Function spaces and pseudo-differential operators on vector bundles

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Osittaisdifferentiaaliyhtälöt muodostavat tärkeän työkalun matemaattisessa mallinnuksessa. Niitä käytetään malleissa, joissa parametriavaruus on riittävän sileä. Ensimmäisiä osittaisdifferentiaaliyhtälöitä olivat aalto- ja lämpöyhtälöt avaruudessa $\mathbb{R}^3$. Alkuaikoina osittaisdifferentiaaliyhtälöiden ratkaisemiseksi käytettiin muun muassa muuttujien separointia, Fourier-sarjoja ja Greenin funktioita. Näiden menetelmien matemaattinen perusteleminen tapahtui vasta myöhemmin.

Lineaaristen osittaisdifferentiaaliyhtälöiden teoriaan kehitettiin 1900-luvulla distribuutio- ja Sobolev-avaruudet. Näiden avaruuksien avulla monet käsitteet täsmällisesti määriteltyä ja menetelmien toimivuus saatiin todistettua. Toinen 1900-luvulla ilmennyt asia oli monistojen tärkeys matemaattisessa mallinnuksessa ja erityisesti fysiikassa. Suhteellisuusteoria osoitti, että aika-avaruutta kannattaa mallintaa monistona. Tällöin myös muut fysiikan osittaisdifferentiaaliyhtälöt kannattaa esitellä monistoilla. Kolmas 1900-luvun keksintö oli pseudo-differentiaalioperaattorit.

Tämän työn tarkoitus on määritellä modernin analyysin työvälineitä monistoilla ja soveltaa näitä elliptisiin osittaisdifferentiaaliyhtälöihin. Painotamme differentiaaligeometrialle tyypillisiä koordinaatistovapaita määritelmiä ja tavoitteenamme on antaa analyysin käsitteiden määritelmät myös koordinaatistovapaasti. Toisena ideana työssä on koota kattavasti modernin analyysin työkaluja yhteen esitykseen. Tämän takia olemme tehneet kompromisseja ja annamme joidenkin lauseiden kohdalla vain viitteen todistukseen.

Määrittelemme ensin differentiaaligeometrian käsitteitä, joiden pohjalta voimme luoda vektorikimpuille L^p-avaruuksien käsitteen. Tämän jälkeen voimme määritellä distribuutioavaruudet ja esitellä L^p-avaruuksien upotukset distribuutioavaruuteen. Distribuutioteoriaan liittyen rakennamme myös yleisen matemaattisen kehyksen, joka osoittaa mitä matemaattisia rakenteita distribuutioteoriaan tarvitaan.

Analyysille tärkeät Sobolev-avaruudet määrittelemme kahdella tavalla: käyttäen derivoinnin kaltaisia operaattoreita ja käyttäen Fourier-muunnosta. Differentiaaligeometrian käsitteistä tangenttikimppu osoittautuu tärkeäksi, koska sen avulla voimme määritellä kovariantin derivaatan ja yleistää Fourier-analyysin koordinaattiriippumattomasti monistoille. Annamme Fourier-muunnoksen hyödyllisyydestä muutaman esimerkin euklidisen avaruuden osittaisdifferentiaaliyhtälöille.

Työn toisella puoliskolla tarkoituksenamme on esitellä pseudo-differentiaalioperaattorit ensin euklidisessa avaruudessa ja sitten kompakteilla reunattomilla monistoilla. Painopisteemme on pseudo-differentiaalioperaattoreiden perusominaisuuksien ja operaattorikalkyylin esittelyssä. Lopuksi sovelluksena pseudo-differentiaalioperaattoreista osoitamme parametriksien olemassaolon elliptisille operaattoreille.

Contents

1 Introduction

The objective of this Master thesis is to give coordinate-free definitions of certain concepts in modern analysis on vector bundles and to apply these tools to elliptic partial differential equations. Topics that we cover are distribution theory, Sobolev spaces and pseudo-differential operators. We study these topics on vector bundles since they are used extensively in physics. Many physical laws are formulated on vector bundles, for example, Maxwell's equations and equations of continuum mechanics are formulated using vector-valued functions. This work can be seen as a survey or an overview of these topics on vector bundles.

We will also include definitions using coordinate charts as they are used in the literature. Therefore, we need to define objects also in the Euclidean space $\mathbb{R}^n$. In that setting, we can more easily give examples of how to apply the theory to the theory of partial differential equations and we have done so to demonstrate the use of the theory. The theory can be developed solely using charts but we have included coordinate-free definitions as they give geometric constructions of the objects and since coordinate-free definitions are often used in differential geometry.

We start by reviewing vector bundles and L^2-spaces in Section 2 and then we will study distributions in Section 3 where we will give a general framework of distribution theory and then provide examples of distribution spaces. In Section 4, we move on to study Sobolev spaces and discuss them on $\mathbb{R}^n$, on open subsets of $\mathbb{R}^n$ and then introduce Fourier analysis and Sobolev spaces on vector bundles.

In Section 5, we will focus on partial differential operators on vector bundles. Our goal is to introduce a coordinate-free definition of partial differential operators on a vector bundle. After this, in Section 6 we will introduce pseudo-differential operators on $\mathbb{R}^n$ and discuss the basic properties. The discussion of pseudo-differential operators on vector bundles is handled in Section 7.

In the last section, we are going to study the Fredholm theory of elliptic operators on compact manifolds. This requires a review of Fredholm operators. Using pseudo-differential calculus, we will prove that elliptic pseudo-differential operators are Fredholm operators. We end the section by giving an application to elliptic regularity and studying the Poisson problem of an elliptic pseudo-differential operator.

Although we will review differential geometry and tensors, we assume that the reader is familiar with ideas of differential geometry. However, one can read only parts where the Euclidean space is used and skip parts where manifolds are needed. Since our exposition is closer to a survey or an overview, we have taken some results without proof and provided only a reference for the proof. This makes it possible to give a wider overview of the techniques. In Section 9, we have pointed out some further topics on the subject.

2 Vector bundles and L^2-spaces

In this section, we give necessary definitions, notation and results of differential geometry that we use later in the text. We start by reviewing manifolds, tensors and smooth locally trivialisable vector bundles. We have used [1, 2, 3] as our references about differential geometry.

2.1 Manifolds

Definition 2.1 (Topological manifold)**.** *A topological manifold M is a topological space which is second countable, Hausdorff and has the following property: Every point $p \in M$ has a neighbourhood U and a function ϕ_U from U to an open set of $\mathbb{R}^n$ such that the function is a homeomorphism. The pair (U, ϕ_U) is called a coordinate chart and values of $\phi_U(p)$ are called coordinates.*

In topological manifolds, change of coordinates between charts is defined to be a function $\phi_{VU} = \phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$ whenever $U \cap V$ is nonempty. Charts ϕ_U and ϕ_V are said to be C^k-compatible if functions ϕ_{UV} and ϕ_{VU} are C^k-mappings. With this, we can define a C^k-differentiable structure on a manifold.

Definition 2.2. *A collection of charts $\mathcal{A} = \{(U, \phi_U), (V, \phi_V), (W, \phi_W), \dots\}$ is said to be a C^k-differentiable structure on a manifold M if the following conditions hold:*

- *The sets $U, V, W, \dots$ form an open cover of M,*
- *every pair of charts is C^k-compatible whenever charts' domains overlap,*
- *and $\mathcal{A}$ is maximal. Meaning that if a coordinate chart (U, ϕ_U) is compatible with charts on $\mathcal{A}$, then $(U, \phi_U) \in \mathcal{A}$.*

Let M be a topological manifold and $\mathcal{A}$ be a C^k-differentiable structure on M, then we say that $(M, \mathcal{A})$ is a C^k-manifold. A differentiable structure on a manifold is uniquely determined by any open cover of compatible charts [3, p. 4]. In this thesis, We consider only C^∞-manifolds and call them smooth manifolds.

Example 2.3. The simplest manifold is the Euclidean space $\mathbb{R}^n$. It is an n-dimensional manifold with the differentiable structure determined by the chart $(\mathbb{R}^n, \mathrm{id})$.

Example 2.4. A smooth function $f : M \to \mathbb{R}^k$ on a manifold is a function such that $f \circ \phi_U^{-1}$ is smooth for every chart ϕ_U. For every smooth function f, the set $\{(x, f(x)) \in M \times \mathbb{R}^k | x \in M\}$ is a manifold with differentiable structure determined by open sets $\tilde{U} = \{(x, f(x)) | x \in U,\ U \text{ is open}\}$ and charts $\phi_V \circ \mathrm{pr}_x$ where $\mathrm{pr}_x : (x, v) \mapsto x$.

For every manifold M, we can construct an associated tangent bundle TM. Elements of the tangent bundle consist of points $x \in M$ together with equivalence classes of curves γ through the point x. The equivalence relation is given by the following condition: The curves $\gamma_1(t)$ and $\gamma_2(t)$ are equivalent if and only if $\gamma_1(0) = \gamma_2(0) = x$

and $\frac{d}{dt}|_{t=0}f(\gamma_1(t)) = \frac{d}{dt}|_{t=0}f(\gamma_2(t))$ for every smooth function f. An equivalence class through a point x is called a vector on T_xM. Every tangent vector determines a linear functional on smooth functions by $l_\gamma(f) = \frac{d}{dt}|_{t=0}f(\gamma(t))$. The linear functional representation provides a natural vector space structure for vectors T_xM. The map $\rho : TM \to M$ taking a tangent vector to its associated point x is called the bundle projection.

We say that a continuous mapping $F : M \to N$ is smooth if for every chart (U, ϕ_U) on M and (V, ϕ_V) on N such that $F(U) \subset V$, the function $\phi_V \circ F \circ \phi_U^{-1}$ is smooth. For every smooth mapping $F : M \to N$ there exists an associated mapping $F_* : TM \to TN$ between the tangent bundles. This mapping is given by $(x, v) \mapsto (F(x), w)$ where w is the linear functional $w(f) = v(f \circ F)$. The mapping is called the differential of F. Let us have charts (U, ϕ) on M and (V, ψ) on V. Then we have a local representation of F as $\tilde{F} = \psi \circ F \circ \phi^{-1}$ and the local representation of the differential is $D\tilde{F}$. With these concepts, we can provide two general ways to construct manifolds. These examples will involve concept of regular submanifolds of M. They are subsets S of M such that every point $x \in S$ has a neighbourhood U and ϕ_U such that the set $U \cap S$ is obtained by vanishing k coordinate functions of ϕ_U. The dimension of the regular submanifold in this case is $\dim(M) - k$, see [1, p. 100]

Example 2.5. Let M, N be manifolds with dimensions m and n, respectively. We say that a point c is a regular value of a smooth mapping $F : M \to N$ if the preimage of c is empty or for every point $x \in F^{-1}(\{c\})$ the differential $F_* : T_xM \to T_{F(x)}N$ is surjective. In this case, the set $F^{-1}(\{c\})$ is called a regular level set. Now it can be shown that a regular level set is a regular submanifold of M and its dimension is $m - n$. Especially the set $F^{-1}(\{c\})$ can be equipped with a smooth structure [1, p. 105]. The most common examples arise when $N = \mathbb{R}$ and F is a smooth function. In this case, it is enough to show that F_* is nonzero, at every point of the level set.

Let the manifold M be the space $\mathbb{R}^3$ with the standard differential structure and let $F(x, y, z) = x^2 + y^2 + z^2$. Then $F(x, y, z) = 1$ is the unit sphere in $\mathbb{R}^3$. Let us show that the differential of F is nonzero when $x \neq 0$. The differential is $DF = \langle 2x, 2y, 2z\rangle$ which is zero only when $\langle x, y, z\rangle = \langle 0, 0, 0\rangle$. Since $F(0, 0, 0) = 0$, we see that $c = 1$ is a regular value of F and thus the sphere S^2 is a manifold.

Example 2.6. If we have a smooth function $F : M \to N$ such that F is injective and the tangent map F_* is injective at every point $p \in M$, then F is called an immersion. It can be shown that $F(M)$ can be given a manifold structure and $F(M)$ is called an immersed submanifold. However, the immersed manifold may not be a manifold with respect to the subspace topology [1, p. 122]. When we require that $F : M \to F(M)$ is a homeomorphism with respect to the subspace topology, then $F(M)$ is a regular submanifold of N. The mapping F is called an embedding and $F(M)$ is called an embedded or regular submanifold.

To show that an immersion is a homeomorphism it is useful to use the following fact: A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Intuitively the above discussion says that an immersion of a compact manifold is a regular manifold. For example the image of S^2 under the map $f(x) = Ax + b$, where A is invertible matrix and $b \in \mathbb{R}^3$, is a manifold. Thus, ellipsoids are manifolds as they are obtained with this way.

2.2 Tensors

Tensors and exterior algebras are standard objects on manifolds. For example, they are used to represent multilinear functions and differential forms. We collect and revise definitions and results that we need later in this thesis. The exposition is based on books [3, 1, 2] and [4]. We focus on the tensor algebra since its construction varies in the literature. However, we discuss shortly exterior algebra as it is needed for definition of differential forms. We will use Einstein's summation convention while discussing tensors.

Definition 2.7 (Dual space and dual pairing)**.** *Let V be a finite dimensional vector space with dimension n. Its dual space V^* is the space of ccontinuous linear functionals to $\mathbb{R}$, that is, $V^* = \mathrm{Hom}(V, \mathbb{R}) := \{f : V \to \mathbb{R} \mid f \text{ is linear and continuous}\}$. We can think of elements of V also as elements of $(V^*)^*$ as there is a natural dual pairing of $v \in V$ and $f \in V^*$ given by $\langle f, v\rangle = f(v)$. We will reserve the notation $\langle \cdot, \cdot \rangle$ for the metric tensor so we will sometimes use a standard abuse of notation and denote dual pairing as $v(f)$.*

We will often use change of coordinates which induces a change of basis so it is necessary to know how to work with bases. Let $\{e_k\}_{k=1}^n$ be a basis for V. Then we define a dual basis to be the set $\{f^k\}_{k=1}^n$ such that $f^k(e_i) = \delta_i^k$ where δ_i^k is Kronecker's delta. We will denote the coordinate vectors of $v \in V$, $f \in V^*$ by $\mathbf{v}, \mathbf{f} \in \mathbb{R}^{n\times 1}$ with respect to the bases $\{e_k\}_{k=1}^n, \{f^k\}_{k=1}^n$. Then, by the definition we have $v(f) = f(v) = \mathbf{f}^\top \mathbf{v}$.

Let $\{\tilde{e}^k\}_{k=1}^n, \{\tilde{f}^k\}_{k=1}^n$ be different bases of V and V^* respectively. Then a change of basis matrix P is a matrix for which the equation $\tilde{\mathbf{v}} = P\mathbf{v}$ holds. Then for dual bases, the matrix $P^{-\top}$ is the change of basis matrix for dual bases. This is seen from the following calculation $(P^{-\top}\mathbf{f})^\top P\mathbf{v} = \mathbf{f}^\top P^{-1}P\mathbf{v} = \mathbf{f}^\top \mathbf{v}$.

It is often useful to write a basis vectors e_i and f^k as a linear combination of another basis. In notation above, this is written as $e_i = a_i^j \tilde{e}_j$, $f^k = b_j^k \tilde{f}^j$. Now P, a_i^j and b_j^k are related: By evaluating P with standard basis vectors, we notice that $a_i^j = P_{ji}$ and similarly we get that $b_j^k = (P^{-\top})_{jk} = (P^{-1})_{kj}$.

Definition 2.8 (Tensor product space)**.** *Let $V_1, V_2, \ldots, V_n, Z$ be vector spaces. A multi- or k-linear function is a function $f : V_1 \times V_2 \times \cdots \times V_n \to Z$ such that f is linear in every argument: For every $1 \le k \le n$ we have*

$$f(v_1, v_2, \ldots, av_k + bw_k, \ldots, v_n) = af(v_1, v_2, \ldots, v_k, \ldots, v_n) + bf(v_1, v_2, \ldots, w_k, \ldots, v_n).$$

We denote the space of Z-valued multilinear functions by $\mathcal{L}(V_1, V_2, \ldots, V_n; Z)$. We *define the tensor product space* $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ *as* $\mathcal{L}(V_1^*, V_2^*, \ldots, V_n^*; \mathbb{R})$. *The space of* (k,l)*-tensors is denoted by* V_l^k *and is defined as* $\underbrace{V \otimes V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{l}$.

There is a natural way to form an element of $V \otimes W$ with elements $v \in V, w \in W$, namely, the tensor product of $v \otimes w$. The intuition behind the following definition is that the tensor product will have following properties

$$\begin{aligned}
(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\
v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\
(rv) \otimes w &= v \otimes (rw) = r(v \otimes w)
\end{aligned}$$

where $r \in \mathbb{R}$ [4]. We want to generalize this for more than two terms so we give the definition in the general case.

Definition 2.9. *A tensor product* $\otimes : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n$ *is defined via*

$$\otimes(v_1, v_2, \ldots, v_n)(f_1, f_2, \ldots, f_n) = \prod_{k=1}^{n} v_k(f_k). \tag{1}$$

The function is readily multilinear. This mapping is usually written as $v_1 \otimes v_2 \otimes \cdots \otimes v_n$. *We can extend this for elements of* $S \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$ *and* $T \in W_1 \otimes W_2 \otimes \cdots \otimes W_m$ *as*

$$(S \otimes T)(v_1^*, v_2^*, \ldots, v_n^*, w_1^*, w_2^*, \ldots, w_m^*) = S(v_1^*, v_2^*, \ldots, v_n^*)T(w_1^*, w_2^*, \ldots, w_m^*) \tag{2}$$

where $v_k^* \in V_k^*$ for $k = 1, 2, \ldots, n$ and $w_j^* \in W_j^*$ for $j = 1, 2, \ldots, m$. Tensor products are vector spaces so they have the concept of a basis as well. We can obtain a basis of tensor product spaces by taking tensor products of bases. Local calculations on manifold are done using of basis representations of tensors. Moreover, tensors and tensor fields can be defined also by giving components and describing the change of basis and frames.

Theorem 2.10 (Basis theorem)**.** *Let* $\{v_i^{(k)}\}_{i=1}^{n_k}$ *be a basis of* V_k *for* $k = 1, 2, \ldots, m$. *Then the following set* $\{v_{i_1}^{(1)} \otimes v_{i_2}^{(2)} \otimes \cdots \otimes v_{i_m}^{(m)} : 1 \le i_k \le n_k\}$ *is basis for* $V_1 \otimes V_2 \otimes \cdots \otimes V_m$. *Let* $\{(f)_{(k)}^i\}_{i=1}^{n_k}$ *be a dual basis of* $\{v_i^{(k)}\}_{i=1}^{n_k}$. *Then if we represent a tensor* T *as* $T^{i_1 i_2 \ldots i_m} v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m}$, *we can calculate the coefficient* $T^{i_1 i_2 \ldots i_m}$ *with the following formula*

$$T^{i_1 i_2 \ldots i_m} = T(f_{(1)}^{i_1}, f_{(2)}^{i_2}, \ldots, f_{(m)}^{i_m}). \tag{3}$$

If we have a different basis $\{w_i^{(k)}\}_{i=1}^{n_k}$ *with* $v_i^{(k)} = a_i^{(k),j} w_j^{(k)}$, *then the components change as*

$$\tilde{T}^{j_1 j_2 \ldots j_m} = T^{i_1 i_2 \ldots i_m} a_{i_1}^{(1),j_1} a_{i_2}^{(2),j_2} \ldots a_{i_m}^{(k),j_m}. \tag{4}$$

Proof. Let us have dual vectors $f_{(k)} = b_{(k),j} f^j_{(k)}$ on spaces V_k^*, then following equalities hold

$$\begin{aligned} T(f_{(1)}, f_{(2)}, \dots . f_{(m)}) &= T(b_{(1),j_1} f^{j_1}_{(1)}, b_{(2),j_2} f^{j_2}_{(2)}, \dots, b_{(m),j_m} f^{j_m}_{(m)}) \\ &= T(f^{j_1}_{(1)} f^{j_2}_{(2)}, \dots, f^{j_m}_{(m)}) b_{(1),j_1} b_{(2),j_2} \dots b_{(m),j_m} \\ &= T^{j_1 j_2, \dots, j_m} v^{(1)}_{j_1}(b_{(1),l_1} f^{l_1}_{(1)}) v^{(2)}_{j_2}(b_{(2),l_2} f^{l_2}_{(2)}) \dots v^{(m)}_{j_m}(b_{(m),l_m} f^{l_m}_{(m)}) \\ &= T^{j_1 j_2, \dots, j_m} v^{(1)}_{j_1} \otimes v^{(2)}_{j_2} \otimes \dots \otimes v^{(m)}_{j_m}(f_{(1)}, f_{(2)}, \dots, f_{(m)}). \end{aligned} \tag{5}$$

This proves that the set spans the vector space $V_1 \otimes V_2 \otimes \dots \otimes V_m$. Linear independence is proven in following way: When we apply dual bases to a linear combination of $T^{i_1 i_2 \dots i_m} v^{(1)}_{i_1} \otimes v^{(2)}_{i_2} \otimes \dots \otimes v^{(m)}_{i_m}$ and when we observe that $v^{(k)}_i(f^j_{(k)}) = \delta^j_i$ for every k, we get

$$\begin{aligned} 0 &= T^{i_1 i_2 \dots i_m} v^{(1)}_{i_1}(f^{j_1}_{(k)}) v^{(1)}_{i_1}(f^{j_1}_{(k)}) \dots v^{(1)}_{i_1}(f^{j_1}_{(k)}) \\ &= T^{i_1 i_2 \dots i_m} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \dots \delta^{j_m}_{i_m} \\ &= T^{j_1 i_2 \dots j_m}. \end{aligned} \tag{6}$$

So the set is linearly independent. Let us have another set of bases $v^{(k)}_i = a^{(k),j}_i w^{(k)}_j$. Then applying algebraic properties of tensor product we get

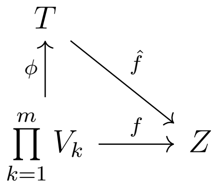
$$\begin{gathered} T^{i_1 i_2 \dots i_m} v^{(1)}_{i_1} \otimes v^{(2)}_{i_2} \otimes \dots \otimes v^{(m)}_{i_m} = \\ T^{i_1 i_2 \dots i_m} (a^{(1),j_1}_{i_1} w^{(1)}_{j_1}) \otimes (a^{(2),j_2}_{i_2} w^{(2)}_{j_2}) \otimes \dots \otimes (a^{(m),j_m}_{i_m} w^{(m)}_{j_m}) = \\ T^{i_1 i_2 \dots i_m} a^{(1),j_1}_{i_1} a^{(2),j_2}_{i_2} a^{(m),j_m}_{i_m} w^{(1)}_{j_1} \otimes w^{(2)}_{j_2} \otimes \dots \otimes w^{(m)}_{j_m}. \end{gathered} \tag{7}$$

We can read the transformation of coefficients from the last line. □

The change of basis is essentially performed by writing old basis vectors as a linear combinations of new basis vectors and using rules of tensor algebra.

Tensors are natural objects for representing multilinear functions. Every Z-valued multilinear function f can be represented as a composition of the tensor product map $\otimes : V_1 \times V_2 \times \dots \times V_m \to V_1 \otimes V_2 \otimes \dots \otimes V_m$ and a linear function $\hat{f} : V_1 \otimes V_2 \otimes \dots \otimes V_m \to Z$, that is, $f = \hat{f} \circ \otimes$. This property is in fact enough to characterize tensors and is used as a definition in some references. Furthermore, there exists an isomorphism $\iota : E \otimes V_1^* \otimes V_2^* \otimes \dots \otimes V_m^* \to \mathcal{L}(V_1, V_2, \dots, V_m; E)$. This isomorphism is used to define vector-valued (k, l)-tensors. We gather these facts to two theorems. In these theorems, we will denote $V_1 \times V_2 \times \dots \times V_m$ by $\prod_{k=1}^{m} V_k$.

Theorem 2.11. *[5, p. 26] Let $T, V_1, V_2, \dots, V_m$ and Z be vector spaces. We say that a pair (T, ϕ), where ϕ is a multilinear function $\phi : \prod_{k=1}^{m} V_k \to T$, has the universal mapping property for multilinear functions on $\prod_{k=1}^{m} V_k$ if for every $f : \prod_{k=1}^{m} V_k \to Z$ there* *exist an unique linear function* $\hat{f} : T \to Z$ *such that the diagram*



commutes. Furthermore, if (T, ϕ) *is such a pair, then there exists an isomorphism* $j : V_1 \otimes V_2 \otimes \cdots \otimes V_m \to T$ *such that* $j \circ \otimes = \phi$.

