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HARDY INEQUALITIES AND ASSOUAD DIMENSIONS

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ABSTRACT. We establish both sufficient and necessary conditions for weighted Hardy inequalities in metric spaces in terms of Assouad (co)dimensions. Our sufficient conditions in the case where the complement is thin are new even in Euclidean spaces, while in the case of a thick complement we give new formulations for previously known sufficient conditions which reveal a natural duality between these two cases. Our necessary conditions are rather straight-forward generalizations from the unweighted case, but together with some examples they indicate the essential sharpness of our results. In addition, we consider the mixed case where the complement may contain both thick and thin parts.

1. INTRODUCTION

Let X be a complete metric measure space. We say that an open set $\Omega \subset X$ *admits a* (p, β)*-Hardy inequality*, if there exists a constant $C > 0$ such that the inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta-p} \, d\mu \leq C \int_\Omega g_u(x)^p d_\Omega(x)^\beta \, d\mu$$

holds for all $u \in \operatorname{Lip}_0(\Omega)$ and for all upper gradients g_u of u. Here $d_\Omega(x) = \operatorname{dist}(x, \Omega^c)$ is the distance from $x \in \Omega$ to the complement $\Omega^c = X \setminus \Omega$, and in the case $X = \mathbb{R}^n$ we have $g_u = |\nabla u|$.

There is a well-known dichotomy concerning domains admitting a Hardy inequality: either the complement of the domain is large (or "thick") or sufficiently "thin". For instance, if an open set $\Omega \subset \mathbb{R}^n$ admits a (p, β)-Hardy inequality, then there exists $\delta > 0$ such that for each ball $B \subset \mathbb{R}^n$ either $\dim_{\mathrm{H}}(2B \cap \Omega^c) > n - p + \beta + \delta$ or $\dim_{\mathrm{A}}(B \cap \Omega^c) < n-p+\beta-\delta$; see [23] (the case $\beta = 0$) and [25]. Here $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension and $\dim_{\mathrm{A}}$ is the (upper) Assouad dimension (see Section 2 for definitions).

Reflecting this dichotomy, sufficient conditions for the validity of a (p, β)-Hardy inequality can be given in both of the above cases. For thick complements, a canonical sufficient condition for the unweighted ($\beta = 0$) p-Hardy inequality in Ω is the uniform p-fatness of Ω^c, or equivalently a uniform Hausdorff content density condition for Ω^c, see [31, 36, 22]. In $\mathbb{R}^n$, uniform p-fatness of Ω^c implies in particular that $\dim_{\mathrm{H}}(2B \cap \Omega^c) > n - p$ for all balls centered at Ω^c. On the other hand, in the case of thin complements the smallness of the (upper) Assouad dimension of the complement ($\dim_{\mathrm{A}}(\Omega^c) < n - p$) is known to be sufficient for the p-Hardy inequality; see [23, 25] and note that in this case these results are based on the works of Aikawa [1, 3]. See also [7, 20, 22, 28, 29] for sufficient conditions for weighted Hardy inequalities and to metric space versions of such results.

The main purpose of this paper is to sharpen the previously known sufficient conditions for the validity of Hardy inequalities in the case where the complement is assumed to be thin. More precisely, we prove the following theorem in the setting of a doubling metric space X supporting certain Poincaré inequalities (cf. Section 2). Here the thinness is formulated in terms of the so-called lower Assouad codimension of Ω^c (a metric space version of the (upper) Assouad dimension, see Section 2).

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Theorem 1.1. *Let $1 \leq p < \infty$ and $\beta < p-1$, and assume that X is an unbounded doubling metric space. If $\beta \leq 0$, we further assume that X supports a p-Poincaré inequality, and if $\beta > 0$ we assume that X supports a $(p-\beta)$-Poincaré inequality. If $\Omega \subset X$ is an open set satisfying*

$$\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p - \beta,$$

then Ω admits a (p,β)-Hardy inequality.

In the unweighted case $\beta = 0$, which is the most important and most interesting, Theorem 1.1 shows that a p-Hardy inequality holds in Ω under the assumptions that X supports a p-Poincaré inequality and $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p > 1$; in a Q-regular space the latter condition is equivalent to $\dim_{\mathrm{A}}(\Omega^c) < Q - p$. In particular, this gives a complete answer to a question of Koskela and Zhong [23, Remark 2.8]. See also Corollary 6.6 for an improvement concerning the boundary values of test-functions in Theorem 1.1.

In $\mathbb{R}^n$, the case $\beta = 0$ of Theorem 1.1 coincides with the above-mentioned results from [23, 25], but our approach gives a completely new proof is this case. For $\beta \neq 0$ the result is new even in Eulidean spaces. Our proof of Theorem 1.1 follows the general scheme of Wannebo [36]: We first prove (p,β)-Hardy inequalities for $\beta < 0$, with a suitable control for the constants in the inequalities for β close to 0, and then elementary — but slightly technical — integration tricks yield the inequalities for $0 \leq \beta < p - 1$.

Another goal of this work is to bring together much of the recent research on Hardy inequalities (see e.g. [20, 22, 23, 25, 26, 27, 28, 29, 30]) in a unified manner in the setting of metric spaces. For instance, it was shown in [28] that an open set $\Omega \subset X$ admits a (p,β)-Hardy inequality if the complement Ω^c satisfies a uniform density condition in terms of a Hausdorff content of codimension $q < p - \beta$. In the present paper, we establish a new characterization for the upper Assouad codimension by means of Hausdorff co-content density, see Corollary 5.2. (In Ahlfors regular spaces, such a characterization was observed in [18, Remark 3.2].) Consequently, we obtain the following sufficient condition for Hardy inequalities in terms of the upper Assouad codimension, which provides a natural counterpart for Theorem 1.1 and shows that there exists a nice "duality" between the sufficient conditions for the cases of thick and thin complements.

Theorem 1.2. *Let $1 < p < \infty$ and $\beta < p - 1$, and assume that X is a doubling metric space supporting a p-Poincaré inequality if $\beta \leq 0$, and a $(p-\beta)$-Poincaré inequality if $\beta > 0$. Let $\Omega \subset X$ be an open set satisfying*

$$\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) < p - \beta,$$

and, in case Ω is unbounded, we require in addition that Ω^c is unbounded as well. Then Ω admits a (p,β)-Hardy inequality.

Since the Euclidean space $\mathbb{R}^n$ is n-regular and supports p-Poincaré inequalities whenever $1 \leq p < \infty$, for $X = \mathbb{R}^n$ the results of Theorems 1.1 and 1.2 can be formulated as follows:

Corollary 1.3. *Let $1 < p < \infty$ and $\beta < p - 1$, and let $\Omega \subset \mathbb{R}^n$ be an open set. If*

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < n - p + \beta \quad \text{or} \quad \underline{\dim}_{\mathrm{A}}(\Omega^c) > n - p + \beta,$$

then Ω admits a (p,β)-Hardy inequality; in the latter case, if Ω is unbounded, then we require that also Ω^c is unbounded.

Here $\overline{\dim}_{\mathrm{A}} = \dim_{\mathrm{A}}$ is the (upper) Assouad dimension and $\underline{\dim}_{\mathrm{A}}$ is the lower Assouad dimension (see Section 2). In fact, Corollary 1.3 holds, with n replaced with Q, in any Q-regular metric space supporting a 1-Poincaré inequality; prime examples of such spaces are the Carnot groups. Let us mention here that in the recent work [8], which has been prepared independently of the present paper, the authors establish similar sufficient conditions for *fractional* Hardy inequalities in $\mathbb{R}^n$.

The sharpness of Theorems 1.1 and 1.2 will be discussed in detail in Section 8, but let us mention here some of the relevant facts. First of all, the bound $p-\beta$ for the codimensions is very natural and sharp. Indeed, we show in Theorem 6.1 that if Ω admits a (p,β)-Hardy inequality, then either $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c) < p-\beta$ or $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p-\beta$, and it is clear that for sufficiently regular Ω^c we have $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c) = \overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c)$. Nevertheless, it is not necessary for a (p,β)-Hardy inequality that either $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) < p-\beta$ or $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p-\beta$, since suitable local combinations of the assumptions in Theorems 1.1 and 1.2 yield sufficient conditions for Hardy inequalities as well (cf. Section 7), and in such cases typically only the bound $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c) < p-\beta$ is satisfied. There is also a corresponding local dimension dichotomy for Hardy inequalities, see Theorem 6.2.

The requirement $p-\beta > 1$ is also sharp in both of the theorems. For instance, the unit ball $B = B(0,1) \subset \mathbb{R}^n$ gives a simple counterexample for the result of Theorem 1.2 in the case $0 < p-\beta \le 1$, since $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\mathbb{R}^n \setminus B) = 0$, but B admits (p,β)-Hardy inequalities only when $p-\beta > 1$. However, under additional conditions on Ω these Hardy inequalities can also be proven in the case $p-\beta \le 1$, see Remark 4.1 and the discussion in Section 8. The unboundedness assumptions of X and Ω^c in Theorems 1.1 and 1.2, respectively, can not be relaxed either, as the examples at the end of Section 8 show. The only assumption whose role is not completely understood at the moment is the $(p-\beta)$-Poincaré inequality in the cases $\beta > 0$ of the theorems; a p-Poincaré inequality is certainly necessary in any of the cases. See Remark 4.1 for a related discussion.

Part of the motivation for the present work stems from the connection between Hardy inequalities and the so-called quasiadditivity property of the variational capacity. Some aspects of such a connection have been visible e.g. in [2, 3, 23, 25], but only recently it was shown in [29] that quasiadditivity of the p-capacity with respect to Ω and the validity of a p-Hardy inequality in Ω are essentially equivalent conditions (under some mild assumptions on the space X or the open set Ω). An assumption equivalent to the dimension bound $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(E) > p$ (or rather $\overline{\dim}_{\mathrm{A}}(E) < n-p$) was used already by Aikawa [1] in connection to the quasiadditivity of the Riesz capacity $R_{1,p}$ with respect to Whitney decompositions of the complement $\mathbb{R}^n \setminus E$. In order to obtain a corresponding result for the variational p-capacity (in metric spaces), the unweighted case $\beta = 0$ of Theorem 1.1 was deduced in [29, Prop. 3] under an additional accessibility condition (note that the lower Assouad codimension is called the *Aikawa codimension* in [29] and in [30], see Section 2 for a discussion). Now Theorem 1.1 makes such an additional condition unnecessary, and thus we have the following corollary to Theorem 1.1 and [29, Thm 1], yielding a complete analogy with the results of Aikawa [1, 3].

Corollary 1.4. *Let $1 < p < \infty$, and assume that X is an unbounded doubling metric space supporting a p-Poincaré inequality. If $\Omega \subset X$ is an open set satisfying $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p$, then the variational p-capacity $\operatorname{cap}_p(\cdot,\Omega)$ is quasiadditive with respect to Whitney covers $\mathcal{W}_c(\Omega)$ for suitably small parameters $c > 0$.*

We refer to [29] for all the relevant definitions. Let us also point out that the proof of the corresponding Hardy inequalities in [29] is more straight-forward than the proof of Theorem 1.1 here, and so the proof from [29] may actually be preferred in the cases where the accessibility condition is known to hold.

The organization of the rest of the paper is as follows. In Section 2 we recall the necessary background material concerning metric spaces and the various notions of dimension. Section 3 contains a proof of the case $\beta < 0$ of Theorem 1.1, and the case $0 \le \beta < p-1$ is then established in the following Section 4. The relation between Hausdorff (co)content density and the upper Assouad codimension is studied in Section 5 with the help of a measure distribution procedure. This section also contains the proof of Theorem 1.2. The necessary conditions for Hardy inequalities are the topic of Section 6. Finally, in Section 7 we discuss the case where the complement contains both thick and thin parts, and in Section 8 we give examples which indicate the sharpness of our assumptions.

For the notation we remark that C and c will denote positive constants whose values are not necessarily the same at each occurrence. If there exist constants $c_1, c_2 > 0$ such that $c_1 F \le G \le c_2 F$, we sometimes write $F \simeq G$ and say that F and G are comparable.

2. Metric spaces and concepts of dimension

We assume throughout this paper that $X = (X, d, \mu)$ is a complete metric measure space, where μ is a Borel measure supported on X, with $0 < \mu(B) < \infty$ whenever $B = B(x, r) := \{y \in X : d(x, y) \le r\}$ is a (closed) ball in X. In addition, we assume that μ is *doubling*, that is, there is a constant $C > 0$ such that whenever $x \in X$ and $r > 0$, we have

$$\mu(B(x, 2r)) \le C\, \mu(B(x, r)).$$

The completeness of X is actually not needed in all of our results, but for simplicity we still keep this as a standing assumption. We also make the tacit assumption that each ball $B \subset X$ has a fixed center x_B and radius $\operatorname{rad}(B)$ (but these need not be unique), and thus notation such as $\lambda B = B(x_B, \lambda \operatorname{rad}(B))$ is well-defined for all $\lambda > 0$. The diameter of a set $E \subset X$ is denoted $\operatorname{diam}(E)$, the distance from a point x to E is $\operatorname{dist}(x, E)$, and χ_E denotes the characteristic function of E.

