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Controlled diffeomorphic extension of homeomorphisms

Pekka Koskela, Zhuang Wang, and Haiqing Xu

Abstract

Let Ω be an internal chord-arc Jordan domain and $\varphi : \mathbb{S} \to \partial\Omega$ be a homeomorphism. We show that φ has finite dyadic energy if and only if φ has a diffeomorphic extension $h : \mathbb{D} \to \Omega$ which has finite energy.

Keywords: Poisson extension, diffeomorphism, chord-arc curve.

1 Introduction

Let $\Omega \subset \mathbb{C}$ be a bounded convex domain and suppose that φ is a homeomorphism from the unit circle $\mathbb{S}$ onto $\partial\Omega$. Then, by [7], the complex-valued Poisson extension h of φ is a homeomorphism from $\bar{\mathbb{D}}$ onto $\bar{\Omega}$. This harmonic map h is a diffeomorphism in $\mathbb{D}$ but its derivatives are not necessarily uniformly bounded. In 2007, G. C. Verchota [10] proved that the derivatives of h may fail to be square integrable but that they are necessarily p-integrable over $\mathbb{D}$ for any $p < 2$. In 2009, T. Iwaniec, G. Martin and C. Sbordone improved on [6] by showing that the derivatives belong to weak-L^2 with sharp estimates. In a related work [1] by K. Astala, T. Iwaniec, G. Martin and J. Onninen, it was shown that if additionally $\partial\Omega$ is a C^1-regular Jordan curve, the square integrability of the derivatives of h is equivalent to the requirement that

$$\int_{\partial\Omega}\int_{\partial\Omega} |\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|||d\xi||d\eta| < +\infty.$$

In this note we give a generalization of the aforementioned results. Towards this end, recall that the Poisson extension of a homeomorphism $\varphi : \mathbb{S} \to \partial\Omega$ may fail to be injective if Ω is not convex. Next, the boundary $\partial\Omega$ of a bounded convex domain Ω is a *chord-arc Jordan curve*: $\partial\Omega$ is a rectifiable Jordan curve and there is a constant C such that for all $w_1, w_2 \in \partial\Omega$,

$$\ell(w_1, w_2) \le C|w_1 - w_2|,$$

where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial\Omega$ joining w_1 to w_2. A domain whose boundary is a chord-arc Jordan curve is called a *chord-arc Jordan domain*.

Hence we are lead to ask for the optimal regularity of homeomorphic extensions $h : \mathbb{D} \to \Omega$ for a given homeomorphism $\varphi : \mathbb{S} \to \partial\Omega$ and a chord-arc Jordan domain Ω.

Before introducing our first result, let us fix a dyadic decomposition $\{\Gamma_{j,k} : j \in \mathbb{N}, k = 1, \cdots, 2^j\}$ of $\mathbb{S}$, such that for a fixed $j \in \mathbb{N}$, $\{\Gamma_{j,k} : k = 1, \cdots, 2^j\}$ is a family of arcs of length $2\pi/2^j$ with $\bigcup_k \Gamma_{j,k} = \mathbb{S}$. The next generation is constructed in such a way that for each $k \in \{1, \cdots, 2^{j+1}\}$, there exists a unique number $k' \in \{1, \cdots, 2^j\}$, satisfying $\Gamma_{j+1,k} \subset \Gamma_{j,k'}$. Here, we call $\Gamma_{j,k'}$ the parent of $\Gamma_{j+1,k}$. Fix $\Gamma_{1,1}$ to be the image of $[0, \pi]$ under the map $\theta \mapsto e^{i\theta}$. We denoted by $\ell(\varphi(\Gamma_{j,k}))$ the arc length of the image arc of $\Gamma_{j,k}$ under the homeomorphism φ.

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be an internal chord-arc Jordan domain and suppose that $\varphi : \mathbb{S} \to \partial\Omega$ is a homeomorphism. Then for $\lambda \in (-1, +\infty)$, the following are equivalent:*

(i) $\int_{\mathbb{D}} |Dh(z)|^2 \log^{\lambda}(e + |Dh(z)|)\, dz < +\infty$ for some diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ;

(ii) $\int_{\mathbb{D}} |Dh(z)|^2 \log^{\lambda}(\frac{2}{1-|z|})\, dz < +\infty$ for some diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ;

(iii) $\int_{\partial\Omega} \int_{\partial\Omega} |\log|\varphi^{-1}(\eta) - \varphi^{-1}(\xi)||^{\lambda+1}\, |d\eta|\, |d\xi| < +\infty$;

(iv) $\sum_{j=1}^{\infty} j^{\lambda} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 < +\infty$;

(v) $\sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^{\lambda}\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) < +\infty$.

A Jordan domain $\Omega \subset \mathbb{C}$ is an *internal chord-arc Jordan domain* if $\partial\Omega$ is rectifiable and there is a constant $C > 0$ such that for all $w_1, w_2 \in \partial\Omega$,

$$\ell(w_1, w_2) \le C\lambda_{\Omega}(w_1, w_2), \tag{1.1}$$

where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial\Omega$ joining w_1 to w_2 and $\lambda_{\Omega}(w_1, w_2)$ is the *internal distance* between w_1, w_2, which is defined as

$$\lambda_{\Omega}(w_1, w_2) = \inf_{\alpha} \ell(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha \subset \Omega$ joining w_1 and w_2; if there is no rectifiable curve joining w_1 and w_2, we let $\lambda_{\Omega}(w_1, w_2) = \infty$; cf. [9, Section 3.1] or [2, Section 2].

Naturally, every chord-arc Jordan domain is also an internal chord-arc domain, but there are internal chord-arc domains that fail to be chord-arc; e.g., the inward cusp domain:

$$\Omega_{\epsilon} = \mathbb{D} \setminus \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, |y| \le x^2\}.$$

The statement of Theorem 1 does not allow for $\lambda \le -1$. Actually, (i)-(v) all hold for $\lambda < -1$, independently of φ. When $\lambda = -1$, (iii) needs to be reformulated via a

double logarithm, after which one still has a list of mutually equivalent conditions that may or may not hold, see [11].

In the case when $\Omega = \mathbb{D}$, Theorem 1 can be formulated via the harmonic Poisson extension.

Theorem 2. *Let $\varphi : \mathbb{S} \to \mathbb{S}$ be a homeomorphism and $h = P[\varphi] : \mathbb{D} \to \mathbb{D}$ be the harmonic Poisson extension of φ. Then for $\lambda \in (-1, +\infty)$ the following are equivalent:*

(i) $\int_{\mathbb{D}} |Dh(z)|^2 \log^{\lambda}(e + |Dh(z)|)\, dz < +\infty$;

(ii) $\int_{\mathbb{D}} |Dh(z)|^2 \log^{\lambda}(\frac{2}{1-|z|})\, dz < +\infty$;

(iii) $\int_{\mathbb{S}} \int_{\mathbb{S}} |\log |\varphi^{-1}(\eta) - \varphi^{-1}(\xi)||^{\lambda+1} |d\eta|\, |d\xi| < +\infty$;

(iv) $\sum_{j=1}^{\infty} j^{\lambda} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 < +\infty$;

(v) $\sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^{\lambda}\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) < +\infty$.