Theorem 2.12. *There exists an unique isomorphism*

$$\iota : E \otimes V_1^* \otimes V_2^* \otimes \cdots \otimes V_m^* \to \mathcal{L}(V_1, V_2, \ldots, V_n; E) \tag{8}$$

such that for every $e \in E, f_1 \in V_1^*, f_2 \in V_2^*, \ldots, f_m \in V_m^*$,

$$\iota(e \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_m)(v_1, v_2, \ldots, v_m) = e \prod_{k=1}^{m} f_k(v_k). \tag{9}$$

Proof. The proof is based on [2, p. 159]. Let $\{f_{(k)}^i\}_{i=1}^{n_k}$ be a basis of V_k^*, let $\{e_i\}_{i=1}^{n_E}$ be a basis of E and $T \in \mathcal{L}(V_1, V_2, \ldots, V_n; E)$. It is enough to show that the set $\{\iota(e_k \otimes f_{(1)}^{j_1}, f_{(2)}^{j_2}, \ldots, f_{(m)}^{j_m}) : \ 1 \le k \le n_e, 1 \le j_k \le n_k\}$ is a basis. By the calculation

$$\begin{aligned} T(a_{(1)}^{i_1} v_{i_1}, a_{(2)}^{i_2} v_{i_2}, \ldots, a_{(m)}^{i_m} v_{i_m}) &= \\ T(v_{i_1}, v_{i_2}, \ldots, v_{i_m}) a_{(1)}^{i_1} a_{(2)}^{i_2} \ldots a_{(m)}^{i_m} &= \\ T_{i_1 i_2 \ldots i_m}^k e_k a_{(1)}^{i_1} a_{(2)}^{i_2} \ldots a_{(m)}^{i_m} &= \\ T_{i_1 i_2 \ldots i_m}^k \iota(e_k \otimes f_{(1)}^{i_1} \otimes f_{(2)}^{i_2} \otimes \cdots \otimes f_{(3)}^{i_m})(v_{(1)}, v_{(2)}, \ldots, v_{(m)}) \end{aligned} \tag{10}$$

we see that the set $\{\iota(e_k \otimes f_{(1)}^{j_1}, f_{(2)}^{j_2}, \ldots, f_{(m)}^{j_m}) : \ 1 \le k \le n_e, 1 \le j_k \le n_k\}$ spans the vector space $\mathcal{L}(V_1, V_2, \ldots, V_n; E)$. The proof of linear independence is similar to the proof used in proof of Theorem 2.10. □

There are two additional operations on (k, l)-tensors that are useful: contraction and interior product. Contraction is way to produce a $(k-1, l-1)$-tensor from a (k, l)-tensor and the interior product is a way to evaluate a tensor with a vector. With the definitions that we use, the interior product is a trivial operator but we need to define it properly. These operators are important for tensor analysis.

Definition 2.13. *Let* $\mu \in \{1, 2, \ldots, k\}$ *and* $\lambda \in \{1, 2, \ldots, l\}$, *then a contraction* $C_{\mu,\lambda}$ *of* (k, l)*-tensor* $T = v_1 \otimes v_2 \otimes \cdots \otimes v_k \otimes f^1 \otimes f^2 \otimes \cdots \otimes f^l$ *is given by*

$$C_{\mu,\lambda}(T) = v_\mu(f^\lambda) v_1 \otimes v_2 \otimes \cdots \otimes \hat{v}_\mu \otimes \cdots \otimes v_k \otimes f^1 \otimes f^2 \otimes \cdots \otimes \hat{f}^\lambda \otimes \cdots \otimes f^l \tag{11}$$

where hat notation $\hat{v}$ means that we omit the term. We extend this by linearity for the whole space. This is well defined since $v_\mu(f^\lambda)$ is invariant under change of basis.

The interior product ι_v is defined as

$$\iota_v(T) = C_{1,1}(v \otimes T) = T(v, \cdot, \dots, \cdot). \tag{12}$$

The indices in $C_{1,1}$ can be changed if needed. We omit these indices unless otherwise stated.

Example 2.14. Let us show how we can represent a matrix as a tensor product. Let the matrix be

$$A = \begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix} \tag{13}$$

with respect to the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$. The element e_1 maps to the element $3e_1 + 5e_2$ so combining this with the element e_1^* of the dual basis we obtain the term $(3e_1 + 5e_2) \otimes e_1^*$. Similarly with e_2, we get the term $(7e_1 + 9e_2) \otimes e_2^*$. So the representation is $\iota^{-1}(A) = (3e_1 + 5e_2) \otimes e_1^* + (7e_1 + 9e_2) \otimes e_2^*$. The evaluation of this representation with a vector v of $\mathbb{R}^2$ is done by interior product with respect to second factor of the tensor product so $A(v) = C_{1,1}(v \otimes \iota^{-1}(A))$.

With tensor algebra, we can construct the exterior algebra. There the role of tensor product is replaced by the wedge product. The exterior algebra is the building block of differential forms which are used to define an integration theory on manifolds. We will only introduce the exterior space and the wedge product. We start by discussing even permutations and anti-symmetric tensors.

Definition 2.15. *A permutation of the set $I_n = \{1, 2, \dots, n\}$ is a bijection $\sigma : I_n \to I_n$. The set of permutations of I_n is denoted by S_n. Every permutation $\sigma \in S_n$ can be composed of swaps which are permutations that changes only two elements. With this knowledge, we can associate a sign to a permutation*

$$\operatorname{sgn}(\sigma) = \begin{cases} 1, \textit{if the permutation is given by composition of even number of swaps} \\ -1, \textit{if the permutation is given by composition odd number of swaps.} \end{cases} \tag{14}$$

Permutations form a group. We can form an associated group action on the space $V^n = V \times V \times \cdots \times V$ given by

$$\sigma(v_1, v_2, \dots, v_n) = (v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)}). \tag{15}$$

This extends immediately as a group action on tensors by $(\sigma T)(w) = T(\sigma w)$ where $T \in V_n^0$ and $w \in V^n$. We say that a tensor $T \in V_n^0$ is alternating if $\sigma T = \operatorname{sgn}(\sigma)T$. We denote the space of alternating n-tensors as $\Lambda^n(V)$ and call it exterior space of V with degree n. We can define the following operator on tensors

$$A_n(T) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma T. \tag{16}$$

Notice that if the tensor T is alternating, then $A_n(T) = T$. Moreover, for an arbitrary tensor $T \in V_n^0$ the tensor $A_n(T)$ is alternating. The wedge product is defined as

$$S \wedge T = A_{k+l}(S \otimes T) \tag{17}$$

where $S \in \Lambda^k(V)$ and $T \in \Lambda^l(V)$. The wedge product has following properties:

Theorem 2.16. *[3, p. 53] Let $S_1, S_2 \in \Lambda^k(V)$, $T_1, T_2 \in \Lambda^l(V)$ and $U \in \Lambda^m(V)$, then the following equations hold:*

- $(S_1 + S_2) \wedge T_1 = S_1 \wedge T_1 + S_2 \wedge T_1$ *and* $S_1 \wedge (T_1 + T_2) = S_1 \wedge T_1 + S_1 \wedge T_2$,
- $(S_1 \wedge T_1) \wedge U = S_1 \wedge (T_1 \wedge U)$ *and*
- $S_1 \wedge T_1 = (-1)^{kl} T_1 \wedge S_1$.

2.3 Vector bundles

A vector bundle is a smooth manifold with a vector space structure on it.

Definition 2.17. *A vector bundle is a triplet (E, M, π) where E and M are smooth manifolds and $\pi : E \to M$ is a smooth mapping and the following properties hold: For every point $p \in M$ the fiber $E_p = \pi^{-1}(p)$ has a real vector space structure of dimension k. Furthermore there exists an atlas $\{U_i\}_{i=1}^{\infty}$ of M and diffeomorphisms $\Psi_i : U_i \times \mathbb{R}^k \to \pi^{-1}(U_i)$ such that for every $p \in M$, the function's $\Psi_i(p, \cdot)$ image is E_p and it is a linear isomorphism.*

Let a map ϕ_U be a chart on $U \subset M$ then we can form a chart Φ_U of E as follows. Let us define an auxiliary map $\mathrm{pr}_\phi : (p, v) \mapsto (\phi(p), v)$. Then the chart Φ_U is given by

$$\Phi_U = \mathrm{pr}_\phi \circ \Psi_U^{-1}. \tag{18}$$

With these maps we can form an atlas of E. The inverse map Φ_U^{-1} is given by

$$\Phi_U^{-1} : \mathbb{R}^n \times \mathbb{R}^k \to E, (x, v) \mapsto \Psi_U(\phi^{-1}(x), v).$$

The map Φ_i^{-1} is linear with respect to v.

Vector bundles can also be characterized by transition mappings and cocycles. Let $\Psi_U : U \times \mathbb{R}^k \to E$, $\Psi_V : V \times \mathbb{R}^k \to E$ be bundle charts and $p \in M$. Then there is a function $g_{VU} : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ such that

$$\Psi_V^{-1} \circ \Psi_U(p, v) = (p, g_{VU}(p)v) \ \ \forall \ (p, v) \in U \cap V \times \mathbb{R}^k. \tag{19}$$

The mapping g_{UV} is called a transition map. Transition maps have the following three properties.

- The functions $g_{VU} : U \to \mathrm{GL}(k, \mathbb{R})$ are smooth.
- For every $p \in U$ we have $g_{UU}(p) = \mathrm{id}$.

- For every $p \in U \cap V \cap W$ we have

$$g_{UW}(p)g_{WV}(p)g_{VU}(p) = \mathrm{id}\,. \tag{20}$$

Knowing transition functions and charts on a manifold is equivalent to knowing the vector bundle structure [3, p. 71]. Let $\phi_U : U \to \tilde{U}$ and $\phi_V : V \to \tilde{V}$ be charts on M. Then we have a formula

$$\begin{aligned}\Phi_V \circ \Phi_U^{-1}(x, v) &= \Phi_V(\Psi_U(\phi_U^{-1}(x), v)) = \mathrm{pr}_{\phi_V}(\Psi_V^{-1}(\Psi_U(\phi_U^{-1}(x), v))) \\ &= \mathrm{pr}_{\phi_V}(\phi_U^{-1}(x), g_{VU}(\phi_U^{-1}(x))v) = (\phi_{VU}(x), g_{VU}(\phi_U^{-1}(x))v)\end{aligned} \tag{21}$$

where $\phi_{VU} = \phi_V \circ \phi_U^{-1}$ is the transition map between charts on M. Let us define $\tilde{g}_{VU} = g_{VU} \circ \phi_U^{-1}$, then we have the change of variables formula

$$\Phi_V \circ \Phi_U^{-1}(x, v) = (\phi_{\tilde{V}\tilde{U}}(x), \tilde{g}_{VU}(x)v). \tag{22}$$

We denote the transformation from $\tilde{U}$ to $\tilde{V}$ by Φ_{VU}. In this thesis, we use only locally trivialisable vector bundles and omit the prefix locally trivialisable. To perform local calculations and change of variables, we need only to know $\tilde{g}_{VU}(x)$. To make the terminology exact, we give following definition of change of variables.

Definition 2.18 (Change of variables)**.** *Let U and V be open sets on $\mathbb{R}^n$. A change of variables formula for an operator P is its pullback under a diffeomorphic mapping ϕ from U to V which means that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\phi^*(P)} & \mathcal{B}(U) \\ \downarrow{\scriptstyle \alpha_{VU}} & & \downarrow{\scriptstyle \beta_{VU}} \\ \mathcal{A}(V) & \xrightarrow{P} & \mathcal{B}(V) \end{array}$$

where $\mathcal{A}(X), \mathcal{B}(X)$ represent function spaces over X and α_{VU}, β_{VU} are isomorphisms associated with the function spaces and ϕ. If points of U and V are denoted by x and y respectively, then there is a standard abuse of the notation to denote the diffeomorphism also with $y(x)$ so for example $\frac{\partial \phi_j}{\partial x^i}$ would be written $\frac{\partial y_j}{\partial x^i}$ [4, p. 51]. In the case of confusion, it is advisable to write mappings explicitly.

Vector bundles are convenient objects for generalizing vector-valued quantities such as vector fields. They can be thought of as a generalization of the concept of a tangent bundle and a section is a generalization of the notion of vector fields to vector bundles.

Definition 2.19 (Section)**.** *A smooth map $s : M \to E$ is called a smooth section if it satisfies the property*

$$\pi \circ s = \mathrm{id} : M \to M. \tag{23}$$

We denote the set of smooth sections by $\Gamma(E)$. We can represent a section s locally by using a chart (U, ϕ) and an associated bundle chart Φ in the following way: Let us have a point $x = (x_1, x_2, \ldots, x_n) \in \phi(U)$, then we can write a section locally as

$$\Phi_U \circ s \circ \phi^{-1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, a_1(x), a_2(x), \ldots, a_k(x)). \tag{24}$$

where functions a_i are smooth functions on M. Thus, when we define basis sections s_i on a chart (U, ϕ) as

$$s_i(x) = \Phi^{-1}(x, e_i) = \Phi_U^{-1}(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 1, 0, \ldots, 0) \tag{25}$$

where e_i is the canonical i^{th} unit basis vector on $\mathbb{R}^k$, then every section can be represented as a sum

$$s(x) = \sum_{i=1}^{k} a^i(x) s_i(x) \quad \forall x \in U_i. \tag{26}$$

We gather the above discussion to a theorem that we will use in the following sections.

Theorem 2.20. *Every smooth section s of a vector bundle E can be represented locally as a sum*

$$s(x) = \sum_{i=1}^{k} a^i(x) s_i(x) \quad \forall x \in U_i. \tag{27}$$

Given two overlapping charts, we want to discuss a change of variables formula for sections. It is a mapping $D\Phi_{VU} : \Gamma(U \times \mathbb{R}^k)$ to $\Gamma(V \times \mathbb{R}^k)$ and is defined so that following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{\phi_{UV}} & U \\ \downarrow{\scriptstyle D\Phi_{VU}(s)} & & \downarrow{\scriptstyle s} \\ V \times \mathbb{R}^k & \xrightarrow{\Phi_{UV}} & U \times \mathbb{R}^k \end{array}$$

We can read from the diagram that $D\Phi_{VU}(s) = \Phi_{UV}^{-1} \circ s \circ \phi_{UV}$ or more explicitly

$$D\Phi_{VU} s(y) = (y, \tilde{g}_{VU}(\phi_{UV}(y)) s(\phi_{UV}(y))). \tag{28}$$

There are at least four common ways to produce vector bundles from two vector bundles E and E': the dual bundle E^*, the Whitney sum $E \oplus E'$, the tensor product bundle $E \otimes E'$ and the exterior space $\Lambda^k(E)$. Let g_{VU} and g'_{VU} be transition functions on E and E' respectively. Then fibers and transition functions h_{VU} of these spaces are given by following identities

$$
\begin{aligned}
(E^*)_p =& E_p^*, \quad h_{VU} = g_{VU}^{-\top} \\
(E \oplus E')_p =& E_p \oplus E'_p, \quad h_{VU} = \begin{pmatrix} g_{VU} & 0 \\ 0 & g'_{VU} \end{pmatrix} \\
(E \otimes E')_p =& E_p \otimes E'_p, \quad h_{VU}(v \otimes v') = g_{VU}(v) \otimes g'_{VU}(v') \\
\Lambda^k(E)_p =& \Lambda^k(E_p)
\end{aligned}
$$

where the tensor product of matrices A, B are define via $A \otimes B(v \otimes w) = A(v) \otimes B(w)$. We have omitted the transition mapping in case of $\Lambda^k(E)$ since $\Lambda^k(E)$ is a subspace of $\otimes^k(E)$.

Example 2.21. A common example of a vector bundle is the tangent bundle (M, E, ρ) mentioned earlier. If $\phi_{VU}(x)$ is a change of coordinates, then the transition function is the Jacobian $\tilde{g}_{VU}(x) = D\phi_{VU}$. The dual bundle construction gives us the cotangent bundle (M, E, π) and its transition function is $(D\phi_{VU})^{-\top}$. From tangent and cotangent bundle, we can construct (k, l)-tensor bundle $T^{k,l}(M)$. Its transition function is given by tensor product rule. However, tensors are often handled as a linear combinations of basis vectors in local coordinates and the change of variables is easier to do by substitution and using properties of the tensor product. For example in the case of $(1, 1)$-tensors: Let us have bases $\{v_i\}_{i=1}^k$ and $\{\tilde{v}_i\}_{i=1}^k$ on TU, dual bases $\{f_i\}_{i=1}^k$, $\{\tilde{f}_i\}_{i=1}^k$ and let us write bases as following linear combinations $\tilde{v}_i = a_i^s \tilde{v}_s$, $f^k = b_t^k \tilde{f}^t$ then

$$
v_i \otimes f^j = (a_i^s \tilde{v}_s) \otimes (b_t^j \tilde{f}^t) = a_i^s b_t^j \tilde{v}_s \otimes \tilde{f}^t. \tag{29}
$$

The coefficients a_i^s and b_t^k can be read from $D\phi_{VU}$ with methods represented in end of Definition 2.7. The bundle of $k-$forms, $\Lambda^k(T^*M)$, is a subbundle of $T^{0,k}(M)$. Since the wedge product is sum of tensor products, we can replace the tensor product with the wedge product in above calculations. The space of sections of this vector bundle is denoted by $\Lambda^k(M) = \Gamma(\Lambda^k(T^*M))$ and its elements are called differential k-forms.

We will use differential forms later as they provides us a notion of integration on oriented manifolds and they are useful in examples of partial differential operators on manifolds. Thus we will introduce the exterior derivative and integration of differential forms.

Definition 2.22 (exterior derivative)**.** *The exterior derivative is a collection of unique operators $d_k : \Lambda^k(M) \to \Lambda^{k+1}(M)$ such that following properties holds for any function $f \in \Gamma(M)$, $X \in \Gamma(TM)$, $\omega_1, \omega_1' \in \Lambda^k(M)$ and $\omega_2 \in \Lambda^j(M)$*

- $d_k(\omega_1 + \omega_2) = d_k\omega_1 + d_k\omega_1'$
- $d_{k+j}(\omega_1 \wedge \omega_2) = d_k\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d_j\omega_2$
- $d_0(f)(X) = X(f)$

- $d_{k+1} d_k \omega_1 = 0$ *for all* $k \in \{1, 2, \ldots, \dim(M)\}$

It is usual to drop the subscript from the operator and denote d_k *as* d*. In the local coordinates, the exterior derivative is determined by the formula*

$$d(f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}. \tag{30}$$

Vector-valued $(0, k)$-tensors are linear functions $f : T_pM \times T_pM \times \cdots \times T_pM \to E_p$ at every point $p \in M$. By Theorem 2.12 we can represent multilinear maps to E_p as tensors of $\otimes^k T^*M \otimes E$ up to a canonical isomorphism. Therefore, we can think of a vector valued tensor as a section of $\Gamma(\otimes^k T^*M \otimes E)$. This can be done similarly with (k, l)-tensors or differential forms $\Lambda^k(M)$.

2.4 L^p-theory and differential operators on vector bundles

A metric tensor on a vector bundle E gives a geometric structure to the bundle. To our needs, it is enough to define a metric tensor as a smooth mapping $g : E \oplus E \to \mathbb{R}$ which is fiber-wise a symmetric, positive definite bilinear form. Locally, with given basis s_i the metric tensor is determined by values of basis sections $g_{ij} = g(s_i, s_j)$. We use also the notation $\langle \cdot, \cdot \rangle$ for the metric tensor.

If we have two or more bundles over M with metrics, then we can induce a metric on a vector bundle that is constructed from them. The constructions can be done inductively so we show only the case of two vector bundles.

Theorem 2.23. *Let us have vector bundles* E_1, E_2 *and with metrics* $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ *respectively, then we have induced metrices on the associated bundles:*

1. *The dual bundle* E_1^* *has the metric given by Riesz's isomorphism mapping: Every metric defines locally an unique symmetric invertible matrix* G *by* $\langle x, y \rangle = x^\top G y$ *where* $x, y \in \mathbb{R}^{n \times 1}$ *are local coordinate vectors. This matrix determines a local isomorphism* $G^\top : E_p \to E_p^*$ *such that* $\langle x, y \rangle = G^\top x(y)$*. We can carry the inner product over to dual vector space via the inverse map* $G^{-\top}$ *via the formula* $\langle v^*, w^* \rangle_{E^*} = \langle G^{-\top} v^*, G^{-\top} w^* \rangle_E$*. We obtain that the asssociated matrix of the dual metric is* $G^{-\top}$ *and locally it is given by* $(G^{-\top})^{ij} e_i \otimes e_j$ *where vectors* e_i *are basis sections of* $E|_U$*.*

2. *The Whitney sum* $E_1 \oplus E_2$ *has the metric given by*

$$\langle (v_1, w_1), (v_2, w_2) \rangle = \langle v_1, v_2 \rangle_1 + \langle w_1, w_2 \rangle_2. \tag{31}$$

   Locally the metric is given by $\sum_{1 \le i,j \le k_1} g_1^{ij} f^i \otimes f^j + \sum_{1 \le i,j \le k_2} g_2^{ij} f^{k_1+i} \otimes f^{k_1+j}$ *where* k_1, k_2 *are dimensions of fibers of* E_1, E_2*, respectively, and* $\{f_i\}_{i=1}^{k_1}, \{f_{k_1+i}\}_{i=1}^{k_2}$ *are basis sections of* E_1^* *and* E_2^**, respectively.*

3. The metric on a tensor product $E_1 \otimes E_2$ is based on the formula

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle \tag{32}$$

and is extended linearly: Let $\{e_i\}_{i=1}^{k_1}$ and $\{f_i\}_{i=1}^{k_2}$ be bases of E_1, E_2 respectively. Then using Einstein's summation convention we define the metric as

$$\langle a^{ij} e_i \otimes f_j, b^{kl} e_k \otimes f_l \rangle = a^{ij} b^{kl} \langle e_i, e_k \rangle \langle f_j, f_l \rangle = g_{ik} g_{jl} a^{ij} b^{kl}. \tag{33}$$

This is well-defined, that is, independent of choice of a basis.

4. We get a metric to differential forms $\Lambda^k(E)$ by observing that differential forms form a subspace of the tensor product $\underbrace{E \otimes E \otimes \cdots \otimes E}_{k}$.

Proof. 1) We need to show bilinearity, symmetry, nondegenerativity and smoothness. In case of the dual bundle, the bilinearity and symmetry are seen from $\langle x, y \rangle = x^\top G^{-1} y$. The nondegenerativity follows from the fact that $\ker G^{-\top} = \{0\}$. The inverse matrix $G^{-\top}$ can be written with Cramer's rule and from that form we see smoothness.

2) Bilinearity and symmetry are also clearly read from the definition in the Whitney sum. The nondegeneracy follows from choosing w_1, w_2 such that $\langle v_1, w_1 \rangle$ and $\langle v_2, w_2 \rangle$ are positive. This is possible since the metrics $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ are nondegenerate. Smoothness can be seen from the local representation.

3) Let us show that the metric given in the tensor bundle case is well-defined: Let $\{\tilde{e}_i\}_{i=1}^{k_1}, \{e_i\}_{i=1}^{k_1}$ be bases for E_1 and $\{\tilde{f}_i\}_{i=1}^{k_2}, \{f_i\}_{i=1}^{k_2}$ be bases for E_2. Then we have $e_i = s_i^p \tilde{e}_p$ and $f_j = t_j^r \tilde{f}_r$ so

$$\begin{aligned}
\langle a^{ij} e_i \otimes f_j, b^{kl} e_k \otimes f_l \rangle &= a^{ij} b^{kl} \langle e_i, e_k \rangle_1 \langle f_j, f_l \rangle_2 \\
&= a^{ij} b^{kl} \langle s_i^{p_1} \tilde{e}_{p_1}, s_k^{p_2} \tilde{e}_{p_2} \rangle_1 \langle t_j^{r_1} \tilde{f}_{r_1}, t_l^{r_2} \tilde{f}_{r_2} \rangle_2 \\
&= (a^{ij} s_i^{p_1} t_j^{r_1})(b^{kl} s_k^{p_2} t_l^{r_2}) \langle \tilde{e}_{p_1}, \tilde{e}_{p_2} \rangle_1 \langle \tilde{f}_{r_1}, \tilde{f}_{r_2} \rangle_2 \\
&= \langle (a^{ij} s_i^{p_1} t_j^{r_1}) \tilde{e}_{p_1} \otimes \tilde{f}_{r_1}, (b^{kl} s_k^{p_2} t_l^{r_2}) \tilde{e}_{p_2} \otimes \tilde{f}_{r_2} \rangle.
\end{aligned} \tag{34}$$

The coefficients $(a^{ij} s_i^{p_1} t_j^{r_1})$ and $(b^{kl} s_k^{p_2} t_l^{r_2})$ are coefficients of tensors $a^{ij} e_i \otimes f_j$ and $b^{kl} e_k \otimes f_l$ in the new basis so the definition does not depend on the chosen basis. Bilinearity, symmetry and smoothness can be easily noticed from the definition. Every tensor is a sum of elements of form $v \otimes w$. To show nondegeneracy, let us use ortonormal bases $\{e_i\}_{i=1}^{k_1}$ and $\{f_i\}_{i=1}^{k_1}$ for E_1, E_2. Each element of $E_1 \otimes E_2$ can be written as sum $a^{ij} e_i \otimes f_j$. If we have a nonzero element v of $E_1 \otimes E_2$ then its reprsentation has some nonzero coefficient $a^{i_1 j_1}$. Let us evaluate, without Einstein's summation convention, an inner product $\langle v, a^{i_1 j_1} e_{i_1} \otimes f_{j_1} \rangle$. Since the bases are ortonormal, the only nonzero term will be $(a^{ij})^2$. This proves the nondegenerativity. □

We need a notion of the support of a section $s \in \Gamma(E)$.

