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Abstract: We study a simple superconvergent scheme which recovers the gradient when solving a second-order elliptic problem in the plane by the usual linear elements. The recovered gradient globally approximates the true gradient even by one order of accuracy higher in the L^2-norm than the piecewise constant gradient of the Ritz–Galerkin solution. A superconvergent approximation to the boundary flux is presented as well.

Keywords: Global superconvergence for the gradient, post-processing of the Ritz–Galerkin scheme, error estimates, boundary flux.

1. Introduction

When a displacement finite element method is used a recovery of the gradient is often done by post-processing the FE-solution to improve the accuracy. We propose a simple post-processing technique which globally improves the approximation for the gradient of the solution to a second order elliptic problem when using linear triangular elements.

This paper can be considered as an extension of the local superconvergence results investigated by the authors in [11]. Another scheme which recovers the gradient at midpoints of sides can be found in [6,14]. For a recovery at centroids of triangles we refer to [13]. For a post-processing technique by convolution for the gradient when using B-splines, see [19]. In the survey article [12] other post-processing techniques can be found.

The paper is organized as follows. In Section 2 the global averaged operator G_h for the gradient of a piecewise linear FE-solution is introduced. In Section 3 its approximation properties are studied. We will show under certain assumptions on triangulations that

$$\|\operatorname{grad} v - G_h(\Pi_h v)\|_{0,\Omega} \leqslant Ch^2 |v|_{3,\Omega} \tag{1.1}$$

for all $v \in H^3(\Omega)$, where $\Pi_h v$ denotes the piecewise linear interpolant of v.

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The global superconvergence result proved in Section 4 reads:

$$\| \text{grad } u - G_h(u_h) \|_{0,\Omega} \leqslant Ch^2 \| u \|_{3,\Omega}, \tag{1.2}$$

where u is a solution of a second-order elliptic equation and u_h is its piecewise linear Ritz–Galerkin approximation. Then we introduce a simple superconvergence technique for calculation the boundary flux. Our technique differs from that presented in [8, p.389] which is based on some ideas of [7].

In Section 5 some results of numerical tests are reported which confirm the theoretical error estimate (1.2). Finally, we notice that the post-processing technique proposed here requires only $\mathcal{O}(m)$ arithmetic operators, where m is the number of nodal points in question.

2. Preliminaries and the averaged gradient

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a polygonal boundary $\partial\Omega$. The usual norm and seminorm in the (product) Sobolev space $(W_p^k(\Omega))^r = W_p^k(\Omega) \times \cdots \times W_p^k(\Omega)$, $k \geqslant 0$, $p \in [1, \infty]$, $r = 1, 2, \ldots$, are denoted by $\| \cdot \|_{k,p,\Omega}$ and $| \cdot |_{k,p,\Omega}$, respectively. We shall omit the subscript p in the case $p = 2$ and we write $H^k(\Omega) = W_2^k(\Omega)$. The notation $(\cdot, \cdot)_{0,\Omega}$ is used for the inner product in $(L^2(\Omega))^r$, $r = 1, 2, \ldots$. All the vectors are supposed to be column vectors. By $\| \cdot \|$ we denote the Euclidean norm. The space $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$, consisting of functions with zero traces. By $P_j(\Omega)$ we mean the space of polynomials of the degree j.

The notations $C, C', \ldots$ are reserved for generic positive constants which may vary with context. Moreover, all our statements will always hold only for a sufficiently small discretization parameter h.

Consider the problem

$$\begin{aligned} -\text{div}(A \text{ grad } u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $A \in (H^\alpha(\Omega))^{2\times 2}$ (for some $\alpha > 2$) is a symmetric uniformly positive definite matrix and $f \in L^2(\Omega)$. The standard Ritz–Galerkin method for (2.1) based on linear triangular elements consists in finding

$$u_h \in V_h = \{ v_h \in H_0^1(\Omega) \mid v_h|_T \in P_1(T) \ \forall T \in \mathcal{T}_h \}$$

for which

$$(A \text{ grad } u_h, \text{grad } v_h)_{0,\Omega} = (f, v_h)_{0,\Omega} \quad \forall v_h \in V_h,$$

where $\mathcal{T}_h$ belongs to a regular family of triangulations of $\bar{\Omega}$ (see [3] for Zlámal's condition); triangles are assumed to be closed.

We denote by N_h the set of all nodal points corresponding to a given triangulation $\mathcal{T}_h$. Let $Y \subset N_h$ be the (finite) set of vertices of $\bar{\Omega}$ and let

$$\begin{aligned} E_h &= N_h \cap (\partial\Omega - Y) \quad &&\text{(boundary nodes except vertices)}, \\ I_h &= N_h \cap \Omega &&\text{(internal nodes)}. \end{aligned}$$

Consequently,

$$N_h = Y \cup E_h \cup I_h,$$

where Y, E_h, I_h are mutually disjoint.