Our main task is actually to prove Theorem 2. Indeed, once we know that Theorem 2 holds, Theorem 1 is obtained via a suitable change of variable, relying on the fact that there is a bi-Lipschitz map from $\bar{\mathbb{D}}$ onto $(\bar{\Omega}, \lambda_{\Omega})$; see Section 3.2.

The paper is organized as follows. In Section 2 we give some relevant facts about the dyadic decomposition of $\mathbb{S}$ and some properties of the N-function $\Phi(t) = t^2 \log^{\lambda}(e + t)$ for $\lambda > -1$. Section 3 contains the full proofs of Theorem 1 and Theorem 2.

2 Preliminaries

2.1 Dyadic decomposition

Since our dyadic decomposition $\{\Gamma_{j,k} : j \in \mathbb{N}, k = 1, \cdots, 2^j\}$ of $\mathbb{S}$ satisfies that $\Gamma_{1,1}$ is the image of $[0, \pi]$ under the map $\theta \mapsto e^{i\theta}$, we may assume that the dyadic decomposition $\{I_{j,k} = [2\pi(k-1)/2^j, 2\pi k/2^j] : j \in \mathbb{N}, k = 1, \cdots, 2^j\}$ of the interval $[0, 2\pi]$ matches with $\{\Gamma_{j,k}\}$ in the sense that each $\Gamma_{j,k}$ is exactly the image of $I_{j,k}$ under the map $\theta \mapsto e^{i\theta}$. Moreover, the dyadic arc $\Gamma_{j,k}$ is called a j-level dyadic arc, and we denote the two end points of $\Gamma_{j,k}$ by $\xi_{j,k}$ and $\xi_{j,k+1}$. Two dyadic arcs which are j-level dyadic arcs for some $j \in \mathbb{N}$ are called brother dyadic arcs if they have a common parent.

For the dyadic decompositions $\{\Gamma_{j,k}\}$ of $\mathbb{S}$ and $\{I_{j,k}\}$ of $[0, 2\pi]$ as above, we have a decomposition of the unit disk $\mathbb{D}$ given by $\{Q_{j,k} : j \in \mathbb{N}, k = 1, \cdots, 2^j\}$ where

$$Q_{j,k} = \left\{re^{i\theta} : 1 - 1/2^{j-1} \le r \le 1 - 1/2^j, \theta \in I_{j,k}\right\}$$

for any $j \in \mathbb{N}$ and $k = 1, \cdots, 2^j$, see Figure 1. It is easy to see that $\operatorname{dist}(Q_{j,k}, \mathbb{S}) = 2^{-j}$ and that if $P : \mathbb{D} \to \mathbb{S}$ is the radial projection map, then $P(Q_{j,k}) = \Gamma_{j,k}$ for all $j \in \mathbb{N}$

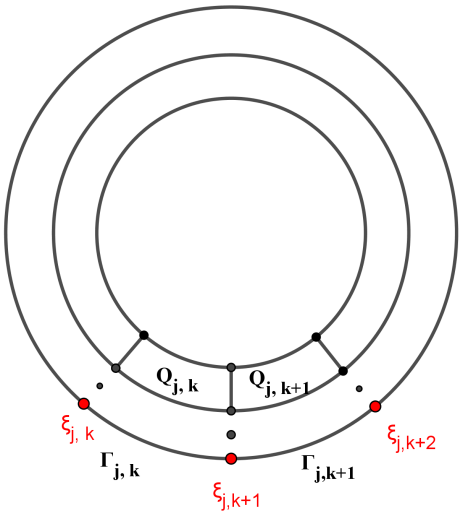

Figure 1: The decompositions of $\mathbb{S}$ and $\mathbb{D}$

and $k = 1, \cdots, 2^j$. There is an uniform constant $C > 0$ such that for any $Q_{j,k}$ with center $x_{j,k} = r_{j,k}e^{i\theta_{j,k}}$, $r_{j,k} = 1 - 3/2^{j+1}$ and $\theta_{j,k} = \pi(2k-1)/2^j$, we have

$$B(x_{j,k}, C^{-1}\mathrm{diam}(Q_{j,k})) \subset Q_{j,k} \subset B(x_{j,k}, C\mathrm{diam}(Q_{j,k})).$$

So for any $Q_{j,k}$, we can find a disk $B_{j,k}$ satisfying $B_{j,k} \subset Q_{j,k} \subset CB_{j,k}$, where C is a constant independent of $Q_{j,k}$.

In order to later estimate the integral of $|Dh|$ over $Q_{j,k}$, we employ the following decomposition of $\mathbb{S}$ via the dyadic arcs $\Gamma_{i,l}$ with $i \leq j$. The idea is to build a dyadic-type annular decomposition around $\Gamma_{j,k}$.

We fix $\Gamma_{j,k}$ and construct the decomposition via the following steps. For simplicity, we assume that the unique brother arc of $\Gamma_{j,k}$ is $\Gamma_{j,k+1}$ and located at the anticlockwise side of $\Gamma_{j,k}$.

Step 1. On the anticlockwise side of $\Gamma_{j,k+1}$, we choose the unique arc $\Gamma_{j-1,m^1_{j-1}}$ so that $\Gamma_{j,k+1} \cap \Gamma_{j-1,m^1_{j-1}}$ is a singleton. If the arc $\Gamma_{j-1,m^1_{j-1}+1}$ shares the common parent with $\Gamma_{j-1,m^1_{j-1}}$, then set $(\Gamma_{j-1,m^1_{j-1}+1}) = \Gamma_{j-1,m^1_{j-1}+1}$. If not, set $(\Gamma_{j-1,m^1_{j-1}+1}) = \emptyset$. Then define $\Gamma_{j-1,m^2_{j-1}}$ and $(\Gamma_{j-1,m^2_{j-1}-1})$ analogously along the clockwise side of $\Gamma_{j,k}$.

