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The Development of Early Mathematical Skills – A Theoretical Framework for a Holistic Model

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ABSTRACT: This article presents a theoretical framework for a holistic model of the development of early mathematical skills in early childhood education. The first aim of this study was to conduct a comprehensive international review of the literature to explore early mathematical skills categories. The literature review yielded three early mathematical skills categories, namely (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. Previous studies have shown that several mathematical skills develop gradually and simultaneously in early ages and that these skills areas are interconnected in mathematical skills learning. Accordingly, the second aim of this study was to build a holistic theoretical model illustrating the interconnections between the three developing skills categories. The mathematical skills gained in early childhood influence later mathematical achievement. Thus, the importance of simultaneous strengthening of versatile early mathematical skills through conscious early learning practices should be addressed in early mathematical learning.

Keywords: *early mathematical skills, numerical skills, spatial thinking skills, mathematical thinking and reasoning skills, holistic model of early mathematical skills*

Introduction

Mathematics is part of everyday life and necessary for being an autonomous citizen; it helps people organise their insights and discoveries about the world in systematic ways, such as when they need to represent quantities and spatial relationships (Cross, Woods,

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Schweingruber & National Research Council, 2009). Cross and colleagues (2009) hold the view that mathematics provides means for reasoning, describing and understanding the world and its phenomena. Because of the importance of mathematics in human life, most governments require that education systems provide basic mathematical understanding and proficiency (van Oers, 2013). The importance of strengthening mathematical skills and stressing mathematical learning in early childhood education is thus clear (Sarama & Clements, 2009).

Contemporary research findings attest to the importance of early mathematical learning. They show that the understanding of complex mathematics and abstract reasoning develops much earlier than was once believed (Clements, Fuson & Sarama, 2019; Cross et al., 2009; Mulligan & Mitchelmore, 2013). Studies have broadened our understanding by showing that humans are born with a number sense (meaning quantity and number knowledge) and spatial sense (Clements & Sarama, 2007) and that the development of mathematical skills begins at a very early age. Studies also show that an early understanding of mathematics correlates positively with both children's future success in mathematics (Aunio & Niemivirta, 2010; Jordan, Kaplan, Ramineni & Locuniak, 2009; Jordan, Mulhern & Wylie, 2009) and overall achievements (Aubrey, Dahl, Godfrey, 2006; Romano, Kohen, Babchishin & Pagini, 2010). Besides, the quality of mathematics education is central to the development of mathematical skills (Aunio, Korhonen, Bashash & Khoshbakht, 2014; MacDonald & Carmichael, 2018).

Broadly speaking, mathematics education is often divided into two parts: numeracy and geometry (e.g. Tsamir, Tirosh & Levenson, 2011). Less attention is given to mathematical reasoning processes, even though children also learn these skills (e.g. Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). In 2007, Clements and Sarama were the first researchers who compiled the development of early mathematical skills through an extensive inclusion of the studies on mathematics learning and teaching in early childhood. Their comprehensive research review included six areas: (1) number and quantitative thinking, (2) geometry and spatial thinking, (3) geometric measurement, (4) patterns and algebraic thinking, (5) data analysis and (6) mathematical processes. Later, Sarama and Clements (2009) combined the latter three areas under the umbrella of 'other content domains and processes'.

Although Clements and Sarama (2007, 2009) presented their comprehensive compilations some time ago, the literature reviews on early childhood mathematics education remains limited. Only two studies (see Fox & Diezmann, 2007; Linder, Ramey & Zambak, 2013) have focused on children's perspectives whereas three other studies (see Antohony & Walshaw, 2009; Linder & Simpson, 2018; MacDonald & Murphy, 2019) have concentrated on teachers' perspectives. Early mathematical skills have thus far been categorised and classified from a numerical standpoint (see Aunio & Räsänen, 2016). A literature review that would address the learning and development of early mathematical skills comprehensively does not yet exist. The new research findings concerning children's early mathematical skills suggest that a comprehensive theoretical model of early mathematical skills illustrating the interconnections of simultaneous development of skills is needed.

Aims

The purpose of this study is to propose a theoretical holistic model of the development of early mathematical skills through the conceptual analysis of the research literature. The first aim is to form early mathematical skills categories and find possible gaps in the existing studies by answering the following research question:

1. In what way are early mathematical skills conceptualised, and can they be categorised based on the existing research literature?

Drawing on the answers to the first research question, the second aim is to build a comprehensive, holistic theoretical model of the development of early mathematical skills through the following question:

2. What kind of holistic framework describing the development of early mathematical skills can be built?

Method

The research literature on early mathematical skills served as data in this investigation. The literature review was conducted, and English-language peer-reviewed articles were identified via electronic searches using the Education Resources Information Center (ERIC). Clements and Sarama's (2007, 2009) extensive pioneering work was used to decide on the initial search terms, which were completed with subject headings and frequently used keywords of the articles with the help of a thesaurus. The search terms included 'mathematics education', 'mathematics instruction', 'mathematics', 'mathematics skills', 'numeracy', 'numerical skills', 'geometric', 'measurement', 'spatial ability', 'spatial relationship', 'algebra', 'early algebra' and 'early childhood education'. Search terms were carefully selected based on earlier research-knowledge about early mathematical skills and with the help of frequently used search terms. Education level was limited to early childhood education. The ERIC identified 492 peer-reviewed articles.

The aim was to attain a comprehensive and high-quality search, which would correspond with and broaden the existing knowledge (see Cooper, 2009; Reed & Baxter, 2009).

The initial analysis of the titles helped in selecting relevant articles written between 2003 and 2018 for detailed examination, yielding 273 articles. The rationale behind the above time frame was to enable accessing contemporary data from the latest studies into the learning and development of early mathematical skills and broadening Clements and Sarama's (2007, 2009) earlier findings. This choice was supported by the changes that have taken place in early mathematics education and pedagogy during the present century (Newton & Alexander, 2013). The identified 273 articles were collected using the RefWorks programme for further processing. The next step was to refine the search by exploring the articles' abstracts. During this analysis, age groups (from infants to 8-year-olds) and article contents were carefully reviewed, and the articles related to early mathematical skills in early childhood education were selected for qualitative content analysis. Also, other relevant materials and sources were searched to avoid gaps in data collection.

Altogether, 134 articles were analysed. The process of qualitative content analysis (see Krippendorff, 2004; Mason, 2009; Patton, 2015; Ryan & Bernard, 2000) focusing on literature analysis (see Cooper, 2009; Wilson, 2009) was followed. Articles were first categorised according to conceptual codes (e.g. number sense, symmetry, geometry and patterning). Second, the concepts were carefully organised into broader thematic clusters. Thirteen thematic clusters emerged from the data (e.g. counting skills, spatial reasoning, geometrical awareness and problem-solving strategies). Third, the thematic clusters were further divided into typologies, and the following three early mathematical skills categories were formed: (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. Figure 1 describes the process of the content analysis evolving from conceptual codes into thematic clusters and further into the broader three themes, answering to the first research question.

Sixty-two of the analysed articles were related to numerical learning, whereas 49 focused on spatial and geometrical learning; only five articles were concerned with measurement. Of the remaining articles (of 134), 18 discussed mathematical thinking and reasoning processes (e.g. algebra, data analysis and patterning) separately, and most of them focused on seven- to eight-year-old children.