We also say that the measure μ is Q-regular, if there is a constant $C \ge 1$ such that

$$C^{-1} r^Q \le \mu(B(x, r)) \le C r^Q$$

for all $x \in X$ and every $0 < r < \operatorname{diam}(X)$.

Given a measurable function $f\colon X \to [-\infty, \infty]$, a Borel measurable non-negative function g on X is an *upper gradient* of f if whenever γ is a compact rectifiable curve in X, we have

$$|f(y) - f(x)| \le \int_\gamma g\, ds.$$

Here x and y are the two endpoints of γ, and the above condition should be interpreted as claiming that $\int_\gamma g\, ds = \infty$ whenever at least one of $|f(x)|, |f(y)|$ is infinite. See e.g. [4, 15, 16] for introduction on analysis on metric spaces based on the notion of upper gradients.

In addition to the doubling property, we will also assume throughout the paper that the space X supports a $(1, p)$-Poincaré inequality (or simply p-Poincaré inequality) for $1 \le p < \infty$, that is, there exist constants $C > 0$ and $\lambda \ge 1$ such that whenever $B = B(x, r) \subset X$ and g is an upper gradient of a measurable function f, we have

$$\fint_B |f - f_B|\, d\mu \le C\, r \left(\fint_{\lambda B} g^p\, d\mu \right)^{1/p}$$

where

$$f_B := \frac{1}{\mu(B)} \int_B f\, d\mu =: \fint_B f\, d\mu.$$

To be more precise, we keep a p-Poincaré inequality as a standing assumption, but as was already seen in Theorems 1.1 and 1.2, we occasionally require even stronger Poincaré inequalities.

Moreover, we will rely in some of our formulations on the fundamental result of Keith and Zhong [19] on the self-improvement of Poincaré inequalities: If $1 < p < \infty$ and a complete doubling metric space X supports a p-Poincaré inequality, then there exist $1 \le p_0 < p$ such that X supports also a p_0-Poincaré inequality, and hence actually p'-Poincaré inequalities for all $p' \ge p_0$.

One consequence of Poincaré inequalities for the geometry of X is that a space supporting a p-Poincaré inequality is *quasiconvex*. This means that there exists $C \ge 1$ such that each pair of points $x, y \in X$ can be joined using a rectifiable curve $\gamma_{x,y}$ of length

$\ell(\gamma_{x,y}) \le Cd(x,y)$; see e.g. [4] for details. From quasiconvexity we obtain the useful fact that if $\Omega \subset X$ is an open set, then $d_\Omega(x) := \operatorname{dist}(x,\Omega^c) \le C \operatorname{dist}(x,\partial\Omega) \le C d_\Omega(x)$ for all $x \in \Omega$.

Let $\Omega \subset X$. A function $u\colon \Omega \to \mathbb{R}$ is said to be *(L-)Lipschitz*, if

$$|u(x) - u(y)| \le L d(x,y) \qquad \text{for all } x, y \in \Omega.$$

We denote the set of all Lipschitz functions $u\colon \Omega \to \mathbb{R}$ by $\operatorname{Lip}(\Omega)$. In addition, $\operatorname{Lip}_0(\Omega)$ (resp. $\operatorname{Lip}_b(\Omega)$) denotes the set of Lipschitz functions with compact (resp. bounded) support in Ω. Recall that the support of a function $u\colon \Omega \to \mathbb{R}$, denoted $\operatorname{spt}(u)$, is the closure of the set where u is non-zero.

The *upper and lower pointwise Lipschitz constants* of a function $u\colon \Omega \to \mathbb{R}$ at $x \in \Omega$ are

$$\operatorname{Lip}(u;x) = \limsup_{y\to x} \sup_{y \in B(x,r)} \frac{|u(x)-u(y)|}{r}$$

and

$$\operatorname{lip}(u;x) = \liminf_{y\to x} \sup_{y \in B(x,r)} \frac{|u(x)-u(y)|}{r},$$

respectively. It is not hard to see that both of these are upper gradients of a (locally) Lipschitz function $u\colon \Omega \to \mathbb{R}$ (cf. [4, Proposition 1.14]). In $\mathbb{R}^n$, on the other hand, $|\nabla u|$ is a (minimal weak) upper gradient of $u \in \operatorname{Lip}(\mathbb{R}^n)$, consult e.g. [4] for the result and the terminology.

Let us now recall the various notions of dimension that will be important for us throughout the paper. Let $E \subset X$. The *(upper) Assouad dimension* of E, denoted $\overline{\dim}_{\mathrm{A}}(E)$ (or simply $\dim_{\mathrm{A}}(E)$), is the infimum of exponents $s \ge 0$ for which there is a constant $C \ge 1$ such that for all $x \in E$ and every $0 < r < R < \operatorname{diam}(X)$, the set $E \cap B(x,R)$ can be covered by at most $C(r/R)^{-s}$ balls of radius r. Notice that for $\operatorname{diam}(E) \le r < \operatorname{diam}(X)$ this condition is trivial. We remark that this upper Assouad dimension is the "usual" Assouad dimension found in the literature. See Luukkainen [32] for the basic properties and a historical account on the (upper) Assouad dimension.

Conversely to the above definition, in [18] the *lower Assouad dimension* of E, $\underline{\dim}_{\mathrm{A}}(E)$, was defined to be the supremum of exponents $t \ge 0$ for which there is a constant $c > 0$ so that if $0 < r < R < \operatorname{diam}(E)$, then for every $x \in E$ at least $c(r/R)^{-t}$ balls of radius r are needed to cover $E \cap B(x,R)$; if $\operatorname{diam}(E) = 0$, we omit the upper bound for R. Closely related concepts have been considered e.g. by Larman [24] and Farser [10], but an important difference in our definition is that we consider all radii $0 < r < \operatorname{diam}(E)$, not just small radii; in the context of Hardy inequalities this turns out to be essential.

For comparison, recall that the *upper Minkowski dimension* of a compact $E \subset X$, denoted $\overline{\dim}_{\mathrm{M}}(E)$, is the infimum of $\lambda \ge 0$ such that the whole set E can be covered by at most $Cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$, and the *lower Minkowski dimension*, $\underline{\dim}_{\mathrm{M}}(E)$, is the supremum of $\lambda \ge 0$ for which at least $cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$ are needed to cover E. It follows immediately that $\underline{\dim}_{\mathrm{A}}(E) \le \underline{\dim}_{\mathrm{M}}(E) \le \overline{\dim}_{\mathrm{M}}(E) \le \overline{\dim}_{\mathrm{A}}(E)$.

Since a doubling metric space is separable, there exists for all $r > 0$ a maximal r-packing of $E \subset X$, that is, a countable collection $\mathcal{B}$ of pairwise disjoint balls $B(x_i, r)$, with $x_i \in E$, such that for each $x \in E$ there is $B \in \mathcal{B}$ intersecting $B(x,r)$. It is obvious that if $\{B_i\}_i$ is a maximal packing of E, then $\{2B_i\}_i$ is a cover of E.

When working in a (non-regular) metric space X, it is often convenient to describe the sizes of sets in terms of *codimensions* rather than dimensions. For instance, the *Hausdorff codimension* of $E \subset X$ (with respect to μ) is the number

$$\operatorname{co}\dim_{\mathrm{H}}(E) = \sup\bigl\{q \ge 0 : \mathcal{H}^{\mu,q}_R(E) = 0\bigr\},$$

where

$$\mathcal{H}^{\mu,q}_R(E) = \inf\Big\{ \sum_k \operatorname{rad}(B_k)^{-q}\mu(B_k) : E \subset \bigcup_k B_k,\ \operatorname{rad}(B_k) \le R \Big\}$$

is the *Hausdorff content of codimension* q; if $\mu(E) > 0$, then we set $\operatorname{co\,dim}_{\mathrm{H}}(E) = 0$. If μ is Q-regular, then we have for all $E \subset X$ that $Q - \operatorname{co\,dim}_{\mathrm{H}}(E) = \dim_{\mathrm{H}}(E)$, the usual Hausdorff dimension.

We define next the Assouad codimensions following [18]:

When $E \subset X$ and $r > 0$, the (open) r-neighborhood of E is the set $E_r = \{x \in X : \operatorname{dist}(x, E) < r\}$. The *lower Assouad codimension*, denoted $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$, is the supremum of all $t \ge 0$ for which there exists a constant $C \ge 1$ such that

$$\frac{\mu(E_r \cap B(x,R))}{\mu(B(x,R))} \le C\Big(\frac{r}{R}\Big)^t$$

for every $x \in E$ and all $0 < r < R < \operatorname{diam}(X)$. Conversely, the *upper Assouad codimension* of $E \subset X$, denoted $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$, is the infimum of all $s \ge 0$ for which there is $c > 0$ such that

$$\frac{\mu(E_r \cap B(x,R))}{\mu(B(x,R))} \ge c\Big(\frac{r}{R}\Big)^s$$

for every $x \in E$ and all $0 < r < R < \operatorname{diam}(E)$. If $\operatorname{diam}(E) = 0$, we omit the upper bound for R.

If μ is Q-regular, then it is not hard to see that

$$\overline{\dim}_{\mathrm{A}}(E) = Q - \underline{\operatorname{co\,dim}}_{\mathrm{A}}(E) \quad \text{and} \quad \underline{\dim}_{\mathrm{A}}(E) = Q - \overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$$

for all $E \subset X$ (cf. [18]).

Remark 2.1. It was shown in [30, Thm. 5.1] that the lower Assouad codimension can also be characterized as the supremum of all $q \ge 0$ for which there exists a constant $C \ge 1$ such that

$$\int_{B(x,r)} \operatorname{dist}(y,E)^{-q}\, d\mu(y) \le C r^{-q} \mu(B(x,r)) \tag{1}$$

for every $x \in E$ and all $0 < r < \operatorname{diam}(X)$. (Here we interpret the integral to be $+\infty$ if $q > 0$ and E has positive measure.)

A concept of dimension defined via integrals as in (1) was used by Aikawa in [1] for subsets of $\mathbb{R}^n$ (see also [3]). Thus, in [29, 30], where the interest originates from such integral estimates, the lower Assouad codimension was called the *Aikawa codimension*. We will see later, especially in Section 6, that this kind of integral estimates arise very naturally in connection to Hardy inequalities.

The next lemma records the fact that the Aikawa condition (1) enjoys self-improvement. This property is a direct consequence of the famous self-improvement result for reverse Hölder inequalities (in $\mathbb{R}^n$ due to Gehring [11]), and we will need this in Section 6 when proving our necessary conditions for Hardy inequalities. For $q > 1$, the result of Lemma 2.2 is contained in the proof of Lemma 2.4 in [23], and in the proof of Proposition 4.3 in [17] the same fact is used in $\mathbb{R}^n$.

Lemma 2.2. *Let $0 < R_0 \le \infty$ and assume that $E \subset X$ satisfies the Aikawa condition* (1) *for every $x \in E$ and all $0 < r < R_0$ with an exponent $q > 0$ and a constant $C_0 > 0$. Then there exist $\delta > 0$ and $C > 0$, depending only on the given data, such that condition* (1) *holds for every $x \in E$ and all $0 < r < R_0$ with the exponent $q + \delta$ and the constant C.*

Proof. The proof is based on the metric space version of the Gehring Lemma; see e.g. [4, Thm. 3.22] or [35, p. 11].

Fix any $0 < s < q$, e.g. $s = q/2$, and let $B_0 = B(x,R)$ with $x \in E$ and $0 < R < R_0$. In addition, let B be a ball such that $B \subset B_0$. If $4B \cap E \neq \emptyset$, then we find a ball B' centered at E and with a radius comparable to $\mathrm{rad}(B)$ such that $B \subset B'$, and thus we have by (1) and doubling that

$$\int_B \mathrm{dist}(y,E)^{-q}\,d\mu \le \int_{B'} \mathrm{dist}(y,E)^{-q}\,d\mu \le C_0\mu(B')\,\mathrm{rad}(B')^{-q}$$
$$\le C\mu(B)\big(\mathrm{rad}(B)^{-s}\big)^{q/s} \le C\mu(B)\bigg(\fint_{2B} \mathrm{dist}(y,E)^{-s}\,d\mu\bigg)^{q/s};$$

in the last inequality we used the fact that $\mathrm{dist}(y,E)^{-s} \ge C\,\mathrm{rad}(B)^{-s}$ for all $y \in 2B$. In particular we obtain the reverse Hölder inequality

$$\bigg(\fint_B \mathrm{dist}(y,E)^{-q}\,d\mu\bigg)^{s/q} \le C\fint_{2B} \mathrm{dist}(y,E)^{-s}\,d\mu. \tag{2}$$

On the other hand, if $4B \cap E = \emptyset$, then $\mathrm{dist}(y,E) \simeq \mathrm{dist}(x,E)$ for all $y \in 2B$, and thus (2) holds in this case as well.