Step 2. Repeat the process in Step 1 with $\Gamma_{j,k+1}$ replaced by $\Gamma_{j-1,m^1_{j-1}}$ if we have $(\Gamma_{j-1,m^1_{j-1}+1}) = \emptyset$ or $\Gamma_{j-1,m^1_{j-1}+1}$ otherwise, and with $\Gamma_{j,k}$ replaced by $\Gamma_{j-1,m^2_{j-1}}$ if $(\Gamma_{j-1,m^2_{j-1}-1}) = \emptyset$ or $\Gamma_{j-1,m^2_{j-1}-1}$ otherwise; unless we get $\Gamma_{j_0,m^1_{j_0}}$, $(\Gamma_{j_0,m^1_{j_0}+1})$, $\Gamma_{j_0,m^2_{j_0}}$ and $(\Gamma_{j_0,m^2_{j_0}-1})$ such that

$$\left(\Gamma_{j_0,m^1_{j_0}} \cup (\Gamma_{j_0,m^1_{j_0}+1})\right) \bigcap \left(\Gamma_{j_0,m^2_{j_0}} \cup (\Gamma_{j_0,m^2_{j_0}-1})\right) \neq \emptyset.$$

This procedure decomposes $\mathbb{S}$:

$$\mathcal{P}(\Gamma_{j,k}) = \big\{\Gamma_{j,k}, \Gamma_{j,k+1}, \Gamma_{j-1,m^1_{j-1}}, (\Gamma_{j-1,m^1_{j-1}+1}), \Gamma_{j-1,m^2_{j-1}}, (\Gamma_{j-1,m^2_{j-1}-1}),$$
$$..., \Gamma_{j_0,m^1_{j_0}}, (\Gamma_{j_0,m^1_{j_0}+1}), \Gamma_{j_0,m^2_{j_0}}, (\Gamma_{j_0,m^2_{j_0}-1})\big\}.$$

where each $(\cdot)$ is either empty or a dyadic arc. Correspondingly, we obtain a decomposition of $[0, 2\pi]$ as

$$\mathcal{P}(I_{j,k}) = \big\{I_{j,k}, I_{j,k+1}, I_{j-1,m^1_{j-1}}, (I_{j-1,m^1_{j-1}+1}), I_{j-1,m^2_{j-1}}, (I_{j-1,m^2_{j-1}-1}),$$
$$..., I_{j_0,m^1_{j_0}}, (I_{j_0,m^1_{j_0}+1}), I_{j_0,m^2_{j_0}}, (I_{j_0,m^2_{j_0}-1})\big\}.$$

Let us illustrate this procedure by an example, see Figure 2 below. We use a binary tree to represent the dyadic decomposition $\{\Gamma_{j,k}\}$ of $\mathbb{S}$. Suppose that we are given a dyadic arc $\Gamma_{4,7}$. Then $\Gamma_{4,8}$ is the unique brother arc of $\Gamma_{4,7}$. In Step 1, we choose $\Gamma_{3,5}$ and $\Gamma_{3.6}$ on the anticlockwise side of $\Gamma_{4,8}$, and $\Gamma_{3,3}$ on the clockwise side of $\Gamma_{4,7}$. In Step 2, we choose $\Gamma_{2,4}$ on the anticlockwise side of $\Gamma_{3,6}$, and $\Gamma_{2,1}$ on the clockwise side of $\Gamma_{3,3}$. There will be no more steps because $\Gamma_{2,1} \cap \Gamma_{2,4} \neq \emptyset$.

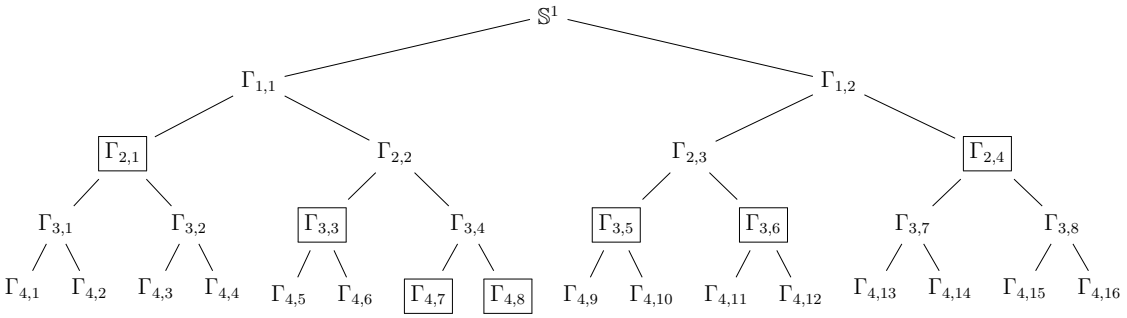

Figure 2: The decomposition of $\mathbb{S}$ around $\Gamma_{4,7}$

Suppose we are given an n-level arc $\Gamma_{n,m}$. We are interested in the number of j-level arcs, $j \geq n$, which can induce $\Gamma_{n,m}$ by the above decomposition method. Here, we say that a dyadic arc $\Gamma_{j,l}$ induces $\Gamma_{n,m}$ by the above decomposition method if and only if $\Gamma_{n,m}$ is in the set $\mathcal{P}(\Gamma_{j,l})$.

To begin, $\Gamma_{n,m}$ can be first induced by its brother (without loss of generality assume that it is $\Gamma_{n,m+1}$), and two couples of $(n+1)$-level arcs. Notice that among all $(n+1)$-level arcs, there is only one couple that can induce both $\Gamma_{n,m}$ and $\Gamma_{n,m+1}$. It follows that $\Gamma_{n,m}$ can be induced by three couples of $(n+1)$-level arcs. Choose one of these $(n+1)$-level arcs. There is only one couple of $(n+2)$-level arcs which induce both $\Gamma_{n,m}$ and the chosen $(n+1)$-level arcs. Then $\Gamma_{n,m}$ can be induced by six couples of $(n+2)$-level arcs. Generally, for $j \geq n$ we have

$$\sharp\{\Gamma : \Gamma \text{ is a } j\text{-level dyadic arc and induce } \Gamma_{n,m}\} \leq 3 \cdot 2^{j-n}. \tag{2.1}$$

2.2 N-functions

A function $\Phi : [0,\infty) \to [0,\infty)$ is an *N-function* if it is a continuous, increasing and convex function satisfying $\Phi(0) = 0$,

$$\lim_{t\to 0+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty.$$

An N-function Φ can be expressed as

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0,\infty) \to [0,\infty)$ is an increasing, right-continuous function with $\phi(0) = 0$ and $\lim_{t\to+\infty} \phi(t) = +\infty$.

For each N-function Φ and $t \geq 0$, set

$$\psi(t) = \sup_{\phi(s)\leq t} s \quad \text{and} \quad \Psi(t) = \int_0^t \psi(s)\, ds.$$

Then we call Ψ the *complementary function* of Φ. The complementary function of an N-function is also an N-function. We call Ψ, Φ a *pair of complementary N-functions*.

