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REGULARITY AND MODULUS OF CONTINUITY OF SPACE-FILLING CURVES

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Abstract. We study critical regularity assumptions on space-filling curves that possess certain modulus of continuity. The bounds we obtain are essentially sharp, as demonstrated by an example.

1. Introduction

In 1890, G. Peano [15] gave the first example of a curve whose range contains the entire unit square. More precisely, he constructed a continuous mapping $g\colon [0,1] \to \mathbb{R}^2$, such that $[0,1]^2 \subset g([0,1])$. Mappings like this are now customarily called Peano curves. Peano's construction allowed also for space-filling: there exists a continuous mapping $g\colon [0,1] \to \mathbb{R}^n$, where $n \in \mathbb{N}$, $n \geq 3$, such that $[0,1]^n \subset g([0,1])$. In what follows, we call such a g a space-filling curve. Once these mappings are known to exist, it is natural to ask how regular they can be. Regularity properties of space-filling curves can be measured in terms of continuity and total oscillation (energy). Despite the age of the topic the research is still active. For some recent advances see [6], [17], [3], [18], [19]. In this paper we consider the interplay of modulus of continuity and energy conditions and our results are not covered by previous results.

For technical reasons, let us consider the case of closed curves $g\colon S^1 \to \mathbb{R}^n$, $n \geq 2$, parameterized by the unit circle S^1. Regarding continuity, a simple dimension estimate shows that a Peano curve cannot be Hölder continuous with any exponent strictly greater that 1/2, and similarly with $1/n$ in the case of a space-filling curve. Hölder continuity exponents 1/2 and $1/n$ are actually possible, see, for instance [1, 6, 17]. Moreover, it is possible to construct an almost everywhere differentiable Peano (space-filling) curve with Hölder continuity exponent arbitrarily close to 1/2 ($1/n$) (see [16, Chapter 5] for one of such constructions).

One is then motivated to ask whether a suitable energy bound would rule out space filling. Indeed, if the component functions of $g\colon S^1 \to \mathbb{R}^n$ are absolutely continuous, then $g(S^1)$ necessarily has finite length; hence it cannot cover a square or a cube. Thus, the case of usual Sobolev regularity is trivial, and one is lead to consider fractional derivatives. The component functions of $g\colon S^1 \to \mathbb{R}^2$ belong to the fractional Sobolev space $W^{\frac{1}{2},2}(S^1)$ exactly when the Poisson extension $Pg\colon B(0,1) \to \mathbb{R}^2$ belongs to the Sobolev space $W^{1,2}(B(0,1);\mathbb{R}^2)$. Such Peano curves do exist [12, Section 5], but according to [12, Theorem D], Pg cannot be Hölder continuous (hence neither can g) under this regularity of Pg. Thus, Hölder continuity

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and finiteness of a suitable energy prevent g from covering a square, actually force $g(S^1)$ to be of area zero. The energy in question is the one corresponding to the Sobolev space $W^{\frac{1}{2},2}(S^1)$.

From above, we see that neither Hölder continuity nor finiteness of $W^{\frac{1}{2},2}(S^1)$-energy is sufficient to rule out square filling, but together they are. Let us analyze the energy assumption in more detail. Given $g\colon S^1 \to \mathbb{R}^2$, this energy is essentially

$$\int_{S^1}\int_{S^1} \frac{\left|f(x)-f(y)\right|^2}{\left|x-y\right|^2} d\mathcal{H}^1_x d\mathcal{H}^1_y.$$

For our purposes, we consider a dyadic version of this energy, given by

$$\mathcal{E}(g;2,0) := \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{U_{i,j}}|^2. \tag{1}$$

Here, $\{I_{i,j} : i \in \mathbb{N}, j = 1, \dots, 2^i\}$ is a dyadic decomposition of S^1, such that for a fixed $i \in \mathbb{N}$, $\{I_{i,j}\colon j = 1, \dots, 2^i\}$ is a family of arcs of length $2\pi/2^i$ with $\cup_j I_{i,j} = S^1$. The next generation is constructed in such a way that for each $j \in \{1, \dots, 2^{i+1}\}$, there exists a unique number $k \in \{1, \dots, 2^i\}$, satisfying $I_{i+1,j} \subset I_{i,k}$. We denote this parent of $I_{i+1,j}$ by $U_{i+1,j}$, and $U_{1,j} = S^1$ for $j = 1, 2$. By g_A, $A \subset S^1$, we denote the mean value $g_A = \fint_A g\, d\mathcal{H}^1 = \frac{1}{\mathcal{H}^1(A)}\int_A g\, d\mathcal{H}^1$. One can check that finiteness of the former energy implies the finiteness of the latter (see Theorem 3.1).

If $g\colon S^1 \to \mathbb{R}^2$ is a 1/2-Hölder continuous Peano curve, then the inner sum in (1) is uniformly bounded from above. Additionally, one can rather easily obtain a positive lower bound for the inner sum in the case of a certain almost everywhere differentiable Peano curve with Hölder exponent arbitrarily close to 1/2. Thus the energy (1) does not appear critical for Hölder continuous maps. Let us consider

$$\mathcal{E}(g;2,\lambda) := \sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{U_{i,j}}|^2$$

for $\lambda \in \mathbb{R}$. A reader familiar with trace theorems for Sobolev spaces should consider our norm as a version of the trace norm of an Orlicz-Sobolev space corresponding to the Orlicz function $\psi(t) = t^2 \log^{\lambda}(e + t)$. The value $\lambda = -1$ is critical for Hölder continuous curves by the discussion above. Our first result implies that 1/2-Hölder continuous Peano curves have finite energy $\mathcal{E}(g;2,\lambda)$ exactly when $\lambda < -1$.

Theorem 1.1. *If $g\colon S^1 \to \mathbb{R}^2$ is a Hölder continuous Peano curve then $\mathcal{E}(g;2,-1) = \infty$.*

In order to deal with space-filling curves $g\colon S^1 \to \mathbb{R}^n$, we modify the above definition by setting

$$\mathcal{E}(g;\nu,\lambda) := \sum_{i=1}^{\infty} i^{\lambda} \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{U_{i,j}}|^{\nu} \tag{2}$$

for $\nu \in]1, n]$ and $\lambda \in \mathbb{R}$. The space-filling analogue of Theorem 1.1 is

Theorem 1.2. *If $g\colon S^1 \to \mathbb{R}^n$ is a Hölder continuous space-filling curve then $\mathcal{E}(g;n,-1) = \infty$.*

Let us return to Peano curves in $W^{\frac{1}{2},2}(S^1)$. This energy that corresponds to $\mathcal{E}(g;2,0)$ in our notation is in principle much larger than $\mathcal{E}(g;2,-1)$, hence one expects a weaker optimal modulus of continuity than Hölder continuity. Indeed, in [7] the optimal modulus of continuity of general mappings $f \in W^{1,2}(B(0,1),\mathbb{R}^n)$ which guarantees that the 2-dimensional Hausdorff measure of $f(S)$ be zero, has been recognized to be

$$|f(x)-f(y)| \le C_0 \exp\left(-C_1 \log^{1/2}\frac{2\pi}{|x-y|}\right), \tag{3}$$

where $C_0, C_1 > 0$ are any constants. To simplify our terminology, we use the term "space-filling" even for Peano curves from now on. Our next results deal with general energies $\mathcal{E}(g;n,\lambda)$.

Theorem 1.3. *Let $g\colon S^1 \to \mathbb{R}^n$ be a space-filling curve, satisfying*

$$|g(x)-g(y)| \le C_0 \exp\left(-C_1 \log^{\alpha}\frac{2\pi}{|x-y|}\right) \tag{4}$$

with some $\alpha \in]0,1]$ and $C_0, C_1 > 0$, whenever $x, y \in S^1$ are close enough ($|x-y| < \epsilon_0$ for some $\epsilon_0 > 0$). Then $\mathcal{E}(g;n,\lambda) = \infty$ for $\lambda = n-1-\alpha n$, if $\alpha \ge \frac{n-1}{n}$, and for $\lambda > n-1-\alpha n$, if $\alpha < \frac{n-1}{n}$.

Note that, when $\alpha = 1$, g is assumed to be Hölder continuous.

Theorem 1.4. *Let $g\colon S^1 \to \mathbb{R}^n$ be a space-filling curve, satisfying*

$$|g(x)-g(y)| \le C_0 \log^{-C_1}\frac{2\pi}{|x-y|} \tag{5}$$

with some $C_0, C_1 > 0$, whenever $x, y \in S^1$ are close enough. Then $\mathcal{E}(g;n,n-1) = \infty$.

Note that by Theorem 3.1, the conclusion $\mathcal{E}(g;\nu,\lambda) = \infty$ in Theorems 1.1–1.4 implies a weaker one that the integral in (7) is infinite. Also, $\mathcal{E}(g;n,\lambda) = \infty$ for any space-filling curve $g\colon S^1 \to \mathbb{R}^n$, when $\lambda > n-1$. This follows from the argument in [11].

In the process of verifying the above results, we actually prove a stronger statement, related to Hausdorff measures. It is given by the following theorem.

Theorem 1.5. *Let $n \ge 2$, $\nu \in]1,n]$, $\lambda \in [-1,\nu-1]$ and $g\colon S^1 \to \mathbb{R}^n$ be a continuous mapping with $\mathcal{E}(g;\nu,\lambda) < \infty$, satisfying (4) with $\alpha = (\nu-1-\lambda)/\nu$, if $\lambda \in [-1,0]$, and $\alpha > (\nu-1-\lambda)/\nu$, if $\lambda \in]0,\nu-1[$. If $\lambda = \nu-1$, we assume that g satisfies (5). Then $\mathcal{H}^\nu(g(S^1)) = 0$.*

Notice that ν is not necessarily an integer. Theorems 1.1–1.5 are essentially sharp, see Sections 3 and 4.

The proof of Theorem 1.5 is based on the following idea. We consider the balls $B(g_{i,j}, r_{i,j})$ with

$$g_{i,j} = g_{I_{i,j}} \quad\text{and}\quad r_{i,j} = \max\left\{|g_{I_{i,j}} - g_{U_{i,j}}|, |g_{I_{i,j}} - g_{I_{i+1,j_1}}|, |g_{I_{i,j}} - g_{I_{i+1,j_2}}|\right\}, \tag{6}$$

where j_1 and j_2 are such that $I_{i,j} = U_{i+1,j_l}$, $l = 1,2$. Our energy assumption gives us control over a weighted double sum of $r^\nu_{i,j}$. Even though these balls cannot necessarily be used to cover $g(S^1)$, the given modulus of continuity allows us to construct a desired cover via this sequence of balls. For related arguments in the special case of quasiconformal mappings see [10, 9]; also see [7, 11] for the setting of Sobolev mappings.