Definition 2.24. *Let s be a section on a vector bundle E. The support of s is denoted by* $\text{supp}(s)$ *and is defined as*

$$\text{supp}(s) = \text{cl}(\{x \in M | s(x) \neq 0\}).$$

Smooth sections with compact support form a vector space which is denoted by $\Gamma_0(E)$.

Orientability of the manifold is a necessary condition for integration of compactly supported n-forms on n-dimensional manifold. A manifold is orientable if there exists a nowhere vanishing n-form on the manifold. We will assume that manifolds are orientable. We will denote the integral of $\omega \in \Lambda_0^n(M)$ with

$$\int_M \omega. \tag{35}$$

For orientable manifolds the Riemannian metric provides a notion of volume form $\text{vol}_n \in \Lambda^n(M)$ which allows us to integrate functions with compact support on a manifold. This is done by the formula

$$\int_M f \,\text{vol}_n \tag{36}$$

where $f \in \Gamma_0(M)$. With these tools, we can define the p-norm of $s \in \Gamma_0(E)$ as

$$\|s\|_p = (\int_M \langle s, s\rangle_E^{\frac{p}{2}} \text{vol}_n)^{\frac{1}{p}}. \tag{37}$$

The space $L^p(E)$ is defined as the completion of $\Gamma_0(E)$ with respect to the p-norm.

Also with the volume form, we can introduce the Hodge star operator. It is needed in the formulation of Hodge Laplacian which is a generalization of Laplace operator.

Definition 2.25 (Hodge star)**.** *The Hodge star operator $* : \Lambda^k(M) \to \Lambda^{n-k}(M)$ is a bijection and is determined by the equation*

$$*\omega \wedge \eta = \langle \omega, \eta\rangle \,\text{vol}_n\,. \tag{38}$$

With vector fields, we can introduce the notion of a connection which is a generalization of differentiation of vector fields. An operator $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ is called connection if for every $v, w \in \Gamma(TM)$, $s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(M)$ operator has the following propeties:

$$\nabla_{v+w} s = \nabla_v s + \nabla_w s \tag{39}$$

$$\nabla_{fv} s = f \nabla_v s \tag{40}$$

$$\nabla_v (s_1 + s_2) = \nabla_v s_1 + \nabla_v s_2 \tag{41}$$

$$\nabla_v (f s_1) = f \nabla_v s_1 + v(f) s_1. \tag{42}$$

By observing that a connection is C^∞-linear with respect to $\Gamma(TM)$ variable and recalling Theorem 2.12, we notice that we can think of connection as an operator from sections to E-valued $(0,1)$- tensors, that is, $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ up an isomorphism. The evaluation by $X \in \Gamma(TM)$ is given by

$$(\nabla s)(X) = \nabla_X s$$

where $s \in \Gamma(E)$.

There exists a canonical connection associated to a metric g on the tangent bundle TM called Levi-Civita connection which we denote by ∇^g or ∇^M. It can be extended to act on vectors defined on curves. There exists an unique operator $\frac{D}{dt}$ associated to every smooth curve $\gamma(t)$ such that $\frac{D}{dt}$ coincides with the Levi-civita connection: Let $V(t)$ be a section on the curve and $\tilde{V}$ be a section of the tangent bundle such that $V(t) = \tilde{V}(\gamma(t))$, then we have the equation

$$\frac{DV(t)}{dt} = \nabla_{W(\gamma(t))}\tilde{V}(\dot{\gamma}(t)) \tag{43}$$

where W is a smooth vector field such that $W(\gamma(t)) = \dot{\gamma}(t)$. We will abuse notation and denote $\frac{D}{dt}$ by $\nabla_{\dot{\gamma}(t)}$. The geodesic of manifolds are defined to be curves $\gamma(t)$ such that the equation

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0 \tag{44}$$

holds. Every point $x \in M$ has $\epsilon_x > 0$ depending on x such that for every $(x, v) \in T_xM$ with $\|v\| < \epsilon_x$ there exist an unique geodesic $\gamma(t)$ with unit length and initial values $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Thus, geodesics produce an operator $\exp : U \subset TM \to M$ which maps the point $(x, v) \in U$ to the point $\gamma(1)$ given by the unique geodesic $\gamma(t)$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$ where the set U is a neighbourhood of the zero section given by the union $\bigcup_{x\in M} U_{x,\epsilon}$ where $U_{x,\epsilon} = \{(x, v) \in T_xM \mid \|v\| < \epsilon_x\}$.

We can also define covariant derivatives along curves for arbitrary connection on a vector bundle. This provides us the notion of parallel transport of a vector v. We say that the vector $V \in E_x$ is parallel transported along curve $\gamma(t) : [0, 1] \to M$ if for $V(t)$ we have $V(0) = V$ and

$$\nabla_{\dot{\gamma}(t)}V(t) = 0 \tag{45}$$

for every $t \in [0, 1]$. We say that a connection is metric compatible or a metric connection with respect to g_E if $V(g_E(s_1, s_2)) = g_E(\nabla_V s_1, s_2) + g_E(s_1, \nabla_V s_2)$ for every $V \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$. Parallel transport associated to a metric connection will preserve the norm of vector. We will assume that connections are metric compatible whenever there is given a metric on a vector bundle.

We would want to extend connections to tensor bundles and tensor products. If we have two vector bundles E_1, E_2 and connections ∇^1, ∇^2 on them, we can create a connection for the vector bundle $E_1 \otimes E_2$. This is done by the formula

$$\nabla_X(s_1 \otimes s_2) = \nabla^1_X s_1 \otimes s_2 + s_1 \otimes \nabla^2_X s_2 \tag{46}$$

and extended by linearity to the whole bundle. We can also introduce a connection for the dual bundle E_1'. This is done by imposing that the connection will commute with contraction, which means that

$$\nabla \circ C_{\mu,\lambda} = C_{\mu,\lambda} \circ \nabla,$$

and imposing that the connection acts on smooth functions by the formula

$$\nabla_X(f) = X(f). \tag{47}$$

These conditions give the following equation for the dual connection ∇^*:

$$X\langle s^*, s\rangle = \langle \nabla_X^* s^*, s\rangle + \langle s, \nabla_X s\rangle. \tag{48}$$

The above condition is equivalent with the requirement that the connection commutes with contraction. This is seen from the following calculation. Let us assume that the connection commutes with the contraction. Then we have

$$X\langle s^*, s\rangle = \nabla_X\langle s^*, s\rangle = \nabla_X C_{1,1}(s^*\otimes s) = C_{11}\nabla_X(s^*\otimes s) = \langle \nabla_X^* s^*, s\rangle + \langle s^*, \nabla_X s\rangle \tag{49}$$

where $s \in E, s^* \in E^*$ and brackets $\langle\cdot,\cdot\rangle$ denote the dual pairing instead of inner product. Now let us assume that formula (48) holds, then following holds

$$\nabla_X C_{1,1}(s^*\otimes s) = \nabla_X\langle s^*, s\rangle = X\langle s^*, s\rangle = \langle \nabla_X^* s^*, s\rangle + \langle s^*, \nabla_X s\rangle = C_{11}\nabla_X(s^*\otimes s) \tag{50}$$

and proves the claim.

We can derive an evaluation formula for $\nabla : \Gamma(\otimes^k T^*M \otimes E) \to \Gamma(\otimes^{(k+1)} T^*M \otimes E)$.

Theorem 2.26. *Let us have vector fields $X_0, X_1, \ldots, X_k \in \Gamma(TM)$ and a section $T \in \Gamma(\otimes^k T^*M \otimes E)$, then we have*

$$\begin{aligned}\nabla T(X_0, X_1, X_2, \ldots, X_k) = &\nabla_{X_0}^E(T(X_1, X_2, \ldots, X_k)) \\ &- \sum_{i=1}^{k} T(X_1, X_2, \ldots, \nabla_{X_0}^M X_i, \ldots, X_k).\end{aligned}$$

Proof. Let us start by proving $\iota_v \nabla_{X_0} T = \nabla_{X_0}^E(\iota_v T) - \iota_{(\nabla_{X_0}^M(v))} T$ where $\iota_v T$ was defined as $C_{1,1}(v \otimes T)$.

$$\begin{aligned}\iota_v \nabla_{X_0} T &= C_{1,1}(v \otimes \nabla_{X_0} T) \\ &= C_{1,1}(\nabla_{X_0}(v \otimes T) - \nabla_{X_0}^M v \otimes T) \\ &= \nabla_{X_0}^E C_{1,1}(v \otimes T) - C_{1,1}(\nabla_{X_0}^M(v) \otimes T) \\ &= \nabla_{X_0}^E(\iota_v T) - \iota_{(\nabla_{X_0}^M(v))} T.\end{aligned} \tag{51}$$

We obtain the result when we apply the result above and notice that

$$\nabla T(X_0, X_1, X_2, \ldots, X_k) = \iota_{X_k}\iota_{X_{k-1}} \ldots \iota_{X_1}\nabla_{X_0} T.$$ □

By the above definition, the composition of connections is well defined and we can define the k-th covariant derivative of a section $\nabla^k : \Gamma(E) \to \Gamma(T^*M^{\otimes k} \otimes E)$ as

$$\nabla^k(s) = \underbrace{\nabla \circ \nabla \circ \cdots \circ \nabla}_{k} s. \tag{52}$$

This will be used in the coordinate-free definition of Sobolev spaces. We can write a recursive formula for evaluating the k-th covariant derivative:

$$\begin{aligned}(\nabla^k s)(X_0, X_1, X_2, \ldots, X_k) = {} & \nabla^E_{X_0} \nabla^{k-1} s(X_1, X_2, \ldots, X_k) \\ & - \sum_{i=1}^{k} \nabla^{k-1} s(X_1, X_2, \ldots, \nabla^M_{X_0} X_i, \ldots, X_k).\end{aligned}$$

3 Distributions

Distribution theory is an important part of modern analysis. Distributions are used extensively in the literature. The main advantage of distributions is that they make it possible to extend operators to wider function spaces. Our objective in this section is to introduce a general framework for distributions and then give examples of distribution spaces.

3.1 General framework

Let us have function spaces V, $V_0 \subset V$ and V_1 over a space M and a bilinear form $B_V : V \times V_1 \to W$ where W is a vector space, often the field $\mathbb{R}$ or $\mathbb{C}$. We will call the space V_1 as the test function space. The space of W-valued distributions $\mathcal{D}'(M, V_1, W)$ is defined to be the space $\mathcal{L}(V_1; W)$. However, we will omit the test function space and vector space W from the notation $\mathcal{D}'(M, V_1, W)$ when they are clear from the context. When $W = \mathbb{R}$, then the space of distributions is the dual space V_1^*.

We have a natural embedding V to $\mathcal{D}'(M)$. The embedding is given by

$$\iota_V : V \to \mathcal{D}'(M) \quad v \mapsto B_V(v, \cdot). \tag{53}$$

It is common to omit the embedding mapping and identify elements of V as elements of $\mathcal{D}'(M, V_1)$. We do the same whenever it is convenient. However, we insert a tilde on elements of V when we mean corresponding element of $\iota_V(V)$.

From the algebraic perspective, the triplet (V, V_1, B_V) captures the necessary structure used in the distribution theory. However from a topological perspective, one has to construct topologies for the space V_1 which is not a trivial task. We assume that V_1 has the structure of a locally convex vector space and that the bilinear form is continuous. The space $\mathcal{D}'(M)$ is often equipped with the weak*-topology. We will review two basic concepts of functional analysis.

Definition 3.1 (Transpose)**.** *Let us have a linear operator $A : V \to W$. Then we can form a linear operator $A' : W' \to V'$ given by $W' \ni w \mapsto w \circ A$.*

Definition 3.2 (Adjoint)**.** *Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be inner product spaces and $A : V \to W$ be a linear operator. Then we say that an operator A^* is an adjoint operator if the following identity holds*

$$\langle Av, w \rangle_W = \langle v, A^* w \rangle_V \tag{54}$$

for all $v \in V, w \in W$. This notion can be extended to bilinear forms by replacing inner products by bilinear forms.

Almost every construction in the distribution theory involves either the adjoint or transpose of an operator. However, the terminology is not standard as some references call a transpose operator as an adjoint operator and vica versa. We use the same terminology as in Tréves' book [6, p. 240, 252].

Let us have a distribution spaces (V, V_1, B_V) and (W, W_1, B_W) and a linear operator $A : V_0 \to W_0$ where $V_0 \subset V$ and $W_0 \subset W$. We wish to extend the domain of A to V. This can be achieved in multiple ways but when we have an adjoint operator $A^* : W_0 \to V_0$ such that

$$B_W(Av, w) = B_V(v, A^* w) \tag{55}$$

holds whenever $v \in V_0$ and $w \in W_0$, then we can use the adjoint to produce an extension operator $\tilde{A} : \mathcal{D}'(V) \to \mathcal{D}'(W)$ via the formula

$$\tilde{A}(v) = v \circ A^*. \tag{56}$$

Operators A and $\tilde{A}$ are connected to each other in the following sense: Operators and embeddings commute whenever the element is in the original domain which means that operators satisfy the equation

$$\iota_W \circ A = \tilde{A} \circ \iota_V. \tag{57}$$

In other words, the diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{\quad A \quad} & W_0 \\ \downarrow{\scriptstyle \iota_v} & & \downarrow{\scriptstyle \iota_W} \\ \mathcal{D}'(M, V) & \xrightarrow{\tilde{A}} & \mathcal{D}'(M, W) \end{array}$$

commutes. Let us prove this: Given elements $v \in V_0$ and $\phi \in W_0$, we have a chain of equalities

$$\begin{aligned} \iota_W(A(v))(\phi) &= B_W(A(v), \phi) \\ &= B_V(v, A^*(\phi)) \\ &= \tilde{A}(\iota_V(v))(\phi). \end{aligned} \tag{58}$$

If the operators A, A^* are continuous, then the extension is continuous as well. Any operator $\tilde{A}$ with property $\iota_W \circ A = \tilde{A} \circ \iota_V$ is a sensible extension operator of A. For example, extension by Cauchy sequences could be also possible since spaces are locally convex vector spaces. However in this work, extension by adjoint operators is the main focus.

With distributions one can speak about weak solutions of an equation. Given an element $w \in W$, we say that $v \in V$ is a weak solution for $A(v) = w$ if $\tilde{A}(\tilde{v}) = \tilde{w}$ holds. If there exists a strong solution $A(v) = w$, then it is also a weak solution by the identity (57).

There are many possible choices in this framework. One can, for example, choose freely bilinear forms and space of test functions. So the structure is flexible and can be adjusted to different situations. The bilinear form is usually derived from the operator A using integration identities and the space of test functions is chosen accordingly. The following subsections give examples of distribution spaces.

3.2 Distribution spaces on Euclidean space

The theory of distributions is usually first developed in the space $\mathbb{R}^n$. We choose compactly supported smooth functions $C_0^\infty(\Omega)$ to be our space of test functions V_1. We will construct a topology for $C_0^\infty(\Omega)$ and its dual. This will be done via Fréchet topology and inductive limit topology.

A locally convex topological vector space X is called Fréchet if it is metrizable with a translation-invariant metric and it is complete. Every Fréchet topology can be constructed from a countable set of separating seminorms $\{p_j\}_{j\in\mathbb{N}}$ for which the induced metric

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x-y)}{1 + p_j(x-y)} \tag{59}$$

is complete. A set of seminorms is called separating if for every nonzero $v \in X$ there exists a seminorm p_j such that $p_j(v) \neq 0$. Thus, giving a locally convex vector space a countably family of separating seminorms determines a Fréchet topology. [7, p. 417,418]

An inductive limit topology is a construction based on a family of Fréchet spaces. Let us have a family of Fréchet spaces $\{X_j\}_{j\in J}$ such that for every $X_{j_1} \subset X_{j_2}$ there exist a space X_{j_3} so that $X_{j_1} \cup X_{j_2} \subset X_{j_3}$ and if $X_{j_1} \subset X_{j_2}$ then the topology on X_{j_1} is finer than the subspace topology induced by X_{j_2}. Then it is possible to construct an inductive limit topology for the union $\bigcup_{j\in J} X_j$. However we will not give the construction here. [7, p. 417,418]

We will equip the space $C_K^\infty(\Omega) = \{u \in C_0^\infty(\Omega) |\, \operatorname{supp} u \subset K\}$ with the Fréchet topology given by following seminorms

$$p_k(u) = \sup_{x \in K, |\alpha| \leq k} |\partial^\alpha u(x)|. \tag{60}$$

Let us have an increasing sequence of compact sets K_j such that $\bigcup_{k=1}^\infty K_j = \Omega$. Then the space $C_0^\infty(\Omega)$ is given the inductive limit topology induced by the sets $\{C_{K_j}^\infty(\Omega)\}_{j=1}^\infty$.

Definition 3.3. *The space of distributions $\mathcal{D}'(\Omega)$ is defined as the space of continuous linear functionals on $C_0^\infty(\Omega)$ with a weak*-topology induced by the seminorms*

$$p_\phi(u) = |u(\phi)|, \tag{61}$$

where $\phi \in C_0^\infty(\Omega)$. [8, p. 27]

The concept of convergence is important when studying topological vector spaces. We give the convergence criteria for the space of test functions and for distributions but omit the proof.

Theorem 3.4 (Convergence criteria)**.** *A sequence $\phi_k \in C_0^\infty(\Omega)$ convergences to an element ϕ if and only if there exists a compact set $K \subset \Omega$ such that* $\operatorname{supp} \phi_k, \phi \subset K$ *and we have*

$$\lim_{k \to \infty} \sup_{x \in K} |\partial^\alpha(\phi_k - \phi)(x)| = 0 \tag{62}$$

for all $\alpha \in \mathbb{N}_0^n$. [7, p. 11]

A linear functional u on the space of test functions is continuous if and only if for all compact sets $K \subset \Omega$ there exist $k \in \mathbb{N}_0$ and $C > 0$ such that

$$|\langle u, \phi \rangle| \leq C \sup_{x \in K, |\alpha| \leq k} \{|\partial^\alpha \phi(x)|\} \tag{63}$$

for all $\phi \in C_0^\infty(\Omega)$ with $\operatorname{supp}(\phi) \subset K$*. [7, p. 18]*

A sequence of distributions u_k converges to a distribution u if and only if the equation

$$\lim_{k \to \infty} \langle u_k - u, \phi \rangle = 0 \tag{64}$$

holds for all $\phi \in C_0^\infty(\Omega)$. [7, p. 26]

The distribution spaces have nice embedding properties with respect to the embeddings. Let us demonstrate this by an example. Consider sets $\Omega' \subset \Omega$. We have a canonical embedding $\iota : \mathrm{C}_0^\infty(\Omega') \to \mathrm{C}_0^\infty(\Omega)$, namely the extension by zero. So we can define an embedding $\iota^* : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega')$ via $\iota^*(f) = f \circ \iota$. This is also an example of a construction by transpose operator.

We want study partial differential operators, which are naturally defined on smooth functions $C^\infty(\Omega)$, and extend them to act on L^p-functions. So in the general framework, we have $V_0 = C^\infty(\Omega)$ and $V = L^p(\Omega)$. The embedding of $L^p(\Omega)$ to $\mathcal{D}'(\Omega)$ is given by the bilinear form

$$u \mapsto \iota_u(\phi) = \langle u, \phi \rangle = \int_\Omega u\phi \, dx. \tag{65}$$

Test functions have many useful properties with respect to this bilinear form which are straightforward to prove by using basic results of integration theory. To state these properties, let us denote $\int_\Omega fg \, dx$ by $\langle f, g\rangle$ and let $\phi, \psi, \chi \in \mathrm{C}_0^\infty(\Omega)$ and $f \in \mathrm{C}^\infty(\Omega)$. The properties are:

1. $\langle \psi + \phi, \chi\rangle = \langle \psi, \chi\rangle + \langle \phi, \chi\rangle$.
2. Let us denote multiplication by f as $\mathcal{M}_f$, then $\langle \psi, \mathcal{M}_f \, \phi\rangle = \langle \mathcal{M}_f \, \psi, \phi\rangle$.
3. The integration by parts yields $\langle \partial^\alpha \psi, \phi\rangle = \langle \psi, (-1)^{|\alpha|}\partial^\alpha(\psi)\rangle$.

We can read the adjoint operators for these basic operators from above identities and extend these operations for distributions.

Definition 3.5 (Operators)**.** *Let $f \in C^\infty(\Omega)$ and $\phi \in C_0^\infty(\Omega)$, then we define multiplication operator $\mathcal{M}_f : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ as $(\mathcal{M}_f \, u)(\phi) = u(\mathcal{M}_f \, \phi)$ and we extend ∂^α for distributions as the operator $\partial^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ given by $(\partial^\alpha u)(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi)$ where $\alpha \in \mathbb{N}_0$.*

The Euclidean structure makes possible to introduce the notion of convolution. It is an useful tool in the analysis of partial differential equations. With it and distribution theory, we can study rigorously fundamental solutions of a partial differential equation. Convolution of two $f, g \in L^1(\mathbb{R}^n)$ functions are defined as

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y) \, dy. \tag{66}$$

Convolution of two $C_0^\infty(\mathbb{R}^n)$ functions is again a smooth function. So we can search for an adjoint identity. Let us define an operator $\mathcal{C}_\phi$ as convolution by ϕ so $\mathcal{C}_\phi(f) = \phi * f$. We have following identity for $\mathcal{C}_\phi$:

$$\langle \mathcal{C}_\phi \psi, \chi\rangle = \langle \psi, \mathcal{C}_{\check{\phi}} \chi\rangle \tag{67}$$

where $\check{\phi}(x) = \phi(-x)$ [8, p. 40]. This operation is used in smoothing and approximation procedures.

To extend the notion of convolution for distributions, we need to impose additional conditions for distributions. For example, we can study distributions that have smooth functions $C^\infty(\Omega)$ as their test functions. The distribution space induced by smooth functions is denoted by $\mathcal{E}'(\Omega)$. We do not discuss the topology of the space of $\mathcal{E}'(\Omega)$. To gain more information, the reader can consult the reference [7] or a standard text on distribution theory such as [9]. Notice that the space $\mathcal{E}'(\Omega)$ is called distributions with compact support in the literature.

We can generalize convolution to a map $* : \mathcal{E}'(\Omega) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\Omega)$. We need the following theorem to make things well-defined. We denote the dual pairing with $\langle \cdot, \cdot \rangle$ in the following discussion.

Theorem 3.6. *[6, p. 284] Let us have a function $\psi(x) : \Omega \to E$ where E is a topological vector space. If ψ is a C^k-function with respect to x, then for every linear functional $T \in E^*$, the function $\langle T, \psi(x) \rangle$ is a C^k-function with respect to x and we have for any $|\alpha| \leq k$ the following identity*

$$\partial_x^\alpha \langle T, \psi(x) \rangle = \langle T, \partial_x^\alpha \psi(x) \rangle. \tag{68}$$

In our case, we take E to be $C_0^\infty(\mathbb{R}^n)$ and ψ can be thought as a function $\psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that $\psi(x, \cdot) \in C_0^\infty(\mathbb{R}^n)$ for every $x \in \Omega$. Let us have a function $\phi \in C_0^\infty(\Omega)$. We can extend ϕ by zero to a function $\tilde{\phi} \in C_0^\infty(\mathbb{R}^n)$. Let $S \in \mathcal{E}'(\Omega)$ and $T \in \mathcal{D}'(\mathbb{R}^n)$, then by Theorem 3.6, the following distribution is well-defined

$$S * T(\phi) = \langle S(x), \langle T(y), \tilde{\phi}(x+y) \rangle \rangle \tag{69}$$

where $T(y)$ denotes the distribution evaluated with respect to function of variable y and similarly for $S(x)$. By the definition of distributional derivative we have

$$\begin{aligned} \partial^\alpha (S * T)(\phi) &= \langle S(x), (-1)^{|\alpha|} \langle T(y), \partial^\alpha \tilde{\phi}(x+y) \rangle \rangle \\ &= \langle S(x), \langle \partial^\alpha T(y), \tilde{\phi}(x+y) \rangle \rangle. \end{aligned} \tag{70}$$

We have proven that $\partial^\alpha (S * T) = S * \partial^\alpha T$. With this knowledge, we can prove the following theorem.

