Functional Type Error Control for Stabilised Space-Time IgA Approximations to Parabolic Problems

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Abstract. The paper is concerned with reliable space-time IgA schemes for parabolic initial-boundary value problems. We deduce a posteriori error estimates and investigate their applicability to space-time IgA approximations. Since the derivation is based on purely functional arguments, the estimates do not contain mesh dependent constants and are valid for any approximation from the admissible (energy) class. In particular, they imply estimates for discrete norms associated with stabilised space-time IgA approximations. Finally, we illustrate the reliability and efficiency of presented error estimates for the approximate solutions recovered with IgA techniques on a model example.

Keywords: Error control · Functional error estimates
Stabilised space-time IgA schemes · Fully-adaptive space-time schemes

Countless usage of the time-dependent systems governed by parabolic partial differential equations (PDEs) in scientific and engineering applications trigger their active investigation in mathematical and numerical modelling. By virtue of the fast development of parallel computers, treating time in the evolutionary equations as yet another dimension in space became quite natural. The so-called *space-time approach* is not restricted with pitfalls of time-marching schemes. On the contrary, it becomes quite useful when efficient parallel methods and their implementation on massively parallel computers are considered (rather than attempt to reiterate all prior work, we refer the reader to [17], whose introductory section contains an extensive overview of various space-time techniques).

Investigation of effective adaptive refinement methods is crucial for the construction of fast and efficient solvers for PDEs. In the same time, the aspect of scheme localisation is strongly linked with reliable and quantitatively efficient a posteriori error estimation tools. The latter one is expected to identify the areas of the considered computational domain with relatively high discretization errors and provide an automated refinement strategy in order to reach the desired accuracy level for the current reconstruction. Local refinement tools of IgA (e.g., T-splines, THB-splines, and LR-splines) have been combined with various a posteriori error estimation techniques, e.g., error estimates (EEs) using the hierarchical basis [4,26], residual-based [2,9,13], and goal-oriented EEs [3,14,27]. Below, we use a different (functional) method providing fully guaranteed EEs in the various weighted norms equivalent to the global energy norm. These estimates include only global constants (independent of the mesh characteristic h) and are valid for any approximation from the admissible functional space. Functional EEs (so-called majorants and minorants) were introduced in [24] and later applied to different mathematical models [18,22]. They provide guaranteed, sharp, and fully computable upper and lower bounds of errors. This approach in combination with IgA approximations generated by tensor-product splines was investigated in [11] for elliptic boundary value problems (BVPs).

In this paper, we derive functional-type a posteriori EEs for time-dependent problems in the context of the space-time IgA scheme introduced in [17]. The latter one exploits the time-upwind test function motivated by the space-time streamline diffusion method (see, e.g., [8,10]) and approximations provided by IgA framework. By exploiting the universality and efficiency of the considered EEs as well as the smoothness of the IgA approximations, we aim at the construction of fully-adaptive, fast and efficient parallel space-time methods that could tackle complicated problems inspired by industrial applications.

This work has the following structure: Sect. 1 defines the problem and discusses its solvability, whereas Sect. 2 presents the stabilised space-time IgA scheme with its main properties. An overview of main ideas and definitions used in the IgA framework can be found in the same section. In Sect. 3, we introduce new functional type a posteriori EEs using the stabilised formulation of parabolic initial BVPs (I-BVPs). Finally, Sect. 4 presents numerical results demonstrating the efficiency of the majorants in the elliptic case.

1 Model Problem

Let $\overline{Q} := Q \cup \partial Q, Q := \Omega \times (0, T)$, denote the space-time cylinder, where $\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}$, is a bounded Lipschitz domain with boundary $\partial\Omega$, and $(0, T)$ is a given time interval, $0 < T < +\infty$. Here, the cylindrical surface is defined as $\partial Q := \Sigma \cup \overline{\Sigma}_0 \cup \overline{\Sigma}_T$ with $\Sigma = \partial\Omega \times (0, T)$, $\Sigma_0 = \Omega \times \{0\}$, and $\Sigma_T = \Omega \times \{T\}$. We discuss our approach to guaranteed error control of space-time approximations with the paradigm of the classical *linear parabolic I-BVP*: find $u : \overline{Q} \rightarrow \mathbb{R}$ satisfying the system

$$\partial_t u - \Delta_x u = f \quad \text{in} \quad Q, \qquad u = 0 \quad \text{on} \quad \Sigma, \qquad u = u_0 \quad \text{on} \quad \overline{\Sigma}_0, \tag{1}$$

where ∂_t is the time derivative, Δ_x denotes the Laplace operator in space, $f \in L^2(Q)$, and $u_0 \in H^1_0(\Sigma_0)$ are the given source function and initial data, respectively. Here, $L^2(Q)$ is the space of square-integrable functions over

Q quipped with the usual norm and scalar product denoted respectively by $\| v \|_Q := \| v \|_{L^2(Q)}$ and $(v, w)_Q := \int_Q v(x,t) w(x,t)\, \mathrm{d}x\mathrm{d}t, \forall v, w \in L^2(Q)$.