Since the function $f(y) = \mathrm{dist}(y,E)^{-s} \in L^1(B_0)$ now satisfies the assumption of the Gehring Lemma [4, Thm. 3.22] for all balls $B \subset B_0$, the proof in [4] shows that there is $\delta > 0$ such that we have for the ball $B_1 = \frac{1}{2}B_0$ that

$$\bigg(\fint_{B_1} \mathrm{dist}(y,E)^{-(q+\delta)}\,d\mu\bigg)^{s/(q+\delta)} \le C\fint_{2B_1} \mathrm{dist}(y,E)^{-s}\,d\mu$$
$$\le C\bigg(\fint_{2B_1} \mathrm{dist}(y,E)^{-q}\,d\mu\bigg)^{s/q} \le C\,\mathrm{rad}(B_1)^s,$$

where we also used Hölder's inequality and the original estimate (1). Moreover, here the constant $C > 0$ is independent of the ball B_1. The claim follows, since for balls B_1 with $R_0/2 \le \mathrm{rad}(B_1) < R_0$ we can consider a cover using smaller balls. □

3. A weighted Hardy inequality for $\beta < 0$

As a first step towards Theorem 1.1, we establish in this section the result in the case $\beta < 0$. The particular form of the constant in the (p,β)-Hardy inequalities below plays an important role in the proof of the general case of Theorem 1.1. Notice that here the test functions are not required to vanish in Ω^c, and recall that we have the standing assumption that X is a complete doubling metric space supporting a p-Poincaré inequality.

Proposition 3.1. *Assume that X is unbounded and let $1 \le p < \infty$ and $\beta < 0$. If $\Omega \subset X$ is an open set with $\underline{\mathrm{co\,dim}}_{\mathrm{A}}(\Omega^c) > p - \beta$, then Ω admits a (p,β)-Hardy inequality, and, in fact, the (p,β)-Hardy inequality holds for all $u \in \mathrm{Lip}_b(\Omega)$.*

Moreover, there exists $\widetilde{\beta} < 0$ such that for $\widetilde{\beta} < \beta < 0$ the constant in the (p,β)-Hardy inequality can be chosen to be $C = |\beta|^{-1}C^* > 0$, where $C^* > 0$ is independent of β.

Proof. Write $E = \Omega^c$. For each $k \in \mathbb{Z}$, let $\mathcal{B}_k = \{B_{k,i}\}$ be a maximal packing of E with balls $B_{k,i} = B(x_{k,i}, 2^k)$, $x_{k,i} \in E$, and write $N_k = \bigcup_i 4B_{k,i}$ and $A_k = N_k \setminus N_{k-1}$. We can then choose for each $B = B_{k,i} \in \mathcal{B}_k$ balls $B^j \in \mathcal{B}_j$, $j \ge k$, such that $B = B^k$ and $4B^j \subset 4B^{j+1}$ for all $j \ge k$. Indeed, if x^j is the center of B^j, there is $B^{j+1} = B(x^{j+1}, 2^{j+1}) \in \mathcal{B}_{j+1}$ such that $x^j \in 2B^{j+1}$, and hence $4B^j \subset B(x^{j+1}, 2\cdot 2^{j+1} + 4\cdot 2^j) = 4B^{j+1}$. In particular, it follows that $B \subset 4B^j$ for all $j \ge k$.

Let $u \in \mathrm{Lip}_b(\Omega)$. Since X is unbounded, we have for each $B_{k,i}$ that $\fint_{4B^j_{k,i}} u \to 0$ as $j \to \infty$. A standard telescoping trick using the p-Poincaré inequality then yields for every $B = B_{k,i} \in \mathcal{B}_k$ that

$$|u_{4B}| \le \sum_{j=k}^{\infty} |u_{4B^j} - u_{4B^{j+1}}| \le C\sum_{j=k}^{\infty} 2^j \bigg(\fint_{4\lambda B^j} g_u^p\,d\mu\bigg)^{1/p}. \tag{3}$$

Comparison of the sum on the right-hand side of (3) with the convergent geometric series $\sum_{j=k}^{\infty} 2^{(k-j)\delta}$, for any $\delta > 0$, shows that there exists a constant $C_1(\delta) > 0$, independent of u and B, and an index $j(B) \geq k$ such that

$$2^{j(B)}\Big(\fint_{4\lambda B^{j(B)}} g_u^p \, d\mu\Big)^{1/p} \geq C_1 |u_{4B}| 2^{(k-j(B))\delta}. \tag{4}$$

Now fix q_1, q_2 such that $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > q_1 > q_2 > p - \beta$, and set $\delta = (q_2 - p + \beta)/p > 0$. We obtain from (4) for each $B \in \mathcal{B}_k$ a ball $B^{j(B)}$ of radius $2^{j(B)}$ satisfying

$$2^{k(q_2-p+\beta)}|u_{4B}|^p \leq C_1 (2^{j(B)})^{q_2+\beta} \mu\big(B^{j(B)}\big)^{-1} \int_{4\lambda B^{j(B)}} g_u^p \, d\mu. \tag{5}$$

Let us now start to estimate the left-hand side of the (p,β)-Hardy inequality. Since $d_\Omega(x) \geq 2^{k-1}$ for $x \in A_k$, we have

$$\begin{aligned}
\int_\Omega |u|^p d_\Omega{}^{\beta-p} \, d\mu &= \sum_{k=-\infty}^{\infty} \int_{A_k} |u|^p d_\Omega{}^{\beta-p} \, d\mu \\
&\leq 2^{p-\beta} \sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \int_{A_k} |u|^p \, d\mu \leq 2^{p-\beta} \sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \sum_{B \in \mathcal{B}_k} \int_{4B} |u|^p \, d\mu \\
&\leq C_2 \sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k} \int_{4B} |u - u_{4B}|^p \, d\mu + C_2 \sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \sum_{B \in \mathcal{B}_k} \int_{4B} |u_{4B}|^p \, d\mu,
\end{aligned} \tag{6}$$

where $C_2 = 2^{p-\beta}C'$ and $C' = 2^{p-1} > 0$ is independent of β. The first sum in the last line of (6) can be estimated with the help of the (p,p)-Poincaré inequality (which is a well-known consequence of the p-Poincaré inequality, see e.g. [13, 15]) and the controlled overlap of the balls $4\lambda B$ for $B \in \mathcal{B}_k$ (with a fixed $k \in \mathbb{Z}$, by doubling). We also rewrite the integral, change the order of summation, and use the simple estimate $d_\Omega(x) \leq 4 \cdot 2^i$ for $x \in A_i$, as follows:

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \sum_{B \in \mathcal{B}_k} \int_{4B} |u - u_{4B}|^p \, d\mu &\leq C \sum_{k=-\infty}^{\infty} 2^{k\beta} \sum_{B\in\mathcal{B}_k} \int_{4\lambda B} g_u^p \, d\mu \\
&\leq C \sum_{k=-\infty}^{\infty} 2^{k\beta} \int_{N_{k+M}} g_u^p \, d\mu = C 2^{-M\beta} \sum_{k=-\infty}^{\infty} 2^{k\beta} \int_{N_k} g_u^p \, d\mu \\
&= C 2^{-M\beta} \sum_{k=-\infty}^{\infty} 2^{k\beta} \sum_{i=-\infty}^{k} \int_{A_i} g_u^p \, d\mu = C 2^{-M\beta} \sum_{i=-\infty}^{\infty} \int_{A_i} g_u^p \, d\mu \sum_{k=i}^{\infty} 2^{k\beta} \\
&= C 2^{-M\beta} \sum_{i=-\infty}^{\infty} \frac{2^{i\beta}}{1-2^\beta} \int_{A_i} g_u^p \, d\mu \leq C \frac{2^{-M\beta}4^{-\beta}}{1-2^\beta} \sum_{i=-\infty}^{\infty} \int_{A_i} g_u^p d_\Omega{}^\beta \, d\mu \\
&\leq C \frac{2^{-(M+2)\beta}}{1-2^\beta} \int_\Omega g_u^p d_\Omega{}^\beta \, d\mu.
\end{aligned} \tag{7}$$

Above the constants $C > 0$ and $M = M(\lambda) \geq 4$ are independent of β; note also how the assumption $\beta < 0$ was needed.

In the last sum of (6) we first use (5) and then change the order of summation:

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k} \int_{4B} |u_{4B}|^p \, d\mu &= \sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k} \mu(4B)|u_{4B}|^p \\
&\le C \sum_{k=-\infty}^{\infty} 2^{-q_2 k} \sum_{B\in\mathcal{B}_k} \mu(4B)(2^{j(B)})^{q_2+\beta} \mu\big(B^{j(B)}\big)^{-1} \int_{4\lambda B^{j(B)}} g_u^p \, d\mu \\
&= C \sum_{j=-\infty}^{\infty} \sum_{\tilde{B}\in\mathcal{B}_j} 2^{j\beta} \int_{4\lambda\tilde{B}} g_u^p \, d\mu \sum_{k\le j} \sum_{\{B\in\mathcal{B}_k : \tilde{B}=B^{j(B)}\}} 2^{-q_2 k} 2^{q_2 j} \mu(4B)\mu(\tilde{B})^{-1} \\
&\le C \sum_{j=-\infty}^{\infty} \sum_{\tilde{B}\in\mathcal{B}_j} 2^{j\beta} \int_{4\lambda\tilde{B}} g_u^p \, d\mu \sum_{k\le j} \sum_{\{B\in\mathcal{B}_k : B\subset 4\tilde{B}\}} \frac{\mu(B)2^{-q_2 k}}{\mu(\tilde{B})2^{-q_2 j}}.
\end{aligned}
\tag{8}
$$

Since the balls B, for $B \in \mathcal{B}_k$, are pairwise disjoint, the assumption $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(E) > q_1 > q_2$ implies (recall here that $E_{2^k} = \{x \in X : \operatorname{dist}(x, E) < 2^k\}$)

$$
\begin{aligned}
\sum_{k\le j} \sum_{\{B\in\mathcal{B}_k : B\subset 4\tilde{B}\}} \frac{\mu(B)2^{-q_2 k}}{\mu(\tilde{B})2^{-q_2 j}} &\le C \sum_{k\le j} \frac{\mu(E_{2^k}\cap 4\tilde{B})}{\mu(4\tilde{B})} \left(\frac{2^k}{2^j}\right)^{-q_1} \left(\frac{2^k}{2^j}\right)^{q_1-q_2} \\
&\le C \sum_{k\le j} \left(\frac{2^k}{2^j}\right)^{q_1-q_2} \le C(q_1, q_2).
\end{aligned}
$$

Thus we obtain from (8), using also the bounded overlap of $4\lambda\tilde{B}$, that

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k} \int_{4B} |u_{4B}|^p \, d\mu &\le C \sum_{j=-\infty}^{\infty} \sum_{\tilde{B}\in\mathcal{B}_j} 2^{j\beta} \int_{4\lambda\tilde{B}} g_u^p \, d\mu \\
&\le C \sum_{j=-\infty}^{\infty} 2^{j\beta} \int_{N_{j+M}} g_u^p \, d\mu \le C \frac{2^{-(M+2)\beta}}{1-2^\beta} \int_\Omega g_u^p d_\Omega{}^\beta \, d\mu,
\end{aligned}
\tag{9}
$$

where the last inequality follows just like in (7). A combination of (6), (7), and (9) thus yields the (p, β)-Hardy inequality

$$
\int_\Omega |u(x)|^p d_\Omega(x)^{\beta-p} \, d\mu \le C \frac{2^{-(M+3)\beta}}{1-2^\beta} \int_\Omega g_u(x)^p d_\Omega(x)^\beta \, d\mu,
\tag{10}
$$

where the constant $C > 0$ is independent of β, but depends on δ (cf. (3)), q_1, q_2, p, and the data associated to X.

We conclude the proof with a closer examination of the constant in (10). First of all, if $-1 < \beta < 0$, then $2^{-(M+3)\beta} \le 2^{M+3}$, and when β is close enough to 0, then $1 - 2^\beta \simeq -\beta$. In addition, the constant C in (10) depends on δ, q_1 and q_2, and hence indirectly on β as well, since $\delta = (q_2 - p + \beta)/p$ and $p - \beta < q_1 < q_2 < \underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c)$. Nevertheless, if e.g. $\big(p - \underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c)\big)/2 < \beta < 0$, then this constant can obviously be chosen to depend only on p and $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c)$ (and the data associated to X). It follows that there exists $\widetilde{\beta} < 0$, depending on p and $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c)$, such that for all $\widetilde{\beta} < \beta < 0$ we have

$$
\int_\Omega |u(x)|^p d_\Omega(x)^{\beta-p} \, d\mu \le \frac{C^*}{|\beta|} \int_\Omega g_u(x)^p d_\Omega(x)^\beta \, d\mu,
$$

where the constant $C^* > 0$ is independent of the particular β. □

4. The case $0 \le \beta < p-1$ of Theorem 1.1

We now turn to the proof of the weighted (p,β)-Hardy inequality in the case $0 \le \beta < p-1$ under the assumption $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p-\beta$. The proof is based on Proposition 3.1, and it combines ideas from [36, 23, 26]. In fact, in the Euclidean case the result can be readily deduced from the (p,β)-Hardy inequalities of Proposition 3.1 with a careful use of [26, Lemma 2.1].

Proof of Theorem 1.1. For $\beta < 0$, the claim follows from Proposition 3.1. and thus we are left with the case $0 \le \beta < p-1$. Since $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p-\beta > 1$, we have by Proposition 3.1 that Ω admits an $(p-\beta,-\beta_0)$-Hardy inequality whenever $0 < \beta_0 < \underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) - p + \beta$; here we need to know that X supports a $(p-\beta)$-Poincaré inequality. Moreover, there exists $\widetilde{\beta}_0 > 0$ such that for $0 < \beta_0 < \widetilde{\beta}_0$ the constant in the $(p-\beta,-\beta_0)$-Hardy inequality is $C^*\beta_0^{-1}$, with $C^* > 0$ independent of β_0. Fix such β_0 to be chosen later.