Theorem 3.7. *Let us assume that we have a constant coefficient partial differential operator P defined on the space $\mathbb{R}^n$ and let $E \in \mathcal{D}'(\mathbb{R}^n)$ be a fundamental solution of P, that is, $PE = \delta_0$ where δ_0 is the distribution defined by $\delta_0(\phi) = \phi(0)$. Let f belong to $\mathcal{E}'(\Omega)$, then $u = f * E$ is distributional solution to $Pu = f$.*

Proof. By direct calculation, we have

$$\begin{aligned} Pu(\phi) &= P(f * E)(\phi) \\ &= f * (PE)(\phi) \\ &= f * \delta_0(\phi) \\ &= \langle f(x), \langle \delta_0(y), \tilde{\phi}(x+y) \rangle \rangle \\ &= \langle f(x), \phi(x) \rangle \\ &= \langle f, \phi \rangle \end{aligned} \tag{71}$$

which concludes the proof. □

Distributions, convolutions and Fourier theory form together a toolbox that is used to study partial differential equations via fundamental solutions. It is possible to build the fundamental solutions by taking Fourier transform of the equation $PE = \delta_0$ and searching for a suitable fundamental solution. The Fourier transform will be studied in the next section.

3.3 Distributions on vector bundles

This section is based on the reference [10]. We start by discussing seminorms for compactly supported smooth sections $\Gamma_0(M,E)$ and convergence with respect to seminorm topology.

Definition 3.8. *Let K be a compact subset of M and let s be a section in $\Gamma_0(M,E)$. The $C^k(K)$-seminorm $\|s\|_{C^k(K)}$ is defined as*

$$\|s\|_{C^k(K)} = \max_{0\leq j\leq k} \max_{x\in K} \|\nabla^j s(x)\|. \tag{72}$$

We say that a sequence $u_n \in \Gamma_0(M,E)$ is convergent to $u \in \Gamma_0(M,E)$ if there exists a compact subset K such that $\operatorname{supp}(u_n), \operatorname{supp}(u) \subset K$ *and for all $k \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \|u_n - u\|_{C^k(K)} = 0. \tag{73}$$

These seminorms define a topology on $\Gamma_0(M,E)$. A characteristic property of the topology is that a linear operator $T : \Gamma_0(M,E) \to \mathbb{R}$ is continuous if and only if $\lim_{n\to\infty} u_n = 0$ implies $\lim_{n\to\infty} T(u_n) = 0$. This topology is independent of metric and connection on E. We denote the space of $\Gamma_0(M,E)$ with above topology as $\mathcal{D}(M,E)$.

There are two suitable choices for the space of test functions, $\mathcal{D}(M,E)$ and $\mathcal{D}(M,E^*)$. If we have an metric tensor on E, then we have a natural bilinear form $L^2(M,E) \times \mathcal{D}(M,E) \to \mathbb{R}$ given by

$$\langle u, \phi \rangle = \int_M \langle u, \phi \rangle_E \ \mathrm{vol}_n . \tag{74}$$

However, the space $\mathcal{D}(M,E^*)$ has a natural bilinear form with respect to $L^2(M,E)$ which does not need any other structures than integration of scalar functions: Let $u \in L^2(M,E)$ and $\phi \in \mathcal{D}(M,E^*)$, then the bilinear form is

$$\langle u, \phi \rangle = \int_M \phi(u) \ \mathrm{vol}_n . \tag{75}$$

Both spaces $\mathcal{D}(M,E)$ and $\mathcal{D}(M,E^*)$ have suitable scalar-valued bilinear forms for distribution theory. We choose to follow the references [10],[11] and choose the test function space to be $\mathcal{D}(M,E^*)$.

Definition 3.9. *We define the space of distributions to consist of continuous linear functionals $T : \mathcal{D}(M,E^*) \to \mathbb{R}$. We denote this space as $\mathcal{D}'(M,E)$. We equip it with the weak*-topology: A sequence T_n converges to an element T if for all $\phi \in \mathcal{D}(M,E^*)$ we have $\lim_{n\to\infty} T_n(\phi) \to T(\phi)$.*

Example 3.10. Differential forms are common objects in the theory of PDEs. Many physical laws can be written using them. For example, Maxwell's law can be written with differential forms. We will introduce three distribution spaces that can be used with differential forms. Using the definition gives that the test functions are sections of $\Gamma_0((\Lambda^k(TM))^*)$ and the bilinear form is

$$\langle u, \phi \rangle = \int_M u(\phi) \operatorname{vol}_n \tag{76}$$

where $u \in \Omega^k(M)$ and $\phi \in \Gamma_0((\Lambda^k(TM))^*)$. However, the discussion in the previous section provided the notion of Hodge star which can also be used to produce a bilinear form for differential forms. Let us use elements of $\Lambda_0^k(M)$ as our test functions. In this case, the bilinear form is given with help of Hodge star as

$$\langle u, \phi \rangle = \int_M u(x) \wedge *\phi(x). \tag{77}$$

Finally, we can also use $\Lambda_0^{n-k}(M)$ as our test function space and following bilinear form

$$\langle u, \phi \rangle = \int_M u \wedge \phi. \tag{78}$$

This way to form distributions leads to the theory of currents[1] which were introduced by de Rham [12, p. 31, 33].

4 Sobolev spaces

Functions in the space $L^2(\Omega)$ can be regular or irregular with respect to differentiability. For example, functions in $C_0^\infty(\Omega)$ are highly regular but on the other hand, there exist continuous integrable nowhere differentiable functions as well. We want to study the regularity of L^2-functions. Sobolev spaces are a way to introduce regularity classes for L^2-functions.

Sobolev spaces are formed by requiring additional properties from L^p-functions or distributions. The following three conditions are characteristic properties for Sobolev spaces $W^{k,p}(\Omega)$:

- They are complete normed spaces, that is, Banach spaces.
- Sobolev spaces are nested:

$$W^{0,p}(\Omega) \supset W^{1,p}(\Omega) \supset W^{2,p}(\Omega) \supset \dots. \tag{79}$$

- The function space of compactly supported classically differentiable sections $C_0^m(X)$ is a subspace of the Sobolev space $W^{m,p}(\Omega)$, that is,

$$C_0^m(\Omega) \subset W^{m,p}(\Omega), \tag{80}$$

[1] Not related to the electrical currents

The first condition assures that the space is topologically well-behaved. The second statement says that the parameter m measures level of regularity. The latter property demands that classically differentiable functions with compact support belongs to the corresponding Sobolev space. Any sensible theory of Sobolev functions on manifolds would need fulfill these three properties.

There are many different constructions of Sobolev spaces on $\Omega \subset \mathbb{R}^n$. Each definition gives us a different way to study problems. Sobolev spaces can be divided into classical and fractional Sobolev spaces. Classical Sobolev spaces are defined by introducing a norm for smooth functions and taking a completion of the vector subspace of $C^\infty(\Omega)$ whose elements have finite Sobolev norms.

Fractional order Sobolev spaces are generalizations of classical Sobolev spaces. These generalizations introduce spaces $W^{s,p}(\Omega)$ where the order s can be a real number. A desirable property for generalization is that when $s = k$, we obtain classical Sobolev spaces $W^{k,p}(\Omega)$. There are many different ways to form these spaces and they utilize additional structures of the set Ω. We gather different constructions to a list and give references for them.

- Hajlasz-Slobodeckij spaces uses the metric and measure space structure to generalize Sobolev spaces [13].
- Interpolation theory uses Banach space structure to introduce $H^s(\Omega)$ spaces for $k < s < k+1$ as interpolation of spaces $H^k(\Omega)$ and $H^{k+1}(\Omega)$ [14].
- Fourier theoretic generalization uses Fourier transform or Fourier series to introduce $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and these spaces are sometimes called Bessel potential spaces. This construction can be extended to suitably regular open sets Ω via an extension map [8, 15].

Each construction can be used in different setting. In fact, Sobolev spaces are just one possible type of function spaces that one can study. One could study Hölder, Lipschitz, B.V., Hardy or some other space as well. But Sobolev spaces are well suited for the analysis of partial differential and pseudo-differential operators.

In this thesis, we discuss only classical Sobolev spaces and Fourier theoretic construction of fractional Sobolev spaces. These constructions can be generalized to compact manifolds. We start by studying Sobolev spaces on $\mathbb{R}^n$ which has additional structures and properties that allow us to define Sobolev spaces using simpler definitions than in the case of general manifold.

4.1 Classical Sobolev spaces on $\mathbb{R}^n$ and on open sets $\Omega \subset \mathbb{R}^n$

In the theory of partial differential equations, methods of topology and functional analysis are widely used. Topological arguments often rely on the completeness of metric spaces. To utilize these methods for smooth functions we need to introduce a norm to the space $C^\infty(\Omega)$. The natural way is to derive a suitable norm utilizing

L^p-norms. The resulting norm will be

$$\|u\|_{k,p} = \Big(\sum_{0\le|\alpha|\le k} \|\partial^\alpha u\|_p^p \Big)^{\frac{1}{p}} \tag{81}$$

and is called Sobolev norm. The subspace of $C^\infty(\Omega)$ functions, whose Sobolev norm is bounded, is transformed into a normed space. However, the resulting space is not complete. For this reason, we need to take completion. This leads to the following definition.

Definition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $k \in \mathbb{N}, p \in [1, \infty)$. The classical Sobolev space $W^{k,p}(\Omega)$ is defined as the completion of the following normed space $(\{u \in C^\infty(\Omega) \mid \|u\|_{k,p} < \infty\}, \|\cdot\|_{k,p})$.*

This is not the most used definition. The usual definition uses weak derivatives to define Sobolev spaces: A function $f \in L^p(\Omega)$ belongs to $W^{k,p}(\Omega)$ if it has partial derivatives $\partial^\alpha f$ up to $|\alpha| \le k$ where derivatives are understood as distributional derivatives. To show the equivalence of these definitions, it is enough to show that the space is complete and the compactly supported smooth functions are dense in that space. The proof will use bump functions and convolution. We gather the result into the following theorem.

Theorem 4.2. *We will use notation from (65) to represent a L^p-function as a distribution. The space $W^{k,p}(\Omega)$ is equivalent to a norm space*

$$\{u \in L^p(\Omega) \mid \forall|\alpha| \le k\ \exists g_\alpha \in L^p(\Omega)\ s.t.\ \partial^\alpha \iota_u = \iota_{g_\alpha}\} \tag{82}$$

equipped with the same norm. Especially smooth functions are dense in the space given above.

Let us show that the above space is indeed a Banach space.

Theorem 4.3. *The space*

$$\{u \in L^p(\Omega) \mid \forall|\alpha| \le k\ \exists g_\alpha \in L^p(\Omega)\ s.t.\ \partial^\alpha \iota_u = \iota_{g_\alpha}\}. \tag{83}$$

with the Sobolev norm is a Banach space.

Proof. Let $(u_n)_{n=1}^\infty$ be a Cauchy sequence. Let us study the distributional derivative $\partial^\alpha u_n$ for fixed α. By estimating other terms below by zero, we obtain following inequality

$$\|\partial^\alpha u_n\|_p \le \|u\|_{k,p} = \Big(\sum_{0\le|\alpha|\le k} \|\partial^\alpha u_n\|_p^p \Big)^{\frac{1}{p}} \tag{84}$$

thus $\partial^\alpha u_n$ is also a Cauchy sequence. Since $L^p(\Omega)$ is complete, there are limiting functions $u_n \to u, \partial^\alpha u_n \to g_\alpha$. Moreover, we have for any $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}\langle \partial^\alpha u, \phi\rangle &= \lim_{n\to\infty} \langle \partial^\alpha (u - u_n), \phi\rangle + \langle \partial^\alpha u_n, \phi\rangle \\ &= \lim_{n\to\infty} \langle \partial^\alpha (u - u_n), \phi\rangle + \langle \partial^\alpha u_n - g_\alpha, \phi\rangle + \langle g_\alpha, \phi\rangle \\ &= \langle g_\alpha, \phi\rangle.\end{aligned} \tag{85}$$

Thus the weak derivative $\partial^\alpha u$ of u exists and is g_α. So there exists a limit function $u \in W^{k,p}(\Omega)$ and thus $W^{k,p}(\Omega)$ is complete. □

The above definitions give quite abstract characterizations of Sobolev spaces. There are more concrete ways to characterize Sobolev spaces. Three possible characterizations are maximal function, difference quotient and ACL characterizations. We do not discuss them but we want to point out that those are useful while studying some problems.

Studying the regularity of Sobolev functions is important for applications. As Sobolev regularity of k increases, it would be useful to know if classical differentiability increase as well. The following theorems give tools for showing regularity in L^p and C^k-sense. The theorems that give results about the regularity of Sobolev functions are often called embedding theorems. There are two basic results: Sobolev-Gagliardo-Nirenberg inequality and Morrey's inequality. To state these result let us assume that Ω is a bounded set with smooth boundary. The Sobolev-Gagliardo-Nirenberg inequality states that

$$\|u\|_{p^*} \leq C\|\nabla u\|_p \tag{86}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. This inequality and an interpolation argument gives the inequality

$$\|u\|_q \leq C\|u\|_{1,p} \tag{87}$$

where $q \in [p, p^*]$. Morrey's inequality can be stated in the following manner: When $p > N$, we have an inequality

$$|u(x) - u(y)| \leq C\|u\|_{1,p}|x-y|^\alpha \quad a.e.\ x, y \in \Omega \tag{88}$$

where $\alpha = 1 - \frac{n}{p}$. These results are usually proven first in the space $\mathbb{R}^n$ and then they are extended to subsets $\Omega \subset \mathbb{R}^n$. This is done by proving that there exists an extension map $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ for which $\|Eu\|_{1,p} \leq C\|u\|_{1,p}$ and $(Eu)|_\Omega = u$ for almost everywhere.

The Sobolev-Gagliardo-Nirenberg inequality gives us tools to trade derivatives for higher integrability and Morrey's theorem says that when a Sobolev function is sufficiently integrable, then it is Hölder continuous. This gives a strategy to show regularity of a function: To show the regularity of a function f it is enough to show that $\partial^\alpha f$ belongs to $W^{1,q}(\Omega)$ with $q > n$. This can be shown by trading Sobolev derivatives for higher integrability and applying Morrey's inequality to $\partial^\alpha f$ to show its continuity. With these tools, we can formulate the Sobolev embedding theorem.

Theorem 4.4 (Embedding theorems)**.** *[16, p. 182] Let* Ω *be an open set of* $\mathbb{R}^n$ *and let* $1 \leq j < k$ *and* $1 \leq p, q < \infty$*.*

1. *if* $k - \frac{n}{p} \geq j - \frac{n}{q}$*. Then we have a continuous inclusion*

   $$W^{k,p}(\Omega) \to W^{j,q}(\Omega). \tag{89}$$

   Furthermore, if inequality is strict and the domain is compact set with a smooth boundary, then the inclusion map is compact.

2. if $k - \frac{n}{p} > j$, then we have a continuous inclusion

$$W^{k,p}(\Omega) \to C^j(\overline{\Omega}) \tag{90}$$

which is a compact map when the domain is a compact set with a smooth boundary.

4.2 Fractional order Sobolev spaces on $\mathbb{R}^n$

In the definition of classical Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ the number p can be any number greater or equal to 1. When we study the case $p = 2$, we can utilize Fourier analytical methods and Fourier transform. This leads to a different characterization of Sobolev spaces, different proofs and a generalization of classical Sobolev spaces. We start by reviewing basic definitions and results about Schwartz spaces and the Fourier transform. We have used the books [8, 7] and lecture notes [17] as our references on Fourier analysis and Schwartz spaces.

Definition 4.5 (Schwartz class)**.** *Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is a vector subspace of the space of smooth functions $C^\infty(\mathbb{R}^n)$ and is given by*

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \mid \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty, \forall \alpha, \beta \in \mathbb{N}_0^n\} \tag{91}$$

where we use the standard multi-index notation. The condition (91) is equivalent to

$$\forall m \in \mathbb{N}_0,\ \forall \alpha \in \mathbb{N}_0^n\ \exists C_{m,\alpha} > 0\ \ s.t.\ |\partial^\alpha \phi(x)| < C_{m,\alpha}(1 + |x|)^{-m}. \tag{92}$$

Every function of Schwartz class is also in $L^p(\mathbb{R}^n)$ for all $p \in [1, \infty]$. Furthermore, it is the most convenient space to perform Fourier analysis on the Euclidean space $\mathbb{R}^n$. This can extended to vector valued functions by requiring that components belong to Schwartz class. We denote vector-valued Schwartz class as $\mathcal{S}(\mathbb{R}^n; \mathbb{R}^k)$. Schwartz class is closed under many useful operators.

Theorem 4.6. *If $u, v \in \mathcal{S}(\mathbb{R}^n)$ and $p(\xi)$ is a polynomial on $\mathbb{R}^n$, then $uv, u * v, \partial^\alpha u$ and $p(\xi)u$ all belongs to Schwartz class and the mappings are continuous. [7, p. 94]*

Now, the Fourier transform can be defined in the Schwartz class and can be shown to be a bijection.

Definition 4.7. *The Fourier transform of $u \in \mathcal{S}(\mathbb{R}^n)$ is defined as*

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x)\, dx. \tag{93}$$

For vector-valued functions the Fourier transform is defined component-wise.

Theorem 4.8. *Let us assume that $u, v \in \mathcal{S}(\mathbb{R}^{\mathrm{n}})$, $\alpha \in \mathbb{N}_0^n$. Then Fourier transform has the following properties:*

- *the Fourier transform $\mathcal{F}(u)$ belongs also in $\mathcal{S}(\mathbb{R}^{\mathrm{n}})$.*
- *Fourier transform is a bijection $\mathcal{F} : \mathcal{S}(\mathbb{R}^{\mathrm{n}}) \to \mathcal{S}(\mathbb{R}^{\mathrm{n}})$ and the inverse is given by*

$$\mathcal{F}^{-1}(u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} u(\xi)\, d\xi. \tag{94}$$

- *Fourier transform is almost an isometry on $\mathcal{S}(\mathbb{R}^{\mathrm{n}})$, that is, the following identity holds*

$$\int_{\mathbb{R}^n} u(x)\overline{v}(x)\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}(u)\overline{\mathcal{F}(v)}(x)\, dx. \tag{95}$$

The Fourier transform can be extended to $L^2(\mathbb{R}^n)$ and these properties hold still [8, p. 97,100].

In Schwartz class, the Fourier transform has good properties with respect to derivatives and multiplying by polynomials. We gather the most important properties of the Fourier transform in the following theorem.

Theorem 4.9. *Let $u, v \in \mathcal{S}(\mathbb{R}^{\mathrm{n}})$, then following algebraic-differential identities hold*

- $\mathcal{F}(\partial^\alpha u)(\xi) = (i\xi)^\alpha\, \mathcal{F}(u)(\xi)$
- $\partial^\alpha(\mathcal{F}(u)(\xi)) = \mathcal{F}((-ix)^\alpha u)(\xi)$.

Furthermore, the convolution operator has following properties:

- $\mathcal{F}(u * v) = \mathcal{F}(u)\, \mathcal{F}(v)$
- $\mathcal{F}(uv) = (2\pi)^{-n}\, \mathcal{F}(u) * \mathcal{F}(v)$.

One possible way to motivate the definition of Schwartz space is to find a space where this theorem is true.

We can define distributions with Schwartz space as the space of test functions. The distribution space will be called the space of tempered distributions and is denoted by $\mathcal{S}'(\mathbb{R}^n)$. The space $L^p(\mathbb{R}^n)$ can be embedded into $\mathcal{S}'(\mathbb{R}^n)$ by the standard inclusion map $f \mapsto \langle f, \phi\rangle = \int_{R^n} f(x)\phi(x)\, dx$. Observe that we do not include complex conjugate in the integral. The usual operations can be extended for tempered distributions as long as we check that required adjoint identities hold true. Especially we are interested in the adjoint of Fourier transform.

Theorem 4.10. *[7, p. 119] Let $u \in L^2(\mathbb{R}^n)$ and $v \in \mathcal{S}(\mathbb{R}^{\mathrm{n}})$, then we have the following identity for the Fourier transform*

$$\langle \mathcal{F}\, u, v\rangle = \langle u, \mathcal{F}\, v\rangle. \tag{96}$$

With this identity, we can readily define the Fourier transform of tempered distributions and it will be used in the definition of fractional Sobolev spaces.

Definition 4.11. *Let $S \in \mathcal{S}'(\mathbb{R}^n)$, then we define the Fourier transform of S as*

$$\mathcal{F}(S)(\phi) = S(\mathcal{F}(\phi)). \tag{97}$$

The isometry property of the Fourier transform, $\langle u, \overline{v}\rangle = (2\pi)^{-n}\langle \mathcal{F}\, u, \overline{\mathcal{F}\, v}\rangle$, is often called the Parseval-Plancherel theorem. It provides an important connection between classical Sobolev spaces and Fourier analysis as following propositions will show.

Proposition 4.12. *Let us have a function $u \in H^k(\mathbb{R}^n)$, then for every $|\alpha| \leq k$*

$$\mathcal{F}(\partial^\alpha u)(\xi) = (-i\xi)^\alpha\, \mathcal{F}(u)(\xi). \tag{98}$$

Proof. Since by the definition $u \in H^k(\mathbb{R}^n)$ belongs to $L^2(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^{\text{n}})$, we can extend u to the space $\mathcal{S}'(\mathbb{R}^n)$. Let $|\alpha| \leq k$ and $\partial^\alpha u = g_\alpha$, then by the definition of the weak derivative we have

$$\begin{aligned} \langle g_\alpha, \overline{\phi}\rangle_2 &= \langle u, (-1)^{|\alpha|}\partial^\alpha \overline{\phi}\rangle_2 \\ \iff \langle \mathcal{F}(g_\alpha), \overline{\mathcal{F}(\phi)}\rangle_2 &= \langle \mathcal{F}(u), \overline{(-1)^{|\alpha|}\mathcal{F}(\partial^\alpha \phi)}\rangle_2 \\ \iff \langle \mathcal{F}(g_\alpha), \overline{\mathcal{F}(\phi)}\rangle_2 &= \langle \mathcal{F}(u), (-1)^{|\alpha|}\overline{(i\xi)^\alpha\, \mathcal{F}(\phi)}\rangle_2 \\ \iff \langle \mathcal{F}(g_\alpha) - (i\xi)^\alpha\, \mathcal{F}(u), \overline{\mathcal{F}(\phi)}\rangle_2 &= 0. \end{aligned} \tag{99}$$

This holds for every $\phi \in \mathcal{S}(\mathbb{R}^{\text{n}})$. Since the Fourier transform maps $\mathcal{S}(\mathbb{R}^{\text{n}})$ bijectively to $\mathcal{S}(\mathbb{R}^{\text{n}})$, $\mathcal{F}(\phi)$ attains all elements of $C_0^\infty(\mathbb{R}^n)$. Therefore, we can apply the Du Bois-Reymond lemma and it shows that $\mathcal{F}(g_\alpha) - (i\xi)^\alpha\, \mathcal{F}(u) = 0$ which concludes the proof. ☐

When the weak derivative exists, then we know that $(i\xi)^\alpha\, \mathcal{F}(u)$ belongs to $L^2(\mathbb{R}^n)$, that is, $\xi^{2\alpha}|\,\mathcal{F}(u)|^2$ is integrable. Now the weak derivative could be defined via (98). This raises a question: When this can be done and what is the necessary and sufficient condition for $\xi^{2\alpha}|\,\mathcal{F}(u)|^2$ to be integrable for all $|\alpha| \leq k$? To answer this question, we will need the following lemma.

Lemma 4.13. *Let us define an auxiliary function*

$$\langle \xi \rangle = (1 + \|\xi\|^2)^{\frac{1}{2}}. \tag{100}$$

Then we will have inequalities for any $|\alpha| \leq k$:

$$\xi^{2\alpha} \leq \sum_{|\beta|\leq k} \xi^{2\beta} \leq \langle \xi\rangle^{2k} \leq C \sum_{|\beta|\leq k} \xi^{2\beta}. \tag{101}$$

Proof. Using multinomial identity, we can write $\langle \xi\rangle^{2k}$ as

$$\langle \xi\rangle^{2k} = (1 + |\xi|^2)^k = \sum_{|\beta|\leq k} C_\beta \xi^{2\beta} \tag{102}$$

where C_β is the multinomial coefficient $\frac{k!}{\beta!(k-|\beta|)!}$. The inequalities will follow when we use estimates $1 \le C_\beta \le C = \max_{|\beta|\le k} C_\beta$. $\square$

Using this lemma, the answer to the above question can be read from the following theorem.