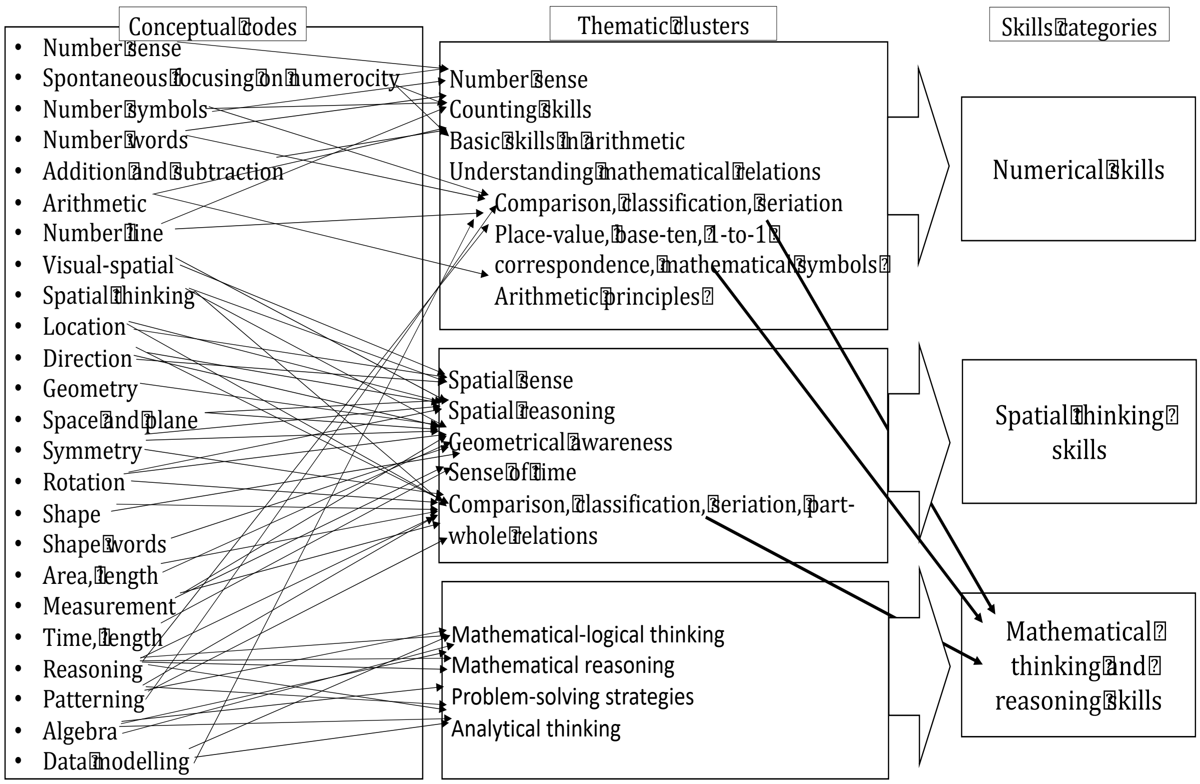

FIGURE 1 The process of the thematic content analysis.

After indicating the conceptual codes and grouping them into skills categories (research question 1), the next step of the analysis was to further explore the connections between the conceptual codes, thematic clusters and skills categories. This was carried out through crosswise comparison. The final phase of the analysis was to build up the holistic model of the early mathematical skills development (research question 2), in which the found interconnections of the skills categories are represented. Throughout the whole research process, standard ethical standards were followed. These included systematic data collection and cautious analysis (e.g. Byrne, 2016; Cooper, 2009; Christians, 2011; Reed & Baxter, 2009; Wilson, 2009).

Results

Three early mathematical skills categories were found in the research literature on mathematics in early childhood education, namely (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. The elements and sub-skills of these categories and the existing gaps in the research literature will be discussed first. Thereafter, the focus will turn to the second research question: The framework for the holistic model of the development of early mathematical skills, which combines early mathematical skills comprehensively.

Numerical skills

The articles related to numerical skills revealed that numerical skills, in general, refer to the number and quantitative thinking and are founded on an intrinsic number sense. As categorised by Clements and Sarama (2007, 2009), the number and quantitative thinking also includes verbal and object counting, early arithmetic strategies, comparing and ordering skills and learning to compose and decompose mathematical symbols. One recent article by Aunio and Räsänen (2016) proposed a model for teaching core numerical skills. This model is based on a series of longitudinal studies and includes similar elements to earlier categorisations by Clements and Sarama (2007, 2009). The model is built around four elements, which include several sub-skills. These are number sense (symbolic and non-symbolic), counting skills (knowledge of quantities and number symbols, number word sequence and enumeration), basic skills in arithmetic (addition, subtraction and arithmetic combinations) and 'understanding mathematical relations' (mathematical-logical principles, arithmetic principles, mathematical symbols, place-value and base-ten system). Next, the elements of numerical skills will be presented.

Number sense

The literature showed that *number sense* requires understanding numbers and quantities in symbolic and non-symbolic number sense (Aunio & Räsänen, 2016; Jordan, Glutting & Ramineni, 2008; Toll, Kroesbergen & Van Luit, 2016) and infants' progress in recognising small quantities (Clements & Sarama, 2007; Kaufmann, 2008; Sarama & Clements, 2009). Although number sense is an innate skill, the ability to differentiate quantities develops gradually in young children and varies from person to person (Hannula-Sormunen, Lehtinen & Räsänen, 2015). Therefore, several articles suggested number sense as the basis for later mathematics skills development (Aunio & Niemivirta, 2010; Kolkman, Kroesbergen & Leseman, 2013). The reviewed articles showed that a strong number sense has a positive influence on children's subsequent mathematical skills, whereas weak number sense explains mathematical learning difficulties (Jordan et al., 2008; Jordan, Kaplan, Nabors & Locuniak, 2006; Merkley & Ansari, 2016; Price & Ansari, 2013; Toll et al., 2016). In addition, recent longitudinal studies indicated that children's spontaneous focusing on numerosity in early childhood predicts their mathematical performance in school (Bojorque, Torbeyns, Hannula-Sormunen, Van Nijlen & Verschaffel, 2017; Hannula, Lepola & Lehtinen, 2010; Hannula-Sormunen et al., 2015; Nanu, McMullen, Munck, Pipari Study Group & Hannula-Sormunen, 2018). These findings support Aunio and Räsänen's (2016) model showing that number sense is a vital part of numerical skills and form a base for mathematical learning.

Counting skills, basic skills in arithmetic and understanding mathematical relations

In the literature, *counting skills* were held to include skills and knowledge of quantities, number words and number symbols, number word sequence and enumeration (e.g. Aunio & Räsänen, 2016; Purpura & Lonigan, 2015). Besides, the literature showed that children need mathematical reasoning skills for fluent counting. As conceptualised by Aunio and Räsänen (2016), 'understanding mathematical relations' include early mathematical-logical principles (comparison, classification, seriation and one-to-one correspondence) as well as place-value logic, including ordinality and cardinality.