Let $u \in \operatorname{Lip}_0(\Omega)$ with an upper gradient g_u, and define

$$v(x) = |u(x)|^{p/(p-\beta)} d_\Omega(x)^{\beta_0/(p-\beta)}.$$

Then v is a Lipschitz-function with a compact support in Ω, $|u(x)|^p = |v(x)|^{p-\beta} d_\Omega(x)^{-\beta_0}$, and, moreover, the function

$$g_v(x) := \tfrac{p}{p-\beta}|u(x)|^{\beta/(p-\beta)} g_u(x) d_\Omega(x)^{\beta_0/(p-\beta)} + \tfrac{\beta_0}{p-\beta}|u(x)|^{p/(p-\beta)} d_\Omega(x)^{(\beta_0-p+\beta)/(p-\beta)} \tag{11}$$

is an upper gradient of v (cf. e.g. [4, Thm. 2.15 and 2.16]); here it is essential that the support of u is a compact set inside Ω. Using the $(p-\beta,-\beta_0)$-Hardy inequality of Proposition 3.1 for v, we obtain

$$\begin{aligned} \int_\Omega |u|^p {d_\Omega}^{\beta-p}\, d\mu &= \int_\Omega |v|^{p-\beta} {d_\Omega}^{-\beta_0-(p-\beta)}\, d\mu \\ &\le C^*\beta_0^{-1} \int_\Omega g_v^{p-\beta} {d_\Omega}^{-\beta_0}\, d\mu. \end{aligned} \tag{12}$$

By (11) and Hölder's inequality (for exponents $\frac{p}{\beta}$ and $\frac{p}{p-\beta}$), we estimate the above integral for g_v as

$$\begin{aligned} \int_\Omega g_v^{p-\beta} {d_\Omega}^{-\beta_0}\, d\mu &\le 2^{p-\beta}\Big(\tfrac{p}{p-\beta}\Big)^{p-\beta} \int_\Omega |u|^\beta g_u^{p-\beta} {d_\Omega}^{\beta_0-\beta_0}\, d\mu \\ &\quad + 2^{p-\beta}\Big(\tfrac{\beta_0}{p-\beta}\Big)^{p-\beta} \int_\Omega |u|^p {d_\Omega}^{\beta_0-p+\beta-\beta_0}\, d\mu \\ &\le C(p,\beta) \int_\Omega \Big(|u|^\beta {d_\Omega}^{\frac{\beta(\beta-p)}{p}}\Big)\Big(g_u^{p-\beta} {d_\Omega}^{\frac{\beta(p-\beta)}{p}}\Big)\, d\mu \\ &\quad + C(p,\beta){\beta_0}^{p-\beta} \int_\Omega |u|^p {d_\Omega}^{\beta-p}\, d\mu \\ &\le C(p,\beta)\Big(\int_\Omega |u|^p {d_\Omega}^{\beta-p}\, d\mu\Big)^{\frac{\beta}{p}} \Big(\int_\Omega g_u^p {d_\Omega}^{\beta}\, d\mu\Big)^{\frac{p-\beta}{p}} \\ &\quad + C(p,\beta){\beta_0}^{p-\beta} \int_\Omega |u|^p {d_\Omega}^{\beta-p}\, d\mu, \end{aligned} \tag{13}$$

where the constant $C(p,\beta) = 2^{p-\beta}\big(p/(p-\beta)\big)^{p-\beta}$ is independent of β_0.

We now choose $0 < \beta_0 < \widetilde{\beta}_0$ to be so small that

$$C^*\beta_0^{-1} C(p,\beta){\beta_0}^{p-\beta} = C^* C(p,\beta){\beta_0}^{p-\beta-1} < \tfrac{1}{2}.$$

This is possible since $p-\beta > 1$ and the factor $C^*C(p,\beta)$ does not depend on β_0. After the insertion of (13) into (12), we observe that under the above choice of β_0, the second term emerging on the right-hand side is less than half of the left-hand side, and thus we obtain

$$\int_\Omega |u|^p d_\Omega{}^{\beta-p}\,d\mu \le C\Big(\int_\Omega |u|^p d_\Omega{}^{\beta-p}\,d\mu\Big)^{\frac{\beta}{p}}\Big(\int_\Omega g_u^p d_\Omega{}^{\beta}\,d\mu\Big)^{\frac{p-\beta}{p}}. \tag{14}$$

The (p,β)-Hardy inequality for u now follows from (14) by dividing with the first factor on the right-hand side (which we may assume to be non-zero), and then taking both sides to power $p/(p-\beta)$. □

Notice that the requirement $p-\beta>1$ is essential in the above proof. This is not merely a technical assumption, since Theorem 1.1 need not hold if $p-\beta\le 1$; see Section 8.

Remark 4.1. It would be interesting to know if there is a more direct proof for the case $\beta\ge 0$, i.e. one avoiding the use of the case $\beta<0$. This is strongly related to the question what Poincaré inequalities are actually needed in Theorem 1.1. The same question applies to Theorem 1.2 as well in the case $\beta>0$ (cf. the proof at the end of Section 5).

Here it is good to recall that when Ω^c (or actually $\partial\Omega$) satisfies additional accessibility conditions from within Ω, then such direct proofs exist. Moreover, under these accessibility conditions the results of Theorems 1.1 and 1.2 can be extended to the case $\beta\ge p-1$, see e.g. [22, Thm. 1.4], [25, Thm. 4.3], and [28, Thm. 4.5].

Remark 4.2. An interesting special case of Theorem 1.1 is that where X is unbounded and the distance function $d_\Omega(x)$ is replaced by the distance to a fixed point $x_0\in X$, i.e., we have the inequality

$$\int_X |u(x)|^p\,d(x,x_0)^{\beta-p}\,d\mu \le C\int_X g_u(x)^p\,d(x,x_0)^{\beta}\,d\mu. \tag{15}$$

It follows from Theorem 1.1 that this inequality holds for all $u\in \mathrm{Lip}_0(X\setminus\{x_0\})$ when $\underline{\operatorname{co\,dim}}_{\mathrm A}(\{x_0\})>p-\beta>1$, and in fact, by Corollary 6.6 below, u need not vanish at x_0, so the inequality actually holds for all $u\in\mathrm{Lip}_b(X)$. On the other hand, Theorem 1.2 implies that for $\overline{\operatorname{co\,dim}}_{\mathrm A}(\{x_0\})<p-\beta$ inequality (15) is valid for all $u\in\mathrm{Lip}_0(X\setminus\{x_0\})$.

Let us mention here that the lower and upper Assouad codimensions of a point are closely related to the *exponent sets* of the point x_0, defined in [5]. Namely, $\underline{\operatorname{co\,dim}}_{\mathrm A}(\{x_0\})=\sup\underline{Q}(x_0)$ and $\overline{\operatorname{co\,dim}}_{\mathrm A}(\{x_0\})=\inf\overline{Q}(x_0)$ (see [5] for the definitions of the Q-sets).

In the Heisenberg group $\mathbb{H}_n$, which is one particular example of a metric space satisfying our general assumptions, an inequality of the type (15) was recently obtained by Yang [37] using a completely different approach. Since $\underline{\operatorname{co\,dim}}_{\mathrm A}(\{0\})=Q:=2n+2$ for $0\in\mathbb{H}_n$, inequality (3.6) in [37] corresponds exactly to inequality (15), for $x_0=0$, under the condition $1<p<\underline{\operatorname{co\,dim}}_{\mathrm A}(\{x_0\})$; notice that Theorem 1.1 in [37] only records the unweighted case $\beta=0$. However, the requirement $p-\beta>1$ is not needed in [37], and the inequality is established even with the sharp constant $\big(p/(Q-p+\beta)\big)^p$. With our techniques there is no hope of obtaining any sharpness for the constants.

Recall also that in Euclidean spaces the corresponding well-known inequality, i.e.

$$\int_{\mathbb{R}^n} |u(x)|^p\,|x|^{\beta-p}\,d\mu \le C\int_{\mathbb{R}^n} |\nabla u(x)|^p\,|x|^{\beta}\,d\mu,$$

with the optimal constant $C=\big(p/|n-p+\beta|\big)^p$, follows easily by using the classical 1-dimensional weighted Hardy inequalities (cf. [14]) on rays starting from the origin. For $p-\beta<n$ this inequality holds for all $u\in\mathrm{Lip}_b(\mathbb{R}^n)$, and for $p-\beta>n$ for all $u\in\mathrm{Lip}_0(\mathbb{R}^n\setminus\{0\})$. See also [34] for related inequalities where the distance is taken to a k-dimensional subspace of $\mathbb{R}^n$, $1\le k<n$.

5. Upper Assouad codimension and thickness

In this section we establish a connection between the upper Assouad codimension and Hausdorff content density conditions, which might also be of independent interest, and as a consequence obtain a proof for Theorem 1.2. The following lemma is a modification of [27, Lemma 4.1], where a corresponding statement was given in terms of Minkowski contents in Euclidean spaces.

Lemma 5.1. *Let $E \subset X$ be a closed set and assume that $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < q$. Then there exists a constant $C > 0$ such that*

$$\mathcal{H}^{\mu,q}_R\big(E \cap B(w,R)\big) \ge C\, R^{-q} \mu(B(w,R)) \tag{16}$$

for every $w \in E$ and all $0 < R < \operatorname{diam}(E)$.

Proof. Let $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < q' < q$ and fix $0 < \delta < 1/2$ to be chosen a bit later. Let also $w \in E$ and $0 < R < \operatorname{diam}(E)$, and denote $B_0 = B(w,R)$ and $r_k = \delta^k R$.

We begin with a maximal packing $\{B_{i_1}\}_{i_1}$ of $\frac12 B_0 \cap E$ with balls $B_{i_1} = B(w_{i_1}, r_1)$, $i_1 \in I_0 \subset \mathbb{N}$, where $w_{i_1} \in \frac12 B_0 \cap E$. Then we have for the r_1-neighborhood of E that $E_{r_1} \cap \frac14 B_0 \subset \bigcup_{i_1} 3B_{i_1}$, and thus doubling and the fact $q' > \overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$ imply

$$\mu(B_0)\Big(\frac{r_1}{R}\Big)^{q'} \le C\mu\big(E_{r_1} \cap \tfrac14 B_0\big) \le C \sum_{i_1} \mu(3B_{i_1}) \le C \sum_{i_1} \mu(B_{i_1}).$$

In particular, there exists a constant $c_0 > 0$, independent of w and R, such that

$$\sum_{i_1} \mu(B_{i_1}) \ge c_0 \delta^{q'} \mu(B_0).$$

We now choose $0 < \delta < 1$ to be so small that $\delta^{q-q'} < c_0$, whence $c_0\delta^{q'} > \delta^q$, and so

$$M_0 := \sum_{i_1} \mu(B_{i_1}) > \delta^q \mu(B_0).$$

We complete the first step of the construction by defining a measure distribution for the balls B_{i_1} by

$$\nu(B_{i_1}) = \mu(B_{i_1})/M_0 < \mu(B_{i_1})\delta^{-q}\mu(B_0)^{-1}. \tag{17}$$

In the next step, we create a similar measure distribution inside the balls B_{i_1}. As above, we find for each $i_1 \in I_0$ pairwise disjoint balls $B_{i_1 i_2} = B(w_{i_1 i_2}, r_2)$, $i_2 \in I_{i_1} \subset \mathbb{N}$, where $w_{i_1 i_2} \in \frac12 B_{i_1} \cap E$ and

$$M_{i_1} := \sum_{i_2} \mu(B_{i_1 i_2}) \ge c_0 \delta^{q'} \mu(B_{i_1}) > \delta^q \mu(B_{i_1}). \tag{18}$$

We define

$$\nu(B_{i_1 i_2}) := \nu(B_{i_1})\mu(B_{i_1 i_2})/M_{i_1} < \mu(B_{i_1 i_2})\delta^{-2q}\mu(B_0)^{-1},$$

where the inequality follows from (17) and (18). Notice that since $B_{i_1} \cap B_{j_1} = \emptyset$ whenever $i_1 \neq j_1$, and clearly $B_{i_1 i_2} \subset B_{i_1}$ for every $i_2 \in I_{i_1}$, we have that all the balls $B_{i_1 i_2}$ are pairwise disjoint.

