Semiclassical Plasma Dynamics in Electroweak Baryogenesis

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Abstract

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This thesis will examine the dynamics of particles during a first-order electroweak phase transition, with regard to the electroweak baryogenesis scenario. We will derive the CP-violating semiclassical force on helicity states in the plasma frame. We will also study the dynamics of the plasma in the Schwinger-Keldysh formalism to find the equations for the time-evolution of the plasma.

Keywords: Cosmology, baryogenesis, electroweak phase transition

Tiivistelmä

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Semiklassinen dynamiikka sähköheikossa baryogeneesissa
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Tämä Pro Gradu-tutkielma käsittelee mahdollista sähköheikko-faasimuunnoksen aikana tapahtuva baryogeneesia. Työn päämääränä on johtaa faasimuunnoksen aikana hiukkasten helisiteetti-tiloihin vaikuttava CP-symmetriaa rikkova, semiklassinen voima plasma-koordinaatistossa. Työssä johdetaan plasman aikakehitysyhtälöt käyttäen Schwinger-Keldysh formalismia plasman dynamiikalle.

Avainsanat: kosmologia, baryogeneesi, sähköheikko faasimuutos

Foreword

Firstly, I would like to express my profound gratitude to my supervisor, Professor Kimmo Kainulainen, for his invaluable guidance, and my second reviewer, Professor Tuomas Lappi, for his helpful advice and comments.

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Jyväskylä, 13 November 2018

Niyati Venkatesan

Contents

1 Introduction

In the Minimal Standard Model of particle physics, each of the particles that make up the matter around us has a corresponding antiparticle that exhibits very similar behaviour. In all perturbative processes, particles and antiparticles are created and destroyed in pairs, so it seems impossible to create matter from nothing. And yet the universe as we see it consists almost entirely of ordinary matter; but why?

The simplest explanation would be that the matter in the universe was always there, so there would be no need for a process to have created it from nothing. However, this explanation is ruled out by the accepted paradigm of ΛCDM cosmology, in which many considerations[1] indicate that the universe underwent a period of exponential expansion, or 'inflation,' which would have diluted any pre-existing matter density to zero. So to explain the existing baryon-antibaryon asymmetry, we need some process to have specifically favoured matter over antimatter at some stage of the universe's development.

This thesis is based on the *electroweak baryogenesis* scenario, in which this asymmetry results from the dynamics of particles during the so-called *electroweak phase transition* through which the fundamental particles first acquired their masses in the early universe.

The asymmetry produced relies on the CP-violating microscopic dynamics of particles in the plasma during the phase transition. In this thesis, we will be computing the CP-violating semiclassical force acting on the particles in the frame of the plasma, which allows us to compute how the distribution of particles evolves and hence the asymmetry generated.

Chapter 2 will introduce the physics of electroweak baryogenesis and the semiclassical formalism used to study it. Chapter 3 will use the semiclassical formalism to derive the semiclassical force on particles during the electroweak phase transition. Chapter 4 will use the Schwinger-Keldysh formalism to verify these results and derive the Liouville equation of the plasma. In Chapter 5, we will integrate over the Liouville equation in the wall frame to obtain the CP-even and -odd fluid equations.

[1] For instance, the horizon problem, the flatness problem and structure formation. [1]

2 Theoretical background

2.1 Baryogenesis

It is an indisputable observational fact that there exists an asymmetry between matter and antimatter in the observable universe. This asymmetry is best quantified by the baryon-to-photon ratio: [2]

$$5.8 \times 10^{-10} < \eta < 6.6 \times 10^{-10} \text{ [95\% CL]}$$

This baryon-antibaryon asymmetry (BAU) is not explained in the context of the Minimal Standard Model of particle physics and the ΛCDM model of cosmology. We know that this cannot simply be explained by the universe's initial conditions, since the period of inflation in the early universe would have wiped it out.

The existing BAU could have been produced either by directly creating an imbalance of baryons over antibaryons - a process known as *baryogenesis* - or creating first an imbalance of leptons over antileptons, known as *leptogenesis*, and subsquently converting this to an excess of baryons.

Sakharov outlined in 1967 the conditions that needed to be met by any microscopic theory for baryogenesis. [3]

1. **Some interaction of elementary particles must violate the conservation of baryon number.** This is understandable enough: to generate the observed asymmetry from nothing, some process must be able to produce an unequal number of baryons and antibaryons.

2. **The C (charge conjugation) and CP (the product of charge conjugation and parity) symmetries are not exact.** Essentially, this means that under some interaction, a matter particle and its antiparticle do not behave exactly alike. This is needed to bias the aforementioned baryon-number violating interactions so that the net number of baryons produced is greater than that of antibaryons.

3. **The universe is required to have been out of thermal equilibrium at some stage.** This means that the reaction that generates the BAU happens at a higher rate than its inverse reaction.

It is difficult to meet these conditions within the Minimal Standard Model. There is some C- and CP-violation in the MSM, but probably not enough. Baryon number violation does not happen in the Minimal Standard Model at a classical level; however, there exist non-perturbative baryon number-violating processes called *sphalerons* in the Standard Model at a quantum level. But Sakharov's third condition is the hardest to meet, since the universe was always very close to thermal equilibrium. We can measure the rate of the universe's expansion by the Hubble constant H, and the rate of electroweak interactions by the reaction width, Γ:

$$H = (\frac{\rho}{3M_p^2})^{\frac{1}{2}} \sim (\frac{T^4}{M_p^2})^{\frac{1}{2}} \sim 10^{-14}(\frac{T}{100\text{GeV}})^2 \tag{1}$$

$$\Gamma \sim n\sigma_{EW} \sim T^3(\frac{\alpha^2}{T^2}) \sim 10^{-5}\frac{T}{100\text{GeV}}. \tag{2}$$

Clearly, since the expansion of the universe is so slow compared to the interactions, the interactions have plenty of time to bring the particles to equilibrium. This means that within the MSM, the early universe was in equilibrium to a precision of 10^9. So we need a dynamical source of these out-of-equilibrium conditions.

There exist different candidate models that fulfil Sakharov's conditions for baryogenesis, such as GUT baryogenesis, Affleck-Dine baryogenesis and electroweak baryogenesis. We will here examine the latter, in which baryogenesis is postulated to have happened due to the electroweak phase transition.

2.2 Electroweak baryogenesis

Electroweak baryogenesis (henceforth EWBG) is the mechanism of generating BAU during the electroweak phase transition. It is of particular interest since it involves experimentally testable Beyond the Standard Model physics, at scales that contemporary colliders can reach. Gravitational waves from the electroweak phase transition could also be observable by the forthcoming LISA detector, which would tell us more about the phase transition itself. [4], [5]

The electroweak phase transition (EWPT) is the transition that took place in the early universe between the zero and non-zero vacuum expectation values (VEV) of the Higgs field. This resulted in the breaking of the electroweak symmetry, leading to the hierarchy between the electromagnetic and weak forces that we observe now. However, the breaking of the electroweak symmetry took place so far back in time that we do not know very much about the transition itself.

We know from thermodynamics that physically observed phase transitions fall into two categories: first-order transitions, where the system needs to cross an energy barrier between the two phases (resulting in a physical boundary between the two phases), and continuous, or second-order, phase transitions. We do not yet know whether the EWPT is first-order or second-order, but if it was a first-order transition, it would be very promising as a source of Sakharov's out-of-equilibrium condition.

If the EWPT was a first-order transition, it would have proceeded via nucleation and expansion of bubbles of the broken symmetry phase into a universe filled with the symmetric phase. This would result in out-of-equilibrium conditions at the bubble wall, where a CP-violating force would act on the particles of the plasma to produce a chiral asymmetry (an excess of left-handed quarks over right-handed anti-quarks). Bubble nucleation is a very sensitive function of T. It is negligible at first, but grows exponentially, and becomes very large at nucleation temperature T_n, corresponding roughly to the point at which the probability of creating at least one bubble per horizon volume is of order one. [6] After this the phase transition completes almost instantly.

Figure 1 depicts two possible forms of the Higgs potential at different temperatures. For temperatures higher than a critical temperature, T_c, the potential has only one minimum, at 0. Thus in the early universe, the vacuum expectation value of the Higgs field v was zero, and particles had no mass. However, as the universe expanded and cooled down, the potential is thought to have changed form and acquired another local minimum, until at the critical temperature, the minima became degenerate. Beyond the critical temperature, the non-zero VEV corresponds to a lower energy than the zero VEV, inducing a symmetry-breaking phase transition. If there was a potential barrier between the two minima, the transition would have been first-order.

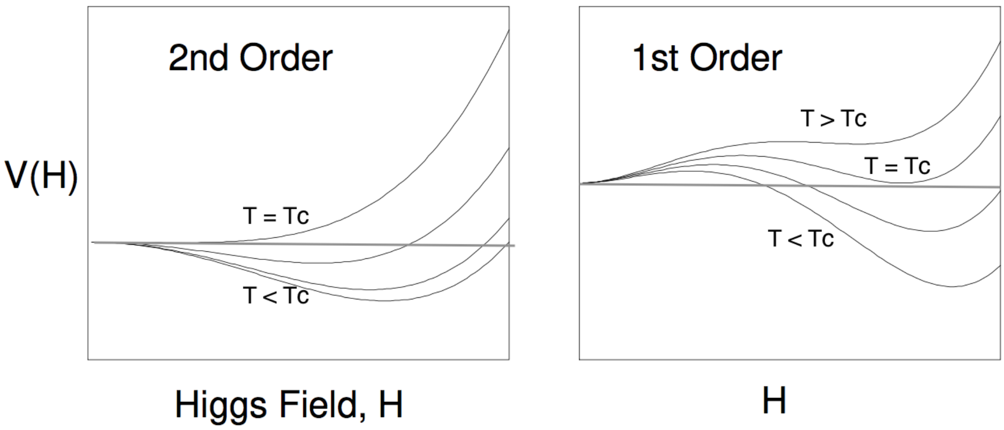

Figure 1. Two alternative forms of the Higgs potential at different temperatures, corresponding to second-order and first-order phase transitions. [7]

In the case of a first-order phase transition, a new dynamical time scale comes into play, related to the passage of the bubble wall: [8]

$$\frac{1}{t_w} \sim \frac{v_w}{L_w} \sim 10^{-2} - 10^{-3}T \tag{3}$$

This time scale is faster than the rate of the interactions that would bring particles back to equilibrium. So during a first-order phase transition, the universe would

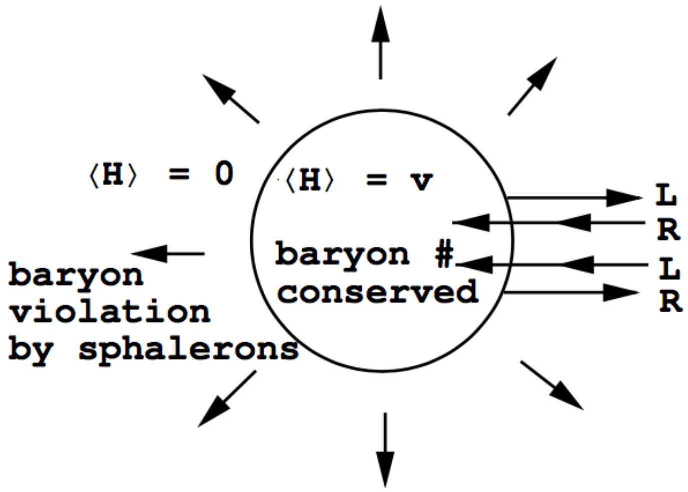

Figure 2. The electroweak baryogenesis scenario. [7]

have been out of equilibrium.

