AHLFORS-REGULAR DISTANCES ON THE HEISENBERG GROUP WITHOUT BILIPSCHITZ PIECES

ENRICO LE DONNE, SEAN LI, AND TAPIO RAJALA

Abstract. We show that the Heisenberg group is not minimal in looking down. This answers Problem 11.15 in *Fractured fractals and broken dreams* by David and Semmes, or equivalently, Question 22 and hence also Question 24 in *Thirty-three yes or no questions about mappings, measures, and metrics* by Heinonen and Semmes.

The non-minimality of the Heisenberg group is shown by giving an example of an Ahlfors 4-regular metric space X having big pieces of itself such that no Lipschitz map from a subset of X to the Heisenberg group has image with positive measure, and by providing a Lipschitz map from the Heisenberg group to the space X having as image the whole X.

As part of proving the above result we define a new distance on the Heisenberg group that is bounded by the Carnot-Carathéodory distance, that preserves the Ahlfors-regularity, and such that the Carnot-Carathéodory distance and the new distance are biLipschitz equivalent on no set of positive measure. This construction works more generally in any Ahlfors-regular metric space where one can make suitable shortcuts. Such spaces include for example all snowflaked Ahlfors-regular metric spaces. With the same techniques we also provide an example of a left-invariant distance on the Heisenberg group biLipschitz to the Carnot-Carathéodory distance for which no blow-up admits nontrivial dilations.

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Contents

1. Introduction

In [DS97] David and Semmes proposed a concept of BPI (big pieces of itself) spaces as a notion of rough self-similarity for metric spaces. The definition of a BPI space requires any two balls of the space to contain big pieces that are biLipschitz equivalent, see Definition 2.1 for the precise definition. Self-similar fractals and Carnot groups are easy examples of BPI spaces. David and Semmes also introduced *BPI equivalence* and a partial order for BPI spaces called *looking down*. Both of them will be defined in Section 2. Two BPI spaces are BPI equivalent if large parts of the two spaces are biLipschitz equivalent. A BPI metric space X looks down on another BPI metric space Y if X and Y have same Hausdorff dimension and there is a closed subset of X that can be mapped to a set of positive measure in Y via a Lipschitz map. BPI equivalence of spaces X and Y implies that X and Y are *look-down equivalent*, meaning that X looks down on Y and Y looks down on X. However, Laakso has shown that the converse is not true in general [Laa02].

The partial ordering of BPI spaces raises the interesting question of what are the possible minimal spaces in this ordering. A space X is *mimimal* in looking down if every space Y on which X looks down is look-down equivalent to X. For example, from the result of Kirchheim [Kir94] we know that Euclidean spaces are minimal in looking down. A quantitative version of Kirchheim's theorem was later given in [Sch09] in which it was shown that if a map $f : [0,1]^n \to X$ has positive Hausdorff n-content, then it has a quantitatively large biLipschitz piece.

David and Semmes asked in Problem 11.15 of [DS97] if the Heisenberg group $\mathbb{H}$ is also minimal in looking down, when equipped with sub-Riemannian distances, also called Carnot-Carathéodory distances. This was also asked as Question 22 of [HS97]. We show that this is not the case.

Theorem 1.1. *The subRiemannian Heisenberg group is not minimal in looking down.*

This theorem has important implications in the development of a theory of rectifiability based on the Heisenberg group. Recall that a metric measure space (X, d, μ) is countably n-rectifiable if there exist a countable set of Borel subsets $A_i \subseteq \mathbb{R}^n$ and Lipschitz maps $f_i : A_i \to X$ such that $\mu\left(X \setminus \bigcup_i f_i(A_i)\right) = 0$ and $\mu \ll \mathcal{H}^n$ where $\mathcal{H}^n$ is the Hausdorff n-measure (we review the definition of Hausdorff measure in the next section). It was shown in [Kir94] that, by further countably decomposing each $f_i(A_i)$ if necessary, one may assume that each f_i is biLipschitz.

One can easily create a definition of being $\mathbb{H}$-rectifiable by letting each A_i be a Borel subset of the Heisenberg group $\mathbb{H}$ and setting $n = 4$, the Hausdorff dimension of $\mathbb{H}$. However, we now see that there exists a metric measure space (X, d, μ) with positive Hausdorff 4-measure that is the Lipschitz image of a subset of $\mathbb{H}$ but is not the countable union of biLipschitz images of subsets of $\mathbb{H}$. Thus, "Lipschitz rectifiability" is strictly weaker than "biLipschitz rectifiability" when using the Heisenberg geometry.

Using the self-similarity of the Carnot-Carathéodory distance d_{cc} it is easy to construct BPI spaces that can be realized as subsets of $\mathbb{H}$ with self-similar type modifications of the distance d_{cc}. A critical part in the proof of Theorem 1.1 is to modify the distance d_{cc} to get a new distance d in such a way that with the d_{cc} distance the space looks down on the space equipped with the distance d, but not the other way. Such distance is constructed using a shortening technique that has been also used in [LD13, LR16] to give examples of distances not satisfying the Besicovitch Covering Property. The result obtained here with the shortening technique is the following.

Theorem 1.2. *Let $(\mathbb{H}, d_{cc})$ be the subRiemannian Heisenberg group. There exists a distance d on $\mathbb{H}$ such that*

(1) $d \le d_{cc}$*;*
(2) $(\mathbb{H}, d)$ *is Ahlfors 4-regular;*
(3) *if $A \subseteq \mathbb{H}$ is a subset with $\mathcal{H}^4_{cc}(A) > 0$, then d and d_{cc} are not biLipschitz equivalent on A.*

Recall that a metric measure space (X, d, μ) is Ahlfors Q-regular for some $Q > 0$ if there exists some $C \ge 1$ so that

$$\frac{1}{C} r^Q \le \mu(B_d(x,r)) \le C r^Q, \qquad \forall\, x \in X,\, \forall\, r \in (0, \mathrm{diam}_d(X)).$$

It is easy to see that if (X, d, μ) is Ahlfors Q-regular, then so is $(X, d, \mathcal{H}^Q_d)$ and so we can just talk about Ahlfors Q-regular metric spaces.

Thus, we construct an Ahlfors 4-regular metric space X onto which $(\mathbb{H}, d_{cc})$ Lipschitz surjects, but for which this surjection has no biLipschitz pieces. Theorem 1.2 answers Question 24 of [HS97] negatively (although the same negative answer is provided by the negative answer to Question 22 given by Theorem 1.1).

It should be noted that this behaviour changes when one requires that the target X is another Carnot group. Indeed, one can then use a similar argument as in [Kir94], with inspiration from [Pau04], to show that Lipschitz maps from the Heisenberg group to another Carnot group with positive 4-measure image have biLipschitz pieces. This statement can also be made quantitative as was done in [Mey13, Li15].

Another situation where Lipschitz maps have biLipschitz pieces is when the spaces are Ahlfors regular, linearly locally contractible topological manifolds and the target has manifold weak tangents, see the work of G.C. David [Dav15] (this David is not the same David of David-Semmes). We note that in Theorem 1.2 the constructed space $(\mathbb{H}, d)$ neither has manifold tangents nor is linearly locally contractible.

The construction of the distance d in Theorem 1.2 relies on the fact that in the Heisenberg group we can shorten the distance between two points that differ only in the vertical component without affecting the distances far away from the two points. By taking this property as an assumption we obtain a more general result.

Theorem 1.3. *Let (X, ρ) be a metric space and $Q > 0$. Assume*

(1) (X, ρ) *is Ahlfors Q-regular;*
(2) *there exists* $\lambda \in (0,1)$ *such that for all* $p \in X$ *and all* $0 < r < \operatorname{diam}_\rho(X)$ *there exist* $q_1, q_2 \in B_\rho(p, r)$ *such that*

$$\rho(q_1, q_2) \geq \lambda r$$

and

$$\rho(p_1, p_2) \leq \rho(p_1, q_1) + \rho(p_2, q_2), \qquad \forall p_1, p_2 \notin B_\rho(p, r). \tag{1.1}$$

Then there exists a distance d on X such that

(1) $d \leq \rho$*;*
(2) (X, d) *is Ahlfors Q-regular;*
(3) *if* $A \subseteq X$ *is a subset with* $\mathcal{H}^Q_\rho(A) > 0$*, then d and ρ are not biLipschitz equivalent on A.*

We will first prove Theorem 1.3 in Section 3. After having proven Theorem 1.3, the proof of Theorem 1.2 follows by showing that there is a metric on $\mathbb{H}$, biLipschitz equivalent to the Carnot-Carathéodory metric, that satisfies (1.1). This will be done in Section 4. Theorem 1.1 will then be proven in Section 5. Other examples of spaces satisfying the condition in Theorem 1.3 are snowflakes of Ahlfors-regular metric spaces, e.g., the real line equipped with the square root of the Euclidean distance, see Theorem 4.1.

David and Semmes also asked in Problem 11.17 of [DS97] for which $s_1, \ldots, s_n \in (0,1]$ the space

$$\left(\mathbb{R}^n, \sum_{i=1}^n |x_i - y_i|^{s_i}\right)$$

is minimal in looking down. Based on our results we can deduce that if $s_1 = \cdots = s_n \neq 1$, then, since the space is a snowflake of Euclidean space, it is not minimal in looking down. We would conjecture that such coordinate-wise snowflakes are minimal in looking down if and only if $s_i = 1$ for all i. We shall not further investigate this problem in this paper.

In the second part of the paper we consider distances on $\mathbb{H}$ that have extra homogeneity structure. For example, we assume that left translations are biLipschitz. We show that with the assumptions of Theorem 1.2 such distances are locally biLipschitz equivalent to the distance d_{cc}.

Theorem 1.4. *Let d be a distance on the Heisenberg group $\mathbb{H}$ such that $d \leq d_{cc}$ and $\mathcal{H}^4_d(B_{cc}(0,1)) > 0$. Assume that the left translations in $\mathbb{H}$ are biLipschitz with respect to d. Then d and d_{cc} are biLipschitz equivalent on compact sets.*

We remark that the assumptions in Theorem 1.4 are necessary. Indeed, if we don't assume $d \leq d_{cc}$, then as a counterexample one can take two sub-Riemannian distances on $\mathbb{H}$ that have two different horizontal bundles. If we don't assume $\mathcal{H}^4_d(B_{cc}(0,1)) > 0$, then a counterexample is given by every Riemannian left-invariant distance. Moreover, the distance $\min\{1, d_{cc}\}$ shows that the conclusion of the theorem may not be global.

Theorems 1.1, 1.2, and 1.4 are stated for the sub-Riemannian distance d_{cc}. However, it follows immediately that proving these theorems for a distance that is biLipschitz equivalent to d_{cc} will also prove the theorems for d_{cc}. Thus, we will actually prove these statements for a different distance d_b, which is biLipschitz equivalent to d_{cc}, that we define at (2.3).

We conclude the paper by showing that for distances that are biLipschitz equivalent to d_{cc} the metric differentiation does not hold in general. Kirchheim's result in [Kir94] can be stated as the fact that every semi-distance d in $\mathbb{R}^n$ that is smaller than the Euclidean distance is metrically differentiable, i.e., at almost every point its blow-up is a homogeneous semi-distance. Similarly, by [Pau01], we know that on Carnot groups semi-distances smaller than d_{cc} are metrically differentiable but only in the horizontal directions. Regarding non-horizontal directions, from [KM03] we know that there is a distance in the Heisenberg group that is a counterexample to metric differentiability, although it is not biLipschitz to d_{cc}. As the last result of this paper we give in Section 6.2 another pair of counterexamples to metric differentiability that are biLipschitz equivalent to d_{cc} and whose blow-ups even fail self-similarity, which is a weaker property than homogeneity. If $\{\delta_\lambda\}_{\lambda>0}$ denotes the standard one-parameter family of isomorphisms of $\mathbb{H}$, see Section 2.1, a (semi-)distance d is *self-similar* if there exists some $\lambda > 1$ for which $d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$, for all $p, q \in \mathbb{H}$. In the following result, by *a blow-up* of a distance d we mean any point-wise limit of the functions

$$(p_1, p_2) \mapsto \frac{1}{\lambda_j} d(q_j \delta_{\lambda_j}(p_1), q_j \delta_{\lambda_j}(p_2)),$$

as $\lambda_j \to 0$ and $q_j \in \mathbb{H}$.

Theorem 1.5 (Failure of Kirchheim-metric differentiation for biLipschitz maps)**.** *There exist two distances d_1, d_2 on $\mathbb{H}$ that are biLipschitz equivalent to d_{cc} such that*

(1) *The distance d_1 is left-invariant, but no blow-up of d_1 is self-similar.*
(2) *No blow-up of d_2 is left-invariant nor self-similar.*

Both Theorem 1.4 and Theorem 1.5 are proved in Section 6.

2. Preliminaries

We begin by recalling the definition of Hausdorff measures on a metric space (X, d). Let $Q > 0$. Then for $A \subseteq X$, one defines

$$\mathcal{H}^Q_d(A) := \lim_{s\to 0^+} \inf\left\{\sum_{i=1}^{\infty} (\mathrm{diam}_d\, E_i)^Q : A \subseteq \bigcup_i E_i \text{ an open cover}, \mathrm{diam}_d(E_i) < s\right\}.$$

We say that $\mathcal{H}^Q$ is the Hausdorff Q-measure of (X, d). It is known that the Hausdorff Q-measure is Borel regular although it may not be locally finite. We shall denote the open ball in the metric space (X, d) by $B_d(x, r) := \{y \in X : d(y, x) < r\}$ and more generally, the open r-neighbourhood of a set $A \subset X$ by $B_d(A, r) := \{y \in X : \mathrm{dist}_d(A, y) < r\}$.

A biLipschitz map f between metric spaces (X, d) and (X', d') is said to be C-*conformally biLipschitz* with scale factor $\lambda > 0$ if f is C-biLipschitz between the metric spaces $(X, \lambda d)$ and (X', d'). Another term, coming from Banach space theory, for the same notion is *quasi-similarity*.

Definition 2.1 (BPI space)**.** An Ahlfors Q-regular metric space (X, d) is said to be a *BPI* ("big pieces of itself") *space* if there exist constants $C \geq 1$ and $\theta > 0$ such that for all $x_1, x_2 \in X$ and $0 < r_1, r_2 < \mathrm{diam}_d(X)$ there is a closed set $A \subseteq B_d(x_1, r_1)$ with $\mathcal{H}^Q_d(A) \geq \theta r_1^Q$ and if there is a C-conformally biLipschitz embedding $f\colon A \to B_d(x_2, r_2)$ with scale factor r_2/r_1.

Definition 2.2 (BPI equivalence)**.** Two BPI spaces (X, d) and (X', d') of the same dimension Q are called *BPI equivalent* if there exist constants $\theta > 0$ and $C > 0$ such that for each $x \in X$, $x' \in X'$ and radii $0 < R < \mathrm{diam}_d(X)$, $0 < R' < \mathrm{diam}_{d'}(X')$ there exist a subset $A \subset B_d(x, R) \subset X$ with $\mathcal{H}^Q_d(A) \geq \theta R^Q$ and a C-conformally biLipschitz embedding $f\colon A \to B_{d'}(x', R')$ with scale factor R'/R.

Definition 2.3 (Looking down)**.** Let (X, d) and (X', d') be BPI metric spaces of Hausdorff dimension Q. The space (X, d) is said to *look down on* (X', d') if there is a closed set $A \subset X$ and a Lipschitz map $f\colon A \to X'$ such that $f(A)$ has positive Hausdorff Q-measure. If also X' looks down on X, then X and X' are called *look-down equivalent*.

2.1. **The Heisenberg group and its distances.** The Heisenberg group $\mathbb{H}$ is the simply connected Lie group whose Lie algebra is generated by three vectors X, Y, Z with only non-zero relation $[X, Y] = Z$. Via exponential coordinates it can be identified as the manifold $\mathbb{R}^3$ equipped with Lie multiplication:

$$p \cdot q = \left(x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - y_p x_q) \right).$$

It follows easily from the definition that the origin $(0, 0, 0) \in \mathbb{H}$ is the identity element and that the center of the group is

$$\mathrm{Z}(\mathbb{H}) = \{(0, 0, z) : z \in \mathbb{R}\}.$$

For each $\lambda > 0$, the Heisenberg group has an automorphism defined as

$$\delta_\lambda(x, y, z) := (\lambda x, \lambda y, \lambda^2 z). \tag{2.1}$$

A left-invariant (semi-)distance d is *homogeneous*, with respect to (2.1), if for all $\lambda > 0$

$$d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q), \qquad \forall p, q \in \mathbb{H}. \tag{2.2}$$

Our main example of homogeneous distance is the following. We introduce the *box norm*

$$\|p\| := \max\left\{|x_p|, |y_p|, \sqrt{|z_p|}\right\}.$$

We define the *box distance* as

$$d_b(p,q) := \left\|p^{-1}q\right\|.$$

Clearly, d_b is left-invariant and it satisfies (2.2). To check that it satisfies the triangle inequality we need to show that

$$\|p \cdot q\| \leq \|p\| + \|q\|.$$

First,

$$|x_{p\cdot q}| = |x_p + x_q| \leq |x_p| + |x_q| \leq \|p\| + \|q\|,$$

and analogously for the y component. Second,

$$\begin{aligned}
\sqrt{|z_{p\cdot q}|} &= \sqrt{\left|z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p)\right|} \\
&\leq \sqrt{|z_p| + |z_q| + |x_p||y_q| + |x_q||y_p|} \\
&\leq \sqrt{\|p\|^2 + \|q\|^2 + 2\|p\|\|q\|} \\
&\leq \|p\| + \|q\|.
\end{aligned}$$

Explicitly, the box distance is

$$d_b(p_1,p_2) = \max\left\{|x_1 - x_2|, |y_1 - y_2|, \sqrt{\left|z_1 - z_2 - \frac{1}{2}(x_1 y_2 - x_2 y_1)\right|}\right\}. \tag{2.3}$$

One can easily show that d_b and d_{cc} are biLipschitz equivalent using the fact that both distances are homogeneous and left-invariant.

