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SHARPNESS OF UNIFORM CONTINUITY OF QUASICONFORMAL MAPPINGS ONTO s-JOHN DOMAINS

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Abstract. We show that a prediction in [8] is inaccurate by constructing quasiconformal mappings onto s-John domains so that the mappings fail to be uniformly continuous between natural distances. These examples also exhibit the sharpeness of the assumptions in [5].

1. Introduction

Recall that a conformal mapping of the unit disk onto a bounded domain Ω is uniformly α-Hölder continuous, $0 < \alpha \leq 1$, if and only if the hyperbolic metric ρ_Ω in Ω satisfies the logarithmic growth condition

$$\rho_\Omega(z_0, z) \leq \frac{1}{\alpha} \log \frac{\operatorname{dist}(z_0, \partial\Omega)}{\operatorname{dist}(z, \partial\Omega)} + C_0, \tag{1.1}$$

where $z_0 = f(0)$ and $C_0 < \infty$. Here dist refers to the Euclidean distance. This result is due to Becker and Pommerenke [2].

Gehring and Martio [3] gave a quasiconformal analogue of the result be Becker and Pommerenke by replacing the hyperbolic metric in (1.1) with the quasihyperbolic metric. Recall that the quasihyperbolic distance between x and x_0 in $\Omega \neq \mathbf{R}^n$ is

$$k_\Omega(x, x_0) = \inf_{\gamma_x} \int_{\gamma_x} \frac{ds}{\operatorname{dist}(z, \partial\Omega)},$$

where the infimum is taken over all rectifiable curves γ_x in Ω which join x to x_0. For $x, y \in \Omega$, there is a (quasihyperbolic) geodesic $[x, y]$ in Ω with

$$k_\Omega(x, y) = \int_{[x,y]} \frac{ds}{\operatorname{dist}(z, \partial\Omega)},$$

see [4]. In particular, they showed that the condition

$$k_\Omega(z_0, z) \leq \frac{1}{\alpha} \log \frac{\operatorname{dist}(z_0, \partial\Omega)}{\operatorname{dist}(z, \partial\Omega)} + C_0, \tag{1.2}$$

guarantees that, given Ω' and a K-quasiconformal mapping $f\colon \Omega' \to \Omega$, the restriction of f to any ball $B \subset \Omega'$ is uniformly Hölder continuous with an exponent β and a constant M that both are independent of B. Under suitable geometric conditions on Ω' they then concluded uniform Hölder continuity in the entire Ω'.

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In [10], Koskela, Onninen and Tyson showed that (1.2) actually implies that a quasiconformal mapping $f\colon \Omega' \to \Omega$ is always uniformly Hölder continuous when Ω' is equipped with the internal distance $d_I(z,w)$ and Ω with the usual Euclidean distance. Recall that $d_I(z,w)$, for a pair of points in a domain G, is the infimum of the lengths of all paths that join z to w in G.

In [8], Hencl and Koskela relaxed (1.2) to

$$k_\Omega(x,x_0) \le \phi\left(\frac{1}{\operatorname{dist}(x,\partial\Omega)}\right), \tag{1.3}$$

under the assumption that

$$\int_1^\infty \frac{dt}{\phi^{-1}(t)} < \infty. \tag{1.4}$$

A uniform continuity estimate with respect to the internal metric in Ω' and the Euclidean metric in Ω was established under the additional assumption that $t \mapsto \Phi(t)^{-a}$ is concave for some $a > n-1$, where

$$\Phi(t) = \psi^{-1}(t) \quad \text{and} \quad \psi(t) = \int_t^\infty \frac{ds}{\phi^{-1}(s)}.$$

This concavity assumption was speculated in [8] to be superfluous.