Continuing the construction in the same way, we find in the k:th step a collection of pairwise disjoint closed balls $B_{i_1 i_2 \dots i_{k-1} i_k} \subset B_{i_1 i_2 \dots i_{k-1}}$, $i_k \in I_{i_1 i_2 \dots i_{k-1}} \subset \mathbb{N}$, with center points $w_{i_1 i_2 \dots i_k} \in E \cap \frac12 B_{i_1 i_2 \dots i_{k-1}}$ and all of radius $r_k = \delta^k R$, such that

$$E_{r_k} \cap \tfrac14 B_{i_1 i_2 \dots i_{k-1}} \subset \bigcup_{i_k} 3B_{i_1 i_2 \dots i_{k-1} i_k}.$$

Thus $q' > \overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$ and the choice of δ imply

$$M_{i_1 i_2 \dots i_{k-1}} := \sum_{i_k} \mu(B_{i_1 i_2 \dots i_{k-1} i_k}) \geq c_0 \delta^{q'} \mu(B_{i_1 i_2 \dots i_{k-1}}) > \delta^q \mu(B_{i_1 i_2 \dots i_{k-1}}). \tag{19}$$

We now distribute the measure for the balls $B_{i_1 i_2 \dots i_{k-1} i_k}$ as follows:

$$\begin{aligned} \nu(B_{i_1 i_2 \dots i_{k-1} i_k}) &:= \nu(B_{i_1 i_2 \dots i_{k-1}}) \mu(B_{i_1 i_2 \dots i_{k-1} i_k}) / M_{i_1 i_2 \dots i_{k-1}} \\ &< \mu(B_{i_1 i_2 \dots i_{k-1} i_k}) \delta^{-kq} \mu(B_0)^{-1}, \end{aligned} \tag{20}$$

where we used (19) and the recursive assumption that

$$\nu(B_{i_1 i_2 \dots i_{k-1}}) < \mu(B_{i_1 i_2 \dots i_{k-1}}) \delta^{-(k-1)q} \mu(B_0)^{-1}.$$

This concludes the general step of the construction.

Next, we define

$$\widetilde{E} = \bigcap_{k=1}^{\infty} \bigcup_{i_1, \dots, i_k} B_{i_1 i_2 \dots i_k},$$

so that $\widetilde{E} \subset E \cap B_0$ is a non-empty compact set (here we need the assumptions that X is complete and E is closed; recall also that balls are assumed to be closed). Using the Carathéodory construction (cf. e.g. [33, pp. 54–55]) for the set function ν, we obtain a Borel regular measure $\tilde{\nu}$ which is supported on $\widetilde{E}$ and satisfies $\tilde{\nu}(B_{i_1 i_2 \dots i_k}) = \nu(B_{i_1 i_2 \dots i_k})$ for all of the balls in the construction (see also [9, pp. 13-14]).

If $x \in \widetilde{E}$ and $0 < r < R$, we choose $k \in \mathbb{N}$ such that $R\delta^k = r_k \leq r < R\delta^{k-1}$. Then there exists a constant $C_1 > 0$ (depending on the doubling constant and δ) such that $B(x,r)$ intersects at most C_1 of the balls $B_{i_1 i_2 \dots i_k}$ from the k:th step of the construction; let these be $B_1', \dots, B_N'$. These balls are pairwise disjoint, contained in $B(x,3r)$, and they cover $\widetilde{E} \cap B(x,r)$, and thus we have by (20), the choice of k, and doubling that

$$\begin{aligned} \tilde{\nu}(B(x,r)) = \sum_{j=1}^{N} \tilde{\nu}(B_j') = \sum_{j=1}^{N} \nu(B_j') &\leq \sum_{j=1}^{N} \mu(B_j') \delta^{-kq} \mu(B_0)^{-1} \\ &\leq C\mu(B(x,r))(r/R)^{-q} \mu(B_0)^{-1}. \end{aligned} \tag{21}$$

Finally, let $\{B(z_i, r_i)\}_i$ be a cover of $E \cap B_0$ with balls of radii $0 < r_i < R$. Using (21), we conclude that

$$1 = \tilde{\nu}\big(E \cap B_0\big) \leq \sum_i \tilde{\nu}(B(z_i, r_i)) \leq C \sum_i \frac{\mu(B(z_i, r_i)) r_i^{-q}}{\mu(B_0) R^{-q}},$$

and so taking the infimum over all such covers yields

$$\mathcal{H}_R^{\mu,q}\big(E \cap B_0\big) \geq C\, R^{-q} \mu(B_0),$$

as desired. □

Consequently, we obtain a characterization for the upper Assouad codimension (of closed sets) in terms of Hausdorff content density:

Corollary 5.2. *Let $E \subset X$ be a closed set. Then $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$ is the infimum of all $q \geq 0$ for which there exists $C \geq 0$ such that* (16) *holds for every $w \in E$ and all $0 < R < \operatorname{diam}(E)$.*

Proof. Let $q \geq 0$ be such that (16) holds for every $w \in E$ and all $0 < R < \operatorname{diam}(E)$, and let $\{B_i\}$ be a maximal packing of $E \cap B(w, R/2)$ with balls of radius $0 < r < R$. Then $\{2B_i\}$ is a cover of $E \cap B(w, R/2)$, and so the doubling condition and (16) imply

$$r^{-q} \mu(E_r \cap B(w,R)) \geq c r^{-q} \sum_i \mu(B_i) \geq c(2r)^{-q} \sum_i \mu(2B_i) \geq c R^{-q} \mu(B(w,R)).$$

Thus $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$ gives a lower bound for exponents satisfying (16).

On the other hand, Lemma 5.1 shows that there can not be a larger lower bound for these q, and thus $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$ is the infimum, as was required. □

Proof of Theorem 1.2. We assumed that $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) < p-\beta$ and $p-\beta > 1$, and thus we can choose $q > 1$ so that $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) < q < p-\beta$. Lemma 5.1 then implies that

$$\mathcal{H}^{\mu,q}_R\big(\Omega^c \cap B(w,R)\big) \geq C\,R^{-q}\mu(B(w,R)) \tag{22}$$

for every $w \in \Omega^c$ and all $0 < R < \operatorname{diam}(\Omega^c)$, with a constant $C > 0$ independent of w and R. By [28, Thm. 4.1], this condition is sufficient for Ω to admit a (p,β)-Hardy inequality, as desired. The following remarks are however in order here:

The condition in [28, Thm. 4.1] actually requires that

$$\mathcal{H}^{\mu,q}_{d_\Omega(x)}\big(\partial\Omega \cap B(x,2d_\Omega(x))\big) \geq C\,d_\Omega(x)^{-q}\mu\big(B(x,2d_\Omega(x))\big) \tag{23}$$

for all $x \in \Omega$, where $d_\Omega(x) = \operatorname{dist}(x,\partial\Omega)$. Recall from Section 2 that the validity of a Poincaré inequality implies that X is quasiconvex, and thus $\operatorname{dist}(x,\partial\Omega) \simeq \operatorname{dist}(x,\Omega^c)$, and so the different distance function causes no problems here. Moreover, inspecting the proofs in [28] one sees that for $\beta \leq 0$ the assumption (23) can be replaced in Lemma 3.1(a) of [28] with the condition (22). One subtlety here is the case when Ω is unbounded, since then (22) is needed for all radii $0 < R < \infty$, and thus we have to assume that in this case Ω^c is unbounded as well. Once Lemma 3.1(a) of [28] is established, the (p,β)-inequalities for $\beta > 0$ follow just like in the proof of [28, Thm. 4.1]; the idea is the same as in the proof of Theorem 1.1 of the present paper.

Let us also remark that the proofs in [28] require the validity of a p_0-Poincaré inequality for $1 \leq p_0 < p$, which is guaranteed by the self-improvement result of Keith and Zhong [19]. □

Remark 5.3. Actually both the *inner boundary density* of (23) and the complement density (22), with an exponent $1 \leq q < p$, are equivalent to the *uniform p-fatness* of Ω^c, see [20]. The deep fact that (also) uniform fatness is a self-improving condition (see [31, 7]) is essential in the necessity part of this claim.

6. Necessary conditions

In this section we extend the previously known necessary conditions for Hardy inequalities to cover also weighted Hardy inequalities in metric measure spaces. In [23] and [30], such conditions were obtained in the unweighted case $\beta = 0$ in metric spaces, and in [25] for weighted inequalities in the Euclidean setting. Let us mention here that actually no Poincaré inequalities are needed to establish the results in this section, so we only need to assume that μ is doubling.

We have the following generalization of [25, Thm. 1.1] and [30, Thm. 6.1]:

Theorem 6.1. *Let $1 \leq p < \infty$ and $\beta \neq p$, and assume that $\Omega \subset X$ admits a (p,β)-Hardy inequality. Then there exists $\varepsilon > 0$, depending only on the given data, such that either*

$$\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c) < p-\beta-\varepsilon \quad \text{or} \quad \underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p-\beta+\varepsilon.$$

In particular, $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c) < p-\beta$ *or* $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p-\beta$.

Here the "given data" means the parameters p and β and the constants in the doubling condition and in the assumed Hardy inequality. Our next result gives a local version of such a dimension dichotomy; see [25, Thm. 5.3] for the Euclidean case and [30, Thm. 6.2] for the case $\beta = 0$.

Theorem 6.2. *Let $1 \le p < \infty$ and $\beta \neq p$, and assume that $\Omega \subset X$ admits a (p,β)-Hardy inequality. Then there exists $\varepsilon > 0$, depending only on the given data, such that for each ball $B_0 \subset X$ either*

$$\operatorname{co\,dim_H}(2B_0 \cap \Omega^c) < p - \beta - \varepsilon$$

or the Aikawa condition (1) *holds with an exponent $q > p - \beta + \varepsilon$ for all $w \in \Omega^c \cap B_0$ and all $0 < r < \operatorname{rad}(B_0)$.*

Here the factor 2 in $2B_0$ is not essential (but convenient), any fixed $L > 1$ can be used instead.

Remark 6.3. We can not in general conclude in Theorem 6.2 that either $\operatorname{co\,dim_H}(2B_0 \cap \Omega^c) < p - \beta - \varepsilon$ or $\underline{\operatorname{co\,dim}}_A(B_0 \cap \Omega^c) > p - \beta + \varepsilon$, since the latter would require the Aikawa condition for all $0 < r < \operatorname{diam}(X)$, and this we can not reach under the assumptions of the theorem. Nevertheless, if we further assume that there is a constant $C > 0$ and an exponent $s > p - \beta$ such that

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \le C\Big(\frac{r}{R}\Big)^s \tag{24}$$

for all $x \in X$ and all $0 < r < R < \operatorname{diam}(X)$, then it is possible to conclude in the setting of Theorem 6.2 that either $\operatorname{co\,dim_H}(2B_0 \cap \Omega^c) < p - \beta - \varepsilon$ or $\underline{\operatorname{co\,dim}}_A(B_0 \cap \Omega^c) > p - \beta + \varepsilon$. The main idea here is that the Aikawa condition, for all $0 < r < \operatorname{rad}(B_0)$, implies that also the condition in the definition of the lower Assouad codimension holds for all $0 < r < R < \operatorname{rad}(B_0)$ with some exponent $t > p - \beta + \varepsilon$ (cf. Remark 2.1), while for other radii $0 < r < R < \operatorname{diam}(X)$ the latter condition follows with the help of the above relative measure bound (24) (possibly with another $\varepsilon > 0$). Notice, in particular, that (24) holds in a Q-regular space for $s = Q$.

Recall that our sufficient conditions for Hardy inequalities were given in terms of $\underline{\operatorname{co\,dim}}_A(\Omega^c)$ and $\overline{\operatorname{co\,dim}}_A(\Omega^c)$. However, in the above necessary conditions it is not possible to replace $\operatorname{co\,dim_H}$ by the (larger) $\overline{\operatorname{co\,dim}}_A$ in either of the theorems, cf. the discussion in Section 8. Also the assumption $\beta \neq p$ is essential in both of the theorems, as the result need not hold for the (p,p)-Hardy inequality, see [25]. On the other hand, for $\beta > p$ the claims reduce to trivialities, since always $\underline{\operatorname{co\,dim}}_A(E) \ge 0$.

One important ingredient in the proofs of these necessary conditions is the following self-improvement result for Hardy inequalities.

Proposition 6.4. *Let $1 \le p < \infty$ and $\beta \in \mathbb{R}$, and assume that $\Omega \subset X$ admits a (p,β)-Hardy inequality. Then there exists $\varepsilon > 0$, depending only on the given data, such that Ω admits $(p,\tilde\beta)$-Hardy inequalities whenever $\beta - \varepsilon \le \tilde\beta \le \beta + \varepsilon$. Moreover, the constant $C > 0$ in all these Hardy inequalities can be chosen to be independent of the particular $\tilde\beta$.*

The proof of Proposition 6.4 is almost identical to the Euclidean case, which follows from the case $s = 0$ of [26, Lemma 2.1], so we omit the details. Notice in addition that while [26, Lemma 2.1] is formulated only for $1 < p < \infty$, the same proof actually works also when $p = 1$.

Another fact that we need in the proofs of Theorems 6.1 and 6.2 is that if a part of the complement of Ω is small enough, then the test functions for the Hardy inequalities need not vanish in that particular part of Ω^c.

Lemma 6.5. *Let $1 \le p < \infty$ and $0 \le \beta < p$, and assume that $\Omega \subset X$ admits a (p,β)-Hardy inequality with a constant $C_0 > 0$. Assume further that $U \subset X$ is an open set such that*

$$\mathcal{H}^{\mu,p-\beta}_{\operatorname{diam}(U)}(U \cap \Omega^c) = 0. \tag{25}$$

Then a (p,β)-Hardy inequality holds for all $u \in \operatorname{Lip}_0(\Omega \cup U)$ with a constant $C_1 = C_1(C_0,p) > 0$.