Let $\pi : (\mathbb{H}, d_b) \to (\mathbb{R}^2, |\cdot|_\infty)$ be the projection onto the xy-plane. One easily sees that this is a 1-Lipschitz homomorphism.

Given a homomorphism $L : \mathbb{H} \to \mathbb{H}$, one can define the Jacobian to be

$$J(L) = \frac{\mathcal{H}^4_{d_b}(L(B_{d_b}(0,1)))}{\mathcal{H}^4_{d_b}(B_{d_b}(0,1))}.$$

Let $f : (\mathbb{H}, d_b) \to (\mathbb{H}, d_b)$ be a Lipschitz map. Pansu proved in [Pan89] that for almost every $x \in \mathbb{H}$ there exists a Lipschitz homomorphism $Df(x) : \mathbb{H} \to \mathbb{H}$ (the Pansu-derivative of f at x) so that

$$Df(x)(g) = \lim_{\lambda \to 0} \delta_{1/\lambda}(f(x)^{-1} f(x\delta_\lambda(g))).$$

This result was extended to Lipschitz maps whose domains are measurable subsets $A \subseteq \mathbb{H}$ by Magnani in [Mag01]. Magnani also used the Pansu-derivative in conjunction with the Jacobian to get the following area formula:

$$\int_{\mathbb{H}} N(f,A,y)\, d\mathcal{H}^4_{d_b}(y) = \int_A J(Df(x))\, d\mathcal{H}^4_{d_b}(x). \tag{2.4}$$

Here, $N(f, A, y)$ is the multiplicity of f with respect to the set A.

2.2. **Shortening distances.** Given a metric space (X, ρ), a symmetric function $c : X \times X \to [0, \infty)$ such that $c \leq \rho$ will be called a *cost function*. We denote by $\mathcal{S}$ all those pairs of points $(x, y) \in X \times X$ such that $c(x, y) < \rho(x, y)$

$$\mathcal{S} := \{(x, y) \in X \times X \ : \ c(x, y) < \rho(x, y)\}.$$

An element in $\mathcal{S}$ will be called *shortcut* (or flight or tunnel). If we have $N \in \mathbb{N}$ and $x_0, x_1, \ldots, x_N \in X$ then the N-tuple $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ will be called an *itinerary* from the extreme points x_0 to x_N and we set $\operatorname{Ext}(\mathbf{x}) := (x_0, x_N)$ and $\ell(\mathbf{x}) := N$. We will denote by $\mathcal{I}$ the collection of all itineraries in X, i.e.,

$$\mathcal{I} := \{(x_0, x_1, \ldots, x_N) \ : \ N \in \mathbb{N},\, x_j \in X\}.$$

The *cost* of an itinerary $\mathbf{x} = (x_0, x_1, \ldots, x_N) \in \mathcal{I}$ is

$$c(\mathbf{x}) := \sum_{i=1}^{N} c(x_{i-1}, x_i).$$

The distance d associated to the cost function c is defined as

$$d(x, y) := \inf\{c(\mathbf{x}) \ : \ \mathbf{x} \in \mathcal{I},\, \operatorname{Ext}(\mathbf{x}) = (x, y)\}. \tag{2.5}$$

Remark 2.4. It is not too hard to verify symmetry and the triangle inequality for d and so d is a semi-distance on X. If there is another distance d' on X such that

$$d'(x, y) \leq c(x, y), \qquad \forall x, y \in X,$$

then by the triangle inequality for d', we also have that

$$d'(x, y) \leq d(x, y), \qquad \forall x, y \in X,$$

and so d is then a distance.

We shall assign to each shortcut a natural number that we call *level* of the shortcut. Namely, a function $L \colon \mathcal{S} \to \mathbb{N}$ will be called a *level function*. Larger levels will usually indicate shortcuts over smaller distances.

3. Breaking biLipschitz equivalence using shortcuts

In this section we prove Theorem 1.3. Let (X, ρ) and λ be as in the assumptions of Theorem 1.3.

3.1. **Constructing the shortcuts.** Let $(\alpha_n)_{n=1}^{\infty}$ be a non-increasing sequence of real numbers in $[0,1)$. The number α_n will be the ratio of the cost of the level n shortcut compared to the original distance of the shortcut.

Let us define the shortcuts one level at a time. We define inductively the level n shortcuts $\mathcal{S}_n \subset X \times X$, for $n \in \mathbb{N}$ as follows. We set $c_E \geq 8$ to be a constant that we now fix. Inductively, let $\mathcal{N}_n := \{x_i\}$ to be a set of points in

$$X \setminus \bigcup_{j=1}^{n-1} \bigcup_{(x,y)\in\mathcal{S}_j} B_\rho(\{x,y\}, 4\lambda^n) \tag{3.1}$$

such that $\rho(x_i, x_j) \geq 4\lambda^n$ and $X \subseteq \bigcup_i B_\rho(x_i, c_E\lambda^n)$. For $n = 1$, the condition (3.1) becomes vacuous as there is no $\mathcal{S}_0$, hence $\mathcal{N}_1$ is just a 4λ- separated set that is also a $c_E\lambda$-net for X. Necessarily, we shall need that $c_E \geq 4$ and in general more restrictions of c_E are necessary to ensure existence of such an $\mathcal{N}_n$. We show later in Lemma 3.2 that there is always a choice of the constants λ and c_E for which the set $\mathcal{N}_n$ exists.

Using assumption (2) of Theorem 1.3 we select for each i points $q_{i,1}, q_{i,2} \in B_\rho(x_i, \lambda^n)$ such that

$$\rho(q_{i,1}, q_{i,2}) \geq \lambda^{n+1}$$

and

$$\rho(p_1, p_2) \leq \rho(p_1, q_{i,1}) + \rho(p_2, q_{i,2}) \qquad \text{for all } p_1, p_2 \notin B_\rho(x_i, \lambda^n). \tag{3.2}$$

Now define the level n shortcuts as

$$\mathcal{S}_n := \{(q_{i,1}, q_{i,2}) \,:\, i\} \cup \{(q_{i,2}, q_{i,1}) \,:\, i\},$$

their corresponding costs as

$$c(q_{i,1}, q_{i,2}) := c(q_{i,2}, q_{i,1}) := \alpha_n \rho(q_{i,1}, q_{i,2})$$

and their level as

$$L(q_{i,1}, q_{i,2}) := L(q_{i,2}, q_{i,1}) := n.$$

Finally, let

$$\mathcal{S} := \bigcup_{n=1}^{\infty} \mathcal{S}_n.$$

We also set defined $c(x,y) = \rho(x,y)$ for pairs $(x,y) \notin \mathcal{S}$. Finally, we define d as in (2.5).

Remark 3.1. The construction gives us uniqueness of shortcuts. That is, if $(x,y) \in \mathcal{S}$, then $(x,z) \notin \mathcal{S}$ for all $z \neq y$. This follows easily from (3.1).

We now prove the existence of the sets $\mathcal{N}_n$ for certain choices of λ and c_E.

Lemma 3.2. *There exists some $\lambda_0 \in (0, 1/4)$ depending only on the Ahlfors regularity of (X,ρ) such that if we set $\lambda \leq \lambda_0$ and $c_E = 8 + \frac{1}{\lambda}$, then we can always find $\mathcal{N}_n$.*

Proof. Let μ be a measure on (X, ρ) so that (X, ρ, μ) is Ahlfors regular (one could use $\mu = \mathcal{H}^Q_\rho$ for instance). We may suppose by taking λ small enough (as we are free to do) and using Ahlfors regularity of (X, ρ, μ) that

$$\mu(B_\rho(x, r/4)) - \mu(B_\rho(y, \lambda r)) - \mu(B_\rho(z, \lambda r)) > 0, \qquad \forall x, y, z \in X, 0 < r < \mathrm{diam}_\rho(X). \tag{3.3}$$

Let $A = \bigcup_{j<n} \bigcup_{(x,y) \in \mathcal{S}_j} \{x, y\}$. By the definition of A, each $x \in A$ comes with a pair $x' \in A$ such that $(x, x') \in \mathcal{S}_l$ for some $l < n$. We claim that

$$\rho(x, y) \geq 2\lambda^{n-1}, \qquad \forall y \in A \setminus \{x, x'\}. \tag{3.4}$$

To see this, taking $y \in A \setminus \{x, x'\}$, there exists $y' \in A$ such that $(y, y') \in \mathcal{S}_k$ for some $k < n$. Let $x^l_i \in \mathcal{N}_l$ such that $x, x' \in B_\rho(x^l_i, \lambda^l)$ and $x^k_j \in \mathcal{N}_k$ such that $y, y' \in B_\rho(x^k_j, \lambda^k)$. We consider two cases. Suppose first that $k = l$. By the $4\lambda^k$ separation of $\mathcal{N}_k$ we then have

$$\rho(x, y) \geq \rho(x^l_i, x^k_j) - \rho(x, x^l_i) - \rho(y, x^k_j) \geq 4\lambda^k - \lambda^k - \lambda^k = 2\lambda^k \geq 2\lambda^{n-1}.$$

Suppose now that $k \neq l$. By symmetry we may assume $k < l$. Then by construction, $x^l_i \notin B_\rho(y, 4\lambda^l)$ and thus

$$\rho(x, y) \geq \rho(x^l_i, y) - \rho(x, x^l_i) \geq 4\lambda^l - \lambda^l = 3\lambda^l \geq 2\lambda^{n-1}.$$

Thus (3.4) is proven.

Let $\{x_i\}$ be a maximal $4\lambda^n$-separated net of $X \setminus B_\rho(A, 4\lambda^n)$. Let $x \in X$. Suppose there exists $y \in A$ such that $\rho(x, y) < 4\lambda^n$. As $\lambda < 1/4$, we get by (3.4) that the number of balls $\{B_\rho(p, 4\lambda^n)\}_{p \in A}$ that intersect $B_\rho(y, \lambda^{n-1})$ is at most 2. This, together with (3.3), gives that

$$B_\rho(y, \lambda^{n-1}) \setminus B_\rho(A, 4\lambda^n) \neq \emptyset.$$

Thus, there exists some $z \in B_\rho(y, \lambda^{n-1}) \setminus B_\rho(A, 4\lambda^n)$. As $\{x_i\}$ is also a $4\lambda^n$ covering of $X \setminus B_\rho(A, 4\lambda^n)$, we get that there exists some x_i such that $\rho(z, x_i) < 4\lambda^n$. Altogether, we get that

$$\rho(x, x_i) \leq \rho(x, y) + \rho(y, z) + \rho(z, x_i) < 4\lambda^n + \lambda^{n-1} + 4\lambda^n = \left(8 + \frac{1}{\lambda}\right)\lambda^n.$$

In the case when $x \notin B_\rho(A, 4\lambda^n)$, we are also done as the set $\{x_i\}$ is a $4\lambda^n$-cover of $X \setminus B_\rho(A, 4\lambda^n)$. □

3.2. **Properties of the new semi-distance.** In this section we point out some properties of the semi-distance d, for example, the fact that it is a distance when α_n are positive. We define the subset of *alternating itineraries*

$$\mathcal{I}_A := \{(x_0, \dots, x_N) \in \mathcal{I} \ : \ N \text{ odd}, (x_{j-1}, x_j) \in \mathcal{S} \iff j \text{ even}\}.$$

Colloquially speaking, for each of these alternating itineraries, one walks at every odd step and flies at every even step. Note that we allow for the stationary walks, i.e., the itinerary can have $x_{j-1} = x_j$, for some j odd. Hence, every itinerary can be modified to be an alternating itinerary with no increase in cost by merging consecutive walks and adding a stationary walk between consecutive shortcuts. Note then that x_i lies in some shortcut if $1 < i < N$.

We also define the subset of *non-self-intersecting itineraries*

$$\mathcal{I}'_A := \{(x_0, \ldots, x_N) \in \mathcal{I}_A \ : \ x_i \neq x_j \text{ for all } i \neq j \text{ except possibly } (i,j) \in \{(0,1),(N-1,N)\}\}.$$

Thus, elements in $\mathcal{I}'_A$ are alternating itineraries that do not revisit a site in X. However, being alternating they are of the form no shortcut, shortcut, no shortcut, shortcut, etc etc. Hence, we need to allow the first (and last) jump to be possibly trivial, if we need to start (or end) with a shortcut.

Our first result is that there is a subitinerary of any alternating itinerary that is itself alternating and non-self-intersecting.

Lemma 3.3. *For all* $\mathbf{x} \in \mathcal{I}_A$*, there exist* $\mathbf{x}' \in \mathcal{I}'_A$ *so that* $\mathrm{Ext}(\mathbf{x}) = \mathrm{Ext}(\mathbf{x}')$ *and* $c(\mathbf{x}') \leq c(\mathbf{x})$*.*

Proof. We will prove that if $\mathbf{x} \in \mathcal{I}_A$ is self-intersecting, then there exists a strictly shorter itinerary $\mathbf{y} \in \mathcal{I}_A$ with $\mathrm{Ext}(\mathbf{y}) = \mathrm{Ext}(\mathbf{x})$ and $c(\mathbf{y}) \leq c(\mathbf{x})$. By iterating this procedure, we get the lemma.

Suppose first that there exist $1 \leq i < j \leq N-1$ so that $x_i = x_j$. There are a few cases to check. If i and j are both even, then we can remove $(x_i, x_{i+1}, ..., x_{j-1})$ from the itinerary to get $\mathbf{y}$. If i and j are both odd, we remove $(x_{i+1}, ..., x_j)$.

If i is odd and j is even, then $j \geq i+3$. If $(x_{i-1}, x_{j+1}) \notin \mathcal{S}$, then we remove $(x_i, x_{i+1}, ..., x_j)$ from the itinerary and are done. If $(x_{i-1}, x_{j+1}) \in \mathcal{S}$, then we replace $(x_i, ..., x_j)$ in the itinerary with (x_{i-1}, x_{j+1}) and are done as $j \geq i+3$. If i is even and j is odd, both (x_{i-1}, x_i) and (x_j, x_{j+1}) are shortcuts. Hence, $x_i = x_j$ implies that $(x_i, x_{i+1}) \in \mathcal{S}$, whence $x_{i-1} = x_{j+1}$ by uniqueness of shortcuts (see Remark 3.1). Then, note that $i-1$ is odd and $j+1$ is even, so we are back in the previous case.

Now suppose there exists $1 < i \leq N$ so that $x_0 = x_i$. In the case i is odd, then we can consider the itinerary $(x_0, x_i, x_{i+1}, \ldots, x_N)$, which is alternating and has no greater cost. We then consider the case i even, so that $(x_{i-1}, x_i) \in \mathcal{S}$. We then we can consider the itinerary $(x_0, x_{i+1}, \ldots, x_N)$, which is alternating and has no greater cost.

The case when there exist $1 \leq i < N$ so that $x_i = x_N$ is similar. $\square$

We can define the level function of an alternating itinerary $\mathbf{x} = (x_0, \ldots, x_N) \in \mathcal{I}_A$ as the function

$$\begin{aligned} L_{\mathbf{x}} : \{1, \ldots, \lfloor N/2 \rfloor\} &\to \mathbb{N} \\ k &\mapsto L(x_{2k-1}, x_{2k}). \end{aligned}$$

We say that a function $f : \{1, \dots, n\} \to \mathbb{R}$ is *decreasing-increasing* if there is some $k \in \{1, \dots, n\}$ for which $f|_{[1,k]}$ is decreasing and $f|_{[k+1,n]}$ is increasing (both not necessarily strictly monotonically). We can then define a further subset of itineraries with decreasing-increasing level functions:

$$\mathcal{I}^* := \{\mathbf{x} \in \mathcal{I}'_A \ : \ L_{\mathbf{x}} \text{ is decreasing-increasing}\}.$$

We first show that, if the level function on a non-self-intersecting alternating itinerary is not decreasing-increasing, then there exists a shorter (non-self-intersecting) alternating itinerary with the same endpoints of no greater cost.