Theorem 4.14. *The following equivalence is true:*

$$u \in H^k(\mathbb{R}^n) \text{ if and only if } \int_{\mathbb{R}^n} \langle\xi\rangle^{2k} |\mathcal{F}(u)(\xi)|^2 \, d\xi < \infty. \tag{103}$$

Proof. Let us assume that weak derivatives exists for $|\alpha| \le k$, then by (98) we have that $(i\xi)^\alpha \mathcal{F}(u)$ is the Fourier transform of the weak derivative and thus $\int_{\mathbb{R}^n} \xi^{2\alpha} |\mathcal{F}(u)|^2(\xi) d\xi$ is finite so by (101) we have

$$\int_{\mathbb{R}^n} \xi^{2\alpha} |\mathcal{F}(u)|^2 \, d\xi \le \int_{\mathbb{R}^n} \langle\xi\rangle^{2k} |\mathcal{F}(u)(\xi)|^2 \, d\xi \le C \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} \xi^{2\alpha} |\mathcal{F}(u)|^2 d\xi < \infty. \tag{104}$$

Now, let us assume that $\int_{\mathbb{R}^n} \langle\xi\rangle^{2k} |\mathcal{F}(u)|^2 \, d\xi$ exists. The above inequality proves that $(i\xi)^\alpha \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^n)$ so we can define the weak derivative by taking the inverse Fourier transform in (98). $\square$

This theorem motivates following definition of space $H^s(\mathbb{R}^n)$.

Definition 4.15. *Let $s \in \mathbb{R}$, then the Sobolev space $H^s(\mathbb{R}^n)$ is defined to be the set*

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)\}. \tag{105}$$

This space can be equipped with the inner product

$$\langle u, v\rangle_s = (2\pi)^{-n} \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \mathcal{F}(u)(\xi) \overline{\mathcal{F}(v)(\xi)} \, d\xi. \tag{106}$$

We want to extend fractional Sobolev spaces to subsets $\Omega \subset \mathbb{R}^n$. There are at least two ways to construct fractional Sobolev spaces for open sets: We can define $H^s(\Omega)$ as a suitable restriction of $H^s(\mathbb{R}^n)$ given by

$$H^s(\Omega) = \{u \in L^2(\Omega) \mid \exists v \in H^s(\mathbb{R}^n) : v|_\Omega = u\}. \tag{107}$$

We can also define fractional local Sobolev spaces by using cut-off functions. We define $H^s_{\text{loc}}(\Omega)$ as

$$H^s_{\text{loc}}(\Omega) = \{u \in L^2(\Omega) \mid \forall \phi \in C_0^\infty(\Omega) : \phi u \in H^s(\mathbb{R}^n)\}. \tag{108}$$

We can extend the regularity theorem for $H^s(\mathbb{R}^n)$.

Theorem 4.16. *[18, p. 215] Let us assume that* $s > \frac{n}{2} + k$*, then* $H^s(\mathbb{R}^n)$ *can be continuously embedded to* $C^k(\mathbb{R}^n)$ *which is equipped with* C^k*-norm. Furthermore, when* Ω *has a k-extension propety, that is,* $H^k(\Omega)$ *can be embedded to* $H^k(\mathbb{R}^n)$*, then if* $k > \frac{n}{2} + j$ *we have* $H^k(\Omega) \subset C^j(\overline{\Omega})$*.*

The theory that extends Fourier analytical methods to $L^p(\mathbb{R}^n)$ spaces is called L^p-multiplier and Littlewood-Paley theory. The main idea in those theories is to study the question: For what multipliers $m(\xi)$ the function $\mathcal{F}^{-1}(m(\xi)\,\mathcal{F}(u)(\xi))$ is L^p-continuous when $u \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is equipped with L^q-norm?

We will end this subsection with two examples of how Fourier analysis and distribution theory can be applied to PDEs.

Example 4.17. Let us study the heat equation on $\mathbb{R}^n \times (0, \infty)$ with initial values in the set $\{(x, 0) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n\}$. The equation is

$$\begin{cases} \partial_t f(x,t) = \Delta_x f(x,t) \\ f(x,0) = h(x). \end{cases} \tag{109}$$

When we take Fourier transform of the equation with respect to x variable and denote $\mathcal{F}_x(f) = \hat{f}_x$, we obtain following ordinary differential equation for f_x

$$\begin{cases} \partial_t \hat{f}_x(\xi,t) = -\|\xi\|^2 \hat{f}_x(\xi,t) \\ \hat{f}_x(\xi,0) = \hat{h}(\xi). \end{cases} \tag{110}$$

The solution of this equation is given by $\hat{f}_x(\xi,t) = \hat{h}(\xi)e^{-t\|\xi\|^2}$. Taking the inverse Fourier transform we notice that the solution $f(x,t)$ is obtained as a convolution $f(x,t) = h(x) * \mathcal{F}_x^{-1}(e^{-t\|\xi\|^2})$. Now the inverse Fourier transform can be calculated from knowing that $\mathcal{F}(e^{\frac{-\|x\|^2}{2}}) = (2\pi)^{\frac{n}{2}} e^{\frac{-\|\xi\|^2}{2}}$. Now

$$\begin{aligned} \mathcal{F}_x^{-1}(e^{-t\|\xi\|^2}) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-t\|\xi\|^2}\, d\xi \\ &= \frac{1}{(2t)^{\frac{n}{2}}} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle \frac{-x}{\sqrt{2t}}, z\rangle} e^{\frac{-\|z\|^2}{2}}\, dz \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-\|x\|^2}{4t}}. \end{aligned} \tag{111}$$

We get that $f(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} h(y) e^{\frac{-\|x-y\|^2}{4t}}\, dy$ is a solution for the heat equation.

Example 4.18. This example is based on the reference [7, p.218]. Fourier analytical methods are an effective way to calculate fundamental solutions of partial differential equations. We demonstrate this via calculating the fundamental solution E for the Laplace operator when the dimension of the Euclidean space is $n \geq 3$. The distribution E fulfills the equation $\Delta E = \delta$. Taking the Fourier transform of both sides of equation, we obtain the equation $-\|\xi\|^2 \hat{E} = 1$. So $\hat{E} = \frac{1}{\|\xi\|^2}$. Finding the inverse transform requires more tools: If a function $f(x)$ is homogeneous of degree d, then the Fourier transform $\hat{f}$ is homogeneous of degree $-n-d$. This can be seen from following calculations

$$\begin{aligned}
\mathcal{F}(f)(\lambda\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\lambda\xi\rangle} f(x)\,dx \\
&= \int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle} f(\frac{z}{\lambda})\lambda^{-n}\,dz \\
&= \lambda^{-n-d} \int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle} f(z)\,dz \\
&= \lambda^{-n-d}\,\mathcal{F}(f)(\xi).
\end{aligned} \tag{112}$$

The second tool that we need is rotational invariance. If the function is rotation invariant, then the Fourier transform is also rotation invariant. This is shown with a similar calculation:

$$\begin{aligned}
\mathcal{F}(f)(R\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,R\xi\rangle} f(x)\,dx \\
&= \int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle} f((R^*)^{-1}z)|\det((R^*)^{-1})|\,dz \\
&= \int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle} f(z)\,dz \\
&= \mathcal{F}(f)(\xi).
\end{aligned} \tag{113}$$

Let us calculate the Fourier transform of $f(x) = \|x\|^{-d}$ for $d \in (0,n)$.

$$\begin{aligned}
\mathcal{F}(f)(\xi) &= \mathcal{F}(f)\left(\frac{\|\xi\|\xi}{\|\xi\|}\right) \\
&= \|\xi\|^{d-n}\,\mathcal{F}(f)\left(\frac{\xi}{\|\xi\|}\right) \\
&= \|\xi\|^{d-n}\,\mathcal{F}(f)(R(0,0,\ldots,1)) \quad \text{for some rotation matrix } R \\
&= \|\xi\|^{d-n}\,\mathcal{F}(f)(0,0,\ldots,1) \\
&= c_d\|\xi\|^{d-n}
\end{aligned} \tag{114}$$

where $c_d = \mathcal{F}(f)((0,0,\ldots,1))$. When we impose that $d-n=-2$ we obtain that $d=n-2$ so the inverse Fourier transform is form of

$$\mathcal{F}^{-1}(\|\xi\|^{-2})(x) = \frac{1}{c_{n-2}}\|x\|^{2-n} \tag{115}$$

so the fundamental solution is $E(x) = \frac{1}{c_{n-2}}\|x\|^{2-n}$. We will omit calculation of c_{n-2}.

To justify the use of the Fourier transforms, we need to show that $\|x\|^{-2}$ is in $\mathcal{S}'(\mathbb{R}^n)$ for $n \geq 3$. When $\phi \in \mathcal{S}(\mathbb{R}^n)$, then we have

$$\int_{\mathbb{R}^n} \|x\|^{-2}\phi(x)\,dx = \int_{B(0,1)} \|x\|^{-2}\phi(x)\,dx + \int_{\mathbb{R}^n\setminus B(0,1)} \|x\|^{-2}\phi(x)\,dx. \tag{116}$$

The second term is observed to be finite when we notice that $\frac{1}{\|x\|^2} \leq 1$ and that any Schwartz function is integrable. The first integral can be estimated using polar coordinates which leads to an estimator $C(\max_{x\in B(0,1)} |\phi(x)|) \int_0^1 \rho^{(n-1)-2}\, d\rho$ which is finite for $n \geq 3$. The case $n = 2$ needs different methods and we refer the reader to look it up for example from [7, p. 218].

Fourier analytical methods can be used similarly to calculate fundamental solutions of other partial differential equations. Moreover, one can prove more general statements about fundamental solutions of PDEs, namely the Malgrange-Ehrenpreis theorem and its following extension.

Theorem 4.19. *[7, p. 198] Every nonzero constant coefficient partial differential operator P on $\mathbb{R}^n$ has a fundamental solution $E \in \mathcal{S}'(\mathbb{R}^n)$ such that $P(E) = \delta$ in $\mathcal{S}'(\mathbb{R}^n)$.*

4.3 Sobolev spaces $H^k(M, E)$ on vector bundles

There are at least two ways to define Sobolev space on vector bundles: The coordinate-free way to define Sobolev spaces is to define them as the completion of $\Gamma(E)$ with respect to Sobolev norm. The second definition is a local definition which is based on coordinate invariance of Sobolev spaces on open sets $\Omega \subset \mathbb{R}$. We give both definitions in this subsection.

Definition 4.20. *Let u be a section of $\Gamma_0(E)$. Then we define $W^{k,p}$-Sobolev norm as*

$$\|u\|_{k,p} = (\sum_{j\leq k} \|\nabla^j u\|_p)^{\frac{1}{p}}. \tag{117}$$

With this norm, we can introduce the Sobolev space $W^{k,p}(M, E)$. Furthermore, we can introduce the space of sections with continuous k-th derivative $C^k(M, E)$.

Definition 4.21. *The Sobolev space $W^{k,p}(M, E)$ is defined as the completion of the set $\{u \in \Gamma_0(E) \mid \|u\|_{k,p} < \infty\}$ with respect of the Sobolev norm $\|\cdot\|_{k,p}$.*

The space $C^k(M, E)$ is defined as the set of continuous sections u such that for every $j \leq k$, function's j-th covariant derivative $\nabla^j u$ exists, is continuous and its L^∞-norm $\|\nabla^k u\|_\infty$ is finite where the L^∞-norm is $\|u\|_\infty = \sup_{x\in M} \|u(x)\|_E$. This space is equipped with norm

$$\|u\|_{j,\infty} = \max_{k\leq j} \|\nabla^k u\|_\infty. \tag{118}$$

This definition has the benefit that it is similar to the definition of L^p spaces and it uses only the method of completing a normed space. Furthermore, the definition uses only coordinate-free operations of manifolds and, thus, gives a coordinate-free way to define Sobolev norm. One can prove that this definition is equivalent with the following local definition.

Definition 4.22 (Local Sobolev spaces)**.** *Let M be a compact manifold and $\pi : E \to M$ a vector bundle of dimension n. Let us have a cover $M = \bigcup_{i=1}^{N} U_k$ such that there exist charts $\Phi_k : \pi^{-1}(U_k) \to V_k \times \mathbb{R}^n$ on the bundle E. As every section $s \in \Gamma(E)$ can represented locally as linear combination of basis sections, we can study push-forward section $(D\Phi_k)s : \Gamma(\pi^{-1}(U_k)) \to \Gamma(V_k \times \mathbb{R}^n)$ which are defined by the local representation $x \mapsto ((D\Phi_i s)_1(x), (D\Phi_i s)_2(x), \ldots, (D\Phi_i s)_n(x))$. We can define a Sobolev $W^{k,p}$-norm as*

$$\|s\|_{W^{k,p}(M,E)} = \sum_{i=1}^{N} \sum_{j=1}^{n} \|(D\Phi_i s)_j\|_{W^{k,p}(V_i)}. \tag{119}$$

A section s belongs to $W^{k,p}(M,E)$ if its above norm $\|s\|_{W^{k,p}(M,E)}$ is finite.

Theorem 4.23. *[16, p.181] Different atlases of M produce topologically equivalent local Sobolev norms. Furthermore, these norms are also equivalent to the coordinate-free norm. So the coordinate-free definition and local definition produce the same Sobolev space. Especially a section of E belongs to a Sobolev space if and only if for every chart, the components belong to the corresponding Sobolev space.*

This theorem implies that we can prove results for Sobolev spaces of vector bundles by proving them for scalar functions and then extend results to sections by using partition of unity. The Sobolev embedding theorem is an example of this.

Theorem 4.24. *[16, p. 182] Let E be a vector bundle over a compact manifold M. Let $j < k$ and $1 \leq p, q < \infty$.*

1. *if $k - \frac{n}{p} \geq j - \frac{n}{q}$, then we have a continuous inclusion*

$$W^{k,p}(M,E) \to W^{j,q}(M,E). \tag{120}$$

   Furthermore, if the inequality is strict, then the inclusion map is compact.

2. *if $k - \frac{n}{p} > j$, then we have a continuous inclusion*

$$W^{k,p}(M,E) \to C^j(M,E) \tag{121}$$

   which is a compact map.

These embeddings are often used to prove regularity results for solutions of equation $Pu = f$. To show the regularity, a common strategy is to prove that the operator has the property

$$Pu \in W^{k,p}(M;E) \implies u \in W^{k+l,p}(M;E). \tag{122}$$

Then the smoothness of solution on a compact base manifold M follows immediately when we notice that $\Gamma(E) = \bigcap_{k \in \mathbb{N}} W^{k,p}(M;E)$ which is consequence of the Sobolev embedding theorem. Thus if $f \in \Gamma(E)$ then $u \in \Gamma(E)$.

4.4 Fourier analysis and $H^s_{loc}(M, E)$ spaces on vector bundles

In the manifold case, we lack the vector space structure that we used to define the Fourier transform on L^2-spaces over the space $\mathbb{R}^n$. However, we get an inner product space structure to our use when we consider sections over the tangent bundle and the cotangent bundle. To use this, we need to lift functions from the manifold to the tangent bundle. We start by defining a micro-local lift and then the Fourier transform between the tangent bundle and the cotangent bundle. This section is based on the article [19].

Throughout this subsection, let assume that (M, g) be a Riemannian manifold, E and F are vector bundles over M with metric tensors $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_F$ and s is a section of $\Gamma(E)$. We need following technical lemma.

Lemma 4.25. *Let us denote the tangent bundle as $\rho : TM \to M$. There is a neighbourhood W of the zero section of TM such that the function (ρ, exp_M) maps W diffeomorphically into neighbourhood of diagonal of $M \times M$. Then there exists a smooth function $\psi : TM \to [0, 1]$ such that $\operatorname{supp}(\psi) \subset W$, $\operatorname{supp}(\psi(x, \cdot))$ is compact for every $x \in M$ and $\psi|_{\tilde{W}} = 1$ for some open set $\tilde{W}$ including the zero section of TM. The function ψ is called a cut-off function.*

With a cut-off function and the parallel transport $\tau_{\gamma(1),\gamma(0)} : E_{\gamma(0)} \to E_{\gamma(1)}$ we can define a microlocal lift as follows.

Definition 4.26. *The micro-local lift of f is given by*

$$f \mapsto f^{\psi}(v) = \psi(v)\tau^{-1}_{\exp(v)} f(\exp(v)), \text{ for any } v \in W \tag{123}$$

and extended by zero for $TM \setminus W$.

We get the original function back when we evaluate the microlocal lift at the zero section of the tangent bundle. So we do not lose information when we lift a function to a tangent bundle. To study function's local properties it is enough to consider properties of the micro-local lift at each fiber. We can now introduce the Fourier transform on vector bundles. We need the following functions space in the theory of Fourier transforms.

Definition 4.27. *The space of smoothing symbols $\mathrm{S}^{-\infty}(TM, E)$ over the tangent bundle is defined as a set of smooth functions $a : TM \to E$ such that for every open set $U \subset M$ and every open set $V \subset \phi(U)$, such that trivializations exist, the following property holds: For every trivialization $\Psi : E|_V \to V \times \mathbb{R}^N$, compact set $K \subset V$, $\alpha, \beta \in \mathbb{N}^d$ and $\mu \in \mathbb{R}$ there exists a constant $C = C_{\alpha,\beta,K,\mu} > 0$ such that the following inequality holds in K:*

$$\|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial \zeta^\beta}\Psi(a(\xi))\| < C(1 + \|\xi\|)^\mu \tag{124}$$

where $\xi \in \rho^{-1}(\phi^{-1}(K))$.

A smoothing symbol restricted to every fiber T_xM belongs to the Schwartz class. So it is useful to define smoothly varying tempered distributions on the tangent bundle $\mathrm{S}^{-\infty}(TM, E)^*$ as functions which relate to every point $x \in M$ a distribution $u \in \mathcal{S}'(T_xM)$ such that $u(f) \in C^\infty(M)$ for every $f \in \mathrm{S}^{-\infty}(TM, E)$.

Notice that we could have replaced the tangent bundle with the cotangent bundle in definitions and the result above. Thus, all definitions can be given also in the case of the cotangent bundle. We need a notion of L^2-space on a fiber. We say that a section $u : TM \to E$ belongs to $L^2_x(TM, E)$ if the integral

$$\int_{T_xM} \langle u(x,\xi), u(x,\xi)\rangle_E \, d\xi \tag{125}$$

is finite for all $x \in M$ where the measure will be given by the volume form associated to the metric's matrix representation at point x. We have the following useful lemma about L^2-spaces.

Lemma 4.28. *Let* $f \in L^2(E)$ *then* $f^\psi(v) \in L^2_x(TM, E)$.

Proof. Let $v \in T_xM$ and let us denote $\psi(x, v)$ by $\phi(v)$, thus $f^\psi(v) = \phi(v)\tau^{-1}_{\exp(v)} f(\exp(v))$. Since parallel transport preserves the inner product, we have

$$\langle \phi(v)\tau^{-1}_{\exp(v)} f(\exp(v)), \phi(v)\tau^{-1}_{\exp(v)} f(\exp(v))\rangle = \phi^2(v)\langle f(\exp(v)), f(\exp(v))\rangle \tag{126}$$

and it is enough to consider $\int_{T_xM} \phi^2(v)\langle f(\exp(v)), f(\exp(v))\rangle_E \, dv$. Since $\mathrm{supp}(\phi)$ is compact, the local presentation of volume form in normal coordinates, g, obtains a minimum $C > 0$ on the set $V = \exp(\mathrm{supp}(\phi))$ where g is determined by $\mathrm{vol}(M) = g dz_1 dz_2 \ldots dz_n$. Let us denote the image of the set V under normal coordinates by V'. Now the estimate

$$\int_V \langle f(x), f(x)\rangle \, \mathrm{vol}_M(x) \geq C \int_{V'} \langle f(z), f(z)\rangle \, dz_1 \, dz_2 \ldots \, dz_n \tag{127}$$

holds and thus $\int_{V'} \langle f(z), f(z)\rangle \, dz_1 \, dz_2 \ldots \, dz_n$ is finite. The claim follows when we expand $\int_{T_xM} \phi^2(v)\langle f(\exp(v)), f(\exp(v))\rangle_E \, dv$ in normal coordinates and use the estimate $|\phi(v)| < M$ for some M:

$$\int_{T_xM} \phi^2(v)\langle f(\exp(v)), f(\exp(v))\rangle_E \, dv \leq M^2 \int_{V'} \langle f(z), f(z)\rangle \, dz_1 \, dz_2 \ldots \, dz_n. \tag{128}$$

□

Definition 4.29. *The Fourier transform over the tangent bundle is a mapping* $\mathcal{F} : \mathrm{S}^{-\infty}(TM, E) \to \mathrm{S}^{-\infty}(T^*M, E)$ *given by*

$$u \mapsto \hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{T_{\pi(\xi)}M} e^{-i\langle \xi, v\rangle} u(v) dv. \tag{129}$$

This operator has an inverse $\mathcal{F}^{-1} : \mathrm{S}^{-\infty}(TM, E) \to \mathrm{S}^{-\infty}(T^*M, E)$ *given by*

$$\hat{u} \mapsto u(v) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{T^*_{\pi(v)}M} e^{i\langle \xi, v\rangle} \hat{u}(\xi) d\xi. \tag{130}$$

The Fourier transform depends only on the fiber. Therefore, we can extend the Fourier transform to an operator $\mathcal{F} : L^2_x(TM, E) \to L^2_x(T^*M, E)$ *and this is well-defined by properties of the Fourier transform on the space* $\mathbb{R}^n$.

We define the Fourier transform $\mathcal{F} : \mathrm{S}^{-\infty}(TM, E)^* \to \mathrm{S}^{-\infty}(T^*M, E)^*$ on smoothly varying distribution as $\mathcal{F}(u)(f) = u(\overline{\mathcal{F}^{-1}(f)})$ where $f \in \mathrm{S}^{-\infty}(T^*M, E)$. Since we study the Fourier transform fiber-wisely, this definition makes sense and the adjoint property follows from the adjoint property on the fiber over the point x. We can define a fiber-wise lift of $\mathcal{D}'(M, E)$ to the space $\mathrm{S}^{-\infty}(TM, E)^*$. Let us fix a point $x \in M$, then we push locally a function $f \in \mathrm{S}^{-\infty}(TM, E)$ to the base manifold via

$$f_{\psi,x}(y) = \psi(\exp_x^{-1}(y)) f(y, 0) \tag{131}$$

where ψ is a cut-off function. The function $f_{\psi,x}$ belongs to the space $\Gamma_0(M, E)$ so we can use it as a test function. We can define the local lift as $u^{\psi}(f)(x) = u(f_{\psi,x})$. This makes possible for us to use the Fourier transform on distributions $\mathcal{D}'(M, E)$.

With the Fourier transform on a vector bundle, we can define fractional Sobolev spaces on a vector bundle using the Fourier transform.

Definition 4.30. *Sobolev spaces*

(i) Let $s \geq 0$*, we define* $H^s_{\mathrm{loc}}(M, E)$ *to be the set*

$$\begin{aligned} H^s_{\mathrm{loc}}(M, E) = &\{u \in L^2(M, E) \mid \langle \xi \rangle^s \mathcal{F}(u^{\psi})(\xi) \in L^2_{\xi}(T^*M, E) \\ &\text{for all cut-off functions } \psi\} \end{aligned} \tag{132}$$

(ii) And more generally we can set $s \in \mathbb{R}$ *and define* $H^s_{\mathrm{loc}}(M, E)$ *as*

$$\begin{aligned} H^s_{\mathrm{loc}}(M, E) = &\{u \in \mathcal{D}'(M, E) \mid \langle \xi \rangle^s \mathcal{F}(u^{\psi})(\xi) \in L^2_{\xi}(T^*M, E) \\ &\text{for all cut-off functions } \psi\} \end{aligned} \tag{133}$$

(iii) When the base manifold M *is compact, then the Sobolev space can be defined as follows. Let* $\{B(x_j, r_j)\}_{j=1}^N$ *be a finite cover of* M *such that* $\psi(\exp_{x_j}^{-1}(y))$ *is positive for every* $y \in B(x_j, r_j)$*, then let us define* $H^s(M, E)$ *as*

$$H^s(M, E) = \{u \in L^2(M, E) \mid \langle \xi \rangle^s \mathcal{F}(u^{\psi})(x_j, \cdot) \in L^2(T^*_{x_j}M)\} \tag{134}$$

and define a norm $\|u\| = \sum_{j=1}^{N} \|\langle \xi \rangle^s \mathcal{F}(u^{\psi})(x_j, \cdot)\|_{L^2_{\xi}}$ *and the associated inner product* $\langle u, v\rangle = \sum_{j=1}^{N} \langle \xi \rangle^s \mathcal{F}(u^{\psi})(x_j, \cdot), \langle \xi \rangle^s \mathcal{F}(v^{\psi})(x_j, \cdot)\rangle_{L^2_{\xi}}$.

5 Partial differential operators

A linear partial differential operator of degree k on the space $\mathbb{R}^n$ is defined as a linear operator which can be written in the form of

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \tag{135}$$

A natural way to extend partial differential operators to vector bundles is to require operators to look like ordinary partial differential operators on every chart. There are also two other ways to define partial differential operators: Definition based on Peetre's theorem and an algebraic definition. In this section, we give all three definitions.