This paper is organized as follows. In the next section, we introduce our basic notation and give the proof of Theorem 1.5. We discuss the connection between our energy $\mathcal{E}(g;\nu,\lambda)$

and the integral energy

$$\int_{S^1}\int_{S^1} \frac{\left|g(x)-g(y)\right|^{\nu}}{\left|x-y\right|^{2}} \log^{\lambda}\left(e+\frac{\left|g(x)-g(y)\right|}{\left|x-y\right|}\right) d\mathcal{H}^1_x d\mathcal{H}^1_y \tag{7}$$

in Section 3. Finally, in Section 4, we construct an example, which shows the essential sharpness of our results.

2. Proof of Theorem 1.5

First, we would like to introduce our basic notation. Given a number $a > 0$, we write $\lfloor a \rfloor$ for the largest integer less or equal to a. If A is an arbitrary set, by $\sharp A$ we mean the total number of elements in A. By $\operatorname{diam}(A)$ and χ_A, we denote the diameter and the characteristic function of the set $A \subset \mathbb{R}^n$, respectively. Given a point $x \in \mathbb{R}^n$ and a non-negative number r, $B(x,r)$ denotes an open ball centred in x and having radius r. If B is a ball, then $r(B)$ stands for its radius. We write $\mathcal{H}^{\nu}_{\delta}(A)$ with $\nu > 0$ and $0 < \delta \le \infty$ for the ν-dimensional Hausdorff content of a set A, while $\mathcal{H}^{\nu}(A)$ denotes its ν-dimensional Hausdorff measure. If we write $L = L(\cdot)$, we mean that the number $L > 0$ depends on the parameters listed in the parentheses. Finally, C denotes a positive constant, which may depend on data and differ from occurrence to occurrence.

In addition to the standard Hausdorff content, we need a weighted Hausdorff content of a set $A \subset \mathbb{R}^n$ given by

$$\lambda^{\nu}_{\infty}(A) = \inf\left\{\sum_{i=1}^{\infty} c_i(\operatorname{diam} E_i)^{\nu} \colon c_i \ge 0 \text{ and } \chi_A \le \sum_{i=1}^{\infty} c_i \chi_{E_i}\right\}$$

for $\nu > 0$. We refer to the collection of pairs $(E_i, c_i)_{i=1}^{\infty}$, such that $c_i \ge 0$, $E_i \subset \mathbb{R}^n$ and $\chi_A \le \sum_{i=1}^{\infty} c_i \chi_{E_i}$, as a weighted covering of the set A. It is known that there exists a constant $\tau > 0$, such that $\mathcal{H}^{\nu}_{\infty}(E) \le \tau \lambda^{\nu}_{\infty}(E)$ for all bounded sets E (see, for instance, [5], Corollary 8.2 and Theorem 9.7).

Before starting the proof, we state two simple lemmas concerning sequences.

Lemma 2.1. *Let $\{a_i\}_{i\in\mathbb{N}}$ be an nondecreasing sequence of real numbers such that $1 \le a_i \le C_0 i^{\beta}$ for each $i \ge i_0$ and some $C_0, \beta > 0$, $i_0 \in \mathbb{N}$. Then there exist constants $c = c(\beta) > 0$ and $k_0 = k_0(\beta, C_0, i_0) \in \mathbb{N}$, such that for any $k \ge k_0$, there are at least $6k$ integers i in the set $\{k, \dots, 8k\}$, satisfying*

$$ca_i \ge \frac{a_{i+1} - a_i}{\log i} i.$$

Proof. Let us fix an integer $k \ge i_0$ and assume that there exist numbers $\{i_1, \dots, i_{k+1}\} \subset \{k, \dots, 8k\}$ such that

$$ca_{i_j} < \frac{a_{i_j+1} - a_{i_j}}{\log i_j} i_j$$

for $c = 32\beta$ and $j = 1, \dots, k+1$. Then

$$\log a_{i_j+1} > \log a_{i_j} + \log\left(1 + \frac{c \log i_j}{i_j}\right) > \log a_{i_j} + \frac{c \log i_j}{2 i_j}$$

for all these j, provided $k > c\log k$. The monotonicity of $\{a_i\}$ and $\left\{\frac{\log i}{i}\right\}$ imply

$$\log a_{8k+1} \geq \log a_k + \frac{c}{2}\sum_{i=7k}^{8k}\frac{\log i}{i} \geq \frac{c\log 8k}{16}.$$

On the other hand, $\log a_{8k+1} \leq \log C_0 + \beta\log(8k+1)$, which is a contradiction when $k \geq k_0$, once $k_0 = k_0(\beta, C_0, i_0)$ is fixed large enough. □

Lemma 2.2. *Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of positive real numbers such that $a_i \leq C_1 e^{C_0 i}$ for each integer $i \geq i_0$ and some $C_0, C_1 > 0$, $i_0 \in \mathbb{N}$. Then there exists a constant $c = c(C_0) > 0$, such that there are infinitely many $i \in \mathbb{N}$ satisfying $ca_i \geq a_{i+1} - a_i$.*

Proof. Trivial. □

Proof of Theorem 1.5. The proof is somewhat different for non-positive values of λ, $\lambda \in]0, \nu-1[$ and $\lambda = \nu - 1$. We provide most of the details for all three cases. The origins of part of the proof for positive values of λ can be found in [4], while the case of negative λ adopts ideas of [7]. We may assume that the modulus of continuity (4) or (5) is satisfied globally, locality of the estimate will only influence the choices of parameters. The proof is broken into 5 steps.

Step 1. In this step we build our fundamental construction. In the end of the step, we decompose S^1 into smaller sets to be estimated separately.

Let $g_{i,j}$ and $r_{i,j}$ be as defined in (6) for $i = 1, 2, \ldots$ and $j = 1, \ldots, 2^i$. Since the energy $\mathcal{E}(g; \nu, \lambda)$ is finite, we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{2^i} i^\lambda r_{i,j}^\nu \leq C\sum_{i=1}^{\infty} i^\lambda \sum_{j=1}^{2^i} |g_{I_{i,j}} - g_{U_{i,j}}|^\nu < \infty. \tag{8}$$

For each pair of indices (i, j) with $r_{i,j} > 0$, we define a collection of weighted balls $\mathcal{S}_{i,j} = \left\{B_{i,j,k} = B(g_{i,j}, r_{i,j}/2^k) \colon k \in \mathbb{N}\right\}$, such that each ball $B = B_{i,j,k}$ has weight $w_B = v_B i^\lambda$, where $v_B = 2^k$. We observe

$$\sum_{B\in\mathcal{S}_{i,j}} w_B r^\nu(B) = i^\lambda r_{i,j}^\nu \sum_{k=1}^{\infty}\frac{1}{2^{k(\nu-1)}} = C_\nu i^\lambda r_{i,j}^\nu,$$

where $C_\nu = 1/(2^{\nu-1} - 1)$. Defining $\mathcal{F} = \cup_{i,j}\mathcal{S}_{i,j}$, we have

$$\sum_{B\in\mathcal{F}} w_B r^\nu(B) \leq C_\nu \sum_{i,j} i^\lambda r_{i,j}^\nu < \infty \tag{9}$$

by (8). Some balls in $\mathcal{F}$ may coincide, however, we treat them as if they were different.

Pick $x \in S^1$ and a corresponding sequence of arcs $I_1(x) \supset I_2(x) \supset \ldots$, such that $x \in I_i(x)$ and $I_i(x) = I_{i,j_i(x)}$ for some $j_i(x) \in \{1, \ldots, 2^i\}$. We denote $g_i(x) = g_{i,j_i(x)}$ and $r_i(x) = r_{i,j_i(x)}$. Then $g_i(x) \to g(x)$, when $i \to \infty$. Indeed, denoting by $\psi\colon [0,\infty[\to [0,\infty[$ the modulus of continuity of g, we obtain

$$|g_i(x) - g(x)| \leq \fint_{I_{i,j_i(x)}} |g(y) - g(x)|dy \leq \fint_{I_{i,j_i(x)}} \psi(|y - x|)dy \leq \psi(\operatorname{diam} I_{i,j_i(x)}) \leq \psi(\pi 2^{-i+1}) \to 0,$$

when $i \to \infty$. Thus, we may assume existence of at least one pair of indices (i, j) with $r_{i,j} > 0$, otherwise $g(S^1)$ would be a one-point set.

We neglect the set $\Gamma = \left\{x \in S^1 \colon g_i(x) = g(x) \text{ for all } i \in \mathbb{N}\right\}$, whose image $g(\Gamma)$ is countable. Let $x \in S^1 \setminus \Gamma$. We define the number $l_0(x) \in \mathbb{N}$ so that there are elements $g_i(x)$ for some $i \in \mathbb{N}$ outside the ball $B(g(x), 2^{-l_0(x)+1})$, and construct a decomposition

$$S^1 \setminus \Gamma = \bigcup_{l \in \mathbb{N}} \Gamma_l, \text{ where } \Gamma_l = \left\{x \in S^1 \setminus \Gamma \colon l_0(x) \leq l\right\}.$$

We denote $G_l = g(\Gamma_l)$, so that $g(S^1 \setminus \Gamma) = \cup_{l \in \mathbb{N}} G_l$.

Step 2. Fix $l_1 \in \mathbb{N}$. The remaining steps will be dedicated to the estimate for the set Γ_{l_1}. In this step, for each point $x \in \Gamma_{l_1}$ we define sequences that characterize how $g_i(x)$ converge to $g(x)$. We establish some important properties of these sequences, implied by the modulus of continuity, satisfied by g.

Let $x \in \Gamma_{l_1}$. For each integer $l \geq l_1$, let $a_l(x) \in \mathbb{N}$ be the smallest number, such that $g_i(x) \in B(x, 2^{-l+1})$ for all $i > a_l(x)$. If $a_{l+1}(x) > a_l(x)$, we define $P_l(x) = \{a_l(x) + 1, \ldots, a_{l+1}(x)\}$ and $p_l(x) = \sharp P_l(x)$, otherwise $P_l(x) = \{a_l(x) + 1\}$ and $p_l(x) = 0$.

Let $l \geq l_1$ and $i_0 = a_{l+1}(x)$. In the case $\lambda \neq \nu - 1$, we have

$$2^{-l} \leq |g_{i_0}(x) - g(x)| \leq \fint_{I_{i_0}(x)} |g(y) - g(x)|\, dy \leq C_0 \exp\left(-C_1 \log^{\alpha}(2^{i_0})\right) = C_0 \exp(-C i_0^{\alpha}),$$

where $C = C_1 \log^{\alpha} 2$, because $|x - y| \leq 2\pi/2^i$, when $x, y \in I_i(x)$. The last estimate implies $a_{l+1}(x) + 1 = i_0 + 1 \leq c_0 l^{1/\alpha}$, thus $i \leq c_0 l^{1/\alpha}$ for each $i \in P_l(x)$, with some constant $c_0 = c_0(\alpha, C_0, C_1) > 0$. Similarly, when $\lambda = \nu - 1$, we obtain $a_l(x) \leq c_1 e^{c_0 l}$ for each $l \geq l_1$, with $c_1 = c_1(C_0, C_1) > 0$ and $c_0 = c_0(C_1) > 0$.

