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Counterexamples to the Kalman Conjectures ⋆

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Abstract: In the paper counterexamples to the Kalman conjecture with smooth nonlinearity basing on the Fitts system, that are periodic solution or hidden chaotic attractor are presented. It is shown, that despite the fact that Kalman's conjecture (as well as Aizerman's) turned out to be incorrect in the case of $n > 3$, it had a huge impact on the theory of absolute stability, namely, the selection of the class of nonlinear systems whose stability can be studied with linear methods.

Keywords: Kalman conjecture, Fitts system, Barabanov system, point-mapping method, hidden attractor

1. INTRODUCTION

In the middle of the past century the theory of absolute stability was rapidly developed [Lurie and Postnikov, 1944, Bulgakov, 1943, Aizerman, 1949, Letov, 1965, Pliss, 1958, LaSalle and Lefschetz, 1961, Yakubovich, 1958, Aizerman and Gantmakher, 1963, Andronov et al., 1966, Gelig et al., 1978].history

For continuous and discontinuous nonlinearities one of the challenging problems is the selection of classes of systems for which it is possible to obtain a necessary and sufficient condition for absolute stability. The history of attempts to solve this problem is connected with the Aizermans' [Aizerman, 1949] and Kalmans' [Kalman, 1957] conjectures about absolute stability of control systems with nonlinearity satysfying Routh-Hurwitz criterion. In the present paper the differences in the behavior of systems with continuous and discontinuous nonlinearities that are counterexamples to the Kalman conjecture are considered.

Aizerman's conjecture was complitely investigated in two-dimensional case [Malkin, 1952, Erugin, 1952, Krasovsky, 1952]. It turned out to be true except for the special case when trajectories tend to infinity.

In 1957 R.E. Kalman, being unaware of Aizerman's research, proposed a statement concerning restrictions on the derivative of nonlinearity to be in the Hurwitz angle.

Kalman conjecture is more rigorous than Aizermans' one, so it turned out to be valid for two- and three-dimensional cases [Leonov et al., 1996]. These cases are natural for applied mechanical problems, so it is necessary to emphasize Kalmans' scientific intuition.

Unlike the continuous-time case, Kalman conjecture is false in general for two-dimensional discrete-time systems [Alli-Oke et al., 2012].

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By now Kalman conjecture remains unsolved in the general case.

2. KALMAN CONJECTURE

Consider the following system with one scalar non-linearity in the Lur'e form

$$\dot{x} = Ax + b\varphi(\sigma), \qquad \sigma = c^*x, \tag{1}$$

where A is a constant $n \times n$ matrix, b and c - constant n-dimensional vectors, all quantities are real, $*$ is the sign of transposition, φ is a smooth scalar function with $\varphi(0) = 0$ and the following condition is satisfied at differentiability points:

$$k_1 \leq \varphi'(\sigma) \leq k_2, \qquad \sigma \in (-\infty, +\infty), \tag{2}$$

where k_1 is a number or $-\infty$, k_2 is a number or $+\infty$.

In 1957, R.E. Kalman formulated the following conjecture: if a linear system $\dot{x} = Ax + kbc^*x$, $k \in [k_1, k_2]$, is globally asymptotically stable, then the system (1) is also globally asymptotically stable. Let us recall that a system is globally asymptotically stable if its zero solution is Lyapunov stable and $\lim_{t\to+\infty} |x(t, x_0)| = 0$ for any $x_0 \in \mathbb{R}^n$.

3. HISTORY

The first counterexample to the Kalman conjecture were obtained due to experiments by Fitts [1966], who studied oscillations in nonlinear feedback systems.

Further attempts to construct counterexamples were mainly related to the consideration of systems with discontinuous piecewise-linear nonlinearities and integration of such systems in sections of linearity [Andronov et al., 1966].

In the beginning of the past century the concept of discontinuous system appeared in study of various applied mechanical problems, e.g. vibrations in a mechanical model with dry friction [den Hartog, 1930], damping flutter in aircraft control systems with dry friction [Keldysh, 1944], autopilot construction problem [Andronov and Bautin, 1955].