Let us recall a class of domains for which growth conditions of the above type are easily verified. First of all, a bounded domain $\Omega \subset \mathbf{R}^n$ is a John domain if there is a constant C and a point $x_0 \in \Omega$ so that, for each $x \in \Omega$, one can find a rectifiable curve $\gamma\colon [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ and with

$$C \operatorname{dist}(\gamma(t), \partial\Omega) \ge l(\gamma([0,t])) \tag{1.5}$$

for each $0 < t \le 1$. John used this condition in his work on elasticity [9] and the term was coined by Martio and Sarvas [11]. Smith and Stegenga [13] introduced the more general concept of an s-John domain, $s \ge 1$, by replacing (1.5) with

$$Cd(\gamma(t), \partial\Omega) \ge l(\gamma([0,t]))^s. \tag{1.6}$$

The recent studies [1, 6, 7] on mappings of finite distortion have generated new interest in the class of s-John domains. Direct integration along a curve from the definition of a John domain gives (1.2) with z_0 replaced by x_0 for the 1-John case and (1.3) with $\phi(t) = Ct^{s-1}$ in the case of an s-John domain, $s > 1$. It is easy to check that the concavity assumption on the associated Φ holds if $s < 1 + \frac{1}{n}$ and that the convergence condition holds when $s < 2$. On the other hand, Guo has established in [5] the above uniform continuity result for s-John domains with $1 \le s < 1 + \frac{1}{n-1}$. Our first result shows that the concavity assumption is not superfluous and that the requirement that $1 \le s < 1 + \frac{1}{n-1}$ cannot be relaxed in the planar case.

Theorem 1.1. *There exist a bounded 2-John domain $\Omega \subset \mathbf{R}^2$, a constant $C < \infty$ and a point $x_0 \in \Omega$ with*

$$k_\Omega(x,x_0) \le C \operatorname{dist}(x,\partial\Omega)^{-\frac{1}{2}} \tag{1.7}$$

for all $x \in \Omega$ so that a quasiconformal mapping $f\colon \Omega' \to \Omega$ fails to be uniformly continuous with respect to the Euclidean metric in Ω and d_I in Ω' for a bounded domain $\Omega' \subset \mathbf{R}^2$.

The above example is somewhat surprising since the modulus of continuity from [8] does not degenerate when the exponent $-a$ in $k_\Omega(x,x_0) \le C \operatorname{dist}(x,\partial\Omega)^{-a}$ tends

$-1/n$ nor does the modulus of continuity in [5] when s tends to $1+\frac{1}{n-1}$ in the s-John condition.

Our second result shows that the concavity condition is necessary in all dimensions and that the value $1+\frac{1}{n-1}$ is critical in the s-John condition.

Theorem 1.2. *For each $n \geq 3$, there exist a bounded domain $\Omega' \subset \mathbf{R}^n$ and a domain $\Omega \subset \mathbf{R}^n$ that is s-John for all $s > 1+\frac{1}{n-1}$ with*

$$k_\Omega(x, x_0) \leq C \operatorname{dist}(x, \partial\Omega)^{-\frac{1}{n}} \log \frac{C}{\operatorname{dist}(x, \partial\Omega)} \tag{1.8}$$

for some constant $C < \infty$, all $x \in \Omega$ and a quasiconformal mapping $f\colon \Omega' \to \Omega$ so that f is not uniformly continuous with respect to the Euclidean metric in Ω and d_I in Ω'.

The domain Ω in Theorem 1.1 cannot be required to be simply connected. More generally, neither the domain in Theorem 1.1 nor in Theorem 1.2 can be required to be quasiconformally equivalent to a uniform domain. For this see [5].

It would be interesting to know whether one could take $s = 1+\frac{1}{n-1}$ and dispose with the logarithmic term in Theorem 1.2.

2. Proofs of the main results

Proof of Theorem 1.1. Our 2-John domain Ω will be constructed inductively as indicated in Figure 1.