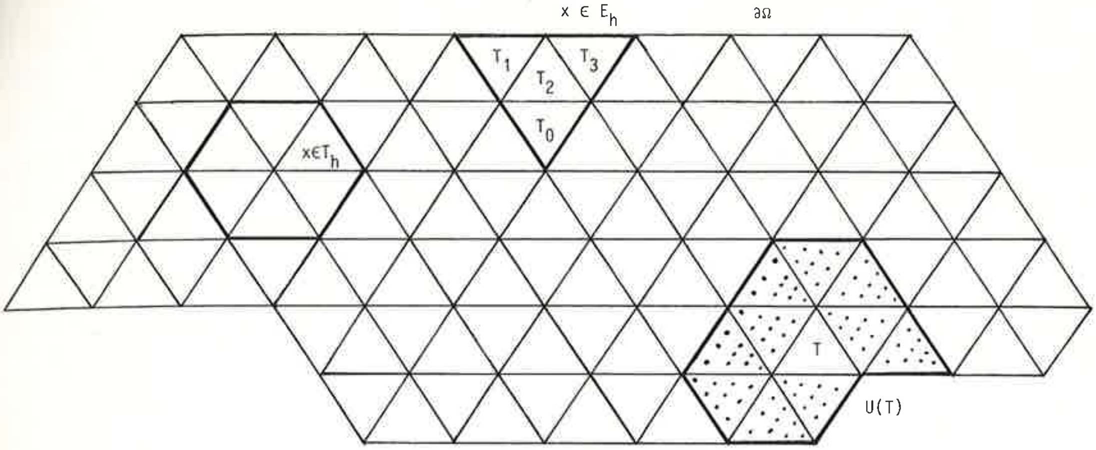

Fig. 1.

We define a *linear interpolant* $\Pi_h w \in W_h$ of $w \in H^1(\Omega) \cap C(\bar{\Omega})$ by setting

$$(\Pi_h w)(x) = w(x) \quad \forall x \in N_h,$$

where

$$W_h = \{ w \in H^1(\Omega) \mid w|_T \in P_1(T) \ \forall T \in \mathcal{T}_h \}.$$

Definition 2.1. A triangulation $\mathcal{T}_h$ is said to be *uniform*, if any two adjacent triangles of $\mathcal{T}_h$ form a parallelogram.

For later use we take a closer look at a polygonal domain Ω with a uniform triangulation. Referring to the notations in Fig. 1 we find that $\bigcap_{i=1}^3 T_i$ is a point $x \in E_h$ and $\bigcap_{i=0}^3 T_i$ is a triangle whose side lying on $\partial\Omega$ has x as midpoint.

For $T \in \mathcal{T}_h$ we define a subset $U(T)$ of $\bar{\Omega}$ by

$$U(T) = \bigcup_{\substack{T' \in \mathcal{T}_h \\ T' \cap T \neq \emptyset}} T',$$

see Fig. 1.

Suppose that we have a uniform triangulation $\mathcal{T}_h$ of $\bar{\Omega}$. Referring to the notations of Fig. 1 we introduce the *averaged gradient*

$$G_h:\ V_h \to W_h \times W_h,$$

uniquely determined by the formulae

$$(G_h(v_h))(x) = \begin{cases} 0, & x \in Y, & \text{(2.2a)} \\ \sum_{i=0}^{3} w_i \operatorname{grad} v_h|_{T_i}, & x \in E_h, & \text{(2.2b)} \\ \frac{1}{6} \sum_{T \cap \{x\} \neq \emptyset} \operatorname{grad} v_h|_T, & x \in I_h, & \text{(2.2c)} \end{cases}$$

where

$$-w_0 = w_1 = w_2 = w_3 = \tfrac{1}{2}. \tag{2.3}$$

Remark 2.1. As $u|_{\partial\Omega} = 0$ (which implies that the tangential derivatives of u vanishes on $\partial\Omega$), we get for $u \in H^3(\Omega)$ that grad $u(x) = 0$ for all $x \in Y$, which justifies (2.2a). For another choice of G_h see Remark 3.6.

3. Approximation properties of the averaged gradient

The aim of this section is to prove the following theorem.

Theorem 3.1. *Let $\{\mathcal{T}_h\}$ be a regular family of uniform triangulations of $\overline{\Omega}$. Then*

$$\|\text{grad } v - G_h(\Pi_h v)\|_{0,\Omega} \leqslant Ch^2 |v|_{3,\Omega} \quad \forall v \in H^3(\Omega) \cap H_0^1(\Omega). \tag{3.1}$$

The proof is based on several auxiliary lemmas. We first give some definitions. Consider the reference uniform triangulation $\hat{\mathcal{T}}$ of a half-plane Z consisting of right-angled triangles with mesh size 1 (see e.g. Fig. 2).

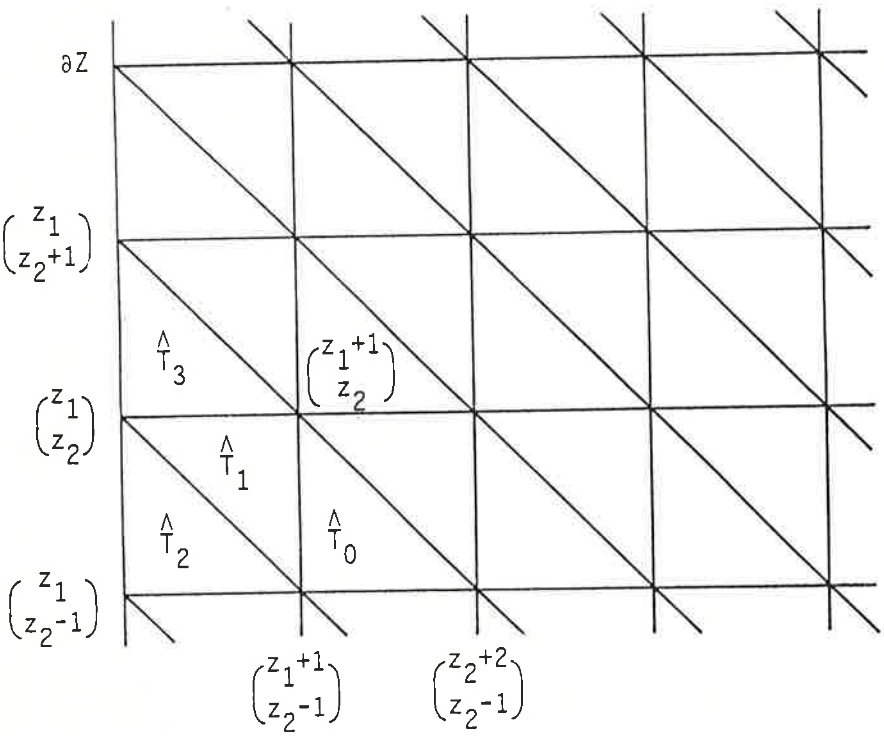

Fig. 2.