The literature review revealed that children learn number word sequence skills, meaning naming numbers and counting those forwards or backwards from a given number during early childhood (Aunio & Räsänen, 2016). Furthermore, understanding ordinality of numbers is part of numerical symbol knowledge and helps children learn the place dependency of a number (Merkley & Ansari, 2016; Slusser & Sarnecka, 2011), requiring awareness that numbers themselves have magnitudes (Jordan et al., 2006). Additionally, Aunio and Räsänen (2016) showed that by comparing and seriating numbers, children become aware of the place-value system, which is a requirement for fluent number word sequence skills. Hence, mastering mental number word sequence skills has been recommended as a foundation for successful arithmetic skills because when adding amounts, children need to be able to count the number word sequence forwards or backwards for subtraction (Aunio & Räsänen, 2016; Jordan, Kaplan et al., 2009; Jordan, Mulhern et al., 2009).

Articles related to counting skills state that in counting numerosities, children use their number word sequence skills to enumerate (Aunio & Räsänen, 2016; Jordan et al., 2006). Besides, successful enumeration requires understanding the logic of one-to-one correspondence, cardinality and ordinality (Sarama & Clements, 2009; Slusser & Sarnecka, 2011). Gaining insights into the interrelationships between number word, number symbol and quantity has been shown to help the child comprehend the principle of ordinality (Sarama & Clements, 2009; Merkley & Ansari, 2016). In learning cardinality, instead, the literature points out that children need to understand that when one starts listing the number word sequence forwards from the first object, the last counted object and its referring number reveal the total amount of the counted objects (Aunio & Räsänen, 2016; Muldoon, Lewis & Francis, 2007; Slusser & Sarnecka, 2011). Thus, based on the reviewed literature, it can be argued that children need the skills above to gain fluent counting skills. Additionally, they need to understand that all kinds of objects can be counted; they also need to know that every countable object is counted only once and that each number word is connected towards one particular object (Clements & Sarama, 2007; Sarama & Clements, 2009). Thus, the literature indicates that to count successfully, children must understand that objects can be counted in any order as long as other counting principles are followed, such as a number word sequence listed in the right order (Aunio & Räsänen, 2016; Muldoon et al., 2007).

Learning *basic skills in arithmetic* requires children to learn that numbers form a coherent system whereby they can be composed, decomposed and commuted (Aunio & Räsänen, 2016; Clements & Sarama, 2007). They, therefore, need to develop an understanding of mathematical relationships (e.g. part-whole) and arithmetic principles to solve arithmetic tasks successfully (Aunio & Räsänen, 2016; Cross et al., 2009). Thus, literature asserted that children need to know that entities or parts need to be counted into sums or wholes and that larger sets are made up of smaller sets. Additionally, children must understand that by adding entities in a different order, they will reach the same answer (commutativity) and that sets decomposed and recombined in different orders will still yield the same answer (associativity) (Aunio & Räsänen, 2016; Cross et al., 2009). Several studies stressed that developing numerical skills widely during early childhood education is crucial to understanding arithmetic principles as they together form the foundation for subsequent mathematical development (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Merkley & Ansari, 2016; Purpura & Lonigan, 2013).

As conceptualised above, numerical learning is founded on number sense and, alongside counting skills, forms a basis for developing basic skills in arithmetic (Figure 2). Besides, 'understanding mathematical relations' (mathematical-logical principles, arithmetic principles, mathematical symbols, place-value and the base-ten system) is essential to counting and arithmetic processes (Aunio & Räsänen, 2016). For the congruence of conceptualisation of this study, these skills will later be referred to as *mathematical thinking and reasoning skills.* Figure 2 shows the skills, sub-skills and the interconnections of the numerical skills.

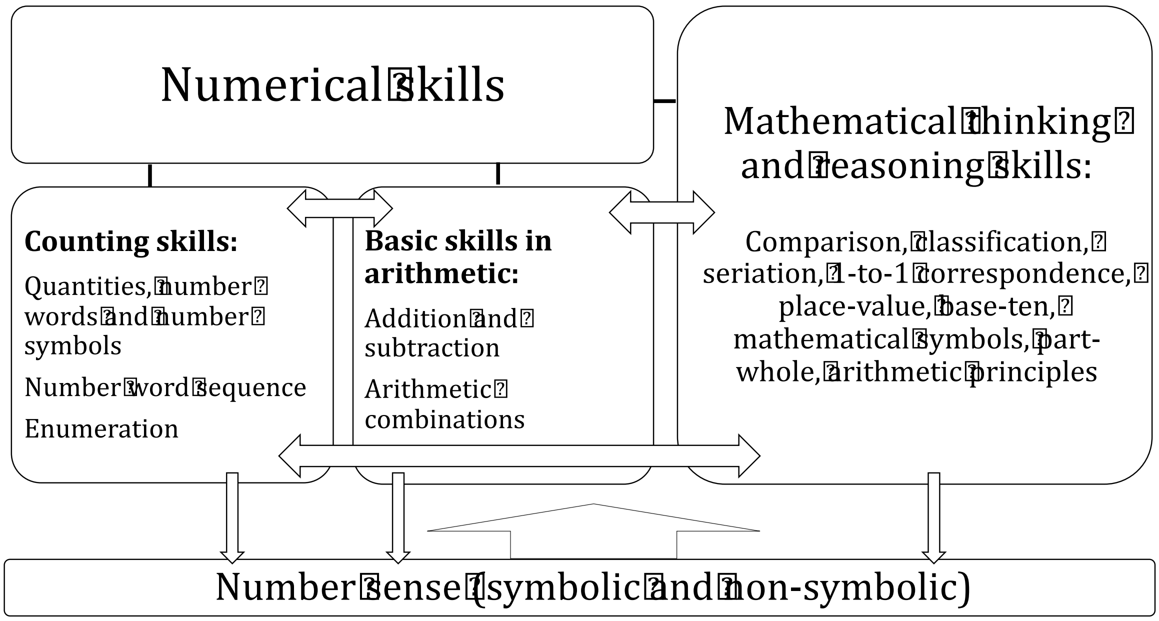

FIGURE 2 Numerical skills (adapted from the original model of Aunio and Räsänen, 2016).

Aunio and Räsänen's (2016) model for teaching core numerical skills has been an important part of categorising early mathematical skills. Such conceptualisation can aid in assessing children's numerical learning reliably and differentiating instructions as required (Purpura & Lonigan, 2015). A common categorisation may also boost global endeavours aimed at increasing teaching competencies in early mathematics education. The need for categorisation can be advocated by contemporary studies, which show that teaching systems and educational backgrounds of early childhood education teachers in different countries can heavily influence children's early numerical learning, although the learning objectives set by the curriculum have been similar (e.g. Aunio et al., 2014). The importance of learning numerical skills during early childhood has also been noted in the literature as the majority of the studies (62 articles) reviewed in this study dealt with this skill area, centred on counting skills.