Fermions obtain their masses through their Yukawa couplings to the Higgs field: $m_f = \frac{1}{\sqrt{2}}\lambda v$. Before the EWPT, particles had no mass, but now the non-zero VEV of the Higgs field means that they acquire masses through the Higgs mechanism. This means that we will be dealing with a fermion with a mass that varies spatially in response to the passage of the bubble wall: massless in the symmetric phase and massive inside the bubbles, in the broken symmetry phase.

Electroweak baryogenesis makes use of the baryon number-violating sphaleron process that happens in the MSM at a quantum level. According to Noether's theorem, continuous symmetries of a system are associated with conserved currents and conserved charges. However, while baryon and lepton number are conserved in the vertices allowed by the Standard model, their conservation is based on global symmetries; there is no local symmetry forbidding their violation and no massless gauge field associated with them. Such symmetries can be violated in extensions of the MSM, or even within the MSM by loop corrections. [9] The sphaleron is one such process, that relates, through the so-called 'chiral anomaly,' the violation of Chern-Simons number in the gauge sector to violation of baryon number conservation. This Chern-Simons number labels different, equivalent gauge vacua (see Figure 3). Thus the electroweak field possesses an infinite number of degenerate vacuum states labelled by different Chern-Simons numbers. It can tunnel between consecutive vacuum states via a sphaleron transition.

Indeed, a patch in the space-time can tunnel from one vacuum to another, SU(2) gauge-rotated vacuum, characterised by a different topological quantum number (the Chern-Simons number). [10]

We can define baryon and lepton number currents, $j^\mu_{B(L)}$, whose space integrals yield respectively the baryon and lepton numbers of the system. In electroweak theory, these currents are not exactly conserved. Instead, they are related to a new 'Chern-Simons current' from the gauge sector:

$$\partial_\mu j_B^\mu = \partial_\mu j_L^\mu = N_f \frac{g^2}{16\pi^2} G_{\mu\nu}^a \widetilde{G}^{\mu\nu a} = N_f \partial_\mu j_{CS}^\mu \tag{4}$$

where

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon_{abc} A_\mu^b A_\nu^c \tag{5}$$

$$\widetilde{G}_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}^a \tag{6}$$

and $N_f = 3$ is the number of quark families. Then the topological quantum number is given by

$$N_{CS} = \int d^4x j_{CS}^\mu = \frac{g^2}{32\pi^2} \int d^4x G_{\mu\nu}^a \widetilde{G}^{\mu\nu a} \tag{7}$$

From (4) it follows that

$$\Delta N_B = \Delta N_L = N_f \Delta N_{CS} \tag{8}$$

so that for every transition between consecutive vacua, the Chern-Simons number changes by one unit, and the baryon and lepton numbers change by three units.

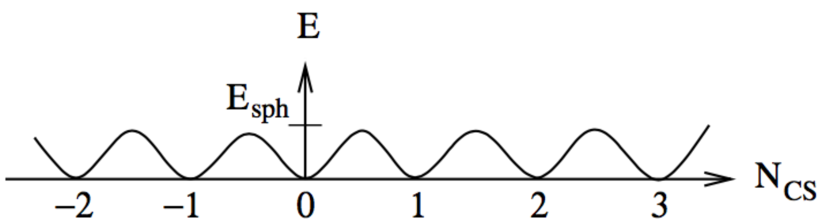

Figure 3. Vacuum structure of the pure-gauge electroweak theory. [11]

The height of the potential barrier shown in Figure 3 is directly related to the VEV of the Higgs field. Thus below the critical temperature, these transitions are exponentially suppressed by a factor of $\exp\left(\frac{E_{sph}}{T}\right)$, where $E_{sph}/T = 40\phi/T$. At temperatures above the electroweak scale, there is no such suppression: these processes have access to thermal energy, and thus happen at a rate of $\Gamma = (18 \pm 3)\alpha_w^5 T^4$. [11]

At high temperatures in the symmetric phase, these processes would be in equilibrium:

$$H = 10^{-14}(\frac{T}{100})^2 \tag{9}$$

$$\Gamma = 100(\frac{T}{100}), \tag{10}$$

so no net asymmetry would be produced. However, during a first-order phase transition, they would try to convert the local seed chiral asymmetry to a baryon asymmetry, which then diffuses into the bubbles. Inside the bubbles, sphaleron processes are exponentially suppressed in the broken phase so the generated BAU is not washed out, if the condition $v > 1.1T$ holds [7].

Electroweak baryogenesis thus presupposes a first-order phase transition, so that the sphaleron interactions are completely suppressed in the broken phase and the created asymmetry cannot be wiped out. Hence it requires physics beyond the Standard Model (known as BSM physics), because in the MSM the EWPT is a crossover for the measured value of the Higgs mass [12]. There is also probably not enough CP-violation in the MSM, so BSM physics would provide an additional source of CP-violation [13].

There exist different models that meet these conditions, such as multi-Higgs models (see for example [14]) and dark matter portal models (see for example [15], [7]).

Here we assume that the EWPT was a thermal transition. A thermal phase transition happens through bubble nucleation. The wall of the bubble moves at a constant speed through the plasma, experiencing friction (which stops it 'running away' and becoming relativistic) and heating up the plasma in its turn (in contrast to a vacuum phase transition, which happens through quantum tunnelling, and in which the wall accelerates without heating up the plasma).[4], [5] The usual 'hot' baryogenesis scenarios typically presuppose a thermal phase transition, in which the wall moves relatively slowly and interacts with the plasma, and particles diffuse through it. Baryogenesis is more efficient for wall speeds much less than the speed of sound in the plasma - that is, for weak subsonic deflagration walls [4].

There are two alternative mathematical formalisms that can be used to study the dynamics of the plasma near walls. We will take a look at these in the next two sections.

2.3 Semiclassical approach

To estimate the BAU produced quantitatively, we need to estimate the seed asymmetry produced at the microscopic level. This follows from the dynamics of the plasma during the passage of the wall, which will be the subject of this thesis. This is most easily studied using the semiclassical formalism.

In the semiclassical formalism, we treat the plasma as a collection of on-shell, single particle states without quantum coherences.

The semiclassical approach can be derived in the WKB approximation, in which we have a slowly-varying potential, i.e. a wall thick enough that $TL_w \gg 1$ (where L_w is the wall thickness). Since $T \sim p$ for typical particles in the plasma, this means that the average wavelength of plasma particles is smaller than the width of the wall, so they see the wall as a classical object.

The main qualitative features of the semiclassical approach can be seen at the level of a simple toy model. Consider a fermionic field with a complex mass, described by the Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}_L m \psi_R - \bar{\psi}_R m^* \psi_L + \mathcal{L}_{int} \tag{11}$$

This gives us the equation of motion, which is a generalisation of the Dirac equation for a complex mass:

$$(i\gamma^\mu \partial_\mu - mP_R - m^* P_L)\psi = 0 \tag{12}$$

where we use the projection operators, $P_{R,L} = \frac{1}{2}(1 \pm \gamma^5)$. This corresponds to the Dirac equation with a complex mass term. The mass of a particle is expressed as $m(u) = m_R + im_L$ and that of an antiparticle is $m^* = m_R - im_L$. So the mass term in the Dirac equation can be written as

$$mP_R + m^* P_L = m_R + i\gamma^5 m_L \tag{13}$$

In this formalism, we solve the equation of motion for a fermion with a spatially-varying mass. We consider bubbles large enough that the bubble wall is planar, and the z-axis is perpendicular to the bubble wall. So in the frame of the bubble wall, the mass of the fermion is dependent only on z:

$$m = |m(z)|e^{i\theta(z)}$$

We will use the fact that (1) the z-axis spin operator, $\widetilde{S}_z = \gamma^0\gamma^3\gamma^5$ commutes with the free Hamiltonian in the frame where the fermion has no parallel momentum, $k_\parallel = 0$, and (2) that the energy is conserved in the wall frame due to the time translation invariance of the system. We can then decompose the wavefunction of the fermion as

$$\psi = \exp\Big(-i\omega t + \vec{k}_{\parallel} \cdot \vec{x}_{\parallel}\Big)\chi_s \otimes \begin{pmatrix} L_s \\ R_s \end{pmatrix}$$

where ω is the wall frame energy, s is the spin in the double-boostead frame, and L_s and R_s are the left-handed and right-handed parts of the spinor. Since we have a slowly-varying potential, we can make a WKB ansatz:

$$L_s = we^{i\int^z k_c dz'}$$

and likewise for R_s. It turns out that the momentum-like variable k_c here is not the physical momentum, but a gauge-dependent *canonical momentum:* it is unique only up to an additive constant. This ultimately reflects the gauge-dependence of the classical vector potential in the electromagnetic Lagrangian.

However, we can use this to find the physical momentum using the Hamiltonian equations of motion, where we identify the physical velocity of the particle with the group velocity of the wavepacket:

$$k_z = \omega v_g = \omega(\frac{\partial \omega}{\partial k_c})_z$$

We can use this to find the semiclassical force that causes the particles to move this way:

$$\begin{aligned} F_z = \frac{dk_z}{dt} &= \omega\frac{dv_g}{dt} \\ &= \omega(\frac{dz}{dt}(\frac{\partial v_g}{\partial z})_{k_c} + \frac{dk_c}{dt}(\frac{\partial v_g}{\partial k_c})_z) \\ &= \omega(v_g(\frac{\partial v_g}{\partial z})_{k_c} - (\frac{\partial \omega}{\partial z})_{k_c}\frac{\partial v_g}{\partial k_c}), \end{aligned} \tag{14}$$

where we have used the Hamiltonian equations of motion,

$$k_c = \frac{\partial \omega}{\partial \dot{z}} \tag{15}$$

$$\frac{dk_c}{dt} = -\frac{\partial \omega}{\partial z}. \tag{16}$$

In the semiclassical regime, where we can define a semiclassical force and group velocity, we can use these to describe the time-evolution of the distribution function according to the Boltzmann equation:

$$\frac{df}{dt} = (\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial z} + F_z \frac{\partial}{\partial k_z})f = C[f] \tag{17}$$

where $C[f]$ is a collision integral due to the interaction term of the Hamiltonian, and incorporates all the two-particle and higher-order interactions.