If $\lambda \neq \nu - 1$, we denote

$$\theta_l(x) = \begin{cases} 1, & \text{if } p_l(x) \leq c l^{\frac{1-\alpha}{\alpha}}, \\ 0, & \text{otherwise,} \end{cases}$$

for $l \geq l_1$ and a constant $c \in \mathbb{N}$, which we will specify later. We use ideas originating from [9, 14] in order to prove that there exists an integer $l' \geq 2l_1$, such that

$$\sum_{k=l_1}^{l} \theta_k(x) \geq \frac{l}{2} \tag{10}$$

for each $l \geq l'$. In other words, at least half of the annuli do not contain too many centres from $(g_i(x))_{i \in \mathbb{N}}$.

Let us assume that (10) does not hold for some $l \geq 2l_1$. We obtain a lower bound for $a_{l+1}(x)$, using the assumption that we have at least $\lfloor l/2 \rfloor - l_1 + 2$ integers $k \in \{l_1, \ldots, l\}$ with $\theta_k(x) = 0$. We have

$$c_0 l^{1/\alpha} \geq a_{l+1}(x) \geq \sum_{\substack{k = l_1, \ldots, l \\ \theta_k(x) = 0}} p_k(x) \geq \sum_{k=l_1}^{\lfloor l/2 \rfloor + 1} c k^{\frac{1-\alpha}{\alpha}} \geq c\alpha 2^{-\frac{1}{\alpha}} l^{\frac{1}{\alpha}} - c\alpha l_1^{\frac{1}{\alpha}},$$

which can not be true, if c is fixed so that $c > c_0 2^{1/\alpha}/\alpha$, and l is large enough. In other words, there exists a number $l' = l'(l_1, \alpha, C_0, C_1) \in \mathbb{N}$, such that (10) holds for all $l \geq l'$. For completeness, in the case $\lambda = \nu - 1$, we put $\theta_l(x) = 1$ for each $l \geq l_1$.

Step 3. In the following two steps, we prove that for a substantial number of integers k, we may find a set of balls $B \in \mathcal{F}$ with radii in a suitable range and centres in $B(g(x), 2^{-k+1})$, such that $\sum r^\nu(B) w_B \geq const \cdot 2^{-\nu k} k^{-1+\delta}$ for some $\delta \geq 0$. When $\lambda \neq \nu - 1$, δ is chosen so that $\alpha = (\nu - 1 - \lambda)/(\nu - 2\delta)$; otherwise, $\delta = 1$. Note that $\delta = 0$, when $\lambda \leq 0$. Without the loss of generality, we may assume that $\delta \leq 1$. In this step, we define the desired collection of balls and provide the estimates for $\sum r^\nu(B) v_B$.

First, we concentrate on the case $\lambda > 0$. Let us take $k \geq l_1$ with $\theta_k(x) = 1$. We find a set of balls $M_k(x) \subset \mathcal{F}$ with radii in the range $[2^{-k-3}/\sharp P_k(x), 2^{-k-2}/\sharp P_k(x)[$, satisfying

(11)
$$\sum_{B \in M_k(x)} r^\nu(B) v_B \geq 2^{-\nu k - 3\nu + 1} \left(\sharp P_k(x)\right)^{1-\nu}.$$

By the definitions of $P_k(x)$ and $r_i(x)$, we have

(12)
$$\sum_{i \in P_k(x)} r_i(x) \geq \frac{2^{-k}}{2}.$$

Therefore,

$$\sum_{\substack{i \in P_k(x) \\ r_i(x) \geq 2^{-k-2}/\sharp P_k(x)}} r_i(x) \geq \frac{2^{-k}}{4}.$$

For each $i \in P_k(x)$, such that $r_i(x) \geq 2^{-k-2}/\sharp P_k(x)$, we define a number $q_{k,i}(x) \in \mathbb{N}$, satisfying

$$\frac{2^{-k+q_{k,i}(x)-1}}{4 \cdot \sharp P_k(x)} \leq r_i(x) < \frac{2^{-k+q_{k,i}(x)}}{4 \cdot \sharp P_k(x)}.$$

Then the collection $M_k(x)$ is defined by

$$M_k(x) = \left\{B_{i,j,q} \colon i \in P_k(x),\ r_i(x) \geq 2^{-k-2}/\sharp P_k(x),\ j = j_i(x),\ q = q_{k,i}(x)\right\}.$$

We have

$$\frac{2^{-k}}{8 \cdot \sharp P_k(x)} \leq r(B) < \frac{2^{-k}}{4 \cdot \sharp P_k(x)} \leq \frac{2^{-k}}{4}$$

for each $B \in M_k(x)$, which gives

(13)
$$\frac{2^{-k}}{8ck^{(1-\alpha)/\alpha}} \leq r(B) < \frac{2^{-k}}{4}$$

for each $B \in M_k(x)$, when $\lambda \in]0, \nu - 1[$. Additionally,

$$\sum_{B \in M_k(x)} r(B) v_B = \sum_{\substack{i \in P_k(x) \\ r_i(x) \geq 2^{-k-2}/\sharp P_k(x)}} r_i(x) \geq \frac{2^{-k}}{4}.$$

Finally,

$$\sum_{B \in M_k(x)} r^\nu(B) v_B \geq \left(\frac{2^{-k}}{8 \cdot \sharp P_k(x)}\right)^{\nu-1} \sum_{B \in M_k(x)} r(B) v_B \geq \frac{2^{-\nu k}}{2^{3\nu-1} \left(\sharp P_k(x)\right)^{\nu-1}},$$

as desired. Note also that this argument works even if $p_k(x) = 0$, because in that case the set $P_k(x)$ consists of one index i with $g_i(x) \in B(g(x), 2^{-k})$. The definition of $r_{i,j}$ implies $r_i(x) > 2^{-k}$.

For non-positive λ, similarly, we take a number $k \geq l_1$ with $\theta_k(x) = 1$. Starting with (12), we proceed a similar proof with $\sharp P_k(x)$ replaced by $ck^{\frac{1-\alpha}{\alpha}}$ (note the case $p_k(x) = 0$).

This gives us a set of balls $M_k(x) \subset \mathcal{F}$ centred in $B(g(x), 2^{-k+1})$ with radii in the range $[c^{-1}2^{-k-3}k^{(\alpha-1)/\alpha}, c^{-1}2^{-k-2}k^{(\alpha-1)/\alpha}[$, and satisfying

$$\sum_{B\in M_k(x)} r^\nu(B)v_B \geq c^{1-\nu}2^{-\nu k-3\nu+1}k^{(\nu-1)(\alpha-1)/\alpha}. \tag{14}$$

When $\theta_k(x) = 0$, for completeness, we put $M_k(x) = \emptyset$.

Step 4. In this step we complete the estimates, announced in the beginning of the previous step, by estimating the index i of each ball $B_{i,j,q} \in M_k(x)$.

Let us consider $\lambda \in]0, \nu-1[$ only. We apply Lemma 2.1 to $\beta = 1/\alpha$, $C_0 = c_0$ and $i_0 = l'$, obtaining an integer $k_1 = k_1(\alpha, c_0, l')$ and a constant $c_1 = c_1(\alpha) > 0$, such that for each $k_0 \geq k_1$ the set $\{k_0, \ldots, 8k_0\}$ contains at least $6k_0$ numbers k with the property

$$i > a_k(x) > \frac{c_1(a_{k+1}(x) - a_k(x))k}{\log k} = \frac{c_1 p_k(x)k}{\log k},$$

whenever $i \in P_k(x)$. Clearly, we may also assume that $\log^\lambda k < k^\delta$ for all $k > k_1$. Fix $k_0 \geq k_1$. By (10), we have at most $4k_0$ integers k from the set $\{k_0, \ldots, 8k_0\}$, such that $p_k(x) > ck^{(1-\alpha)/\alpha}$. Together with (11), all these facts imply that, for more than k_0 integers k in the set $\{k_0, \ldots, 8k_0\}$, we have either $p_k(x) \neq 0$ and

$$\sum_{B\in M_k(x)} r^\nu(B)w_B \geq C2^{-\nu k}\frac{k^\lambda}{\log^\lambda k}p_k^{\lambda-\nu+1}(x) \geq C2^{-\nu k}k^{\lambda-(\nu-1-\lambda)(1-\alpha)/\alpha-\delta} = \tilde{C}2^{-\nu k}k^{-1+\delta};$$

or $p_k(x) = 0$ and

$$r^\nu(B)w_B \geq r^\nu(B)v_B \geq \tilde{C}2^{-\nu k} \geq \tilde{C}2^{-\nu k}k^{-1+\delta}$$

for the only ball B in $M_k(x)$. Here $\tilde{C}$ is some positive constant, which depends on ν, C_0, C_1, α and λ.

Let us obtain an analogue of this for $\lambda \in [-1, 0]$. For each $k \geq l_1$ with $\theta_k(x) = 1$ we observe by (14) and $a_{k+1}(x) + 1 \leq c_0 k^{1/\alpha}$ that

$$\sum_{B\in M_k(x)} r^\nu(B)w_B \geq C2^{-\nu k}k^{[(\alpha-1)(\nu-1)+\lambda]/\alpha} = \tilde{C}2^{-\nu k}k^{-1},$$

where we may have redefined the value of $\tilde{C}$, but it still depends on ν, α, λ, C_0 and C_1 only. By (10), we obtain more than k_0 integers $k \in \{k_0, \ldots, 8k_0\}$ with the required property.

Finally, in the case $\lambda = \nu - 1$, we apply Lemma 2.2. There exists $k \geq k_0$, such that $i > a_k(x) \geq C(a_{k+1}(x) - a_k(x)) = Cp_k(x)$, whenever $i \in P_k(x)$. Combining this with (11), we obtain

$$\sum_{B\in M_k(x)} r^\nu(B)w_B \geq C2^{-\nu k} \tag{15}$$

for this k (note $p_k(x) = 0$ again).

Step 5. In this step we complete the estimates for Γ_{l_1} and the proof.

