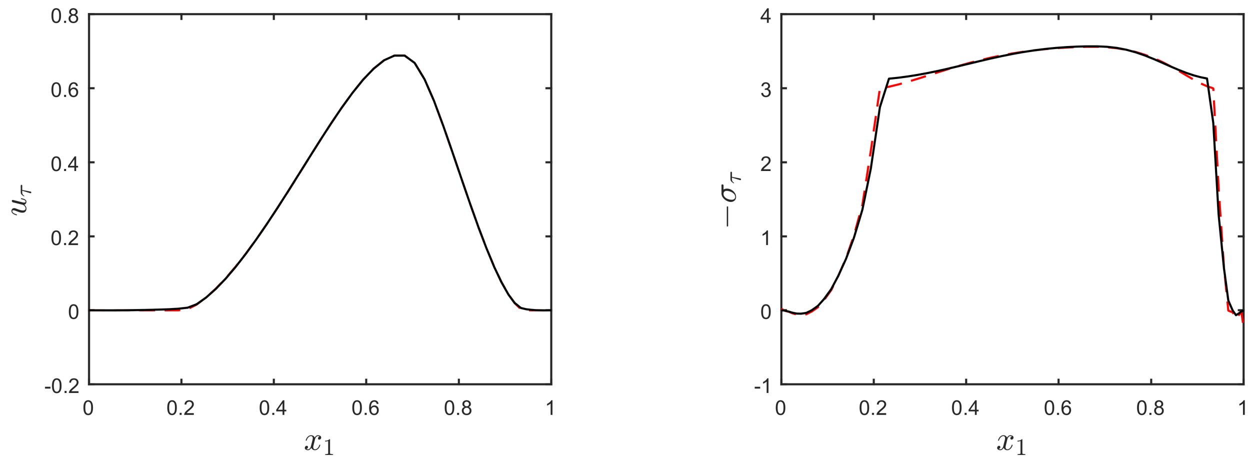

FIGURE 6 Left: Target tangential velocity u_1^{ref} (dashed line) and identified approximate solution with $\varepsilon = 10^{-2}$ (solid line). Right: Target tangential stress σ_τ^{ref} on S (dashed line) and the tangential stress for the identified approximate solution with $\varepsilon = 10^{-2}$ (solid line).