Let $\hat{E}$ and $\hat{I}$ be the sets of the boundary and internal nodes of $\hat{\mathscr{T}}$, respectively. Setting

$$\hat{W} = \{\hat{v} \in C(\bar{Z}) \mid \hat{v}|_{\hat{T}} \in P_1(\hat{T})\ \forall \hat{T} \in \hat{\mathscr{T}}\},$$

we define the reference averaged gradient $\hat{G}$: $\hat{W} \to \hat{W} \times \hat{W}$ in the way analogous to (2.2b) and (2.2c). Let us introduce the reference interpolant $\hat{\Pi}\hat{v} \in \hat{W}$ for $\hat{v} \in C(\bar{Z})$ given by

$$\hat{\Pi}\hat{v}(\hat{x}) = \hat{v}(\hat{x}) \quad \forall \hat{x} \in \hat{E} \cup \hat{I}.$$

Then a direct calculation (cf. [11, p.108]) leads to the following lemma.

Lemma 3.2. *The equality*

$$\hat{G}(\hat{\Pi}\hat{p}) = \text{grad } \hat{p} \quad \forall \hat{p} \in P_2(\bar{Z}) \tag{3.2}$$

is valid.

Lemma 3.3. *Let $\hat{T} = \hat{T}_i$ for some $i \in \{0, 1, 2\}$ and let $\hat{U} = U(\hat{T})$. Then*

$$\|\text{grad } \hat{v} - \hat{G}(\hat{\Pi}\hat{v})\|_{0,\infty,\hat{T}} \leqslant \hat{C}\,|\hat{v}|_{3,\hat{U}} \quad \forall \hat{v} \in H^3(\hat{U}). \tag{3.3}$$

Proof. Take any $\hat{v} \in H^3(\hat{U})$. As the function $\hat{G}(\hat{\Pi}\hat{v})|_{\hat{T}}$ is linear we have from (2.2), (2.3) and [3, p.123],

$$\begin{aligned} \|\hat{G}(\hat{\Pi}\hat{v})\|_{0,\infty,\hat{T}} &= \max_{\hat{x} \in \hat{T}} \|\hat{G}(\hat{\Pi}\hat{v})(\hat{x})\| = \|\hat{G}(\hat{\Pi}\hat{v})(\hat{y})\| \\ &\leqslant 2\,\|\text{grad}(\hat{\Pi}\hat{v})\|_{0,\infty,\hat{U}} \leqslant 2\,\|\text{grad } \hat{v}\|_{0,\infty,\hat{U}}, \end{aligned} \tag{3.4}$$

where $\hat{y}$ is a convenient vertex of $\hat{T}$. Let $j \in \{1, 2\}$ and $\hat{x} \in \hat{T}$ be arbitrarily fixed and define the linear functional ϕ by

$$\phi(\hat{v}) = \big((\text{grad } \hat{v} - \hat{G}(\hat{\Pi}\hat{v}))(\hat{x})\big)_j, \quad \hat{v} \in H^3(\hat{U}).$$

Applying the Sobolev imbedding theorem $H^3(\hat{U}) \hookrightarrow C^1(\hat{U})$, we get from (3.4)

$$|\phi(\hat{v})| \leqslant \|\text{grad } \hat{v} - \hat{G}(\hat{\Pi}\hat{v})\|_{0,\infty,\hat{T}} \leqslant 3\,\|\text{grad } \hat{v}\|_{0,\infty,\hat{U}} \leqslant C\,\|\hat{v}\|_{3,\hat{U}}.$$

Thus ϕ is continuous and by (3.2) it vanishes for all quadratics. Now (3.3) follows from the Bramble–Hilbert lemma [2,3]. □

Henceforth, for any $T \in \mathscr{T}_h$, $T \cap Y = \emptyset$, we define a linearly affine continuous one-to-one mapping F_T: $\mathbb{R}^2 \to \mathbb{R}^2$ so that $F_T(\hat{T}_i) = T$, where $i \in \{0, 1, 2\}$ is the number of vertices of T belonging to E_h. (For instance, if $T \cap \partial\Omega = \emptyset$, $T \in \mathscr{T}_h$, then its original is $\hat{T}_0$.) Moreover, we assume that vertices of $\hat{T}_i$ which belong to $\hat{E}$ and $\hat{I}$ are mapped into E_h and I_h, respectively. Clearly, the mappings F_T are of the form $F_T(\hat{x}) = B_T\hat{x} + b_T$, $\hat{x} \in \mathbb{R}^2$, where $b_T \in \mathbb{R}^2$ and B_T are regular 2×2 matrices satisfying

$$\|B_T\| \leqslant Ch, \qquad \|B_T^{-1}\| \leqslant Ch^{-1}, \tag{3.5}$$

when $\mathscr{T}_h$ belongs to some regular family of the uniform triangulations $\mathscr{M} = \{T_h\}$.