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Andronov et al. [1966] introduced the idea of trajectory "sewing" and developed all the necessary "ingredients" of discontinuous systems theory. Later, the elements of the discontinuous systems theory were rigorously formulated in [Wazewski, 1961, Filippov, 1988].

At first the development of the theory of absolute stability was related to the names of its founders Lurie and Postnikov [1944]. They tried to solve the problem of absolute stability of automatic control systems using Lyapunov function method. Popov [1961, 1973] developed original and effective criterion in the form of frequency sufficient condition for absolute stability. The conjecture that sufficient conditions of absolute stability, obtained by using of frequency methods, are also necessary conditions was refuted by Yakubovich [1967], who constructed an absolutely stable system, for which the Popov's frequency condition is not satisfied, and later by Pyatnitsky [1973]. Important results by Yakubovich [1962] and Kalman [1963] resulted in a well-known Kalman-Yakubovich-Popov lemma (see [Barabanov et al., 1996]).

Also it was natural to generalize various concepts of Lyapunov's stability theory and frequency approach to the discontinuous systems theory. The first corresponding results and new different approaches were obtained by representatives of the scientific school of V.A. Yakubovich ([Yakubovich, 1967, 1975, Gelig et al., 1978, Barabanov et al., 1996]).

Mention that similar results independently obtained in [Shevitz and Paden, 1994].

Later results given in [Gelig et al., 1978] were developed and new methods of stability analysis of discontinuous control systems were presented. Note that only sufficient conditions for absolute stability of discontinuous systems were stated [Gelig et al., 1978].

Now let us consider two counterexamples to Kalman conjecture and verify the fulfillment of analytical sufficient conditions for global asymptotic stability of corresponding systems.

4. FITTS COUNTEREXAMPLE

As already mentioned, the first counterexample to the Kalman conjecture was proposed by Fitts [Fitts, 1966], who performed the computer simulation of system (1) with the transfer function

$$W(p) = \frac{p^2}{((p+\beta)^2+0.9^2)((p+\beta)^2+1.1^2)} \tag{3}$$

and the cubic nonlinearity $\varphi(\sigma) = K\sigma^3$. As a result of the simulation, Fitts discovered periodic solutions of the system (1) for the values of parameters $m_1 = 0.9$, $m_2 = 1.1$, $K = 10$ and $\beta \in (0.01, 0.75)$. However, later N.E. Barabanov showed in [Barabanov, 1988] that the results of the experiments were incorrect for a part of the parameters that Fitts considered, specifically, for $\beta \in (0.572, 0.75)$. The Kalman conjecture was further discussed and doubts in the counterexamples of Fitts and Barabanov were raised in [Bernat and Llibre, 1996, Glutsyuk, 1998, Meisters, 1996].

4.1 Fitts' counterexample variation

Let us present the following novel variation of Fitts' counterexample. Consider system (1) with $n = 4$ defined by transfer function (3) from Fitts' counterexample with the nonlinearity

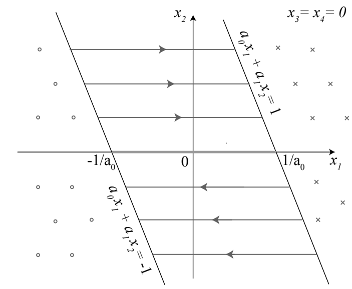

Fig. 1. Sliding mode manifold $\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_3 = x_4 = 0, -1 \le a_0x_1 + a_1x_2 \le 1\}$ for the Fitts system (4). Arrowed lines define the motion on the surface, thick green line defines the rest segment.

$\varphi(\sigma) = \text{sign}(\sigma)$. Deriving the system from transfer function (3), one obtains [Leonov, 2001]:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + \text{sign}(-x_3), \end{aligned} \tag{4}$$

where $a_0 = (m_1^2+\beta^2)(m_2^2+\beta^2)$, $a_1 = 2\beta(m_1^2+m_2^2+2\beta^2)$, $a_2 = m_1^2+m_2^2+6\beta^2$, $a_3 = 4\beta$.