Spatial thinking skills

The literature review illustrated that geometry and spatial thinking skills base on an innate spatial sense (e.g. Clements & Sarama, 2007) and generally speaking include spatial thinking, geometry and geometric measurement (Battista, 2007; Clements & Sarama, 2007; Jones & Tzekaki, 2016; Sarama & Clements, 2009). Most articles indicated that learning these skills requires mathematical reasoning skills, such as comparison, classification, seriation and part-whole understanding (Battista, 2007; Clements &

Sarama, 2007; Jones & Tzekaki, 2016; Sarama & Clements, 2009). However, inconsistencies in categorising geometry and spatial thinking skills were detected in the data. While some categorisations refer to the skill area as 'geometry and measurement' (see Cross et al., 2009), others that follow the two-part mathematics education division (numbers and geometry) call the area 'geometry' (Tsamir et al., 2011). Some articles on measurement discussed geometric measurement, although two of them also discussed time measurement (Meaney, 2011; Russell, Alexis & Clayton, 2010). Such findings indicate that a new classification of this broad skill area is necessary. Because of the discrepancies in categorisations, this study conceptualises these spatial thinking skills, covering the development of the skills aimed at comprehending spatial environment. Spatial thinking skills are founded on spatial sense and include spatial reasoning, geometrical awareness and sense of time and require *mathematical thinking and reasoning skills*. Learning measurement is categorised as a sub-skill that enhances understanding in the main skills area. The next section presents the conceptual arguments of this new classification.

Spatial sense

The reviewed articles showed that babies and toddlers have the requisite abilities to locate both themselves and objects in space by using either landmarks or geometric cues (Clements & Sarama, 2007; Cross et al., 2009). Spatial sense evolves, and early spatial thinking skills develop through motor learning, in which perceptual-motor skills, especially the development of depth perception, play an essential role (Donnelly, Mueller & Gallahue, 2017). A few articles hold the view that perceptual-motor skills help in understanding spatial relationships (location and direction, depth perception and motion perception) and becoming aware of one's own body (body actions and positions) (Clements & Sarama, 2007; Donnelly et al., 2017). Additionally, one recent article indicated that some connections to the spatial thinking development are similar to cognitive processes development of motor areas (Asunta, Viholainen, Ahonen, Rintala & Cantell, 2016). The literature review, in general, revealed that *spatial sense* forms an essential basis for spatial and geometric learning (Cross et al., 2009; Howse & Howse, 2014).

Spatial reasoning, geometrical awareness and sense of time

Spatial reasoning was conceptualised in the literature as the ability to visualise, explore and reflect on spatial objects and images as well as their relationships and transformations (Battista, 2007). It connects the web of skills, such as the ability to structure, think visually and form mental images (Mulligan, Woolcott, Mitchelmore & Davis, 2018). According to Clements and Sarama (2007), spatial reasoning includes location, navigation and spatial orientation. The literature shows that the development of spatial reasoning begins at an early age when toddlers learn to use visual information to function, for example, for reaching places, people and objects (Cross et al., 2009). It also suggests that this development continues throughout early childhood. Several studies indicated that children gradually learn to build mental images of the surrounding environment and create simple maps to navigate their way to their target (Clements & Sarama, 2007; Krüger, Kaiser, Mahler, Bartels & Krist, 2014; Tsubota & Chen, 2012). Many studies also indicated that young children can use maps and spatial coding successfully (Chen, 2007; Fantozzi, Cottino & Gennarelli, 2013; Hribar, Haun & Call, 2012; Vasilyeva & Bowers, 2006, 2010). Additionally, the literature showed that through spatial reasoning, children come to understand symmetry, asymmetry, mirror images, rotation and patterns (Hawes, Moss, Caswell, Naqvi & MacKinnon, 2017; Krüger et al., 2014; Ramful, Ho & Lorie, 2015; Sarama & Celements, 2009). In these reasoning processes, children also use mathematical thinking and reasoning skills, such as comparison and part-whole relationships.

Contemporary articles addressed that enhancing children’s spatial reasoning is important, mainly because acquired skills increase conceptual understanding, visuospatial reasoning and use of spatial language (Hawes et al., 2017; Jones & Tzekaki, 2016). The literature also showed that the development of spatial reasoning skills is influenced by language: Children first learn to express directions, then positions and finally abstract attributes (Lyytinen, 2014; Owens, 2014). However, part of the literature also indicated that the relationship between spatial thinking and mathematics is not straightforward (Clements & Sarama, 2009; Kyttälä, Aunio, Lepola & Hautamäki, 2014). Several articles, nevertheless, stressed that spatial reasoning is proved to be a fundamental part of early mathematical learning because children´s spatial abilities (e.g. visuospatial skills) predict their mathematics achievements (Carlson, Rowe & Curby, 2013; Simms, Clayton, Cragg, Gilmore & Johnson, 2016). Additionally, some studies indicated that spatial reasoning is critical to science, technology, engineering, arts and mathematics (STEAM) education (e.g. Mulligan et al., 2018). Importantly, studies emphasised that spatial reasoning is spatial and not visual, although visual input is essential for the full development of spatial reasoning (Clements & Sarama, 2007; Hribar et al., 2012; Sarama & Clements, 2009). These conceptual arguments require acknowledging spatial reasoning as a component of spatial thinking skills rather than underestimating the value of this complex content area by conceptualising it as being part of geometry.

Regardless of the need for a separate classification, the data also revealed that spatial reasoning is needed in *geometrical awareness*, as the literature stresses the relationship between spatial sense and geometric reasoning (Howse & Howse, 2014). Geometrical awareness connects concepts and ways of reasoning and presenting spatial environments through physical attributes, such as shape and space (Battista, 2007; Hawes et al., 2017; Howse & Howse, 2014). During early childhood, children first learn to name the common shapes verbally, then gradually gain skills to use shape-related attributes and finally express geometrical features by using complex terminology (Cross et al., 2009; Skoumpourdi, 2016). Additionally, the literature reviewed in this study asserted that the same progress applies to the depiction of shapes, starting from drawing a circle at the age of three to gaining skills to draw a star at the age of six, and the progress is influenced by the maturation of fine motor skills (Payne & Isaacs, 2017; Sarama & Clements, 2009; Villarroel & Sanz, 2017).

This literature review indicated that the aforementioned skills tend to become somewhat sophisticated by the age of eight (Kaur, 2015); this is the time when children know several geometric concepts and can depict a variety of shapes and figures in both two and three dimensions (Dagli & Halat, 2016; Skoumpourdi, 2016; Villarroel & Sanz, 2017). The studies indicated that understanding the two- and three-dimensionality of shapes and figures requires that children can compare and classify features but also understand part-whole relationships (Clements & Sarama, 2007; Hallowell, Okamoto, Romo & La Joy, 2015). Some studies suggested that to learn the principles of rotation, pattern and seriation related to two- and three-dimensionality, rigid ways of presenting shapes should be avoided (Clements & Sarama, 2007; Sinclair & Moss, 2012). Additionally, recent studies have suggested that the diversity of two- and three-dimensional learning fosters spatial-knowledge development and spatial reasoning (Hawes et al., 2017; Jones & Tzekaki, 2016). These conceptual arguments illustrate that versatile skills related to geometrical awareness are a fundamental part of spatial thinking skills as they promote understanding of certain aspects of the spatial environment.