By $H^k(Q), k \geq 1$, we denote spaces of functions having generalised square-summable derivatives of the order k with respect to (w.r.t.) space and time. Next, we introduce the Sobolev spaces $H_0^1(Q) := \{ w \in H^1(Q) : w|_\Sigma = 0 \}$, $H^1_{0,\overline{0}}(Q) := \{ w \in H_0^1(Q) : w_{\Sigma_T} = 0 \}, V_0 := H^1_{0,\underline{0}}(Q) := \{ w \in H_0^1(Q) : w_{\Sigma_0} = 0 \}$, and $V_0^{\Delta_x} := H_0^{\Delta_x,1}(Q) := \{ w \in H_0^1(Q) \, : \, \Delta_x w \in L^2(Q) \}$. Moreover, we use auxiliary Hilbert spaces for vector-valued functions

$$H^{\mathrm{div}_x,0}(Q) := \{ \boldsymbol{y} \in [L^2(Q)]^d \; : \; \mathrm{div}_x \boldsymbol{y} \in L^2(Q) \} \text{ and}$$
$$H^{\mathrm{div}_x,1}(Q) := \{ \boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q) \; : \; \partial_t \boldsymbol{y} \in [L^2(Q)]^d \}$$

equipped with respective semi-norms $\|\boldsymbol{y}\|^2_{H^{\mathrm{div}_x,0}} := \|\mathrm{div}_x \, \boldsymbol{y}\|^2_Q$ and $\|\boldsymbol{y}\|^2_{H^{\mathrm{div}_x,1}} := \|\mathrm{div}_x \, \boldsymbol{y}\|^2_Q + \|\partial_t \, \boldsymbol{y}\|^2_Q$.

Further in the paper, C_{F} stands for the constant in the Friedrichs inequality $\|w\|_Q \leq C_{\mathrm{F}} \|\nabla_x w\|_Q, \forall w \in H_0^{1,0}(Q) := \{ w \in L^2(Q) \colon \nabla_x w \in [L^2(Q)]^2, w|_\Sigma = 0 \}$. From [15, Theorem 2.1] it follows that, if $f \in L^2(Q)$ and $u_0 \in H_0^1(\Sigma_0)$, the problem (1) is uniquely solvable in $V_0^{\Delta_x}$, and the solution u depends continuously on t in the $H_0^1(\Omega)$-norm. Moreover, according to [15, Remark 2.2], $\| \nabla_x u(\cdot, t) \|^2_\Omega$ is an absolutely continuous function of $t \in [0, T]$ for any $u \in V_0^{\Delta_x}$. If $u_0 \in L^2(\Sigma_0)$, then the problem has a unique solution u in the wider class $H_0^{1,0}(Q)$, and it satisfies the generalised formulation

$$(\nabla_x u, \nabla_x w)_Q - (u, \partial_t w)_Q =: a(u,w) = l(w) := (f, w)_Q + (u_0, w)_{\Sigma_0} \qquad (2)$$

for all $w \in H^1_{0,\overline{0}}(Q)$, where $(u_0, w)_{\Sigma_0} := \int_{\Sigma_0} u_0(x)\, w(x,0) dx = \int_\Omega u_0(x)\, w(x,0) dx$. According to the well-established arguments (see [15,28]), without loss of generality, we can 'homogenise' the problem, i.e., consider (2) with $u_0 = 0$.

Our main goal is to derive fully computable estimates for space-time IgA approximations of this class of problems. For this purpose, we use the functional approach to a posteriori EE. Initially, their simplest form has been obtained for a heat equation in [23]. Numerical properties of above-mentioned EE w.r.t. the time-marching and space-time method are discussed in [6,20,21].

2 Stabilized Formulation of the Problem and Its Discretization

For the convenience of the reader, we first recall the general concept of the IgA approach, the definition of B-splines (NURBS) and their use in the geometrical representation of the space-time cylinder Q, as well as in the construction of the IgA trial spaces, used to approximate solutions satisfying (2).

Let $p \geq 2$ denote a degree of polynomials used for the IgA approximations and n denote the number of basis functions used to construct a B-spline curve. A *Knot-vector* is a non-decreasing set of coordinates in a parameter domain, written as $\Xi = \{\xi_1, \ldots, \xi_{n+p+1}\}, \xi_i \in \mathbb{R}$, where $\xi_1 = 0$ and $\xi_{n+p+1} = 1$. The knots can be repeated, and the multiplicity of the i-th knot is indicated by m_i. Throughout the paper, we consider only so-called open knot vectors, i.e., $m_1 = m_{n+p+1} = p+1$. For $\widehat{Q} := (0,1)$, $\widehat{\mathcal{K}}_h$ denotes a locally quasi-uniform mesh, where each element $\widehat{K} \in \widehat{\mathcal{K}}_h$ is constructed by the distinct neighbouring knots. The global size of $\widehat{\mathcal{K}}_h$ is denoted by $\hat{h} := \max_{\widehat{K} \in \widehat{\mathcal{K}}_h} \{\hat{h}_{\widehat{K}}\}$, where $\hat{h}_{\widehat{K}} := \mathrm{diam}(\widehat{K})$.