Proof. Let $u \in \mathrm{Lip}_0(\Omega \cup U)$ with an upper gradient g_u. By the definition of $\mathcal{H}^{\mu,p-\beta}_{\mathrm{diam}(U)}$, there then exist, for a fixed $j \in \mathbb{N}$, balls $B_i^j = B(w_i, r_i)$ with $w_i \in \mathrm{spt}(u) \cap U \cap \Omega^c$ and $r_i \le \mathrm{diam}(U)$, $i = 1, \dots, N_j$, so that $\mathrm{spt}(u) \cap U \cap \Omega^c \subset \bigcup_{i=1}^{N_j} B_i^j$ and

$$\sum_{i=1}^{N_j} \mu(B_i^j) {r_i}^{-p+\beta} \le \|u\|_\infty^{-p} 2^{-j}. \tag{26}$$

Let $B_0 = B(x_0, R_0)$ be a ball such that $\mathrm{spt}(u) \cap U \cap \Omega^c \subset \frac{1}{2} B_0$. Iteration of the doubling condition shows that then there exists $Q > 0$ and a constant $C > 0$ such that $\mu(B_i^j)/\mu(B_0) \ge C(r_i/R_0)^Q$ for all i and j; see for instance [4, Lemma 3.3]. Moreover, since one can always choose a larger Q in this condition, we may assume that $Q > p - \beta$. Thus it follows from (26) that, for each $j \in \mathbb{N}$, all the radii r_i (of the balls B_i^j) satisfy $r_i^{Q-p+\beta} \le C 2^{-j}$, where the constant $C > 0$ may depend on u and B_0, but is independent of j. In particular, $r_i \to 0$ uniformly as $j \to \infty$, and hence we may in addition assume that the covers $\{B_i^j\}_{i=1}^{N_j}$ are nested, i.e., $\bigcup_{i=1}^{N_{j+1}} B_i^{j+1} \subset \bigcup_{i=1}^{N_j} B_i^j$ for all $j \in \mathbb{N}$.

We now define cut-off functions $\psi_j(x) = \min_i\{1, {r_i}^{-1} d(x, 2B_i^j)\}$. Each function ψ_j has an upper gradient g_{ψ_j} satisfying $g_{\psi_j}^p \le \sum_i {r_i}^{-p} \chi_{3B_i^j}$ (cf. [4, Cor. 2.20]). Set $u_j = \psi_j u$. Then $u_j \in \mathrm{Lip}_0(\Omega)$, and $g_{u_j} = g_{\psi_j}|u| + g_u$ is an upper gradient of u_j (cf. [4, Thm. 2.15]). In addition, since the covers were assumed to be nested and $r_i \to 0$ uniformly as $j \to \infty$, we have that $u_j \le u_{j+1}$ for each $j \in \mathbb{N}$ and $u_j \to u$ pointwise in Ω.

Since $\beta \ge 0$, we have $d_\Omega(y)^\beta \le r_i^\beta$ for all $y \in B_i^j$, and thus the (p,β)-Hardy inequality for the functions u_j and estimate (26) imply that

$$\begin{aligned}
\int_\Omega |u_j|^p {d_\Omega}^{\beta-p}\, d\mu &\le C \left[\|u\|_\infty^p \int_\Omega g_{\psi_j}^p {d_\Omega}^\beta\, d\mu + \int_\Omega g_u^p {d_\Omega}^\beta\, d\mu \right] \\
&\le C \left[\|u\|_\infty^p \sum_{i=1}^{N_j} \mu(B_i^j) {r_i}^{-p+\beta} + \int_\Omega g_u^p {d_\Omega}^\beta\, d\mu \right] \\
&\le C 2^{-j} + C \int_\Omega g_u^p {d_\Omega}^\beta\, d\mu,
\end{aligned}$$

where $C = C(C_0, p) > 0$. The claim now follows by monotone convergence, since $u_j(x) \to u(x)$ in Ω. □

Let us record here the following consequence of the previous lemma, which gives an improvement to Theorem 1.1:

Corollary 6.6. *Let $1 \le p < \infty$ and $\beta < p - 1$, and assume that X and Ω are as in Theorem 1.1, in particular that $\underline{\mathrm{co\,dim}}_{\mathrm{A}}(\Omega^c) > p - \beta$. Then a (p,β)-Hardy inequality holds for all $u \in \mathrm{Lip}_b(\Omega)$.*

Proof. For $\beta < 0$ the claim follows directly from Proposition 3.1. For $\beta \ge 0$ we have by Theorem 1.1 that Ω admits a (p,β)-Hardy inequality. We now choose $U = (\Omega^c)_1 = \{x \in X : d(x, \Omega^c) < 1\}$. Since $p - \beta < \underline{\mathrm{co\,dim}}_{\mathrm{A}}(\Omega^c) \le \mathrm{co\,dim}_{\mathrm{H}}(\Omega^c)$ and $\Omega^c \subset U$, it follows in particular that $\mathcal{H}^{\mu,p-\beta}_{\mathrm{diam}(U)}(U \cap \Omega^c) = 0$. Thus Lemma 6.5 implies that actually the (p,β)-Hardy inequality holds for all $\mathrm{Lip}_0(\Omega \cup U) = \mathrm{Lip}_b(X)$, and the claim follows. □

Lemma 6.5 and the self-improvement of the Aikawa condition from Lemma 2.2 now yield the following result, which is essentially a "weighted" version of [23, Lemma 2.4]. For Euclidean spaces, a similar result can be found in [25, Lemma 5.2], but note that there the proof is different and especially avoids the use of Gehring's Lemma, thus making the proof therein more self-contained. The approach of [25] could be used in the present setting of metric spaces as well, but we chose instead to follow the outline of the proofs from [23] for the sake of brevity and also to emphasize the role of the self-improvement result of Lemma 2.2.

Lemma 6.7. *Let $1 \le p < \infty$, $0 \le \beta < p$, and assume that $\Omega \subset X$ admits a (p,β)-Hardy inequality. Assume further that $B_0 = B(x_0,R) \subset X$ is an open ball such that $\mathcal{H}^{\mu,p-\beta}_R(2B_0 \cap \Omega^c) = 0$. Then there exists $\delta > 0$, depending only on the given data, such that the Aikawa condition* (1) *holds with the exponent $q = p - \beta + \delta$ for all $w \in \Omega^c \cap B_0$ and all $0 < r < R$.*

Proof. Let $w \in \Omega^c \cap B_0$ and $0 < r < R/2$, and denote $U = 2B_0$ and $B = B(w,r)$, so that $2B \subset 2B_0$. Define $\varphi(x) = r^{-1}d\big(x, X \setminus 2B\big)$. Then φ is a Lipschitz function with a compact support in $\Omega \cup U$, $\varphi \ge 1$ in B, and $g_\varphi = r^{-1}\chi_{2B}$ is an upper gradient of φ. By Lemma 6.5 the (p,β)-Hardy inequality holds for φ, and since $d_\Omega \le 2r$ in $2B$ and $\beta \ge 0$, we obtain

$$\int_B \operatorname{dist}(y, \Omega^c \cap B_0)^{\beta-p}\,d\mu(y) \le \int_{2B} \varphi^p d_\Omega{}^{\beta-p}\,d\mu \le C_1 \int_{2B} g_\varphi^p d_\Omega{}^{\beta}\,d\mu \le C_2\mu(B) r^{-p+\beta}.$$

In particular, the Aikawa condition (1) holds with $q = p - \beta > 0$ (for $R/2 \le r < R$ the claim follows by covering $B(x,r)$ with smaller balls). By Lemma 2.2 there then exists $\delta > 0$ such that the Aikawa condition holds with $p - \beta + \delta$, proving the claim. □

Remark 6.8. Since $\delta > 0$ in Lemma 6.7 depends only on the data associated to X, Ω, and the (p,β)-Hardy inequality, we have the following uniformity result: If (q,β)-Hardy inequalities hold for all $p_1 < q < p_2$ with a constant C_1, we can choose $\delta > 0$ in Lemma 6.7 to be independent of the particular q; more precisely, then $\delta = \delta(p_1, p_2, \beta, C_1, \Omega, X) > 0$.

We have now established enough tools to prove Theorem 6.2. The proof follows the lines of the proofs of [23, Corollary 2.7] and [30, Thm. 6.2], but we present the main ideas here for the convenience of the reader.

Proof of Theorem 6.2. Let $B_0 = B(x_0,R) \subset X$. It is clear that if $\beta > p$, we choose $\delta < \beta - p$ and then the Aikawa condition (1) holds with the exponent $q = p - \beta + \delta < 0$, and so we only need to consider the case $\beta < p$. First of all, we may assume that $\beta \ge 0$. Indeed, if this is not the case, we have, by [26, Thm. 2.2], that Ω admits a $(p-\beta, 0)$-Hardy inequality with a constant depending only on the data, and now we may consider this instead of the original (p,β)-Hardy inequality. Notice that even though [26, Thm. 2.2] is written in Euclidean spaces, the proof applies almost verbatim in metric spaces.

By the self-improvement of Hardy inequalities from Proposition 6.4, we find $\varepsilon_1 > 0$ and $C_1 > 0$ such that Ω admits $(p,\tilde\beta)$-Hardy inequalities for all $\beta \le \tilde\beta \le \beta + \varepsilon_1$, and moreover the constant in all these inequalities can be taken to be C_1. In addition, we require that $\varepsilon_1 \le p - \beta$.

Let then $0 < \varepsilon < \varepsilon_1/2$ to be specified later. If $\operatorname{co\,dim}_{\mathrm{H}}(2B_0 \cap \Omega^c) < p - \beta - \varepsilon$, the claim holds, and thus we may assume that $\operatorname{co\,dim}_{\mathrm{H}}(2B_0 \cap \Omega^c) \ge p - \beta - \varepsilon$. It follows that

$$\mathcal{H}^{\mu,q}_R(2B_0 \cap \Omega^c) = 0 \quad \text{for } q = p - \beta - 2\varepsilon.$$

As Ω admits a $(p, \beta + 2\varepsilon)$-Hardy inequality and $p \ge \beta + \varepsilon_1 > \beta + 2\varepsilon$, we may use Lemma 6.7 to conclude that there exists $\delta > 0$, independent of the particular choice of $\varepsilon < \varepsilon_1/2$ (cf. Remark 6.8), such that the Aikawa condition (1) holds with the exponent $q = p - (\beta + 2\varepsilon) + \delta = p - \beta - 2\varepsilon + \delta$. We now choose $\varepsilon < \min\{\varepsilon_1/2, \delta/3\}$, and the claim follows. □

The global dimension dichotomy in Theorem 6.1 follows along the same lines as above: If Ω admits a (p,β)-Hardy inequality for $0 \le \beta < p$, and if in addition $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c) \ge p - \beta - \varepsilon$, we obtain from Lemma 6.7 that

$$\int_B d_\Omega(x)^{-p+\beta+2\varepsilon-\delta}\,dx \le C\mu(B) r^{-p+\beta+2\varepsilon-\delta}$$

for *any* ball $B = B(w,r)$ with $w \in \Omega^c$ and $0 < r < \operatorname{diam}(X)$, where C and δ are independent of B and the particular ε. Choosing $\varepsilon > 0$ as in the proof of Theorem 6.2 shows that the Aikawa condition (1) holds with an exponent $q > p - \beta + \varepsilon$ for all $w \in \Omega^c$ and all $0 < r < \operatorname{diam}(X)$. Hence we conclude from Remark 2.1 that indeed $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > p - \beta + \varepsilon$.

7. Combining thick and thin parts of the complement

Theorem 1.1 gives a sufficient condition for Hardy inequalities in the case where the complement of Ω is thin. Conversely, Theorem 1.2 gives such a condition in the case where the complement is thick (everywhere and at all scales). Nevertheless, requiring the whole complement to be either thick or thin rules out all cases where the complement contains both large and small pieces; an easy (and well-understood) example is the punctured ball $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$. In the next proposition we show how it is possible to combine the results of Theorems 1.1 and 1.2 for this kind of domains. A slightly different approach to Hardy inequalities in such domains, for $\beta = 0$, was given in [29, Section 5], and in the Euclidean case earlier results for weighted inequalities can be found in [25]; both of these require additional accessibility properties for Ω. On the other hand, the results from [25] also cover the case $\beta \geq p - 1$, where such extra conditions are known to be indispensable (cf. Section 8).

Proposition 7.1. *Let $1 < p < \infty$ and $\beta < p - 1$. If $\beta \leq 0$, we assume that X supports a p-Poincaré inequality, and if $\beta > 0$ we assume that X supports a $(p-\beta)$-Poincaré inequality. Let $\Omega_0 \subset X$ be an open set satisfying $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) < p - \beta$. If $F \subset \overline{\Omega_0}$ is a closed set with $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(F) > p - \beta$, then $\Omega = \Omega_0 \setminus F$ admits a (p,β)-Hardy inequality.*

Moreover, a (p,β)-Hardy inequality (in Ω) actually holds for all $u \in \operatorname{Lip}_0(\Omega_0)$, i.e. the test functions need not vanish in $F \cap \Omega_0$.