Here

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, c = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{5}$$

Sliding mode manifold for the system (4) is given by:

$$\begin{aligned} D_{\text{fitts}} = \{&(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_3 = x_4 = 0, \\ &-1 \le a_0x_1 + a_1x_2 \le 1\}, \end{aligned}$$

Moreover, a sliding mode is described by the equations

$$\dot{x}_1 = x_2, \dot{x}_2 = 0, \dot{x}_3 = 0, \dot{x}_4 = 0, \tag{6}$$

so for the point $(x_{01}, x_{02}, 0, 0) \in D_{\text{fitts}}$ one gets $x_1(t) = x_{02}t + x_{01}, x_2(t) \equiv x_{02}$. The rest segment is

$$\Lambda_{\text{fitts}} = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_2 = x_3 = x_4 = 0, \tag{7}$$

$$-\tfrac{1}{a_0} \le x_1 \le \tfrac{1}{a_0}\}. \tag{8}$$

4.2 Stability of Rest Segment

If there are trajectories that tend to some periodic solution of the system or infinity, then one can say that the system is not globally asymptotically stable.

Hidden and self-excited classification. Since is not proven that system (4) is globally asymptotically stable, one can expect the existence of a nontrivial attractor in the phase space. First let us recall the definition of attractor.

Consider system

$$\dot{\mathrm{x}} = f(\mathrm{x}, t), \tag{9}$$

where $\mathrm{x} \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^n$ Define by $\mathrm{x}(t, \mathrm{x}_0)$ a solution of (9) such that $\mathrm{x}(0, \mathrm{x}_0) = \mathrm{x}_0$.

Definition 1. For system (9), a bounded closed invariant set K is

(i) a *(local) attractor* if it is a locally attractive set (i.e. $\lim_{t\to+\infty} \text{dist}(K, \mathrm{x}(t, \mathrm{x}_0)) = 0 \ \forall \mathrm{x}_0 \in K(\varepsilon)$, where $K(\varepsilon)$ is a certain ε-neighborhood of set K),

(ii) a *global attractor* if it is a globally attractive set (i.e. $\lim_{t\to+\infty} \mathrm{dist}(K, \mathrm{x}(t, \mathrm{x}_0)) = 0 \ \forall \mathrm{x}_0 \in \mathbb{R}^n$),

where $\mathrm{dist}(K, \mathrm{x}) = \inf_{v\in K} ||v - \mathrm{x}||$ is the distance from the point $\mathrm{x} \in \mathbb{R}^n$ to the set $K \subset \mathbb{R}^n$ (see, e.g. [Leonov et al., 2015]).

Since the whole phase space is a global attractor and any finite union of attractors is again an attractor, it is reasonable to consider only minimal global and local attractors, i.e. the smallest bounded closed invariant set possessing the property (ii) or (i).

Localization and analysis of attractors is one of the main tasks of the investigation of dynamical systems. While trivial attractors (stable equilibrium points) can be easily found analytically, the search of periodic and chaotic attractors can turn out to be a challenging problem. For numerical localization of an attractor one needs to choose an initial point in the basin of attraction and observe how the trajectory, starting from this initial point, after a transient process visualizes the attractor. Leonov and Kuznetsov introduced in [Leonov et al., 2011, Leonov and Kuznetsov, 2011, Kuznetsov and Leonov, 2014] a classification of attractors based on the simplicity of finding the basins of attraction in the phase space.

Definition 2. An attractor is called a *self-excited attractor* if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise, it is called a *hidden attractor*.

Self-excited attractors can be easily visualized because its basin of attraction is connected with an unstable equilibrium and, therefore, can be localized numerically. For a hidden attractor, its basin of attraction is not connected with equilibria and, thus, the search and visualization of hidden attractors in the phase space may be a difficult task.

Further using a special computational package [Piiroinen and Kuznetsov, 2008] and Andronov's point-mapping method [Andronov and Maier, 1947], it will be shown that in system (4), for certain values of the β parameter it is possible to localize hidden attractors. Also it will be shown that this hidden attractors coexist with periodic solutions.