The *univariate B-spline basis functions* $\widehat{B}_{i,p} : \widehat{Q} \to \mathbb{R}$ are defined by means of Cox-de Boor recursion formula and are $(p - m_i)$-times continuously differentiable across the i-th knot with multiplicity m_i. The scope of this paper is limited to a single-patch domain. The *multivariate B-splines* on $\widehat{Q} := (0,1)^{d+1}, d = \{1,2,3\}$, is defined as a tensor-product of the univariate ones. In multidimensional case, we define the knot-vector dependent on the coordinate direction $\varXi^\alpha = \{\xi_1^\alpha, \ldots, \xi^\alpha_{n^\alpha+p^\alpha+1}\}, \xi_i^\alpha \in \mathbb{R}$, where $\alpha = 1, \ldots, d+1$ indicates the direction (in space or time). Furthermore, we introduce set of multi-indices $\mathcal{I} = \big\{\, i = (i_1, \ldots, i_{d+1}) : i_\alpha = 1, \ldots, n_\alpha,\ \alpha = 1, \ldots, d+1 \big\}$ and multi-index $p := (p_1, \ldots, p_{d+1})$ indicating the order of polynomials. Then, multivariate B-spline basis functions are defined as $\widehat{B}_{i,p}(\boldsymbol{\xi}) := \prod_{\alpha=1}^{d+1} \widehat{B}_{i_\alpha, p_\alpha}(\xi^\alpha)$, where $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^{d+1}) \in \widehat{Q}$. The *univariate and multivariate NURBS basis* functions are defined in $\widehat{Q}$ by means of B-spine basis functions, i.e., for given p and any $i \in \mathcal{I}$ $\widehat{R}_{i,p} : \widehat{Q} \to \mathbb{R}$ is generated as $\widehat{R}_{i,p}(\boldsymbol{\xi}) := \frac{w_i\, \widehat{B}_{i,p}(\boldsymbol{\xi})}{W(\boldsymbol{\xi})}$. Here, $W(\boldsymbol{\xi})$ is a weighting function $W(\boldsymbol{\xi}) := \sum_{i \in \mathcal{I}} w_i\, \widehat{B}_{i,p}(\boldsymbol{\xi})$, where $w_i \in \mathbb{R}^+$.

The physical space-time domain $Q \subset \mathbb{R}^{d+1}$ is defined by the geometrical mapping of the parametric domain $\widehat{Q} := (0,1)^{d+1}$:

$$\varPhi : \widehat{Q} \to Q := \varPhi(\widehat{Q}) \subset \mathbb{R}^{d+1}, \quad \varPhi(\boldsymbol{\xi}) := \sum_{i \in \mathcal{I}} \widehat{R}_{i,p}(\boldsymbol{\xi})\, \mathbf{P}_i, \tag{3}$$

where $\{\mathbf{P}_i\}_{i \in \mathcal{I}} \in \mathbb{R}^{d+1}$ are the control points. For simplicity, we assume the same polynomial degree for all coordinate directions, i.e., $p_\alpha = p$ for all $\alpha = 1, \ldots, d{+}1$. By means of geometrical mapping (3), the mesh $\mathcal{K}_h$ discretising Q is defined as $\mathcal{K}_h := \big\{ K = \varPhi(\widehat{K}) : \widehat{K} \in \widehat{\mathcal{K}}_h \big\}$. The global mesh size is denoted by

$$h := \max_{K \in \mathcal{K}_h} \{\, h_K \,\}, \quad h_K := \|\nabla \varPhi\|_{L^\infty(K)} \hat{h}_{\widehat{K}}. \tag{4}$$

Moreover, we assume that $\mathcal{K}_h$ is quasi-uniform mesh, i.e., there exists a positive constant C_u independent of h, such that $h_K \le h \le C_u\, h_K$.

The finite dimensional spaces on Q are constructed by a push-forward of the NURBS basis functions $V_h := \mathrm{span}\big\{ \phi_{h,i} := \widehat{R}_{i,p} \circ \varPhi^{-1} \big\}_{i \in \mathcal{I}}$, where the geometrical mapping $\varPhi$ is invertible in Q, with smooth inverse on each element $K \in \mathcal{K}_h$ (see [1,25]). The subspace $V_{0h} := V_h \cap V_{0,\underline{0}}(Q)$, where $V_{0,\underline{0}} := V_0 \cap H^1_{0,\underline{0}}(Q)$ is introduced for the functions satisfying homogeneous boundary condition (BC).