We return to the case $\lambda \neq \nu - 1$. For $k \in \{k_0, \ldots, 8k_0\}$, let us define

$$S_k = \left\{x \in \Gamma_{l_1} : \sum_{B\in M_k(x)} r^\nu(B)w_B \geq \tilde{C}2^{-\nu k}k^{-1+\delta}\right\} \text{ and } \tilde{M}_k = \bigcup_{x\in S_k} M_k(x).$$

The previous argument implies

$$\sum_{k=k_0}^{8k_0} \chi_{g(S_k)} \geq k_0 \chi_{G_{l_1}}.$$

We finish the proof for $\lambda \in]0, \nu-1[$. The techniques used are the ones applied in [4] in cases "C" and "D". If $k_1, k_2 \in \{k_0, \ldots, 8k_0\}$ are such that $k_1 \geq k_2 + 1 + \log_2 c + \frac{1-\alpha}{\alpha} \log_2 k_2$, then

$$\frac{2^{-k_1}}{4} \leq \frac{2^{-k_2}}{8ck_2^{(1-\alpha)/\alpha}};$$

thus, $\tilde{M}_{k_1} \cap \tilde{M}_{k_2} = \emptyset$ by (13). This means that there exists a constant C', independent of k_0, such that $\tilde{M}_{k_1} \cap \tilde{M}_{k_2} = \emptyset$ for each $k_1, k_2 \in \{k_0, \ldots, 8k_0\}$ with $k_1 - k_2 \geq \lfloor C' \log k_0 \rfloor \geq 1$, once k_0 is large enough. We prove next that there are not so many numbers $k \in \{k_0, \ldots, 8k_0\}$, satisfying (16). Let us assume that there are at least $\lfloor k_0/2 \rfloor$ collections $\tilde{M}_k$, $k \in \{k_0, \ldots, 8k_0\}$, with

$$\sum_{B \in \tilde{M}_k} r^\nu(B) w_B > k^{\delta/2-1}. \tag{16}$$

The previous conclusion implies that there are at least $\frac{\lfloor k_0/2 \rfloor}{\lfloor C' \log k_0 \rfloor}$ collections $\tilde{M}_k$ with the property (16), consisting of different balls each. Therefore, we obtain

$$\sum_{B \in \mathcal{F}} r^\nu(B) w_B \geq \frac{\lfloor k_0/2 \rfloor}{\lfloor C' \log k_0 \rfloor} 8^{\delta/2-1} k_0^{\delta/2-1},$$

which is a contradiction with (9), once $k_0 \geq 2$ is large enough. Therefore, there are at most $\lfloor k_0/2 \rfloor - 1$ collections $\tilde{M}_k$, satisfying (16), which yields

$$\sum_{\substack{k=k_0,\ldots,8k_0 \\ \sum_{B \in \tilde{M}_k} r^\nu(B) w_B \leq k^{\delta/2-1}}} \chi_{g(S_k)} \geq \lfloor k_0/2 \rfloor \chi_{G_{l_1}}.$$

Let us pick $k \in \{k_0, \ldots, 8k_0\}$ with $S_k \neq \emptyset$, such that (16) is not true. By the Besicovitch covering theorem, we cover $g(S_k)$ with a finite collection of balls $\{B_{k,i} = B(g(x_{k,i}), 2^{-k+1})\}_{i \in I_k}$ with $x_{k,i} \in S_k$, such that $\sum_i \chi_{B_{k,i}}(y) \leq N$ for each $y \in \mathbb{R}^n$ and some $N = N(n) \in \mathbb{N}$. We observe by the definition of S_k

$$\sharp I_k \cdot 2^{-\nu k + 2\nu} \leq 4^\nu \frac{k^{1-\delta}}{\tilde{C}} \sum_{i \in I_k} \sum_{B \in M_k(x_{k,i})} r^\nu(B) w_B \leq \frac{4^\nu N}{\tilde{C}} k^{1-\delta} \sum_{B \in \tilde{M}_k} r^\nu(B) w_B \leq \frac{4^\nu N}{\tilde{C}} k^{-\delta/2}, \tag{17}$$

where we have used the fact that all balls in $M_k(x_{k,i})$, with $i \in I_k$, are centred in $B_{k,i}$. Since $\bigcup_k \left\{ \left(B_{k,i}, \lfloor k_0/2 \rfloor^{-1} \right) : i \in I_k \right\}$ is a weighted covering of the set G_{l_1}, we obtain ($I_k = \emptyset$, if $S_k = \emptyset$)

$$\mathcal{H}^\nu_\infty(G_{l_1}) \leq \tau \lambda^\nu_\infty(G_{l_1}) \leq \frac{\tau}{\lfloor k_0/2 \rfloor} \sum_{\substack{k=k_0,\ldots,8k_0 \\ \sum_{B \in \tilde{M}_k} r^\nu(B) w_B \leq k^{\delta/2-1}}} \sharp I_k \cdot 2^{-\nu k + 2\nu} \leq \frac{4^\nu N \tau}{\tilde{C} \lfloor k_0/2 \rfloor} \sum_{k=k_0}^{8k_0} k^{-\delta/2}$$

$$\leq \frac{2^{2\nu+5} N \tau}{\tilde{C}} k_0^{-\delta/2}.$$

Finally, letting k_0 go to infinity, we obtain $\mathcal{H}^\nu_\infty(G_{l_1}) = 0$. This implies $\mathcal{H}^\nu(G_{l_1}) = 0$ and $\mathcal{H}^\nu(g(S^1 \setminus \Gamma)) = 0$.

In the case of non-positive λ, the proof is more trivial. We simply notice that $\tilde{M}_{k_1} \cap \tilde{M}_{k_2} = \emptyset$ for each $k_1 \neq k_2$. As in (17), using the Besicovitch covering theorem and the definition of S_k, we find a covering $\{B_{k,i}\}_{i\in I_k}$ of the set $g(S_k)$, such that

$$\sum_{i\in I_k} \operatorname{diam}^\nu(B_{k,i}) = \sharp I_k \cdot 2^{-\nu k+2\nu} \leq \frac{4^\nu N}{\tilde{C}} k \sum_{B\in\tilde{M}_k} r^\nu(B) w_B.$$

We conclude

$$\mathcal{H}^\nu_\infty(G_{l_1}) \leq \frac{\tau}{k_0} \sum_{k=k_0}^{8k_0} \sharp I_k \cdot 2^{-\nu k+2\nu} \leq \frac{4^\nu N\tau}{\tilde{C} k_0} \sum_{k=k_0}^{8k_0} k \sum_{B\in\tilde{M}_k} r^\nu(B) w_B \leq \frac{2^{2\nu+3} N\tau}{\tilde{C}} \sum_{\substack{B\in\mathcal{F} \\ r(B)\leq 2^{-k_0}}} r^\nu(B) w_B \to 0,$$

when $k_0 \to \infty$.

Finally, in the case $\lambda = \nu - 1$, we apply the Besicovitch covering theorem to obtain a collection of balls $\{B_i = B(g(x_i), 2^{-k_i+1})\}_{i\in I}$ with $x_i \in \Gamma_{l_1}$ and $k_i \geq k_0$, covering the set G_{l_1}, such that $\sum_i \chi_{B_i}(y) \leq N$ and (15) holds with $x = x_i$ and $k = k_i$ for each $i \in I$. We estimate

$$\mathcal{H}^\nu_\infty(G_{l_1}) \leq \sum_{i\in I} 2^{-\nu k_i+2\nu} \leq C \sum_{i\in I} \sum_{B\in M_k(x_i)} r^\nu(B) w_B \leq CN \sum_{\substack{B\in\mathcal{F} \\ r(B)\leq 2^{-k_0}}} r^\nu(B) w_B \to 0,$$

when $k_0 \to \infty$. □

3. How to compute the energy

In this section, we use a fixed dyadic decomposition of S^1 again. We denote by $\mathcal{D}_i$ the set of 2^i dyadic intervals of S^1 of generation i, having length $2^{1-i}\pi$. In this section, we denote the parent interval of $I \in \mathcal{D}_i$ by $U(I) \in \mathcal{D}_{i-1}$. Recall that, the parent interval of I is the dyadic interval of previous generation which contains I.

Theorem 3.1. *Let $g : S^1 \to \mathbb{R}^n$, $n \geq 2$, be continuous. Let $\lambda \in \mathbb{R}$ and $\nu > 1$. If*

$$\int_{S^1}\int_{S^1} \frac{|g(x) - g(y)|^\nu}{|x-y|^2} \log^\lambda\left(e + \frac{|g(x) - g(y)|}{|x-y|}\right) d\mathcal{H}^1_x d\mathcal{H}^1_y < \infty,$$

then $\mathcal{E}(g; \nu, \lambda) < \infty$.

Proof. We may assume that $|g(x)| \leq 1$ for all $x \in S^1$. Let

$$A_k(y) = \left\{x \in S^1 : 2^{-k+1}\pi < |x-y|_{S^1} \leq 2^{-k+2}\pi\right\}$$

for each $k \in \mathbb{N}$, where $|\cdot|_{S^1}$ is the length of the shorter arc of S^1 connecting x and y. In the following we will use the notation

$$\Delta g(x, y) := \frac{|g(x) - g(y)|}{|x-y|}$$

and

$$A_{I,k} := \fint_I \fint_{A_k(y)} \Delta g(x, y)\, d\mathcal{H}^1_x d\mathcal{H}^1_y,$$

where I is some dyadic interval and $k \in \mathbb{N}$. Here and in what follows, we use the convention that $\fint_{A_1(y)} \psi(x, y) d\mathcal{H}^1_x = 0$ for all $y \in S^1$ and any function $\psi\colon S^1 \times S^1 \to \mathbb{R}$.

Let $\varphi : [0,\infty[\to [0,\infty[$ and $\tilde{\varphi} : [0,\infty[\to [0,\infty[$ be two increasing continuous functions such that φ is convex on $[0,\infty[$ and $\varphi(t) = t^\nu \log^\lambda(e+t)$, $\tilde{\varphi}(y) = y^{\frac{1}{\nu}} \log^{-\lambda/\nu}(e+y)$, for $t > t_0$ and $y > \varphi(t_0)$, where t_0 is a suitably fixed number. In addition, we require that $\tilde{\varphi}(\varphi(t)) \geq Ct$ for all $t \geq t_0$ and some constant $C > 0$. Let $I \in \mathcal{D}_i$. Using these facts and Jensen's inequality we obtain

$$
\begin{aligned}
(18)\quad & \fint_I \int_{U(I)} \left|g(x) - g(y)\right| \, d\mathcal{H}^1_x d\mathcal{H}^1_y \leq C \sum_{k=i}^{\infty} 2^{i-2k} \fint_I \fint_{A_k(y)} \Delta g(x,y) \, d\mathcal{H}^1_x d\mathcal{H}^1_y \\
&= C \sum_{\substack{k=i \\ A_{I,k} \leq t_0}}^{\infty} 2^{i-2k} \fint_I \fint_{A_k(y)} \Delta g(x,y) \, d\mathcal{H}^1_x d\mathcal{H}^1_y + C \sum_{\substack{k=i \\ A_{I,k} > t_0}}^{\infty} 2^{i-2k} \fint_I \fint_{A_k(y)} \Delta g(x,y) \, d\mathcal{H}^1_x d\mathcal{H}^1_y \\
&\leq C t_0 \sum_{k=i}^{\infty} 2^{i-2k} + C \sum_{\substack{k=i \\ A_{I,k} > t_0}}^{\infty} 2^{i-2k} \tilde{\varphi}\left(\varphi \left(\fint_I \fint_{A_k(y)} \Delta g(x,y) \, d\mathcal{H}^1_x d\mathcal{H}^1_y \right)\right) \\
&\leq C 2^{-i} + C \sum_{\substack{k=i \\ A_{I,k} > t_0}}^{\infty} 2^{i-2k} \tilde{\varphi}\left(\fint_I \fint_{A_k(y)} \varphi\left(\Delta g(x,y)\right) \, d\mathcal{H}^1_x d\mathcal{H}^1_y \right) \\
&\leq C \sum_{k=i}^{\infty} 2^{i-2k} \left(\fint_I \fint_{A_k(y)} \varphi\left(\Delta g(x,y)\right) \, d\mathcal{H}^1_x d\mathcal{H}^1_y \right)^{1/\nu} \log^{-\lambda/\nu} \left(e + \fint_I \fint_{A_k(y)} \varphi\left(\Delta g(x,y)\right) \, d\mathcal{H}^1_x d\mathcal{H}^1_y \right) \\
&+ C 2^{-i}.
\end{aligned}
$$