The data also revealed that learning all aspects of spatial environment requires *a sense of time*. In environmental phenomena, the space-time relationship is conceptualised as part of the spatial environment (see Golledge, Marsh & Battersby, 2008), hence part of spatial thinking skills. However, the reviewed articles showed that sense of time was widely ignored in the literature on early mathematical skills since it was addressed by three studies only. These articles emphasised that children use spatial thinking skills while learning time-related reasoning (Meaney, 2011; Russell et al., 2010) and structural reasoning while familiarising themselves with clock faces (Mulligan & Mitchelmore, 2013). The literature review revealed that global categorisations mainly focus on geometry, measurement and spatial thinking skills. This attention may explain why learning about the concept of time was ignored in early mathematical skills. However, the Finnish curricula (Finnish National Agency for Education, 2017; Finnish National Board of Education, 2016) suggest that time should be explored through seasons and other time-related concepts and children should be familiarised with chronological order. Russell and colleagues´ (2010) study supports this view by showing that young children are capable of future-oriented thinking. Additionally, only a few articles stressed that language learning can influence understanding the concept of time, as children first learn to speak about the past and the future then gradually learn to express time in more specific ways (Lyytinen, 2014). The findings of the review illustrate that the sense of time has been neglected in the studies while it should have been acknowledged as part of spatial thinking skills.

The literature indicated that spatial environment reasoning requires the aforementioned skills (spatial reasoning, geometrical awareness and sense of time) as well as understanding basics of measurement (Battista, 2007; Solomon, Vasilyeva, Huttenlocher & Levine, 2015). In several mathematical theories, measurement relates to geometry and includes length, area, volume and mass (Battista, 2007; Clements & Sarama, 2007; Jones & Tzekaki, 2016). The articles argued that measurement enables children to learn about the size of the objects but also to determine locations in space (Clements & Sarama, 2007; Fantozzi et al., 2013). Additionally, a few studies highlighted that spatial reasoning required understanding conservation, although it is somewhat demanding at an early age and very often leads to wrong judgements between the measured items (Cross et al., 2009). It was also shown that children learn space-time reasoning through measurement (Meaney, 2011). The findings above support earlier assertions that understanding versatile spatial relationships supports human endeavours, such as architecture, art, engineering and science (Cross et al., 2009; Mulligan et al., 2018). Thus, the findings of this study indicate that measurement needs to be classified as a sub-skill, which is a requirement for learning several spatial relationships.

The conceptual arguments about *the elements of spatial thinking skills*, showing the discrepancies and gaps in earlier categorisations, indicate the necessity of a new classification among these skills. This need is supported by the knowledge that teaching geometry and spatial thinking skills are usually minimised in early childhood education (Clements & Sarama, 2011; Sarama & Clements, 2009; Tsamir et al., 2011). Therefore, this study suggests that this skill cluster be called ‘spatial thinking skills’, according to the model in Figure 3.

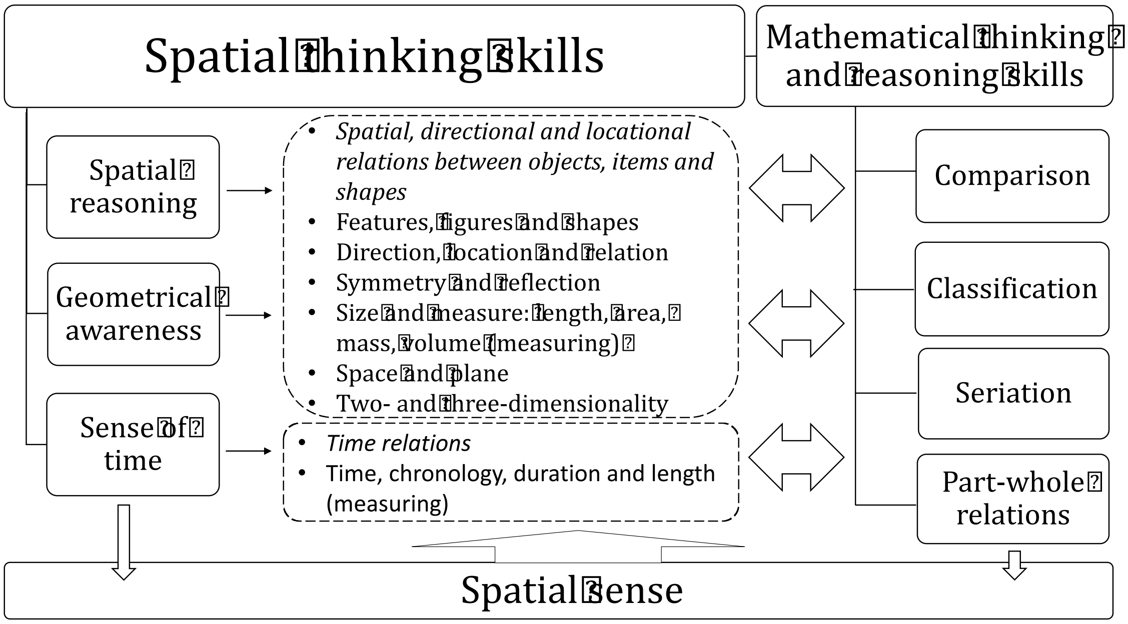

FIGURE 3 Spatial thinking skills.

The model illustrates that spatial thinking skills are based on intrinsic spatial sense and includes spatial reasoning, geometrical awareness and sense of time. These skills are connected and include several sub-skills (e.g. measurement), which develop during early childhood. Additionally, learning spatial thinking skills requires *mathematical thinking and reasoning skills*. Together these skills support a child´s learning about versatile aspects of the spatial environment. Therefore, it is proposed that this suggested model be used as a guiding model for teaching spatial thinking skills in early childhood education.

Mathematical thinking and reasoning skills

A significant part of early mathematical skills involves developing the reasoning process used in learning about the phenomena surrounding us and the world. Mathematical reasoning processes generally require an understanding of the associations and interrelationships between objects, items, number symbols, quantities and their features. The articles reviewed in this study revealed that alongside analogical and statistical reasoning, children learn quantitative, numerical and spatial reasoning (Mulligan, 2015; Mulligan & Mitchelmore, 2013; Mulligan, Mitchelmore, English & Crevensten, 2013; Mulligan et al., 2018). In some articles, these skills were conceptualised as early algebraic thinking and reasoning skills (Carraher & Schliemann, 2007), covering the beginning of primary education (Warren, Trigueros & Ursini, 2016), and as algebraic thinking later at school. Clements and Sarama (2007, 2009) categorise these skills using number and quantity thinking, geometry and spatial thinking, patterns and structures (algebraic thinking), data analysis and mathematical processes. Mulligan and Mitchelmore (2013), on the other hand, define these processes as spatial and quantitative reasoning, deduction and induction, analogical reasoning, and statistical reasoning. Mathematical reasoning processes require several parallel logical thinking and reasoning skills and given that the literature review revealed an absence of a common classification in early mathematics education, it is suggested here that the cluster of these skills be conceptualised as *mathematical thinking and reasoning skills*. The conceptual arguments will be presented next.