Now, if there are CP-violating terms in equation (17), they will lead to a difference in the distribution of particles and antiparticles, which will seed the macroscopic asymmetry.

The semiclassical approach agrees with the results obtained from the Schwinger-Keldysh approximation for slowly-varying backgrounds; it has been shown that the semiclassical approximation works unexpectedly well right up to $\lambda \sim \frac{2}{3}L_w$ [16].

As we said above, the semiclassical approximation rests on the assumption of a thick wall, whose validity is model-dependent. If the wall is thick, the electroweak baryogenesis scenario we are dealing with is based on a fundamentally classical mechanism: particles see the wall as a classical object and experience a CP-violating semiclassical force. However, if the wall is thin, non-local baryogenesis happens through CP-violating quantum reflection [17]. We will not be considering this case here.

Thicker walls are usually correlated with lower wall velocities, ensuring that particles can diffuse efficiently through the walls, which is favourable for baryogenesis. However, there is some optimisation called for here, since thinner walls are correlated with higher wall speeds and stronger phase transitions, ensuring that the sphaleron process is shut off in the broken phase and that baryon-antibaryon asymmetry is preserved. This is also a requirement for effective baryogenesis [7].

The obvious question now is whether the thick wall assumption, and hence the WKB approximation, is physically valid. The width of the wall is non-trivially dependent on the form of the Higgs potential (naïvely, one may expect that high potential barriers between the two minima are correlated with strongly first-order transitions and thin walls). Precise calculation of the wall width also requires finding the actual nucleation temperature, T_n, and solving the Higgs field equation of motion at this point; however, this is an extremely complex and model-dependent problem, and hence beyond the scope of this thesis.

In general, the wall width is of the order of $1 - 20T_n^{-1}$. Thus the assumption of a thick wall typically holds in most models, except possibly for highly infrared bosons [18].

2.4 A note on the outer product notation

A 4×4 matrix can be written as the outer product of two 2×2 matrices. We use the convention in which

$$\begin{aligned}\gamma^0 &= \mathbf{1}_2 \otimes \rho^1 \\ \gamma^k &= i\sigma^k \otimes \rho^2 \\ \gamma^5 &= -\mathbf{1}_2 \otimes \rho^3,\end{aligned}$$

where the first term in each outer product expansion corresponds to the spin degree of freedom, and the second to the chirality.

2.5 A note on frames

One difficulty we face is that there are several different frames relevant for calculations. Collision terms and self-energies are most easily computed in the frame of the plasma particles themselves, so we would like to eventually have our force in this frame, and in terms of plasma frame variables.

However, this is not the frame in which it is most convenient to perform the intervening calculations. Calculations are simplified in the frame of the bubble wall, where there is no time evolution, so the system is static and energy is a good quantum number. But this frame still does not allow us to exploit the full planar symmetry of the system: when we deal with large bubble walls, we can consider them to be planar, and fermion masses only vary perpendicular to the bubble wall.

Thus it is useful to boost to a frame in which particles have no momentum parallel to the wall, since the spin along the z-axis is a good quantum number in that frame. We will perform the calculation in this frame, and then perform successive Lorentz inverse boosts to the wall and plasma frames. We calculate the Lorentz boosts below.

2.5.1 From the wall frame to the plasma frame

The wall frame variables are related to those in the plasma frame by the usual Lorentz transformation, which reads

$$\omega^w = \gamma_w(\omega^{pl} - v_w k_z^{pl}) \tag{18}$$

$$k_z^w = \gamma_w(k_z^{pl} - v_w \omega^{pl}). \tag{19}$$

We deduce the spinor representation of the boost from the fact that it must give us the Dirac equation in the plasma frame:

$$\begin{aligned} S_w(\gamma^0\omega^{pl} - \gamma^3 k_z^{pl})S_w^{-1} &= \gamma^0\omega^w - \gamma^3 k_z^w \\ &= \gamma^0(\gamma_w(\omega^{pl} - v_w k_z^{pl})) - \gamma^3(\gamma_w(k_z^{pl} - v_w\omega^{pl})). \end{aligned} \tag{20}$$

S_w^{-1} is obtained from S_w by replacing v_w with $-v_w$. So we make the ansätze

$$S_w = a + bv_w\alpha_z \tag{21}$$

$$S_w^{-1} = a - bv_w\alpha_z, \tag{22}$$

and substitute these in (20), which gives

$$(a^2 + b^2v_w^2)(\gamma^0\omega^{pl} - \gamma^3 k_z^{pl}) - 2abv_w(\gamma^3\omega^{pl} - \gamma^0 k_z^{pl}) \tag{23}$$

$$= \gamma_w(\gamma^0\omega^{pl} - \gamma^3 k_z^{pl}) + \gamma_w v_w(\gamma^3\omega^{pl} - \gamma^0 k_z^{pl}). \tag{24}$$

This implies

$$a^2 + b^2v_w^2 = \gamma_w \tag{25}$$

$$2ab = -\gamma_w. \tag{26}$$

Solving equations (25)-(26) gives us

$$a = \pm\frac{\sqrt{\gamma_w + 1}}{\sqrt{2}} \tag{27}$$

$$b = \mp\frac{\sqrt{\gamma_w - 1}}{v_w\sqrt{2}}. \tag{28}$$

We can see from (26) that $2ab$ has to be negative, so a and b have to have different signs. We also know that for $v_w = 0$, $S_w = S_w^{-1} = 1$, so we choose the positive value for a and the negative value for b. This gives us the Lorentz boosts

$$S_w = \frac{\sqrt{\gamma_w + 1} - \sqrt{\gamma_w - 1}\alpha_z}{\sqrt{2}} \tag{29}$$

$$S_w^{-1} = \frac{\sqrt{\gamma_w + 1} + \sqrt{\gamma_w - 1}\alpha_z}{\sqrt{2}}, \tag{30}$$

which correspond to the spinor space representation of equations (18) and (19).

We can easily check that these boosts satisfy the normalisation condition, $S_w S_w^{-1} = 1$, or

$$a^2 - b^2 v_w^2 = 1 \tag{31}$$

This is the spinor space representation of the Lorentz boost. It represents the same boost as equations (18)-(19), but it acts on Dirac spinors in spinor space instead of 4-vectors in physical space. We can use this to transform the spinors and spin operators from the wall frame to the plasma frame.

2.5.2 From the $k_\parallel = 0$ frame to the wall frame

The wall frame and $p_\parallel = 0$ frame variables are related by a Lorentz boost, where we require that $\widetilde{k}_\parallel = 0$ in the boosted frame and k_z is unchanged. Since k^2 is invariant under a Lorentz boost, this is equivalent to requiring that $\widetilde{\omega} = \mathrm{sgn}(\omega^w)\sqrt{(\omega^w)^2 - (\vec{k}_\parallel)^2}$. This Lorentz boost must give us the Dirac equation for the $k_\parallel = 0$ frame:

$$S_\parallel(\gamma^0\omega^w - \vec{\gamma}\cdot\vec{k}_\parallel - \gamma^3\cdot k_z^w)S_\parallel^{-1} = \gamma^0\widetilde{\omega} - \gamma^3\widetilde{k}_z. \tag{32}$$

We know that the Lorentz boost has to commute with γ^3 and γ^5, so we assume it to be of the form $a + b\gamma^0\vec{\gamma}\cdot\vec{v}_\parallel$. The inverse transform has $\vec{v}_\parallel \to -\vec{v}_\parallel$. So we use the fact that $S_\parallel^{-1}S_\parallel = 1$ and (32) to determine the coefficients a and b, which results in

$$S_\parallel = \frac{\omega^w + \widetilde{\omega} - \gamma^0\vec{\gamma}\cdot\vec{k}_\parallel}{\sqrt{2\widetilde{\omega}(\omega^w + \widetilde{\omega})}}. \tag{33}$$

This is the spinor space representation of the Lorentz boost from the $k_\parallel = 0$ frame to the wall frame. Clearly, unlike the boost in the last section, this boost is momentum-dependent.

3 Semiclassical approach

In slowly varying backgrounds, quantum systems usually allow for a semiclassical expansion in gradients of the background. Thus we can use this method for a thick wall, where the bubble profile (and hence the mass of a fermion at the wall) varies slowly enough in space. We are following the usual WKB procedure, but for a relativistic system.

3.1 $k_\parallel = 0$ frame

We start by boosting from the wall frame to the $k_\parallel = 0$ frame, in which the calculations are most easily performed, to find the group velocity and the force. In the next section, we will perform the inverse-boost back to the wall frame.

The wall frame Dirac equation is

$$(\gamma^0\omega^w - \vec{\gamma}\cdot\vec{k_\parallel} + i\gamma^3\partial_z - m_R - i\gamma^5 m_L)\psi_{\omega^w\vec{k}_\parallel} = 0. \tag{34}$$

Following [19], we use (33) to boost to the $p_\parallel = 0$ frame, where $\widetilde{\omega} = \mathrm{sgn}(\omega^w)\sqrt{\widetilde{\omega}^2 - |\vec{k_\parallel}|^2}$, $\widetilde{k}_z = k_z^w$ and $\vec{\widetilde{k}}_\parallel = 0$.

Applying this boost gets rid of the $\gamma_\parallel$ term, giving us the Dirac equation in the $k_\parallel = 0$ frame:

$$(\gamma^0\widetilde{\omega} - \gamma^3\widetilde{k}_z - m_R - i\gamma^5 m_L)\psi = 0.$$

In this frame, helicity along the z-axis (represented by the operator $\widetilde{S}_z = \gamma^0\gamma^3\gamma^5$) is a good quantum number.

Following [20], we use the Weyl basis, in which the Dirac field can be decomposed into two left- and right-chiral Weyl fields by the projection operators, $P_{R,L} = \frac{1}{2}(1 \pm \gamma^5)$.

$$\psi \equiv e^{-i\widetilde{\omega}t}\chi_s \otimes \begin{pmatrix} L_s \\ R_s \end{pmatrix},$$

where $\sigma_3\chi_s = s\chi_s$. Using the outer product expansion of the γ matrices, the Dirac equation reduces to

$$\begin{pmatrix} \widetilde{\omega} + is\partial_z & -m \\ -m & \widetilde{\omega} - is\partial_z \end{pmatrix} \begin{pmatrix} R_s \\ L_s \end{pmatrix} = 0, \tag{35}$$

which gives us two coupled equations,

$$(\widetilde{\omega} - is\partial_z)L_s = mR_s \tag{36}$$
$$(\widetilde{\omega} + is\partial_z)R_s = m^* L_s. \tag{37}$$

Substituting for R_s from equation (36) in equation (37), we get

$$((\widetilde{\omega} + is\partial_z)\frac{1}{m}(\widetilde{\omega} - is\partial_z) - m^*)L_s = 0. \tag{38}$$

We make a WKB ansatz for L_s:

$$L_s = we^{\int^z \widetilde{k}_c(z')dz'}.$$

Substituting this in (38) and using the fact that we have normalised s^2 to 1, we get the equations

$$\widetilde{\omega}^2 - m^2 - \widetilde{k}_c^2 + \frac{\widetilde{w}''}{w} - \frac{\widetilde{|m|}'}{|m|}\frac{\widetilde{w}'}{w} + (s\widetilde{\omega} + \widetilde{k}_c)\widetilde{\theta}' = 0 \tag{39}$$

$$2\widetilde{k}_c\widetilde{w}' + \widetilde{k}_c' w - \frac{\widetilde{|m|}'}{|m|}(s\widetilde{\omega} + \widetilde{k}_c)w - \widetilde{\theta}'\widetilde{w}' = 0, \tag{40}$$

where $\widetilde{\theta}'$ denotes the z-derivative of θ in this frame, and so on.