Let E, F be vector bundles with $\dim(E) = k$ and $\dim(F) = m$. Let s be a section of E and $\sum_{j=1}^{n} a^j(x) s_j(x)$ be its local representation. For an arbitrary linear operator $P : \Gamma(E) \to \Gamma(F)$, we obtain

$$P(\sum_{j=1}^{n} a^j(x) s_j(x)) = \sum_{j=1}^{k} P(a^j(x) s_j(x)). \tag{136}$$

We need to take account that E and F have different bases so the most general linear transformation rule available is

$$P(a^l(x) s_l(x)) = \sum_{i=1}^{m} P_{li}(a^l(x)) \tilde{s}^i(x) \tag{137}$$

where P_{li} is linear operator on $\Omega \subset \mathbb{R}^n$ and $\tilde{s}^j$ is another local basis. We get the following definition.

Definition 5.1 (Partial differential operators in local coordinates)**.** *We say that P is a partial differential operator $\Gamma(E) \to \Gamma(F)$ if for all local bases $s^j(x)$ and $\tilde{s}^j(x)$ we have*

$$P(\sum_{i=1}^{k} a^i(x) s_i(x)) = \sum_{i=1}^{k} \sum_{j=1}^{m} P_{ij}(a^i(x)) \tilde{s}^j(x). \tag{138}$$

where P_{ij} are ordinary partial differential operators. If we represent coefficients of local basis as column vectors, then the definition can be written as

$$P\big(\sum_{i=1}^{k} a^i(x) s_i(x)\big) = \begin{bmatrix} P_{11} & \dots & P_{1k} \\ \vdots & \ddots & \vdots \\ P_{m1} & \dots & P_{mk} \end{bmatrix} \begin{bmatrix} a^1(x) \\ \vdots \\ a^k(x) \end{bmatrix}. \tag{139}$$

This definition is convenient for local calculations and looks similar to the Euclidean case. To check that this definition is coordinate invariant, we need to introduce the change of variable formula. We show how to derive it.

Let U and V be charts on M and let $P : \Gamma(\tilde{\mathrm{U}}, \mathbb{R}^k) \to \Gamma(\tilde{\mathrm{U}}, \mathbb{R}^m)$ be a partial differential operator. Then the natural constrain for $\tilde{P} : \Gamma(\tilde{\mathrm{V}}, \mathbb{R}^k) \to \Gamma(\tilde{\mathrm{V}}, \mathbb{R}^m)$ is that the diagram

$$\begin{array}{ccc} \Gamma(\tilde{\mathrm{V}}, \mathbb{R}^k) & \xrightarrow{D\Phi_{\tilde{\mathrm{U}}\,\tilde{\mathrm{V}}}} & \Gamma(\tilde{\mathrm{U}}, \mathbb{R}^k) \\ \big\downarrow \tilde{P} & & \big\downarrow P \\ \Gamma(\tilde{\mathrm{V}}, \mathbb{R}^m) & \xrightarrow{D\Phi_{\tilde{\mathrm{U}}\,\tilde{\mathrm{V}}}} & \Gamma(\tilde{\mathrm{U}}, \mathbb{R}^m) \end{array}$$

commutes. Thus, the operator is given by $\tilde{P} = D\Phi^{-1}_{\tilde{\mathrm{U}}\,\tilde{\mathrm{V}}} \circ P \circ D\Phi_{\tilde{\mathrm{U}}\,\tilde{\mathrm{V}}}$. From this, we can read that the pull-back operator is also a partial differential operator.

Definition 5.2. *The degree of a partial differential operator is the largest degree among degrees of P_{ij} of all local representations.*

We see from the change of variables formula that the degree of the partial differential operator is independent of choice of charts. Furthermore, the degree has the following property.

Proposition 5.3. *Let $f \in C^\infty(M)$, P be a partial differential operator of degree k and s be section, then the operator*

$$[P, f](s) = P(fs) - fP(s) \tag{140}$$

is a partial differential operator and degree $k - 1$.

Proof. By the linearity of operators, it is enough to show this for an operator consisting only of $a_\alpha(x)\partial^\alpha$:

$$\begin{aligned} a_\alpha(x)\partial^\alpha(fs) - a_\alpha(x)f(x)\partial^\alpha s &= a_\alpha(x)\left(\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta f\partial^{\alpha-\beta}s - f\partial^\alpha s\right) \\ &= a_\alpha(x)\left(\sum_{1\le\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta f\partial^{\alpha-\beta}s\right) \end{aligned} \tag{141}$$

since the inequality $|\alpha - \beta| \le k - 1$ holds in the sum, we have shown the claim. □

This gives motivation for the algebraic definition. We define a sequence of operator spaces $\mathrm{PDO}^k(E, F)$. We do this inductively. Let

$$\mathrm{PDO}^0(E, F) = \{P \in \mathrm{Hom}(E, F) | [P, u] = 0 \ \forall u \in C^\infty(M)\} \tag{142}$$

and then we define a space $PDO^k(E, F)$ as

$$PDO^k(E, F) = \{P \in \mathrm{Hom}(E, F) | [P, u] \in \mathrm{PDO}^{k-1}(E, F) \ \forall u \in C^\infty(M)\}. \tag{143}$$

With these spaces the definition of a partial differential operator is shortly as follows:

Definition 5.4 (Algebraic definition). *We say that P is PDO operator of the degree k if $P \in PDO^k(E,F)$.*

This is a way to define partial differential operators in a coordinate-free manner. The degree of a partial differential operator can be calculated by using algebra. Let us demonstrate this by confirming that the exterior derivative and co-differential, $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ which is defined as $\delta = (-1)^k *^{-1} d*$, are operators with degree one.

Example 5.5. Let us study degree of the exterior derivative:

$$[d,f](\omega) = d(f\omega) - fd(\omega) = fd\omega + df \wedge \omega - fd\omega = df \wedge \omega. \tag{144}$$

where $f \in \Gamma(M)$. Since wedge product is $C^\infty(M)$ linear, $df \wedge \omega$ is in PDO^0. For the co-differential we have equalities

$$\begin{aligned}[][\delta,f](\omega) &= (-1)^k *^{-1} d*(f\omega) - f(-1)^k *^{-1} d*\omega \\ &= (-1)^k *^{-1} df \wedge *\omega + f(-1)^k *^{-1} d*\omega - f(-1)^k *^{-1} d*\omega \\ &= (-1)^k *^{-1} df \wedge *\omega.\end{aligned} \tag{145}$$

Now by $C^\infty(M)$-linearity of wedge product and Hodge star, we obtain that the operator $[\delta, f]$ belongs to PDO^0.

When we equip k-forms $\Lambda_0^k(M)$ with a Hodge inner product $\int_M \omega \wedge *\eta$. Then the co-differential δ will be an adjoint of the exterior derivative d, that is, $\langle d\omega, \eta\rangle = \langle \omega, \delta\eta\rangle$. This can seen from following calculation: Let $\omega \in \Lambda^{k-1}, \eta \in \Lambda_0^k(M)$, then Stokes' theorem implies that

$$0 = \int_{\partial M} (\omega \wedge *\eta) = \int_M d(\omega \wedge *\eta) = \int_M d\omega \wedge *\eta + (-1)^{k-1} \int_M \omega \wedge d*\eta \tag{146}$$

and thus

$$\int_M d\omega \wedge *\eta = \int_M \omega \wedge (-1)^k d*\eta = \int_M \omega \wedge *(-1)^k *^{-1} d*\eta. \tag{147}$$

This shows that δ is an adjoint of d. With these operators, we can define the Hodge Laplacian $\Delta = d\delta + \delta d$ which can be seen to be formally self-adjoint by straighforward calculation. The degree of the Hodge Laplacian is two as can be concluded from the following theorem.

Theorem 5.6. *Let us have operators $P \in \mathrm{PDO}^k(M,E,F)$ and $Q \in \mathrm{PDO}^l(M,F,G)$, then the composition QP belongs to $\mathrm{PDO}^{k+l}(M,E,G)$.*

Proof. Let $P \in \mathrm{PDO}^k(M,E,F)$ and $Q \in \mathrm{PDO}^l(M,F,G)$ be partial differential operator, then we have the identity

$$\begin{aligned}[][QP,f]u &= QP(fu) - fQP(u) \\ &= Q(fP(u) + [P,f]u) - fQP(u) \\ &= fQP(u) + [Q,f](Pu) + Q([P,f]u) - fQP(u) \\ &= [Q,f](Pu) + Q([P,f])u.\end{aligned} \tag{148}$$

We see that the $[QP, f]$ is a sum of composition of partial differential operators with lower degrees. When one of the operator is degree 0, then one of the terms vanishes in the sum. The theorem follows when we apply this identity $k + l$ times. □

Partial differential operators have the property that they are local operators which means that $\text{supp}(Ps) \subset \text{supp}(s)$. This is easily seen from the local definition. Let us prove that this holds also for the operators in $PDO^m(E, F)$.

Proposition 5.7. *Let P be a partial differential operator of degree m and s be a section $\Gamma(E)$ then*

$$\text{supp}(Ps) \subset \text{supp}(s) \tag{149}$$

holds.

Proof. We follow the idea of the reference [20, p. 423]. Let us prove the claim by induction. In the case $m = 0$, the operator $P \in PDO^m(E, F)$ has property

$$P(fu) = fP(u) \quad \forall u \in \Gamma(E), f \in C^\infty(M). \tag{150}$$

For any open set $\mathcal{O} \supset \text{supp}(u)$, we can find a smooth bump function ϕ such that $\text{supp}(\phi) \subset \mathcal{O}$ and $\phi|_{\text{supp}(u)} = 1$. Using the equation (150) with $f = \phi$ we see that

$$\text{supp}(P(u)) = \text{supp}(P(\phi u)) \subset \text{supp}(\phi). \tag{151}$$

We can take $\text{supp}\,\phi$ as close to $\text{supp}\,u$ as we want which is enough to conclude the claim in the case $m = 0$.

Now assume that the claim holds $m = k$. Then for any $m = k+1$, $P \in PDO^{k+1}(E, F)$ and $f \in C^\infty(M)$ we have

$$P(fu) = [P, f]u + fP(u) \tag{152}$$

by the induction step we have that $\text{supp}([P, f]u) \subset \text{supp}(u)$ because $[P, f]$ is an operator of degree k. Using same bump functions ϕ and the same argument again, we conclude that

$$\text{supp}(Pu) = \text{supp}(P(\phi u)) \subset \text{supp}(u) \cup \text{supp}(\phi) \tag{153}$$

which finishes the proof. □

This result shows that the value of a partial differential operator is determined by its values on a neighbourhood of a point: Let u and v agree on a small neighbourhood of x then the x belongs to the complement of the support of $u - v$. The proposition will imply that $\text{supp}(P(u - v))^\complement \supset \text{supp}(u - v)^\complement$ so $Pu(x) = Pv(x)$. Thus, Pu is determined by local information of u.

In fact, the property in Proposition 5.7 is enough for a linear operator to be a partial differential operator locally. When the manifold is compact, then the operator is a partial differential operator globally. This is known as Peetre's theorem. So we can use the condition in Proposition 5.7 as a definition for partial differential operators.

Theorem 5.8 (Peetre's theorem). *[21, p. 196] Let M be a smooth manifold and E_1, E_2 be vector bundles over M with dimensions k and m, respectively. Let us suppose that P is a linear operator such that, for all sections s, we have* $\operatorname{supp} P(s) \subset \operatorname{supp} s$.

Then for any point $a \in M$ there exists a neighbourhood (U, ϕ) such that the vector bundles have trivializations and the pull back operator, $\tilde{P} : \Gamma_0(\tilde{\mathrm{U}}, \mathbb{R}^k) \to \Gamma_0(\tilde{\mathrm{U}}, \mathbb{R}^m)$, can be represent as a partial differential operator in the analytical sense: For all sections $(a_1(x), a_2(x), \ldots, a_n(x)) \in \Gamma_0(\tilde{\mathrm{U}}, \mathbb{R}^k)$ we have

$$\tilde{P}\left(\begin{bmatrix} a^1(x) \\ \vdots \\ a^n(x) \end{bmatrix} \right) = \begin{bmatrix} P_{11} & \ldots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{m1} & \ldots & P_{mn} \end{bmatrix} \begin{bmatrix} a^1(x) \\ \vdots \\ a^n(x) \end{bmatrix} \tag{154}$$

where P_{ij} are partial differential operators in $\tilde{\mathrm{U}} \in \mathbb{R}^n$.

We have shown that the analytical definition implies the algebraic property and that algebraic property implies the support property. Therefore, Peetre's theorem shows that all three definitions will be equivalent on compact manifolds. If we study noncompact manifolds, then we can find operators that have the property $\operatorname{supp} P(s) \subset \operatorname{supp} s$ but the degrees of local partial differential operator representations do not have a global upper bound.

6 Pseudo-differential operators

Main objects in the theory of pseudo-differential operators are the graded operator algebra $\Psi^\infty(M; E, F)$ together with the graded symbol space $S^\infty(M, \mathrm{Hom}(E, F))$. The theory can be divided into two parts: Establishment of a pseudo-differential calculus and applications of the calculus. The most important properties of pseudo-differential calculus are L^2 and Sobolev continuity, the composition theorem, the existence of adjoint operator and the asymptotic summation property. Our object is to establish pseudo-differential calculus and to state rigorously above properties. Applications of pseudo-differential operators are given in Section 8.

There are three different definitions of pseudo-differential operators in the literature: local, axiomatic and coordinate-free. Each way provides a different point of view to pseudo-differential operators. We will include a local and coordinate-free definitions in this thesis. The axiomatic definition takes properties of pseudo-differential calculus and uses them as a definition. By including local and coordinate-free definitions, we have tried to give a coherent picture of the theory. All three definitions will lead to the same calculus of pseudo-differential operators. Since the local definition uses pseudo-differential operators on $\mathbb{R}^n$, we start by introducing pseudo-differential operators in the space $\mathbb{R}^n$ which is an important case by it own.

6.1 Pseudo-differential operators in $\mathbb{R}^n$

The theory of pseudo-differential operators arose from harmonic and Fourier analysis. In the space $\mathbb{R}^n$, pseudo-differential operators can be viewed as a generalization of partial differential operators in the Fourier analytical framework. Therefore, we will introduce pseudo-differential operators through the Fourier analysis of Schwartz spaces where we have the following theorem about partial differential operators.

Proposition 6.1. *Let* $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha$ *be a partial differential operator with coefficients in the* $\mathcal{S}(\mathbb{R}^n)$*. The symbol of* P *is defined as* $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)(i\xi)^\alpha$*. Let* $u \in \mathcal{S}(\mathbb{R}^n)$*, then we have a formula that connects the symbol and the operator:*

$$P(x, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} p(x, \xi)\hat{u}(\xi)\, d\xi. \tag{155}$$

Proof. By the properties of the Schwartz class we have that $(i\xi)^\alpha \hat{u}$ is in the Schwartz class and therefore it is also absolutely integrable. Thus, we can take partial derivatives to under of the integral sign in the following calculation [22, p. 154].

$$\begin{aligned} P(x, D)u(x) &= \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha u(x) \\ &= \sum_{|\alpha| \leq m} (2\pi)^{-n} a_\alpha(x)\partial_x^\alpha \int_{\mathbb{R}^n} e^{i<x,\xi>}\hat{u}(\xi)\, d\xi \\ &= (2\pi)^{-n} \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} a_\alpha(x)\partial_x^\alpha e^{i<x,\xi>}\hat{u}(\xi)\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} p(x, \xi)\hat{u}(\xi)\, d\xi. \end{aligned} \tag{156}$$

This shows the proposition. □

Pseudo-differential operators are generalization of this representation. In the case of partial differential operator with constant coefficients, the symbol $p(x, \xi)$ is a polynomial with respect to ξ variable. However for pseudo-differential operators, we allow more general symbols to be used in the identity (155). The function spaces that we will use, are called symbol spaces. We follow the notation used in the reference [8] with minor changes.

Definition 6.2 (Symbol spaces)**.** *Let* $d \in \mathbb{R}$*,* $0 \leq \delta \leq 1, 0 \leq \rho \leq 1$ *and* Σ *be an open subset of the* $\mathbb{R}^l$ *for some* $l \in \mathbb{N}$*. The space* $S^d_{\rho,\delta}(\Sigma; \mathbb{R}^{m\times k})$ *is called matrix valued symbol space of degree* d *and type* ρ, δ *and is defined as subspace of matrix valued smooth function* $C^\infty(\Sigma \times \mathbb{R}^n; \mathbb{R}^{m\times k})$ *such that for any* $p(X, \xi) \in S^d_{\rho,\delta}(\Sigma; \mathbb{R}^{m\times k})$ *and for any* *compact $K \subset \Sigma$ and for any $\alpha \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^l$ following inequality holds for some $c_{\alpha,\beta,K}$*

$$\|\partial_X^\beta \partial_\xi^\alpha p(X,\xi)\| \leq c_{\alpha,\beta,K}(1+\|\xi\|)^{d-\rho|\alpha|+\delta\beta}. \tag{157}$$

There are many suitable choices for Σ for developing the theory but in this thesis we use $\Sigma = \mathbb{R}^n$ or $\Sigma = \Omega \subset \mathbb{R}^n$. Furthermore, we will restrict ourselves to the case $(\rho,\delta) = (1,0)$ and omit the subscripts. However, large part of the results will apply also for ρ, δ such that $0 \leq 1-\rho \leq \delta \leq \rho \leq 1$. When we study scalar valued symbol spaces, then we denote simply symbol space as $S^d(\Sigma)$. One can also use sets $\mathbb{R}^n \times \mathbb{R}^n$ or $\Omega_1 \times \Omega_2$ as a choice for Σ in the definition of symbol spaces. We will not discuss these symbol spaces in the main text but introduce them shortly in Appendix. We define the following spaces as well:

$$\begin{aligned} S_{1,0}^{\infty}(\Sigma;\mathbb{R}^{m\times k}) &= \bigcup_{d\in\mathbb{R}} S_{1,0}^d(\Sigma;\mathbb{R}^{m\times k}) \\ S_{1,0}^{-\infty}(\Sigma;\mathbb{R}^{m\times k}) &= \bigcap_{d\in\mathbb{R}} S_{1,0}^d(\Sigma;\mathbb{R}^{m\times k}). \end{aligned} \tag{158}$$

The symbol space $S_{1,0}^{-\infty}(\Sigma;\mathbb{R}^{m\times k})$ is called the space of smoothing symbols. Symbol spaces have the following elementary properties.

Theorem 6.3 (Properties of symbol spaces)**.** *Let $d, d' \in \mathbb{R}$ and $p, q \in S^d(\Sigma;\mathbb{R}^{m\times k})$ and $r \in S^{d'}(\Sigma;\mathbb{R}^{k\times l})$ then we have:*

1. *$p + q \in S^d(\Sigma;\mathbb{R}^{m\times k})$*
2. *$pr \in S^{d+d'}(\Sigma;\mathbb{R}^{m\times l})$.*

We will need symbol series in the theory and the concept of asymptotic sum is the correct notion to use. The definition of asymptotic sum is based on the following theorem.

Theorem 6.4. *[8, p. 166] Let $\{d_j\}_{j=0}^\infty$ be a decreasing sequence of real numbers such that d_j tends to $-\infty$. Then for any sequence of symbols $p_{d_j} \in S^{d_j}(\Sigma;\mathbb{R}^{m\times k})$ there exist a symbol $p(X,\xi) \in S^{d_0}(\Sigma;\mathbb{R}^{m\times k})$ such that for any $k \in \mathbb{N}$*

$$p(X,\xi) - \sum_{j=0}^{k} p_{m_j}(X,\xi) \in S^{d_{k+1}}(\Sigma;\mathbb{R}^{m\times k}). \tag{159}$$

We define the symbol p provided by the above theorem as an asymptotic sum of p_{d_j} and denote this by $p \sim \sum p_{d_j}$. Asymptotic sums are used in stating results about pseudo-differential operators and in the construction of a parametrix. With symbol spaces, we can now define pseudo-differential operators on the Schwartz space.

Definition 6.5 (Pseudo-differential operators)**.** *Let $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R}^k)$ and let us have a symbol $p \in S^d(\mathbb{R}^n;\mathbb{R}^{m\times k})$. Then an operator* $\mathrm{Op}(p)$ *is defined as*

$$Op(p)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\,d\xi. \tag{160}$$

We say that an operator is a pseudo-differential operator if it can be represented as (160). The operator space induced by $S^d(\mathbb{R}^n;\mathbb{R}^{m\times k})$ is denoted by $\Psi^d(\mathbb{R}^n;\mathbb{R}^{m\times k})$ and is called space of pseudo-differential operators of degree d. Let us check that a pseudo-differential operator is well defined on the Schwartz space.

Proposition 6.6. *Let u and p be as in definition above, then* $\mathrm{Op}(p)u \in \mathcal{S}(\mathbb{R}^n;\mathbb{R}^m)$.

Proof. [23, p. 40] We need to show that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \mathrm{Op}(p)u| < \infty \tag{161}$$

for all $\alpha,\beta \in \mathbb{N}_0^n$. By using integration by parts and calculating we have

$$\begin{aligned}
|x^\alpha \partial_x^\beta \mathrm{Op}(p)u| &= |x^\alpha \partial_x^\beta \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\,d\xi| \\
&= |\int_{\mathbb{R}^n} x^\alpha \partial_x^\beta (e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi))\,d\xi| \\
&= |\int_{\mathbb{R}^n} x^\alpha \sum_{\gamma\le\beta} \binom{|\beta|}{\gamma} \partial_x^\gamma (e^{i\langle x,\xi\rangle}) \partial_x^{\beta-\gamma} p(x,\xi)\hat{u}(\xi))\,d\xi| \\
&= |\int_{\mathbb{R}^n} \frac{1}{i^{|\alpha|}} \partial_\xi^\alpha e^{i\langle x,\xi\rangle} \sum_{\gamma\le\beta} \binom{|\beta|}{\gamma} \partial_x^{\beta-\gamma}(p(x,\xi))(i\xi)^\gamma \hat{u}(\xi))\,d\xi| \\
&= |\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sum_{\gamma\le\beta} \binom{|\beta|}{\gamma} \partial_\xi^\alpha (\partial_x^{\beta-\gamma}(p(x,\xi))(i\xi)^\gamma \hat{u}(\xi)))\,d\xi| \\
&= |\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sum_{\gamma\le\beta}\sum_{\delta\le\alpha} \binom{|\alpha|}{\delta}\binom{|\beta|}{\gamma} (\partial_\xi^\delta \partial_x^{\beta-\gamma}(p(x,\xi))\partial_\xi^{\alpha-\delta}(i\xi)^\gamma \hat{u}(\xi)))\,d\xi|.
\end{aligned} \tag{162}$$

Now the symbol condition (157) gives us that $|\partial_\xi^\delta \partial_x^{\beta-\gamma}(p(x,\xi))| \le C(1+|\xi|)^{m-|\delta|}$ and we obtain that

$$\begin{aligned}
&\le \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sum_{\gamma\le\beta}\sum_{\delta\le\alpha} \binom{|\alpha|}{\delta}\binom{|\beta|}{\gamma} |(\partial_\xi^\delta \partial_x^{\beta-\gamma}(p(x,\xi))\partial_\xi^{\alpha-\delta}(i\xi)^\gamma \hat{u}(\xi)))|\,d\xi \\
&\le \int_{\mathbb{R}^n} |e^{i\langle x,\xi\rangle} \sum_{\gamma\le\beta}\sum_{\delta\le\alpha} \binom{|\alpha|}{\delta}\binom{|\beta|}{\gamma} C(1+|\xi|)^{m-|\delta|}\partial_\xi^{\alpha-\delta}(i\xi)^\gamma \hat{u}(\xi)))|\,d\xi.
\end{aligned} \tag{163}$$

By the properties of Schwartz class $(1+|\xi|)^{m-|\delta|}\partial_\xi^{\alpha-\delta}((i\xi)^\gamma \hat{u}(\xi))$ is bounded by the estimate $C_0(1+|\xi|)^{-M}$ for some M that is large enough so the integral is finite. This concludes the proof. □

Symbols that are polynomials in ξ and bounded in x with fixed ξ are easily seen to satisfy the symbol condition (157). Especially this is the case when the symnol is given by $p(x,\xi) = \sum\limits_{|\alpha|\leq m} a_\alpha(x)(i\xi)^\alpha$ with $a_\alpha(x) \in \mathcal{S}(\mathbb{R}^n)$. Thus, we have following theorem.

Theorem 6.7. *A linear partial differential operators on the space $\mathbb{R}^n$ with coefficients in the Schwartz class are pseudo-differential operators.*

There does not exist the notion of the Schwartz class on open sets so we can not use it to define pseudo-differential operators. However there are compactly supported smooth functions $C_0^\infty(\Omega;\mathbb{R}^k)$ that we can use. Furthermore, we can take the zero extension of compactly supported function and calculate the Fourier transform of the extension. This provides a suitable object for the definition of pseudo-differential operator on open sets.