4.3 Trajectories computation

We performed numerical simulation in the vicinity $\varepsilon = 0.1$ of the rest segment (7) in the subspace (x_1, x_4) while $x_2 = x_3 = 0$. In our experiment for integration of solutions we used computational package from [Piiroinen and Kuznetsov, 2008]. Trajectories with initial point in the vicinity tended to a periodic solution (see Fig. 2).

4.4 Trajectories sewing

Let's write down the solutions of linear systems $\dot{\mathrm{x}} = A\mathrm{x} + b$ and $\dot{\mathrm{x}} = A\mathrm{x} - b$ given by (5) in the corresponding regions $\Sigma^+ = \{\mathrm{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 < 0\}$, $\Sigma^- = \{\mathrm{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 > 0\}$. Trajectories of (4) in three regions of phase space as the solutions of the linear systems may be obtained analytically without using numerical methods for solving ordinary differential equations and sewing them when switching modes. This gives the trajectory released from the point $(x_{01}, x_{02}, x_{03}, x_{04}) = (10, 10, 10, 10)$ for parameters values $m_1 = 0.9, m_2 = 1.1, \beta = 0.03$ on the time interval $t \in [0, 500]$ and precision of 32 digits. Calculation shows that this trajectory attracts to the periodic orbit (see Fig. 3).

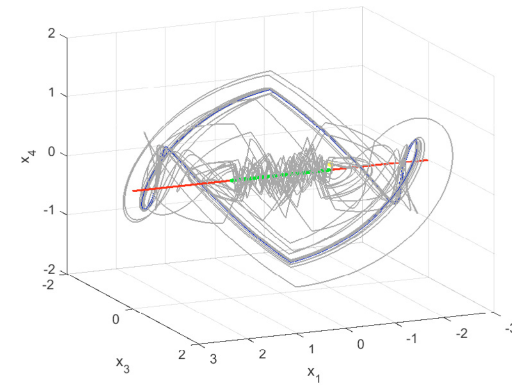

Fig. 2. Trajectory (gray) released from the point (-1.105017,0,0,0.05) (yellow) from vicinity of rest segment Λ_{fitts} tends to periodic solution (blue). $\beta = 0.03$

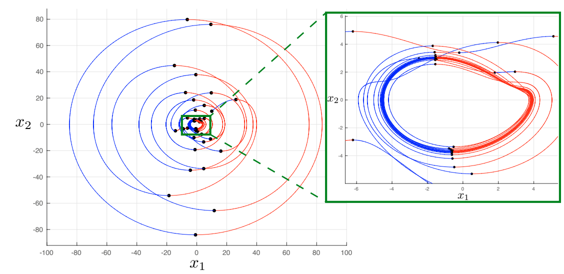

Fig. 3. Modeling of the system (4) for $\beta = 0.03$. Trajectories of the system $\dot{x} = Ax + b$ (red) are being sewed with the trajectories of the system $\dot{x} = Ax - b$ (blue) at the switching mode points (black).

4.5 Point-mapping method

This result can be clarified using Andronov's point-mapping method [Andronov and Maier, 1947]. Note that periodic solution of the system (4) consists of two parts: $\mathrm{x}^+(t, \mathrm{x}_0^+) \in \Sigma^+$, $t \in [0, T_+^{\mathrm{sw}}]$ (mode I) and $\mathrm{x}^-(t, \mathrm{x}_0^-) \in \Sigma^-$, $t \in [0, T_-^{\mathrm{sw}}]$ (mode II). Wherein $\mathrm{x}^\pm(0, \mathrm{x}_0^\pm) = \mathrm{x}_0^\pm = (x_{01}^\pm, x_{02}^\pm, 0, x_{04}^\pm)$, where $x_{04}^+ < 0$, $x_{04}^- > 0$ and $\mathrm{x}^\pm(T_\pm^{\mathrm{sw}}, \mathrm{x}_0^\pm) = \mathrm{x}_0^\mp$. Therefore the following equality holds:

$$\mathrm{x}^-(T_-^{\mathrm{sw}}, \mathrm{x}_0^-) = \mathrm{x}_0^+ = \mathrm{x}^+(-T_+^{\mathrm{sw}}, \mathrm{x}_0^-). \tag{10}$$