To the lowest order, we can set all derivative terms to 0 and get the dispersion relation $\widetilde{\omega}^2 = \widetilde{k}_c^2 + m^2$. This corresponds to the vacuum dispersion relation, which holds if we neglect interactions. We define $p_0 = \text{sgn}(\widetilde{k}_z)\sqrt{\widetilde{\omega}^2 - m^2}$. Then we solve the equation to the next order. The resulting momentum-like variable, k_c, is the gauge-dependent canonical momentum, defined up to an arbitrary additive constant, α'.

$$p_0^2 - \widetilde{k}_c^2 + (s\widetilde{\omega} + p_0)\widetilde{\theta}' = 0$$
$$\widetilde{k}_c = p_0 + \frac{s\widetilde{\omega} + p_0}{2p_0}\theta' + \alpha',$$

where the factor of α' indicates that this is just a canonical momentum and not the physical, gauge-invariant momentum.

For antiparticles, the replacement $m \to m^*$ means that $\widetilde{\theta}'$ is replaced by $-\widetilde{\theta}'$. So we can combine the two equations into

$$\widetilde{k}_c = p_0 + s_{CP}\frac{s\widetilde{\omega} + p_0}{2p_0}\widetilde{\theta}' + \alpha', \tag{41}$$

where $s_{CP} = 1$ for particles and -1 for antiparticles. The same procedure applied to R_s gives

$$\widetilde{k}_c = p_0 + s_{CP}\frac{s\widetilde{\omega} - p_0}{2p_0}\widetilde{\theta}' + \alpha'. \tag{42}$$

We invert equation (41) to obtain

$$\widetilde{\omega} = \sqrt{(\widetilde{k}_c - \alpha_{CP})^2 + m^2} - s_{CP}\frac{s\widetilde{\theta}'}{2}, \tag{43}$$

where $\alpha_{CP} \equiv \alpha\prime \pm s_{CP}\frac{\widetilde{\theta}'}{2}$ in the left- (right-) chiral sector.

We then compute the group velocity of the wavepacket, which we identify to be the velocity of the WKB particle:

$$\begin{aligned} v_g &= (\partial_{\widetilde{k}_c}\widetilde{\omega})_x \\ &= \frac{\widetilde{k}_c - \alpha_{CP}}{\sqrt{(\widetilde{k}_c - \alpha_{CP})^2 + m^2}} \\ &\approx \frac{p_0}{\widetilde{\omega}}(1 - \frac{s_{CP}\widetilde{\theta}'}{2})(\frac{1}{\widetilde{\omega}} - \frac{\widetilde{\omega}}{p_0^2}) \\ &= \frac{p_0}{\widetilde{\omega}}(1 + \frac{s_{CP}\widetilde{\theta}' m^2}{2p_0^2\,\widetilde{\omega}}). \end{aligned} \tag{44}$$

From the physical velocity, we can define the physical momentum $\widetilde{k}_z$ as $\widetilde{k}_0 v_g$, or

$$\widetilde{k}_z = p_0(1 + \frac{s_{CP}\widetilde{\theta}' m^2}{2p_0^2\widetilde{\omega}}).$$

The semiclassical force which causes the particles to move this way is given by

$$
\begin{aligned}
F_z = \widetilde{\omega}\dot{v}_g = \widetilde{\omega}\dot{z}\frac{\partial}{\partial z}v_g = \widetilde{\omega}\frac{\partial v_g}{\partial x}v_g \\
= \frac{|m||m|'}{\widetilde{\omega}} + s_{CP}\frac{s(m^2\widetilde{\theta}')'}{2\widetilde{\omega}^2} \\
= \frac{(m^2)'}{2\widetilde{\omega}} \pm \frac{s(m^2\widetilde{\theta}')'}{2\widetilde{\omega}^2}
\end{aligned} \tag{45}
$$

where $+(-)$ refers to particles (antiparticles). This force has a leading order term that is the same for particles and antiparticles (CP-even), and a first-order term in θ' that is CP-violating. It is this CP-violating term that causes the separation of charges in the Boltzmann equation, which will eventually seed the baryon-antibaryon asymmetry.

3.2 Wall frame

We now boost to the frame of the bubble wall. In this frame, the energy is a good quantum number, but the spin along the z-axis is no longer a good quantum number.

3.2.1 Dynamics

In the last section, we solved the boosted Dirac equation to get the canonical momentum in the boosted frame. Now we can use equations (41), (43) and the fact that $(\omega^w)^2 = \widetilde{\omega}^2 - \vec{k}_\parallel^2$ to obtain an expression for the energy in the wall frame. (We keep in mind that since the z-coordinate does not change, θ'_w is the same as $\widetilde{\theta}'$.)

$$
\omega^w = \sqrt{(\widetilde{k}_c - \alpha_{CP})^2 + \vec{k}_\parallel^2 + m^2} - s_{CP}\frac{s\theta'_w}{2}\frac{\sqrt{(\widetilde{k}_c - \alpha_{CP})^2 + m^2}}{\sqrt{(\widetilde{k}_c - \alpha_{CP})^2 + k_\parallel^2 + m^2}} \tag{46}
$$

We can then compute the physical momentum:

$$
\begin{aligned}
k_z^w &= \omega^w(\partial_{k_c}\omega)_z^w \\
&= p_0 \pm \gamma_\parallel \frac{sm^2\theta'}{2p_0\omega^w},
\end{aligned} \tag{47}
$$

where p_0 is the same as before: $p_0 = \sqrt{\widetilde{\omega}^2 - m^2} = \sqrt{(\omega^w)^2 - k_\parallel^2 - m^2}$.

The physical momentum is related to the canonical momentum by

$$
(k_c^w - \alpha_{CP}) = k_z^w(1 \pm \frac{s\theta'}{2\omega_{0z}}), \tag{48}
$$

which gives us the dispersion relation in terms of physical variables:

$$\omega^w = \omega_0^w \mp \frac{s\theta' m^2}{2\omega_0^w \omega_{0z}^w}, \tag{49}$$

where we define

$$\omega_{0z}^w = \sqrt{(k_z^w)^2 + m^2} \tag{50}$$
$$\omega_0^w = \sqrt{(\vec{k}^w)^2 + m^2}. \tag{51}$$

The corresponding physical force is given by

$$F_w = -\frac{(m^2)'}{2\omega^w} \pm \gamma_{\parallel} \frac{s(m^2\theta')'}{2(\omega_w)^2}, \tag{52}$$

which is related to F_z by $F_z = \gamma_{\parallel} F_w$. Clearly, this agrees with the result we would get from a direct Lorentz transformation along a direction perpendicular to the force.

3.2.2 Helicity states

The spins of fermions in the wall frame no longer point along the z-axis, but roughly (not exactly) along the direction of momentum, which could have a parallel component. These states are not easy to work with, and we would rather use the helicity basis. The helicities $h = \pm 1$ measure whether spin is in the same direction as the momentum, or opposite to it. We are interested in the helicity states because helicity commutes with the free Hamiltonian. Thus they are our closest approximation to physical on-shell states. Moreover, for relativistic particles, helicity is close to chirality, and it is the chirality of particles that is relevant for electroweak interactions and hence for evaluating the collision terms.

The operator $\widetilde{S}_z = \gamma^0\gamma^3\gamma^5$ measures spin in the z-direction in the $k_{\parallel} = 0$ frame, and transforms under the Lorentz boost according to

$$\begin{aligned} S_z^w &= S_{\parallel}^{-1} \widetilde{S}_z S_{\parallel} \\ &= \gamma_{\parallel}[\widetilde{S}_z - i(\vec{v}_{\parallel} \times \vec{\alpha})_z]. \end{aligned} \tag{53}$$

Clearly, S_z^w commutes with the differential operator in equation (34), so the conserved spin states are its eigenstates.

We have F_w in terms of the eigenstates of $\widetilde{S}_z$, but we want to find out how it acts on the helicity states in the wall frame.

The explicit spinors for the spin states in the wall frame are

$$u(p, s) = S_{\|}^{-1} \widetilde{u}(\widetilde{p}_z, s).$$

These obey

$$\widetilde{S}_z \widetilde{u}(\widetilde{p}_z, s) = s\widetilde{u}(\widetilde{p}_z, s)$$

$$S_z^w u(p_z^w, s) = su(p_z^w, s).$$

It is easy to see that these states are correctly normalised:

$$\begin{aligned} u^\dagger u &= \widetilde{u}^\dagger S_{\|}^{-2} \widetilde{u} \\ &= 2\gamma_{\|}\widetilde{\omega} = 2\omega^w. \qquad (54) \\ \bar{u}u &= \bar{\widetilde{u}}\widetilde{u} = 2m. \qquad (55) \end{aligned}$$