An N-function Φ is said to satisfy the Δ_2-condition if there is a constant $C_\Phi > 0$, called a *doubling constant* of Φ, such that

$$\Phi(2t) \leq C_\Phi \Phi(t), \ \forall \ \ t \geq 0.$$

Proposition 1. *If an N-function Φ satisfies the Δ_2-condition, then for any constant $c > 0$, there exist $c_1, c_2 > 0$ such that*

$$c_1\Phi(t) \leq \Phi(ct) \leq c_2\Phi(t) \quad \text{for all} \quad t \geq 0,$$

where c_1 and c_2 depend only on c and the doubling constant C_Φ. Therefore, we obtain that if $A \approx B$, then $\Phi(A) \approx \Phi(B)$.

Lemma 1. ([8, Theorem 4.2]) *The complementary function of an $N-$function Φ satisfies the Δ_2-condition on $[0,+\infty)$ if there is a constants $l > 1$ such that*

$$\Phi(t) \leq \frac{1}{2l}\Phi(lt) \ \ \textit{for any } t \geq 0.$$

Given an $N-$function Φ, we denote by L^*_Φ the collection of all measurable functions f such that $\int_{\mathbb{R}^n} \Phi(af) < \infty$ for some $a > 0$.

For a measurable function f on $\mathbb{R}^2$, we define the Hardy-Littlewood maximal function of f by setting

$$M_f(x) = \sup \fint_B |f(z)|\, dz = \sup \frac{1}{|B|} \int_B |f(z)|\, dz,$$

where the supremum is taken over all open disks B that contain x.

Proposition 2. ([3, Theorem 2.1]) *Let Φ and Ψ be a pair of complementary N-functions. The following two conditions are equivalent:*

(i) there exists positive constants C and b such that

$$\int_{\mathbb{R}^n} \Phi(bM_f)(z)dz \leq C \int_{\mathbb{R}^n} \Phi(|f|)(z)dz, \quad \forall\ f \in L^*_\Phi,$$

(ii) Ψ satisfies the Δ_2-condition on $[0, +\infty)$.

Example 1. Denote by Φ the function $\Phi(t) = t^2 \log^\lambda(e + t)$ for $\lambda > -1$. An elementary computation shows that Φ is increasing, continuous and convex on $[0, +\infty)$ with $\lim\limits_{t\to 0+} \frac{\Phi(t)}{t} = 0$ and $\lim\limits_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty$. So Φ is an $N-$function. Moreover, both the function Φ and its complementary function satisfy the Δ_2-condition. Hence we know that Φ satisfies Proposition 1 and 2. We will use Proposition 1 frequently in Section 3.1.

Actually, a direct computation shows that the above Φ satisfies the Δ_2-condition on $[0, +\infty)$. In order to check that the complementary $N-$function of Φ satisfies the Δ_2-condition, by Proposition 1, we only need to find a constant $l > 1$ so that

$$2\log^\lambda(e + t) \leq l \log^\lambda(e + lt),\ \forall\ t \geq 0. \tag{2.2}$$

In fact, if $\lambda \geq 0$ we can take $l = 2$. By monotonicity of $\log^\lambda(e + \cdot)$, we have that inequality (2.2) holds for any $t \geq 0$. If $-1 < \lambda < 0$, we take $l = 2^{1/(1+\lambda)}$. By monotonicity we have

$$\log(e + lt) \leq l \log(e + t),\ \forall t \geq 0.$$

Together with $l = (2/l)^{1/\lambda}$, it follows that (2.2) holds for all $t \geq 0$.

3 Proofs of Theorem 1 and Theorem 2

In this section, the notation $A \lesssim B$ means that there is a constant $C > 0$ so that $A \leq C \cdot B$. Here and in this section, the notation C denotes a positive constant which may differ from line to line. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

3.1 Proof of Theorem 2

Proof of (iv)⇔(v). Our claim is obvious when $\lambda = 0$.

Case $\lambda > 0$. Assume (iv) holds. Since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ for any $\Gamma_{j,k}$ and $\lambda > 0$, we have

$$\log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) \leq \log^\lambda\left(e + 2^{j-1}\pi\right) \lesssim j^\lambda.$$

Therefore, we get

$$\sum_{j=1}^{\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2\log^{\lambda}\left(e+\frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right)\lesssim\sum_{j=1}^{\infty}j^{\lambda}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2<+\infty,$$

which gives the implication (iv)⇒(v). For the other direction, assume that (v) holds. In order to estimate the logarithmic term from below, we set

$$\chi(j,k)=\begin{cases}1, & \text{if } \ \ell(\varphi(\Gamma_{j,k}))>2^{-\frac{3j}{4}};\\ 0, & \text{otherwise.}\end{cases} \tag{3.1}$$

Then

$$\begin{aligned}\sum_{j=1}^{\infty}j^{\lambda}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2 &= \sum_{j=1}^{\infty}j^{\lambda}\sum_{k=1}^{2^j}\chi(j,k)\ell(\varphi(\Gamma_{j,k}))^2\\ &\quad+\sum_{j=1}^{\infty}j^{\lambda}\sum_{k=1}^{2^j}(1-\chi(j,k))\ell(\varphi(\Gamma_{j,k}))^2=:P_1+P_2.\end{aligned}$$

If $\ell(\varphi(\Gamma_{j,k}))>2^{-\frac{3j}{4}}$, then we have

$$\log^{\lambda}\left(e+\frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right)\geq\log^{\lambda}(e+2^{\frac{1}{4}j})\gtrsim j^{\lambda}.$$

Hence, we obtain from (v) that

$$P_1\lesssim\sum_{j=1}^{\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2\log^{\lambda}\left(e+\frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right)<+\infty.$$

For P_2, we have

$$P_2\leq\sum_{j=1}^{\infty}j^{\lambda}\sum_{k=1}^{2^j}2^{-\frac{3}{2}j}=\sum_{j=1}^{+\infty}2^{-\frac{1}{2}j}j^{\lambda}<+\infty.$$

In conclusion, $P_1+P_2<+\infty$, and (iv) follows.

Case $\lambda<0$. Assume that (v) holds. Since $\ell(\varphi(\Gamma_{j,k}))\leq 2\pi$ for any $\Gamma_{j,k}$ and $\lambda<0$, we have

$$\log^{\lambda}\left(e+\frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right)\geq\log^{\lambda}\left(e+2^{j-1}\pi\right)\gtrsim j^{\lambda}.$$

Therefore, we get

$$\sum_{j=1}^{\infty}j^{\lambda}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2\lesssim\sum_{j=1}^{\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2\log^{\lambda}\left(e+\frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right)<+\infty,$$

which gives us the implication (v)⇒(iv). For the other direction, assume that (iv) holds. Using the equation (3.1), if $\ell(\varphi(\Gamma_{j,k})) > 2^{-3j/4}$, we have

$$\log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) \leq \log^\lambda(e + 2^{\frac{1}{4}j}) \lesssim j^\lambda.$$

Therefore, we obtain

$$P_1' := \sum_{j=1}^{\infty}\sum_{k=1}^{2^j}\chi(j,k)\ell(\varphi(\Gamma_{j,k}))^2\log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) \lesssim \sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2 < +\infty.$$

Moreover, since $\ell(\varphi(\Gamma_{j,k})) \geq 0$ and $\lambda < 0$, we always have

$$\log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) \leq 1.$$

Hence, we obtain

$$\begin{aligned} P_2' &:= \sum_{j=1}^{\infty}\sum_{k=1}^{2^j}(1-\chi(j,k))\ell(\varphi(\Gamma_{j,k}))^2\log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{2^j} 2^{-\frac{3}{2}j} = \sum_{j=1}^{\infty} 2^{-\frac{1}{2}j} < +\infty. \end{aligned}$$

Then $P_1' + P_2' < +\infty$ which gives us (v).