Raising the last term on the RHS of (18) to the power of ν, multiplying it by i^λ and summing over all $I \in \mathcal{D}_i$ and over all $i \in \mathbb{N}$, we get some finite number. Now we concentrate on the first term on the RHS of (18). For simplicity we denote

$$
V_{I,k} := \fint_I \fint_{A_k(y)} \varphi\left(\Delta g(x,y)\right) \, d\mathcal{H}^1_x d\mathcal{H}^1_y
$$

in the following. We start with Hölder's inequality:

$$
\begin{aligned}
(19)\quad & 2^i \sum_{k=i}^{\infty} 2^{-2k} \left(V_{I,k}\right)^{1/\nu} \log^{-\lambda/\nu}\left(e + V_{I,k}\right) \leq 2^i \left[\sum_{k=i}^{\infty} 2^{-k\frac{\nu}{\nu-1}} \right]^{(\nu-1)/\nu} \left[\sum_{k=i}^{\infty} 2^{-\nu k} V_{I,k} \log^{-\lambda}\left(e + V_{I,k}\right) \right]^{1/\nu} \\
&\leq C \left[\sum_{k=i}^{\infty} 2^{-\nu k} V_{I,k} \log^{-\lambda}\left(e + V_{I,k}\right) \right]^{1/\nu}.
\end{aligned}
$$

From here on we split the argument into two cases according to the value of λ. Assume first that $\lambda \leq 0$. Since $\left|g\right| \leq 1$ and $\varphi(t) \leq \varphi(t_0) + t^\nu \log^\lambda\left(e+t\right)$, we have, for $k \geq 1$,

$$
e + \fint_I \fint_{A_k(y)} \varphi\left(\frac{\left|g(x) - g(y)\right|}{\left|x - y\right|} \right) dx dy \leq e + \varphi(t_0) + 2^{\nu k} \leq 2^{(\nu + C_1)k},
$$

where C_1 depends on $\varphi(t_0)$ and ν only. Therefore,

$$
(20)\qquad \log^{-\lambda}\left(e + V_{I,k}\right) \leq C k^{-\lambda}.
$$

We estimate now the contribution of the first term on the right hand side of (18) to the sum (2). Using estimates (18), (19) and (20) and the definitions of I and $A_k(y)$ we obtain the estimate

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{I\in\mathcal{D}_i}\left(2^i\sum_{k=i}^{\infty}2^{-2k}\left(V_{I,k}\right)^{1/\nu}\log^{-\lambda/\nu}\left(e+V_{I,k}\right)\right)^{\nu} i^{\lambda}\\
&\leq C\sum_{i=1}^{\infty}\sum_{I\in\mathcal{D}_i}\sum_{k=i}^{\infty}2^{-\nu k}i^{\lambda}V_{I,k}\log^{-\lambda}\left(e+V_{I,k}\right)\\
&\leq C\sum_{k=1}^{\infty}\sum_{i=1}^{k}\sum_{I\in\mathcal{D}_i}2^{-\nu k}i^{\lambda}k^{-\lambda}\fint_I\fint_{A_k(y)}\varphi\left(\Delta g(x,y)\right)dxdy\\
&\leq C\sum_{k=1}^{\infty}\sum_{i=1}^{k}2^i i^{\lambda}2^{-k}k^{-\lambda}\sum_{I\in\mathcal{D}_i}\int_I\int_{A_k(y)}\frac{\left|g(x)-g(y)\right|^{\nu}}{\left|x-y\right|^2}\log^{\lambda}\left(e+\frac{\left|g(x)-g(y)\right|}{\left|x-y\right|}\right)d\mathcal{H}^1_x d\mathcal{H}^1_y\\
&+C\sum_{k=1}^{\infty}\sum_{i=1}^{k}\sum_{I\in\mathcal{D}_i}2^{-\nu k}i^{\lambda}k^{-\lambda}\varphi(t_0)\\
&\leq C\sum_{k=1}^{\infty}\underbrace{\left[\sum_{i=1}^{k}2^i i^{\lambda}\right]}_{\approx 2^k k^{\lambda}}2^{-k}k^{-\lambda}\int_{S^1}\int_{A_k(y)}\frac{\left|g(x)-g(y)\right|^{\nu}}{\left|x-y\right|^2}\log^{\lambda}\left(e+\frac{\left|g(x)-g(y)\right|}{\left|x-y\right|}\right)d\mathcal{H}^1_x d\mathcal{H}^1_y\\
&+C\sum_{k=1}^{\infty}2^{-(\nu-1)k}\varphi(t_0)\\
&\leq C\sum_{k=1}^{\infty}\int_{S^1}\int_{A_k(y)}\frac{\left|g(x)-g(y)\right|^{\nu}}{\left|x-y\right|^2}\log^{\lambda}\left(e+\frac{\left|g(x)-g(y)\right|}{\left|x-y\right|}\right)d\mathcal{H}^1_x d\mathcal{H}^1_y+C\\
&=C\int_{S^1}\int_{S^1}\frac{\left|g(x)-g(y)\right|^{\nu}}{\left|x-y\right|^2}\log^{\lambda}\left(e+\frac{\left|g(x)-g(y)\right|}{\left|x-y\right|}\right)d\mathcal{H}^1_x d\mathcal{H}^1_y+C<\infty.
\end{aligned}
$$

This finishes the proof in the case $\lambda \leq 0$.

We may now assume that $\lambda > 0$. In order to estimate the logarithmic term from above, we define

$$
\chi(I,k)=\begin{cases}1, & \text{if } V_{I,k}\geq 2^{k\frac{\nu-1}{2}}\\ 0, & \text{otherwise.}\end{cases}
$$

We start with the estimate (19) as in previous calculation to obtain the estimate

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{I\in\mathcal{D}_i}\left(2^i\sum_{k=i}^{\infty}2^{-2k}\left(V_{I,k}\right)^{1/\nu}\log^{-\lambda/\nu}\left(e+V_{I,k}\right)\right)^{\nu} i^{\lambda}\\
&\quad\le C\sum_{k=1}^{\infty}\sum_{i=1}^{k}\sum_{I\in\mathcal{D}_i}2^{-\nu k}i^{\lambda}V_{I,k}\log^{-\lambda}\left(e+V_{I,k}\right)\\
&\quad\le C\sum_{k=1}^{\infty}\sum_{i=1}^{k}\sum_{I\in\mathcal{D}_i}\chi(I,k)2^{-\nu k}i^{\lambda}V_{I,k}\log^{-\lambda}\left(e+V_{I,k}\right)\\
&\quad+C\sum_{k=1}^{\infty}\sum_{i=1}^{k}\sum_{I\in\mathcal{D}_i}(1-\chi(I,k))2^{-\nu k}i^{\lambda}V_{I,k}\log^{-\lambda}\left(e+V_{I,k}\right)=:P_1+P_2.
\end{aligned}$$

In order to estimate P_1 we may proceed in the same way as in the case $\lambda\le 0$. For P_2 we use the definition of χ as follows

$$P_2\le C\sum_{k=1}^{\infty}\sum_{i=1}^{k}\sum_{I\in\mathcal{D}_i}(1-\chi(I,k))2^{-\nu k}2^{k\frac{\nu-1}{2}}i^{\lambda}\le C\sum_{k=1}^{\infty}2^{-\nu k}2^{k\frac{\nu-1}{2}}\sum_{i=1}^{k}2^i i^{\lambda}\le C\sum_{k=1}^{\infty}2^{-k\frac{\nu-1}{2}}k^{\lambda}\le C.$$

Therefore, we obtain

$$P_1+P_2\le C\int_{S^1}\int_{S^1}\frac{\left|g(x)-g(y)\right|^{\nu}}{\left|x-y\right|^2}\log^{\lambda}\left(e+\frac{\left|g(x)-g(y)\right|}{\left|x-y\right|}\right)d\mathcal{H}^1_x d\mathcal{H}^1_y+C.$$

□

4. Example

This section is dedicated to the following example. Note that ν is not necessarily an integer.

Example 1. Let $n\ge 2$, $1<\nu\le n$ and $\lambda\in[-1,\nu-1]$. If $\lambda\ne\nu-1$, assume $0<1/\beta<\frac{\nu-1-\lambda}{\nu}$; otherwise, take $\beta>\frac{\nu}{\nu-1}$. There exists a mapping $f:S^1\to\mathbb{R}^n$ with

$$\int_{S^1}\int_{S^1}\frac{\left|f(x)-f(y)\right|^{\nu}}{\left|x-y\right|^2}\log^{\lambda}\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right)d\mathcal{H}^1_x d\mathcal{H}^1_y<\infty \tag{21}$$

and $\mathcal{H}^{\nu}(f(S^1))>0$, which is continuous with modulus of continuity $\varphi(t)=C_0\exp\left(-C_1\log^{1/\beta}\frac{1}{t}\right)$, if $\lambda\ne\nu-1$, and $\varphi(t)=C_0\exp\left(-C_1\log^{\frac{1}{\beta}}\log\frac{1}{t}\right)$, otherwise; where $C_0>0$ and $C_1>0$ depend on the given parameters.

In the following subsections, we concentrate on the case $\lambda\ne\nu-1$. The remaining case is addressed in Subsection 4.9 in the end of this section.

4.1. **Construction.** The following construction is a modification to the example given in [8]. We will define the map $f:I\to\mathbb{R}^n$, where $I\in\mathbb{R}$ is an interval. It is easy to modify the construction to a map $f:S^1\to\mathbb{R}^n$.

Put $R_j=\exp\left(-j^{\beta}\right)$ and $r_j=2^n\exp\left(-(j+1)^{\beta}\right)$. Let n_0 be a large fixed integer, which will be determined later. We utilize the numbers R_j and r_j, with $j\ge n_0$, in a Cantor-type construction used in defining the mapping. We start with an interval I'_{n_0} of length $2r_{n_0}$. Divide I'_{n_0} into 2^n

intervals of length $2R_{n_0+1}$. We denote these intervals I_{i,n_0+1}, where $i = 1, 2, \dots, 2^n$, and denote the corresponding midpoints of these intervals by a_{i,n_0+1}. Define also intervals I'_{i,n_0+1} with the same centers and lengths $2r_{n_0+1}$. To ensure that $R_j > r_j$, we shall choose n_0 large enough. We use also the following notation $A_{i,n_0+1} := I_{i,n_0+1} \setminus I'_{i,n_0+1}$. We continue inductively. Assuming that we have defined the intervals $I'_{i,j}$ with lengths $2r_j$, $i = 1, 2, \dots, 2^{n(j-n_0)}$, we proceed dividing each interval $I'_{i,j}$ into 2^n intervals of length $2R_{j+1}$. These intervals are denoted by $I_{i',j+1}$, and their midpoints are $a_{i',j+1}$, where $i' = 1, 2, \dots, 2^{n(j+1-n_0)}$. Again, we choose the intervals $I'_{i',j+1}$ with midpoints $a_{i',j+1}$ and lengths $2r_{j+1}$, and define $A_{i',j+1} := I_{i',j+1} \setminus I'_{i',j+1}$.