Equation (52) gives us the force on the projected spin-states. Since it depends linearly on spin, we can compute the force on helicity states as a projection:

$$F_w^h = \sum_s \langle u(p, h)|F_w|u(p, h)\rangle, \qquad (56)$$

where $u(p, h)$ are the helicity eigenstates. This is proportional to

$$s_h^w = \langle u(p, h)|S_z|u(p, h)\rangle. \qquad (57)$$

We can write $u(p, h)$ as a linear combination of $u(p, s)$ states:

$$u(p, h) = \sum c_s^h u(p, s)$$

and hence find c_s from

$$c_s^h = \frac{\bar{u}(p, s)u(p, h)}{2m}$$

The expectation value of the wall frame spin is given by

$$
\begin{aligned}
s_h &= \sum_s s|c_s^h|^2 \\
&= \sum_s s\frac{\bar{u}(p,h)u(p,s)\bar{u}(p,s)u(p,h)}{4m^2} \\
&= \sum_s s\frac{\bar{u}(p,h)S_\parallel^{-1}\widetilde{u}(\widetilde{p}_z,s)\bar{\widetilde{u}}(\widetilde{p}_z,s)S_\parallel u(p,h)}{4m^2} \\
&= \sum_s \frac{\bar{u}(p,h)S_\parallel^{-1}\widetilde{S}_z\widetilde{u}(\widetilde{p}_z,s)\bar{\widetilde{u}}(\widetilde{p}_z,s)S_\parallel u(p,h)}{4m^2} \\
&= \frac{\bar{u}(p,h)S_\parallel^{-1}\widetilde{S}_z\sum_s(\widetilde{u}(\widetilde{p}_z,s)\bar{\widetilde{u}}(\widetilde{p}_z,s))S_\parallel u(p,h)}{4m^2} \\
&= \frac{\bar{u}(p,h)S_\parallel^{-1}\widetilde{S}_z S_\parallel S_\parallel^{-1}(\not{\widetilde{p}}_z+m)S_\parallel u(p,h)}{4m^2} \\
&= \frac{\bar{u}(p,h)S_\parallel^{-1}\widetilde{S}_z S_\parallel(\not{p}+m)u(p,h)}{4m^2} \\
&= \frac{\bar{u}(p,h)S_\parallel^{-1}\widetilde{S}_z S_\parallel u(p,h)}{2m} \\
&= \langle S_\parallel^{-1}\widetilde{S}_z S_\parallel\rangle_h \\
&= \gamma_\parallel\langle\widetilde{S}_z - i\vec{v}_\parallel\times\vec{\alpha}\rangle_h \\
&= \gamma_\parallel\langle\widetilde{S}_z\rangle_h \\
&= \gamma_\parallel h\frac{p_z}{p}
\end{aligned} \tag{58}
$$

where we used the Dirac equation and equation (53), and the fact that

$$
\begin{aligned}
\langle\widetilde{S}_z\rangle &= \bar{u}(p,h)\gamma^0\gamma^3\gamma^5 u(p,h) \\
&= h\frac{p_z}{p},
\end{aligned} \tag{59}
$$

since the spin points along the direction of the momentum, which means that its z-component is proportional to the z-momentum and normalised to 1 for the total momentum, and the helicity tells us whether it is directed along the direction of the momentum or opposite to it. Substituting this in equation (52), we get the force on the wall frame helicity states:

$$
F_w^h = -\frac{(m^2)'_w}{2\omega^w} \pm \frac{hk_z^w}{|\vec{k}^w|}\gamma_\parallel\frac{(m^2\theta')'_w}{2\omega^w\widetilde{\omega}}. \tag{60}
$$

This is an intermediate result of some interest, that has been discussed in the literature (see for example [19], [21]); we will be using it to derive the wall frame transport equations in Chapter 5. Our eventual goal is to derive the force on the plasma frame helicity states. However, it turns out that this is not so easily done in the semiclassical formalism; we will eventually derive the plasma frame force through the Schwinger-Keldysh formalism, in Section 4.3.

3.3 Plasma frame

We are interested in the dynamics in the plasma frame because this frame is the most convenient when it comes to evaluating the collision terms, and hence studying the time evolution of the system according to the Boltzmann equation.

However, deriving the force in this frame is conceptually a more difficult problem. In this frame, the energy of the particle is no longer a good quantum number due to interactions with the wall, so we can no longer use the relation $k_z = \omega v_g$.

There are three momentum-like variables in the problem: the *canonical momentum,* that is gauge-dependent but plays the role of the momentum in the Hamiltonian equations of motion; the *adiabatic momentum,* that obeys the Einstein dispersion relation, $\omega^2 = p^2 + m^2$; and the *physical momentum,* that is related to the group velocity of the wavepacket and describes how the particles actually move in space.

The wall frame and the plasma frame are related by an inverse Lorentz boost along the z-axis, and it is not immediately clear which of these momenta transforms under the Lorentz transformation equations. Thus we do not perform the derivation here, but defer it to the section on the Schwinger-Keldysh formalism, where the derivation is relatively straightforward, since we deal directly with the spinors.

4 Schwinger-Keldysh formalism

Now that we have derived the force using the WKB approximation in the wall frame, we shall verify that the same results can be obtained using a more fundamental approach. Moreover, the Schwinger-Keldysh formalism allows us to derive the dynamics of the plasma from first principles, using quantum field theory for out-of-equilibrium systems [22].

The statistical properties of an out-of-equilibrium system can be described by a propagator, that splits into four parts. [23]

$$iG^F(u,v) \equiv \langle \mathcal{T}[\psi(u)\bar{\psi}(v)]\rangle \tag{61}$$

$$iG^<(u,v) \equiv \langle \bar{\psi}(v)\psi(u)\rangle \tag{62}$$

$$iG^>(u,v) \equiv \langle \psi(u)\bar{\psi}(v)\rangle \tag{63}$$

$$iG^{\bar{F}}(u,v) \equiv \langle \mathcal{T}[\bar{\psi}(u)\psi(v)]\rangle, \tag{64}$$

where G^F represents the usual causal Feynman propagator, $G^{\bar{F}}$ is the anti-chronal anti-Feynman propagator, $G^<$ is the absolutely-ordered negative frequency Wightman propagator, and $G^>$ is the absolutely-ordered positive frequency Wightman propagator. [24]

These different propagators are not independent, but related by [8]

$$G^F(u,v) = \theta(u_0 - v_0)G^>(u,v) + \theta(v_0 - u_0)G^<(u,v) \tag{65}$$

$$G^{\bar{F}}(u,v) = \theta(u_0 - v_0)G^<(u,v) + \theta(v_0 - u_0)G^>(u,v), \tag{66}$$

since the Feynman propagator propagates positive frequencies forwards in time and negative frequencies backwards, and the anti-chronological propagator does the reverse.

Of these, the one of greatest interest to us is the two-point Wightman function, $iG^<_s(u,v) = \langle\bar{\psi}(v)\psi(u)\rangle$. The Hermitian version of the Wightman function is given by

$$\bar{G}^<_s(u,v) = i\gamma^0 G^<_s(u,v). \tag{67}$$

This two-point Wightman function is translationally invariant in thermal equilibrium, i.e. it depends only on the internal relative coordinate, $r = u - v$. However, under out-of-equilibrium conditions, it is no longer translationally invariant; it also depends on the external variable, $X = u + v$. The Wigner transformation separates out these internal (or microscopic) degrees of freedom from the macroscopic degrees of freedom:

$$G_s^<(k, X) = \int d^4 r e^{ik\cdot r} G_s^<(X + \frac{r}{2}, X - \frac{r}{2}), \tag{68}$$

where k is the conjugate momentum to r, related by

$$k = \frac{\partial \hat{H}}{\partial \dot{r}}$$

For a collisionless system, the generalised Dirac equation corresponding to Lagrangian (11) can be written in the form

$$(i\partial\!\!\!/_u - m_R(u) - i m_I(u)\gamma_5)\psi(u) = 0. \tag{69}$$

We can perform a Wigner transformation to get the Dirac equation in the mixed representation: [19]

$$(\gamma^0 \hat{k}_0 - \hat{k}_z \gamma^3 - \hat{k}_\parallel \cdot \vec{\gamma} - \hat{m}_0 - i\hat{m}_5 \gamma^5) i\gamma^0 G_s^< = 0, \tag{70}$$

where we have defined the operators

$$\hat{k}_{z(0)} = k_{z(0)} \mp \frac{i}{2}\partial_{z(0)} \tag{71}$$

$$\hat{m}_{0(5)} = m_{R(I)} + \frac{i}{2} m'_{R(I)} \partial_{k_z} + \mathcal{O}(m'' \partial^2_{k_z}), \tag{72}$$

where $-(+)$ refers to $\hat{k}_z(\hat{k}_0)$.

We note that k_0 here refers to the frequency of the wavepacket, which is positive for particles and negative for antiparticles. In the semiclassical limit, where we can think of the plasma as a collection of on-shell particles, its magnitude can be equated to the on-shell energy:

$$k_0 = \pm\omega.$$

We can solve this equation to get the two-point function. Once again, it turns out that the calculation is most conveniently performed in the frame where $k_\parallel = 0$, where the problem is simplified from a 4-dimensional problem to a 2-dimensional one. But for some purposes, we need the results in other frames, so in sections 4.2 and 4.3 we will perform successive Lorentz boosts to the wall frame and hence to the frame of the plasma. Our ultimate objective will be to find the kinetic equation governing the time-evolution of the distribution of fermions in the plasma frame.

4.1 $k_{\|} = 0$ frame

Once again, it is convenient to perform the calculation in this frame, since the fact that spin along the z-axis is conserved reduces the number of degrees of freedom. The subsequent Lorentz transformations mix these degrees of freedom.

4.1.1 Dirac equation

In this frame, $k_{\|} = 0$ and the z-component of the spin is a good quantum number. The Wigner-transformed Dirac equation becomes

$$(\hat{k}_0 + \hat{k}_z\gamma^0\gamma^3 - \hat{m}_0\gamma^0 + \hat{m}_5\gamma^0\gamma^5)i\gamma^0\widetilde{G}_s^< = 0. \tag{73}$$

Henceforth we drop the $<$ superscript. Since spin along the z-axis is conserved, we can decompose the Wightman function as

$$i\gamma^0\widetilde{G}_s = \frac{1}{4}(1 + s\sigma^3) \otimes \rho^\mu \widetilde{g}_\mu^s, \tag{74}$$

where $\rho^\mu = (\mathbf{1}_2, \vec{\rho})$ are the Pauli spin matrices, and correspond to the chirality degree of freedom. Thus the four-dimensional equation reduces to two dimensions in this frame. ρ_μ define a four-dimensional basis, while $\widetilde{g}_\mu^s$ are parameters; $\widetilde{g}_0^s$ measures the number density of particles in phase space, and $\widetilde{g}_3^s$ measures particle flux.

We use the outer product decomposition of the gamma matrices to make the identifications $\gamma^0 \to \rho^1$, $-i\gamma^0\gamma^5 \to \rho^2$, $\gamma^0\gamma^3 \to -s\rho^3$.

To the desired order, this gives us

$$(\hat{k}_0 - s\rho^3\hat{k}_z - \hat{m}_0\rho^1 - \hat{m}_5\rho^2)(\widetilde{g}_0^s + \rho^a\widetilde{g}_a^s) = 0, \tag{75}$$

where

$$\hat{k}_{z(0)} \equiv k_{z(0)} \mp \frac{i}{2}\partial_{z(0)} \tag{76}$$

$$\hat{m}_{0(5)} \equiv m_{R(I)} + \frac{i}{2}m'_{R(I)}\partial_{k_z}. \tag{77}$$

We multiply respectively by $\mathbf{1}_2$ and ρ^i and take the traces to get

$$\begin{aligned}
\hat{k}_0\widetilde{g}_0^s - \hat{m}_0\widetilde{g}_1^s \; - \hat{m}_5\widetilde{g}_2^s \; - s\hat{k}_z\widetilde{g}_3^s &= 0\\
\hat{k}_0\widetilde{g}_1^s + is\hat{k}_z\widetilde{g}_2^s - i\hat{m}_5\widetilde{g}_3^s - \hat{m}_0\widetilde{g}_0^s &= 0\\
\hat{k}_0\widetilde{g}_2^s - is\hat{k}_z\widetilde{g}_1^s + i\hat{m}_0\widetilde{g}_3^s - \hat{m}_5\widetilde{g}_0^s &= 0\\
\hat{k}_0\widetilde{g}_3^s + i\hat{m}_5\widetilde{g}_1^s \; - i\hat{m}_0\widetilde{g}_2^s - s\hat{k}_z\widetilde{g}_0^s &= 0.
\end{aligned}$$

The real part of these equations contains no time-derivatives and hence gives us four non-dynamical constraints, which restrict the phase-space structure of the Wightman function. [23] The imaginary part gives us four kinetic equations, which govern the time-evolution of the Wightman function. Since we have a slowly-varying potential, we can expand the operators and truncate them to the lowest order in derivatives; this corresponds to expanding in powers of $\hbar$. It is enough to write out the constraint equation to zeroth order, and the kinetic equation to first order in derivatives.