By combining the cases $\lambda = 0$, $\lambda > 0$ with $\lambda < 0$ above, we finish the proof of (iv)⇔(v). □

Proof of (iv)⇒(i). The harmonic extension of φ is given by the Poisson integral formula:

$$h(z) = \frac{1}{2\pi}\int_{\mathbb{S}} \frac{1-|z|^2}{|z-\zeta|^2}\varphi(\zeta)\,|d\zeta|.$$

We can therefore compute the z-derivative by differentiating this kernel:

$$h_z(z) = \frac{\partial h}{\partial z}(z) = \frac{1}{2\pi}\int_{\mathbb{S}} \frac{\zeta}{(z-\zeta)^2}\varphi(\zeta)\,|d\zeta| = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}}{(z-e^{i\theta})^2}\varphi(e^{i\theta})\,d\theta. \tag{3.2}$$

We may write

$$\varphi(e^{i\theta}) = \varphi(1)\cdot e^{if(\theta)}$$

where $f : [0,2\pi] \to [0,2\pi]$ is continuous and increasing with $f(0) = 0$ and $f(2\pi) = 2\pi$. This allows us to rewrite (3.2) as

$$h_z(z) = \frac{\varphi(1)}{2\pi i}\int_0^{2\pi}\frac{\partial}{\partial\theta}\left[\frac{1}{(z-e^{i\theta})}\right]e^{if(\theta)}\,d\theta = \frac{\varphi(1)}{2\pi i}\int_0^{2\pi} e^{if(\theta)}d\left(\frac{1}{(z-e^{i\theta})}\right). \tag{3.3}$$

Here we use the Lebesgue-Stieltjes integration on the right-hand side of (3.3). By integrating the right-hand side of (3.3) by parts, we obtain the estimate

$$h_z(z) = \frac{\varphi(1)}{2\pi i}\int_0^{2\pi} \frac{1}{(z-e^{i\theta})} d\left(e^{if(\theta)}\right).$$

We therefore have

$$|h_z(z)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|z-e^{i\theta}|} \left|d\left(e^{if(\theta)}\right)\right|.$$

Denote by μ_f the Lebesgue-Stieltjes measure of the continuous increasing function f. Then we have $\left|d\left(e^{if(\theta)}\right)\right| \le d(f(\theta)) = d\mu_f(\theta)$. Therefore, we obtain the formula

$$|h_z(z)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|z-e^{i\theta}|} d\mu_f(\theta). \tag{3.4}$$

Using our decomposition of the unit disk $\mathbb{D}$ and the decomposition $\mathcal{P}(I_{j,k})$ of $[0,2\pi]$ in Section 2.1, for $\Phi(t) = t^2\log^{\lambda}(e+t)$, we get

$$\begin{aligned}
\int_{\mathbb{D}} \Phi(|h_z(z)|)\, dz &= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi(|h_z(z)|) dz \\
&\le \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi\left(\frac{1}{2\pi}\sum_{I\in\mathcal{P}(I_{j,k})}\int_I \frac{1}{|z-e^{i\theta}|} d\mu_f(\theta)\right) dz \\
&= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi\left(\frac{1}{2\pi}\sum_{n\le j}\sum_{m}\int_{I_{n,m}} \frac{1}{|z-e^{i\theta}|} d\mu_f(\theta)\right) dz, \qquad (3.5)
\end{aligned}$$

where we abuse the notation: sum over those m which belong to $\{1,\cdots,2^n\}$ and satisfy $I_{n,m}\in\mathcal{P}(I_{j,k})$. It is easy to check that $\#\{m\}\le 3$. For any $I_{n,m}\in\mathcal{P}(I_{j,k})$, we know that $|z-e^{i\theta}|\approx 2^{-n}$ for any $z\in Q_{j,k}$ and $\theta\in I_{n,m}$. Since μ_f is the Lebesgue-Stieltjes measure of our continuous increasing function f, for any interval $I_{n,m}$, we get

$$\int_{I_{n,m}} d\mu_f(\theta) = |f(I_{n,m})| = \ell(\varphi(\Gamma_{n,m})).$$

Hence we obtain

$$\begin{aligned}
\int_{\mathbb{D}} \Phi(|h_z(z)|)\, dz &\le \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi\left(\sum_{n\le j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\right) dz \\
&\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{-2j}\Phi\left(\sum_{n\le j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\right). \qquad (3.6)
\end{aligned}$$

Since $\ell(\varphi(\Gamma_{n,m})) \leq 2\pi$, we have

$$\sum_{n\leq j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \leq 2\pi\sum_{n\leq j}\sum_{m}\frac{1}{2^{-n}} \lesssim 2^j.$$

Using the same idea as in Proof of (iv)⇔(v), we have the estimate

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}2^{-2j}\Phi\left(\sum_{n\leq j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\right) \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}2^{-2j}j^{\lambda}\left(\sum_{n\leq j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\right)^2 + C.$$

This allows us to estimate (3.6) as

$$\int_{\mathbb{D}}\Phi(|h_z(z)|)\,dz \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}2^{-2j}j^{\lambda}\left(\sum_{n\leq j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\right)^2 + C. \tag{3.7}$$

For fixed $\varphi(\Gamma_{n,m})$ and fixed $j \geq n$, the estimate (2.1) gives us $\#\{k\} \leq 3\cdot 2^{j-n}$. By applying the Hölder inequality on the right-hand side of inequality (3.7) and Fubini's theorem for series, we obtain

$$\begin{aligned}
\int_{\mathbb{D}}\Phi(|h_z(z)|)\,dz &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}2^{-\frac{3}{2}j}j^{\lambda}\sum_{n\leq j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))^2}{2^{-\frac{3}{2}n}} + C\\
&\lesssim \sum_{n=1}^{+\infty}\sum_{m=1}^{2^n}\frac{\ell(\varphi(\Gamma_{n,m}))^2}{2^{-\frac{3}{2}n}}\left(\sum_{j\geq n}\sum_{k}2^{-\frac{3}{2}j}j^{\lambda}\right) + C\\
&\lesssim \sum_{n=1}^{+\infty}\sum_{m=1}^{2^n}\frac{\ell(\varphi(\Gamma_{n,m}))^2}{2^{-\frac{1}{2}n}}\underbrace{\left(\sum_{j\geq n}2^{-\frac{1}{2}j}j^{\lambda}\right)}_{\approx 2^{-\frac{1}{2}n}n^{\lambda}} + C\\
&\lesssim \sum_{n=1}^{+\infty}\sum_{m=1}^{2^n}n^{\lambda}\ell(\varphi(\Gamma_{n,m}))^2 + C < +\infty.
\end{aligned}$$