Lemma 3.4. *There is a constant $C > 0$ such that for any $T \in \mathscr{T}_h \in \mathscr{M}$, $T \cap Y = \emptyset$ the inequality*

$$\|\text{grad } v - G_h(\Pi_h v)\|_{0,T} \leqslant Ch^2\,|v|_{3,U} \quad \forall v \in H^3(U), \tag{3.6}$$

is valid, where $U = U(T)$.

Proof. As $T \cap Y = \emptyset$ and $\mathcal{T}_h$ is uniform, U is mapped by F_T^{-1} onto $\hat{U} = \hat{U}(\hat{T}_i) \subset \overline{Z}$ for some $i \in \{0, 1, 2\}$. Define $\hat{v} \in H^1(\hat{U})$ by $\hat{v}(\hat{x}) = v(F_T(\hat{x}))$, $v \in H^1(\Omega)$, $\hat{x} \in \hat{U}$. Hence,

$$\operatorname{grad} v(x) = \left(B_T^{-1}\right)^{\mathrm{T}} \operatorname{grad} \hat{v}\left(F_T^{-1}(x)\right) \quad \forall v \in H^3(U) \quad \forall x \in U, \tag{3.7}$$

where $(\cdot)^{\mathrm{T}}$ denotes the transposition. As $(\Pi_h v)\hat{} = \hat{\Pi}\hat{v}$, we find that a similar formula holds also for the averaged gradient (cf. [11])

$$(G_h(\Pi_h v))(x) = \left(B_T^{-1}\right)^T \left(\hat{G}(\hat{\Pi}\hat{v})\right)\left(F_T^{-1}(x)\right) \quad \forall v \in H^3(U) \quad \forall x \in U. \tag{3.8}$$

Thus, employing the substitution $x = F_T(\hat{x})$ and (3.3), we obtain

$$\begin{aligned}\|\operatorname{grad} v - G_h(\Pi_h v)\|_{0,T}^2 &\leqslant \|B_T^{-1}\|^2 \|\operatorname{grad} \hat{v} - \hat{G}(\hat{\Pi}\hat{v})\|_{0,\hat{T}}^2 |\det B_T| \\ &\leqslant \tfrac{1}{2}\|B_T^{-1}\|^2 \|\operatorname{grad} \hat{v} - \hat{G}(\hat{\Pi}\hat{v})\|_{0,\infty,\hat{T}}^2 |\det B_T| \leqslant \tfrac{1}{2}\hat{C}^2 \|B_T^{-1}\|^2 |\det B_T| |\hat{v}|_{3,\hat{U}}^2.\end{aligned}$$

Now the lemma follows from (3.5) and inequality (see [3, p.118])

$$|\hat{v}|_{3,\hat{U}} \leqslant C\|B_T\|^3 |\det B_T|^{-1/2} |v|_{3,U}. \quad \square \tag{3.9}$$

Lemma 3.5. *There is a constant $C > 0$ such that for any $T' \in \mathcal{T}_h \in \mathcal{M}$, $T' \cap Y \neq \emptyset$*

$$\|\operatorname{grad} v - G_h(\Pi_h v)\|_{0,T'} \leqslant Ch^2 |v|_{3,\Omega} \quad \forall v \in H^3(\Omega) \cap H_0^1(\Omega). \tag{3.10}$$

Proof. Define the linear interpolation function $L_h v \in (P_1(T'))^2$ by

$$(L_h v)(x) = \operatorname{grad} v(x) \tag{3.11}$$

for all vertices x of T'. Hence [3, p.121],

$$\|\operatorname{grad} v - L_h v\|_{0,T'} \leqslant Ch^2 |\operatorname{grad} v|_{2,T'} \leqslant Ch^2 |v|_{3,\Omega}. \tag{3.12}$$

Let y be that vertex of T' which is also a vertex of $\overline{\Omega}$. Thus, from (2.2a), (3.11) and from the fact that $\operatorname{grad} v(y) = 0$, we infer $(G_h(\Pi_h v))(y) = (L_h v)(y) = 0$. Consequently,

$$\begin{aligned}\|L_h v - G_h(\Pi_h v)\|_{0,T'}^2 &\leqslant \operatorname{meas} T' \, \|L_h v - G_h(\Pi_h v)\|_{0,\infty,T'}^2 \\ &\leqslant Ch^2 \|(\operatorname{grad} v - G_h(\Pi_h v))(x)\|^2\end{aligned} \tag{3.13}$$

for a suitable vertex x of T' which is distinct from y. Let $T \in \mathcal{T}_h$ be such a triangle containing x for which $T \cap Y = \emptyset$. Then, from (3.13), (3.7), (3.8), (3.3) and (3.5), we get

$$\begin{aligned}\|L_h v - G_h(\Pi_h v)\|_{0,T'} &\leqslant Ch \|\operatorname{grad} v - G_h(\Pi_h v)\|_{0,\infty,T} \\ &\leqslant Ch \|B_T^{-1}\| \, \|\operatorname{grad} \hat{v} - \hat{G}(\hat{\Pi}\hat{v})\|_{0,\infty,\hat{T}} \leqslant C' |\hat{v}|_{3,\hat{U}}.\end{aligned} \tag{3.14}$$