4.1.2 Constraint equations

The constraint equations are given by

$$\begin{aligned}
\widetilde{g}_0^s &= s\frac{k_z}{k_0}\widetilde{g}_3^s + \frac{m_R}{k_0}\widetilde{g}_1^s \quad + \frac{m_I}{k_0}\widetilde{g}_2^s\\
\widetilde{g}_3^s &= s\frac{k_z}{k_0}\widetilde{g}_0^s + \widetilde{m}_I'\partial_{k_z}\widetilde{g}_1^s - \frac{\widetilde{m}_R'}{2k_0}\partial_{k_z}\widetilde{g}_2^s\\
\widetilde{g}_1^s &= \frac{m_R}{k_0}\widetilde{g}_0^s - \frac{s}{2k_0}\partial_z\widetilde{g}_2^s - \frac{\widetilde{m}_I'}{2k_0}\partial_{k_z}\widetilde{g}_3^s\\
\widetilde{g}_2^s &= \frac{m_I}{k_0}\widetilde{g}_0^s + \frac{s}{2k_0}\partial_z\widetilde{g}_1^s + \frac{\widetilde{m}_R'}{2k_0}\partial_{k_z}\widetilde{g}_3^s.
\end{aligned} \tag{A}$$

We can use the slowly-varying background again to solve these iteratively. For example, to first order,

$$\begin{aligned}
\widetilde{g}_1^s &= \frac{m_R}{k_0}\widetilde{g}_0^s \; - \frac{s}{2k_0}\partial_z\widetilde{g}_2^s\\
\widetilde{g}_2^s &= \frac{m_I}{k_0}\widetilde{g}_0^s \; + \frac{s}{2k_0}\partial_z\widetilde{g}_1^s\\
\widetilde{g}_3^s &= s\frac{k_z}{k_0}\widetilde{g}_0^s.
\end{aligned}$$

We can solve for $\widetilde{g}_1^s$ and $\widetilde{g}_2^s$ by substituting for them again to first order. The mass of the particle is given by

$$m = |m(z)| \exp(i\theta(z)),$$

while that of the antiparticle is given by m^*. The mass can also be expressed as $m = m_R + im_I = |m|(\cos\theta + i\sin\theta)$ while $m^* = m_R - im_I = |m|(\cos\theta - i\sin\theta)$.

Thus

$$\begin{aligned} m_I' &= \quad m_R\widetilde{\theta}' \\ m_R' &= -m_I\widetilde{\theta}'. \end{aligned}$$

So

$$m_R m_I' - m_I m_R' = m^2\widetilde{\theta}'. \tag{78}$$

Using this, we get from (A), to first order:

$$\begin{aligned} \widetilde{g}_1^s &= -\frac{s}{2\widetilde{k}_0}\partial_z\widetilde{g}_2^s + \frac{m_R}{\widetilde{k}_0}\widetilde{g}_0^s - \frac{m_I'}{2\widetilde{k}_0}\partial_{\widetilde{k}_z}\widetilde{g}_3^s \\ &\approx -\frac{s}{2\widetilde{k}_0}\partial_z(\frac{m_I}{2\widetilde{k}_0}\widetilde{g}_0^s) + \frac{m_R}{\widetilde{k}_0}\widetilde{g}_0^s - \frac{m_I'}{2\widetilde{k}_0}\partial_{\widetilde{k}_z}(s\frac{\widetilde{k}_z}{\widetilde{k}_0}\widetilde{g}_0^s) \\ &= (-s\frac{m_I'}{\widetilde{k}_0^2} + \frac{m_R}{\widetilde{k}_0})\widetilde{g}_0^s - s\frac{m_I}{2\widetilde{k}_0^2}\partial_z\widetilde{g}_0^s - s\frac{m_I'\widetilde{k}_z}{2\widetilde{k}_0^2}\partial_{\widetilde{k}_z}\widetilde{g}_0^s. \end{aligned} \tag{79}$$

Similarly,

$$\begin{aligned} \widetilde{g}_2^s &= \frac{s}{2\widetilde{k}_0}\partial_z\widetilde{g}_1^s + \frac{m_I}{\widetilde{k}_0}\widetilde{g}_0^s + \frac{m_R'}{2\widetilde{k}_0}\partial_{\widetilde{k}_z}\widetilde{g}_3^s \\ &\approx \frac{s}{2\widetilde{k}_0}\partial_z(\frac{m_R}{k_0}\widetilde{g}_0^s) + \frac{m_I}{\widetilde{k}_0}\widetilde{g}_0^s + \frac{m_R'}{2\widetilde{k}_0}\partial_{\widetilde{k}_z}(s\frac{\widetilde{k}_z}{\widetilde{k}_0}\widetilde{g}_0^s) \\ &= (s\frac{m_R'}{\widetilde{k}_0^2} + \frac{m_I}{\widetilde{k}_0})\widetilde{g}_0^s + s\frac{m_R}{2\widetilde{k}_0^2}\partial_z\widetilde{g}_0^s + s\frac{m_R'\widetilde{k}_z}{2\widetilde{k}_0^2}\partial_{\widetilde{k}_z}\widetilde{g}_0^s, \end{aligned} \tag{80}$$

and finally:

$$\widetilde{g}_3^s = s\frac{\widetilde{k}_z}{\widetilde{k}_0}\widetilde{g}_0^s + \frac{m_R m_I' - m_I m_R'}{2\widetilde{k}_0^2}\partial_{\widetilde{k}_z}\widetilde{g}_0^s$$
$$= s\frac{\widetilde{k}_z}{\widetilde{k}_0}\widetilde{g}_0^s + \frac{m^2\widetilde{\theta}'}{2\widetilde{k}_0^2}\partial_{\widetilde{k}_z}\widetilde{g}_0^s. \tag{81}$$

Combining equations (79) and (80), we also get

$$m_R\widetilde{g}_1^s + m_I\widetilde{g}_2^s = \frac{m^2}{\widetilde{k}_0}\widetilde{g}_0^s + \frac{sm^2\widetilde{\theta}'}{\widetilde{k}_0^2}\widetilde{g}_0^s - \frac{\widetilde{k}_z}{\widetilde{k}_0}\frac{sm^2\widetilde{\theta}'}{2\widetilde{k}_0}\partial_{\widetilde{k}_z}\widetilde{g}_0^s. \tag{82}$$

4.1.3 Dispersion relation

Substituting for the $\widetilde{g}_i^s$s to first order in (79), we get

$$k_0\widetilde{g}_0^s - \frac{k_z^2}{k_0}\widetilde{g}_0^s + m_R(\frac{s}{2k_0}\partial_z\widetilde{g}_2^s) - \frac{m_R^2}{k_0}\widetilde{g}_0^s - m_I(\frac{s}{2k_0}\partial_z\widetilde{g}_1^s) - \frac{m_I^2}{k_0}\widetilde{g}_0^s = 0.$$

Substituting again for $\widetilde{g}_1^s$ and $\widetilde{g}_2^s$ to first order gives us

$$(\widetilde{k}_0^2 - \widetilde{k}_z^2 - m^2 + \frac{sm^2\theta'}{2\widetilde{k}_0})\widetilde{g}_0^s = 0. \tag{83}$$

This is an algebraic equation, which implies that there exists a spectral solution for $\widetilde{g}_0^s$ of the form

$$\widetilde{g}_0^s = \sum_{\pm}\frac{\pi}{2Z_{s\pm}}n_s\delta(\widetilde{k}_0 \mp \widetilde{\omega}_{s\pm}), \tag{84}$$

where $\widetilde{\omega}_{s\pm}$ is a solution of the dispersion relation,

$$\widetilde{k}_0^2 - \widetilde{k}_z^2 - m^2 + \frac{sm^2\theta'}{2\widetilde{k}_0} = 0. \tag{85}$$

The fact that we can find such a solution for $\widetilde{g}_0^s$ implies that to the required order, we can think of the plasma as a collection of single-particle excitations. Thus for slowly-varying backgrounds, the use of the semiclassical approximation in this frame is justified.

4.1.4 Kinetic equation

The kinetic equations are given by the imaginary parts of the Wigner-transformed Dirac equation. The dynamics of the plasma are fully described by one of the four kinetic equations, along with the four kinetic equations. Following [25], we choose the first one, since the quantity we are most interested in is $\widetilde{g}_0^s$, which is related to the particle density in phase space.

This gives us

$$\partial_{\widetilde{t}}\widetilde{g}_0^s + s\partial_{\widetilde{z}}\widetilde{g}_3^s - m_R'\partial_{\widetilde{k}_z}\widetilde{g}_1^s - m_I'\partial_{\widetilde{k}_z}\widetilde{g}_2^s = 0. \tag{86}$$

Substituting for $\widetilde{g}_i^s$ in terms of $\widetilde{g}_0^s$, we get

$$(\partial_{\widetilde{t}} + \frac{\widetilde{k}_z}{\widetilde{k}_0}\partial_{\widetilde{z}} + (-\frac{m^2}{\widetilde{k}_0} + \frac{sm^2\theta'}{2\widetilde{k}_0^2})\partial_{\widetilde{k}_z})\widetilde{g}_0^s = 0. \tag{87}$$

This can clearly be identified with the Liouville equation, in which

$$v_g = \frac{\widetilde{k}_z}{\widetilde{k}_0} \tag{88}$$

$$F_z = -\frac{m^2}{\widetilde{k}_0} + \frac{sm^2\theta'}{2\widetilde{k}_0^2}. \tag{89}$$

This agrees with our computation in the semiclassical formalism. Thus so far, the two calculations agree.

4.2 Wall frame

We now boost to the frame of the wall, where $k_{\|}$ is no longer zero. In this frame, the energy of the wall is a good quantum number. The time-derivative of the distribution function is still zero.

4.2.1 Connection between wall frame and $k_\parallel = 0$ variables

We can use the Lorentz boost (33) to calculate the connections between the Wightman functions in the $k_\parallel = 0$ frame and the wall frame.