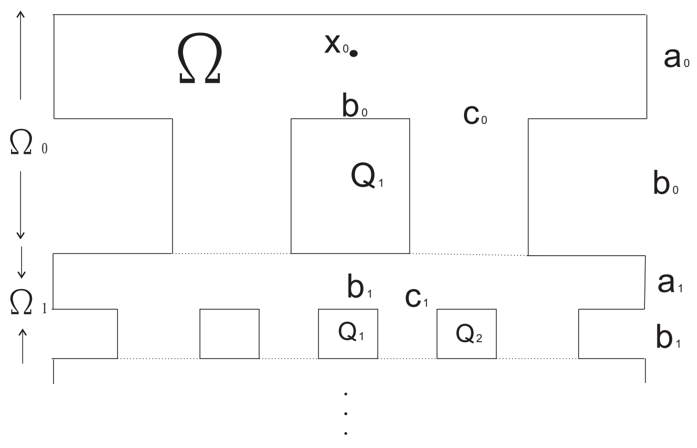

Figure 1. The 2-John domain Ω.

Set $a_j = 2^{-2(j+1)}$, $b_j = 2^{-j}$ and $c_j = 2^{-2(j+1)}$. For $j = 0$, we let the Ω_0-part consist of a rectangle of length 1 and width a_0 centered at the origin, and two rectangular "legs" of width c_0 and length b_0. The two rectangular "legs" are obtained in the following manner: first remove the central square Q_1 of side-length b_0; then set the distance between Q_1 and the vertical boundary of Ω_0 to be c_0. Next, for $j = 1$, we let the Ω_1-part consist of a rectangle of length 1 and width a_1 and four rectangular "legs" of width c_1 and length b_1. The four rectangular "legs" are obtained in a similar fashion as before: first remove 3 squares of side-length b_1; then make them equi-distributed, i.e. the gap between two consecutive squares is c_1; finally set the distance between Q_2 and the vertical boundary of Ω_1 to be c_1. We continue the process. Let the Ω_j-part consist of a rectangle of length 1 and width a_j and 2^j rectangular "legs" of width c_j and length b_j. The rectangular "legs" are obtained by removing $2^{j+1}-1$ equi-distributed squares of side-length b_j in a similar way as before. Among these removed squares, we label from middle to the right-most as Q_1, Q_2, $\dots$, Q_{2^j} respectively. According to our construction, the distance between two consecutive removed squares is c_j and the distance between Q_{2^j} and the vertical boundary of Ω_j is also c_j. Finally, our domain Ω is the union of all Ω_j's. It is clear from the construction that Ω is 2-John and symmetric with respect to the y-axis.

Let $x_0 = (0,0)$ be the point marked in Figure 1. It is easy to check that the assumption (1.7) is satisfied.

We next construct our source domain Ω' and a quasiconformal mapping $g\colon \Omega' \to \Omega$, which is not uniformly continuous with respect to the metrics $d(x,y) = |x-y|$ in Ω and d_I in Ω'. Actually, we construct a quasiconformal mapping $f\colon \Omega \to \Omega'$ whose (quasiconformal) inverse has the desired properties.

The idea is demonstrated in Figure 2: we scale the upper part of each Ω_j by $\frac{1}{j}$ and replace the associated 2^{j+1} rectangular "legs" by the same number of new "legs". The vertical distance between the scaled upper parts of Ω_j and Ω_{j+1} is set to be $2j^{-2}$. We also make the domain Ω' symmetric with respect to y-axis. Since the distance between two consecutive legs in Ω_j is 2^{-j}, the distance between the tops of two consecutive "legs" in Ω'_j is $\frac{2^{-j}}{j}$. For the bottoms, the distance is approximately $\frac{2^{-j-1}}{j+1}$.

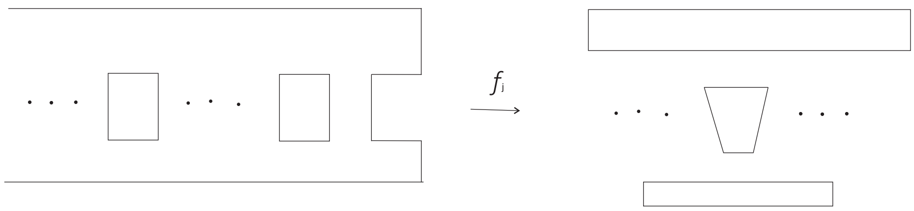

Figure 2. Ω and Ω' in the step j.