Definition 6.8. *Let $p(x,\xi) \in S^d(\Omega;\mathbb{R}^{m\times k})$, $u \in C_0^\infty(\Omega;\mathbb{R}^k)$ and $\hat{u}$ be the Fourier transform of the zero extension of u. We define $\mathrm{Op}(p)$ to be an operator given by*

$$\mathrm{Op}(p)u(x) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)d\xi. \tag{164}$$

There are also alternative definitions of pseudo-differential operators. If the open set is a smooth domain, one can use the Fourier analytical methods on manifolds. This approach is treated in the next section. One can also write the Fourier transform explicitly and obtain a formal integral

$$Pu(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\Omega} e^{i(x-y)\cdot\xi} p(x,\xi)u(y)\,dy\,d\xi. \tag{165}$$

In this formal integral, we can replace $S^d(\Omega;\mathbb{R}^{m\times k})$ with $S^d(\Omega\times\Omega;\mathbb{R}^{m\times k})$ and use symbols of form $p(x,y,\xi)$. However, this leads to oscillatory integrals and we discuss briefly this approach in Appendix.

6.2 Symbol map and its properties

An arbitrary pseudo-differential operator does not have as good properties as we would hope for. Therefore, we need to consider two classes of pseudo-differential operators: properly supported pseudo-differential operators and smoothing operators. Properly supported operators have good properties with respect to the symbol map. Moreover, they have an adjoint on the space $C_0^\infty(\Omega)$ and they form an algebra under the composition of pseudo-differential operators.

Definition 6.9. *A support $\mathrm{supp}\, P \subset \Omega\times\Omega$ is the complement of the largest open set of form $\omega_1\times\omega_2$ such that $\omega_1,\omega_2 \subset \Omega$ and $Pu = 0$ in $\mathcal{D}'(\omega_1)$ for every function $u \in C_0^\infty(\omega_2)$. We say that P is properly supported if the projections $\mathrm{pr}_1 : \mathrm{supp}\, P \to \Omega$ and $\mathrm{pr}_2 : \mathrm{supp}\, P \to \Omega$ are proper maps, that is, an inverse image of a compact set is a compact set. [15, p. 180]*

A smoothing operator is an operator determined by an element of smoothing symbols $S^{-\infty}_{1,0}(\Omega)$. The space of smoothing operators is denoted by $\Psi^{-\infty}(\mathbb{R}^n)$. These operators have good regularity properties since they map Sobolev spaces to the space of compactly supported smooth functions and their Schwartz kernel is smooth which means that there exists a function $K \in C^\infty(\Omega \times \Omega)$ such that

$$Pu(x) = \int_\Omega K(x,y)u(y)\,dy \tag{166}$$

where $u \in \mathrm{C}^\infty_0(\Omega)$ [15, p.179]. More about Schwartz kernels can be found in Appendix.

We have the following theorems about properly supported operators.

Theorem 6.10. *[15, p. 181] An operator P is properly supported if and only if there exists a decomposition*

$$Pu = \sum_j \varphi_j P(\phi_j u) \tag{167}$$

for some $\varphi_j, \phi_j \in C^\infty_0(\Omega)$.

Theorem 6.11. *A properly supported pseudo-differential operator can be extended to be a mapping $P : C^\infty(\Omega) \to C^\infty(\Omega)$. [15, p. 181]*

It can be shown that every pseudo-differential operator has a decomposition into a properly supported and a smoothing pseudo-differential operator.

Theorem 6.12. *Every pseudo-differential operator $P \in \Psi^d(\Sigma; \mathbb{R}^{m\times k})$ can be represented as a sum of a properly supported pseudo-differential operator and a smoothing operator.*

Smoothing operators form a vector subspace of $\Psi^d(\Omega; \mathbb{R}^{m\times k})$. So we can study equivalence classes of symbol space $S^d_{1,0}(\Omega; \mathbb{R}^{m\times k})/S^{-\infty}_{1,0}(\Omega; \mathbb{R}^{m\times k})$ and corresponding operator classes $\Psi^d_{1,0}(\Omega; \mathbb{R}^{m\times k})/\Psi^{-\infty}_{1,0}(\Omega; \mathbb{R}^{m\times k})$. The symbol mapping, which we define next, has nice properties with respect to these equivalence classes.

The symbol of a pseudo-differential operator can not be defined in the same way as the symbol of partial differential operators since the function $e^{i\langle x,\xi\rangle}$ does not have a compact support with respect of x. However, with any function $\phi \in \mathrm{C}^\infty_0(\Omega)$ the function $\phi(x)e^{i\langle x,\xi\rangle}$ is compactly supported. Evaluating P with functions of the form $\phi(x)u(x)$ is process known as localization. In fact, it can be shown that a localization of any continuous linear mapping $P : \mathrm{C}^\infty_0(\Omega) \to C^\infty(\Omega)$ can be represent as pseudo-differential operator [15, p. 167]. We choose to use the term symbol instead of localization. The symbol mapping is defined as follows.

Definition 6.13. *Let P be a scalar pseudo-differential operator of degree d and ϕ be a cut-off function near x, then a symbol of operator P is $\sigma_{P,\phi}$ and is defined via*

$$(x,\xi) \mapsto P(\phi(y)e^{i\langle \xi, y-x\rangle})|_{y=x} = e^{-i\langle \xi,x\rangle} P(\phi(y)e^{i\langle \xi,y\rangle})|_{y=x}. \tag{168}$$

The mapping $\sigma_{P,\phi}$ belongs to $S^d_{1,0}(\mathbb{R}^n)$.

We have an approximation theorem for symbol map.

Theorem 6.14. *[15, p.171,173,182] Let us have $P \in \Psi^d(\Omega)$, $\phi \in C_0^\infty(\Omega)$ and $\sigma_{P,\phi}(x,\xi)$, then we have the following regularity result*

$$\sigma_{\mathrm{Op}(p),\phi} - \sum_{|\alpha|<N} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x,\xi) \partial_x^\alpha \phi(x) \in S^{m-N}(\Omega). \tag{169}$$

This gives us an asymptotic sum

$$\sigma_{\mathrm{Op}(p),\phi} \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x,\xi) \partial_x^\alpha \phi(x). \tag{170}$$

If the operator is properly supported then ϕ can be chosen to be identically 1 *on the whole space, and the symbol is given by*

$$p(x,\xi) = e^{-i\langle \xi, x\rangle} P(e^{i\langle \xi, y\rangle})(x). \tag{171}$$

The theorem above and theorem 6.12 imply that the symbol map and the quantization map $p(x,\xi) \to P(x,D)$ are inverses to each other when they considered mappings between $S^m(\Omega)/S^{-\infty}(\Omega)$ and $\Psi^m(\Omega)/\Psi^{-\infty}(\Omega)$.

We have introduced the total symbol and discussed its properties. However, there is a notion of a principal symbol that is useful in applications. Instead of studying equivalent classes $S^m(\Omega)/S^{-\infty}(\Omega)$ we will study equivalence classes $S^m(\Omega)/S^{m-1}(\Omega)$.

Definition 6.15. *Let us have an operator $P \in \Psi^m(\Omega)$ and an element $q \in S^m(\Omega)$ such that*

$$\sigma_{P,\phi} - q \in S^{m-1}(\Omega). \tag{172}$$

The principal symbol $\sigma_m(P)$ is defined as the equivalence class of q in $S^m(\Omega)/S^{m-1}(\Omega)$.

The following examples are based on the Proposition 1.5 in the reference [19, p. 5].

Example 6.16. If we have two symbols p_1, p_2 which are polynomial in the second variable ξ and belong in the same equivalence class $S_{1,0}^d(\Omega)/S_{1,0}^{-\infty}(\Omega)$. Then the symbol condition implies that

$$|p_1(x,\xi) - p_2(x,\xi)| < \frac{C}{1+\|\xi\|} \tag{173}$$

and thus $p_1 - p_2$ is bounded in ξ variable. However, only bounded polynomials are the constant functions. So when we let ξ to tend infinity in the estimate (173), we obtain that $p_1 - p_2 = 0$, that is, $p_1 = p_2$.

Example 6.17. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is homogenous of degree s if $f(\lambda\xi) = \lambda^s f(\xi)$. Let us study symbols that are homogenous of degree s with respect to variable ξ. We have following inequality for homogenous symbols

$$|f(x,\xi)| = \|\xi\|^s f(x, \frac{\xi}{\|\xi\|}) \leq C(x)(1 + \|\xi\|)^s \tag{174}$$

where $C(x)$ is the maximum of $|f(x,\xi)|$ on $\{x\} \times S^{n-1}$. By observing that $\partial^\alpha f(x,\xi)$ is homogenous of degree $s - |\alpha|$, we see that symbols $p(x,\xi)$ whose $C(x)$ estimate is bounded, belongs to $S^s(\Omega)$.

If we have two homogenous symbols that differ only by smoothing symbols, we can show that they are equal by similar argument as the previous example: Let us have $\xi \in S^{n-1}$. Then the assumption $p_1 - p_2 \in S^{-\infty}(\Omega)$ leads to an estimate

$$|p_1(x,\xi) - p_2(x,\xi)| = |\lambda^{-s}||p_1(x,\lambda\xi) - p_2(x,\lambda\xi)| \leq |\lambda|^{-s}(1 + \|\lambda\xi\|)^d \tag{175}$$

for all $d \in \mathbb{R}$. Choosing small enough d, we get that $|\lambda|^{-s}(1+\|\lambda\xi\|)^d \to 0$ when $\lambda \to \infty$, so $p_1(x,\xi) = p_2(x,\xi)$ when $\xi \in S^{n-1}$. Since a homogenous function is determined by values on the sphere S^{n-1}, we obtain that $p_1 = p_2$ everywhere.

The space of classical symbol of degree d is defined as symbols $p \in S^d(\Omega)$ such that there exists sequence of symbols p_{d-l} for $l \in \mathbb{N}_0$ such that p_{d-l} is homogenous of degree $d - l$ and $p \sim \sum_{l \in \mathbb{N}_0} p_{d-l}$. We denote that space with $S^d_{\text{cl}}(\Omega)$. Now if we use inductively our uniqueness result on principal symbols $\sigma_{d-l}(p - \sum_{0 \leq k < l} \sigma_{d-k}(p))$, we obtain that classes in $S^d_{\text{cl}}(\Omega) / S^{-\infty}(\Omega)$ consists only of one element.

6.3 Pseudo-differential calculus

Pseudo-differential operators have a similar operator calculus as partial differential operators have. They have adjoint operators, the composition of operators and an invariance under change of variables. These results rely on finding a suitable asymptotic sum for the desired object and they are proved often with symbols of form $p(x,y,\xi)$. We start by stating adjoint operator theorems.

Theorem 6.18. *[24, p. 43] Let us have an operator $P \in \Psi^d(\mathbb{R}^n)$. Then there is an adjoint operator P^* such that*

$$\langle Pu, v\rangle = \langle u, P^* v\rangle \tag{176}$$

holds for all $u, v \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, we can extend this for vector-valued operators.

This gives us a way to extend a pseudo-differential operator to act on tempered distribution. In the case of open sets, we need to limit ourselves to properly supported pseudo-differential operators.

Theorem 6.19. *[15, p. 184,193] a properly supported pseudo-differential operator P has an adjoint P^* such that P^* is a pseudo-differential operator with symbol p^* such that*

$$p^* \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{p(x,\xi)}. \tag{177}$$

Since P is properly supported, it maps $\mathrm{C}_0^\infty(\Omega)$ functions to $\mathrm{C}_0^\infty(\Omega)$ functions and the adjoint will also be properly supported. Therefore, we can extend the operator P to act on distributions $\mathcal{D}'(\Omega)$. Furthermore, the principal symbol of P^* is the adjoint of principal symbol of P:

$$\sigma_m(P^*) = \sigma_m(P)^*. \tag{178}$$

With help of the adjoint, we can calculate the Fourier transform of $\phi P u$ where $\phi \in \mathrm{C}_0^\infty(\Omega)$.

$$\begin{aligned} \mathcal{F}(\phi P u)(\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \phi P u(x)\, dx \\ &= \int_{\mathbb{R}^n} P^*(\phi e^{-i\langle x,\xi\rangle}) u(x)\, dx \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} e^{-i\langle x,-\xi\rangle} P^*(\phi e^{i\langle x,-\xi\rangle}) u(x)\, dx \\ &\overset{\text{Def 6.13}}{=} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} p^*_\phi(x,-\xi) u(x) dx \end{aligned} \tag{179}$$

so the Fourier transform of $\phi P u$ is a weighted Fourier transform of u.

A composition of partial differential operators is again a partial differential operator. A similar theorem holds for pseudo-differential operator with a minor change: One of the operators has to be properly supported. This is not a major obstacle as the decomposition Theorem 6.12 says that every pseudo-differential operator is a sum of a properly supported and a smoothing operator.

Theorem 6.20. *[15, p. 196,208] Let Ω be an open set, $p_1(x,\xi) \in S^{d_1}(\Omega)$ and $p_2(x,\xi) \in S^{d_2}(\Omega)$ such that one of the operators $P_1 = \mathrm{Op}(p_1)$ and $\mathrm{Op}(p_2)$ is properly supported. Then we have*

$$\mathrm{Op}(p_2(x,\xi))\,\mathrm{Op}(p_1(x,\xi)) = \mathrm{Op}(p_3(x,\xi)) \tag{180}$$

with $p_3 \in S^{d_1+d_2}(\Omega)$ such that

$$p_3(x,\xi) \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha p_2(x,\xi) \partial_x^\alpha p_1(x,\xi) \tag{181}$$

Especially, we have

$$\sigma_{d_1+d_2}(P_3) = \sigma_{d_2}(P_2)\sigma_{d_1}(P_1). \tag{182}$$

If neither of the operators is properly supported, then $P_2(\phi P_1)$ is still well-defined for every $\phi \in \mathrm{C}_0^\infty(\Omega)$ and it has a symbol

$$\sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha p_2(x,\xi)\partial_x^\alpha(\phi(x)p_1(x,\xi)). \tag{183}$$

This theorem transforms the operator space $\Psi^\infty(\Omega)/\Psi^{-\infty}(\Omega)$ to an algebra which has good properties with respect to the algebra $S^\infty(\Omega)/S^{-\infty}(\Omega)$.

Pseudo-differential operators are also invariant under diffeomorphisms so pseudo-differential operators are coordinate invariant objects in sense of Theorem 6.21. Moreover, the principal symbol has especially good properties under the change of variables as it is a coordinate invariant function of the cotangent bundle.

Theorem 6.21. *[15, p. 207,208].*

Let $\phi : U \to V$ be a diffeomorphism and $P = \mathrm{Op}(p)$ be a pseudo-differential operator on V, then the pullback operator $\phi^*(P)$ obtained by the following commutative diagram

$$\begin{array}{ccc} \mathrm{C}_0^\infty(U;\mathbb{R}^k) & \xrightarrow{\phi^*(P)} & C^\infty(U;\mathbb{R}^l) \\ \big\downarrow{\scriptstyle D\Phi_{VU}} & & \big\downarrow{\scriptstyle D\Phi_{VU}} \\ \mathrm{C}_0^\infty(V;\mathbb{R}^k) & \xrightarrow{P} & C^\infty(V;\mathbb{R}^l) \end{array}$$

is a pseudo-differential operator on U and the symbol $\tilde{p}$ of the operator $\phi^*(P)$ has a representation

$$\tilde{p}(x,\xi) \sim p(\phi(x), J(x)^{-\top}\xi) + \sum_{\substack{2|\beta|\le|\alpha| \\ 2\le|\alpha|}} w_{\alpha\beta}(x)\partial_\xi^\alpha p(x, J(\phi)^{-\top}\xi) \tag{184}$$

up to a smoothing symbol where coefficients $w_{\alpha\beta}(x)$ depend only on ϕ and $J(\phi)$ denotes the Jacobian of ϕ. We obtain an identity

$$\sigma_m(\phi^*(P))(x,\xi) = \sigma_m(P)(x, J(x)^{-\top}\xi) \tag{185}$$

for principal symbols.

Earlier we stated in Theorem 6.6 that said that pseudo-differential operators are continuous mappings from Schwartz class to Schwartz class with respect to seminorm topology. With the pseudo-differential calculus, the result can be extended for L^2 and Sobolev norms. The theorem can be stated purely with Sobolev norms but we formulate also the L^2-continuity since it can be used to prove the general statement.

Theorem 6.22 (L^2 continuity theorem)**.** *Let $P \in \Psi^0(\mathbb{R}^n)$ be a zeroth order pseudo-differential operator and $u \in \mathcal{S}(\mathbb{R}^n)$, then we have the following inequality for some $C > 0$*

$$\|Pu\|_2 < C\|u\|_2. \tag{186}$$

Thus, since $C_0^\infty(\mathbb{R}^n)$ *is a subset of* $\mathcal{S}(\mathbb{R}^n)$ *and dense in* $L^2(\mathbb{R}^n)$ *we can extend* P *to be an operator* $P : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Theorem 6.23 (Sobolev continuity theorem). *Let* $P \in \Psi^d_{1,0}(\mathbb{R}^n)$ *be a pseudo-differential operator in* $\mathbb{R}^n$. *Then we have an estimate*

$$\|Pu\|_{s-d} \le C\|u\|_s \tag{187}$$

where $s \in \mathbb{R}^n$ *and* $u \in \mathcal{S}(\mathbb{R}^n)$. *So the operator* P *can be extended to be an operator* $P : H^s(\mathbb{R}^n) \to H^{s-d}(\mathbb{R}^n)$. *When* $\Omega \subset \mathbb{R}^n$ *is an open set, we can extend an operator* $P \in \Psi^d_{1,0}(\Omega)$ *to be an operator* $P : H^s_{\text{comp}}(\Omega) \to H^{s-d}_{\text{loc}}(\Omega)$. *The space* $H^s_{\text{comp}}(\Omega)$ *is defined as the space* $\bigcup\limits_{l\in\mathbb{N}} H^s_{K_l}(\Omega)$ *with the inductive limit topology where the spaces* $H^s_{K_l}(\Omega)$ *are given by* $H^s_{K_l}(\Omega) = \overline{\{u \in H^s(\Omega) \mid \operatorname{supp} u \subset K_l\}}$ *and* $\{K_l\}_{l\in\mathbb{N}}$ *is an increasing sequence of compact sets such that* $\Omega = \bigcup\limits_{l\in\mathbb{N}} K_l$. *[8, p. 169]*

Remark 6.24. When the operator P is a smoothing operator, then by the definition $P \in \Psi^d(\Omega)$ for all $d \in \mathbb{R}$. By using theorems 3.12. and 6.19 from the book [8], we obtain that $\mathcal{E}'(\Omega) = \cup_{s\in\mathbb{R}} H^s_{\text{comp}}(\Omega)$ and we can conclude that for every $u \in \mathcal{E}'(\Omega)$ we have $Pu \in H^s_{\text{loc}}(\Omega)$ for all $s \in \mathbb{R}$. This observation and the fact that $\bigcap\limits_{s\in\mathbb{R}} H^s_{\text{loc}}(\Omega) \subset C^\infty(\Omega)$ implies that smoothing operators map compactly supported distributions to smooth functions. When P is properly supported, the result can be extended to state that $P : \mathcal{D}'(\Omega) \to C^\infty(\Omega)$ [15, p. 184].

7 Pseudo-differential operators on manifolds

In this section, we are going to give a definition of pseudo-differential operators on manifolds using coordinate invariance and using a geometric construction. For this, we need to define symbols spaces on manifolds. The change of variable theorem shows that the principal symbol transforms like an object on the cotangent bundle. It suggests that symbol spaces should be defined on the cotangent bundle.

Definition 7.1. *[19, p. 3] Let* $d \in \mathbb{R}$ *and* $0 \le \delta \le 1, 0 \le \rho \le 1$. *Furthermore, let* $E \to M$ *be a vector bundle with metric and* $\pi : T^*M \to M$ *be the cotangent bundle. Then the space* $S^d_{\rho,\delta}(M, E)$ *is called* E*-valued symbol space of degree* d *and type* ρ, δ *and is defined as subspace of smooth functions* $\Gamma(\pi^*(E))$ *such that a function* $a(\xi)$ *belongs to* $S^d(M, E)$ *if for every open set* $U \subset M$, *for every trivilization* (x, ζ) *of* T^*M *and for every trivilization* $\Psi : E|_V \to V \times \mathbb{R}^n$ *with* $V \subset U$ *following holds: For any compact set* $K \subset V$ *and for any* $\alpha \in \mathbb{N}^n_0, \beta \in \mathbb{N}^n_0$ *the inequality*

$$\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial \zeta^\beta}\Psi(a(\xi(x,\zeta)))\|_E \le C_{\alpha,\beta,K}(1 + \|\xi(x,\zeta)\|_{T^*M})^{d+\delta|\alpha|-\rho|\beta|} \tag{188}$$

holds for some $C_{\alpha,\beta,K} > 0$ *and every* $(x, \zeta) \in \pi^{-1}(V)$.

The change of variable formula showed us that the pullback of a pseudo-differential operator on the space $\mathbb{R}^n$ is coordinate invariant under diffeomorphisms. This suggests that pseudo-differential operators can be defined with the help of local trivializations of E and F. Informally this means that operator P is a pseudo-differential operator if it is a pseudo-differential operator locally.

Definition 7.2. *Let E, F be vector bundles over M and ϕ be a chart on U and $\Phi_{U,E}$ and $\Phi_{U,F}$ be associated trivializations of E and F, respectively. Let $P : \Gamma_0(E) \to \Gamma(F)$ be a continuous operator and $\tilde{s} \in \Gamma(\phi(U); \mathbb{R}^k)$. Then we say that P is a pseudo-differential operator of order μ if*

$$\tilde{P}(\tilde{s})(x) = D\Phi_{U,F} \circ P(D\Phi_{U,E}^{-1}(\tilde{s})) \circ \phi^{-1}(x) \tag{189}$$

is a pseudo-differential operator on $\Phi^\mu(\phi(U); \mathbb{R}^k, \mathbb{R}^l)$ for every chart (U, ϕ). This definition is based on the reference [25, p. 85] with adaption to the case of vector bundles.

We can construct a pseudo-differential operator from a symbol defined on charts with the following procedure: Let $\phi_j : U_j \subset M \to V_j \subset \mathbb{R}^n$ be a cover of the manifold M and denote by $\Phi_{U,j,E}$ and $\Phi_{U,j,F}$ the associated charts on E and F and ψ_j be a partition of unity subordinate to the cover $\{(U_j, \phi_j)\}_j$ such that $\sum_j \psi_j^2 = 1$, then we can define pseudo-differential operator associated to the symbol a as

$$\mathrm{Op}(a)u(x) = \sum_j \psi_j(x)(D\Phi_{U,j,F})^{-1}(\mathrm{Op}(a_j)(D\Phi_{U,j,E}(\psi_j u))). \tag{190}$$

where a_j is the representation of the symbol a on chart V_j and $\mathrm{Op}(a_j)$ is the operator defined in $\Omega \subset \mathbb{R}^n$. [25, p. 86]

We can give a similar definition of pseudo-differential operators as we gave in the case of Schwartz's space. This will be based on the Fourier transform on vector bundles which was represented in section 4.4. We will use the paper [19] as our reference. We will use symbol spaces over the vector bundle $\mathrm{Hom}(E, F)$.

Definition 7.3. *[19, p. 6] Let M be a Riemannian manifold, E, F be vector bundles with metrics and ψ be any smooth cut-off function associated with the connection as in lemma (4.25). Let $a \in S^d(M, \mathrm{Hom}(E, F))$, then a pseudo-differential operator A_ψ is defined as*

$$A_\psi f(x) = \frac{1}{(2\pi)^n} \int_{T_x^* M} a(\xi) \widehat{f^\psi}(\xi)\, d\xi \tag{191}$$

where $x \in M$, $f \in \Gamma(M, E)$ and f^ψ is the lift of f to the tangent bundle.

Theorem 7.4. *[19, p. 7] The geometric definition of a pseudo-differential operator produces also a pseudo-differential operator in local sense. That is, A_ψ is locally a pseudo-differential operator.*

We can also define a symbol map coordinate-free manner. We will follow the reference [19] with minor changes. We need following space

$$TM \diamond T^*M = \{(v,\xi) \in TM \times T^*M \mid \rho(v) = \pi(\xi)\} \tag{192}$$

and an auxiliary function $\varphi_\psi : TM \diamond T^*M \to \mathbb{C}$ given by

$$\varphi_\psi(v,\xi) = \psi(v)e^{i\langle \xi, v\rangle} \tag{193}$$

where ψ is a cut-off function. The expression $\varphi_\psi(\exp^{-1}_{\pi(\xi)}(x),\xi)$ can be extended for every $x \in M$ by requiring that $\varphi_\psi(\exp^{-1}_{\pi(\xi)}(x),\xi) = 0$ whenever $\exp^{-1}_{\pi(\xi)}(x)$ is not defined. Thus, the function $\varphi_\psi(\exp^{-1}_{\pi(\xi)}(x),\xi)$ will be a smooth function on M.