Lemma 3.4. *Suppose* $\mathbf{x} = (x_0, \dots, x_N) \in \mathcal{I}'_A$ *and there exists* $j \in 2\mathbb{N} - 1$ *such that*

$$L(x_{j+2}, x_{j+3}) \geq \max(L(x_j, x_{j+1}), L(x_{j+4}, x_{j+5})).$$

Then the itinerary $\mathbf{x}' = (x'_0, \dots, x'_{N-2}) \in \mathcal{I}'_A$ *where*

$$x'_k = \begin{cases} x_k & k \in \{0, \dots, j+1\}, \\ x_{k+2} & k \in \{j+2, \dots, N-2\}, \end{cases}$$

satisfies $\operatorname{Ext}(\mathbf{x}) = \operatorname{Ext}(\mathbf{x}')$ *and* $c(\mathbf{x}') \leq c(\mathbf{x})$.

Proof. That $\operatorname{Ext}(\mathbf{x}) = \operatorname{Ext}(\mathbf{x}')$ is obvious from construction. Consider the subitinerary $\mathbf{y} = (x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4})$. Suppose first that $(x_{j+1}, x_{j+4}) \in \mathcal{S}$. Consequently, since $(x_{j+4}, x_{j+5}) \in \mathcal{S}$ and since shortcuts are unique (see Remark 3.1), we must have that $x_{j+5} = x_{j+1}$, which contradicts the hypothesis that $\mathbf{x}$ is non-self-intersecting.

Thus, we may suppose that $(x'_{j+1}, x'_{j+2}) = (x_{j+1}, x_{j+4}) \notin \mathcal{S}$ and so $\mathbf{x}' \in \mathcal{I}'_A$. We claim that $c(x_{j+1}, x_{j+4}) \leq c(\mathbf{y})$, which proves the lemma. Let $x \in \mathcal{N}_n$ be the point for which the shortcut (x_{j+2}, x_{j+3}) was found in $B_\rho(x, \lambda^n)$. Then $n = L(x_{j+2}, x_{j+3})$.

We claim that $x_{j+1}, x_{j+4} \notin B_\rho(x, \lambda^n)$. By symmetry we only need to show that $x_{j+1}, \notin B_\rho(x, \lambda^n)$. By assumption, we have $L(x_j, x_{j+1}) \leq n$. So first suppose $L(x_j, x_{j+1}) < n$. Then x was found in the complement of

$$B_\rho(x_j, 2\lambda^n) \cup B_\rho(x_{j+1}, 2\lambda^n),$$

which implies $x_{j+1}, \notin B_\rho(x, \lambda^n)$. If instead $L(x_j, x_{j+1}) = n$, then let $y \in \mathcal{N}_n$ be the point for which the shortcut (x_j, x_{j+1}) was found in $B_\rho(y, \lambda^n)$. We may assume that $x \neq y$, as otherwise $\{x_j, x_{j+1}\} = \{x_{j+2}, x_{j+3}\}$, which contradicts $\mathbf{x}$ being non-self-intersecting. Hence, we have that $\rho(x, y) \geq 4\lambda^n$ and so

$$B_\rho(x, \lambda^n) \cap B_\rho(y, \lambda^n) = \emptyset.$$

As $x_{j+1} \in B_\rho(y, \lambda^n)$, we get that $x_{j+1} \notin B_\rho(x, \lambda^n)$.

The conclusion follows from (3.2) if we set $x_i = x$, $q_{i,1} = x_{j+2}$, $q_{i,2} = x_{j+3}$, $p_1 = x_{j+1}$, and $p_2 = x_{j+4}$. Indeed, we have

$$c(x_{j+1}, x_{j+4}) = \rho(x_{j+1}, x_{j+4}) \overset{(3.2)}{\leq} \rho(x_{j+1}, x_{j+2}) + \rho(x_{j+3}, x_{j+4}) \leq c(\mathbf{y}),$$

where we used that $(x_{j+1}, x_{j+4}) \notin \mathcal{S}$. □

Lemma 3.5. *For any* $\mathbf{x} = (x_0, \dots, x_N) \in \mathcal{I}_A$*, there exists* $\mathbf{x}' \in \mathcal{I}^*$ *such that* $\mathrm{Ext}(\mathbf{x}) = \mathrm{Ext}(\mathbf{x}')$*,* $c(\mathbf{x}') \leq c(\mathbf{x})$*, and*

$$\# L_{\mathbf{x}'}^{-1}(k) \leq 2, \qquad \forall k \in \mathbb{N}. \tag{3.5}$$

Moreover, if $\mathbf{x} \in \mathcal{I}'_A$*, then* $\mathbf{x}$ *and* $\mathbf{x}'$ *have the same first and last shortcuts.*

Proof. Given an initial $\mathbf{x} \in \mathcal{I}_A$, we may suppose it is in $\mathcal{I}'_A$ by Lemma 3.3. We then iterate Lemma 3.4 until we get an itinerary $\mathbf{x}' = (x_0, \dots, x_N)$ for which there are no indices that satisfy the hypothesis of Lemma 3.4. As the length of the itinerary shrinks by 2 with each application of Lemma 3.4, we get that we have to stop after some finite number of iterations. It is elementary to see that if $L_{\mathbf{x}'} : \{1, \dots, \lfloor N/2 \rfloor\} \to \mathbb{N}$ satisfies

$$L_{\mathbf{x}'}(i+1) < \max(L_{\mathbf{x}'}(i), L_{\mathbf{x}'}(i+2)), \qquad \forall i \in \{1, \dots, \lfloor N/2 \rfloor - 2\},$$

then $L_{\mathbf{x}'}$ is decreasing-increasing, which means that $\mathbf{x}' \in \mathcal{I}^*$.

Now suppose $\# L_{\mathbf{x}'}^{-1}(k) \geq 3$ for some $k \in \mathbb{N}$. Hence, there are 3 jumps of level k. Since $\mathbf{x}'$ is decreasing-increasing, two of these jumps are consecutive. By symmetry we assume that the third jump is later. Namely, we have that there exists some $i, j \in 2\mathbb{N}$ so that $j + 2 < i$ and

$$L(x_j, x_{j+1}) = L(x_{j+2}, x_{j+3}) = L(x_i, x_{i+1}) = k.$$

Notice that since $\mathbf{x}'$ is decreasing-increasing, then $L(x_{j+4}, x_{j+5}) \leq k$. Therefore, the index j satisfy the hypothesis of Lemma 3.4. But this contradicts the assumption on $\mathbf{x}'$.

Finally, if we originally already had $\mathbf{x} \in \mathcal{I}'_A$, we only repeatedly applied Lemma 3.4. Then, since each application of Lemma 3.4 keeps the first and last shortcut of $\mathbf{x}$ unchanged the resulting itinerary $\mathbf{x}'$ has the same first and last shortcut as $\mathbf{x}$. □

Proposition 3.6. *Suppose* $\alpha_n > 0$ *for all* $n \in \mathbb{N}$*. Then the function* d *is a distance on* X*.*

Proof. The validity of the triangle inequality follows from the definition of the distance as defined in (2.5). Symmetry is due to the symmetry of the cost function. What needs to be checked is that $x \neq y$ implies $d(x, y) > 0$. In order to show this, suppose that $x, y \in X$ with $\rho(x, y) > 0$. Let $n \in \mathbb{N}$ be such that

$$4\frac{\lambda^n}{1-\lambda} \leq \frac{1}{2}\rho(x, y).$$

Let (α_n) be the sequence of positive numbers used to construct the cost function in Section 3.1. Consider the positive number

$$\varepsilon := \min\left(\frac{1}{2}\min_{k \in [1, n-1]} \alpha_k \lambda^{k+1}, \frac{1}{4}\rho(x, y)\right).$$

Let $\mathbf{x} = (x_0, \dots, x_N) \in \mathcal{I}^*$ with $\mathrm{Ext}(\mathbf{x}) = (x, y)$, $c(\mathbf{x}) \le d(x, y) + \varepsilon$, and $\# L_{\mathbf{x}}^{-1}(k) \le 2$ for all $k \in \mathbb{N}$, which exists by Lemma 3.5 (remember that using stationary walks every itinerary can be modified to be an alternating itinerary of no greater cost, because of triangle inequality).

On the one hand, if $L_{\mathbf{x}}^{-1}([1, n-1]) = \emptyset$, then the alternating itinerary does not have shortcuts at odd steps and it has them at even steps only of level greater that n and with multiplicity at most 2. Hence, we get

$$\begin{aligned} d(x,y) &\ge c(\mathbf{x}) - \varepsilon \ge \sum_{j \text{ odd}} \rho(x_{j-1}, x_j) - \varepsilon \ge \rho(x,y) - \sum_{j \text{ even}} \rho(x_{j-1}, x_j) - \frac{1}{4}\rho(x,y) \\ &\ge \frac{3}{4}\rho(x,y) - 2\sum_{k=n}^{\infty} 2\lambda^k \ge \frac{3}{4}\rho(x,y) - 4\frac{\lambda^n}{1-\lambda} \ge \frac{1}{4}\rho(x,y) > 0, \end{aligned}$$

where we used that a point in a shortcuts at level k has ρ-distance less than λ^k from the center of the ball in which the shortcut was found. On the other hand, if $L_{\mathbf{x}}^{-1}([1, n-1]) \neq \emptyset$, then, if $(x_{\ell-1}, x_\ell)$ is a shortcut at level $l < n$ of $\mathbf{x}$, we have

$$d(x,y) \ge c(\mathbf{x}) - \varepsilon \ge c(x_{\ell-1}, x_\ell) - \frac{1}{2}\min_{k \in [1, n-1]} \alpha_k \lambda^{k+1} \ge \frac{1}{2}\min_{k \in [1,n-1]} \alpha_k \lambda^{k+1} > 0,$$

where we used that $c(x_{\ell-1}, x_\ell) = \alpha_l \rho(x_{\ell-1}, x_\ell) \ge \alpha_l \lambda^l$. In both cases $d(x,y) > 0$ as needed. $\square$

Lemma 3.7. *Let $x \in X$ and $0 < r < \lambda^n$ with $n \in \mathbb{N}$. There exists at most one pair $\{q_1, q_2\}$ such that $(q_1, q_2) \in \mathcal{S}$, $L(q_1, q_2) < n$ and $\{q_1, q_2\} \cap B_d(x, r) \neq \emptyset$.*

Proof. Suppose to the contrary that there exist two disjoint $(q_1, q_2), (\tilde{q}_1, \tilde{q}_2) \in \mathcal{S}$ with

$$L(q_1, q_2), L(\tilde{q}_1, \tilde{q}_2) < n \tag{3.6}$$

and $q_1, \tilde{q}_1 \in B_d(x, r)$. Then

$$d(q_1, \tilde{q}_1) \le d(q_1, x) + d(x, \tilde{q}_1) < 2r \le 2\lambda^n.$$

Let $\mathbf{x} = (x_0, \dots, x_N) \in \mathcal{I}^*$ with $x_0 = q_1$, $x_N = \tilde{q}_1$ and

$$c(\mathbf{x}) < 2\lambda^n. \tag{3.7}$$

We now consider a possibly slightly longer itinerary $\mathbf{y}$ by attaching the shortcuts (q_1, q_2) and $(\tilde{q}_1, \tilde{q}_2)$ to $\mathbf{x}$ if they were not used in $\mathbf{x}$. In other words, in two steps

$$\tilde{\mathbf{y}} = (\tilde{y}_0, \dots, \tilde{y}_{\tilde{N}}) = \begin{cases} \mathbf{x}, & \text{if } (x_1, x_2) = (q_1, q_2) \\ (q_2, q_2, q_1, x_1, \dots, x_N), & \text{otherwise} \end{cases}$$

and

$$\mathbf{y} = \begin{cases} \tilde{\mathbf{y}}, & \text{if } (x_{N-2}, x_{N-1}) = (\tilde{q}_2, \tilde{q}_1) \\ (\tilde{y}_0, \dots, \tilde{y}_{\tilde{N}-1}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_2), & \text{otherwise.} \end{cases}$$

By construction $\mathbf{y} \in \mathcal{I}'_A$. Therefore by applying Lemma 3.5 to $\mathbf{y}$ we know that there exists $\mathbf{y}' \in \mathcal{I}^*$ such that $c(\mathbf{y}') \leq c(\mathbf{y})$, where $\mathrm{Ext}(\mathbf{y}') = \mathrm{Ext}(\mathbf{y})$ and the first and last shortcuts of $\mathbf{y}'$ and $\mathbf{y}$ are the same. We conclude that the itinerary $\mathbf{x}$ may be replaced with no extra cost by an itinerary $\mathbf{x}'= (x'_0, \ldots, x'_{N'})$ that is decreasing-increasing, starts at q_1 and ends at $\tilde{q}_1$ and that, due to (3.6), we have $L_{\mathbf{x}'}^{-1}([n, \infty]) = \emptyset$.

We remark that the itinerary $\mathbf{x}'$ cannot have only stationary walks. Indeed, otherwise the itinerary cannot move away from $\{q_1, q_2\}$, since distinct shortcuts are separated. Let $j \in 2\mathbb{Z}$ be the smallest even integer so that $x'_j \neq x'_{j+1}$.

Now there exist points $z_1, z_2 \in \{q_2, \tilde{q}_2, x'_2, x'_3, \ldots, x'_{N'-2}\}$ such that $(x'_j, z_1), (x'_{j+1}, z_2) \in \mathcal{S}$ and $k_1, k_2 < n$, where $k_1 := L(x'_j, z_1)$ and $k_2 := L(x'_{j+1}, z_2)$.

Let a, b be the centers of the balls in which the shortcuts (x'_j, z_1), (x'_{j+1}, z_2) were found with radii λ^{k_1} and λ^{k_2}, respectively. Let us distinguish two cases. Assume first that $k_1 = k_2 =: k$, so that a and b are $4\lambda^k$ separated. Hence, we have

$$\rho(x'_j, x'_{j+1}) \geq \rho(a, b) - \rho(a, x'_j) - \rho(b, x'_{j+1}) \geq 4\lambda^k - \lambda^k - \lambda^k \geq 2\lambda^k \geq 2\lambda^n.$$

Suppose now $k_1 \neq k_2$, say that $k_1 < k_2$, the other case is similar. Recall that b was found outside $B(x'_j, 4\lambda^{k_2})$ in the construction of the shortcuts. Hence, we have

$$\rho(x'_j, x'_{j+1}) \geq \rho(b, x'_j) - \rho(b, x'_{j+1}) \geq 4\lambda^{k_2} - \lambda^{k_2} \geq 3\lambda^{k_2} \geq 3\lambda^n.$$

In either case we have

$$c(\mathbf{x}) \geq c(\mathbf{x}') \geq \rho(x'_j, x'_{j+1}) \geq 2\lambda^n,$$

which is in contradiction with (3.7). $\square$

The next lemma will be used for the proof of the Ahlfors Q-regularity in the next section.

Lemma 3.8. *For all $x \in X$ and $r > 0$ there exist $y_1, y_2 \in X$ such that*

$$B_\rho(x, r) \subseteq B_d(x, r) \subseteq B_\rho\left(\{y_1, y_2\}, \left(2 + 8/(\lambda - \lambda^2)\right) r\right). \tag{3.8}$$

Proof. The first inclusion $B_\rho(x, r) \subseteq B_d(x, r)$ follows from the fact that by construction $d \leq \rho$.

Let us show the second inclusion. Suppose first that $r \geq 1$. Let $z \in B_d(x, r)$. By Lemma 3.5 there exists $\mathbf{x} = (x_0, \ldots, x_N) \in \mathcal{I}^*$ with $\mathrm{Ext}(\mathbf{x}) = (x, z)$, $c(\mathbf{x}) \leq r$, and $\# L_{\mathbf{x}}^{-1}(k) \leq 2$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned}
\rho(x, z) \quad &\leq \quad \sum_{j=1}^{N} \rho(x_{j-1}, x_j) \leq \sum_{j \text{ odd}} \rho(x_{j-1}, x_j) + \sum_{j \text{ even}} \rho(x_{j-1}, x_j) \\
&\leq \quad c(\mathbf{x}) + 4 \sum_{k=1}^{\infty} \lambda^k \leq r + 4 \frac{\lambda}{1 - \lambda} \leq \left(2 + 4 \frac{\lambda^{-1}}{1 - \lambda}\right) r,
\end{aligned}$$

since $\lambda < 1 \leq r$, and hence (3.8) holds with $y_1 = y_2 = x$.

Now suppose that $r<1$ and let $n \in \mathbb{N} \cup \{0\}$ be such that

$$\lambda^{n+1} \le r < \lambda^n.$$

By Lemma 3.7 there exists at most one pair $\{y_1, y_2\}$ such that $(y_1, y_2) \in \mathcal{S}$ and

$$L(y_1, y_2) < n \quad \text{and} \qquad B_d(x,r) \cap \{y_1, y_2\} \neq \emptyset. \tag{3.9}$$

If such pair $\{y_1, y_2\}$ does not exist, we define $y_1 = y_2 = x$. Take $z \in B_d(x,r)$. By Lemma 3.5 there exists $\mathbf{x} = (x_0, \dots, x_N) \in \mathcal{I}^*$ with $\operatorname{Ext}(\mathbf{x}) = (x,z)$, $c(\mathbf{x}) < r$, and $\#L_{\mathbf{x}}^{-1}(k) \le 2$ for all $k \in \mathbb{N}$. Note that as $c(\mathbf{x}) < r$, we get that $x_i \in B_d(x,r)$ for all i.