Recall the labelled squares Q_i, $i = 1, \dots, 2^j$ introduced in Ω_j. We denote by $\tilde{Q}_i$ the "leg" next to Q_i, on the right. We will construct a quasiconformal mapping f_j from the (translated) rectangle $\tilde{Q}_i$ to the (translated) new "leg" Q'_i as in Figure 3. Q'_i consists of two parts A' and B'. The distance between the bottom line segment $0\mathbf{a}$ and the top line segment in the x-direction is

$$m_i^j = \frac{[2^{-j-1} + 2^{-2(j+1)}] \cdot i}{j} - \frac{[2^{-j-1} + 2^{-2(j+1)}] \cdot i}{j+1}.$$

It is clear that $m_i^j \approx \frac{i \cdot 2^{-j}}{j^2}$ when j is large. The distance of the top and the bottom in y-direction is $\frac{2}{j^2}$. In Figure 3, $\mathbf{a} = \big(\frac{2^{-2(j+1)}}{j+1}, 0\big)$, $\mathbf{p} = \big(\frac{i \cdot 2^{-j}}{2j^2}, \frac{1}{j^2}\big)$ and $\mathbf{q} = \big(\frac{i \cdot 2^{-j}}{2j^2} + \frac{2^{-(j+1)}}{j}, \frac{1}{j^2}\big)$. We will write down below a quasiconformal mapping $f_j\colon A \to A'$ such that f_j maps the bottom line segment of A linearly to $0\mathbf{a}$ and the top line segment of A affinely to $\mathbf{pq}$, respectively. The line $0\mathbf{p}$ is of the form $y = k_1 x$, where

$$k_1 = \frac{1/j^2}{i \cdot 2^{-j}/(2j^2)} = \frac{2^{j+1}}{i} \geq 1.$$

Similarly, the line $\mathbf{aq}$ is of the form $y = k_2(x - \frac{2^{-2(j+1)}}{j+1})$, where

$$k_2 = \frac{1/j^2}{\frac{i \cdot 2^{-j}}{2j^2} + \frac{2^{-(j+1)}}{j} - \frac{2^{-2(j+1)}}{j+1}} \approx \frac{2^j}{i+j}.$$

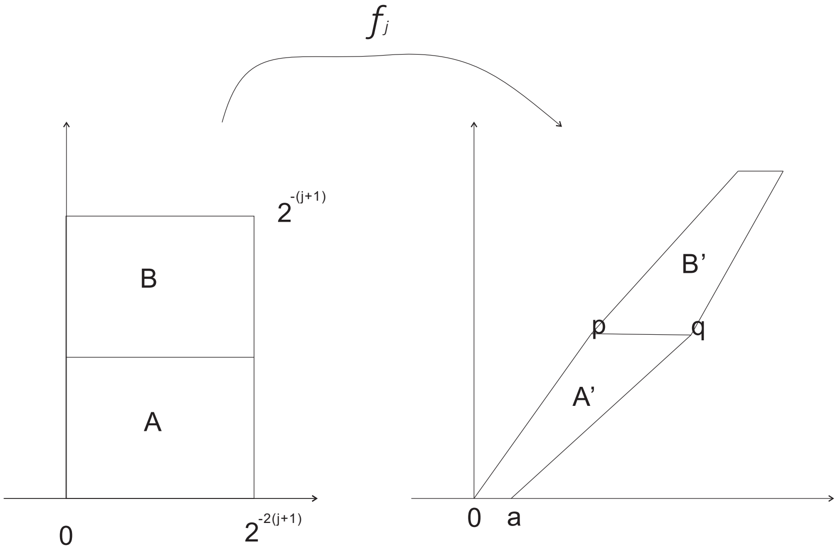

Figure 3. The quasiconformal mapping from $\tilde{Q}_i$ to Q'_i.