In order to provide efficient discretization method, we test (1) with the time-upwind test-function

$$\lambda\, w + \mu\, \partial_t w, \quad w \in V_{0,\underline{0}}^{\nabla_x \partial_t} := \{ w \in V_{0,\underline{0}}^{\Delta_x} : \nabla_x \partial_t w \in L^2(Q) \}, \quad \lambda, \mu \ge 0. \tag{5}$$

and arrive at the stabilised weak formulation for $u \in V_0$, i.e.,

$$\big(\partial_t u, \lambda\, w + \mu\, \partial_t w\big)_Q + \big(\nabla_x u, \nabla_x(\lambda\, w + \mu\, \partial_t w)\big)_Q$$
$$=: a_s(u, w) = l_s(w) := (f, \lambda\, w + \mu\, \partial_t w)_Q, \quad \forall w \in V_{0,\underline{0}}^{\nabla_x \partial_t}. \qquad (6)$$

In [17], it was shown that *stable discrete space-time IgA scheme* corresponds to the case, when $\lambda = 1$ and $\mu = \delta_h = \theta h$ in (5) with $\theta > 0$ and global mesh-size h (cf. (4)) both for the fixed and moving spatial computational domains. Hence, (6) implies the discrete stabiliaed space-time problem: find $u_h \in V_{0h}$ satisfying

$$(\partial_t u_h, w_h + \delta_h \partial_t w_h)_Q + \big(\nabla_x u_h, \nabla_x(w_h + \delta_h \partial_t w_h)\big)_Q$$
$$=: a_{s,h}(u_h, w_h) = l_{s,h}(w_h) := (f, w_h + \delta_h\, \partial_t w_h)_Q, \quad \forall w_h \in V_{0h}. \; (7)$$

The V_{0h}-coercivity of $a_h(\cdot,\cdot) : V_{0h} \times V_{0h} \to \mathbb{R}$ w.r.t. the norm

$$|\!|\!| w_h |\!|\!|_{s,h}^2 := \|\nabla_x w_h\|_Q^2 + \delta_h\, \|\partial_t w_h\|_Q^2 + \|w_h\|_{\Sigma_T}^2 + \delta_h\, \|\nabla_x w_h\|_{\Sigma_T}^2 \qquad (8)$$

follows from [17, Lemma 1] or [16, Lemma 3]. Moreover, one can show a boundedness property of the bilinear form $a_{h,s}(\cdot,\cdot)$ in appropriately chosen norms. Combining these coercivity and boundedness properties of $a_{h,s}(\cdot,\cdot)$ with the consistency of the scheme (7) and approximation results for the IgA spaces implies a corresponding a priori EE presented in Theorem 1 below.

Theorem 1. *Let $u \in H_0^s(Q) := H^s(Q) \cap H_0^{1,0}(Q), s \in \mathbb{N}, s \geq 2$, be the exact solution of (2) and $u_h \in V_{0h}$ be the solution of (7) with some fixed parameter θ. Then, the following a priori EE*

$$\|u - u_h\|_{s,h} \leq C\, h^{r-1}\, \|u\|_{H^r(Q)} \qquad (9)$$

holds, where $r = \min\{s, p+1\}, C > 0$ is a generic constant independent of h.

Proof: See, e.g., [17, Theorem 8]. □

3 Error Majorant

In this section, we derive error majorants for stabilised weak formulation of parabolic I-BVPs. The functional nature of these majorants allows obtaining a posteriori EEs for $u \in V_{0,\underline{0}}^{\Delta_x}$ and any $v \in V_{0,\underline{0}}^{\Delta_x}$. The error $e = u - v$ is measured in terms of

$$|\!|\!| e |\!|\!|_{s,\nu_i}^2 := \nu_1\, \|\nabla_x e\|_Q^2 + \nu_2\, \|\partial_t e\|_Q^2 + \nu_3\, \|\nabla_x e\|_{\Sigma_T}^2 + \nu_4\, \|e\|_{\Sigma_T}^2, \qquad (10)$$

where $\{\nu_i\}_{i=1,\ldots,4}$ are the positive weights introduced in the derivation process.

To obtain guaranteed error bounds of $|\!|\!| e |\!|\!|_{s,\nu_i}^2$, we apply a method similar to the one developed in [21,23] for parabolic I-BVPs. For the derivation process, we consider space of smoother functions $V_{0,\underline{0}}^{\nabla_x \partial_t}$ (cf. (5)) equipped with the norm

$\|w\|_{V_{0,\underline{0}}^{\nabla_x\partial_t}} := \sup_{t\in[0,T]} \|\nabla_x w(\cdot,t)\|^2_Q + \|w\|^2_{V_{0,\underline{0}}^{\Delta_x}}$, where $\|w\|^2_{V_{0,\underline{0}}^{\Delta_x}} := \|\Delta_x w\|^2_Q + \|\partial_t w\|^2_Q$, which is dense in $V_{0,\underline{0}}^{\Delta_x}$. According to [15, Remark 2.2], norms $\|\cdot\|_{V_{0,\underline{0}}^{\nabla_x\partial_t}} \approx \|\cdot\|_{V_{0,\underline{0}}^{\Delta_x}}$.