Definition 7.5. *[19, p. 9] We define the symbol map $\sigma_\psi P : T^*M \to \pi^*(\mathrm{Hom}(E,F))$ to be the element determined by*

$$\sigma_\psi P(\xi)V = A(\varphi_\psi(\exp^{-1}_{\pi(\xi)}(\cdot),\xi)\tau_{(\cdot),\pi(\xi)}V)(\pi(\xi)) \tag{194}$$

where $\psi : T^*M \to \mathbb{R}$ is a cut-off function, and $V \in E_{\pi(\xi)}$.

The principal symbol of P is defined as in the case of $\mathbb{R}^n$: A principal symbol is the equivalence class $\sigma_m(P) \in S^m(M,\mathrm{Hom}(E,F))/S^{m-1}(M,\mathrm{Hom}(E,F))$ given by a representative $q \in S^m(\Omega)$ such that

$$\sigma_{P,\phi} - q \in S^{m-1}(M,\mathrm{Hom}(E,F)). \tag{195}$$

The symbol map and the quantization map are compatible in the following sense: The composition of the symbol map and quantization map produces the same element up to a smoothing element. The exact statement is in the following theorem.

Theorem 7.6. *[19, p. 13] Let $P \in \Psi^\mu_{1,0}(M;E,F)$ be a pseudo-differential operator on a Riemannian manifold M and $p = \sigma_{P,\psi} \in S^\mu_{1,0}(M,\mathrm{Hom}(E,F))$ be its symbol. Then P and $\mathrm{Op}(p)$ coincide modulo smoothing operators. Furthermore, the symbol map and quantization map are inverse of each other when they are considered be maps between $\Psi^\infty(M;E,F)/\Psi^{-\infty}(M;E,F)$ and $S^\infty(M;\mathrm{Hom}(E,F))/S^{-\infty}(M;\mathrm{Hom}(E,F))$.*

This theorem does also imply that every locally defined pseudo-differential operator is given by a geometric quantization of some symbol up to a smoothing symbol. Thus, giving a converse of theorem 7.4 and providing a geometric characterization of locally pseudo-differential operators.

Many other results mentioned for pseudo-differential operators on the Euclidean space $\mathbb{R}^n$ hold also for pseudo-differential operators on vector bundles over compact manifolds. We gather those results as a list in the following theorem to avoid unnecessary repetition.

Theorem 7.7. *Let M be a compact manifold and E, F be vector bundles with metrics over M. The following properties for pseudo-differential operators $P \in \Psi^d(E_1,E_2)$ and $Q \in \Psi^{d'}(E_2,E_3)$ are true.*

1. *Given a decreasing sequence* d_j *and* $P_j \in \Psi^{d_j}(M,E)$ *there exist* $P_0 \in \Psi^{d_0}(M,E)$ *such that* $P_0 - \sum_{j=0}^{N} P_j \in \Psi^{d_{N+1}}(M,E)$ *so the asymptotic sum property holds. [19, p. 4]*
2. *The operator* P *has an adjoint* $P^* \in \Psi^d(E_2, E_1)$ *such that*

$$\langle Pu, v\rangle_{E_2} = \langle u, P^* v\rangle_{E_1} \tag{196}$$

where $u \in \Gamma(M, E_1)$ *and* $v \in \Gamma(M, E_2)$*. [19, p. 19]*

3. *The composition* $Q \circ P$ *is well-defined and we have* $Q \circ P \in \Psi^{d+d'}(E_1, E_3)$ *and for principal symbols we have* $\sigma_{d'+d}(QP) = \sigma_{d'}(Q)\sigma_d(P)$*. [19, p. 23]*
4. *A pseudo-differential operator* $P \in \Psi^d(M; E_1, E_2)$ *is a continuous operator from* $H^s_{\text{loc}}(M, E_1)$ *to* $H^{s-d}_{\text{loc}}(M, E_2)$ *[8, p. 206].*

8 Applications to elliptic operators

In this section, we give applications of the theory to elliptic operators. Elliptic operators appear in the theory of partial differential equations and physics. Two of the best known examples are the Laplace equation $\Delta u = 0$ and the Poisson's equation $\Delta u = f$ on $\Omega \subset \mathbb{R}^3$. Examples of elliptic partial differential equations in physics include equations for electrostatic and thermal equilibrium. These examples are scalar equations. An example of a vector-valued elliptic partial differential equation is the Navier-Cauchy equation in linear elasticity.

Another area of mathematics where elliptic operators are used is geometry where elliptic operators can be used to study the geometric and topological properties of manifolds. Important concepts of this subject are elliptic complexes, Hodge theory and index theory. Famous theorems in this area are Atiyah-Singer theorem, its corollaries and the Hodge decomposition theorem. These results need more background on topology and vector bundles so we do not discuss these in this thesis but an interested reader can explore these topics further.

Our objective is to present applications of the parametrix to an elliptic operator on compact manifolds without boundary and discuss what they provide for the Poisson equations. We start by giving a short review of Fredholm operators.

8.1 Fredholm operators

We follow the content of Grubb's book [8]. Let V and W be Hilbert spaces and T be an operator from V to W.

Definition 8.1. *An operator $T : V \to W$ is called Fredholm operator if its kernel*

$$\ker(T) = \{v \in V | T(v) = 0\} \tag{197}$$

and its cokernel

$$\text{coker}(T) = W/\mathcal{R}(T), \tag{198}$$

where $\mathcal{R}(T)$ is range of T, are finite dimensional. The index of the operator T is defined as

$$\text{index}(T) = \dim\ker(T) - \dim\text{coker}(T). \tag{199}$$

From the definition of a Fredholm operator it will follow that range of a bounded Fredholm operator is closed [8, p. 209]. Fredholm operators have following convenient characterization.

Theorem 8.2. *[8, p. 210] An operator $T : V \to W$ is a Fredholm operator if and only if there exist bounded operators $S_1 : W \to V$ and $S_2 : W \to V$ such that operators*

$$S_1 T - I_V, \quad TS_2 - I_W \tag{200}$$

are compact.

Furthermore, Fredholm operators behave well with respect to compositions and compact operators.

Theorem 8.3. *[8, p. 211] If the operators $T_1 : \mathcal{H}_1 \to \mathcal{H}_2$ and $T_2 : \mathcal{H}_2 \to \mathcal{H}_3$ are Fredholm and $K : \mathcal{H}_1 \to \mathcal{H}_2$ is a compact operator then following statements hold.*

1. *The operator $T_2 \circ T_1$ is a Fredholm operator and*

$$\text{index}(T_2 \circ T_2) = \text{index}(T_1) + \text{index}(T_2). \tag{201}$$

2. *The operator $T_1 + K$ is Fredholm and*

$$\text{index}(T_1 + K) = \text{index}(T_1). \tag{202}$$

8.2 Elliptic operators and parametrix

We start to study elliptic equations on manifolds without a boundary of the form

$$Pu = f. \tag{203}$$

We will study the kernel and the cokernel of the operator as they give information about solutions. The main result for elliptic operators is that they are Fredholm. This tells us that the Poisson equation has solutions for most functions f and the Laplace operator has a finite dimensional kernel. As a byproduct of the theory, we get a way to transform the elliptic equation to an integral equation of the form

$$u + Ru = Qf \tag{204}$$

which can be studied as an integral equation. There are many different ways to define elliptic operators depending on the symbol space but we limit ourselves to classical symbols as they are the most common symbol class and the ellipticity condition gives a concrete way to check that an operator is elliptic.

Definition 8.4 (Elliptic operator). *Let $P \in \Psi^d(M, E, E)$ be an operator with classical symbol, then it is called elliptic if its principal symbol $\sigma_d(P)(x, \xi)$ is invertible for any $x \in M, \xi \neq 0$.*

Definition 8.5 (Parametrix). *Let P be a pseudo-differential operator of order d then we say that P has a parametrix Q if Q satisfies following conditions*

$$\begin{aligned} PQ &= I + R_1 \\ QP &= I + R_2. \end{aligned} \tag{205}$$

where R_1 and R_2 are smoothing operators.

Theorem 8.6. *Let an operator P be elliptic. Then it has a parametrix.*

Proof. Let us study an elliptic operator P. By the decomposition theorem it can be represented as a sum of a smoothing operator P' and a properly supported operator $\tilde{P}$. Moreover, let us have a cut-off function $\chi(x, \xi)$ the near zero section of TM such that $\chi(\xi) = 1$ for $\|\xi\| \leq 1$. We can define a symbol $q(x, \xi) = p(x, \xi)^{-1}(1 - \chi(x, \xi))$ such that the operator $Q(x, \xi) = \mathrm{Op}(q)$ is properly supported. Then by the operator calculus we have

$$\begin{aligned} \sigma_0(QP) &= \sigma_0(Q(\tilde{P} + P')) && \mod S^{-\infty} \\ &= (1 - \chi(x, \xi))p(x, \xi)^{-1}p(x, \xi) + \sigma_0(QP') && \mod S^{-\infty} \\ &= (1 - \chi(x, \xi))I - r' && \mod S^{-\infty} \\ &= I - r && \mod S^{-\infty} \end{aligned} \tag{206}$$

where $r \in S^{-1}(M, E, E)$. We used the fact that $\chi(x, \xi)I \in \Psi^{-\infty}(M, E, E)$ and that $QP' \in \Psi^{-\infty}(M, E, E)$ holds for every smoothing operator P'. Again, we will choose r so that $R = \mathrm{Op}(r)$ is properly supported. Thus, powers of R are well defined. Let us consider an operator $\big(\sum_{k=1}^{N} R^k\big)Q$ which has following property:

$$\Big(\sum_{k=1}^{N} R^k\Big)QP = \Big(\sum_{k=1}^{N} R^k\Big)(1 - R) = I - R^{N+1} \mod \Psi^{-\infty}(M, E, E). \tag{207}$$

So there exists an operator $Q_L = \sum_k R^k Q$ such that $Q_L P - I \in \Psi^{-\infty}(M, E, E)$ which proves the existence of the left inverse. Similarly, we obtain a right inverse Q_R. When we apply following trick

$$Q_L - Q_R \sim Q_L \circ (P \circ Q_R) - (Q_L \circ P) \circ Q_R \sim 0 \mod \Psi^{-\infty}(M, E, E) \tag{208}$$

we will see that $Q_L \sim Q_R \mod \Psi^{-\infty}(M, E, E)$ and we can choose $Q = Q_L$ as our parametrix. □

Existence of a parametrix implies that elliptic operators are Fredholm on compact manifolds. This fact has many useful applications. We will prove this in the following theorem.

Theorem 8.7. *Let $P : H^s(M, E) \to H^{s-d}(M, E)$ be an elliptic pseudo-differential operator of order d on a compact manifold. Then following statements hold: The kernel* $\ker(P)$ *is a finite dimensional subspace $V \subset \Gamma(M, E)$ and does not depend on s. Furthermore, there exists a finite dimensional subspace $W \subset \Gamma(M, E)$ such that $H^{s-d}(M, E) = \mathcal{R}(P) \oplus W$ and the space W is orthogonal complement to the range in sense of inner product of $H^{s-d}(M, E)$.*

Proof. Since P is elliptic, there exists a parametrix Q such that $QP = I + R$ where R is a smoothing operator. Observe that R maps $H^s(M, E)$ to $H^{s+1}(M, E)$. Thus by Rellich's theorem the inclusion map $H^{s+1}(M, E) \to H^s(M, E)$ is compact so $R : H^s(M, E) \to H^s(M, E)$ is also compact. Therefore, $I + R$ is Fredholm and $(I + R)u$ has a finite dimensional kernel. The kernel of P is included in the kernel of $QP = I + R$ so this proves that the kernel of P is finite dimensional. Let $u \in \ker(P)$, then $Pu = 0$ implies that

$$0 = QPu = (I + R)u \tag{209}$$

which leads to $u = -Ru$ so $u \in \Gamma(M, E)$ since smoothing operators map distributions to smooth sections. The space of smooth sections belongs to every $H^s(M, E)$ so the kernel $\ker(P)$ and its dimension do not depend on s. This concludes the proof about kernel of P. The range of operator $PQ = I + R'$ belongs to the range of P. Since the operator $I + R'$ has a finite codimension, the operator P has also a finite codimension. We have proven that P has finite codimension and has a finite dimensional kernel so P is Fredholm and thus the range $\mathcal{R}(P)$ is a closed subspace.

In a Hilbert space $H^{s-d}(M, E)$ a closed subspace $\mathcal{R}(\mathcal{P})$ induces an orthogonal decomposition $H^{s-d}(M, E) = \mathcal{R}(P) \oplus W$. Let us take a point v from W so $\langle Pu, v\rangle = 0$ for all $u \in H^s(M, E)$. This implies that $\langle u, P^*v\rangle = 0$ holds for every $u \in H^s(M, E)$ and thus $P^*v = 0$ and so $W \subset \ker(P^*)$. Now we show that $\ker(P^*) \subset W$. Let us take $v \in \ker(P^*)$, by direct calculation we obtain $0 = \langle u, P^*v\rangle = \langle Pu, v\rangle$ for every $u \in H^s(M, E)$, so $v \in W$ and $W = \ker(P^*)$. By operator calculus we have that $\sigma_m(P^*) = \sigma_m(P)^*$. Therefore the adjoint P^* is also elliptic. The earlier reasoning about $\ker(P)$ gives us that $\ker(P^*)$ is finite dimensional as well. This proves that W is finite dimensional subset of $\Gamma(M, E)$. □

Remark 8.8. Observe that the proof of the above theorem relies on Rellich's theorem that assumes that the space M is compact. Therefore, the theorem does not extend to the space $\mathbb{R}^n$. The second part shows that the equation $Pu = f$ has a solution for all $f \in H^s(M, E)$ expect for elements orthogonal to a finite dimensional subspace.

Theorem 8.9 (Elliptic regularity)**.** *Let P be a properly supported elliptic operator of degree d. If $f \in H^s(M, E)$ and u is a solution for the equation*

$$Pu = f \tag{210}$$

then $u \in H^{s+d}(M, E)$.

Proof. Let us apply the parametrix Q to the equation, we obtain that $Qf = u + Ru$ so $u = Qf - Ru$. Now observe that $Ru \in \Gamma(M, E)$ and $Qf \in H^{s+d}(M, E)$ and therefore $u \in H^{s+d}(M, E)$. □

This theory can be extended to the case when the manifold has a boundary and the boundary conditions fulfill certain conditions and the operator will be a Fredholm operator.

9 Conclusions and further topics

We have given an overview of certain parts of modern analysis with the language of differential geometry. Our presentation is limited and we comment shortly about further topics about the area. This helps the reader to solidify her or his knowledge and give motivation for reading more about the topics.

Our representation of differential geometry is very terse, abstract and we did not give many examples. We reviewed only definitions of differential geometry such as metric tensor, connections and covariant derivative. To gain more intuition about these objects, we recommend that the reader studies Riemannian geometry as it contains more constructions involving these concepts. Furthermore, we omitted many applications of differential forms that connect analysis to geometry, for example, the Hodge theory. Furthermore, the theory of Fourier analysis has been extended to Lie groups so the pseudo-differential operators can be studied in that framework as in the reference [26].

The theory of pseudo-differential operators relies heavily on harmonic analysis. Another approach to the thesis could have been to study more closely the Fourier analysis. The possible topics include singular integrals, oscillatory integrals and L^p-multipliers. Proving properties of pseudo-differential operators requires more tools from harmonic analysis, especially notions of amplitudes and oscillatory integrals. A careful study of these topics and proofs would give the reader a better grasp on techniques of Fourier analysis.

The tools that we represented can be used in the theory of partial differential equations. We demonstrated that fundamental solutions can be calculated using Fourier analysis but we gave only two examples of how to calculate fundamental solutions. One can develop the theory further and calculate fundamental solutions for other partial differential equations as well. Pseudo-differential operators can also be applied to partial differential equations. For example, the Calderón projection can be used to solve boundary value problems and micro-local analysis yields useful energy estimates in many cases.

As we mentioned in the previous section, there are applications of the theory to differential geometry. One of the proofs of Atiyah-Singer theorem uses pseudo-differential operators as a tool. This theorem gives a way to study geometric objects with tools of analysis. For example, one can give a proof of Gauss-Bonnet theorem using the Atiyah Singer theorem. Another avenue that one can take is to study partial differential equations arising from differential geometry such as the Yamabe equation or the minimal surface equation.

10 Appendix

In the theory of pseudo-differential operators, more general symbol spaces, oscillatory integrals and the Schwartz kernel theorem are used extensively in the research literature. The simplest definition of pseudo-differential operators does not need these concepts so we omitted them in the main text. However, in this appendix, we discuss briefly these concepts and how they relate to the subject.

Elements $p(x, y, \xi)$ of the symbol space of $S^m(X \times Y, \mathbb{R}^n)$ are called often amplitudes to distinguish them from symbols of form $p(x, \xi) \in S^m(X \times \mathbb{R}^n)$. We can define the following quantization map to assign an amplitude to an operator

$$P(u)(x) = \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} e^{i\phi(x,\xi)} p(x, y, \xi) u(y) \, dy \, d\xi \tag{211}$$

where $\phi(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$ is called the phase function. However, this integral does not convergence as a normal integral but is well-defined as an oscillatory integral. The following theorem provides a necessary result for a definition an oscillatory integral.

Theorem 10.1. *[27, p.90] Let $X \subset \mathbb{R}^n$ be an open set, $a \in S^m(X, \mathbb{R}^n)$ be a symbol, $u \in C_0^\infty(X)$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(0) = 1$. Let us suppose that a smooth function $\phi(x, \xi) \in C^\infty(X \times (\mathbb{R}^n \setminus \{0\}))$ is a real valued function which is homogeneous of order 1 with respect of ξ. If $\phi(x, \xi)$ does not have critical points when $\xi = 0$ as function of (x, ξ). Then the distribution*

$$I_\phi(a)u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_X e^{i\phi(x,\xi)} a(x, \xi) \chi(\epsilon\xi) u(x) \, dx \, d\xi \tag{212}$$

is well-defined and is a continuous function of $a \in S^m(X, \mathbb{R}^n)$. Furthermore the linear functional $u \mapsto I_\phi(au)$ defines a distribution.

When the expression

$$\int\limits_{\mathbb{R}^n}\int\limits_{X} e^{i\phi(x,\xi)} p(x,\xi) u(x)\, dx\, d\xi \tag{213}$$

is said to be understood as an oscillatory integral, then it should be interpreted as in equation (212).

The above theorem gives us the knowledge that the oscillatory integral (212) defines a distribution and the operator (211) is well-defined. The natural follow up question is: What are the regularity properties of the operator (211)? It can be proven that the obtained function is $C^\infty(X)$ when ϕ has certain regularity properties.

Theorem 10.2. *Let us assume that $u(y) \in C_0^\infty(Y)$ and the real valued function $\phi(x, y, \xi) \in C^\infty(X \times X \times (\mathbb{R}^n \setminus \{0\}))$ is homogeneous of order 1 with respect of ξ and for fixed x it does not have critical points as function of (y, ξ). Then the function f that is defined as oscillatory integral*

$$f(x) = \int\limits_{\mathbb{R}^n}\int\limits_{Y} e^{i\phi(x,y,\xi)} p(x, y, \xi) u(y)\, dy\, d\xi \tag{214}$$

is well defined and it is a smooth function with respect to x which belongs to $C^\infty(X)$. [27, p.99]

In the reference, the theorem above is represented in a stronger form discussing the case when $u \in C^k(Y)$. Especially, the theorem means that we can take derivatives under the integral signs in (214).

These more general operators are needed while proving composition property of pseudo-differential operators [8, p. 168,178]. The assumption holds especially for the function $\phi(x, y, \xi) = (x - y) \cdot \xi$ which is the function ϕ used in the definition of pseudo-differential operators. This theorem shows also that pseudo-differential operators map compactly supported smooth functions to smooth functions. Moreover, oscillatory integrals have other familiar results such as Fubini's theorem for exchanging the order of integration [24, p. 31].

Pseudo-differential operators are sometimes defined via the Schwartz kernel of the operator. This refers to Schwartz's kernel theorem which gives a connection between distributions and linear operators.

Theorem 10.3 (Schwartz's kernel theorem)**.** *Let $P : C_0^\infty(X) \to \mathcal{D}'(Y)$ be a linear operator. Then there exists a distribution $\mathcal{D}'(X \times Y)$ such that*

$$< Pu, \phi >=< K, u \otimes v > \tag{215}$$

and vica versa.

The proof of the theorem relies on showing that every function $C_0^\infty(X \times Y)$ can be approximated by functions of the form $u_n(x)v_n(y)$ where $u_n \in C_0^\infty(X), v_n \in C_0^\infty(Y)$. A pseudo-differential operator can then be defined as a linear operator whose Schwartz kernel is form of

$$u(x, y) \mapsto \int_{\mathbb{R}^n} \int_{X \times Y} e^{i(x-y)\cdot\xi} p(x, y, \xi) u(x, y)\, dx\, dy\, d\xi \tag{216}$$

where this formal integral is understood as an oscillatory integral. However in literature, it is common to use shorthand notation

$$\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x, y, \xi)\, d\xi \tag{217}$$

for the Schwartz's kernel. This may cause confusions as the integral (217) may not exist even as an oscillatory integral and requires distributional interpretation. One way to keep track of distributions and formal integrals, is first to fill the formal integral with a test function with variables that are not integration variables and then adding an integration over those variables. For example, in case of (217) the integral is not taken with respect to x and y so we need fill in a test function of form $u(x, y)$ and then we add an integration over x and y and then this integral is understood as oscillatory integral as in (216). One has to be careful when referring something as distribution in the context of oscillatory integrals. So it is highly advisable to check which distribution is meant while reading literature.

References

[1] Loring W Tu. *An introduction to manifolds.* Springer Science & Business Media, 2010.

[2] L.W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes.* Graduate Texts in Mathematics. Springer International Publishing, 2017.

[3] SS Chern-H Chen-KS Lam, SS Chern, and WH Chen. *Lectures on differential geometry.* World Scientific, Singapore, 2000.

[4] S.B. Sontz. *Principal Bundles: The Classical Case.* Universitext. Springer International Publishing, 2015.

[5] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra.* Addison-Wesley series in mathematics. Avalon Publishing, 1994.

[6] F. Treves. *Topological Vector Spaces, Distributions and Kernels.* Dover books on mathematics. Academic Press, 2006.

[7] D. Mitrea. *Distributions, Partial Differential Equations, and Harmonic Analysis.* Universitext. Springer New York, 2013.

[8] G. Grubb. *Distributions and Operators*. Graduate Texts in Mathematics. Springer Science + Business Media, 2009.

[9] Friedrich Gerard Friedlander, Friedrich Gerard Friedlander, Mark Suresh Joshi, and M Joshi. *Introduction to the Theory of Distributions*. Cambridge University Press, 1998.

[10] Christian Bär. Geometric wave equations. *Preprint Lecture Notes, University of Potsdam https://www. math. uni-potsdam. de/fileadmin/user_ upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/Waves. pdf*, 2017.

[11] Christian Bär, Nicolas Ginoux, and Frank Pfäffle. Wave equations on lorentzian manifolds and quantization. *arXiv preprint arXiv:0806.1036*, 2008.

[12] Tien-Cuong Dinh and Nessim Sibony. Introduction to the theory of currents. *Preprint*, 2005.

[13] Piotr Hajlasz. *Sobolev spaces on metric-measure spaces*, volume 338 of *Contemporary Mathematics*. American Mathematical Soc., 2003.

[14] Michael E Taylor. Partial differential equations i: Basic theory (applied mathematical sciences), 2011.

[15] Bent E Petersen. *Introduction to the Fourier transform & pseudo-differential operators*. Pitman Advanced Publishing Program, 1983.

[16] K. Wehrheim. *Uhlenbeck Compactness*. EMS series of lectures in mathematics. European Mathematical Society, 2004.

[17] Michael Ruzhansky. Introduction to pseudo-differential operators, 2014.

[18] Michael Renardy and Robert C Rogers. *An introduction to partial differential equations*, volume 13. Springer Science & Business Media, 2006.

[19] Markus J. Pflaum. The normal symbol on Riemannian manifolds. *New York J. Math.*, 4:97–125, 1998.

[20] Liviu I Nicolaescu. *Lectures on the Geometry of Manifolds*. World Scientific, 2007.

[21] Donald W Kahn. *Introduction to global analysis*, volume 91. Academic Press, 1980.

[22] Frank Jones. *Lebesgue integration on Euclidean space*. Jones & Bartlett Learning, 2001.

[23] Xavier Saint Raymond. *Elementary introduction to the theory of pseudodifferential operators*. CRC Press, 1991.

[24] Helmut Abels. Pseudodifferential operators, 2011.

[25] Lars Hörmander. *The analysis of linear partial differential operators III: Pseudo-differential operators*. Springer-Verlag, 1985.

[26] Michael Ruzhansky and Ville Turunen. *Pseudo-differential operators and symmetries: background analysis and advanced topics*, volume 2. Springer Science & Business Media, 2009.

[27] Lars Hörmander. Fourier integral operators. i. *Acta Math.*, 127:79–183, 1971.