We are looking for a quasiconformal mapping of the form $f_j^i(x,y) = (\tilde{g}_j(y)x + g_j(y), k_1 g_j(y))$, where $\tilde{g}_j(y) = k_1 g_j'(y)$ for all $y \in [0, 2^{-(j+1)}]$ and g_j is a smooth increasing function. Clearly, such a mapping f_j maps horizontal line segments to horizontal line segments. We further require that it maps the left side of A to $0\mathbf{p}$ and the right side of A to $\mathbf{aq}$, $g_j(0) = 0$, $g_j(2^{-j}) = \frac{1}{j^2}$ and $\tilde{g}_j(0) = \frac{1}{j+1}$. By definition,

$$f_j^i(2^{-2(j+1)}, y) = (\tilde{g}_j(y) \cdot 2^{-2(j+1)} + g_j(y), k_1 g_j(y)).$$

The further requirements are satisfied if $\tilde{g}_j = k_1 g_j'$,

$$g_j(y) = k_2 \cdot k_1^{-1} \tilde{g}_j(y) \cdot 2^{-2(j+1)} + \frac{k_2}{k_1} g_j(y) - \frac{k_2}{j} \cdot 2^{-2(j+1)}, \tag{2.1}$$

$$g_j(0) = 0, g_j(2^{-j}) = \frac{1}{j^2} \quad \text{and} \quad \tilde{g}_j(0) = \frac{1}{j+1}. \tag{2.2}$$

One can easily solve the above system of equations by setting $g_j(y) = a \cdot e^{a_j^i y + c} - b$, where

$$a_i^j = 2^{2(j+1)} \frac{k_1 - k_2}{k_1 k_2}, \quad b = \frac{1}{k_1 (j+1) a_j^i}$$

and the constants b and c are chosen such that

$$a \cdot e^c = b \quad \text{and} \quad a \cdot e^{a_j^i 2^{-j} + c} - b = \frac{1}{j^2}.$$

We next show that f_j^i is a quasiconformal mapping. A direct computation gives us

$$Df_j^i(x,y) = \begin{bmatrix} \tilde{g}_j(y) & \tilde{g}_j'(y)x + g_j'(y) \\ 0 & k_1 g_j'(y) \end{bmatrix}.$$

We only need to show that $\tilde{g}_j'(y)x + g_j'(y) \leq M k_1 g_j'(y)$, for some constant M independent of i and j, and for all $x, y \in A$. Since $k_1 \geq 1$, it suffices to bound $\tilde{g}_j'(y)x$. By definition,

$$\tilde{g}_j(y) = k_1 g_j'(y) = k_1 a a_j^i e^{a_j^i y + c}$$

and

$$\tilde{g}_j'(y) = k_1 a_j^i g_j'(y).$$

Hence we only need to find a uniform bound on $x \cdot a_j^i$. For this, we first note that k_2 is bounded from below by $\frac{1}{2}$ and $\frac{k_1-k_2}{k_1} \le 1$. Since $x \in [0, 2^{-2(j+1)}]$, we have

$$a_j^i x \le \frac{k_1 - k_2}{k_1 k_2} \cdot 2^{2(j+1)} x \le 2.$$

This implies that $\tilde{g}_j'(y)x + g_j'(y) \le 3k_1 g_j'(y)$ and so f_j^i is quasiconformal. Notice that $f_j^i(x,0) = (\frac{x}{j+1}, 0)$, so that, after suitable translations, f_j^i matches with our scaling on the top of Ω_{j+1}. In a similar manner, one can write down a quasiconformal mapping from B to B' such that it coincides with f_j^i on $\mathbf{pq}$ and is linear on each line segment. In fact, the quasiconformal mapping just slightly differs from the reflection of f_j with respect to the line segment $\mathbf{pq}$ (since the length of $0\mathbf{a}$ is approximately the same as the length of the top line segment when $j \to \infty$ and the picture is exactly a reflection with respect to $\mathbf{pq}$). When a suitable coordinate system is fixed, it is clear that the mappings $f_j^{i_1}$ and $f_j^{i_2}$ only differ by a translation in x-direction and hence the desired global quasiconformal mapping f_j from Ω_j to Ω_j' follows by gluing all f_j^i's and the scaling maps.