Let u_n be a sequence in $V_{0,\underline{0}}^{\nabla_x\partial_t}$. We consider the corresponding stabilised identity

$$a_s(u_n, w) = (f_n, \lambda\, w + \mu\, \partial_t w)_Q, \text{ where } f_n = (u_n)_t - \Delta_x u_n \in L^2(Q). \tag{11}$$

By subtracting $a_s(v_n, w), v_n \in V_{0,\underline{0}}^{\nabla_x\partial_t}$, from (11), and by setting $w = e_n = u_n - v_n \in V_{0,\underline{0}}^{\nabla_x\partial_t}$, we arrive at the so-called 'error-identity'

$$\begin{aligned}
&\lambda\,\|\nabla_x e_n\|^2_Q + \mu\,\|\,\partial_t e_n\|^2_Q + \tfrac{1}{2}\,(\mu\,\|\nabla_x e_n\|^2_{\Sigma_T} + \lambda\|e_n\|^2_{\Sigma_T})\\
&= \lambda\Big((f_n - \partial_t v_n, e_n)_Q - (\nabla_x v_n, \nabla_x e_n)_Q\Big)\\
&\quad + \mu\Big((f_n - \partial_t v_n, \partial_t e_n)_Q - (\nabla_x v_n, \nabla_x\,\partial_t e_n)_Q\Big),
\end{aligned}$$

which is used in the derivation of the majorants of (10) in Theorems 2 and 3.

Theorem 2. *For any $v \in V_{0,\underline{0}}^{\Delta_x}$ and $\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q)$, the following estimate holds:*

$$\begin{aligned}
\|e\|^2_{s,\nu_i} \leq \overline{\mathrm{M}}(v, \boldsymbol{y}; \gamma, \alpha_i) &:= \gamma\Big\{\lambda\big((1+\alpha_1)\,\|\mathbf{r}_{\mathrm{d}}\|^2_Q\\
&+ (1+\tfrac{1}{\alpha_1})\,C_{\mathrm{F}}^2\,\|\mathbf{r}_{\mathrm{eq}}\|^2_Q\big) + \mu\big((1+\alpha_2)\,\|\mathrm{div}_x\mathbf{r}_{\mathrm{d}}\|^2_Q + (1+\tfrac{1}{\alpha_2})\,\|\mathbf{r}_{\mathrm{eq}}\|^2_Q\big)\Big\},
\end{aligned} \tag{12}$$

where $\nu_1 = (2-\frac{1}{\gamma})\,\lambda, \nu_2 = (2-\frac{1}{\gamma})\,\mu, \nu_3 = \mu, \nu_4 = \lambda$, C_{F} is the Friedrichs constant, $\mathbf{r}_{\mathrm{eq}}$ and $\mathbf{r}_{\mathrm{d}}$ are residuals defined by relations

$$\mathbf{r}_{\mathrm{eq}}(v, \boldsymbol{y}) := f - \partial_t v + \mathrm{div}_x\,\boldsymbol{y} \text{ and } \mathbf{r}_{\mathrm{d}}(v, \boldsymbol{y}) := \boldsymbol{y} - \nabla_x v, \tag{13}$$

$\lambda, \mu > 0$ are weights introduced in (5), $\gamma \in \big[\frac{1}{2}, +\infty)$, and $\alpha_i > 0, i = 1, 2$.

Proof: The detailed proof can be found in [16, Theorem 2], where we use the 'error-identity' and the density of space $V_{0,\underline{0}}^{\nabla_x\partial_t}$ in $V_{0,\underline{0}}^{\Delta_x}$ to obtain (12). □

The next theorem assumes higher regularity on the approximations v and $\boldsymbol{y}$.

Theorem 3. *For any $v \in V_{0,\underline{0}}^{\nabla_x\partial_t}$ and $\boldsymbol{y} \in H^{\mathrm{div}_x,1}(Q)$, we have the estimate*

$$\begin{aligned}
\|e\|^2_{s,\nu_i} \leq \overline{\mathrm{M}}^{\mathrm{II}}(v, \boldsymbol{y}; \zeta, \beta_i, \epsilon) &:= \epsilon\,\mu\|\mathbf{r}_{\mathrm{d}}\|^2_{\Sigma_T} + \zeta\,\Big(\lambda\big((1+\beta_1)\big((1+\beta_2)\,\|\mathbf{r}_{\mathrm{d}}\|^2_Q\\
&+ (1+\tfrac{1}{\beta_2})C_{\mathrm{F}}^2\,\|\mathbf{r}_{\mathrm{eq}}\|^2_Q\big) + (1+\tfrac{1}{\beta_1})\,\tfrac{\mu^2}{\lambda^2}\,\|\partial_t\mathbf{r}_{\mathrm{d}}\|^2_Q\big) + \mu\,\|\mathbf{r}_{\mathrm{eq}}\|^2_Q\Big),
\end{aligned} \tag{14}$$

where $\nu_1 = (2-\frac{1}{\zeta})\,\lambda, \nu_2 = (2-\frac{1}{\zeta})\,\mu, \nu_3 = \mu\,(1-\frac{1}{\epsilon}), \nu_4 = \lambda$, where C_{F} is the Friedrichs constant, $\mathbf{r}_{\mathrm{eq}}(v, y)$ and $\mathbf{r}_{\mathrm{d}}(v, y)$ are residuals in (13), $\lambda, \mu > 0$ are parameters in (5), $\zeta \in \big[\frac{1}{2}, +\infty), \epsilon \in [1, +\infty)$, and $\beta_i > 0, i = 1, 2$.