Let $d > 0$ satisfy $2^n d^\nu = 1$. For each $j \in \mathbb{N}$ we define $\varphi_j : [0, \infty[\to [0, d^j]$ by

$$\varphi_j(r) = \begin{cases} d^j, & r \le r_j \\ d^j \frac{\log \frac{R_j}{r}}{\log \frac{R_j}{r_j}}, & r_j < r < R_j \\ 0, & r \ge R_j. \end{cases} \tag{22}$$

For the derivative of φ_j, $j \in \mathbb{N}$ and $j \ge n_0$, we have the estimate

$$\left|\varphi'_j(r)\right| \le d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \frac{1}{r} \le C d^j j^{1-\beta} \frac{1}{r}, \tag{23}$$

for $r_j < r < R_j$, since for large j we have

$$\log \frac{R_j}{r_j} = (j+1)^\beta - j^\beta - n \log 2 \ge \beta j^{\beta-1} - n \log 2 \ge C j^{\beta-1}.$$

Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Let v_i, $i = 1, 2, \dots, 2^n$, be the elements of the set $\{\sum_{k=1}^n a_k e_k : a_k = \pm 1\}$ in some fixed order. For all integers $k > 2^n$ we define $v_k = v_{k-2^n}$.

For all indices $j = n_0 + 1, n_0 + 2, \dots$ and $i = 1, 2, \dots, 2^{n(j-n_0)}$ and all $x \in I'_{n_0}$ we define $\tilde{\varphi}_{ij}(x) = (1-d) v_i \varphi_j \left(\left|x - a_{i,j}\right|\right)$. In addition, we set

$$f_j = \sum_{i=1}^{2^{n(j-n_0)}} \tilde{\varphi}_{ij}.$$

Finally, we define

$$f = \sum_{j=n_0+1}^{\infty} f_j.$$

It is clear that the above sum converges at every point.

In the following subsections we prove (21) and establish the correct modulus of continuity, but first, we explain why $\mathcal{H}^\nu(f(I'_{n_0})) > 0$. More precisely, we show that $f(I'_{n_0})$ contains a Cantor set with positive ν-dimensional Hausdorff measure. Define Q^i_j, $j = n_0 + 1, n_0 + 2, \dots$ and $i = 1, \dots, 2^{n(j-n_0)}$, to be the closed cube with sidelength d^j, centre $f(a_{i,j})$ and sides parallel to the coordinate axes. The set $C = \cap_{j=n_0+1}^{\infty} \cup_{i=1}^{2^{n(j-n_0)}} Q^i_j$ is a Cantor set in $\mathbb{R}^n$ with $\mathcal{H}^\nu(C) > 0$, see [2, Theorem 8.6] or [13, 4.13]. Since $f(I'_{n_0})$ contains the centres of each Q^i_j, it must also contain the set C by continuity.

4.2. **Finiteness of energy.** Next, we show that for every $x \in A_{i,j}$ with fixed $j \geq n_0$ and $i \in \{1, \ldots, 2^{n(j-n_0)}\}$, the mapping f satisfies the estimate

$$\int_{I'_{n_0}} \frac{\left|f(x)-f(y)\right|^{\nu}}{\left|x-y\right|^2} \log^{\lambda}\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right) dy$$

$$\leq C 2^{n(-j+n_0)} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \frac{1}{\left|x-a_{i,j}\right|} \log^{\lambda}\left(\frac{1}{\left|x-a_{i,j}\right|}\right) + C_{n_0} \tag{24}$$

with some positive constants C, C_{n_0}, independent of x, i and j. To prove this, we split the interval I'_{n_0} into a finite number of pieces and show the estimate for each of these pieces separately. This is done in the following subsections. The estimate (24) then follows by summing all the obtained estimates together. We first take care of the case $\lambda \geq 0$ and then indicate the required changes for $\lambda < 0$.

Fix $x \in A_{i,j}$. We may assume that $a_{i,j} = 0$ and that $x \geq 0$. We also ensure that n_0 is large enough to guarantee $r_j < R_j/2$. Additionally, notice that

$$\log \frac{1}{R_j} \leq \log \frac{1}{x} \leq \log \frac{1}{r_j} \tag{25}$$

and

$$C^{-1} j^{\beta} \leq (j-1)^{\beta} \leq (j+1)^{\beta} \leq C j^{\beta}.$$

Both of these estimates are elementary and we will use them without an explicit mention.

4.3. **Points close to x.** Here, we integrate over the interval

$$[a,b] := [\max\{x/2, \gamma r_j\}, \min\{2x, 3R_j/2, \zeta\}],$$

where $\gamma = \frac{2^{n+1}-1}{2^{n+1}}$ and ζ is the right endpoint of I'_{n_0}. We assume that n_0 is so large that $\left|\varphi'_{k+1}(R_{k+1})\right| \geq \left|\varphi'_k(r_k)\right|$ for all $k = n_0+1, \ldots$. If $x/2 < r_j$, we estimate, using (23),

$$\frac{|f(x)-f(y)|}{|x-y|} \leq \left|\varphi'_{j+1}\left(\frac{R_{j+1}}{2}\right)\right| \leq d^{j+1} \frac{2}{R_{j+1}} \log^{-1} \frac{R_{j+1}}{r_{j+1}} \leq C d^j \frac{1}{x} \log^{-1} \frac{R_j}{r_j} \leq \frac{1}{x} - e$$

for each $y \in [a,b]$ and large enough j. Otherwise,

$$\frac{|f(x)-f(y)|}{|x-y|} \leq \left|\varphi'_j\left(\frac{x}{2}\right)\right| \leq d^j \frac{2}{x} \log^{-1} \frac{R_j}{r_j} \leq \frac{1}{x} - e.$$

Therefore, we have

$$\int_{[a,b]} \frac{\left|f(x)-f(y)\right|^{\nu}}{\left|x-y\right|^2} \log^{\lambda}\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right) dy \leq C d^{j\nu} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \frac{1}{x^{\nu}} \log^{\lambda}\left(\frac{1}{x}\right) \int_{[0,2x]} \left|x-y\right|^{\nu-2} dy$$

$$\leq C 2^{-jn} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \frac{1}{x} \log^{\lambda}\left(\frac{1}{x}\right),$$

as desired. For the remaining points $y \in I'_{n_0} \setminus [a,b]$, we have $|x-y| > 2^{-n-1}x$.

4.4. **Parent set and its descendants.** In this subsection we integrate over the set $I_{i',j-1} \setminus [a,b]$, where i' is such that $A_{i,j} \subset I'_{i',j-1}$. This set consists of $A_{i,j} \setminus [a,b]$, of its parent set $A_{i',j-1} \setminus [a,b]$, of its $2^n - 1$ siblings $A_{i'',j} \setminus [a,b]$, contained in $I'_{i',j-1}$, and of 2^n intervals $I'_{i'',j}$ and $I'_{i,j} \setminus [a,b]$ of length $2r_j$, embraced by those $A_{i'',j}$ and $A_{i,j}$. In the case $j = n_0 + 1$ the only difference is the absence of the set $A_{i',j-1} \setminus [a,b]$. For a point y in any of these sets, we have

$$\log^{\lambda}\left(e + \frac{|f(x) - f(y)|}{|x-y|}\right) \le C \log^{\lambda} \frac{1}{x}; \tag{26}$$

thus, it remains to estimate the integral of the quotient $|f(x) - f(y)|^{\nu}/|x-y|^2$. This is done differently for each of the mentioned sets.

4.4.1. *Interval containing x.* For $y \in A_{i,j} \setminus [a,b]$ we have the estimate

$$|f(x) - f(y)| \le \sqrt{n} d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \left|\log \frac{R_j}{x} - \log \frac{R_j}{|y|}\right| \le \sqrt{n} d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \left|\log \frac{|y|}{x}\right|. \tag{27}$$

Furthermore,

$$\int_{A_{i,j}\setminus[a,b]} \frac{\left|\log \frac{|y|}{x}\right|^{\nu}}{|x-y|^2} dy = \frac{1}{x} \int_{A_{i,j}\setminus[a,b]} \frac{\left|\log \frac{|y|}{x}\right|^{\nu}}{|1 - y/x|^2} \frac{dy}{x} \le \frac{1}{x}\left(\int_{-\infty}^{\frac{1}{2}} \frac{\left|\log |t|\right|^{\nu}}{(1-t)^2} dt + \int_2^{\infty} \frac{|\log t|^{\nu}}{(1-t)^2} dt\right) \le \frac{C}{x}.$$

Combining this estimate with (26) and (27), we obtain the desired estimate.

4.4.2. *Siblings of $A_{i,j}$.* Let us fix an index i'', so that both $A_{i,j}$ and $A_{i'',j}$ are contained in the same interval $I'_{i',j-1}$ of generation $j-1$. Then

$$\begin{aligned} |f(x) - f(y)| \le |f(x) - f(R_j)| + |f(y) - f(R_j)| &\le \sqrt{n} d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \left(\log \frac{R_j}{x} + \log \frac{R_j}{|y - a_{i'',j}|}\right) \\ &= \sqrt{n} d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \log \frac{R_j^2}{x|y - a_{i'',j}|} \end{aligned} \tag{28}$$

for each $y \in A_{i'',j} \setminus [a,b]$. Note also that $|x - y| \ge R_j/4$ for each of these y. Hence, substituting $t = \frac{x|y - a_{i'',j}|}{R_j^2}$ in each of the two intervals, we obtain

$$\int_{A_{i'',j}\setminus[a,b]} \frac{\left(\log \frac{R_j^2}{x|y - a_{i'',j}|}\right)^{\nu}}{|x-y|^2} dy \le \frac{16}{R_j^2} \int_{A_{i'',j}} \left(\log \frac{R_j^2}{x|y - a_{i'',j}|}\right)^{\nu} dy \le \frac{C}{x} \int_0^1 |\log t|^{\nu} dt \le \frac{C}{x}.$$

The desired estimate follows from the last inequality, (26) and (28). Note that there are $2^n - 1$ suitable i''.