By the analogy with Sec. 4.4 for solutions $\mathrm{x}^\pm(t, \mathrm{x}_0^\pm)$, the solution can be found analytically. For parameters $m_1 = 0.9$, $m_2 = 1.1, \beta = 0.03$ the values, found with the help of MATLAB software, are given in Tab. 1. Using the coordinates of the initial point $(x_{01}^-, x_{02}^-, 0, x_{04}^-)$ we can localize orbitally asymptotically stable periodic solution (see Fig. 8). Note that this periodic solution coexists with periodic solution obtained in 4.3.

4.6 Strange attractor

Now we are going to use continuation method for numerical localization of nonperiodic strange attractor in the system (4). It is often used for hidden attractors localization [Leonov et al., 2010, Bragin et al., 2011, Leonov and Kuznetsov, 2011, 2013]. In this method, a sequence of systems is considered and each corresponds to a specially chosen parameter with values in a certain interval. It is assumed that for the first (initial) system the initial data for numerical localization of periodic (or chaotic) solutions can be obtained analytically. Thus, we can

Table 1. Coordinates of the point on the periodic solution of the system (4) for $\beta = 0.03$ and the duration of the modes I and II.

x_{01}^{-}	$-0.62520516260693109534342362490723$
x_{02}^{-}	$-3.7324097072650610465825278562594$
x_{04}^{-}	$3.4754169728697120793989274111636$
T_{+}^{sw}	$6.0861163299591904401929427933543$
T_{-}^{sw}	$3.2558143241394617470571435917368$

consider a system with an initial self-excited attractor as an initial system. Then we can numerically trace the transformation of the initial solution in the transition from one system to another. At the same time, the initial data for the solution of the next system is the endpoint of the solution of the previous system. The latter system corresponds to a system for which a hidden attractor is sought. As a result, if there is no loss of stability bifurcation, then it is possible to find hidden attractor.

Consider an interval $\beta \in [0.03, 0.1]$ and choose the partition with the step 0.0175. For fixed $m_1 = 0.9$, $m_2 = 1.1$ and for each $\beta = \beta^j = 0.03 + 0.0175j$, $j = 0, \ldots, 4$ we will integrate the solution $\mathrm{x}^j(t)$ of the system (4) on the time interval $[0, T]$, $T = 2000$.

We use as initial data for the system with $\beta = \beta^{j+1}$ the endpoint of the solution with $\beta = \beta^j$, i.e. $\mathrm{x}^{j+1}(0) := \mathrm{x}^j(T)$. Here we can integrate the solutions both using the procedure described in Sec. 4.3 and special computational package described in [Piiroinen and Kuznetsov, 2008] for modeling solutions in Filippov sense. Using the second option and performing the continuation method we localized strange nonperiodic attractor (see Fig. 6,Fig. 7). Also note that this attractor coexist with periodic solution (see Fig. 9, Fig. 10).

This strange attractor (as well as periodic solution for $\beta = 0.03$) remains under the reverse scenario of discontinuous Aizerman – Pyatnitsky approximation [Aizerman and Pyatnitskiy, 1974], i.e. transition from nonlinearity $\varphi(\sigma) = \psi_0(\sigma) = \operatorname{sign}\sigma$ to $\varphi(\sigma) = \psi_N(\sigma)$ nonlinearity where

$$\varphi(\sigma) = \psi_N(\sigma) \equiv \begin{cases} -1, & \sigma \le -N, \\ \frac{1}{N}\sigma, & -N \le \sigma \le N, \\ 1, & \sigma \ge N \end{cases} \tag{11}$$

for sufficiently small values of N (e.g. for $N = 0.05$) in the system (4).