In this manner, the domain Ω' is well-defined. We can define the quasiconformal mapping $g\colon \Omega' \to \Omega$ by setting $g|_{\Omega_j'} = f_j^{-1}$. Moreover, g cannot be uniformly continuous since for each $j \in \mathbf{N}$, it maps a rectangle of length $\frac{1}{j}$ linearly to a rectangle of length 1. ☐

Proof of Theorem 1.2. We will give the detailed constructions of our domains and quasiconformal mapping for $n = 3$ and indicate how to pass them to all dimensions at the end of the proof. The idea of the 3-dimensional construction is similar to the one above and we simply fatten the "Ω_0" part of the planar domain in Figure 1 along the third direction; see Figure 4 below.

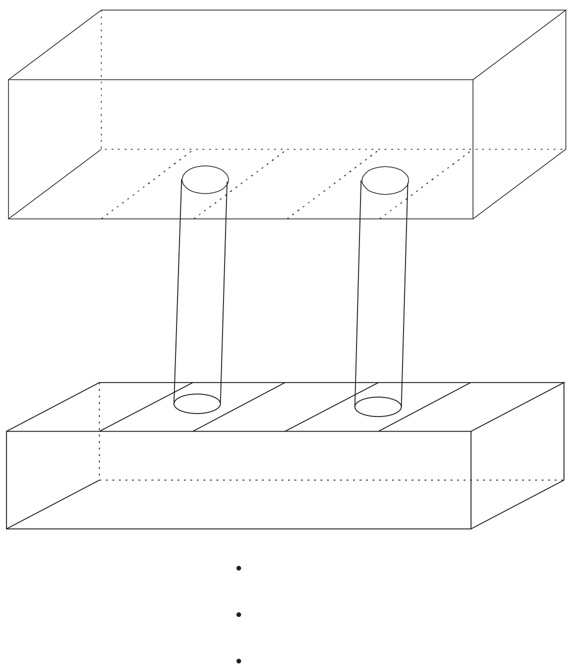

Figure 4. The first part of our domain Ω.

The top part of Figure 4 consists of a rectangle of length 1, width $\frac{1}{2^2}$ and height $\frac{1}{2^3}$. In the bottom, the rectangle has length 1, width $\frac{1}{2^4}$ and height $\frac{1}{2^6}$. We attach four cylindrical "legs" of height $2 \cdot 2^{-2}$ between these rectangles. The radius of the cylinder is about 2^{-3} and the distance between them is about 2^{-2}.

We can proceed our construction in the following manner. At step j, the top part consists of a rectangle of length 1, width 2^{-2j} and height 2^{-3j}. In the bottom, the rectangle has length 1, width $2^{-2(j+1)}$ and height $2^{-3(j+1)}$. We attach 2^{2j} equi-distributed cylindrical "legs" of height 2^{-2j} between them. The radius of the cylinder is about 2^{-3j} and the distance between two consecutive cylinders is about $h_j = j{\cdot}2^{-2j}$. It is clear from our construction that Ω is an s-John domain for any $s \in (1 + \frac{1}{2}, \infty)$.

Let x_0 be the central point in the first rectangle of Ω. It is easy to check that the assumption (1.8) is satisfied.

Our source domain Ω' is obtained by a similar scaling procedure as in the proof of Theorem 1.1. To be more precise, at step j, we scale the top rectangle by $\frac{1}{j^2}$ and replace the associated 2^j cylindrical "legs" by the same number of new "legs". The vertical distance between the scaled top rectangle and the bottom rectangle is set to be $h'_j = \frac{2}{j^2}$.