Suppose first that $\mathbf{x}$ does not contain the shortcut (y_1, y_2). Then as $x_i \in B_d(x,r)$, all the levels of $\mathbf{x}$ are at least n and so we have

$$\begin{aligned}
\rho(x,z) &\le \sum_{j=1}^{N} \rho(x_{j-1}, x_j) \le \sum_{j \text{ odd}} \rho(x_{j-1}, x_j) + \sum_{j \text{ even}} \rho(x_{j-1}, x_j) \\
&\le c(\mathbf{x}) + 4 \sum_{k=n}^{\infty} \lambda^k < r + 4\frac{\lambda^n}{1-\lambda} \le \left(1 + \frac{4}{\lambda - \lambda^2}\right) r,
\end{aligned}$$

since $\lambda^{n+1} \le r$. Now suppose by symmetry that $d(y_1, x) \le d(y_2, x)$. Then there exists an itinerary $\mathbf{y} \in \mathcal{I}^*$ with $\operatorname{Ext}(\mathbf{y}) = (x, y_1)$, $c(\mathbf{y}) < r$, $\#L_{\mathbf{y}}^{-1}(k) \le 2$ for all $k \in \mathbb{N}$. By changing y_1 to y_2 if necessary, we may assume that $\mathbf{y}$ does not contain the shortcut (y_1, y_2). Then a similar analysis gives that

$$\rho(x, y_1) < \left(1 + \frac{4}{\lambda - \lambda^2}\right) r,$$

and so

$$\rho(z, y_1) < 2\left(1 + \frac{4}{\lambda - \lambda^2}\right) r.$$

Thus, (3.8) holds.

Now suppose $\mathbf{x}$ contains the shortcut (y_1, y_2) at x_{i-1}, x_i for i even. Then the itinerary $\mathbf{x}' = (x_i, x_i, x_{i+1}, ..., x_N)$ is in $\mathcal{I}^*$. As $x_j \in B_d(x,r)$ for all j and $\mathbf{x}$ is non-self-intersecting, we have by (3.9) that the levels of all the shortcuts of $\mathbf{x}'$ are at least n. Now by a similar analysis as above, we get

$$\operatorname{dist}_\rho(\{y_1, y_2\}, z) \le \rho(x_i, z) < \left(1 + \frac{4}{\lambda - \lambda^2}\right) r. \qquad \square$$

3.3. **Ahlfors Q-regularity of (X,d).** Recall that the function d is a distance if in the constructions we took $\alpha_n > 0$, see Proposition 3.6. In Theorem 1.3 d is required to be a genuine distance, and in the proof of the theorem we can choose for example $\alpha_n = 1/n$. However, in the proof of Theorem 1.1 in order to prove that (X,d) is a BPI space we need all the α_n to be comparable. Combining this with the requirement that $\alpha_n \to 0$ as $n \to \infty$, we are forced to define $\alpha_n = 0$. Hence, in general, the function d is a semi-distance, thus we consider the quotient space, which we denote by $(\bar{X}, \bar{d})$.

We now give the proof of the Ahlfors Q-regularity of the space (X,d), assuming that (X,ρ) is Ahlfors Q-regular.

Notice that the projection map $(X,\rho) \to (\bar{X},\bar{d})$ is 1-Lipschitz. Thus we can push forward the measure $\mathcal{H}^Q_\rho$, which is then a Borel regular measure on $(\bar{X},\bar{d})$. In what follows, we shall not make distinction between (X,d) and $(\bar{X},\bar{d})$, nor between $\mathcal{H}^Q_\rho$ and its push forward measure, since all our arguments are set-wise.

Assuming that (X,ρ) is Ahlfors Q-regular, there exists a constant $C < \infty$ such that

$$\frac{1}{C} r^Q \leq \mathcal{H}^Q_\rho(B_\rho(x,r)) \leq C r^Q,$$

for all $x \in X$ and $0 < r < \mathrm{diam}_\rho(X)$. Hence by Lemma 3.8 we have

$$\frac{1}{C} r^Q \leq \mathcal{H}^Q_\rho(B_d(x,r)) \leq C2(2 + 8/(\lambda - \lambda^2))^Q r^Q,$$

for all $x \in X$ and $0 < r < \mathrm{diam}_\rho(X)$. Thus (X,d) is also Ahlfors Q-regular.

3.4. **No biLipschitz pieces.** Let $A \subseteq X$ be such that $\mathcal{H}^Q_\rho(A) > 0$. Our aim is to show that d and ρ are not biLipschitz equivalent on A. For this purpose take a density-point x of A. Then for any $\epsilon > 0$ there exists $r_\epsilon > 0$ such that

$$B_\rho(x,r) \subset B_\rho(A,\epsilon r), \qquad \text{for all } r \in (0,r_\epsilon).$$

Now, for all $n \in \mathbb{N}$ there exists $(q_{n,1},q_{n,2}) \in \mathcal{S}$ with $L(q_{n,1},q_{n,2}) = n$ such that

$$\{q_{n,1},q_{n,2}\} \subset B_\rho(x,2c_E\lambda^n).$$

If $3c_E\lambda^n < r_\epsilon$, there exist $x_{n,1},x_{n,2} \in A$ such that

$$\rho(x_{n,1},q_{n,1}) \leq 3c_E\epsilon\lambda^n \qquad \text{and} \qquad \rho(x_{n,2},q_{n,2}) \leq 3c_E\epsilon\lambda^n.$$

Then

$$\rho(x_{n,1},x_{n,2}) \geq \rho(q_{n,1},q_{n,2}) - \rho(x_{n,1},q_{n,1}) - \rho(x_{n,2},q_{n,2}) \geq \lambda^{n+1} - 6c_E\epsilon\lambda^n$$

and

$$\begin{aligned} d(x_{n,1},x_{n,2}) &\leq d(q_{n,1},q_{n,2}) + d(x_{n,1},q_{n,1}) + d(x_{n,2},q_{n,2}) \\ &\leq \alpha_n\rho(q_{n,1},q_{n,2}) + \rho(x_{n,1},q_{n,1}) + \rho(x_{n,2},q_{n,2}) \\ &\leq 2\alpha_n\lambda^n + 6c_E\epsilon\lambda^n. \end{aligned}$$

Therefore we have

$$\frac{d(x_{n,1},x_{n,2})}{\rho(x_{n,1},x_{n,2})} \leq \frac{2\alpha_n\lambda^n + 6c_E\epsilon\lambda^n}{\lambda^{n+1} - 6c_E\epsilon\lambda^n} = \frac{2\alpha_n + 6c_E\epsilon}{\lambda - 6c_E\epsilon}.$$

As $\alpha_n \to 0$, by letting n be sufficiently large and ϵ be sufficiently small, we get that $\alpha_n + 6c_E\epsilon$ is sufficiently small and so the distances d and ρ are not biLipschitz equivalent on A. This concludes the proof of Theorem 1.3.

4. Existence of shortcuts

We will now verify that the shortcuts necessary to employ Theorem 1.3 can be made in the subRiemannian Heisenberg group and in any snowflaked Ahlfors regular metric space.

4.1. **Shortcuts in the Heisenberg group.**

Proof of Theorem 1.2. We will verify that the assumptions of Theorem 1.3 hold in the Heisenberg group with $\lambda = \frac{1}{2}$. Let $p \in \mathbb{H}$ and $r > 0$. By left-translation invariance of the distance d_b in $\mathbb{H}$ we may assume that $p = (0,0,0)$. Take $q_1 = (0,0,0)$ and $q_2 = (0,0,r^2/4)$. Now let $p_1, p_2 \notin B_{d_b}(0,r)$. Since $d_b(q_1,q_2) = \sqrt{r^2/4} = r/2$, by the triangle inequality we have that $d_b(p_1,q_1) \geq r/2$ and $d_b(p_2,q_2) \geq r/2$. Write $p_1 = (x_1,y_1,z_1)$ and $p_2 = (x_2,y_2,z_2)$. Then the equation for the box distance is given by (2.3). Trivially, we have

$$|x_1 - x_2| \leq |x_1| + |x_2| \leq d_b(p_1,q_1) + d_b(p_2,q_2)$$

and

$$|y_2 - y_1| \leq |y_1| + |y_2| \leq d_b(p_1,q_1) + d_b(p_2,q_2).$$

By using the triangle inequality and the estimate $r^2/4 \leq d_b(q_1,p_1)d_b(q_2,p_2)$ we also get

$$\begin{aligned}\left|z_1 - z_2 - \frac{1}{2}(x_1y_2 - x_2y_1)\right| &\leq |z_1| + |z_2 - \frac{1}{4}r^2| + \frac{1}{4}r^2 + \frac{1}{2}|x_1||y_2| + \frac{1}{2}|x_2||y_1| \\ &\leq d_b(p_1,q_1)^2 + d_b(p_2,q_2)^2 + d_b(q_1,p_1)d_b(q_2,p_2) \\ &\quad + \frac{1}{2}d_b(q_1,p_1)d_b(q_2,p_2) + \frac{1}{2}d_b(q_1,p_1)d_b(q_2,p_2) \\ &= d_b(p_1,q_1)^2 + 2d_b(q_1,p_1)d_b(q_2,p_2) + d_b(p_2,q_2)^2 \\ &= \left(d_b(q_1,p_1) + d_b(q_2,p_2)\right)^2.\end{aligned}$$

Thus we have

$$d_b(p_1,p_2) \leq d_b(q_1,p_1) + d_b(q_2,p_2)$$

as required by the assumptions of Theorem 1.3. □

4.2. **Shortcuts in snowflaked Ahlfors regular metric spaces.**

Theorem 4.1. *Let (X,d) be an Ahlfors Q-regular metric space with $Q > 0$ and let $\delta \in (0,1)$. Then the snowflaked metric space (X,d^δ) satisfies the assumptions of Theorem 1.3. Consequently, there exists a distance d' on X such that $d' \leq d^\delta$, (X,d') is Ahlfors Q/δ-regular, and for any $A \subseteq X$ with $\mathcal{H}^{Q/\delta}_{d^\delta}(A) > 0$, we have that d' and d^δ are not biLipschitz equivalent on A.*

Proof. First of all, it is trivial that (X, d^δ) is Ahlfors Q/δ-regular. Let us then check the assumption (2) of Theorem 1.3. Since (X, d) is Q-regular, there exists $C > 1$ such that

$$\frac{1}{C} r^Q \leq \mathcal{H}^Q_d(B_d(x, r)) \leq C r^Q \tag{4.1}$$

for all $x \in X$ and $0 < r < \operatorname{diam}_d(X)$. We shall set $\lambda := (2C)^{-2\delta/Q}(1-\delta)^\delta$. Take $p \in X$ and $0 < r < \operatorname{diam}_d(X)$. Define $q_1 = p$ and take

$$q_2 \in B_d(p, (1-\delta) r^{\frac{1}{\delta}}) \setminus B_d(p, (2C)^{-2/Q}(1-\delta) r^{\frac{1}{\delta}}).$$

Such q_2 exists since the annulus from where the point is taken has positive measure by (4.1) and is hence non-empty. In particular, $q_1, q_2 \in B_{d^\delta}(p, r)$ and

$$d(q_1, q_2)^\delta \geq \left((2C)^{-2/Q}(1-\delta)\right)^\delta r = \lambda r.$$

Now, take $p_1, p_2 \notin B_d\left(p, r^{\frac{1}{\delta}}\right)$. We get that

$$d(q_1, p_1), d(q_2, p_2) \geq \delta r^{1/\delta},$$

and so

$$d(q_1, q_2) \leq (1-\delta)\, r^{1/\delta} \leq \left(\frac{1}{\delta} - 1\right) \min(d(q_1, p_1), d(q_2, p_2)).$$

First assume that $d(p_1, q_1) \leq d(p_2, q_2)$. Then we get

$$\begin{aligned}
d(p_1, p_2)^\delta &\leq (d(p_1, q_1) + d(q_1, q_2) + d(p_2, q_2))^\delta \\
&\leq ((1 + (\frac{1}{\delta} - 1)) d(p_1, q_1) + d(p_2, q_2))^\delta \\
&= (\frac{1}{\delta} d(p_1, q_1) + d(p_2, q_2))^\delta \\
&\leq d(p_2, q_2)^\delta + \delta d(p_2, q_2)^{\delta - 1} \frac{1}{\delta} d(p_1, q_1) \\
&\leq d(p_1, q_1)^\delta + d(p_2, q_2)^\delta
\end{aligned}$$

verifying (1.1). In the penultimate inequality, we used a Taylor expansion of $x \mapsto x^\delta$ centered at $d(p_2, q_2)$ and the fact that $x \mapsto x^\delta$ is concave so that the higher order terms of the Taylor expansion are always negative. An analogous calculation takes care of the case $d(p_2, q_2) \leq d(p_1, q_1)$. □

5. A BPI space using self-similar shortcuts in the Heisenberg group

In this section we prove Theorem 1.1. The idea is to consider a regular subset $K \subset \mathbb{H}$, to specify in a self-similar way the shortcuts taken in the construction of Section 3 and to make all the shortcuts to have zero cost. This will produce a semi-distance d on K and, after factoring, the desired distance for Theorem 1.1. By taking the shortcuts to have zero cost we get that the similitude mappings used in the selection of shortcuts will almost be similitude mappings also for the new semi-distance d. This will allow us to show that the quotient metric space $(\bar{K}, \bar{d})$ (obtained by identifying points of zero distance in (K, d)) is BPI. Then the facts that $(\mathbb{H}, d_b)$ looks down on $(\bar{K}, \bar{d})$ and that $(\bar{K}, \bar{d})$ does not look down on $(\mathbb{H}, d_b)$ follow, after some work, via Theorem 1.3.

5.1. **Defining a self-similar tiling.** In this section we shall construct a self-similar tiling of the Heisenberg group, in the spirit of Strichartz's tilings, see [Str92, Str94] and [BHIT06]. Define the similitude mappings as

$$S_{i,j,k}(p) = \left(\frac{i}{2}, \frac{j}{2}, \frac{k}{4}\right) \cdot \delta_{\frac{1}{2}}(p), \qquad i, j \in \{0, 1\}, k \in \{0, 1, 2, 3\}.$$

Relabel the similitudes by $\{S_i \,:\, i = 1, \dots, 16\} = \{S_{i,j,k}\}$ and denote by K the attractor of $\{S_{i,j,k}\}$, i.e., the nonempty compact set (see [Hut81] for details) satisfying

$$K = \bigcup_{i=1}^{16} S_i(K). \tag{5.1}$$

Let us show that K has nonempty interior. We will use the map

$$\pi : (\mathbb{H}, d_b) \to \mathbb{R}^2$$

that is the projection onto the xy-plane and is a 1-Lipschitz homomorphism, when we endow $\mathbb{R}^2$ with the ℓ_∞-distance. We split the iterated function system $\{S_{i,j,k}\}$ to the *horizontal component* $\{S_{i,j,0} \,:\, i, j \in \{0, 1\}\}$ and the *vertical component* $\{S_{0,0,k} \,:\, k \in \{0, 1, 2, 3\}\}$. First of all, the π-projection of the horizontal component of the iterated function system has the unit square as the attractor and the attractor of the vertical component is $\{(0, 0)\} \times [0, 1]$. Secondly, since the dilation and the group operation commute, we may consider separately the horizontal and vertical components of the iterated function system:

$$\begin{aligned} S_{i,j,k}(x, y, z) &= \left(0, 0, \frac{k}{4}\right) \cdot \left(\frac{i}{2}, \frac{j}{2}, 0\right) \cdot \delta_{\frac{1}{2}}\left((0, 0, z) \cdot (x, y, 0)\right) \\ &= \left(0, 0, \frac{k}{4} + \frac{z}{4}\right) \cdot \left(\frac{i}{2}, \frac{j}{2}, 0\right) \cdot \delta_{\frac{1}{2}}(x, y, 0) \\ &= S_{0,0,k}(0, 0, z) \cdot S_{i,j,0}(x, y, 0). \end{aligned} \tag{5.2}$$

In this way we see that

$$K = \{(x, y, z + t) \,:\, (x, y, z) \in \widetilde{K},\, t \in [0, 1]\}, \tag{5.3}$$

where $\widetilde{K}$ is the attractor of the horizontal component. The set $\widetilde{K}$ has the form

$$\widetilde{K} = \overline{\{(x, y, \varphi(x, y)) \,:\, (x, y) \in [0, 1]^2\}} \tag{5.4}$$

with some Borel function $\varphi\colon [0, 1]^2 \to \mathbb{R}$. Observe that φ is bounded since K is compact. Also, since 0 is the fixed point of $S_{0,0,0}$ and $S_{i,j,0}(\widetilde{K})$ do not contain 0 if $i \neq 0$ or $j \neq 0$, the function φ is continuous at 0. Therefore by (5.3) the attractor

K contains a small ball near 0 and thus K has nonempty interior. Because of the nonempty interior and the self-similar structure (K, d_b) is Ahlfors 4-regular.