An analogous estimate follows for $h_{\bar{z}}$ by the same reasons. Thus we finish the proof of (iv)⇒(i). □

Proof of (iv)⇒(ii). Recall the estimate (3.4) for $|h_z|$ and formula (3.5). From (3.5) and the fact that $\log^{\lambda}(\frac{2}{1-|z|}) \approx j^{\lambda}$ for any $z \in Q_{j,k}$, we have

$$\int_{\mathbb{D}}|h_z(z)|^2\log^{\lambda}\left(\frac{2}{1-|z|}\right)dz = \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}}|h_z(z)|^2\log^{\lambda}(\frac{2}{1-|z|})\,dz$$

$$\approx \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} j^\lambda \int_{Q_{j,k}} |h_z(z)|^2\,dz.$$

By replacing $\Phi(t)=t^2\log^\lambda(e+t)$ by t^2 in (3.6), we obtain

$$\int_{Q_{j,k}} |h_z(z)|^2\,dz \lesssim 2^{-2j}\Big(\sum_{n\le j}\sum_m \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\Big)^2.$$

Hence we have

$$\int_{\mathbb{D}} |h_z(z)|^2 \log^\lambda\big(\frac{2}{1-|z|}\big)\,dz \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{-2j} j^\lambda \left(\sum_{n\le j}\sum_m \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\right)^2.$$

Consequently, we may apply the very same arguments that we used above to estimate the right-hand side of (3.7) so as to arrive at

$$\int_{\mathbb{D}} |h_z(z)|^2 \log^\lambda\big(\frac{2}{1-|z|}\big)\,dz \lesssim \sum_{n=1}^{+\infty}\sum_{m=1}^{2^n} n^\lambda \ell(\varphi(\Gamma_{n,m}))^2 < +\infty.$$

One may similarly deal with $h_{\bar z}$ to finish the proof of (iv)⇒(ii). □

Proof of (iv)⇔(iii). We first divide the double integral on $\mathbb{S}$ into two parts:

$$\begin{aligned}&\int_{\mathbb{S}}\int_{\mathbb{S}} \big|\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\big|^{\lambda+1}\,|d\eta||d\xi| \\ &= \int_{\mathbb{S}}\int_{\{\eta\in\mathbb{S}:1\le|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le 2\}} + \int_{\mathbb{S}}\int_{\{\eta\in\mathbb{S}:0\le|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|<1\}} = I+II. \qquad (3.8)\end{aligned}$$

Since $\lambda>-1$, we have $0\le|\log t|^{\lambda+1}\le(\log 2)^{\lambda+1}$ for $1\le t\le 2$. Therefore we obtain $0\le I\le C$. So our job is to estimate II.

Fubini's theorem changes II to

$$\begin{aligned}II &= \int_{\mathbb{S}}\int_{\{\eta\in\mathbb{S}:|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|<1\}}\int_0^1 \Lambda(t)\chi_{\{|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le t<1\}}dt|d\eta||d\xi| \\ &= \int_{\mathbb{S}}\int_0^1 \ell(\{\eta\in\mathbb{S}:|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le t\})\Lambda(t)dt|d\xi|, \qquad (3.9)\end{aligned}$$

where $\Lambda(t)=(\lambda+1)\log^\lambda\big(\frac{1}{t}\big)/t$. Here we used the fact that

$$|\log t|^{\lambda+1}=\log^{\lambda+1}\left(\frac{1}{t}\right)=\int_t^1\Lambda(t)dt,\ \text{for any}\ \ 0<t<1.$$

Then we have

$$\begin{aligned} II &= \sum_{j=0}^{+\infty}\int_{\mathbb{S}}\int_{2^{-j-1}}^{2^{-j}} \ell(\{\eta : |\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le t\})\Lambda(t)d\xi dt. \\ &\le C+\sum_{j=1}^{+\infty}\int_{\mathbb{S}} \ell(\{\eta : |\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le 2^{-j}\})|d\xi|\int_{2^{-j-1}}^{2^{-j}}\Lambda(t)dt. \end{aligned} \tag{3.10}$$

If $\xi\in\varphi(\Gamma_{j,k})$ and $|h^{-1}(\xi)-h^{-1}(\eta)|\le 2^{-j}$, then the arc length of the shorter arc from $h^{-1}(\xi)$ to $h^{-1}(\eta)$ is at most $2^{-j}\pi$ and hence it follows that $\eta\in\cup_{n=k-1}^{k+1}\varphi(\Gamma_{j,n})$. This means that for $j\ge 1$, we have

$$\begin{aligned} &\sum_{k=1}^{2^j}\int_{\varphi(\Gamma_{j,k})}\ell(\{\eta : |\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le 2^{-j}\})|d\xi| \\ &\qquad\le \sum_{k=1}^{2^j}\ell\big(\cup_{n=k-1}^{k+1}\varphi(\Gamma_{j,n})\big)\ell(\varphi(\Gamma_{j,k})\lesssim\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2. \end{aligned} \tag{3.11}$$

By the mean value theorem, we have

$$\int_{2^{-j-1}}^{2^{-j}}\Lambda(t)dt=\log^{\lambda+1}(2^{j+1})-\log^{\lambda+1}(2^j)\approx j^\lambda,\quad j\ge 1. \tag{3.12}$$

Combining (3.8), (3.9), (3.10), (3.11) and (3.12), we therefore deduce

$$\int_{\mathbb{S}}\int_{\mathbb{S}}|\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)||^{\lambda+1}|d\xi||d\eta|\lesssim\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2j^\lambda+C.$$

We are left to deal with the converse direction. We divide the interval $[0,1]$ in (3.9) into intervals $[\pi/8,1]$ and $[2^{-j+1}\pi,2^{-j+2}\pi]$ for $j\ge 5$. Then we have

$$II\ge\sum_{j=5}^{+\infty}\int_{\mathbb{S}}\ell(\{\eta : |\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le 2^{-j+1}\pi\})|d\xi|\int_{2^{-j+1}\pi}^{2^{-j+2}\pi}\Lambda(t)dt. \tag{3.13}$$