4.4.3. *Children of $A_{i,j}$.* We consider one of 2^n sets $A_{i_0,j+1}$, such that $A_{i_0,j+1} \subset I'_{i,j}$. If $y \in A_{i_0,j+1} \setminus [a,b]$, we have

$$\begin{aligned} |f(x)-f(y)| \le& |f(x)-f(r_j)| + |f(y)-f(r_j)| \le \sqrt{n} d^j \left(1 - \frac{\log \frac{R_j}{x}}{\log \frac{R_j}{r_j}}\right) + \sqrt{n} d^{j+1} \frac{\log \frac{R_{j+1}}{|y-a_{i_0,j+1}|}}{\log \frac{R_{j+1}}{r_{j+1}}} \\ &\le \sqrt{n} d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \left[\log \frac{R_j}{r_j} - \log \frac{R_j}{x} + \log \frac{R_{j+1}}{|y-a_{i_0,j+1}|}\right] \\ &\le \sqrt{n} d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \log \frac{x}{2^n |y-a_{i_0,j+1}|} \end{aligned} \tag{29}$$

Using the inequality $|x-y| \ge 2^{-n-1}x$ for $y \notin [a,b]$, we further estimate with the substitute $t = \frac{2^n|y-a_{i_0,j+1}|}{x}$:

$$\int_{A_{i_0,j+1}\setminus[a,b]} \frac{\left(\log \frac{x}{2^n|y-a_{i_0,j+1}|}\right)^\nu}{|x-y|^2} dy \le \frac{2^{2n+2}}{x} \int_{A_{i_0,j+1}} \left(\log \frac{x}{2^n|y-a_{i_0,j+1}|}\right)^\nu \frac{dy}{x} \le \frac{C}{x} \int_0^1 |\log t|^\nu dt \le \frac{C}{x}.$$

Together with (26) and (29), this gives us (24).

4.4.4. *Parent of $A_{i,j}$.* We fix the unique i' with $A_{i,j} \subset I'_{i',j-1}$. Similarly to (29), interchanging the roles of y and x and shifting the index, we obtain

$$\begin{aligned} |f(x)-f(y)| \le& |f(x)-f(R_j)| + |f(y)-f(R_j)| \le \sqrt{n} d^{j-1} \left(\log \frac{R_{j-1}}{r_{j-1}}\right)^{-1} \log \frac{|y-a_{i',j-1}|}{2^n x} \\ \le& C d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \log \frac{|y-a_{i',j-1}|}{2^n x} \le C d^j \left(\log \frac{R_j}{r_j}\right)^{-1} \log \frac{2|y|}{x} \end{aligned}$$

for $y \in A_{i',j-1} \setminus [a,b]$, since $|a_{i',j-1}| \le r_{j-1}$ and $|y| \ge R_j$. We also have $y \ge 3x/2$, when $y > 0$. Therefore,

$$\int_{A_{i',j-1}\setminus[a,b]} \frac{\left(\log \frac{2|y|}{x}\right)^\nu}{|x-y|^2} dy = \frac{1}{x} \int_{A_{i',j-1}\setminus[a,b]} \frac{\left(\log \frac{2|y|}{x}\right)^\nu}{|1-y/x|^2} \frac{dy}{x} \le \frac{1}{x} \left(\int_{-\infty}^{-1} \frac{\log^\nu |2t|}{(1-t)^2} dt + \int_{\frac{3}{2}}^{\infty} \frac{\log^\nu (2t)}{(1-t)^2} dt\right) \le \frac{C}{x}.$$

The conclusion is as in the previous subsections.

4.4.5. *Grandchildren of $A_{i,j}$, nephews of $A_{i,j}$ and their descendants.* As before, we fix i_0 with $A_{i_0,j+1} \subset I'_{i,j}$. Let us integrate over the set $I'_{i_0,j+1}$. Here we use the fast decay of the numbers r_j. The estimate is the following:

$$\int_{I'_{i_0,j+1}} \frac{\left|f(x)-f(y)\right|^\nu}{\left|x-y\right|^2} dy \le C d^{j\nu} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \frac{1}{x} \frac{r_{j+1}}{x} \left(\log \frac{R_j}{r_j}\right)^\nu \le C 2^{-jn} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \frac{1}{x},$$

since

$$\begin{aligned} \frac{r_{j+1}}{x} \left(\log \frac{R_j}{r_j}\right)^\nu \le \frac{r_{j+1}}{r_j} \left(\log \frac{R_j}{r_j}\right)^\nu \le& C j^{\nu(\beta-1)} \exp\left(-(j+2)^\beta + (j+1)^\beta\right) \\ \le& C j^{\nu(\beta-1)} \exp\left(-\beta(j+1)^{\beta-1}\right) \le C. \end{aligned} \tag{30}$$

The estimate (24) follows from (26).

Similar computations imply the conclusion for the intervals $I'_{i'',j'}$ such that $A_{i,j} \cup A_{i'',j} \subset I'_{i',j-1}$ for i' as above; because $|x-y| \geq R_j$ for $y \in I'_{i'',j}$ and $|I'_{i'',j}| = 2r_j$.

4.5. **Earlier generations.** Now we integrate over the set $I'_{n_0} \setminus I_{i',j-1}$ where $I_{i',j-1} \supset I_{i,j}$. Note that if $y \in [R_j + \sum_{i=k+1}^{j-1}(R_i - r_i), R_j + \sum_{i=k}^{j-1}(R_i - r_i)]$ for some $k \leq j-2$, then there is an interval $I_{l,k}$ of generation k, which contains both x and y. This implies $\left|f(x) - f(y)\right| \leq 2\sqrt{n}\, d^k$.

Define numbers $A_k = \sum_{i=k+1}^{j-1}(R_i - r_i)$ and $B_k = \sum_{i=k}^{j-1}(R_i - r_i)$. We choose $n_0 \in \mathbb{N}$ so large that the inequality

$$e + \sqrt{n}\frac{2^{-l+1}}{A_l} \leq A_l^{-1} \tag{31}$$

holds for each $l \geq n_0$.

If $j = n_0 + 1$, there are no intervals of earlier generations; and if $j = n_0 + 2$, then the only intervals of earlier generations are the intervals of the generation $n_0 + 1$, which are treated in (34) and in Section 4.4.4. Using (31), we get the following estimate for all $j \geq n_0 + 3$:

$$\begin{aligned}
&\int_{[R_j + (R_{j-1} - r_{j-1}), R_j + \sum_{i=n_0+1}^{j-1}(R_i - r_i)]} \frac{\left|f(x) - f(y)\right|^\nu}{\left|x-y\right|^2} \log^\lambda\left(e + \frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right) dy \\
&\leq C \sum_{k=n_0+1}^{j-2} \int_{[R_j + A_k, R_j + B_k]} \frac{d^{k\nu}}{\left|x-y\right|^2} \log^\lambda(A_k^{-1}) dy \leq C \sum_{k=n_0+1}^{j-2} 2^{-kn} \log^\lambda(A_k^{-1}) \int_{A_k}^\infty \frac{dt}{t^2} \\
&\leq C \sum_{k=n_0+1}^{j-2} 2^{-kn} \log^\lambda(A_k^{-1}) \frac{1}{A_k} \leq C \sum_{k=n_0}^{j-2} 2^{-kn} \log^\lambda(R_{k+1}^{-1}) \frac{1}{R_{k+1}}.
\end{aligned} \tag{32}$$

Next, we show that for all large k we have

$$2^{-kn} \log^\lambda(R_k^{-1}) \frac{1}{R_k} \geq 2 \cdot 2^{(-k+1)n} \log^\lambda(R_{k-1}^{-1}) \frac{1}{R_{k-1}}.$$

This is rather easy:

$$\frac{2^{-kn} \log^\lambda(R_k^{-1}) \frac{1}{R_k}}{2^{(-k+1)n} \log^\lambda(R_{k-1}^{-1}) \frac{1}{R_{k-1}}} \geq \frac{1}{2^n} \exp\left(k^\beta - (k-1)^\beta\right) \left(\frac{k}{k-1}\right)^{\beta\lambda} \geq \frac{1}{2^n} \exp\left(\beta(k-1)^{\beta-1}\right) \geq 2,$$

as soon as k is large enough. Now the sum on the right hand side of (32) is seen to be smaller than

$$C \sum_{k=n_0}^{j-2} 2^{k-j+2} 2^{-jn} \log^\lambda(R_{j-1}^{-1}) \frac{1}{R_{j-1}} \leq C 2^{-jn} \log^\lambda(R_{j-1}^{-1}) \frac{1}{R_{j-1}}.$$

Besides, similarly to (30), we observe that $\frac{R_j}{R_{j-1}} \log^\nu \frac{R_j}{r_j} \le C$, which implies the desired estimate

$$\int_{[R_j+(R_{j-1}-r_{j-1}),R_j+\sum_{i=n_0+1}^{j-1}(R_i-r_i)]} \frac{\left|f(x)-f(y)\right|^\nu}{\left|x-y\right|^2} \log^\lambda\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right) dy$$
$$\le C2^{-jn} \log^\lambda\left(\frac{1}{x}\right) \log^{-\nu} \frac{R_j}{r_j}\frac{1}{x}. \tag{33}$$

The integral over the set $[-R_j - B_{n_{0+1}}, -R_j - (R_{j-1} - r_{j-1})]$ is estimated in the same way as the integral over the set $[R_j + (R_{j-1} - r_{j-1}), R_j + B_{n_{0+1}}]$, since we have the same estimates for $\left|f(x) - f(y)\right|$ and $\left|x - y\right|$.

The interval $[-R_j - B_{n_0+1}, R_j + B_{n_0+1}]$ does not cover I'_{n_0}. It remains to estimate the integral over the set $I'_{n_0} \setminus [-R_j - B_{n_0+1}, R_j + B_{n_0+1}]$:

$$\int_{I'_{n_0}\setminus[-R_j-B_{n_0+1},R_j+B_{n_0+1}]} \frac{\left|f(x)-f(y)\right|^\nu}{\left|x-y\right|^2} \log^\lambda\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right) dy \tag{34}$$
$$\le C\left|I'_{n_0}\right| B_{n_0+1}^{-2} \log^\lambda\left(e+B_{n_0+1}^{-1}\right) \le C_{n_0}.$$

4.6. **Case $\lambda < 0$.** We have to make several changes to above arguments. Our aim is to divide the points in I'_{n_0} into two sets so that we can estimate the logarithmic term on the left hand side of (24).

Let $I_{i',j-1}$ be the interval of generation $j-1$ which contains $A_{i,j}$. The length of this interval is at most $2R_{j-1}$. First, we consider all the points $y \in I_{i',j-1}$ such that

$$\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|} \ge R_{j-1}^{-\frac{\nu-1}{\nu}}. \tag{35}$$

Notice that the logarithms of r_j and R_{j-1} are comparable, then by (25) and (35) we have

$$\log\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|} \ge C\log\frac{1}{R_{j-1}} \ge C\log\frac{1}{r_j} \ge C\log\frac{1}{x}.$$

Using this to estimate the logarithm on the set of points satisfying (35) we obtain the analogues of estimates in Sections 4.3 and 4.4.