5.2. **Constructing the shortcuts.** For a multi-index $\mathtt{i} = (i_1, ..., i_k) \in \{1, \dots, 16\}^k$, we shall use the standard notation $S_{\mathtt{i}}$ for the composition

$$S_{\mathtt{i}} := S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}.$$

With $k = 0$ we interpret $\{1, \dots, 16\}^k$ to consist of only one element, call it $\emptyset$, and $S_\emptyset$ is then understood to be the identity map.

As we are working in the Heisenberg group, we may take $\lambda = 1/2$ by the proof of Theorem 1.2. We define $x_{\mathtt{i}} = S_{\mathtt{i}}(\frac{1}{2}, \frac{1}{2}, 0)$ for all $\mathtt{i}$ and set $\mathcal{N}_n := \{x_{\mathtt{i}}\}_{\mathtt{i} \in \{1,\dots,16\}^{n-3}}$. We define the shortcuts at level n as

$$\mathcal{S}_n := \{(x_{\mathtt{i}}, S_{\mathtt{i}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{256})) \,:\, \mathtt{i} \in \{1, \dots, 16\}^{n-3}\}.$$

We also set $L(x, y) = n$ for $(x, y) \in \mathcal{S}_n$. Note that levels start from $n = 3$. Note that the construction of the shortcuts for the third level requires a separation of at least $\lambda^{3+1} = 1/16$. The $1/256 = 1/16^2$ then comes from the 1/2-snowflake behavior in the z-coordinate. Of course we are also free to take anything larger than 1/256, but it has to be less than 1/64 as it has to lie in a ball of radius $\lambda^3 = 1/8$ around $(1/2, 1/2, 0)$.

We then define the total set of shortcuts as

$$\mathcal{S} := \bigcup_{n=3}^{\infty} \mathcal{S}_n.$$

We define the cost as $c(x, y) = 0$ for all $(x, y) \in \mathcal{S}$. In other words we set $\alpha_n = 0$ for all n. We also set $c(x, y) = d_b(x, y)$ for all $(x, y) \notin \mathcal{S}$. Let us check that the construction of Section 3 works with this choice of shortcuts and costs. This will be established by the following three lemmas for $\lambda = 1/2$ and $c_E \geq 8$ some sufficiently large number.

The first lemma shows that the points $x_{\mathtt{i}}$ near which we find the level n shortcuts can be found outside a $4\lambda^n$-neighborhood of shortcut points of lower levels.

Lemma 5.1. *For all $n \geq 3$, we have*

$$\mathcal{N}_n \cap \left(\bigcup_{m<n} \bigcup_{(x,y) \in \mathcal{S}_m} B_{d_b}(\{x, y\}, 2^{-n+2}) \right) = \emptyset.$$

Proof. Let $A_k \subset [0, 1]^2$ be the centers of the dyadic subcubes of level k. Note that for each k we have that

$$\pi(\{S_{\mathtt{i}}(\frac{1}{2}, \frac{1}{2}, t) \,:\, \mathtt{i} \in \{1, \dots, 16\}^k, t \in \mathbb{R}\}) = A_k.$$

As π is 1-Lipschitz, it suffices to prove that

$$A_n \cap B_{\mathbb{R}^2_\infty}\left(\bigcup_{m<n} A_m, 2^{-n-1}\right) = \emptyset.$$

But this follows from the geometry of $(\mathbb{R}^2, \|\cdot\|_\infty)$. Note that we need the sets $B_{\mathbb{R}^2_\infty}$ to be open, which is fine. $\square$

The next lemma says that the sets $\mathcal{N}_n$ themselves are $4\lambda^n$-separated.

Lemma 5.2. *For all $n \geq 3$ the set $\mathcal{N}_n$ is 2^{-n+2}-separated (note: $2^{-n+2} = 4 \cdot 2^{-n}$, which is needed for the construction).*

Proof. As shown in the previous lemma, the image of $\mathcal{N}_n$ under π is precisely the centers of the dyadic subcubes of $[0,1]$ of level $n-3$. Let $x, y \in \mathcal{N}_n$ and suppose $\pi(x) \neq \pi(y)$. Then

$$d_b(x,y) \geq \|\pi(x) - \pi(y)\| \geq 2^{-n+3}.$$

Now suppose $\pi(x) = \pi(y)$ but $x \neq y$. Then using (5.2) we see that the z-coordinate of x and y are points in the center of the level $n-3$ 4-dic subintervals of $[\varphi(\pi(x)), \varphi(\pi(x))+1]$, where φ is the function in (5.4). Thus, they differ by no less than 4^{-n+3} and so

$$d_b(x,y) \geq \sqrt{4^{-n+3}} = 2^{-n+3} > 2^{-n+2}. \qquad \square$$

Finally, we show that the level n shortcut points form a $c_E\lambda^n$-covering of K for sufficiently large c_E. This finishes all the properties needed to construct the shortcuts.

Lemma 5.3. *There exists some absolute constant $c_E > 0$ so that*

$$K \subseteq \bigcup_{\mathtt{i} \in \{1,\ldots,16\}^{n-3}} B_{d_b}(x_{\mathtt{i}}, c_E 2^{-n}), \qquad \forall n \geq 3. \tag{5.5}$$

Proof. We prove the claim by induction. As K is bounded, we easily get (5.5) for $n = 3$ by choosing some c_E large enough. Now assume that (5.5) holds for some $n \geq 3$. Then by the self-similarity of K as exhibited in (5.1) we get

$$\begin{aligned} K &= \bigcup_{i=1}^{16} S_i(K) \subseteq \bigcup_{i=1}^{16} S_i\Big(\bigcup_{\mathtt{i} \in \{1,\ldots,16\}^{n-3}} B_{d_b}(x_{\mathtt{i}}, c_E 2^{-n})\Big) \\ &= \bigcup_{\mathtt{i} \in \{1,\ldots,16\}^{n-2}} B_{d_b}(x_{\mathtt{i}}, c_E 2^{-n-1}) \end{aligned}$$

Thus (5.5) holds for $n+1$. $\square$

Recall that taking zero costs for shortcuts, i.e. $\alpha_n = 0$ for all n, in the construction of Section 3 is allowed, but we then obtain only a semi-distance d on K and we need to consider the quotient space $(\bar{K}, \bar{d})$. From the proof of Theorem 1.2 we see that $\lambda = \frac{1}{2}$ works in the Heisenberg group.

Now the conclusions of Theorem 1.3 hold for the constructed semi-distance d. That is, the identity map $\mathrm{id}\colon (K, d_b) \to (K, d)$ (and hence the quotient projection $\pi_\sim\colon (K, d_b) \to (\bar{K}, \bar{d})$) is Lipschitz, but not biLipschitz on any set of positive measure, and the space (K, d) is Ahlfors regular. In particular, $(\mathbb{H}, d_b)$ looks down on $(\bar{K}, \bar{d})$. In order to show that $(\mathbb{H}, d_b)$ is not minimal in looking down, we still need to prove that $(\bar{K}, \bar{d})$ is a BPI space and that $(\bar{K}, \bar{d})$ does not look down on $(\mathbb{H}, d_b)$.

5.3. (K, d) **is a BPI space.** We begin with the following lemma.

Lemma 5.4. *Let* $\mathbf{x} \in \mathcal{I}^*$ *with* $\ell(\mathbf{x}) > 1$ *and* $n = \min\{k \in \mathbb{N} : L_{\mathbf{x}}^{-1}(k) \neq \emptyset\}$. *There exists* $\mathbf{x}' = (x'_0, ..., x'_N) \in \mathcal{I}^*$ *such that* $\mathrm{Ext}(\mathbf{x}) = \mathrm{Ext}(\mathbf{x}')$, $c(\mathbf{x}') \leq c(\mathbf{x})$, *and for any* $j \in \mathbb{N}$ *such that* $\min(L(x'_{2j-1}, x'_{2j}), L(x'_{2j+1}, x'_{2j+2})) = m > n$, *then*

$$d_b(x'_{2j}, x'_{2j+1}) \leq 2\lambda^m.$$

Proof. As usual, we will show that if $\mathbf{x}$ does not already satisfy the conclusion of the lemma, then we can find a strictly shorter itinerary in $\mathcal{I}^*$ with same endpoints and no greater cost that $\mathbf{x}$. The lemma then follows by iterating this procedure until one cannot.

Fix j such that $\min(L(x'_{2j-1}, x'_{2j}), L(x'_{2j+1}, x'_{2j+2})) = m > n$. We may suppose without loss of generality that there exist $k \in L_{\mathbf{x}}{}^{-1}(n)$ such that $k > j + 1$, that is, $L_{\mathbf{x}}$ is still decreasing from j to $j + 1$. Thus, $L(x_{2j+1}, x_{2j+2}) = m$.

Let $x \in \mathcal{N}_m$ be the point for which the shortcut (x_{2j+1}, x_{2j+2}) is found in $B_{d_b}(x, \lambda^m)$. First suppose that $(x_{2j}, x_{2j+3}) \in \mathcal{S}$. Then by uniqueness of shortcuts, $x_{2j+4} = x_{2j}$, which contradicts the fact that $\mathbf{x}$ is non-self-intersecting.

Thus, we may suppose $(x_{2j}, x_{2j+3}) \notin \mathcal{S}$. Then as $L(x_{2j+3}, x_{2j+4}) \leq m$, we get that $x_{2j+3} \notin B_{b_d}(x, \lambda^m)$ as $\mathbf{x}$ is non-self-intersecting. If $d_b(x_{2j}, x_{2j+1}) > 2\lambda^m$, then we get that $x_{2j} \notin B_{b_d}(x, \lambda^m)$ by the triangle inequality. Thus, applying (1.1) with $q_1 = x_{2j+1}$, $q_2 = x_{2j+2}$, $p_1 = x_{2j}$, and $p_2 = x_{2j+3}$, we get that we can replace $(x_{2j}, x_{2j+1}, x_{2j+2}, x_{2j+3})$ in $\mathbf{x}$ with (x_{2j}, x_{2j+3}) to get an itinerary in $\mathcal{I}^*$ with lower cost and two fewer points with the same extremal points. $\square$

The following lemma says that one can connect $x, y \in S_{\mathtt{i}}(K)$ by an itinerary that does not go too far out.

In this section we write $|\mathtt{i}| = k$ if $\mathtt{i} \in \{1, \ldots, 16\}^k$.

Lemma 5.5. *There exists some* $C > 0$ *so that for all multi-indices* $\mathtt{i}$, *for all* $x, y \in S_{\mathtt{i}}(K)$ *and all* $\epsilon \in (0, 1)$, *there exists an itinerary* $\mathbf{x} = (x_0, \ldots, x_n) \in \mathcal{I}^*$ *such that* $c(\mathbf{x}) \leq (1 + \epsilon)d(x, y)$, $\mathrm{Ext}(\mathbf{x}) = (x, y)$, *and* $x_0, \ldots, x_n \in B_{d_b}(S_{\mathtt{i}}(K), C2^{-|\mathtt{i}|})$.

Proof. We claim that there exists some constant $M \in \mathbb{N}$ depending only on $c_E > 0$ of Lemma 5.3 such that if $\epsilon > 0$, $\mathtt{i} \in \{1, \ldots, 16\}^k$ for $k \geq M$, and $x, y \in S_{\mathtt{i}}(K)$, then there exists some itinerary $\mathbf{x} \in \mathcal{I}^*$ such that

(1) $c(\mathbf{x}) \leq (1 + \epsilon)d(x, y)$,

(2) $\mathrm{Ext}(\mathbf{x}) = (x, y)$,
(3) $\# L_{\mathbf{x}}^{-1}(k) \leq 2, \qquad \forall k \in \mathbb{N}$,
(4) $d_b(x_{j-1}, x_j) \leq 2^{1-m}$ for all j odd such that

$$m = \min(L(x_{j-2}, x_{j-1}), L(x_j, x_{j+1})) > n,$$

where $n = \min\{k : L_{\mathbf{x}}^{-1}(k) \neq \emptyset\}$,
(5) $d_b(x_{j-1}, x_j) \leq 2^{M/2-k}$ for all j odd,
(6) $L_{\mathbf{x}}^{-1}([0, k-M]) = \emptyset$.

Suppose the claim holds and for $x, y \in S_{\mathtt{i}}(K)$ with $\mathtt{i} \in \{1, \dots, 16\}^k$ for some $k \geq M$, take such an itinerary $\mathbf{x} = (x_0, \dots, x_N) \in \mathcal{I}^*$. We will bound $\sum_{i=1}^N d_b(x_{i-1}, x_i)$ by $C2^{-k}$ for some C depending only on M. The lemma then follows for all $\mathtt{i} \in \{1, \dots, 16\}^k$ with $k \geq M$ from the triangle inequality and for all $\mathtt{i} \in \{1, \dots, 16\}^k$ with $k < M$ by the fact that there are only finitely many such $\mathtt{i}$.

By the fifth property, $d_b(x_0, x_1)$ and $d_b(x_{N-1}, x_N)$ are both less than $2^{M/2}2^{-k}$ so we only need to care about the values $d_b(x_{i-1}, x_i)$ when $2 \leq i \leq N-1$. By the third and the sixth property, we have

$$\sum_{i \text{ even}} d_b(x_{i-1}, x_i) \leq 2 \sum_{j=k-M+1}^{\infty} 2^{-j} \leq 2^{M+1}2^{-k}, \tag{5.6}$$

and we so only need to care about $d_b(x_{i-1}, x_i)$ when $3 \leq i \leq N-2$ is odd.

We split the sum into two

$$\sum_{3 \leq i \leq N-2 \text{ odd}} d_b(x_{i-1}, x_i) = \sum_{I} d_b(x_{i-1}, x_i) + \sum_{II} d_b(x_{i-1}, x_i)$$

where I represent all the indices i for which $\min(L(x_{i-2}, x_{i-1}), L(x_i, x_{i+1})) = n$ and II are the rest. By the third property and the fact that $\mathbf{x} \in \mathcal{I}^*$, the number of summands in the I sum is at most three. Together with the fifth property, we get that

$$\sum_{I} d_b(x_{i-1}, x_i) \leq 3 \cdot 2^{M/2-k} \leq 2^{M/2+2}2^{-k}.$$

Finally, the fourth property says that the distance $d_b(x_{i-1}, x_i)$ for the odd indices where $\min(L(x_{i-2}, x_{i-1}), L(x_i, x_{i+1})) > n$ are controlled by

$$\max(d_b(x_{i-2}, x_{i-1}), d_b(x_i, x_{i+1})),$$

since $d_b(x_{j-1}, x_j) \geq 2^{1-m}$, if $L(x_{j-1}, x_j) = m$. Thus, we get that

$$\sum_{II} d_b(x_{i-1}, x_i) \leq 2 \cdot \sum_{i \text{ even}} d_b(x_{i-1}, x_i) \overset{(5.6)}{\leq} 2^{M+2}2^{-k},$$

which finishes the bound on the summation of $\sum_{i=1}^N d_b(x_{i-1}, x_i)$.

Let us prove the claim. Let $c_E > 0$ be the constant from Lemma 5.3. Let $M \in \mathbb{N}$ be the minimal even number such that

$$2^{M/2} > 2c_E \tag{5.7}$$

and

$$2^{M-1} - 2^{M/2} - 2^{M/2+6} > c_E. \tag{5.8}$$

As $x, y \in S_{\mathtt{i}}(K)$, with $|\mathtt{i}| = k$, we get that

$$d(x,y) \leq d_b(x,y) \leq c_E 2^{-k}. \tag{5.9}$$

By an application of Lemma 3.5 on some itinerary with cost no more than $(1+\epsilon)d(x,y)$, we get an itinerary $\mathbf{x} = (x_0, \ldots, x_N)$ that satisfies the first three properties. We then apply Lemma 5.4 on $\mathbf{x}$ (and still calling the result $\mathbf{x}$) to get that the fourth property is satisfied.

Suppose $d_b(x_{j-1}, x_j) > 2^{M/2-k}$ for some odd j. Then

$$c(\mathbf{x}) \geq d_b(x_{j-1}, x_j) \geq 2^{M/2-k} \overset{(5.9)\wedge(5.7)}{>} 2d(x,y),$$

a contradiction. Thus, the fifth condition is satisfied.

Now suppose $L_{\mathbf{x}}^{-1}([0, k-M]) \neq \emptyset$. Suppose first that $\#L_{\mathbf{x}}^{-1}([0, k-M/2]) \geq 2$. Then as $\mathbf{x} \in \mathcal{I}^*$, we then have that there exists some j so that $\min\{L(j), L(j+1)\} \geq k - M/2$. Thus, $(x_{2j}, x_{2j+1}) \notin \mathcal{S}$ and so

$$c(\mathbf{x}) \geq c(x_{2j}, x_{2j+1}) \geq 4 \cdot 2^{M/2-k} \overset{(5.9)\wedge(5.7)}{>} 2d(x,y),$$

which is a contradiction.