Since this result (in this generality) is new even in Euclidean spaces, let us formulate this special case as a corollary:

Corollary 7.2. *Let $1 < p < \infty$ and $\beta < p - 1$, and assume that $\Omega_0 \subset \mathbb{R}^n$ is an open set satisfying $\underline{\dim}_{\mathrm{A}}(\Omega^c) > n - p + \beta$. If $F \subset \overline{\Omega_0}$ is a closed set with $\overline{\dim}_{\mathrm{A}}(F) < n - p + \beta$, a (p,β)-Hardy inequality holds in $\Omega = \Omega_0 \setminus F$ for all $u \in \operatorname{Lip}_0(\Omega_0)$.*

Proof of Proposition 7.1. Again, it suffices to prove the claim for $\beta < 0$, with a suitable control for the constant, since then the claim for $\beta \geq 0$ follows just as in the proof of Theorem 1.1 from Section 4. For $\beta < 0$, the idea is to modify the proof of Proposition 3.1. In the present case we can no longer "chain to infinity" as in (3), but we can instead use for sufficiently large balls centered at F the fact that a (p,β)-Hardy inequality holds in Ω_0.

Let $\mathcal{W}(\Omega_0)$ be a Whitney-type cover of Ω_0 with balls $B(x, c d_\Omega(x))$, $x \in \Omega$, where $0 < c < 1/2$ is such that the balls $4\lambda B$, $B \in \mathcal{W}(\Omega)$, have a uniformly bounded overlap (see e.g. [6]); here λ is the dilatation constant from the Poincaré inequality. Let $B_1, B_2, \dots \in \mathcal{W}(\Omega_0)$ be such that $F \cap \Omega_0 \subset \bigcup_i B_i$.

Without loss of generality, let us first consider $E := B_1 \cap F$, where $B_1 = B(x,R)$. Choose $k_0 \in \mathbb{Z}$ so that $4 \cdot 2^{k_0} < R \leq 4 \cdot 2^{k_0+1}$, and let $\mathcal{B}_k$, N_k, and A_k for $k \leq k_0$ be just as in the proof of Proposition 3.1 for this set E. Then $4B \subset 4B_1$ for all $B \in \mathcal{B}_{k_0}$.

Now let $u \in \operatorname{Lip}_0(\Omega)$. We divide each $\mathcal{B}_k$, $k \leq k_0$, into two subsets as follows: We set for $B \in \mathcal{B}_k$ that $B \in \mathcal{B}_k^{(1)}$ if $|u_{4B_1}| \leq \frac{1}{2}|u_{4B}|$, and otherwise $B \in \mathcal{B}_k^{(2)}$. For convenience, we also set $\mathcal{B}_{k_0+1} = \mathcal{B}_{k_0+1}^{(1)} = \mathcal{B}_{k_0+1}^{(2)} := \{B_1\}$. For $B \in \mathcal{B}_k^{(1)}$, $k \leq k_0$, we then have

$$|u_{4B}| \leq |u_{4B} - u_{4B_1}| + |u_{4B_1}| \leq |u_{4B} - u_{4B_1}| + \tfrac{1}{2}|u_{4B}|,$$

and thus

$$\tfrac{1}{2}|u_{4B}| \le \sum_{j=k}^{k_0-1} |u_{4B^j} - u_{4B^{j+1}}| + |u_{4B^{k_0}} - u_{4B_1}|,$$

where $B = B^k$. This is analogous to estimate (3), and, indeed, the balls $B \in \mathcal{B}_k^{(1)}$ can be treated just like in the proof of Proposition 3.1, yielding as in (8) and (9) that

$$\sum_{k=-\infty}^{k_0+1} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k^{(1)}} \int_{4B} |u_{4B}|^p \, d\mu \le C \int_{4\lambda B_1} g_u^p {d_\Omega}^\beta \, d\mu. \tag{27}$$

On the other hand, for the balls $B \in \mathcal{B}_k^{(2)}$, $k \le k_0$, we have by Hölder's inequality that

$$|u_{4B}| < 2|u_{4B_1}| \le 2\biggl(\fint_{4B_1} |u|^p \, d\mu\biggr)^{1/p},$$

and since $d_{\Omega_0}(y) \simeq R$ for all $y \in 4B_1$, we obtain

$$|u_{4B}|^p \le C\frac{R^{p-\beta}}{\mu(B_1)} \int_{4B_1} |u|^p {d_{\Omega_0}}^{\beta-p} \, d\mu. \tag{28}$$

Now pick q such that $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(F) > q > p - \beta$, whence by definition

$$\frac{\mu(E_{2^k})}{\mu(B_1)} \le C\frac{2^{kq}}{R^q} \qquad \text{for all } k \le k_0 + 1.$$

Estimate (28) and the bounded overlap of the balls $4B$, for $B \in \mathcal{B}_k^{(2)}$ with a fixed k, then imply

$$\begin{aligned} \sum_{k=-\infty}^{k_0+1} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k^{(2)}} \int_{4B} |u_{4B}|^p \, d\mu &\le C \int_{4B_1} |u|^p {d_{\Omega_0}}^{\beta-p} \, d\mu \sum_{k=-\infty}^{k_0+1} \frac{2^{k(\beta-p)}}{R^{\beta-p}} \frac{\mu(E_{2^k})}{\mu(B_1)} \\ &\le C \int_{4B_1} |u|^p {d_{\Omega_0}}^{\beta-p} \, d\mu \sum_{k=-\infty}^{k_0+1} \frac{2^{k(q+\beta-p)}}{R^{q+\beta-p}} \le C \int_{4B_1} |u|^p {d_{\Omega_0}}^{\beta-p} \, d\mu, \end{aligned} \tag{29}$$

since $q + \beta - p > 0$.

Following the proof of Proposition 3.1, we can now combine (27) and (29), and we obtain as in (6) that

$$\begin{aligned} \int_{4B_1} |u|^p {d_\Omega}^{\beta-p} \, d\mu &\le C \sum_{k=-\infty}^{k_0+1} \int_{A_k} |u|^p {d_\Omega}^{\beta-p} \, d\mu \\ &= \sum_{k=-\infty}^{k_0+1} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k} \int_{4B} |u - u_{4B}|^p \, d\mu + C \sum_{k=-\infty}^{k_0+1} 2^{k(\beta-p)} \sum_{B\in\mathcal{B}_k} \int_{4B} |u_{4B}|^p \, d\mu \\ &\le C \int_{4\lambda B_1} g_u^p {d_\Omega}^\beta \, d\mu + C \int_{4B_1} |u|^p {d_{\Omega_0}}^{\beta-p} \, d\mu. \end{aligned} \tag{30}$$

Note that here $A_{k_0+1} = 4B_1 \setminus N_{k_0}$, and that the first integral in the second line can be estimated just like in (7).

Similar estimates hold of course for all balls $B_1, B_2, \dots$, with constants independent of i. Moreover, if $x \in \Omega \setminus \bigcup_i 4B_i$, then $d_\Omega(x) \ge C d_{\Omega_0}(x)$: If $d_\Omega(x) = d_{\Omega_0}(x)$ the claim is trivial, so we may assume that $d_\Omega(x) = d(x, w)$ for some $w \in F$. Pick $B_i \ni w$. Since $x \notin 4B_i$ and $\operatorname{rad}(B_i) \ge \tilde{c}d(w, \Omega_0)$ with $0 < \tilde{c} < 1$, it follows that

$$d_\Omega(x) = d(x, w) \ge \operatorname{rad}(B_i) \ge \tilde{c}d(w, \Omega_0) \ge \tilde{c}d(x, \Omega_0) - \tilde{c}d(x, w) = \tilde{c}d(x, \Omega_0) - \tilde{c}d_\Omega(x),$$

and the claim follows. In particular $d_\Omega(x)^{\beta-p} \le C d_{\Omega_0}(x)^{\beta-p}$ for these x, and thus estimate (30) for each B_i, the (p,β)-Hardy inequality for Ω_0, and the bounded overlap of the balls $4\lambda B_i$ (and thus of $4B_i$) yield

$$\begin{aligned}
\int_\Omega |u|^p d_\Omega{}^{\beta-p}\,d\mu &\le \int_{\Omega\setminus\bigcup_i 4B_i} |u|^p d_\Omega{}^{\beta-p}\,d\mu + \sum_i \int_{4B_i} |u|^p d_\Omega{}^{\beta-p}\,d\mu \\
&\le C\int_\Omega |u|^p d_{\Omega_0}{}^{\beta-p}\,d\mu + C\sum_i \int_{4\lambda B_i} g_u^p d_\Omega{}^{\beta}\,d\mu + C\sum_i \int_{4B_i} |u|^p d_{\Omega_0}{}^{\beta-p}\,d\mu \\
&\le C\int_{\Omega_0} |u|^p d_{\Omega_0}{}^{\beta-p}\,d\mu + C\sum_i \int_{4\lambda B_i} g_u^p d_\Omega{}^{\beta}\,d\mu \\
&\le C\int_{\Omega_0} g_u^p d_{\Omega_0}{}^{\beta}\,d\mu + C\int_{\bigcup_i 4\lambda B_i} g_u^p d_\Omega{}^{\beta}\,d\mu \\
&\le C\int_{\Omega_0} g_u^p d_{\Omega_0}{}^{\beta}\,d\mu + C\int_{\Omega} g_u^p d_\Omega{}^{\beta}\,d\mu \le C\int_{\Omega} g_u^p d_\Omega{}^{\beta}\,d\mu;
\end{aligned}$$

we also used the facts that $d_{\Omega_0}(x)^\beta \le d_\Omega(x)^\beta$ for all $x \in \Omega$ (since $\beta < 0$) and that $\mu(F) = 0$.

The above constant $C > 0$ naturally depends on β, but for $-1 < \beta < 0$ close enough to 0, the dependence can be reduced again to the form $C = |\beta|^{-1}C^*$, where $C^* > 0$ is independent of β; the details are exactly the same as in the proof of Proposition 3.1. Hence (p,β)-Hardy inequalities, for $0 \le \beta < p-1$, now follow along the same lines as in the proof of the corresponding case of Theorem 1.1 (cf. Section 4).

Finally, regarding the boundary values, we see that in the above proof of the case $\beta < 0$ it is not necessary for u to vanish in F, and thus this case indeed holds for all $u \in \mathrm{Lip}_0(\Omega_0)$. On the other hand, in the case $\beta \ge 0$ we can apply Lemma 6.5 with $U = \Omega_0$ (since $p - \beta < \underline{\mathrm{co\,dim}}_{\mathrm{A}}(F) \le \mathrm{co\,dim}_{\mathrm{H}}(\Omega_0 \cap \Omega^c)$), and it follows again that a (p,β)-Hardy inequality holds for all $u \in \mathrm{Lip}_0(\Omega \cup \Omega_0) = \mathrm{Lip}_0(\Omega_0)$. This concludes the proof. □

8. Sharpness of the results

We close the paper with an examination of the sharpness of our results. In particular, we consider the necessity of the assumptions in our main theorems.

It was already mentioned in the introduction that the bound $p-\beta$ is very natural for the dimensions in all of the sufficient and necessary conditions, and can not be improved. Moreover, the bound $p-\beta$ for the lower Assouad codimension appears both in the sufficient and necessary conditions in the cases where the complement is thin, and so it is obvious that $\underline{\mathrm{co\,dim}}_{\mathrm{A}}$ is the optimal concept of dimension in this setting. However, when (a part of) the complement is thick, the sufficient conditions are given in terms of the upper Assouad codimension, while in the necessary conditions the Hausdorff codimension is used, and so these conditions do not quite meet. This raises the question whether it could be possible to improve the bounds in either sufficient or necessary conditions by using a different concept of dimension. Since the possibility to combine thick and thin parts in the sufficient conditions (Proposition 7.1) immediately rules out such improvements in the global results, the sharpness of the conditions (in terms of dimensions) is established if we show that (i) $\mathrm{co\,dim}_{\mathrm{H}}(2B_0 \cap \Omega^c)$ can not be replaced by $\overline{\mathrm{co\,dim}}_{\mathrm{A}}(2B_0 \cap \Omega^c)$ in Theorem 6.2, and that on the other hand (ii) the local bound $\mathrm{co\,dim}_{\mathrm{H}}(B \cap \Omega^c) < p - \beta$, for all balls $B = B(w,r)$ with $w \in \Omega^c$, does not suffice for the (p,β)-Hardy inequality in Ω.

The following construction yields a counterexample for both (i) and (ii).

Example 8.1. We consider here, for simplicity, the unweighted case $\beta = 0$ in $\mathbb{R}^n$, with μ being the usual Lebesgue measure. Then $|\nabla u|$ is an optimal upper gradient for each Lipschitz function u.

Denote $w_j = (2^{-j}, 0, \dots, 0) \in \mathbb{R}^n$ for $j \in \mathbb{N}$ and $0 = (0, 0, \dots, 0) \in \mathbb{R}^n$, and let $F = \{w_j : j \geq 2\} \cup \{0\} \subset \mathbb{R}^n$ and $\Omega_1 = B(0,2) \setminus F \subset \mathbb{R}^n$. Since $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(B(0,2)^c) = 0$ and $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(F) = n$, we have by Proposition 7.1 that Ω_1 admits a p-Hardy inequality whenever $1 < p < n$ (and Ω_1 does not admit an n-Hardy inequality).