Proof: By using analogous density arguments and integral manipulations with the 'error-identity', we obtain (14) (see also [16, Theorem 3]). □

Corollary 1 presents majorants for $\lambda = 1$ and $\mu = \delta_h$, where $\delta_h = \theta\, h$, $\theta > 0$.

Corollary 1

(i) If $v \in V_{0,\underline{0}}^{\Delta_x}$ and $\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q)$, Theorem 2 yields the estimate

$$\begin{aligned}\|\!|e|\!\|^2_{s,\nu_i} \leq \overline{\mathrm{M}}_{\delta_h}(v,\boldsymbol{y};\gamma,\alpha_i) &:= \gamma\,\Big((1+\alpha_1)\,\|\mathbf{r}_\mathrm{d}\|^2_Q \\ &+ (1+\tfrac{1}{\alpha_1})\,C_\mathrm{F}^2\,\|\mathbf{r}_\mathrm{eq}\|^2_Q + \delta_h\,\big((1+\alpha_2)\,\|\mathrm{div}_x\mathbf{r}_\mathrm{d}\|^2_Q + (1+\tfrac{1}{\alpha_2})\,\|\mathbf{r}_\mathrm{eq}\|^2_Q\big)\Big),\end{aligned} \tag{15}$$

where $\nu_1 = (2-\frac{1}{\gamma}), \nu_2 = (2-\frac{1}{\gamma})\,\delta_h, \nu_3 = \delta_h\ \nu_4 = 1$.

(ii) If $v \in V_{0,\underline{0}}^{\nabla_x\partial_t}$ and $\boldsymbol{y} \in H^{\mathrm{div}_x,1}(Q)$, then Theorem 3 yields

$$\begin{aligned}\|\!|e|\!\|^2_{s,\nu_i} \leq \overline{\mathrm{M}}^{\mathrm{II}}_{\delta_h}(v,\boldsymbol{y};\zeta,\beta_i,\epsilon) &:= \epsilon\,\delta_h\,\|\mathbf{r}_\mathrm{d}\|^2_{\Sigma_T} + \zeta\,\Big((1+\beta_1)\big((1+\beta_2)\,\|\mathbf{r}_\mathrm{d}\|^2_Q \\ &+ (1+\tfrac{1}{\beta_2})C_\mathrm{F}^2\,\|\mathbf{r}_\mathrm{eq}\|^2_Q\big) + (1+\tfrac{1}{\beta_1})\delta_h^2\,\|\partial_t\mathbf{r}_\mathrm{d}\|^2_Q + \delta_h\,\|\mathbf{r}_\mathrm{eq}\|^2_Q\Big),\end{aligned} \tag{16}$$

where $\nu_1 = (2-\frac{1}{\zeta}), \nu_2 = (2-\frac{1}{\zeta})\,\delta_h, \nu_3 = \delta_h$, and $\nu_4 = 4$. In (i) and (ii), $\mathbf{r}_\mathrm{d}$ and $\mathbf{r}_\mathrm{eq}$ are defined in (13), C_F is the Friedrichs constant, δ_h is discretisation parameter, $\gamma, \zeta \in \big[\frac{1}{2}, +\infty\big), \epsilon \in [1, +\infty)$, and $\beta_i > 0, i = 1, 2$.

4 Numerical Example

In the final section of this work, we present an example demonstrating the numerical behaviour of the derived majorants for the static case of the parabolic I-BVP. In fact, the space-time approach treats the parabolic problem as yet another elliptic problem in $\mathbb{R}^{d+1}$ with strong convection in $(d+1)$-th direction. Therefore, for the simplicity of presentation, we consider the Poisson Dirichlet problem

$$-\Delta_x u = f \quad \text{in} \quad Q := (0,1)^2 \in \mathbb{R}^2, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{17}$$

Let $u_h \in V_{0h}$, where $V_h \equiv \mathcal{S}_h^{p,p} := \big\{\hat{V}_h \circ \Phi^{-1}\big\}$ and $\hat{V}_h \equiv \hat{\mathcal{S}}_h^{p,p}$, be generated with NURBS of degree $p = 2$. Due to the restriction on the knots-multiplicity of $\hat{\mathcal{S}}_h^{p,p}$, we have $u_h \in C^{p-1}$. Then, $u_h(x) := \sum_{i\in\mathcal{I}} \underline{\mathrm{u}}_{h,i}\,\phi_{h,i}$, where $\underline{\mathrm{u}}_h := \big[\underline{\mathrm{u}}_{h,i}\big]_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ is a vector of degrees of freedom (DOFs) defined by a system

$$\mathrm{K}_h\,\underline{\mathrm{u}}_h = \mathrm{f}_h, \quad \mathrm{K}_h := \big[(\nabla_x\phi_{h,i}, \nabla_x\phi_{h,j})_Q\big]_{i,j\in\mathcal{I}}, \quad \mathrm{f}_h := \big[(f, \phi_{h,i})_Q\big]_{i\in\mathcal{I}}. \tag{18}$$