Given any $j\ge 5$ and $1\le k\le 2^j$, the inequality

$$|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\le\ell(\Gamma_{j,k})\le\pi 2^{1-j}$$

holds for all $\eta,\xi\in\varphi(\Gamma_{j,k})$. Thus we have

$$\sum_{k=1}^{2^j}\int_{\varphi(\Gamma_{j,k})}\ell(\{\eta : |h^{-1}(\xi)-h^{-1}(\eta)|\le\pi 2^{1-j}\})|d\xi|\ge\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2. \tag{3.14}$$

Finally, the estimates (3.8), (3.12), (3.13) and (3.14) yield

$$\int_{\mathbb{S}}\int_{\mathbb{S}} |\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)||^{\lambda+1}|d\xi||d\eta| + C \gtrsim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^{\lambda}.$$

□

Proof of (ii)⇒(iv). Since φ is uniformly continuous, there exists $j_0 \geq 3$ such that $\ell(\varphi(\Gamma_{j,k})) \leq \pi/3$ whenever $j \geq j_0$ and $k = 1, \cdots, 2^j$. Fix such j, k and let $\xi \in \Gamma_{j,k-1}$, $\eta \in \Gamma_{j,k+1}$ ($\Gamma_{j,0} = \Gamma_{j,2^j}$, $\Gamma_{j,2^j+1} = \Gamma_{j,1}$). Since $h \in C(\bar{\mathbb{D}})$ is a diffeomorphism, we have that

$$|h(\xi) - h(\eta)| \leq \int_{\gamma} |Dh|\, ds.$$

for any rectifiable curve $\gamma \subset \mathbb{D}$ joining ξ to η. Moreover, $|h(\xi_{j,k}) - h(\xi_{j,k+1})| \leq |h(\xi) - h(\eta)|$ because φ is homeomorphic and $\ell\big(\varphi(\Gamma_{j,k-1} \cup \Gamma_{j,k} \cup \Gamma_{j,k+1})\big) \leq \pi$. Denote by $\hat{\xi}$ the midpoint of the arc $\Gamma_{j,k}$. Let $t_1 > 0$ be the smallest t for which $\partial B(\hat{\xi}, t) \cap \Gamma_{j,k+1} \neq \emptyset$ and let t_2 be correspondingly the largest such t. Write $\gamma_t = \partial B(\hat{\xi}, t) \cap \mathbb{D}$ for $t_1 \leq t \leq t_2$. Then there exists an absolute constant C so that

$$\bigcup_{t\in[t_1,t_2]} \gamma_t \subset CQ_{j,k} \cap \mathbb{D}.$$

Now

$$|h(\xi_{j,k}) - h(\xi_{j,k+1})| \leq \int_{\gamma_t} |Dh|\, ds$$

for $t \in [t_1, t_2]$ and by integrating with respect to t we obtain

$$(t_2 - t_1)|h(\xi_{j,k}) - h(\xi_{j,k+1})| \leq \int_{t_1}^{t_2}\int_{\gamma_t} |Dh|\, ds\, dt \leq \int_{CQ_{j,k}\cap\mathbb{D}} |Dh(z)|\, dz. \tag{3.15}$$

Notice that $t_2 - t_1$ is uniformly comparable to $2^{-j} \approx \ell(\Gamma_{j,k})$ when $j \geq j_0$ and $k \in \{1, \cdots, 2^j\}$. It follows from (3.15) that

$$|h(\xi_{j,k}) - h(\xi_{j,k+1})| \lesssim \frac{1}{\ell(\Gamma_{j,k})} \int_{CQ_{j,k}\cap\mathbb{D}} |Dh(z)|\, dz. \tag{3.16}$$

Recall that we have $\ell(\varphi(\Gamma_{j,k})) \leq \pi/3$. Thus $\ell(\varphi(\Gamma_{j,k}))$ is also uniformly comparable to $|\varphi(\xi_{j,k}) - \varphi(\xi_{j,k+1})|$. We can therefore conclude from (3.16) that for any $j \geq j_0$ and $1 \leq k \leq 2^j$, the following inequality holds:

$$\ell(\varphi(\Gamma_{j,k})) \lesssim \frac{1}{\ell(\Gamma_{j,k})} \int_{CQ_{j,k}\cap\mathbb{D}} |Dh(z)|\, dz. \tag{3.17}$$

Fix $p \in (1,2)$ and $q \in (2,+\infty)$ with $1/p + 1/q = 1$. By applying the Hölder inequality to (3.17), we have

$$\ell(\varphi(\Gamma_{j,k}))^p \lesssim \frac{1}{\ell(\Gamma_{j,k})^p} \int_{CQ_{j,k}} H(z)dz \left(\int_{CQ_{j,k}\cap\mathbb{D}} \log^{-\frac{\lambda q}{2}} \left(\frac{1}{1-|z|} \right) dz \right)^{p/q},$$

where $H(z) = |Dh(z)|^p \log^{\lambda p/2} \left(\frac{1}{1-|z|} \right) \chi_{\mathbb{D}}(z)$. Moreover by changing to polar coordinates we have the estimate

$$\begin{aligned} \int_{CQ_{j,k}\cap\mathbb{D}} \log^{-\frac{\lambda q}{2}} \left(\frac{1}{1-|z|} \right) dz &\lesssim \ell(\Gamma_{j,k}) \int_0^{C\ell(\Gamma_{j,k})} \log^{-\frac{\lambda q}{2}} \left(\frac{1}{s} \right) ds \\ &\lesssim \ell(\Gamma_{j,k})^2 \log^{-\frac{\lambda q}{2}} \left(\frac{1}{\ell(\Gamma_{j,k})} \right) \\ &\approx \ell(\Gamma_{j,k})^2 j^{-\lambda q/2}. \end{aligned}$$

It follows that

$$\ell(\varphi(\Gamma_{j,k}))^p j^{\lambda p/2} \lesssim \ell(\Gamma_{j,k})^p \fint_{CQ_{j,k}\cap\mathbb{D}} H(z)dz. \tag{3.18}$$

Next by the inclusion relationship between $B_{j,k}$ and $Q_{j,k}$ and the definition of Hardy-Litterwood maximal function we have

$$\fint_{CQ_{j,k}} H(z)dz \le \fint_{CB_{j,k}} H(z)dz \le \fint_{B_{j,k}} M_H(z)dz \le \fint_{Q_{j,k}} M_H(z)dz. \tag{3.19}$$

Combining (3.18) with (3.19) and then applying Jensen's inequality, we arrive at

$$\ell(\varphi(\Gamma_{j,k}))^2 j^{\lambda} \lesssim \int_{Q_{j,k}} M_H^{2/p}(z)dz, \quad \forall j \ge j_0 \text{ and } 1 \le k \le 2^j.$$

Then the $L^{2/p}$-boundedness of the Hardy-Littlewood maximal operator implies

$$\begin{aligned} \sum_{j=j_0}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^{\lambda} &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} M_H^{2/p}(z)\, dz \le \int_{\mathbb{R}^2} M_H^{2/p}(z)\, dz \\ &\lesssim \int_{\mathbb{D}} |Dh(z)|^2 \log^{\lambda} \left(\frac{1}{1-|z|} \right) dz. \end{aligned}$$