As $|\det B_T|^{-1/2} \leqslant Ch^{-1}$, the relations (3.14), (3.9) and (3.5) yield

$$\|L_h v - G_h(\Pi_h v)\|_{0,T'} \leqslant Ch^2 |v|_{3,\Omega}.$$

This together with (3.12) gives (3.10). $\square$

Proof of Theorem 3.1. Squaring and summing the formulae (3.6) and (3.10), we get (note that Y is finite)

$$\begin{aligned}\|\operatorname{grad} v - G_h(\Pi_h v)\|_{0,\Omega}^2 &\leqslant Ch^4 \Bigg(|v|_{3,\Omega}^2 + \sum_{T \cap Y = \emptyset} |v|_{3,U(T)}^2 \Bigg) \\ &\leqslant Ch^4 \Bigg(|v|_{3,\Omega}^2 + 13 \sum_{T \cap Y = \emptyset} |v|_{3,T}^2 \Bigg) \leqslant C'h^4 |v|_{3,\Omega}^2,\end{aligned} \tag{3.15}$$

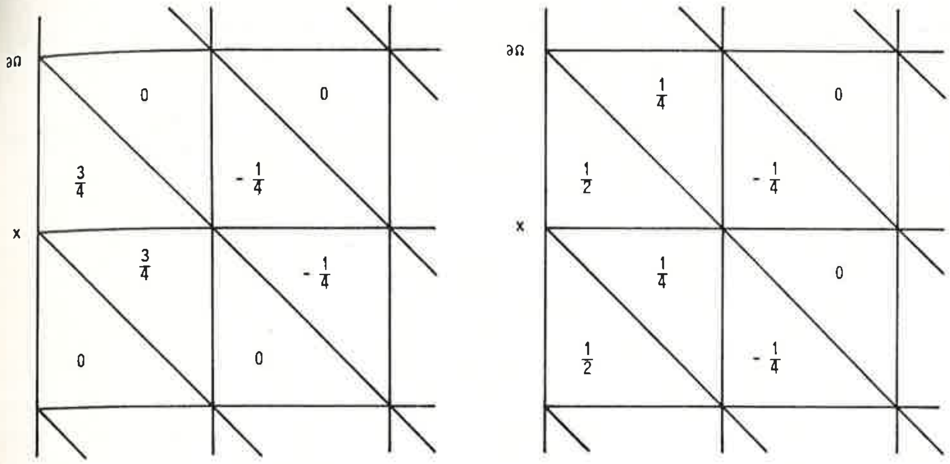

Fig. 3.

since any $T \in \mathscr{T}_h$ is contained in at most 13 sets $U(T^i)$, $i = 1, \ldots, k$ ($k \leqslant 13$), where T^2, $T^3, \ldots, T^k$ are neighbouring triangles to $T^1 = T$ (see Fig. 1). □

Remark 3.6. Theorem 3.1 can be easily modified also for other choices of G_h. The values of some other convenient weights for nodes from E_h are marked in Fig. 3 (Lemma 3.2 remains valid for them). As one can easily verify, also the choice of weights $w^1 = \cdots = w^6 = \frac{1}{6}$ in (2.1c) may be appropriately altered.

4. Global superconvergence estimates to the gradient and boundary flux

At first we shall show that for the weak solution u of (2.1) and its Ritz–Galerkin approximation u_h it is

$$\|\text{grad } u - G_h(u_h)\|_{0,\Omega} \leqslant Ch^2 \|u\|_{3,\Omega}, \tag{4.1}$$

whereas

$$\|\text{grad } u - \text{grad } u_h\|_{0,\Omega} \leqslant Ch\|u\|_{2,\Omega}$$

is the best possible rate.

Remark 4.1. In the proof of (4.1) we shall utilize the fact that

$$\|u_h - \Pi_h u\|_{1,\Omega} \leqslant Ch^2 \|u\|_{3,\Omega}. \tag{4.2}$$

This important result has been studied by several authors. We refer to [17,18] for the case of linear triangular elements, where Ω is a rectangle and the triangulations $\mathscr{T}_h$ are uniform consisting of right-angled isosceles triangles. The results of [17,18] have later been improved (see [13]) for any regular family of uniform triangulations of a polygonal domain including the effect of numerical quadrature (giving rise the term $\|f\|_{2,\Omega}$):

$$\|u_h - \Pi_h u\|_{1,\Omega} \leqslant Ch^2(\|u\|_{3,\Omega} + \|f\|_{2,\Omega}). \tag{4.3}$$

Note that the inequalities (4.2) and (4.3) are true also for quasi-uniform triangulations [13]. For related estimates to (4.2) and (4.3) see also recent results in [1,4].

Theorem 4.2. *Let the solution u of the problem* (2.1) *be in $H^3(\Omega)$. Then for a regular family of uniform triangulations of polygonal domain Ω the bound* (4.1) *is valid.*

Proof. Using the analogous arguments which we applied in proving (3.4), we find that

$$\| G_h(u_h - \Pi_h u) \|_{0,T} \leqslant 2 \| \mathrm{grad}(u_h - \Pi_h u) \|_{0,U(T)} \quad \forall T \in \mathcal{T}_h.$$

As any $T \in \mathcal{T}_h$ is contained in at most 13 sets $U(T^i)$ (cf. (3.15)), it is

$$\| G_h(u_h - \Pi_h u) \|_{0,\Omega} \leqslant 2\sqrt{13} \| \mathrm{grad}(u_h - \Pi_h u) \|_{0,\Omega}. \tag{4.4}$$

Making use of Theorem 3.1, (4.2) and (4.4), we come to

$$\| \mathrm{grad}\ u - G_h(u_h) \|_{0,\Omega}$$
$$\leqslant \| \mathrm{grad}\ u - G_h(\Pi_h u) \|_{0,\Omega} + \| G_h(u_h - \Pi_h u) \|_{0,\Omega} \leqslant Ch^2 \| u \|_{3,\Omega}. \quad \square$$

Remark 4.3. Sufficient assumptions guaranteeing $u \in H^3(\Omega)$ for domains having corners have been established by many authors [9,10,16,20]. For instance, if $f \in H^1(\Omega)$ in (2.1) and the angle of some corner is less than $\frac{1}{2}\pi$ then we have the H^3-regularity of u in a neighbourhood of the angular point (see e.g. [10, p.277]). If f belongs to some weighted Sobolev space, we get the H^3-regularity also for the right angle when considering the Poisson equation (see [10, p.280]). For instance, if $f \in C^2(\overline{\Omega})$ and $f(y) = 0\ \forall y \in Y$, then $u \in H^3(\Omega)$ provided Ω is a rectangle (see [16], p. 185]). Although the above assumptions are very restrictive, the post-processing (2.2) can give good numerical results even when $u \notin H^3(\Omega)$—cf. Section 5.