Then using continuation method we consider nonlinearity $\varphi(\sigma) = \chi_\varepsilon(\sigma) \equiv \psi_N(\sigma) + \varepsilon(\tanh(\sigma/N) - \psi_N(\sigma))$ for ε increasing from 0 to 1 with the step 0.1 to implement transition from piecewise-differentiable nonlinearity (11) (corresponding to $\varphi(\sigma) = \chi_0(\sigma) = \psi_N(\sigma)$) to smooth nonlinearity $\varphi(\sigma) = \chi_1(\sigma) = \tanh(\sigma/N)$. During this transition the local strange attractor obtained in the previous steps is preserved (see Fig. 4–5).

Thus in the system (1) with $\varphi(\sigma) = \tanh(\sigma/N)$ for sufficient small values of N there is strange attractor and for $k_1 < 0$ and $k_2 = +\infty$ Kalman conjecture is wrong.

5. BARABANOV SYSTEM

In 1988 N.E. Barabanov constructed the following counterexample to the Kalman conjecture:

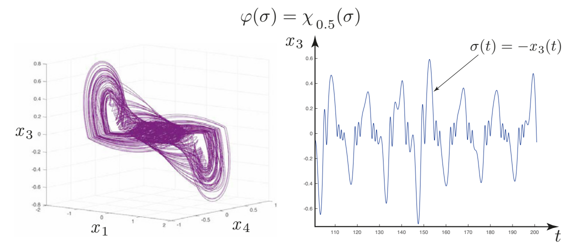

Fig. 4. Strange attractor in the system (4) for $\beta = 0.1$ and $\varphi(\sigma) = \chi_\varepsilon(\sigma) \equiv \psi_N(\sigma) + \varepsilon(\tanh(\sigma/N) - \psi_N(\sigma))$, $N = 0.01$, $\varepsilon = 0.5$.

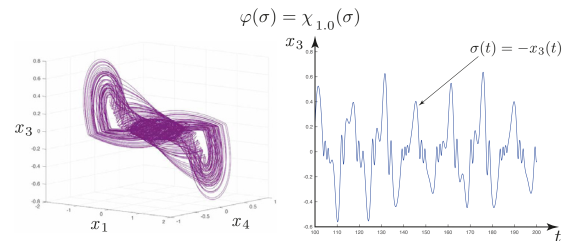

Fig. 5. Strange attractor in the system (4) for $\beta = 0.1$ and $\varphi(\sigma) = \chi_\varepsilon(\sigma) \equiv \psi_N(\sigma) + \varepsilon(\tanh(\sigma/N) - \psi_N(\sigma))$, $N = 0.01$, $\varepsilon = 1$.

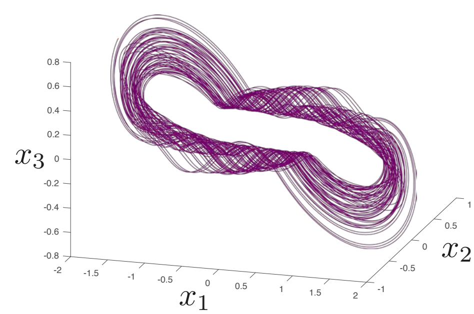

Fig. 6. Projection of the strange attractor in the system (4) for $\beta = 0.1$ in the subspace (x_1, x_2, x_3).

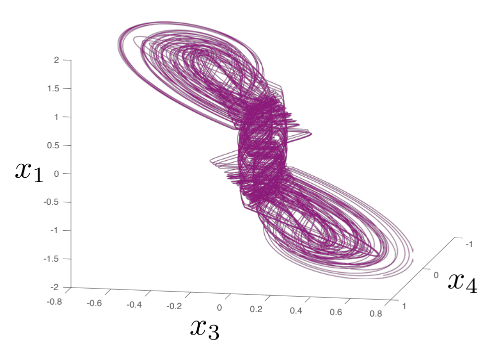

Fig. 7. Projection of the strange attractor in the system (4) for $\beta = 0.1$ in the subspace (x_1, x_3, x_4).