Cluster of logical thinking, reasoning, analysing and problem-solving skills

According to Carraher and Schliemann (2007), children come to understand that certain patterns, functions and their relationships need to be considered in mathematical thinking and reasoning processes. Several articles shared this conceptualisation, addressing that children learn to search for certain structures and regularities to order, predict and create cohesion (McGarvey, 2013; Sarama & Clements, 2009; Schultz, Gopnik & Glymour, 2007). Children practise these skills, for instance, by comparing, sorting, classifying, ordering and seriating objects according to specific features (Carraher & Schliemann, 2007; Warren et al., 2016). Hence, children learn to discover associations and repeatable sequences and reach conclusions about quantities and features by categorising objects. Clements and Sarama (2007) suggested that young children can usually sort things by using a single attribute, although they can reclassify them by using different attributes. Additionally, children can collect and find patterns and functions in graphs or diagrams, which assist in learning the principles of data analysis (Sarama & Clements, 2009). A few studies supported this view by showing that young children are capable of data modelling (English, 2013; Mulligan, 2015).

The reviewed articles in this study revealed that mathematical thinking and reasoning processes guide children on how to use problem-solving and reasoning strategies that are not innate but emerge early in their lives (Clements & Sarama, 2007). Children are also capable of learning structuring process through word problems and numerical and geometrical processes (Mulligan & Mitchelmore, 2013); they are also capable of learning multiple problem-solving methods (Tsamir, Tirosh, Tabach & Levenson, 2010). Additionally, recent studies indicated that children learn somewhat sophisticated thinking during early childhood, such as functional relationships reasoning (Blanton, Brizuela, Gardiner, Sawrey & Newman-Owens, 2015; Lee, Collins & Melton, 2016). These studies also showed that children are capable of processing reasoning and using the required concepts (Lee et al., 2016). Current studies also advanced prior understandings by showing that the development of mathematical thinking and reasoning skills positively correlates with general mathematical achievement (see Lee et al., 2016; Warren et al., 2016). However, the studies also emphasised that children who do not achieve basic mathematical thinking and reasoning competencies during early childhood will often face problems in mathematics later at school (Lee et al., 2016; Warren et al., 2016).

Based on the conceptualisation presented in this study, a cluster of mathematical thinking and reasoning skills (Figure 4) is an essential component of early mathematical skills. These skills enable children to understand and consider patterns, functions and their relationships in mathematical processes while reasoning, solving problems and learning analytical thinking and mathematical-logical thinking.

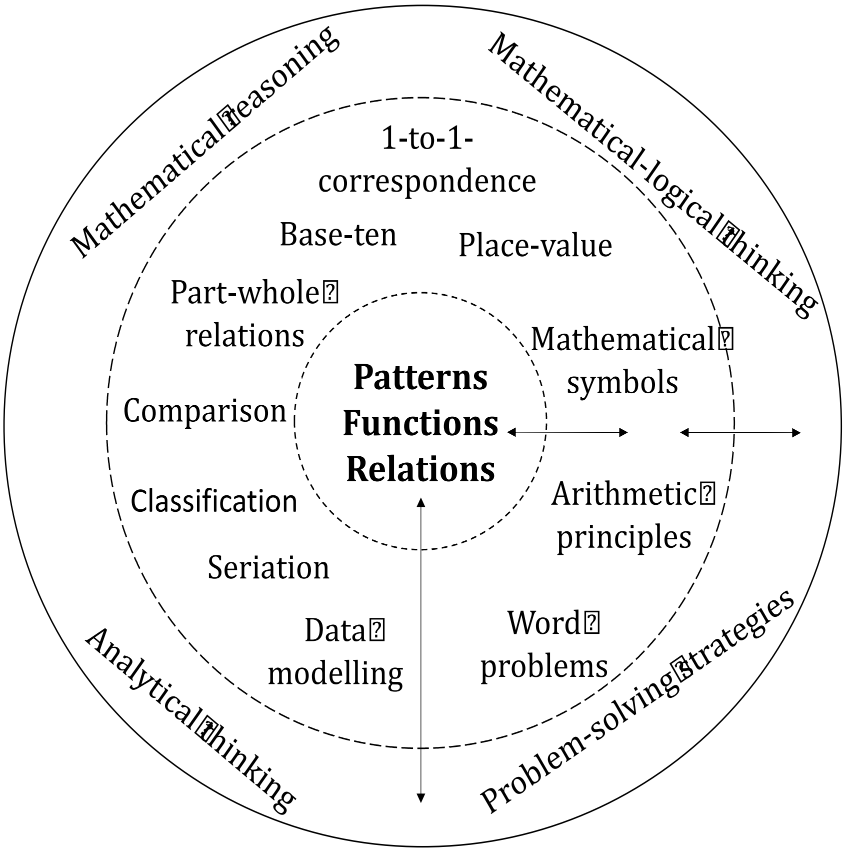

FIGURE 4 Mathematical thinking and reasoning skills.

Figure 4 illustrates how children use versatile mathematical thinking and reasoning skills while learning different aspects of mathematics. Skills are, therefore, presented in a cluster displaying their interconnections. Mathematical thinking and reasoning skills should not be acknowledged as part of numerical skills (Figure 2) and spatial thinking skills (Figure 3) only. It is also crucial to support the development of these skills as a separate skill set.

Holistic model of early mathematical skills

The literature reviewed in this study was utilised in forming early mathematical skills categories, namely (1) numerical skills, (2) spatial thinking skills and (3) mathematical thinking and reasoning skills. The review revealed that these categories include several skills and sub-skills, which are essential to gaining strong mathematical skills necessary for life on the one hand (Clements & Sarama, 2007; Cross et al., 2009; van Oers, 2013) and school mathematics on the other (Aunio & Niemivirta, 2010; Kolkman et al., 2013; Lee et al., 2016; Nanu et al., 2018; Romano et al., 2010; Toll et al., 2016; Warren et al., 2016). Several articles stressed how each skills category influences gaining strong mathematical skills during early childhood (e.g. Hannula & Lehtinen, 2005; Hawes et al., 2017; Howse & Howse, 2014; Kolkman et al., 2013; Mulligan & Mitchelmore, 2013; Simms et al., 2016). Therefore, for children to gain strong mathematical skills, it is vital to ensure the simultaneous development of all early mathematical skills.

Interconnections of skills categories

The reviewed literature revealed that not only do early mathematical skills develop simultaneously but learning new skills is also connected between the skills categories. Some studies stressed that learning about spatial environment requires numerical learning and vice versa. While learning measurement, children connect length and number (Clements & Sarama, 2007; Fantozzi et al., 2013; Solomon et al., 2015) and mass and number (Cheeseman, McDonough & Ferguson, 2014). Several studies showed that children use spatial reasoning while learning counting strategies and basics of arithmetic (Kyttälä et al., 2014; Laski et al., 2013; Laski & Siegler, 2014; Toll et al., 2016). The studies also stressed that children use numerical skills while learning spatial reasoning (e.g. rotation) (Hawes et al., 2017) or spatial contents (e.g. understanding area or time) (Chigeza & Sorin, 2016; Fantozzi et al., 2013; Mulligan & Mitchelmore, 2013; Mulligan et al., 2013). The literature showed that similar to number sense, strong spatial sense positively correlates with a child´s mathematical learning, illustrating that visuospatial system is needed to strengthen number sense (Bobis, 2008; Toll et al., 2016), represent magnitudes (Sarama & Celements, 2009) and understand arithmetic principles (e.g. composing and decomposing) (Laski et al., 2013).