Furthermore, we show that the post-processing (2.2) may be applied to compute the boundary flux $\partial u / \partial n$ which we approximate by $n \cdot G_h(u_h)$ on $\partial\Omega$, where n is the outward unit normal to $\partial\Omega$. To this end we introduce the estimate (cf. (4.2))

$$| u_h - \Pi_h u |_{1,\infty,\Omega} \leqslant Ch^2 |\log h| \, \| u \|_{3,\infty,\Omega}, \tag{4.5}$$

which has been derived in [13,15] for the Poisson equation on a bounded convex domain Ω. The same bound was further obtained in [6] even for $A \in (W^1_\infty(\Omega))^{2\times 2}$ (see (2.1)). Note that for non-convex domains an interior W^1_∞-estimate analogous to (4.5) is known [5].

Similarly to Section 3 we prove the following lemma.

Lemma 4.4. *Let $\mathcal{M} = \{\mathcal{T}_h\}$ be a regular family of uniform triangulations of $\overline{\Omega}$. Then*

$$\| \mathrm{grad}\ u - G_h(\Pi_h v) \|_{0,\infty,\Omega} \leqslant Ch^2 | v |_{3,\infty,\Omega} \quad \forall v \in W^3_\infty(\Omega) \cap H^1_0(\Omega). \tag{4.6}$$

Proof. Choose $T \in \mathcal{T}_h \in \mathcal{M}$ such that $T \cap Y = \emptyset$. Then by (3.7), (3.8), and Lemma 3.3

$$\| \mathrm{grad}\ u - G_h(\Pi_h v) \|_{0,\infty,T} \leqslant \| B_T^{-1} \| \, \| \mathrm{grad}\ \hat{v} - \hat{G}(\hat{\Pi}\hat{v}) \|_{0,\infty,\hat{T}}$$
$$\leqslant C \| B_T^{-1} \| \, | \hat{v} |_{3,\hat{U}} \leqslant C' \| B_T^{-1} \| \, | \hat{v} |_{3,\infty,\hat{U}}.$$

Since (see [3, p.118])

$$| \hat{v} |_{3,\infty,\hat{U}} \leqslant C \| B_T \|^3 \, | v |_{3,\infty,U},$$

it follows from (3.5) that

$$\|\text{grad } v - G_h(\Pi_h v)\|_{0,\infty,T} \leqslant Ch^2 |v|_{3,\infty,U} \leqslant Ch^2 |v|_{3,\infty,\Omega}. \tag{4.7}$$

Next, let $T' \in \mathscr{T}_h$ be such that $\{y\} \in T' \cap Y$. Then (see [3, p.121])

$$\|\text{grad } v - L_h v\|_{0,\infty,T'} \leqslant Ch^2 |\text{grad } v|_{2,\infty,T'} \leqslant Ch^2 |v|_{3,\infty,\Omega}, \tag{4.8}$$

where L_h is defined by (3.11). As $(L_h v)(y) = (G_h(\Pi_h v))(y)$, there exists an appropriate vertex x $(x \neq y)$ of T' such that

$$\begin{aligned}\| L_h v - G_h(\Pi_h v)\|_{0,\infty,T'} &= \|(\text{grad } v - G_h(\Pi_h v))(x)\| \\ &\leqslant \|\text{grad } v - G_h(\Pi_h v)\|_{0,\infty,T} \leqslant Ch^2 |v|_{3,\infty,\Omega},\end{aligned} \tag{4.9}$$

where $T \in \mathscr{T}_h$ contains x and $T \cap Y = \emptyset$. Now, the combination of (4.7), (4.8) and (4.9) leads to the estimate (4.6). □

The last theorem shows that $n \cdot G_h(u_h)|_{\partial\Omega}$ produce higher-order correct approximation to the boundary flux than $(\partial u_h/\partial n)|_{\partial\Omega}$.

Theorem 4.5. *Let* $u \in W^3_\infty(\Omega)$ *be the solution of* (2.1) *with* $A \in (W^1_\infty(\Omega))^{2\times 2}$. *Then for a regular family of uniform triangulations of a convex polygon* Ω *it is*

$$\begin{aligned}\|\partial u/\partial n - n \cdot G_h(u_h)\|_{0,\infty,\partial\Omega} &\leqslant \|\text{grad } u - G_h(u_h)\|_{0,\infty,\Omega} \\ &\leqslant Ch^2 |\log h| \, \|u\|_{3,\infty,\Omega}.\end{aligned}$$

Proof. The first inequality is obvious. Since $u_h - \Pi_h u$ is piecewise linear, we find likewise (3.4) that

$$\begin{aligned}\|G_h(u_h - \Pi_h u)\|_{0,\infty,T} &\leqslant 2\|\text{grad}(u_h - \Pi_h u)\|_{0,\infty,U(T)} \\ &\leqslant 2|u_h - \Pi_h u|_{1,\infty,\Omega}\end{aligned}$$

for all $T \in \mathscr{T}_h$. Thus the use of (4.5) and (4.6) yields

$$\begin{aligned}&\|\text{grad } u - G_h(u_h)\|_{0,\infty,T} \\ &\quad\leqslant \|\text{grad } u - G_h(\Pi_h u)\|_{0,\infty,T} + \|G_h(\Pi_h u - u_h)\|_{0,\infty,T} \\ &\quad\leqslant Ch^2 |\log h| \, \|u\|_{3,\infty,\Omega}. \quad \square\end{aligned}$$

5. Numerical tests

The averaged gradient proposed in this paper has been compared with the gradient of the Ritz–Galerkin solution based on linear elements.