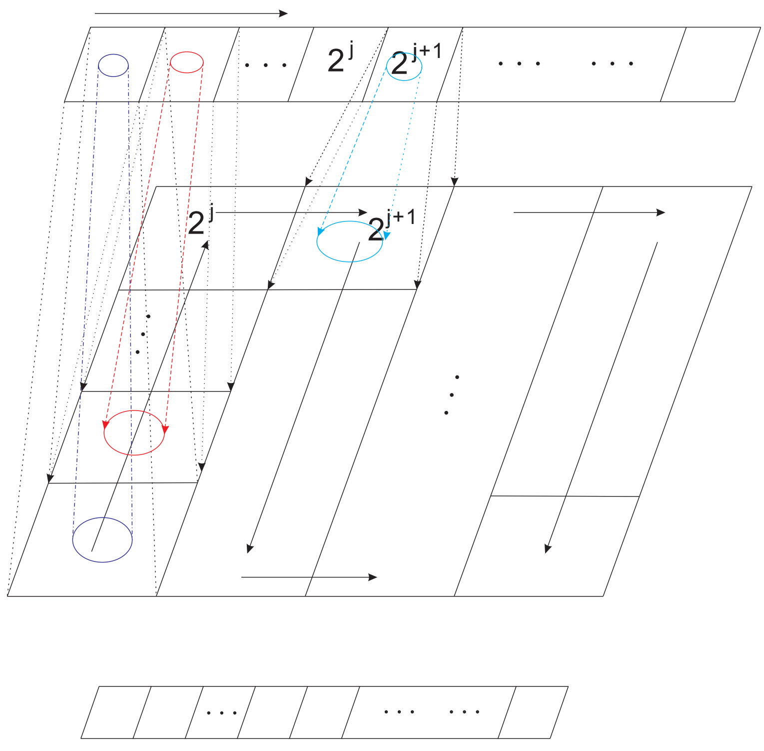

Figure 5. The new "legs" at step j.

We next explain how to select the new "legs", see Figure 5 for a top view. In Figure 5, the top rectangle has length $\frac{1}{j^2}$ and width $\frac{2^{-2j}}{j^2}$. It consists of 2^{2j} squares of side-length $\frac{2^{-2j}}{j^2}$. The bottom rectangle has length $\frac{1}{(j+1)^2}$ and width $\frac{2^{-2(j+1)}}{(j+1)^2}$. The vertical distance between these rectangles is h'_j. We insert a square S_j of side-length $\frac{1}{j^2}$ in the middle of the two rectangles, i.e. the (vertical) distance between S_j and either of the rectangles is $\frac{1}{j^2}$. We divide S_j into 2^{2j} subsquares of side-length $\frac{2^{-j}}{j^2}$. Next, we set up a one-to-one correspondence between the 2^{2j} squares in the top rectangle and the subsquares in S_j. To be more precise, we first construct 2^{2j} affine "rectangles" between each square in the top rectangle and each subsquare in S_j and then we insert a "cylindrical leg" inside each affine "rectangle", see Figure 5 for the order of the affine "rectangles". The radius of the top circle of the "cylindrical leg" is set to be $\frac{2^{-3j}}{j^2}$ and the radius of the bottom circle is $\frac{2^{-j}}{j^2}$. Since the 2^{2j} affine "rectangles" have disjoint interiors, the 2^{2j} "cylindrical legs" are pairwise disjoint. As in the proof of Theorem 1.1, we use a similar construction between S_j and the bottom rectangle.

Reasoning as in the proof of Theorem 1.1, we only need to write down quasiconformal mappings between these "legs". Note that our construction implies that all the 2^{2j} "cylindrical legs" are bi-Lipschitz equivalent, with a constant independent of j. So finally we reduce the problem to the existence of a quasiconformal mapping g as in Figure 6.