The majorant corresponding to (17) can be presented as

$$\overline{\mathrm{M}}(u_h, \boldsymbol{y}_h) := (1+\beta)\,\|\boldsymbol{y}_h - \nabla_x u\|^2_Q + (1+\tfrac{1}{\beta})\,C_\mathrm{F}^2\,\|\mathrm{div}_x\boldsymbol{y}_h + f\|^2_Q, \tag{19}$$

where $\beta > 0$ and $\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q)$. The approximation space for $\boldsymbol{y}_h \in Y_h \equiv \mathcal{S}_h^{q,q} := \left\{\hat{Y}_h \circ \Phi^{-1}\right\}$ is generated by the push-forward of $\hat{Y}_h := \hat{\mathcal{S}}_h^{q,q} \oplus \hat{\mathcal{S}}_h^{q,q}$, where $\mathcal{S}_h^{\hat{q},q}$ is the space of NURBS functions of degree q for each of the components of $\boldsymbol{y}_h = (y_h^{(1)}, y_h^{(2)})^{\mathrm{T}}$. The best EE is obtained by optimisation of $\overline{\mathrm{M}}(u_h, \boldsymbol{y}_h)$ w.r.t. $\boldsymbol{y}_h := \sum_{i \in \mathcal{I}} \underline{\mathbf{y}}_{h,i}\, \boldsymbol{\psi}_{h,i}$. Here $\boldsymbol{\psi}_{h,i}$ is the basis function of the space Y_h, and $\underline{\mathbf{y}}_h := \left[\underline{\mathbf{y}}_{h,i}\right]_{i \in \mathcal{I}} \in \mathbb{R}^{2|\mathcal{I}|}$ is a vector of DOFs of $\boldsymbol{y}_h$ defined by a system

$$\left(C_{\mathrm{F}}^2\, \mathrm{Div}_h + \beta\, \mathrm{M}_h\right)\, \underline{\mathbf{y}}_h = -C_{\mathrm{F}}^2\, \mathrm{z}_h + \beta\, \mathrm{g}_h, \tag{20}$$

where

$$\mathrm{Div_h} := \left[(\mathrm{div}_x \boldsymbol{\psi}_i, \mathrm{div}_x \boldsymbol{\psi}_j)_Q\right]_{i,j=1}^{2|\mathcal{I}|}, \qquad \mathrm{z}_h := \left[\left(f, \mathrm{div}_x \boldsymbol{\psi}_j\right)_Q\right]_{j=1}^{2|\mathcal{I}|},$$
$$\mathrm{M_h} := \left[(\boldsymbol{\psi}_i, \boldsymbol{\psi}_j)_Q\right]_{i,j=1}^{2|\mathcal{I}|}, \qquad \mathrm{g}_h := \left[\left(\nabla_x v, \boldsymbol{\psi}_j\right)_Q\right]_{j=1}^{2|\mathcal{I}|}.$$

According to [11], the most effective results for the majorant reconstruction (with uniform refinement) is obtained, when q is set substantially higher than p. We assume that $q = p + k, k \in \mathbb{N}^+$. In the same time, when u_h is reconstructed on the mesh $\mathcal{T}_h$, we use a coarser one $\mathcal{T}_{Kh}, K \in \mathbb{N}^+$, to recover the flux $\boldsymbol{y}_{Kh}$.

Example 1. We consider a basic example with $u = (1 - x_1)\, x_1^2\, (1 - x_2)\, x_2$, $f = -\big(2\,(1 - 3\,x_1)\,(1 - x_2)\,x_2 - 2\,(1 - x_1)\,x_1^2\big)$, and homogenous Dirichlet BC. For the uniform refinement, we set $p = 2$, i.e., $u_h \in S_h^{2,2}$, and compare two different settings: (a) $\boldsymbol{y}_h \in S_{Kh}^{q,q} \oplus S_{Kh}^{q,q}, q = 5, k = 3, K = 3$ and (b) $\boldsymbol{y}_h \in S_{Kh}^{q,q} \oplus S_{Kh}^{q,q}, q = 9, k = 7, K = 7$. The upper and lower parts of Tables 1 and 2, correspond to the cases (a) and (b), respectively. In the case (a), the time spent on the reconstruction of $\boldsymbol{y}_h$ (i.e., $t_{\mathrm{as}}(\boldsymbol{y}_h) + t_{\mathrm{sol}}(\boldsymbol{y}_h)$) is about 3 times higher than the time $t_{\mathrm{as}}(u_h) + t_{\mathrm{sol}}(u_h)$. However, for the case (b), the assembling time of the systems $\mathrm{Div_h}$ and $\mathrm{M_h}$ (denoted by $t_{\mathrm{as}}(\boldsymbol{y}_h)$) takes approximately 1/10-th of the assembling time for $\mathrm{K_h}$ ($t_{\mathrm{as}}(u_h)$). Moreover, solving the system (20) ($t_{\mathrm{sol}}(\boldsymbol{y}_h)$) takes only 1/500-th part of the time spent on solving (18) ($t_{\mathrm{sol}}(u_h)$). The efficiency of the obtained functional majorant is illustrated by $I_{\mathrm{eff}}(\overline{\mathrm{M}}) = 1.0936$ (see the fifth column of Table 1).