Example 5.1. Assume $\Omega = (0, 1) \times (0, 1)$ and choose f such that

$$u(x, y) = x^\alpha(1 - x) \sin \pi y, \quad \alpha > 0,$$

is the exact solution of the problem

$$\begin{aligned}-\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{5.1}$$

Table 1

h^{-1}	$\|\gamma_h\|_{0,\Omega}$	$\|\delta_h\|_{0,\Omega}$	$\|n\cdot\gamma_h\|_{0,\infty,\partial\Omega}$	$\|n\cdot\delta_h\|_{0,\infty,\partial\Omega}$
4	0.332371	0.134338	0.426777	0.241622
8	0.174089	0.039866	0.220132	0.066431
16	0.088090	0.010663	0.106613	0.016991
32	0.044177	0.002740	0.051667	0.004235

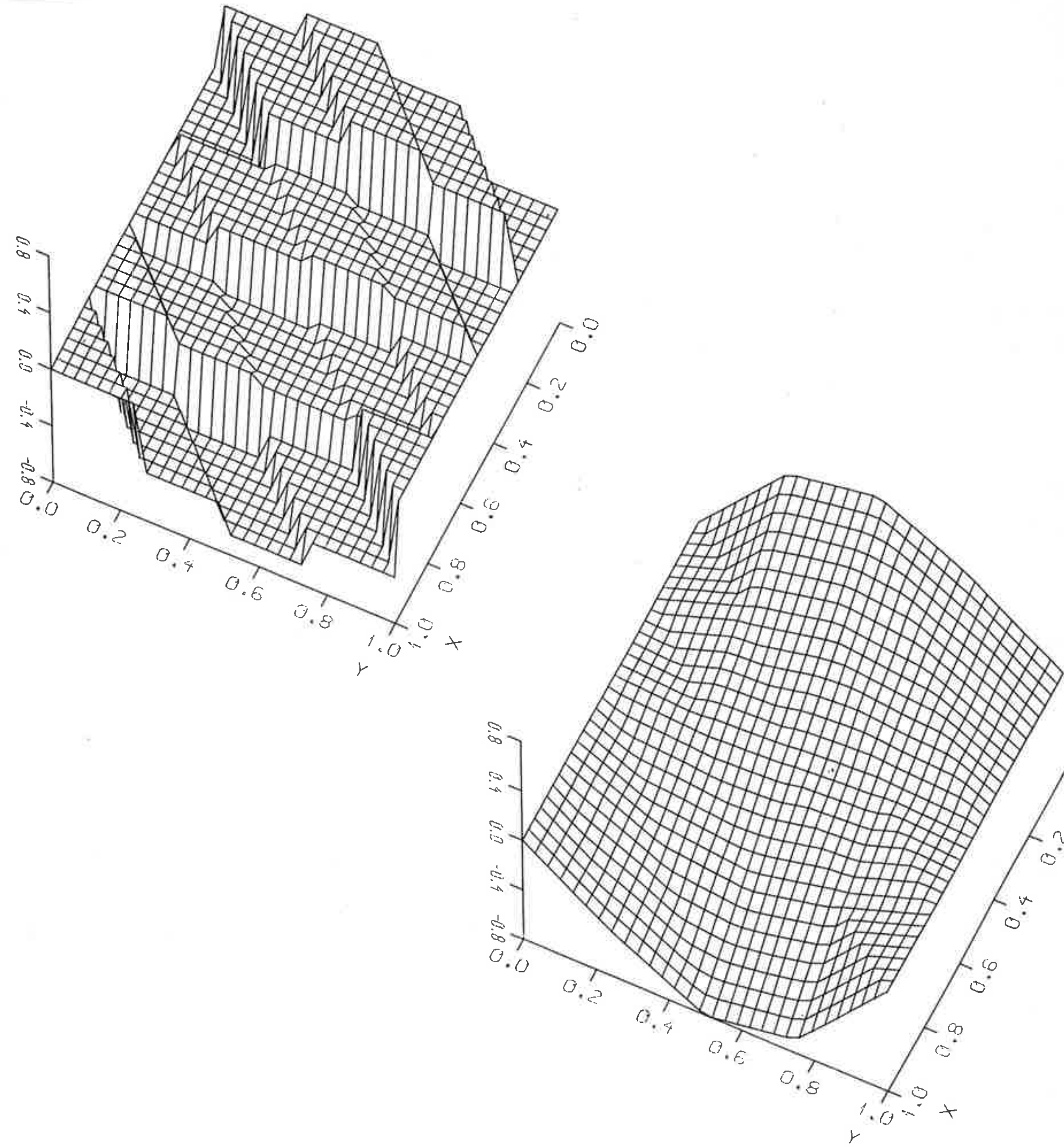

Fig. 4. First component of FE-grad and post-proc grad.