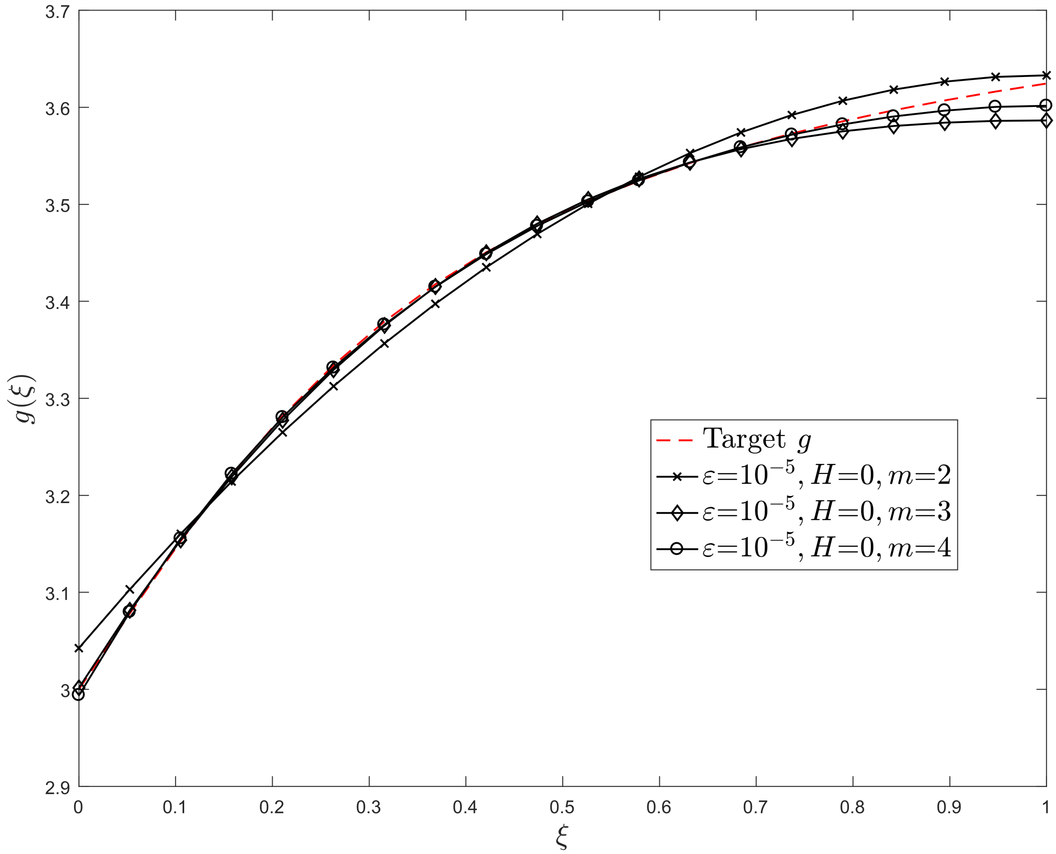

FIGURE 7 Target slip bound g (dashed line) and reconstructed slip bounds g^*_ε ($\varepsilon = 10^{-5}$) for different values of m.

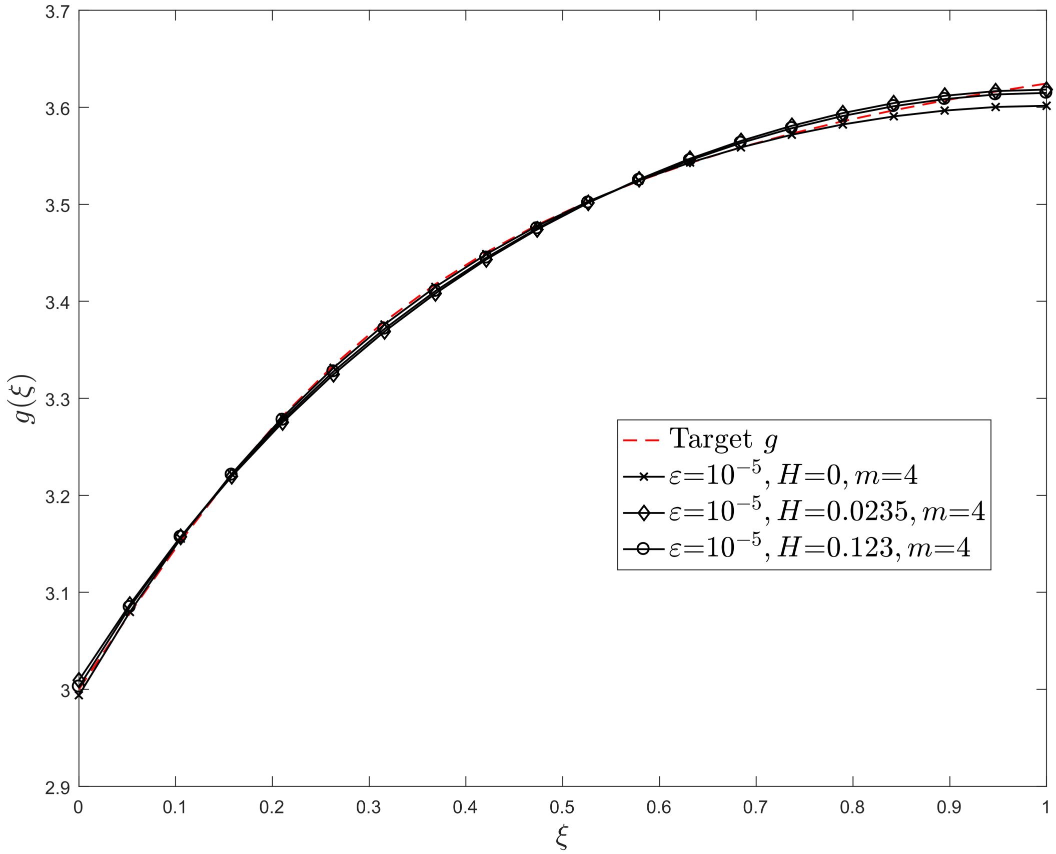

FIGURE 8 Target slip bound g (dashed line) and reconstructed slip bounds g_ε^* for different values of H.