Now suppose that $\#L_{\mathbf{x}}^{-1}([0, k-M/2]) = 1$. Let $L(x_{2j-1}, x_{2j}) \leq k - M$. Then we have by the triangle inequality

$$\begin{aligned} d_b(x,y) \geq d_b(x_{2j-1}, x_{2j}) - \sum_{\ell=0}^{2j-3} d_b(x_\ell, x_{\ell+1}) - d_b(x_{2j-2}, x_{2j-1}) \\ - d_b(x_{2j}, x_{2j+1}) - \sum_{\ell=2j+1}^{N} d_b(x_\ell, x_{\ell+1}) = (*). \end{aligned}$$

We have by the third and fourth property that

$$\begin{aligned} \sum_{\ell=0}^{2j-3} d_b(x_\ell, x_{\ell+1}) &\leq 8 \sum_{s=k-M/2}^{\infty} 2^{-s} = 2^{M/2-k+5} \\ \sum_{\ell=2j+1}^{N} d_b(x_\ell, x_{\ell+1}) &\leq 8 \sum_{s=k-M/2}^{\infty} 2^{-s} = 2^{M/2-k+5}. \end{aligned} \tag{5.10}$$

Similarly as in the case $\#L_{\mathbf{x}}^{-1}([0, k-M/2]) \geq 2$, since $(x_{2j-2}, x_{2j-1}), (x_{2j}, x_{2j+1}) \notin \mathcal{S}$, we have

$$\begin{aligned} d_b(x_{2j-2}, x_{2j-1}) + d_b(x_{2j}, x_{2j+1}) &= c(x_{2j-2}, x_{2j-1}) + c(x_{2j}, x_{2j+1}) \\ &\leq 2d(x,y) < 2^{M/2-k}. \end{aligned}$$

Altogether, we get that

$$(*) \overset{(5.10)}{\geq} 2^{M-k-1} - 2^{M/2-k} - 2^{M/2-k+6} \overset{(5.8)}{>} c_E 2^{-k}.$$

But this is a contradiction because $x, y \in S_{\mathtt{i}}(K)$ and so $d_b(x, y) \leq c_E 2^{-k}$. $\square$

We can now prove the following lemma that says that there exists large subset of every $S_{\mathtt{i}}(K)$ that can be connected optimally by itineraries only in $S_{\mathtt{i}}(K)$.

Lemma 5.6. *There exists some multi-index* $\mathtt{j}$ *such that the following property holds. For any* $\epsilon > 0$, $k \in \mathbb{N}$, $\mathtt{i} \in \{1, \dots, 16\}^k$, *and any two* $x, y \in S_{\mathtt{i}}(S_{\mathtt{j}}(K))$, *there exists an itinerary* $\mathbf{x} = (x_0, \dots, x_N) \in \mathcal{I}^*$ *with* $c(\mathbf{x}) \leq (1+\epsilon) d(x, y)$, $\mathrm{Ext}(\mathbf{x}) = (x, y)$, *and* $x_0, \dots, x_N \in S_{\mathtt{i}}(K)$.

Proof. Let $C > 0$ be the constant from the previous lemma. As K has nonempty interior we may choose $x \in \mathrm{int}(K)$ and $h > 0$ so that $B_{d_b}(x, h) \subset K$. As K is compact, there then exists some $\mathtt{j}$ so that

$$B_{d_b}(S_{\mathtt{j}}(K), C2^{-|\mathtt{j}|}) \subseteq B_{d_b}(x, h) \subseteq K.$$

Now let $x, y \in S_{\mathtt{i}}(S_{\mathtt{j}}(K))$ for some arbitrary $\mathtt{i} \in \{1, \dots, 16\}^k$. Then there exists an itinerary $\mathbf{x} = (x_0, \dots, x_n) \in \mathcal{I}^*$ such that $c(\mathbf{x}) \leq (1+\epsilon) d(x, y)$, $\mathrm{Ext}(\mathbf{x}) = (x, y)$, and each of the points of $\mathbf{x}$ is contained

$$B_{d_b}(S_{\mathtt{i}}(S_{\mathtt{j}}(K)), C2^{-|\mathtt{i}|-|\mathtt{j}|}) = S_{\mathtt{i}}(B_{d_b}(S_{\mathtt{j}}(K), C2^{-|\mathtt{j}|})) \subseteq S_{\mathtt{i}}(K). \quad \square$$

Lemma 5.7. *Let* $\mathtt{j}$ *be from Lemma 5.6. Then for all* $k \in \mathbb{N}$ *and* $\mathtt{i} \in \{1, \dots, 16\}^k$, *we have that*

$$d(S_{\mathtt{i}}(x), S_{\mathtt{i}}(y)) = 2^{-|\mathtt{i}|} d(x, y), \qquad \forall x, y \in S_{\mathtt{j}}(K). \tag{5.11}$$

Proof. If $\mathbf{x} = (x_0, \dots, x_N)$ is an itinerary from x to y, then $S_{\mathtt{i}}(\mathbf{x})$ is an itinerary from $S_{\mathtt{i}}(x)$ to $S_{\mathtt{i}}(y)$ with $c(S_{\mathtt{i}}(\mathbf{x})) = 2^{-|\mathtt{i}|} c(\mathbf{x})$, since $(x_i, x_{i+1}) \in \mathcal{S}$ if and only if $(S_{\mathtt{i}}(x_i), S_{\mathtt{i}}(x_{i+1})) \in \mathcal{S}$, $\alpha_n = 0$ for all n, and $d_b(S_{\mathtt{i}}(x_i), S_{\mathtt{i}}(x_{i+1}) = 2^{-|\mathtt{i}|} d_b(x_i, x_{i+1})$. Thus, we get that $d(S_{\mathtt{i}}(x), S_{\mathtt{i}}(y)) \leq 2^{-|\mathtt{i}|} d(x, y)$.

For any $\epsilon > 0$ and $S_{\mathtt{i}}(x), S_{\mathtt{i}}(y) \in S_{\mathtt{i}}(S_{\mathtt{j}}(K))$, we get from Lemma 5.6 that there exists an itinerary $\mathbf{x} = (x_0, \dots, x_N)$ from $S_{\mathtt{i}}(x)$ to $S_{\mathtt{i}}(y)$ such that $c(\mathbf{x}) \leq (1 + \epsilon) d(S_{\mathtt{i}}(x), S_{\mathtt{i}}(y))$ and $x_j \in S_{\mathtt{i}}(K)$. Thus, applying $S_{\mathtt{i}}^{-1}$ to $\mathbf{x}$, we get an itinerary $\mathbf{x}'$ from x to y such that $x'_j \in K$ and $c(\mathbf{x}') = 2^{|\mathtt{i}|} c(\mathbf{x})$. Thus,

$$d(x, y) \leq c(\mathbf{x}') = 2^{|\mathtt{i}|} c(\mathbf{x}) \leq 2^{|\mathtt{i}|}(1+\epsilon) d(S_{\mathtt{i}}(x), S_{\mathtt{i}}(y)).$$

Taking $\epsilon \to 0$ then gives the lemma. $\square$

We can now prove that $(\bar{K}, \bar{d})$ is BPI. Let $\mathtt{j}$ be the multi-index from Lemma 5.6, $p_1, p_2 \in K$, and $0 < r_1, r_2 < \mathrm{diam}_d(K)$. Now by self-similarity in the d_b-distance, there exist a constant c_1 and two multi-indices $\mathtt{i}_1, \mathtt{i}_2$ such that $S_{\mathtt{i}_j}(K) \subset B_{d_b}(p_j, r_j)$, r_j is comparable to $2^{-|\mathtt{i}_1|}$, and $\mathcal{H}^4_{d_b}(S_{\mathtt{i}_j}(K)) \geq c_1 \mathcal{H}^4_{d_b}(B_{d_b}(p_j, r_j))$ for $j \in \{1, 2\}$.

By Lemma 3.8 we also have $S_{\mathtt{i}_j}(K) \subset B_d(p_j, r_j)$ and that the ratio of $\mathcal{H}^4_{d_b}$ and $\mathcal{H}^4_d$ is bounded away from 0 and from above by constants depending on λ.

Therefore by defining $A = S_{\mathtt{i}_1}(S_{\mathtt{j}}(K)) \subset B_d(p_1, r_1)$, we have $\mathcal{H}^4_d(A) \geq c_2 \mathcal{H}^4_d(B_d(p_1, r_1))$ for some constant c_2. Define the map $f\colon A \to B_d(p_2, r_2)$ as $f = S_{\mathtt{i}_2} \circ S_{\mathtt{i}_1}^{-1}$. Then for any $p, q \in A$, we have that $p' = S_{\mathtt{i}_1}^{-1}(p)$ and $q' = S_{\mathtt{i}_1}^{-1}(q)$ are both in $S_{\mathtt{j}}(K)$. Thus,

$$\begin{aligned} d(f(p), f(q)) &= d(S_{\mathtt{i}_2}(p'), S_{\mathtt{i}_2}(q')) \\ &\overset{(5.11)}{=} 2^{-|\mathtt{i}_2|} d(p', q') \\ &\overset{(5.11)}{=} 2^{|\mathtt{i}_1|-|\mathtt{i}_2|} d(S_{\mathtt{i}_1}(p'), S_{\mathtt{i}_1}(q')) \\ &= 2^{|\mathtt{i}_1|-|\mathtt{i}_2|} d(p, q). \end{aligned}$$

Since $2^{|\mathtt{i}_1|-|\mathtt{i}_2|}$ is comparable to r_2/r_1, we are done with showing that $(\bar{K}, \bar{d})$ is BPI.

5.4. (K, d) **does not look down on** $(\mathbb{H}, d_b)$**.** By contradiction, suppose that $(\bar{K}, \bar{d})$ does look down on $(\mathbb{H}, d_b)$. Then there would exist a closed set $A \subset \bar{K}$ and a Lipschitz map $f\colon (A, \bar{d}) \to (\mathbb{H}, d_b)$ with $\mathcal{H}^4(f(A)) > 0$. Since $d \leq d_b$, also $f \circ \pi_\sim\colon (\pi_\sim^{-1}(A), d_b) \to (\mathbb{H}, d_b)$ is L-Lipschitz, where $\pi_\sim$ is the quotient projection. In the following we write $f \circ \pi_\sim$ as f when we work in $\pi_\sim^{-1}(A)$. Then f is Pansu-differentiable almost everywhere in $\pi_\sim^{-1}(A)$, see [Mag01]. Moreover, the Pansu-differential $Df(x)$ is bijective on a set $A' \subset A$ of positive measure by the area formula:

$$0 < \mathcal{H}^4_{d_b}(f(A)) \overset{(2.4)}{\leq} \int_A J(Df(x))\, d\mathcal{H}^4_{d_b}(x).$$

Since for all $n, m \in \mathbb{N}$ the set

$$B_{n,m} = K \setminus \bigcup_{k=n}^{\infty} \bigcup_{\mathtt{i} \in \{1,\dots,16\}^k} S_{\mathtt{i}}(K \cap B_{d_b}(0, \frac{1}{m}))$$

has $\mathcal{H}^4_{d_b}$-measure zero as a porous set, the set

$$A'' = A' \setminus \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} B_{n,m} = \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{\mathtt{i} \in \{1,\dots,16\}^k} A' \cap S_{\mathtt{i}}(K \cap B_{d_b}(0, \frac{1}{m})) \tag{5.12}$$

has positive measure. Let $x \in A''$ be a density point of A''. Since $x \in A''$, by the definition (5.12) there exists a sequence $(n_m)_{m=1}^{\infty}$ of integers with $n_m \to \infty$ as $m \to \infty$ and a sequence of multi-indices $\mathtt{i}_m$ with $|\mathtt{i}_m| = n_m$ such that $x \in S_{\mathtt{i}_m}(K)$ for all $m \in \mathbb{N}$ and

$$d_b(x, x_m) < \frac{1}{m} 2^{-n_m},$$

for all $m \in \mathbb{N}$, where $x_m = S_{\mathtt{i}_m}(0)$.

Let $K_m = \delta_{2^{n_m}}(x^{-1}A) \cap K$. Then the functions $f_m\colon (K_m, d) \to (\mathbb{H}, d_b)$ defined as

$$f_m(p) = \delta_{2^{n_m}}(f(x)^{-1} f(x \delta_{2^{-n_m}}(p)))$$

satisfy

$$\begin{aligned}
d_b(f_m(p), f_m(q)) &= d_b(\delta_{2^{n_m}}(f(x)^{-1}f(x\delta_{2^{-n_m}}(p))), \delta_{2^{n_m}}(f(x)^{-1}f(x\delta_{2^{-n_m}}(q)))) \\
&= 2^{n_m} d_b(f(x\delta_{2^{-n_m}}(p)), f(x\delta_{2^{-n_m}}(q))) \\
&\leq 2^{n_m} L d(x\delta_{2^{-n_m}}(p), x\delta_{2^{-n_m}}(q)) \\
&\leq L d(S_{\mathtt{i}_m}^{-1}(x\delta_{2^{-n_m}}(p)), S_{\mathtt{i}_m}^{-1}(x\delta_{2^{-n_m}}(q))) \\
&= L d(\delta_{2^{n_m}}(x_m^{-1}x)p, \delta_{2^{n_m}}(x_m^{-1}x)q) \\
&\leq L\left(d(\delta_{2^{n_m}}(x_m^{-1}x)p, p) + d(p,q) + d(q, \delta_{2^{n_m}}(x_m^{-1}x)q)\right),
\end{aligned} \tag{5.13}$$

where the first inequality follows from the fact that f is L-Lipschitz and the second inequality from the fact that

$$S_{\mathtt{i}_n}(\mathcal{S}) \subset \mathcal{S}.$$

Notice that

$$d(\delta_{2^{n_m}}(x_m^{-1}x)p, p) \leq d_b(\delta_{2^{n_m}}(x_m^{-1}x)p, p) \to 0 \tag{5.14}$$

as $m \to \infty$ since $d_b(\delta_{2^{n_m}}(x_m^{-1}x), 0) \to 0$ as $m \to \infty$. The convergence in (5.14) holds uniformly for $p \in K$ by the compactness of K.

Since x is a density point of A'' and hence of A, we have that for all $p \in K$ there exists a sequence $(p_m)_{m=1}^{\infty}$ with $p_m \in K_m$ and $d_b(p_m, p) \to 0$ as $m \to \infty$. Along this sequence by the fact that $Df(x)$ is homogeneous we get

$$\begin{aligned}
d_b(f_m(p_m), Df(x)(p_m)) &= d_b(\delta_{2^{n_m}}(f(x)^{-1}f(x\delta_{2^{-n_m}}(p_m))), \delta_{2^{n_m}}(Df(x)(\delta_{2^{-n_m}}(p_m)))) \\
&= 2^{n_m} d_b(f(x)^{-1}f(x\delta_{2^{-n_m}}(p_m)), Df(x)(\delta_{2^{-n_m}}(p_m))) \to 0,
\end{aligned}$$

as $m \to \infty$. Hence also

$$d_b(f_m(p_m), Df(x)(p)) \leq d_b(f_m(p_m), Df(x)(p_m)) + d_b(Df(x)(p_m), Df(x)(p)) \to 0, \tag{5.15}$$

as $m \to \infty$.

Combining the estimates (5.15), (5.13) and (5.14) with the fact that $d(p_m, p) \leq d_b(p_m, p) \to 0$ we get

$$\begin{aligned}
d_b(Df(x)(p), Df(x)(q)) &\leq d_b(f_m(p_m), Df(x)(p)) + d_b(f_m(q_m), Df(x)(q)) \\
&\qquad + d_b(f_m(p_m), f_m(q_m)) \\
&\leq d_b(f_m(p_m), Df(x)(p)) + d_b(f_m(q_m), Df(x)(q)) \\
&\qquad + L\left(d(\delta_{2^{n_m}}(x_m^{-1}x)p_m, p_m) + d(q_m, \delta_{2^{n_m}}(x_m^{-1}x)q_m)\right) \\
&\qquad + L\left(d(p_m, p) + d(p,q) + d(q, q_m)\right) \\
&\to L d(p,q), \qquad \text{as } m \to \infty.
\end{aligned}$$

Hence $Df(x)$ is also Lipschitz from (K, d) to $(\mathbb{H}, d_b)$. Since $Df(x)\colon (\mathbb{H}, d_b) \to (\mathbb{H}, d_b)$ is biLipschitz, also the identity map $\mathrm{id}\colon (K, d) \to (\mathbb{H}, d_b)$ is Lipschitz, but we have shown that this is not the case in Theorem 1.2.

6. BiLipschitz equivalent distances on the Heisenberg group

In the previous sections we constructed and studied distances that were not biLipschitz equivalent on large sets. In this final section we turn to study distances that are biLipschitz equivalent. First we prove Theorem 1.4 showing that adding to Theorem 1.2 the assumption that the left-translations are biLipschitz for the distance d forces the distances d_{cc} and d to be biLipschitz equivalent on compact sets. After this we prove Theorem 1.5 giving examples of distances on the Heisenberg group that are biLipschitz equivalent with d_{cc} having no self-similar tangents.

6.1. **BiLipschitz left-translations: Proof of Theorem 1.4.** Since d_{cc} is biLipschitz equivalent to the box distance d_b, up to multiplying d by a constant we assume that $d \leq d_b$.