Moreover, we have that

$$\sum_{j=1}^{j_0-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^{\lambda} \le \sum_{j=1}^{j_0-1} j^{\lambda} \left(\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) \right)^2 \lesssim \sum_{j=1}^{j_0-1} j^{\lambda} \le C.$$

Hence we conclude

$$\sum_{j=0}^{+\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2 j^\lambda = \sum_{j=1}^{j_0-1}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2 j^\lambda + \sum_{j=j_0}^{+\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2 j^\lambda$$
$$\lesssim C + \int_{\mathbb{D}} |Dh(z)|^2 \log^\lambda\left(\frac{1}{1-|z|}\right) dz.$$

□

Proof of (i)⇒(v). Set $K(z) = |Dh(z)|\chi_{\mathbb{D}}(z)$. As in the proof of (ii) ⇒ (iv), we have

$$\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \lesssim \fint_{CQ_{j,k}} K(z)dz \quad \text{for any } j \geq j_0 \text{ and } 1 \leq k \leq 2^j.$$

Thus Jensen's inequality for the function $\Phi(t) = t^2\log^\lambda(e+t)$ and inequality (3.19) with H replaced by K imply that

$$\sum_{j=j_0}^{+\infty}\sum_{k=1}^{2^j}\ell(\Gamma_{j,k})^2\Phi\left(\frac{\ell(\phi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\right) \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\ell(\Gamma_{j,k})^2 \fint_{Q_{j,k}} \Phi(M_K)(z)dz$$
$$\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi(M_K)(z)dz \leq \int_{\mathbb{R}^2}\Phi(M_K)(z)dz. \tag{3.20}$$

Here in the second inequality, we used the fact that $\ell(\Gamma_{j,k})^2 \approx |Q_{j,k}|$. Therefore Proposition 2 gives

$$\int_{\mathbb{R}^2}\Phi(M_K)(z)dz \lesssim \int_{\mathbb{R}^2}\Phi(K)(z)dz. \tag{3.21}$$

Moreover, we have that

$$\sum_{j=1}^{j_0-1}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2\log^\lambda\left(e+\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\right) \lesssim \sum_{j=1}^{j_0-1}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2 j^\lambda \lesssim \sum_{j=1}^{j_0-1} j^\lambda \leq C.$$

Combining (3.20), (3.21) with the above inequality gives

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^2\log^\lambda\left(e+\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\right) \lesssim C + \int_{\mathbb{D}}|Dh(z)|^2\log^\lambda(e+|Dh(z)|)dz.$$

□

3.2 Proof of Theorem 1

Proof. From [2, Lemma 3.2], we know that any internal chord-arc Jordan domain $\Omega \subset \mathbb{C}$ is a bounded John disk whose boundary $\partial\Omega$ satisfies (1.1). Note also that in bounded John disks, the internal distance of any two boundary points is finite, [9, Remark 6.6]. By using the arc length parametrization of $\partial\Omega$ and the property (1.1), we see that there is a bi-Lipschitz map $g : \mathbb{S} \to (\partial\Omega, \lambda_\Omega)$. Then applying [2, Theorem 4.7] and the fact that the internal distance in the unit disk is the same as the Euclidean distance, we know that g extends to a bi-Lipschitz map $\tilde{g} : \bar{\mathbb{D}} \to (\bar{\Omega}, \lambda_\Omega)$. Moreover, the bi-Lipschitz map $\tilde{g}$ is a diffeomorphism in $\mathbb{D}$. Indeed, by the construction of the map $\tilde{g}$ in [2, Theorem 4.7], $\tilde{g} = \psi_1 \circ \tilde{f}$, where $\psi_1 : \mathbb{D} \to \Omega$ is a conformal map and $\tilde{f} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is a homeomorphic extension of $f = \psi_1^{-1} \circ g$ from [5, Lemma 2.10]. By the construction of $\tilde{f}$ in [5, Lemma 2.10], $\tilde{f} = \psi_2^{-1} \circ \bar{f} \circ \psi_2$, where $\psi_2 : \bar{\mathbb{D}} \to \bar{\mathbb{H}^2}$ is a Möbius map and $\bar{f} : \bar{\mathbb{H}^2} \to \bar{\mathbb{H}^2}$ is the Beurling-Ahlfors extension of the quasisymmetric map $\psi_2 \circ f \circ \psi_2^{-1}$. Since conformal maps, Möbius maps and Beurling-Ahlfors extensions of quasisymmetric maps are diffeomorphic (cf. [4, Chapter 8]), we obtain that the map $\tilde{g}$ is a diffeomorphism.

In the statements (i) and (ii) in Theorem 1, we let $h = \tilde{g} \circ P[g^{-1} \circ \varphi]$, where $P[g^{-1} \circ \varphi]$ is the Poisson extension of $g^{-1} \circ \varphi$. Hence h is a diffeomorphism. Since $\tilde{g}$ is bi-Lipschitz with respect to the internal distance and the internal distance λ_Ω is the same as the Euclidean distance locally in Ω, we obtain that there is a constant $C > 0$ so that $1/C \leq |D\tilde{g}(z)| \leq C$ for any $z \in \mathbb{D}$. Hence the convergence of the integrals of $|Dh|$ in (i) and (ii) in Theorem 1 is equivalent to the statements (i) and (ii) for $P[g^{-1} \circ \varphi]$ in Theorem 2.

By [2, Lemma 3.2, Lemma 3.4 and Lemma 3.6], we know that for any $z_1, z_2 \in \partial\Omega$, the shorter arc γ_{z_1,z_2} of $\partial\Omega$ joining z_1 and z_2 satisfies

$$\operatorname{diam}(\gamma_{z_1,z_2}) \approx \lambda_\Omega(z_1, z_2).$$

Hence the arc length $\ell(\cdot)$ on $\partial\Omega$ with respect to the Euclidean distance is comparable to the arc length $\ell_{\lambda_\Omega}(\cdot)$ on $\partial\Omega$ with respect to the internal distance λ_Ω. Combining with the bi-Lipschitz property of g, we obtain that $\ell(\varphi(\Gamma_{j,k}))/C \leq \ell(g^{-1} \circ \varphi(\Gamma_{j,k})) \leq C\ell(\varphi(\Gamma_{j,k}))$ for any $\Gamma_{j,k}$ and that we can use a change of variable via g to conclude the equivalence of (iii) for φ^{-1} and $\varphi^{-1} \circ g$ in Theorem 1 and 2. Hence the statements (iii)-(v) for φ in Theorem 1 is equivalent to the statements (iii)-(v) for $g^{-1} \circ \varphi$ in Theorem 2. □

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