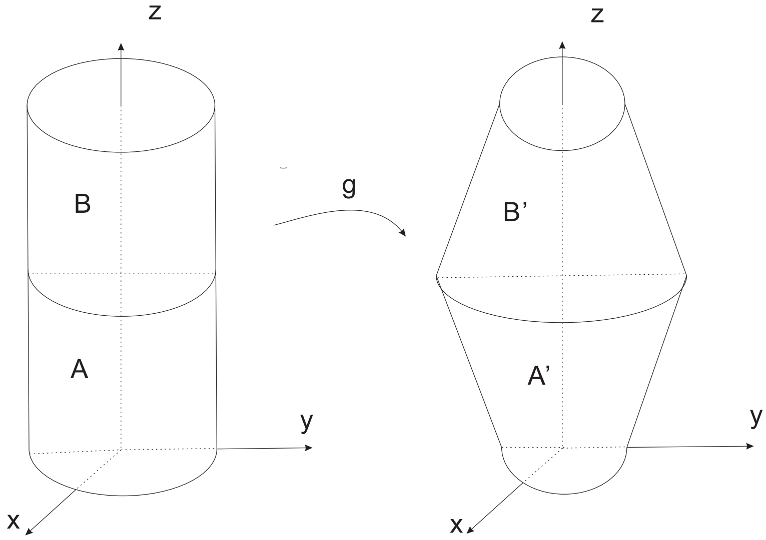

Figure 6. The quasiconformal mapping from a "cylinder" to a "double cone".

We will use the coordinate system marked in Figure 6 and write down a quasiconformal mapping g from A onto A' such that g is a scaling between the bottom and top disks. Set

$$g(x,y,z) = (g_1(z)x, g_1(z)y, g_2(z)).$$

We require that $g_1(0) = \frac{1}{j^2}$, $g_1(h_j) = \frac{2^{2j}}{j^2}$, $g_2(0) = 0$, $g_2(h_j) = h'_j$ and $g'_2(z) = g_1(z)$ for all $z \in [0, h_j]$. It is easy to check that with these requirements, g will be a quasiconformal mapping that maps A to A' such that g is the desired scaling between the bottom and top disks. One can use a map g_2 of the form $g_2(z) = a_j(e^{b_j z} - 1)$, where $a_j \approx \frac{2^{-2j}}{j^2}$ and $b_j \approx 2^{2j}$.

As in the planar case, the global quasiconformal mapping $f\colon \Omega' \to \Omega$ is obtained by gluing all these g's and the corresponding scaling mappings. Moreover, reasoning as in the planar case, we can easily conclude that f cannot be uniformly continuous with respect to the metrics $d(x,y) = |x-y|$ in Ω and d_I in Ω'.

The construction of the general n-dimensional case can be proceeded in a similar manner. At step j, Ω_j consists of a n-dimensional rectangle of length $a_1 = 1$ and (other) edge-lengths $a_2 = \cdots = a_{n-1} = 2^{-(n-1)j}$, $a_n = 2^{-nj}$ and 2^j "cylindrical legs" of length $h_j = j \cdot 2^{-(n-1)j}$. The radius of the cylinder is 2^{-nj}. So Ω is an s-John domain for any $s \in (1 + \frac{1}{n-1}, \infty)$.

The source domain Ω' is obtained by a similar scaling procedure as before. To be more precise, at step j, we scale the top rectangle by $\frac{1}{j^2}$ and replace the associated 2^j cylindrical "legs" by the same number of new "legs". The vertical distance between the scaled top rectangle and the bottom rectangle is set to be $h'_j = \frac{2}{j^2}$.

We use a similar idea as before to obtain new "legs" between the top rectangle and bottom rectangle as in Figure 5. Namely, we insert a $(n-1)$-dimensional cube of edge-length $\frac{1}{j^2}$ and then divide it into $2^{(n-1)j}$ subcubes of edge-length $\frac{2^{-j}}{j^2}$. Then attach $2^{(n-1)j}$ affine "rectangles" in a similar manner as before. Inside each affine "rectangle", we insert a "cylindrical leg". The radius of the top of the "cylindrical leg" is $\frac{2^{-nj}}{j^2}$ and the radius of the bottom is $\frac{2^{-j}}{j^2}$. Reasoning as before, one essentially only needs to write down a quasiconformal mapping g between these "legs".

The global quasiconformal mapping $f\colon \Omega' \to \Omega$ is obtained by gluing all these g's and the corresponding scaling mappings. Moreover, reasoning as in the planar case, we can easily conclude that f cannot be uniformly continuous with with respect to the metrics $d(x,y) = |x-y|$ in Ω and d_I in Ω'. □

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