Using the Baire Category Theorem one can show that there exists $L > 1$ such that, if we restrict to a compact set K, then the distance d is L-biLipschitz homogeneous: each left translation by $g \in K$ is L-biLipschitz on K (see [LD11, Lemma 6.7] applied with the Heisenberg group as X and G). Suppose that the claim of the theorem is not true. Hence, by the left-biLipschitz invariance of the distances, for all $N \in \mathbb{N}$ there exists a point $p \in B_{d_b}(0, \frac{1}{2N})$ with $d_b(0,p) > LNd(0,p)$.

Write $r_N := d_b(0,p) > 0$. We claim that we have that

$$\bigcup_{n=0}^{N} B_{d_b}(p^n, \frac{1}{2}r_N) \subset B_d(0, 2r_N). \tag{6.1}$$

Indeed, if $q \in B_{d_b}(p^n, \frac{1}{2}r_N)$, for some $n \leq N$, then

$$\begin{aligned} d(0,q) &\leq d(0,p) + d(p,p^2) + \ldots + d(p^{n-1}, p^n) + d(p^n, q) \\ &\leq LNd(0,p) + d_b(p^n, q) \\ &\leq d_b(0,p) + r_N/2 < 2r_N. \end{aligned}$$

Moreover, for $i < j < N$, we have that

$$d_b(p^i, p^j) = d_b(0, p^{j-i}) \geq d_b(0,p) = r_N.$$

Let $\{q_i\}_{i \in I_N}$ be a maximal $4r_N$-separated net of points with respect to distance d in $B_{d_b}(0,1)$. First, by (6.1) for all $i \in I_N$ we have that $\{B_{d_b}(q_i p^n, \frac{1}{2}r_N)\}_{n=0}^N$ is a disjointed collection of subsets of $B_{d_b}(q_i, 2r_N)$. Second, $\{B_d(q_i, 2r_N)\}_{i \in I_N}$ is a disjointed collection of subsets of $B_{d_b}(0,2)$. Hence

$$\# I_N N \mathcal{H}^4_{d_b}(B_{d_b}(0, \frac{1}{2}r_N)) \leq \mathcal{H}^4_{d_b}(B_{d_b}(0,2)).$$

Since $\{B_d(q_i, 8r_N)\}_{i\in I_N}$ covers $B_{d_b}(0,1)$, by definition of Hausdorff measure we deduce

$$\begin{aligned}\mathcal{H}^4_d(B_{d_b}(0,1)) &\leq \liminf_{N\to\infty} \#I_N(16r_N)^4\\ &\leq \liminf_{N\to\infty} \frac{\mathcal{H}^4_{d_b}(B_{d_b}(0,2))}{N\mathcal{H}^4_{d_b}(B_{d_b}(0,\frac{1}{2}r_N))}(16r_N)^4\\ &= \liminf_{N\to\infty}\frac{64^4}{N} = 0.\end{aligned}$$

This contradicts the assumption $\mathcal{H}^4_d(B_{d_b}(0,1)) > 0$. □

6.2. **Distances without self-similar tangents.** In this final section we prove Theorem 1.5. Namely, we construct two distances d_1, d_2 on $\mathbb{H}$ that are biLipschitz equivalent to d_{cc} such that

(1) the distance d_1 is left-invariant and for all $\lambda_j \to 0$ such that the distances

$$(p,q) \mapsto \frac{1}{\lambda_j} d_1(\delta_{\lambda_j}(p), \delta_{\lambda_j}(q))$$

converge point-wise to some ρ, the distance ρ is not self-similar;

(2) for all $\lambda_j \to 0$ and $q_j \in \mathbb{H}$ such that the distances

$$(p,q) \mapsto \frac{1}{\lambda_j} d_2(q_j\delta_{\lambda_j}(p), q_j\delta_{\lambda_j}(q))$$

converge point-wise to some ρ, the distance ρ is not self-similar nor left-invariant.

We will first construct the distance d_1 and at the end indicate how the construction can be modified to obtain the distance d_2.

The distance d_1 is defined via (2.5). The initial distance is d_b, which is biLipschitz to d_{cc}, and the shortcuts are defined by first taking a sequence of shortcuts from the origin to points in the vertical direction and then left-translating the shortcuts to start from every point of the space. Since we want none of the tangents to admit nontrivial dilations, we have to be careful in defining the sequence of shortcuts.

Let us define the set of shortcuts from the origin as

$$\mathcal{S}_0 = \left\{\left(0, (0,0,4^{-n})\right) \ :\ n \in a^{-1}(\{1\})\right\},$$

where $a\colon \mathbb{N} \to \{0,1\}$ is a function determining whether a shortcut is taken on scale 4^{-n}. If we were to take $a(n) = 1$ for all n, then the tangents would be self-similar.

The full set of shortcuts is then defined as

$$\mathcal{S} = \{(pq_1, pq_2) \ :\ p \in \mathbb{H},\ (q_1,q_2) \vee (q_2,q_1) \in \mathcal{S}_0\}$$

and the cost function $c\colon \mathbb{H}\times\mathbb{H} \to [0,\infty)$ for $(p,q) \in \mathcal{S}$ as

$$c(p,q) = \frac{1}{2} d_b(p,q).$$

The distance d_1 is then defined as the d in (2.5).

Since $\frac{1}{2}d_b \leq d_1 \leq d_b$, the function d_1 is a distance and it is biLipschitz equivalent with d_b, and so with d_{cc}. By the left-invariance of the set of shortcuts $\mathcal{S}$, the distance d_1 is also left-invariant.

Since we want to avoid self-similarity, we define the function a so that every word written in the alphabet $\{0,1\}$ appears consecutively in the sequence $(a(n))_{n\in\mathbb{N}}$ only some limited number of times. This is achieved for example by defining

$$a(i) := \begin{cases} 1, & \text{if there exists } k \text{ odd and } l \in \mathbb{N} \text{ such that } i = (k\ell \prod_{h<\ell} p_h + 1)p_\ell \\ 0, & \text{otherwise,} \end{cases}$$

where p_ℓ is the ℓ:th prime number.

Most of the remainder of the section will be devoted to proving that with this selection of a no blow-up of d_1 is self-similar. On the level of a the needed property is stated in the next lemma.

Lemma 6.1. *Let $\ell \geq 1$. There exists some $m \geq 1$ so that for any $i \geq 1$, there exists some $j \in \{i, i+1, \dots, i+m\ell\}$ such that $a(j) \neq a(j+m\ell)$.*

Proof. Let us write

$$P_\ell = \left\{ (k\ell \prod_{h<\ell} p_h + 1)p_\ell \; : \; k \in \mathbb{N} \right\}.$$

We claim that $\{P_\ell\}_{\ell\in\mathbb{N}}$ is a disjointed collection of sets. In order to see this take $0 < \ell < \ell' < \infty$ and notice that on one hand for every $k \in \mathbb{N}$ we have $p_\ell \mid (k\ell \prod_{h<\ell} p_h + 1)p_\ell$. On the other hand, since $p_\ell \mid \prod_{h<\ell'} p_h$, we have $p_\ell \nmid (k\ell' \prod_{h<\ell'} p_h + 1)p_{\ell'}$ for all $k \in \mathbb{N}$.

Now let $\ell \geq 1$ be given. Define $m = \prod_{h\leq\ell} p_h$. Then $P_\ell = \{p_\ell + m\ell k \; : \; k \in \mathbb{N}\}$. Let $i \geq 1$ and select $j \in \{i, i+1, \dots, i+m\ell\}$ such that $j \equiv p_\ell \pmod{m\ell}$. Then by definition, $j \in P_\ell$. By the fact that the sets $P_{\ell'}$ are pairwise disjoint we have from the definition of a that

$$a(m\ell k + p_\ell) = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

Thus $a(j) \neq a(j + \ell m)$. $\square$

The next lemmas will be used to connect the blown up distances to the distance d_1, and in particular to a.

Lemma 6.2. *Let $\mathbf{x} = (x_0, \dots, x_N)$ be an itinerary such that $x_0 = 0$ and $x_N \in \mathrm{Z}(\mathbb{H})$. Then there exists another itinerary $\mathbf{y} = (y_0, \dots, y_M)$ such that $\mathrm{Ext}(\mathbf{y}) = \mathrm{Ext}(\mathbf{x})$, $y_i^{-1}y_{i+1} \in \mathrm{Z}(\mathbb{H})$ for all i, and $c(\mathbf{y}) \leq c(\mathbf{x})$.*

Note that as $x_0 = y_0 = 0$, the condition that $y_i^{-1}y_{i+1} \in \mathrm{Z}(\mathbb{H})$ for all i is equivalent to $y_i \in \mathrm{Z}(\mathbb{H})$.

Proof. For the itinerary $\mathbf{x} = (x_0, \ldots, x_N)$, we define $d_k = x_{k-1}^{-1} x_k$. Then $x_N = x_0 d_1 \cdots d_N$. Let

$$A = \{k : d_k \in \mathrm{Z}(\mathbb{H})\}.$$

We can define a bijection $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$ that maps $\{1, \ldots, |A|\}$ to A and preserves the ordering of A^c (thus, σ^{-1} shifts A to the beginning in any order). Note that $x_N = x_0 d_{\sigma(1)} \cdots d_{\sigma(N)}$ as we are moving only the d_k that are central.

We now define the itinerary $(y_0, \ldots, y_{|A|+1})$ where $y_0 = 0$, $y_{|A|+1} = x_N$, and $y_i = d_{\sigma(1)} \cdots d_{\sigma(i)}$. As we only rearranged elements that are in the center, we get that

$$y_{|A|}^{-1} y_{|A|+1} = (x_0 d_{\sigma(1) \cdots \sigma(|A|)})^{-1} x_0 d_{\sigma(1)} \cdots d_{\sigma(N)} = d_{\sigma(|A|+1)} \cdots d_{\sigma(N)}$$

is precisely the product (in order) of all the noncentral d_k.

It remains to show that $c(\mathbf{y}) \leq c(\mathbf{x})$. We have that

$$c(x_{k-1}, x_k) = c(y_{\sigma^{-1}(k)-1}, y_{\sigma^{-1}(k)}), \qquad \forall k \in A. \tag{6.2}$$

As $d_k \notin \mathrm{Z}(\mathbb{H})$ for $k \notin A$, we get that $(x_{k-1}, x_k) \notin \mathcal{S}$ for $k \notin A$ and so

$$\sum_{k \notin A} c(x_{k-1}, x_k) = \sum_{k \notin A} d_b(x_{k-1}, x_k) = \sum_{k=|A|+1}^{N} \|d_{\sigma(k)}\| \geq d_b(y_{|A|}, y_{|A|+1}) \geq c(y_{|A|}, y_{|A|+1}). \tag{6.3}$$

Thus, by (6.2) and (6.3) we get that

$$c(\mathbf{y}) = \sum_{k=1}^{|A|+1} c(y_{k-1}, y_k) \leq \sum_{k=1}^{N} c(x_{k-1}, x_k) = c(\mathbf{x}). \qquad \square$$

Lemma 6.3. *There exists a continuous function $f\colon [1,4] \to [\frac{1}{2}, 1]$ with the properties that $f(t) > \frac{1}{2}$ for all $t \in (1,4)$ and*

$$d_1(0, (0, 0, t4^{-n})) \geq f(t) d_b(0, (0, 0, t4^{-n})) \tag{6.4}$$

for all $n \in \mathbb{N}$ and $t \in (1,4)$.

Proof. We claim that

$$f(t) = \min\left(\frac{1}{\sqrt{t}}, \frac{1}{2}\sqrt{\frac{2t}{t+1}}\right)$$

works. It is immediate from definition that $f(t) > 1/2$ for $t \in (1,4)$.

Let $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ be an itinerary from 0 to $(0, 0, t4^{-n})$ where $t \in (1,4)$. By Lemma 6.2, we may suppose that $x_{i+1}^{-1} x_i \in \mathrm{Z}(\mathbb{H})$. Let ℓ_k be the absolute value of the z-coordinate of $x_k^{-1} x_{k-1}$. Let ℓ_M be the maximum of the ℓ_k's. Then we have that

$$\sum \ell_k \geq t4^{-n}. \tag{6.5}$$

Suppose first that $\ell_M \geq 4^{-n+1}$, then

$$c(\mathbf{x}) \geq \frac{1}{2} \ell_M^{1/2} \geq 2^{-n} = \frac{1}{\sqrt{t}} \sqrt{t} 2^{-n} \geq f(t) d_b(0, (0, 0, t4^{-n})),$$

and we are done. Then suppose $\ell_M \in \left[\frac{1+t}{2}4^{-n}, 4^{-n+1}\right)$. Then as $t \in (1,4)$, we get that $(x_{M-1}, x_M) \notin \mathcal{S}$. This gives

$$c(\mathbf{x}) \geq \ell_M^{1/2} \geq \sqrt{\frac{1+t}{2}}2^{-n} = \sqrt{\frac{1+t}{2t}}\sqrt{t}2^{-n} \geq \frac{1}{\sqrt{t}}\sqrt{t}2^{-n},$$

and we are done. Finally, suppose that $\ell_M < \frac{1+t}{2}4^{-n}$. By maximality of ℓ_M we then have $\ell_k < \frac{1+t}{2}4^{-n}$ for all k. Thus we have that

$$2c(\mathbf{x}) \geq \sum_{k=1}^N \ell_k^{1/2} \geq \sqrt{\frac{2}{t+1}}2^n \sum_{k=1}^N \ell_k \overset{(6.5)}{\geq} \sqrt{\frac{2t}{t+1}}\sqrt{t}2^{-n},$$

and we are done. □

Lemma 6.4. *For all $n \in a^{-1}(\{0\})$ and $t \in (\frac{1}{2}, 2)$, we have that*

$$d_1(0,(0,0,t4^{-n})) \geq \frac{1}{\sqrt{3}}d_b(0,(0,0,t4^{-n})). \tag{6.6}$$

Proof. The proof is largely analogous to the proof of Lemma 6.3.

Let $(x_0, x_1, \ldots, x_N)$ be an itinerary from 0 to $(0,0,t4^{-n})$ where $t \in (1/2, 2)$ and assume that $a(n) = 0$. By Lemma 6.2, we may suppose that $x_{i+1}^{-1}x_i \in \mathrm{Z}(\mathbb{H})$. Let ℓ_k and ℓ_M be as in Lemma 6.3, so that we have (6.5).

Suppose first that $\ell_M \geq 4^{-n+1}$. Then as $t \in (1/2, 2)$, we have

$$c(\mathbf{x}) \geq \frac{1}{2}\ell_M^{1/2} \geq 2^{-n} \geq \frac{1}{\sqrt{2}}\sqrt{t}2^{-n} \geq \frac{1}{\sqrt{3}}d_b(0,(0,0,t4^{-n})),$$

and we are done. Then suppose $\ell_M \in \left[\frac{1+4t}{8}4^{-n}, 4^{-n+1}\right)$. Then as $t \in (1/2, 2)$ and $a(n) = 0$, we get that $(x_{M-1}, x_M) \notin \mathcal{S}$. This gives

$$c(\mathbf{x}) \geq \ell_M^{1/2} \geq \sqrt{\frac{1+4t}{8}}2^{-n} \geq \frac{1}{\sqrt{2}}\sqrt{t}2^{-n},$$

and we are done. Finally, suppose that $\ell_M < \frac{1+4t}{8}4^{-n}$. Thus, we have that $\ell_k < \frac{1+4t}{8}4^{-n}$ for all k. Therefore

$$2c(\mathbf{x}) \geq \sum_{k=1}^N \ell_k^{1/2} \geq \sqrt{\frac{8}{4t+1}}2^n \sum_{k=1}^N \ell_k \overset{(6.5)}{\geq} \sqrt{\frac{8t^2}{4t+1}}2^{-n} \geq \frac{2}{\sqrt{3}}\sqrt{t}2^{-n}, \tag{6.7}$$

and we are done. □

Lemma 6.5. *For every $\epsilon > 0$, there exists some $\eta \in (0, 1/2)$ such that if $|t| < \eta$ and $a(n) = 1$, then*

$$d_1(0,(0,0,(1+t)4^{-n})) \leq \left(\frac{1}{2}+\epsilon\right)d_b(0,(0,0,(1+t)4^{-n})).$$

Proof. One has that

$$d_b(0,(0,0,(1+t)4^{-n})) = \sqrt{1+t}\,d_b(0,(0,0,4^{-n})).$$

Consider the itinerary $\mathbf{x} = ((0,0,0),(0,0,4^{-n}),(0,0,(1+t)4^{-n}))$. Then

$$c(\mathbf{x}) \leq \frac{1}{2}d_b(0,(0,0,4^{-n})) + d_b(0,(0,0,t4^{-n})) = \left(\frac{1}{2}+\sqrt{|t|}\right)d_b(0,(0,0,t4^{-n})).$$

Thus, we need that

$$\frac{1}{2}+\sqrt{|t|} \leq \left(\frac{1}{2}+\epsilon\right)\sqrt{1+t}.$$

One sees easily that by taking η small enough, we can satisfy this inequality. □

With the help of the above lemmas we conclude by proving:

Proposition 6.6. *No blow-up of d_1 is self-similar.*

Proof. Assume to the contrary that there exists a sequence $(\lambda_j)_{j\in\mathbb{N}}$, with $\lambda_j \to 0$ such that the distances

$$(p,q) \mapsto \frac{1}{\lambda_j}d_1(\delta_{\lambda_j}(p),\delta_{\lambda_j}(q))$$

converge point-wise to some ρ, and the distance ρ is self-similar with some constant $\lambda > 1$.