For the points $y \in I_{i',j-1}$, for which (35) is not satisfied, we have the estimate

$$\int_{\{y\in I_{i',j-1}\,:\,(35)\text{ is not true}\}} \frac{\left|f(x)-f(y)\right|^\nu}{\left|x-y\right|^2} \log^\lambda\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right) dy \le C.$$

When considering the earlier generations in Section 4.5, we had to estimate the sum

$$\sum_{k=n_0+1}^{j-2} \int_{[R_j+\sum_{i=k+1}^{j-1}(R_i-r_i),R_j+\sum_{i=k}^{j-1}(R_i-r_i)]} \frac{\left|f(x)-f(y)\right|^\nu}{\left|x-y\right|^2} \log^\lambda\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right) dy. \tag{36}$$

Again, for $\lambda < 0$ we split the integration intervals into two sets. We use the notation introduced in Subsection 4.5. In the set of points $y \in [R_j + A_k, R_j + B_k]$, where

$$\frac{\left|f(x) - f(y)\right|}{\left|x - y\right|} \leq R_k^{-\frac{\nu-1}{2\nu}},$$

the sum (36) reduces to

$$C \sum_{k=n_0+1}^{j-2} R_k^{\frac{\nu-1}{2}} \leq C \sum_{k=1}^{j-2} \exp\left(-\frac{\nu-1}{2} k^{\beta}\right) \leq C.$$

On the other hand, on the set of points y, such that

$$\frac{\left|f(x) - f(y)\right|}{\left|x - y\right|} > R_k^{-\frac{\nu-1}{2\nu}},$$

we may estimate the logarithm term in (36) from above and then use the same arguments as in Section 4.5 to obtain the desired estimate.

4.7. f **has finite energy.** Now we show that the required energy is finite. If $-1 < \lambda < \nu - 1$, then using (24) we obtain the estimate

$$\begin{aligned}
&\int_{I'_{n_0}} \int_{I'_{n_0}} \frac{\left|f(x) - f(y)\right|^{\nu}}{\left|x - y\right|^2} \log^{\lambda}\left(e + \frac{\left|f(x) - f(y)\right|}{\left|x - y\right|}\right) dy dx \\
&\leq \sum_{j=n_0+1}^{\infty} \sum_{i=1}^{2^{(j-n_0)n}} \int_{A_{i,j}} \int_{I'_{n_0}} \frac{\left|f(x) - f(y)\right|^{\nu}}{\left|x - y\right|^2} \log^{\lambda}\left(e + \frac{\left|f(x) - f(y)\right|}{\left|x - y\right|}\right) dy dx \\
&\leq C \sum_{j=n_0+1}^{\infty} \sum_{i=1}^{2^{(j-n_0)n}} \int_{A_{i,j}} 2^{-jn} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \frac{1}{\left|x - a_{i,j}\right|} \log^{\lambda} \frac{1}{\left|x - a_{i,j}\right|} + C_{n_0} dx \\
&\leq C \sum_{j=n_0+1}^{\infty} \sum_{i=1}^{2^{(j-n_0)n}} 2^{-jn} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \left(\log^{\lambda+1} \frac{1}{\left|r_j\right|} - \log^{\lambda+1} \frac{1}{\left|R_j\right|}\right) + C_{n_0} \left|I'_{n_0}\right| \\
&\leq C \sum_{j=n_0+1}^{\infty} \left(\log \frac{R_j}{r_j}\right)^{-\nu} \left(\log^{\lambda+1} \frac{1}{\left|r_j\right|} - \log^{\lambda+1} \frac{1}{\left|R_j\right|}\right) + C_{n_0} \\
&\leq C \sum_{j=n_0+1}^{\infty} j^{-\nu(\beta-1)} \left(\log^{\lambda+1}\left(\exp\left((j+1)^{\beta}\right)\right) - \log^{\lambda+1}\left(\exp\left(j^{\beta}\right)\right)\right) + C_{n_0} \\
&\leq C \sum_{j=n_0+1}^{\infty} j^{\nu-\nu\beta} \left(\left((j+1)^{\beta(\lambda+1)}\right) - \left(j^{\beta(\lambda+1)}\right)\right) + C_{n_0} \leq C \sum_{j=n_0+1}^{\infty} j^{\nu-1-\nu\beta+\beta(\lambda+1)} + C_{n_0} \\
&< \infty,
\end{aligned}$$

if $\nu - 1 + \beta(\lambda - \nu + 1) < -1$, that is $1/\beta < \frac{\nu-1-\lambda}{\nu}$. Using a linear approximation for the logarithm in a calculation similar to the one above, we obtain the same result also for $\lambda = -1$.

4.8. **Modulus of continuity.** We will show that the mapping f has a modulus of continuity $\varphi(t) = C' \exp\left(-C \log^{1/\beta} \frac{1}{t}\right)$, where C and C' are some positive constants. First, let $x, y \in [a_{i,j} + r_j, a_{i,j} + R_j]$ for some $j \in \{n_0 + 1, \ldots\}$ and $i \in \left\{1, \ldots, 2^{n(j-n_0)}\right\}$. We may assume again that $a_{i,j} = 0$. Then

$$\left|f(x) - f(y)\right| = (1-d)\sqrt{n}\frac{d^j}{\log \frac{R_j}{r_j}}\left|\log \frac{x}{y}\right|. \tag{37}$$

Assume that $x > y$ and $d^{m-1} r_j \le x - y < d^m r_j$ for some $m \in \mathbb{Z}$. If $m \ge 0$, we get

$$\begin{aligned}\left|f(x) - f(y)\right| &= (1-d)\sqrt{n}\frac{d^j}{\log \frac{R_j}{r_j}}\left|\log\left(1 + \frac{x-y}{y}\right)\right| \le C\frac{d^j}{\log \frac{R_j}{r_j}}\frac{x-y}{r_j} \le C\frac{d^j}{\log \frac{R_j}{r_j}} d^m \\ &\le C\frac{d^{j+m}}{\log \frac{R_j}{r_j}} \exp\left(C_0\left(\log \frac{1}{\left|x-y\right|}\right)^{1/\beta}\right) \exp\left(-C_0\left(\log \frac{1}{\left|x-y\right|}\right)^{1/\beta}\right),\end{aligned}$$

where we fix $C_0 \in \left]0, \min\left\{\log d^{-1}, (\log d^{-1})^{1-1/\beta}\right\}\right[$. In addition, when j is large enough, we have

$$\begin{aligned}\frac{d^{j+m}}{\log \frac{R_j}{r_j}} \exp\left(C_0\left(\log \frac{1}{\left|x-y\right|}\right)^{1/\beta}\right) &\le \frac{d^{j+m}}{C j^{\beta-1}} \exp\left(C_0\left(\log \frac{d^{-m+1}}{r_j}\right)^{1/\beta}\right) \\ &\le C d^{j+m} \exp\left(C_0\left((-m+1)\log d + (j+1)^\beta\right)^{1/\beta}\right) \\ &\le C d^{j+m} \exp\left(C_0 m^{1/\beta} \log^{1/\beta} d^{-1} + C_0 j + C_0\right) \\ &\le C \exp\left(m \log d + C_0 m^{1/\beta} \log^{1/\beta} d^{-1} + (C_0 + \log d) j\right) \\ &\le C \exp\left(m \log d + C_0 m^{1/\beta} \log^{1/\beta} d^{-1}\right) \exp\left((C_0 + \log d) j\right) \le 1,\end{aligned}$$

by the choice of C_0 and the fact that $\beta > 1$.

If $m < 0$, then (37) implies

$$\begin{aligned}\left|f(x) - f(y)\right| &= C\frac{d^j}{\log \frac{R_j}{r_j}} \log\left(1 + \frac{x-y}{y}\right) \le C d^j j^{1-\beta} \log\left(1 + \frac{x-y}{r_j}\right) \le C d^j j^{1-\beta} \log\left(1 + d^m\right) \\ &\le C 2^{-j} j^{1-\beta} \left|m\right|. \end{aligned} \tag{38}$$

By the choice of m, we have

$$\frac{R_j}{r_j} \ge d^{m-1},$$

which yields

$$|m| \le C j^{\beta-1},$$

and

$$\log \frac{1}{\left|x-y\right|} \le 2^\beta j^\beta.$$

These estimates with (38) give

$$\left|f(x)-f(y)\right| \le C' \exp\left(-\frac{\log 2}{2}\left(\log\frac{1}{\left|x-y\right|}\right)^{1/\beta}\right).$$

If x and y are in neighbouring intervals, then we have

$$\left|f(x)-f(y)\right| \le \left|f(x)-f(c)\right| + \left|f(c)-f(y)\right| \le 2C\exp\left(-C\left(\log\frac{1}{\left|x-y\right|}\right)^{1/\beta}\right).$$

Here c is the point where the two intervals meet.

If x and y are points which are not in same or neighbouring intervals, then there is the smallest j such that a component interval of $A_{i,j}$ lies between x and y. Thus $R_j/2 \le \left|x-y\right|$. That is, $-j \le -C\left(\log\frac{1}{|x-y|}\right)^{1/\beta}$. In this case, there is an interval $I_{i',j-1}$ (or I'_{n_0} in the case $j-1=n_0$) containing both x and y. This implies $\left|f(x)-f(y)\right| \le \sqrt{n}d^{j-1}$. Therefore, we get again the continuity estimate $\left|f(x)-f(y)\right| \le d^{-1}\sqrt{n}\exp\left(-C\left(\log\frac{1}{|x-y|}\right)^{1/\beta}\right)$.

4.9. **Case $\lambda = \nu - 1$.** As before, this example is from [8], where it was presented in a slightly different context. We make the following modifications to the construction from the previous subsections. We define $R_j = \exp(-\exp(j^\beta))$, $r_j = 2^n\exp(-\exp((j+1)^\beta))$ and

$$\varphi_j(r) = \begin{cases} d^j, & r \le r_j \\ d^j \dfrac{\log\frac{\log\frac{1}{r}}{\log\frac{1}{R_j}}}{\log\frac{\log\frac{1}{r_j}}{\log\frac{1}{R_j}}}, & r_j < r < R_j \\ 0, & r \ge R_j, \end{cases}$$

in place of (22).

Verification of the properties is similar to what is done in the previous subsections, but is more tedious. The details are left to the reader. When proving (21), instead of (24), one should establish

$$\int_{I'_{n_0}} \frac{\left|f(x)-f(y)\right|^\nu}{\left|x-y\right|^2}\log^{\nu-1}\left(e+\frac{\left|f(x)-f(y)\right|}{\left|x-y\right|}\right)dy$$
$$\le C2^{n(-j+n_0)}\left(\log\frac{\log\frac{1}{r_j}}{\log\frac{1}{R_j}}\right)^{-\nu}\frac{1}{\left|x-a_{i,j}\right|}\log^{-1}\left(\frac{1}{\left|x-a_{i,j}\right|}\right) + C(n_0)$$

for each $j = n_0+1,\ldots$, $i = 1,\ldots,2^{n(j-n_0)}$ and $x \in A_{i,j}$.

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