Table 2

h^{-1}	$\Vert\gamma_h\Vert_{0,\Omega}$	$\Vert\delta_h\Vert_{0,\Omega}$	$\Vert n\cdot\gamma_h\Vert_{0,\infty,\partial\Omega}$	$\Vert n\cdot\delta_h\Vert_{0,\infty,\partial\Omega}$
4	0.250312	0.124575	0.500345	0.328147
8	0.133086	0.034031	0.277955	0.098464
16	0.067719	0.008633	0.138006	0.025873
32	0.034046	0.002182	0.067914	0.006582

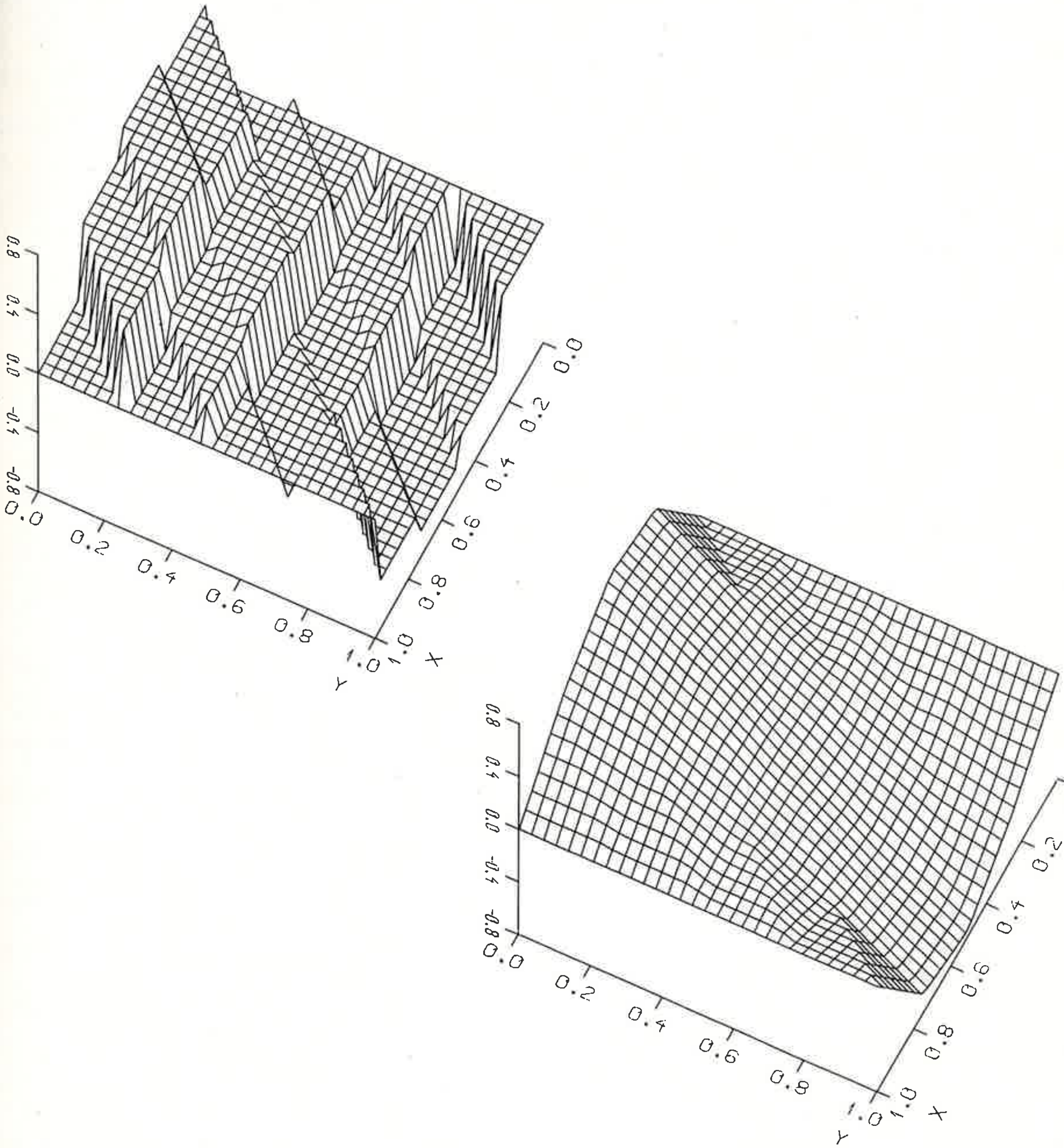

Fig. 5. Second component of FE-grad and post-proc. grad.

In Table 1 we have listed the values of the L^2-norms for $\gamma_h = \text{grad } u - \text{grad } u_h$ and $\delta_h = \text{grad } u - G_h(u_h)$ for different discretization parameters h and $\alpha = 1$.

Table 1 confirms the theoretical results. The growth of the CPU-time due to the post-processing is essentially negligible. In Example 5.1 the time requirements of the Ritz–Galerkin procedure are 0.153, 1.04, 6.49, 41.7, 273 seconds whereas those of the post-processing are 0.031, 0.098, 0.355, 1.32, 5.24 seconds, respectively.

In Figs. 4 and 5 the gradient of the Ritz–Galerkin approximation (piecewise constant) and the corresponding post-processed approximation (piecewise linear) have been illustrated ($h = \frac{1}{4}$).

Table 2 shows the errors for $\alpha = \frac{7}{4}$, i.e. $u \notin H^3(\Omega)$.

Remark 5.2. The post-processing method presented here can also be applied in three-dimensional problems and time-dependent problems.

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