The wall frame and $p_\parallel = 0$ frame Wightman functions are related by

$$\widetilde{G}_s = S_\parallel G^w_s S^{-1}_\parallel. \tag{90}$$

These Wightman functions can be decomposed as

$$\begin{aligned} i\gamma^0\widetilde{G}_s &= \frac{1}{4}((\mathbf{1}+s\sigma^3) \otimes \rho^\mu)\widetilde{g}^s_\mu \\ i\gamma^0 G^w_s &= \frac{1}{4}(\quad \sigma^c \quad \otimes \rho^d)g^w_{cd}, \end{aligned}$$

where the Pauli matrices $\rho^\mu = (\mathbf{1}, \vec{\rho})$ form a 4-dimensional basis spanning the chirality subspace, and $(\sigma^c \otimes \rho^d)$ form a 16-dimensional basis, with the inner product defined by

$$\frac{1}{4}\,\mathrm{Tr}\Big((\sigma^a \otimes \rho^b)(\sigma^c \otimes \rho^d)\Big) = \delta^{ac}\delta^{bd}. \tag{91}$$

The problem clearly has more degrees of freedom in the wall frame, where the spin along the z-axis is no longer conserved.

We can now calculate the desired connections as

$$g^w_{ab} = \sum_{c,d} \frac{1}{4}\,\mathrm{Tr}\Big[S_\parallel((\mathbf{1} + s\sigma^3) \otimes \rho^c)S_\parallel(\sigma^a \otimes \rho^b)\Big]\widetilde{g}^s_c. \tag{92}$$

The Lorentz transformation matrix $S_\parallel$ and its inverse $S^{-1}_\parallel$ can be expanded in outer product form as follows:

$$S_\parallel = \alpha(\mathbf{1} \otimes \mathbf{1}) + \beta(\vec{\sigma} \cdot \vec{k}_\parallel \otimes \rho^3) \tag{93}$$

$$S^{-1}_\parallel = \alpha(\mathbf{1} \otimes \mathbf{1}) - \beta(\vec{\sigma} \cdot \vec{k}_\parallel \otimes \rho^3), \tag{94}$$

where

$$\alpha = \frac{\omega^w + \widetilde{\omega}}{\sqrt{2\widetilde{\omega}(\omega^w + \widetilde{\omega}}} \tag{95}$$

$$\beta = -\frac{1}{\sqrt{2\widetilde{\omega}(\omega^w + \widetilde{\omega})}}. \tag{96}$$

Substituting these in (92), we get

$$\begin{aligned} g^w_{ab} =& \sum_c \frac{1}{4} \text{Tr } [(\alpha(\mathbf{1} \otimes \mathbf{1}) + \beta(\vec{\sigma} \cdot \vec{k}_\parallel \otimes \rho^3))((1 + s\sigma^3) \otimes \rho^c)(\alpha(\mathbf{1} \otimes \mathbf{1}) \\ &+ \beta(\vec{\sigma} \cdot \vec{k}_\parallel \otimes \rho^3))(\sigma^a \otimes \rho^b)]\widetilde{g}^s_c \\ =& \sum_c \frac{1}{4} \text{Tr } [\alpha^2((\mathbf{1} + s\sigma^3) \otimes \rho^c) + \beta^2(\vec{\sigma} \cdot \vec{k}_\parallel(\mathbf{1} + s\sigma^3)\vec{\sigma} \cdot \vec{k}_\parallel \otimes \rho^3\rho^c\rho^3) \\ &+ \alpha\beta((1 + \sigma^3)\vec{\sigma} \cdot \vec{k}_\parallel \otimes \rho^c\rho^3) + \alpha\beta(\vec{\sigma} \cdot \vec{k}_\parallel(1 + s\sigma^3) \otimes \rho^3\rho^c)(\sigma^a \otimes \rho^b)]\widetilde{g}^s_c. \end{aligned} \tag{97}$$

We can calculate the Dirac matrix products:

$$(1 + s\sigma^3)(\sigma^1 k_x + \sigma 2k_y) = \sigma^1 k_x + \sigma 2k_y + si(\sigma^2 k_1 - \sigma^1 k_2) \tag{98}$$

$$(\sigma^1 k_x + \sigma 2k_y)(1 + s\sigma^3) = \sigma^1 k_x + \sigma 2k_y - si(\sigma^2 k_1 - \sigma^1 k_2) \tag{99}$$

$$(\sigma^1 k_x + \sigma^2 k_y)(1 + s\sigma^3)(\sigma^1 k_x + \sigma^2 k_y) = k_1^2 + k_2^2 - s\sigma^3(k_1^2 + k_2^2), \tag{100}$$

and in the other subspace,

$$\begin{aligned} \rho^3\rho^c =& \quad \mathbf{1},\ c = 3 \\ =& \quad \rho^3,\ c = 0 \\ =& i\epsilon_{3ck}\rho^k,\ c = 1, 2; \end{aligned}$$

$$\begin{aligned} \rho^c\rho^3 =& \quad \mathbf{1},\ c = 3 \\ =& \quad \rho^3,\ c = 0 \\ =& i\epsilon_{c3k}\rho^k,\ c = 1, 2; \end{aligned}$$

and

$$\begin{aligned}\rho^3\rho^c\rho^3 &= \ \sigma^3,\ c=3\\ &= \ \ \mathbf{1},\ c=0\\ &= -\sigma^c,\ c=1,2.\end{aligned}$$

Now we use equation (91) to find the non-vanishing inner product elements. We make use of the relations

$$\alpha^2 - \beta^2\vec{k}_\parallel^2 = S_\parallel S_\parallel^{-1} = 1 \tag{101}$$

$$\alpha^2 + \beta^2\vec{k}_\parallel^2 = \gamma_\parallel \tag{102}$$

$$\alpha\beta = \frac{\gamma_\parallel}{2\omega^w}. \tag{103}$$

The connection matrix turns out to be

$$g^w_{ab} = \begin{pmatrix} \gamma_\parallel\widetilde{g}^s_0 & \widetilde{g}^s_1 & \widetilde{g}^s_2 & \gamma_\parallel\widetilde{g}^s_3 \\ \gamma_\parallel v_1\widetilde{g}^s_3 & \gamma_\parallel s v_2\widetilde{g}^s_2 & -\gamma_\parallel s v_2\widetilde{g}^s_1 & \gamma_\parallel v_1\widetilde{g}^s_0 \\ \gamma_\parallel v_2\widetilde{g}^s_3 & -\gamma_\parallel s v_1\widetilde{g}^s_2 & \gamma_\parallel s v_1\widetilde{g}^s_1 & \gamma_\parallel v_2\widetilde{g}^s_0 \\ s\widetilde{g}^s_0 & s\gamma_\parallel\widetilde{g}^s_1 & s\gamma_\parallel\widetilde{g}^s_2 & s\widetilde{g}^s_3 \end{pmatrix} \tag{104}$$

4.2.2 Dirac equation

The Wigner-transformed Dirac equation in the wall frame reads

$$(\hat{k}^w_0 + \gamma^0\gamma^3\hat{k}^w_z + \gamma^0\vec{\gamma}\cdot\vec{k}_\parallel - \gamma^0\hat{m}^w_0 + i\gamma^0\gamma^5\hat{m}^w_5)i\gamma^0 G^w_s = 0. \tag{105}$$

This can be expanded as

$$[\mathbf{1}_2\otimes\mathbf{1}_2\hat{k}^w_0 - (\sigma^i\otimes\rho^3)\hat{k}^w_i - (\mathbf{1}_2\otimes\rho^1)\hat{m}^w_0 - (\mathbf{1}_2\otimes\rho^2)\hat{m}^w_5]\frac{1}{4}g^w_{ab}(\sigma^a\otimes\rho^b) = 0. \tag{106}$$

The most general decomposition of the wall frame Wightman function, as used in the last section, is

$$i\gamma^0 G^w_s = \frac{1}{4}(\sigma^c\otimes\rho^d)g^w_{cd}. \tag{107}$$

As before, we get the constraint equation from the real part and the kinetic equation from the imaginary part. Once again, we are primarily interested in the equation of motion for g^w_{00}, which corresponds to the particle density in the phase space. The other components can always be obtained from g^w_{00} using the constraint equations. Thus it is enough to choose one constraint and one kinetic equation.

This gives us (omitting the wall frame superscripts for clarity):

$$k_0 g_{00} - k_i g_{i3} - m_R g_{01} - m_I g_{02} = 0 \tag{108}$$

$$\partial_i g_{i3} - m'_R \partial_{k_z} g_{01} - m'_I \partial_{k_z} g_{02} = 0, \tag{109}$$

where we used the fact that there is no time-dependence in the wall frame.

Note that the symmetry of the problem allows us to set the $\partial_\parallel$ terms to zero; this is because the mass is only dependent on z, which allowed us to boost to a frame where the spin along the z axis is a good quantum number. Thus the final kinetic equation is

$$\partial^w_z g_{33} - m'_R \partial_{k_z} g_{01} - m'_I \partial_{k_z} g_{02} = 0. \tag{110}$$

4.2.3 Dispersion relation

The dispersion relation can be obtained from equation (108). Using the connections as given in equation (104), this becomes

$$k_0 \gamma_\parallel \widetilde{g}^s_0 - k_z(s\widetilde{g}^s_3) - \vec{k}_\parallel \cdot \vec{v}_\parallel \gamma_\parallel \widetilde{g}^s_0 - m_R \widetilde{g}^s_1 - m_I \widetilde{g}^s_2 = 0. \tag{111}$$

Based on equations (81), (82) and the fact that $g_{00} = \gamma_\parallel \widetilde{g}^s_0$, this simplifies to the dispersion relation

$$[k_0^2 - \vec{k}^2 - m^2 + \gamma_\parallel \frac{s m^2 \theta'}{k_0}] g_{00} = 0. \tag{112}$$

This agrees with the dispersion relation computed from the semiclassical approach (equation (49)). Thus so far, the two computations agree.

4.3 Plasma frame

In the rest frame of the plasma, energy is no longer a good quantum number, since the mass now has a time dependence:

$$m = m(z^w) = m(\gamma_w(z^{pl} - v_w t^{pl})).$$

Due to stationarity, this means that $\partial_t^{pl} = -v_w \partial_z^{pl}$.

We note that the derivatives transform according to the inverse of the coordinate transformation, so that $\partial_z^{pl} = \gamma_w(\partial_z^w - v_w \partial_t^w) = \gamma_w \partial_z^w$. We can verify that this transformation is invertible: $\partial_z^w = \gamma_w(\partial_z^{pl} + v_w \partial_t^{pl}) = \gamma_w(1 - v_w^2)\partial_z^{pl} = \frac{1}{\gamma_w}\partial_z^{pl}$.

4.3.1 Connections

We can use the Lorentz boost to calculate the connections between the wall frame and the plasma frame.