Next, we replace each point w_j with a ball $B_j = B(w_j, 2^{-2j})$, that is, we consider the domain $\Omega = B(0,2) \setminus \big(\bigcup_{j\geq 2} B_j \cup \{0\}\big)$. Then also Ω admits p-Hardy inequalities whenever $1 < p < n$; this can be seen as follows:

Write $\Omega' = B(0,2) \setminus \big(\bigcup_{j\geq 2} 2B_j \cup \{0\}\big)$ and $\Omega'' = \bigcup_{j\geq 2} 2B_j \setminus B_j$, so that $\Omega = \Omega' \cup \Omega''$. For all $x \in \Omega' \subset \Omega_1$ we have $d_{\Omega_1}(x) \geq d_\Omega(x) \geq \frac{1}{2} d_{\Omega_1}(x)$, and thus, using the p-Hardy inequality in Ω_1, we have for all $u \in \operatorname{Lip}_0(\Omega) \subset \operatorname{Lip}_0(\Omega_1)$ that

$$\int_{\Omega'} |u|^p d_\Omega{}^{-p}\,dx \leq C \int_{\Omega_1} |u|^p d_{\Omega_1}{}^{-p}\,dx \leq C \int_{\Omega_1} |\nabla u|^p\,dx = C \int_\Omega |\nabla u|^p\,dx. \tag{31}$$

On the other hand, for all $x \in \Omega''$ the complement of Ω near x satisfies the density condition $\mathcal{H}^n(\Omega^c \cap B(x, 2d_\Omega(x))) \geq C d_\Omega(x)^n$, and so it follows that even a stronger pointwise 1-Hardy inequality holds for these points (cf. [12, 20]): If $u \in \operatorname{Lip}_0(\Omega)$ and $x \in \Omega''$, then $|u(x)| \leq C d_\Omega(x) M_{2d_\Omega(x)} |\nabla u|(x)$, where $M_{2d_\Omega(x)}$ is the restricted Hardy–Littlewood maximal operator. The L^p-boundedness of $M_{2d_\Omega(x)}$ then yields

$$\int_{\Omega''} |u(x)|^p d_\Omega(x)^{-p}\,dx \leq \int_{\Omega''} \big(M_{2d_\Omega(x)} |\nabla u|(x)\big)^p\,dx \leq C \int_\Omega |\nabla u(x)|^p\,dx, \tag{32}$$

and so the p-Hardy inequality for Ω, for every $1 < p < n$, follows by combining (31) and (32).

Nevertheless, while it is clear that $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c \cap B(w,r)) = \underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c \cap B(w,r)) = 0$ whenever $w \in \Omega^c$, and thus in particular $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c \cap B(w,r)) < p$ in accordance with Theorem 6.2, this example shows that $\operatorname{co\,dim}_{\mathrm{H}}$ cannot be replaced with $\overline{\operatorname{co\,dim}}_{\mathrm{A}}$ in the theorem: Indeed, we see that for all balls B centered at the origin, e.g. for $B = B(0,1/2)$, we have $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c \cap B) = n$ (i.e. $\underline{\dim}_{\mathrm{A}}(\Omega^c \cap B) = 0$), since for all $j \geq 3$ the set $\Omega^c \cap 2^j B_j \subset B$ can be covered by the ball B_j, and here the ratio of the radii of the balls has no positive lower bound. Hence neither of the estimates $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c \cap 2B) < p$ and $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c \cap B) > p$ holds here when $1 < p < n$, even though Ω admits a p-Hardy inequality. We conclude that in general, the bound $\operatorname{co\,dim}_{\mathrm{H}}(2B_0 \cap \Omega^c) < p - \beta$ is optimal in Theorem 6.2, and clearly the same conclusion holds for Theorem 6.1 as well. Thus point (i) is established.

For point (ii), we notice that Ω^c is not *uniformly perfect*, that is, there are relatively large annuli around the balls B_j which do not intersect Ω^c. Since uniform perfectness of Ω^c is equivalent to the validity of an n-Hardy inequality in $\Omega \subset \mathbb{R}^n$ (see [21]), we conclude that Ω does not admit an n-Hardy inequality. On the other hand, $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c \cap B(w,r)) = 0 < n$ whenever $w \in \Omega^c$ and $r > 0$, and so this example shows that the uniformity provided by the upper Assouad codimension in the sufficient conditions of Theorem 1.2 and Proposition 7.1 is essential, and can not be replaced with the condition that $\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c \cap B) < p - \beta$ for all balls centered at Ω^c. This yields point (ii).

Let us next take a look at the requirement $p - \beta > 1$ in Theorems 1.1 and 1.2. It was already mentioned in the Introduction that for Theorem 1.2, the unit ball $B = B(0,1) \subset \mathbb{R}^n$ shows the necessity of this condition, since $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\mathbb{R}^n \setminus B) = 0$, but B admits (p,β)-Hardy inequalities only when $p - \beta > 1$. In fact, it is now understood that in the case $p - \beta \leq 1$ it is the thickness of the boundary (rather than the complement) that plays a role in Hardy inequalities. For instance, the planar domain Ω bounded by the usual von Koch -showflake curve of dimension $\lambda = \log 4/\log 3$ admits a (p,β)-Hardy inequality if (and only if) $\beta < p - 2 + \lambda$, i.e. exactly when $p - \beta > \overline{\operatorname{co\,dim}}_{\mathrm{A}}(\partial\Omega)$ (cf. [22]). However, the requirement $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\partial\Omega) < p - \beta$ alone is not sufficient for a (p,β)-Hardy inequality if $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\partial\Omega) < 1$, as is shown by [22, Examples 7.3 and 7.4], but certain accessibility conditions are required in addition; cf. [22, 28] and see also Remark 4.1.

In the case of Theorem 1.1, the unbounded domain $\widetilde{\Omega}^s$ indicated at the end of [25, Example 6.3] serves as an example where $1 = \underline{\operatorname{co\,dim}}_{\mathrm{A}}(\mathbb{R}^n \setminus \widetilde{\Omega}^s)$, but the domain does not admit any (p,β)-Hardy inequalities when $p - \beta \leq 1$. Nevertheless, all known counterexamples here are such that $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) \leq 1$ (and thus in the examples in $\mathbb{R}^n$ we have $\dim_{\mathrm{A}}(\Omega^c) \geq n-1$), and so it could be asked if the requirement $p - \beta > 1$ could actually be removed (or weakened) if $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) > 1$. Under additional accessibility conditions Hardy inequalities can be obtained in the range $p - \beta \leq 1$ in the case of thin boundaries as well; see [25] for the Euclidean case.

Finally, the unboundedness of X in Theorem 1.1 can not be relaxed either, as the simple example $X = [-1,1]^2$, $\Omega = X \setminus \{0\}$ shows. Namely, here we can consider functions $u_j \in \operatorname{Lip}_0(\Omega)$ which have value one in $X \setminus B(0, 2^{-j})$ and $|\nabla u_j| \simeq 2^j$ in $B(0,2^{-j}) \setminus B(0, 2^{-j-1})$ (and $|\nabla u_j|$ vanishes elsewhere). These functions show that (p,β)-Hardy inequalities fail whenever $p - \beta \leq n = \underline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c)$. On the other hand, if $\Omega = \mathbb{R}^2 \setminus B(0,1)$, then $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) = 0$ but Ω does not admit a $(2,0)$-Hardy inequality, and this shows that it is essential in Theorem 1.2 that the complement of an unbounded Ω is unbounded as well.

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References

[1] H. Aikawa, Quasiadditivity of Riesz capacity, *Math. Scand.* 69 (1991), no. 1, 15–30.
[2] H. Aikawa, Quasiadditivity of capacity and minimal thinness, *Ann. Acad. Sci. Fenn. Ser. A I Math.* 18 (1993), no. 1, 65–75.
[3] H. Aikawa and M. Essén, Potential theory—selected topics, Lecture Notes in Mathematics, 1633, Springer-Verlag, Berlin, 1996.
[4] A. Björn and J. Björn, Nonlinear potential theory on metric spaces, EMS Tracts in Mathematics, 17, European Mathematical Society (EMS), Zürich, 2011.
[5] A. Björn, J. Björn and J. Lehrbäck, Sharp capacity estimates for annuli in weighted $\mathbb{R}^n$ and in metric spaces, *Math. Z.* (in press), doi:10.1007/s00209-016-1797-4
[6] A. Björn, J. Björn and N. Shanmugalingam, Sobolev extensions of Hölder continuous and characteristic functions on metric spaces, *Canadian J. Math.* 59 (2007), no. 6, 1135–1153.
[7] J. Björn, P. MacManus and N. Shanmugalingam, Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces, *J. Anal. Math.* 85 (2001), 339–369.
[8] B. Dyda and A.V. Vähäkangas, A framework for fractional Hardy inequalities, *Ann. Acad. Sci. Fenn. Math.* 39 (2014), no. 2, 675–689.
[9] K. Falconer, Fractal Geometry. Mathematical foundations and applications, John Wiley & Sons, Ltd., Chichester, 1990.
[10] J. Fraser, Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc.* (to appear) arXiv:1301.2934
[11] F.W. Gehring, The L^p-integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* 130 (1973), 265–277.
[12] P. Hajłasz, Pointwise Hardy inequalities, *Proc. Amer. Math. Soc.* 127 (1999), no. 2, 417–423.
[13] P. Hajłasz and P. Koskela, Sobolev met Poincaré, *Mem. Amer. Math. Soc.* 145 (2000), no. 688.
[14] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities, 2nd ed, Cambridge University Press, Cambridge, 1952.
[15] J. Heinonen, Lectures on analysis in metric spaces, Universitext, Springer-Verlag, New York, 2001.
[16] J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson Sobolev spaces on metric measure spaces: an approach based on upper gradients, New Mathematical Monographs, Cambridge University Press (forthcoming).

[17] L. Ihnatsyeva and A.V. Vähäkangas, Hardy inequalities in Triebel–Lizorkin spaces II. Aikawa dimension, *Ann. Mat. Pura Appl. (4)*, to appear.

[18] A. Käenmäki, J. Lehrbäck and M. Vuorinen, Dimensions, Whitney covers, and tubular neighborhoods, *Indiana Univ. Math. J.* 62 (2013), No. 6, 1861–1889.

[19] S. Keith and X. Zhong, The Poincaré inequality is an open ended condition, *Ann. of Math. (2)* 167 (2008), no. 2, 575–599.

[20] R. Korte, J. Lehrbäck and H. Tuominen, The equivalence between pointwise Hardy inequalities and uniform fatness, *Math. Ann.* 351 (2011), no. 3, 711–731.

[21] R. Korte and N. Shanmugalingam, Equivalence and self-improvement of p-fatness and Hardy's inequality, and association with uniform perfectness, *Math. Z.* 264 (2010), no. 1, 99–110.

[22] P. Koskela and J. Lehrbäck, Weighted pointwise Hardy inequalities, *J. London Math. Soc.* 79 (2009), no. 3, 757–779.

[23] P. Koskela and X. Zhong, Hardy's inequality and the boundary size, *Proc. Amer. Math. Soc.* 131 (2003), no. 4, 1151–1158.

[24] D.G. Larman, A new theory of dimension, *Proc. London Math. Soc.* 17 (1967), 178–192.

[25] J. Lehrbäck, Weighted Hardy inequalities and the size of the boundary, *Manuscripta Math.* 127 (2008), no. 2, 249–273.

[26] J. Lehrbäck, Self-improving properties of weighted Hardy inequalities, *Adv. Calc. Var.* 1 (2008), no. 2, 193–203.

[27] J. Lehrbäck, Necessary conditions for weighted pointwise Hardy inequalities, *Ann. Acad. Sci. Fenn. Math.* 34 (2009), no. 2, 437–446.

[28] J. Lehrbäck, Weighted Hardy inequalities beyond Lipschitz domains, *Proc. Amer. Math. Soc.* 142 (2014), no. 5, 1705–1715.

[29] J. Lehrbäck and N. Shanmugalingam, Quasiadditivity of variational capacity, *Potential Anal.* 40 (2014), no. 3, 247–265.

[30] J. Lehrbäck and H. Tuominen, A note on the dimensions of Assouad and Aikawa, *J. Math. Soc. Japan* 65 (2013), no. 2, 343–356.

[31] J. Lewis, Uniformly fat sets, *Trans. Amer. Math. Soc.* 308 (1988), no. 1, 177–196.

[32] J. Luukkainen, Assouad dimension: antifractal metrization, porous sets, and homogeneous measures, *J. Korean Math. Soc.* 35 (1998), no. 1, 23–76.

[33] P. Mattila, Geometry of sets and measures in Euclidean spaces, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge Univ. Press, Cambridge, 1995.

[34] S. Secchi, D. Smets and M. Willem, Remarks on a Hardy-Sobolev inequality, *C. R. Math. Acad. Sci. Paris* 336 (2003), no. 10, 811–815.

[35] J.O. Strömberg and A. Torchinsky, Weighted Hardy spaces. Lecture Notes in Mathematics, 1381. Springer-Verlag, Berlin, 1989.

[36] A. Wannebo, Hardy inequalities. *Proc. Amer. Math. Soc.* 109 (1990), no. 1, 85–95.

[37] Q.H. Yang, Hardy type inequalities related to Carnot–Carathéodory distance on the Heisenberg group. *Proc. Amer. Math. Soc.* 141 (2013), no. 1, 351–362.

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