A few earlier studies indicated that in terms of evolving spatial reasoning, neurological maturation influences learning relative magnitude between two numerals, leading to misjudgements between the counted objects at an early age (Ansari, Garcia, Lucas, Hamon & Dhital, 2005). Additionally, the literature showed that to understand the connections between numerical and spatial topics fully, children need conceptualisation skills, which are very often related to learning and development of language (Kyttälä et al., 2014;

Lyytinen, 2014; Owens, 2014). The findings above suggest that children use numerical skills for learning spatial thinking skills and vice versa. Due to the strong relationship between numerical and spatial learning, several studies suggested that spatial-numerical congruence should be better addressed in numerical learning (Laski & Siegler, 2014; Patro & Haman, 2012; Zhang et al., 2017). A finding of Toll's research group (2016) supported this view, showing that concerning number sense, visuospatial working memory strongly influences subsequent mathematical achievement.

In addition to discovering the interconnections between numerical skills and spatial thinking skills, mathematical thinking and reasoning skills were found to be connected to both skills areas. In their categorisation of numerical skills, Aunio and Räsänen (2016) already acknowledged this connection, although they conceptualised it in their model as 'understanding mathematical relations'. According to the literature, children learn the general aspects of variables and entities, such as understanding arithmetic principles, one-to-one correspondence, mathematical symbols (e.g. +, -, <, >, =) and part-whole relationships through mathematical thinking, reasoning skills (i.e. composing and decomposing), place-value and base-ten system (Carraher & Schliemann, 2007; Sarama & Clements, 2009; Slusser & Sarnecka, 2011; Warren et al., 2016). Articles also asserted that children learn the functional relationships between the quantities (Blanton et al., 2015; Brizuela, Blanton, Sawrey, Newman-Owens & Gardiner, 2015; Purpura & Lonigan, 2013; Sarama & Clements, 2009). Additionally, some articles indicated that learning mathematical thinking and reasoning skills (e.g. data modelling) fosters numerical learning (Mulligan, 2015). The review illustrated that children use several mathematical thinking and reasoning processes while learning numerical skills (see Figure 2) but learn mathematical thinking and reasoning skills through numerical learning (see Figure 4).

The literature review also revealed a connection between spatial thinking skills and mathematical thinking and reasoning skills (see Figure 3). The literature addressed that children use mathematical thinking and reasoning skills while learning spatial thinking skills. Several studies showed that children compare, classify, seriate and apply their understanding of part-whole relationships while learning to understand various aspects of spatial environment (Clements & Sarama, 2007; Hallowell et al., 2015; Hawes et al., 2017; Jones & Tzekaki, 2016; Ramful et al., 2015; Zacharos, Antonopoulos & Ravanis, 2011). Additionally, the literature showed that children apply logical thinking skills while reasoning spatial environment (Chen, 2007; Tsubota & Chen, 2012; Vasilyeva & Bowers, 2006, 2010). Again, the same connection would be applied the other way round, as the articles indicated that children use spatial reasoning while learning to understand patterns, functions and their relationships (Mulligan & Mitchelmore, 2013; Mulligan et al., 2013) or arithmetic principles (e.g. decomposition) (Laski et al., 2013). Importantly, one article emphasised that it is crucial to separate visual learning from spatial reasoning, mainly because poorly visualised mathematics software can hardly promote mathematical thinking and reasoning skills; this would mean that children are then not using spatial reasoning skills but instead limited to visual thinking (Ginsburg, Jamalian & Creighan, 2013).

The literature reviewed in this study indicated the interconnections between the three skills categories and asserted that the connection between the categories was bidirectional. Further, it indicated that because of bidirectional connections, children draw on the skills of all categories while learning mathematics. For example, using visuospatial skills while learning arithmetic principles (Kyttälä et al., 2014; Laski et al., 2013) connects numerical skills, spatial thinking skills and mathematical thinking and reasoning skills. Thus, these findings illustrate that early mathematical skills learning and development are interconnected across the three categories.

Simultaneous development of interconnected early mathematical skills

Versatile interconnected mathematical skills begin to develop simultaneously at an early age in a child's life. The literature addresses that children use several different skills when learning certain aspects of mathematics (English, 2013; Hawes et al., 2017; Laski et al., 2013; Mulligan et al., 2018; Sarama & Clements, 2009; Warren et al., 2016). In this study, the skills areas were first categorised separately to build a framework for the holistic learning and development of early mathematical skills. Subsequently, it was necessary to show the interconnections of these skills categories to be able to illustrate the complex learning and development of early mathematical skills fully. The holistic theoretical model of early mathematical skills (Figure 5) was constructed by analysing the literature, noting the early mathematical skills that children learn and develop in early childhood, including the connections and foundations of these skills. The elements of early mathematical skills, namely (1) numerical skills (including number sense), (2) spatial thinking skills (including spatial sense) and (3) mathematical thinking and reasoning skills and their interrelationships are presented in Figure 5.

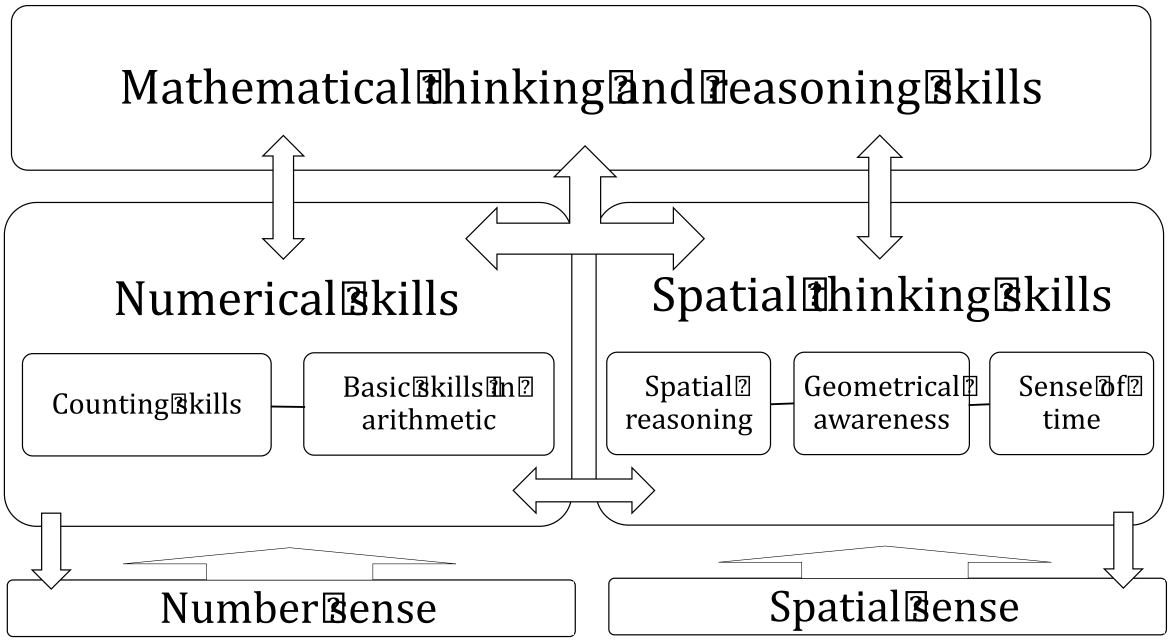

FIGURE 5 Holistic model of the development of early mathematical skills.