Let us now find a contradiction by using the assumed self-similarity. For this purpose let us first take a point $(0,0,s^2) \in \mathbb{H}$ appearing as limit of points to which there is a shortcut from the origin. In other words, take

$$s \in [1,2^4] \cap \bigcap_{j=1}^{\infty} \overline{\bigcup_{i\geq j}\{\lambda_i^{-1}2^{-4(k+1)} \,:\, k\in\mathbb{N}\}}. \tag{6.8}$$

We claim $\lim_{j\to\infty}\frac{1}{\lambda_j}d_1(0,(0,0,\lambda_j^2 s^2)) = \frac{1}{2}d_b(0,(0,0,s^2))$.

First, note that $a(4(k+1)) = 1$ for all $k\in\mathbb{N}$. Indeed, in the definition of a, if we take $l=1$, then $p_l = 2$. By definition of a, we have that $a(2(k+1)) = 1$ for all *odd* k, which implies that $a(4(k+1)) = 1$ for all $k\in\mathbb{N}$.

Let $\epsilon > 0$. By definition of s, there exist $j_m, k_m \to \infty$ so that $\lambda_{j_m}^{-1}2^{-4(k_m+1)} \to s$. Thus, for m sufficiently large, we have

$$|\lambda_{j_m}s - 2^{-4(k_m+1)}| < \frac{\epsilon^2}{100}\lambda_{j_m}.$$

If $\epsilon < 1$, then as $s\in[1,2^4]$, we get by the previous inequality and the triangle inequality that

$$\sqrt{|\lambda_{j_m}^2 s^2 - 4^{-4(k_m+1)}|} = \sqrt{|\lambda_{j_m}s - 2^{-4(k_m+1)}||\lambda_{j_m}s + 2^{-4(k_m+1)}|} < \epsilon\lambda_{j_m}. \tag{6.9}$$

As $a(4(k_m+1)) = 1$, one gets that $d_1(0,(0,0,4^{-4(k_m+1)})) = \frac{1}{2}d_b(0,(0,0,4^{-4(k_m+1)}))$. As d_1 satisfies the triangle inequality and the inequality $d_1 \le d_b$, we get for sufficiently large m that

$$\begin{aligned}
\lambda_{j_m}^{-1} d_1(0,(0,0,\lambda_{j_m}^2 s^2)) &\le \lambda_{j_m}^{-1}(d_1(0,(0,0,4^{-4(k_m+1)})) + d_1(0,(0,0,\lambda_{j_m}^2 s^2 - 4^{-4(k_m+1)}))) \\
&\le \frac{1}{2\lambda_{j_m}} d_b(0,(0,0,4^{-4(k_m+1)})) + d_b(0,(0,0,s^2 - \lambda_{j_m}^{-2}4^{-4(k_m+1)})) \\
&\overset{(6.9)}{\le} \frac{1}{2} d_b(0,(0,0,\lambda_{j_m}^{-2}4^{-4(k_m+1)})) + \epsilon \\
&\le \frac{1}{2}(d_b(0,(0,0,s^2)) + d_b((0,0,s^2),(0,0,\lambda_{j_m}^{-2}4^{-4(k_m+1)}))) + \epsilon \\
&\overset{(6.9)}{\le} \frac{1}{2} d_b(0,(0,0,s^2)) + 2\epsilon.
\end{aligned}$$

As this holds for all $\epsilon > 0$, we get $\lim_{m\to\infty} \lambda_{j_m}^{-1} d_1(0,(0,0,\lambda_{j_m}^2 s^2)) \le \frac{1}{2} d_b(0,(0,0,s^2))$. A similar argument gives the opposite inequality.

Thus, we indeed have

$$\rho(0,(0,0,s^2)) = \lim_{j\to\infty} \frac{1}{\lambda_j} d_1(0,(0,0,\lambda_j^2 s^2)) = \frac{1}{2} d_b(0,(0,0,s^2)).$$

Let us then use the function f of Lemma 6.3 to show that there exists $\ell \in \mathbb{N}$ such that $\lambda = 2^\ell$. Supposing this is not the case, we have $\lambda = t2^\ell$ for some $t \in (1,2)$ and $\ell \in \mathbb{N}$. By (6.8) s is of the form $s = \lim_{m\to\infty} \lambda_{i_m}^{-1} 2^{-4(k_m+1)}$, with $i_m, k_m \to \infty$. Then, by the continuity of the function f we have

$$\begin{aligned}
\rho(0,(0,0,\lambda^2 s^2)) &= \lim_{j\to\infty} \frac{1}{\lambda_j} d_1(0,(0,0,\lambda^2\lambda_j^2 s^2)) \\
&= \lim_{m\to\infty} \frac{1}{\lambda_{i_m}} d_1(0,(0,0,t^2\lambda_{i_m}^2 s^2 4^\ell)) \\
&= \lim_{m\to\infty} \frac{1}{\lambda_{i_m}} d_1\left(0,\left(0,0,t^2\left(\frac{\lambda_{i_m} s}{2^{-4(k_m+1)}}\right)^2 4^\ell 4^{-4(k_m+1)}\right)\right) \\
&\overset{(6.4)}{\ge} \lim_{m\to\infty} \frac{1}{\lambda_{i_m}} f\left(t^2\left(\frac{\lambda_{i_m} s}{2^{-4(k_m+1)}}\right)^2\right) d_b(0,(0,0,\lambda^2\lambda_{i_m}^2 s^2)) \\
&= f(t^2) \lim_{j\to\infty} \frac{1}{\lambda_j} d_b(0,(0,0,\lambda^2\lambda_j^2 s^2)) \\
&= f(t^2)\lambda d_b(0,(0,0,s^2)) > \frac{1}{2}\lambda d_b(0,(0,0,s^2)) = \lambda\rho(0,(0,0,s^2)),
\end{aligned}$$

contradicting the fact that ρ is self-similar with the dilation λ.

Therefore $\lambda = 2^\ell$ for some $\ell \in \mathbb{N}$. Now we employ the properties of the function a. Let $m \in \mathbb{N}$ be the constant from Lemma 6.1. Since ρ is self-similar with factor 2^ℓ, it is self-similar also with factor $2^{\ell m}$. By Lemma 6.5, we have that there exists some η such that $(1+\eta)^N = 4$ for some $N \in \mathbb{N}$ and if $a(n) = 1$, then

$$d_1(0,(0,0,(1+t)4^{-n})) \leq 0.51 d_b(0,(0,0,(1+t)4^{-n})), \qquad \forall t \in (-\eta,\eta). \tag{6.10}$$

Take $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have

$$\frac{\rho(0,(0,0,4^i(1+\eta)^k s^2))}{\lambda_j^{-1} d_1(0,(0,0,4^i(1+\eta)^k \lambda_j^2 s^2))} \in \left(1-\frac{1}{100}, 1+\frac{1}{100}\right), \tag{6.11}$$

for all $(i,k) \in \{0,1,\dots,2m\ell\} \times \{0,\dots,N-1\}$. Fix some $j \geq j_0$ large enough so that for the $n \in \mathbb{Z}$ such that $(\lambda_j s)^2 \in [4^{-n-1}, 4^{-n})$, we get that $n \geq 2m\ell$. Now by Lemma 6.1, we have that there exists some $i \in \{0,\dots,m\ell\}$ such that $a(n-i) \neq a(n-i-m\ell)$. We may suppose without loss of generality that $a(n-i) = 1$ so $a(n-i-m\ell) = 0$. Note that j, n, and i are now fixed.

From the definition of n we have that $4^i \lambda_j^2 s^2 \in [4^{-n+i-1}, 4^{-n+i})$. Also, $(1-\eta)4^{-n+i}$ and $(1+\eta)4^{-n+i}$ differ by a multiplicative factor larger than $1+\eta$. As

$$\{(1+\eta)^k : k \in \{0,...,N-1\}\}$$

increases from 1 to $\frac{4}{1+\eta} > 4(1-\eta)$ in multiplicative increments of $1+\eta$, we then have by the pigeonhole principle that there exists some $k \in \{0,\dots,N-1\}$ such that $4^i(1+\eta)^k \lambda_j^2 s^2 \in ((1-\eta)4^{-n+i}, (1+\eta)4^{-n+i})$. Thus, because $a(n-i) = 1$, we have that

$$\begin{aligned}
\rho(0,(0,0,4^i(1+\eta)^k s^2)) &\overset{(6.11)}{\leq} 1.01 \lambda_j^{-1} d_1(0,(0,0,4^i(1+\eta)^k \lambda_j^2 s^2)) \\
&\overset{(6.10)}{\leq} 0.52\, \lambda_j^{-1} d_b(0,(0,0,4^i(1+\eta)^k \lambda_j^2 s^2)) \\
&= 0.52\, d_b(0,(0,0,4^i(1+\eta)^k s^2)).
\end{aligned} \tag{6.12}$$

On the other hand, because $a(n-i-m\ell) = 0$ and

$$4^{i+m\ell}(1+\eta)^k \lambda_j^2 s^2 \in \left(\frac{1}{2}4^{-n+i+m\ell}, 2 \cdot 4^{-n+i+m\ell}\right),$$

we have that

$$\begin{aligned}
\rho(0,(0,0,4^{i+m\ell}(1+\eta)^k s^2)) &\overset{(6.6)\wedge(6.11)}{\geq} \frac{99}{100}\frac{1}{\sqrt{3}}\, d_b(0,(0,0,4^{i+m\ell}(1+\eta)^k s^2)) \\
&\geq 0.55\, d_b(0,(0,0,4^{i+m\ell}(1+\eta)^k s^2)), \\
&= 0.55 \cdot 2^{m\ell}\, d_b(0,(0,0,4^i(1+\eta)^k s^2)).
\end{aligned}$$

Then by the self-similarity of ρ with ratio $2^{m\ell}$ we have

$$\begin{aligned}
\rho(0,(0,0,4^i(1+\eta)^k s^2)) &= 2^{-m\ell} \rho(0,(0,0,4^{i+m\ell}(1+\eta)^k s^2)) \\
&\geq 0.55\, d_b(0,(0,0,4^i(1+\eta)^k s^2)).
\end{aligned}$$

This contradicts (6.12). $\square$

In order to obtain the distance d_2 of Theorem 1.5, we use only a subset of shortcuts used in the definition of the distance d_1. Let D_n denote the centers of the dyadic cubes in $\mathbb{R}^2$ of side length 2^{-n}.

Define the level n shortcuts as the symmetrization of

$$\tilde{\mathcal{S}}_n = \{((x,y,z),(x,y,z)q) : (x,y) \in D_n, z \in \mathbb{R}, q = (0,0,\pm 4^{-n})\}.$$

We then construct the set of shortcuts as

$$\tilde{\mathcal{S}} = \bigcup_{n \in a^{-1}(\{1\})} \tilde{\mathcal{S}}_n.$$

As in the construction of d_1, the cost function $\tilde{c} : \mathbb{H} \times \mathbb{H} \to [0,\infty)$ for $(p,q) \in \tilde{\mathcal{S}}$ is

$$\tilde{c}(p,q) = \frac{1}{2} d_b(p,q).$$

The distance d_2 is then obtained as the distance d in (2.5), but now with using $\tilde{c}$. Since $\tilde{\mathcal{S}} \subset \mathcal{S}$ and thus $\tilde{c} \geq c$, we have

$$\frac{1}{2} d_b \leq d_1 \leq d_2 \leq d_b.$$

We will also need the following lemmas.

Lemma 6.7. *There exists some absolute $\delta > 0$ so that if for any $n \in \mathbb{N}$, if $(x,y) \in B_{\mathbb{R}^2_\infty}(D_n + (2^{-n-1}, 2^{-n-1}), \delta 2^{-n})$ and $t \in (1/2, 2)$, then*

$$d_2((x,y,z),(x,y,z+t4^{-n})) \geq \frac{1}{\sqrt{3}} d_b(0,(0,0,t4^{-n})), \qquad \forall z \in \mathbb{R}.$$

Proof. Let $(a,b) \in D_n + (2^{-n-1}, 2^{-n-1})$ be so that $(x,y) \in B_{\mathbb{R}^2_\infty}((a,b), \delta 2^{-n})$ (δ to be chosen later) and let $\mathbf{x} = (x_0, ..., x_N)$ be an itinerary from (x,y,z) to $(x,y,z+t4^{-n})$.

Suppose first that there is some $\pi(x_j) \notin B_{\mathbb{R}^2_\infty}((a,b), 2^{-n-1})$. As any non-vertical movement is not a shortcut, we have from the fact that $\pi(x_0) = \pi(x_N) \in B_{\mathbb{R}^2_\infty}((a,b), \delta 2^{-n})$ that if we choose δ sufficiently small, then

$$c(\mathbf{x}) \geq (1-2\delta)2^{-n} \geq \frac{1}{\sqrt{3}} \sqrt{2} \cdot 2^{-n} \geq \frac{1}{\sqrt{3}} \sqrt{t4^{-n}} = \frac{1}{\sqrt{3}} d_b(0,(0,0,t4^{-n})).$$

This would prove the statement of the lemma. Thus, we may suppose that the projection of $\mathbf{x}$ under π does not go outside $B_{\mathbb{R}_2^\infty}((a,b), 2^{-n-1})$.

But now the proof is reduced to that of the proof of Lemma 6.4. Indeed, by the hypothesis of this subcase, the itinerary $\mathbf{x}$ cannot contain any level n shortcuts and so the $c(\mathbf{x})$ bound, which lower bounds $\tilde{c}(\mathbf{x})$, is enough. □

Lemma 6.8. *For every $\epsilon > 0$ there exists $\eta \in (0, 1/2)$ so that if $|t| < \eta$ and $a(n) = 1$, then for all $(x,y) \in B_{\mathbb{R}^2_\infty}(D_n, \eta 2^{-n})$ and $z \in \mathbb{R}$ we have*

$$d_2((x,y,z),(x,y,z+(1+t)4^{-n})) \leq \left(\frac{1}{2} + \epsilon\right) d_b(0,(0,0,(1+t)4^{-n})).$$

This follows by essentially the same proof as Lemma 6.5.

Since the shortcuts are horizontally located on the centers of the dyadic cubes, we have from an easy argument using Lemmas 6.7 and 6.8 that no blow-up of d_2 is left-invariant. Indeed, similarly as in the proof of Proposition 6.6, we can find a shortcut of distance $s \in [1, 2^4]$ at the limit, that is $s \in \bigcap_{j=1}^{\infty} \overline{\bigcup_{i \geq j}\{\lambda_i^{-1} 2^{-4(k+1)} : k \in \mathbb{N}\}}$. This time, instead of finding a limit point $(0, 0, s^2)$ of shortcuts from the origin, we find by compactness a pair of points (x, y, z), $(x, y, z+s^2)$ in $B_{d_b}(0, 2^5)$ appearing as the limit of endpoints of a sequence of shortcuts. Then one can get a subsequence $\lambda_{i_m} \to \infty$ so that $s = \lim_{m\to\infty} \lambda_{i_m}^{-1} 2^{-4(k_m+1)}$. Remembering that $a(4(k_m + 1)) = 1$, Lemma 6.8 tells us that

$$\rho((x, y, z), (x, y, z + s^2)) \leq \left(\frac{1}{2} + \epsilon\right) d_b(0, (0, 0, s^2)).$$

One then considers the points $(s/2, s/2, 0)$ and $(s/2, s/2, s^2)$. Note that $s/2 = \lim_{m\to\infty} \lambda_{i_m}^{-1} 2^{-4(k_m+1)-1}$. Thus, for m sufficiently large, $\lambda_{i_m} \cdot (s/2, s/2) \in B(D_{4(k_m+1)} + (2^{-4(k_m+1)-1}, 2^{-4(k_m+1)-1}), \delta 2^{-4(k_m+1)})$ and so we get that

$$\rho((x + s/2, y + s/2, x + 0), (x + s/2, y + s/2, z + s^2)) \geq \frac{1}{\sqrt{3}} d_b(0, (0, 0, s^2)).$$

In order to see that no blow-up of d_2 is self-similar we argue similarly as for the distance d_1. First suppose that a blow-up is self-similar with some constant $\lambda > 1$. As noted above, we find by compactness a pair of points (x, y, z), $(x, y, z+s^2)$ in $B_{d_b}(0, 2^5)$ with $s \in [1, 2^4]$ appearing as the limit of endpoints of a sequence of shortcuts. Observe that Lemma 6.3 holds also for d_2 since $d_2 \geq d_1$. As in the case of d_1, it then follows via Lemma 6.3 that $\lambda = 2^\ell$ for some $\ell \in \mathbb{N}$. A contradiction with self-similarity then follows again by the properties of the function a. This concludes the proof of Theorem 1.5.

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