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_4, \\ \dot{x}_3 &= x_1 - 2x_4 - \varphi(x_4), \\ \dot{x}_4 &= x_1 + x_3 - x_4 - \varphi(x_4), \end{aligned} \tag{12}$$

where $\varphi = \operatorname{sign}(\sigma)$. In this case

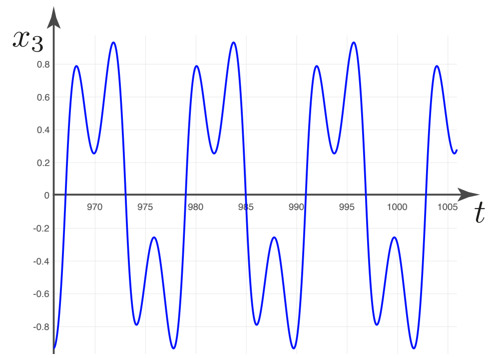

Fig. 8. Periodic solution of the system (4) for $\beta = 0.03$.

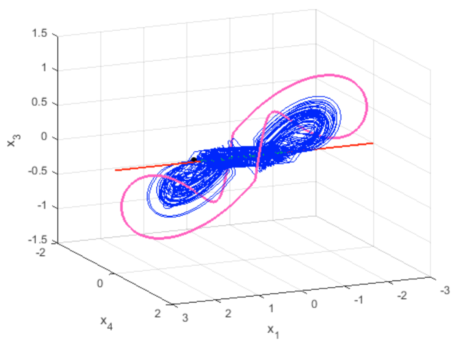

Fig. 9. Coexisting periodic solution and chaotic attractor in (4) for $\beta = 0.1$ in the subspace (x_1, x_3, x_4)

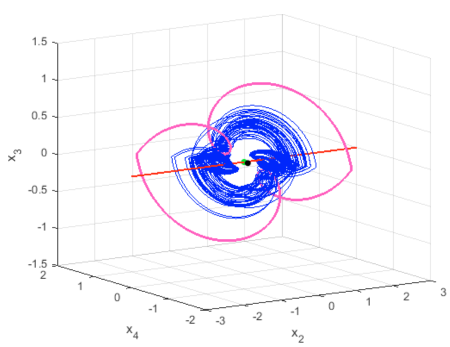

Fig. 10. Coexisting periodic solution and chaotic attractor in (4) for $\beta = 0.1$ in the subspace (x_2, x_3, x_4)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sliding mode manifold for system (12) is

$$D_{\text{bar}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = 0, x_2 = C_1, \\ x_1 = C_1 t + C_2, x_3 = C_3 e^{-t}, -1 \le x_1 + x_3 \le 1\}$$

Rest segment for the system (12) is

$$\Lambda_{\text{bar}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 = x_3 = x_4 = 0, \\ -1 \le x_1 \le 1\}.$$

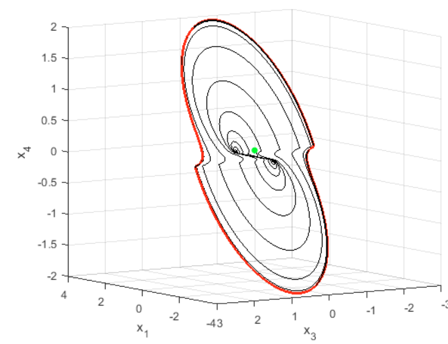

Fig. 11. Trajectory (black) released from the point $(0,0,0,\varepsilon)$ (green) and tends to the periodic solution (red) in the subspace (x_1, x_3, x_4). $T = 5000$.

5.1 Trajectories computation

For the Barabanov system the trajectories released from the vicinity of the rest segment have been numerically found by a simulation. In our experiment vicinity radius is $\varepsilon = 0.1$ in the space (x_1, x_4) (in this case we take $x_2 = x_3 = 0$). Resulting trajectories tended to the periodic solutions of the system (12), see Fig. 11.

6. CONCLUSION

In this paper we presented counterexamples to the Kalman conjecture with smooth nonlinearity basing on the Fitts system, that are periodic solution or hidden chaotic attractor. However, despite the fact that Kalman's conjecture (as well as Aizerman's) turned out to be incorrect in the case of $n > 3$, it had a huge impact on the theory of absolute stability, namely, the selection of the class of nonlinear systems whose stability can be studied with linear methods.

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