As Figure 5 illustrates, different elements of early mathematical skills are completely interconnected, and there exist bi- and multi-directional relationships among these elements. The reviewed articles in this study showed that number sense (symbolic and non-symbolic) and spatial sense develop from an early age. They also showed that they form the foundation for learning numerical skills (Aunio & Niemivirta, 2010; Bobis, 2008; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Kolkman et al., 2013) and spatial thinking skills (Cross et al., 2009; Howse & Howse, 2014). The literature review also revealed that learning numerical skills requires spatial thinking skills and vice versa (Bobis, 2008; Laski & Siegler, 2014; Sarama & Clements, 2009, Zhang et al., 2017). Additionally, it showed that during early childhood, children learn mathematical thinking and reasoning skills and that they use these to learn numerical skills and spatial thinking skills and vice versa (Carraher & Schliemann, 2007; Clements & Sarama, 2007; Mulligan & Mitchelmore, 2013; Sarama & Clements, 2009). Furthermore, the review showed that the three skills areas are interlinked while children learn mathematics (e.g. Laski et al., 2013).

The articles reviewed in this study showed that understanding the simultaneous development of mathematical skills and the interconnections of the skills categories is essential for promoting the holistic development of mathematical skills in early childhood. Reaching a balanced approach to teaching early mathematical skills to children requires that teachers need to be aware of what and how they teach. Several articles showed that when children become able to simultaneously develop their mathematical skills in versatile ways (e.g. learning numeracy with visuospatial aid), their mathematical learning improves holistically (Cheeseman et al., 2014; Cheun & McBridge, 2017; Colliver, 2018; Laski & Siegler, 2014; Lee et al., 2016; MacDonald & Lowric, 2011; McNeil, Fyfe & Dunwiddie, 2015; Zur & Gelman, 2004). The contemporary studies also indicated that children learn somewhat sophisticated mathematical thinking during early childhood when they are taught appropriate mathematical curricular contents (Blanton et al., 2015; Lee et al., 2016). The aims and results of this present study helped in forming a theoretical framework for a holistic model of the development of early mathematical skills in early childhood. The model is suggested to be considered in curriculum reforms and used as a guide on teaching early mathematical skills, as it is grounded on the understanding that early mathematical skills are interconnected and develop simultaneously.

Discussion

This literature review draws a holistic theoretical model of the learning and development of early mathematical skills. Besides, the categorisation of the core mathematical skills showed that children learn these skills simultaneously, and these apects should, therefore, be acknowledged in early childhood education. This understanding illustrates that there exist comprehensive compilations of early mathematical skills development (Clements & Sarama, 2007; Sarama & Clements, 2009) and many independent studies that focus on certain mathematical skills areas (i.e. Cheeseman et al., 2014; Hannula & Lehtinen, 2005; Krüger et al., 2014; Mulligan, 2015). This literature-based holistic model study into early mathematical skills aims to promote discussions of versatile mathematical skills which children learn and develop during early childhood.

The literature review shows that from birth to the age of eight, children learn and develop several mathematical skills, which form the bedrock of the skills that they will learn later (e.g. Aunio & Räsänen, 2016; Hannula-Sormunen et al., 2015; Sarama & Clements, 2009; Simms et al., 2016; Toll et al., 2016). By the age of eight, for example, children have fully developed counting strategies, learned the basics of arithmetic (McNeil et al., 2015) and acquired spatial and geometrical principles (Clements & Sarama, 2007; Kaur, 2015; Sarama & Clements, 2009). They know how to use visuospatial skills for mapping (Cross et al., 2009; Vasleyeva & Bowers, 2006, 2010). Furthermore, they can describe the qualities and features that can help them readily distinguish directions and locations (Clements & Sarama, 2007; Owens, 2014). They also understand certain measures, such as time, length, mass, volume and area (Battista, 2007; Jones & Tzekaki, 2016; Meaney, 2011). Besides, they can draw on mathematical thinking and reasoning skills to solve mathematical problems (Carraher & Schlieman, 2007; Mulligan & Mitchelmore, 2013).

The literature review also shows that versatile mathematical skills develop simultaneously at an early age (e.g. Aunio & Räsänen, 2016; Howse & Howse, 2014; Jones & Tzekaki, 2016; Merkley & Ansari, 2016; Mulligan et al. 2018). Moreover, newer studies clearly illustrate strong interconnections between different early mathematical skills (e.g. Chigeza & Sorin, 2016; Hawes et al., 2017; Kyttälä et al., 2014; Laski & Siegler, 2014; Mulligan et al., 2013; Zhang et al., 2017). Early mathematical skills have thus far been categorised and classified only from a numerical perspective (Aunio & Räsänen, 2016). Hence, in addition to numerical skills, it is essential to categorise all early mathematical skills and build a holistic theoretical framework, illustrating the interconnections of these developing skills areas. The practical aim of the model, therefore, is to raise teachers' awareness that mathematical skills in early childhood education are interconnected and develop simultaneously and encourage them to seek a holistic approach to their teaching. Building an understanding of the interconnections of early mathematical skills can also illustrate how significant it is that children's mathematical skills develop systematically, simultaneously and holistically. Acting accordingly will assure effective planning for balanced mathematical programmes for early childhood education settings.

The literature review illustrates that the number of studies on the development of mathematical thinking and reasoning skills is limited and that more studies are needed to understand this skills category better. Additionally, researchers are advised to extend their investigations into younger age groups, as most of the studies are focused on children aged seven or eight years. More extensive research evidence would illustrate the importance of these skills and perhaps advance our knowledge of individual variations within them. Given the current limited number of studies, more explorations into the learning and development of the sense of time are also necessary to deepen our understanding of this skills area. Additionally, studies exploring the connections between the skills categories would broaden our understanding of the holistic learning and development of early mathematical skills.

One perceived limitation of this study is that the literature search was confined to specific search terms and the ERIC database. Using other databases (e.g. PsycINFO) and more diverse search terms could have yielded a more comprehensive number of relevant articles for review and analysis.

Conclusion

To support children´s mathematical skills, the findings in this study illustrate the need for more research into the practical implications of early mathematics teaching practices, teacher education and professional development (see Piasta, Logan, Pelatti, Capps &

Petrill, 2015). Additionally, the findings suggest that a comprehensive analysis investigating how early mathematical skills are conceptualised and categorised in curricula could provide valuable insights. Collectively, these efforts can broaden our understanding of teaching mathematical skills in early childhood education.

The arguments presented above were validated in the 2009 Conference of European Research in Mathematics Education, which reiterated the importance of mathematics teaching and research for children aged between three and eight years (Tsamir et al., 2011). This literature review suggests that to recognise the importance of early mathematical learning and mathematics teaching, future researchers should focus on (1) specific skills areas that need to be better understood (sense of time, mathematical thinking and reasoning skills); (2) connections between skills development aimed at broadening current understanding and (3) practical implementations to promote the learning of versatile early mathematical skills.

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