Table 1. Assembling and solving time for systems (18) and (20) w.r.t. the last 2 refinement steps.

DOFs(u_h)	DOFs($\boldsymbol{y}_h$)	$t_{\mathrm{as}}(u_h)$	$t_{\mathrm{as}}(\boldsymbol{y}_h)$	$t_{\mathrm{sol}}(u_h)$	$t_{\mathrm{sol}}(\boldsymbol{y}_h)$
$\boldsymbol{y} \in S^{5,5} \oplus S^{5,5}, k = 3, K = 3$					
1 004 004	42 431	4.2770	**10.9889**	17.0640	**43.3740**
4 010 004	205 031	17.1461	**45.0032**	143.2929	**328.5911**
$\boldsymbol{y} \in S^{9,9} \oplus S^{9,9}, k = 7, K = 7$					
1 004 004	441	4.3506	**0.4213**	17.1396	**0.0456**
4 010 004	1 161	17.4620	**1.7268**	142.9116	**0.2667**

Table 2. The error, the majorant, the corresponding efficiency index, and the e.o.c. (error order or convergence) p w.r.t. the last 2 refinement steps.

$\|\nabla_x(u-u_h)\|_Q^2$	$\overline{\mathrm{M}}$	I_{eff}	p
$\boldsymbol{y} \in S^{5,5} \oplus S^{5,5}, k=3, K=3$			
6.229382e−07	7.450215e−07	**1.0936**	2.0113
1.557344e−07	1.862552e−07	**1.0936**	2.0056
$\boldsymbol{y} \in S^{9,9} \oplus S^{9,9}, k=7, K=7$			
6.229382e−07	6.363897e−07	**1.0107**	2.0113
1.557344e − 07	1.557499e − 07	**1.0000**	2.0056

We now consider an adaptive refinement strategy, i.e., THB-Splines [7,12, 26] in combination with the functional EE (19). We use the so-called Dörfler's marking [5] with a parameter $\theta = 0.6$. We start with the following setting: $u_h \in S_h^{2,2}$ is THB-Splines basis (with one level and 36 basis functions of degree 2), and $\boldsymbol{y}_h \in S_{3h}^{5,5} \oplus S_{3h}^{5,5}$ is THB-Splines basis (with one level and 81 basis functions of degree 5). We execute 16 refinement steps to obtain the error illustrated in Table 3 (where only the last two refinement steps are shown). The time spent on the assembling, solving, and generating corresponding EEs is illustrated in Table 4. By using 3 times courser mesh in the refinement of the basis for $\boldsymbol{y}_h$, we have managed to spare the effort of reconstructing the optimal $\boldsymbol{y}_h$ and speed up the over-all reconstruction of the majorant. In the current configuration, we obtain the following ratios of the times spent on reconstruction of $\boldsymbol{y}_h$ and u_h, i.e., $\frac{t_{\mathrm{as}}(u_h)}{t_{\mathrm{as}}(\boldsymbol{y}_h)} \approx 19$ and $\frac{t_{\mathrm{sol}}(u_h)}{t_{\mathrm{sol}}(\boldsymbol{y}_h)} \approx 658$.

Table 3. Assembling and solving time for systems (18) and (20) w.r.t. the last 2 refinements of total 16 steps.

DOFs(u_h)	DOFs($\boldsymbol{y}_h$)	$t_{\mathrm{as}}(u_h)$	$t_{\mathrm{as}}(\boldsymbol{y}_h)$	$t_{\mathrm{sol}}(u_h)$	$t_{\mathrm{sol}}(\boldsymbol{y}_h)$
55 005	145	12.1002	**0.9302**	0.6105	**0.0022**
107 444	132	17.7635	**0.9269**	1.1858	**0.0018**

Table 4. Error, majorant, its efficiency index, and e.o.c. w.r.t. the last 2 refinements of total 16 steps.

$\|\nabla_x(u-u_h)\|_Q^2$	$\overline{\mathrm{M}}$	I_{eff}	p
3.263617e−06	3.373925e−06	**1.0338**	1.4511
2.187249e−06	2.202933e−06	**1.0072**	1.6345

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