The wall frame and plasma frame Wightman functions are related by

$$G_s^w = S_w G_s^{pl} S_w^{-1}. \tag{113}$$

The most general decomposition for the Wightman functions is

$$\begin{aligned} i\gamma^0 G_s^w &= \frac{1}{4}(\sigma^c \otimes \rho^d) g_{cd}^w \\ i\gamma^0 G_s^{pl} &= \frac{1}{4}(\sigma^a \otimes \rho^b) g_{ab}^{pl}, \end{aligned} \tag{114}$$

with the inner product

$$\frac{1}{4}\,\mathrm{Tr}\Big((\sigma^a \otimes \rho^b)(\sigma^c \otimes \rho^d)\Big) = \delta^{ac}\delta^{bd}, \tag{115}$$

where $\sigma^a = (\mathbf{1}, \vec{\sigma})$ and $\rho^b = (\mathbf{1}, \vec{\rho})$ refer to the Pauli spin matrices; ρ^a denotes the chirality degree of freedom, while σ^b refers to the spin. We can calculate the connections to be

$$g_{ab}^{pl} = \sum_{c,d} \frac{1}{4}\,\mathrm{Tr}\Big[S_w(\Lambda)(\sigma^c \otimes \rho^d) S_w(\Lambda)(\sigma^a \otimes \rho^b)\Big] g_{cd}^w. \tag{116}$$

S_w is given by (29). We can decompose it as

$$S_w = \frac{1}{\sqrt{2}}(\sqrt{\gamma_w + 1}(\mathbf{1} \otimes \mathbf{1}) + \sqrt{\gamma_w - 1}(\sigma^3 \otimes \rho^3)). \tag{117}$$

Substituting these in equation (116), we get

$$\begin{aligned} g^{pl}_{ab} &= \frac{1}{8}\sum_{c,d} \text{Tr}\ [(\sqrt{\gamma_w + 1}(\mathbf{1} \otimes \mathbf{1}) + \sqrt{\gamma_w - 1}(\sigma^3 \otimes \rho^3))(\sigma^c \otimes \rho^d)(\sqrt{\gamma_w + 1}(\mathbf{1} \otimes \mathbf{1}) \\ &\quad + \sqrt{\gamma_w - 1}(\sigma^3 \otimes \rho^3))(\sigma^a \otimes \rho^b)]g^w_{cd} && (118) \\ &= \frac{1}{8}\sum_{c,d} \text{Tr}\ [((\gamma_w + 1)(\sigma^c \otimes \rho^d) - \gamma_w v_w(\sigma^3\sigma^c \otimes \rho^3\rho^d + \sigma^c\sigma^3 \otimes \rho^d\rho^3) \\ &\quad + (\gamma_w - 1)(\sigma^3\sigma^c\sigma^3 \otimes \rho^3\rho^d\rho^3))(\sigma^a \otimes \rho^b)]g^w_{cd}. && (119) \end{aligned}$$

We calculate the Dirac matrix products as before, paying special attention to the cross-terms involving $\sigma^{\|}$:

$$\begin{aligned} \sigma^3\sigma^i \otimes \rho^3\rho^i + \sigma^i\sigma^3 \otimes \rho^i\rho^3 &= i\epsilon_{3ij}\sigma^j \otimes i\epsilon_{3ij}\rho^j + i\epsilon_{i3j}\sigma^j \otimes i\epsilon_{i3j}\rho^j \\ &= -2\sigma^j \otimes \rho^j, \end{aligned} \tag{120}$$

and

$$\begin{aligned} \sigma^3\sigma^i \otimes \rho^3\rho^j + \sigma^i\sigma^3 \otimes \rho^j\rho^3 &= i\epsilon_{3ij}\sigma^j \otimes i\epsilon_{3ji}\rho^j + i\epsilon_{i3j}\sigma^j \otimes i\epsilon_{3ji}\rho^j \\ &= +2\sigma^j \otimes \rho^i. \end{aligned} \tag{121}$$

Using these results, we find that the connection is

$$g^{pl}_{ab} = \begin{pmatrix} \gamma_w(g^w_{00} + v_w g^w_{33}) & g^w_{01} & g^w_{02} & \gamma_w(g^w_{03} + v_w g^w_{30}) \\ g^w_{10} & \gamma_w(g^w_{11} - v_w g^w_{22}) & \gamma_w(g^w_{12} + v_w g^w_{21}) & g^w_{13} \\ s g^w_{20} & \gamma_w(g^w_{21} + v_w g^w_{12}) & \gamma_w(g^w_{22} - v_w g^w_{11}) & g^w_{23} \\ \gamma_w(g^w_{30} + v_w g^w_{03}) & g^w_{31} & g^w_{32} & \gamma_w(g^w_{33} + v_w g^w_{00}) \end{pmatrix}. \tag{122}$$

We can use the connection matrix to find the relation between g^{pl}_{00} and g^w_{00}:

$$\begin{aligned} g^{pl}_{00} &= \gamma_w(g^w_{00} + v_w g^w_{33}) \\ &= \gamma_w(g^w_{00} + v_w s\tilde{g}^s_3) \\ &= \gamma_w(g^w_{00} + v_w(\frac{k^w_z}{k^w_0}g^w_{00} + s\frac{m^2\theta'_w\gamma_{\|}}{2(k^w_0)^2}\partial_{k_z}g^w_{00})) \\ &= \frac{k^{pl}_0}{k^w_0}g^w_{00} + v_w\gamma_w\gamma_{\|}\frac{sm^2\theta'_w}{2(k^w_0)^2}\partial^w_{k_z}g^w_{00}, \end{aligned} \tag{123}$$

which allows us to write g^w_{00} as

$$g^w_{00} = \frac{k^w_0}{k^{pl}_0} g^{pl}_{00} - v_w \gamma_{\|} \frac{sm^2 \theta'_{pl}}{2k^w_0 k^{pl}_0} \partial^w_{k_z} g^w_{00}. \tag{124}$$

4.3.2 Dirac equation

Consider the Dirac equation,

The Wightman function G^w_s transforms as $\gamma^0 G^w_s = S^{-1}_w (\gamma^0 G^{pl}_s) S^{-1}_w$, while the differential operator transforms as $D^w_s = S^{-1}_w D^{pl}_s S_w = S^{-1}_w D^{pl}_s \gamma^0 S^{-1}_w \gamma^0$. (Note that due to the Clifford algebra, $S_w \gamma^0 = \gamma^0 S^{-1}_w$.)

Thus the Dirac equation becomes

$$D^{pl}_s \gamma^0 (-i\gamma^0 G^{pl}_s) = 0,$$

and hence the constraint equation is

$$\mathrm{Re}\,\mathrm{Tr}\Big(D^{pl}_s \gamma^0 (-i\gamma^0 G^{pl}_s)\Big) = 0.$$

We now boost the Wigner transformed Dirac equation from the wall frame to the plasma frame. We get

$$S^{-1}_w [\gamma^0 \hat{k}^w_0 - \gamma^i \hat{k}^w_i + \frac{i}{2}\gamma^i \cdot \partial^w_i - \hat{m}^w_0 - i\gamma^5 \hat{m}^w_5] S_w \gamma^0 (-i S_w \gamma^0 G^{pl}_s S_w) = 0.$$

We substitute for the boost:

$$S_w = a + b v_w \alpha_z \tag{125}$$

$$S^{-1}_w = a - b v_w \alpha_z, \tag{126}$$

where

$$a = \frac{\sqrt{\gamma_w + 1}}{\sqrt{2}} \tag{127}$$

$$b = -\frac{\sqrt{\gamma_w - 1}}{\sqrt{2}}. \tag{128}$$

We use the fact that $a^2 + b^2 v_w^2 = \gamma_w$, $2ab = -\gamma_w$ and $S_w S_w^{-1} = 1$. We note that the parallel terms, $\hat{m}_0^w$ and $\hat{m}_5^w$ are unchanged by the boost. Using the Clifford algebra again, we get

$$\begin{aligned}
(a - bv_w\alpha_z)\gamma^0\hat{k}_0^w(a + bv_w\alpha_z) &= (a^2 + b^2v_w^2)\gamma^0\hat{k}_0^w + 2abv_w\gamma^3\hat{k}_0^w \\
&= \gamma_w(\gamma^0 k_0^w - \gamma^3 v_w k_0^w) \qquad (129)\\
(a - bv_w\alpha_z)\gamma^3\hat{k}_z^w(a + bv_w\alpha_z) &= (a^2 + b^2v_w^2)\gamma^3\hat{k}_z^w + 2abv_w\gamma^0\hat{k}_z \\
&= \gamma_w(\gamma^3(k_z^w - \frac{i}{2}\partial_z^w) + v_w\gamma^0(k_z^w - \frac{i}{2}\partial_z^w)) \qquad (130)\\
(a - bv_w\alpha_z)\hat{m}_0^w(a + bv_w\alpha_z) &= (a^2 - b^2v_w^2)\hat{m}_0^w \\
&= \hat{m}_0^w \qquad (131)\\
(a - bv_w\alpha_z)(i\gamma^5\hat{m}_5^w)(a + bv_w\alpha_z) &= (a^2 - b^2v_w^2)(i\gamma^5\hat{m}_5^w) \\
&= i\gamma^5\hat{m}_5^w \qquad (132)
\end{aligned}$$

Thus we get

$$\begin{aligned}
\gamma^0(\gamma_w(k_0^w + v_w k_z^w - \frac{i}{2}v_w\partial_z^w)) - \gamma^3(\gamma_w(k_z^w + v_w k_0^w - \frac{i}{2}\partial_z^w)) - \hat{m}_0^w - i\gamma^5\hat{m}_5^w = 0 \\
\Leftrightarrow \gamma^0(k_0^{pl} + \frac{i}{2}\partial_t^{pl}) - \gamma^3(k_z^{pl} - \frac{i}{2}\partial_z^{pl}) - \hat{m}_0^w - i\gamma^5\hat{m}_5^w = 0 \\
\Leftrightarrow \gamma^0\hat{k}_0^{pl} - \gamma^3\hat{k}_z^{pl} - \hat{m}_0^w - i\gamma^5\hat{m}_5^w = 0. \qquad (133)
\end{aligned}$$

Henceforth we will omit the plasma frame superscripts for clarity.

4.3.3 Constraint equation

The constraint equation is given by the real part of the Dirac equation. It is sufficient for our purposes to calculate it to lowest order in gradients:

$$k_0 g_{00}^{pl} - k_i g_{i3}^{pl} - m_R g_{01}^{pl} - m_I g_{02}^{pl} = 0.$$

We use the connections to the wall frame and the $p_\parallel = 0$ frame to calculate the constraint on g_{00}^{pl}, which measures the number density of the fermion under consideration in phase space.

Substituting for g_{ab}^{pl} in terms of the wall frame components, we find that

$$k_0\gamma_w(g_{00}^w + v_w g_{33}^w) - k_1 g_{13}^w - k_2 g_{23}^w - k_3\gamma_w(g_{33}^w + v_w g_{00}^w)m_R^w g_{01}^w - m_I^w g_{02}^w = 0.$$

Using the connections, we find that

$$[\gamma_w(k_0 - v_w k_z) - \gamma_w(k_z - v_w k_0)\frac{k_z^w}{k_0^w} - \frac{\vec{k}_\|^2}{k_0^w} - \frac{m^2}{k_0^w} + s\frac{\gamma_\| m^2 \theta_w'}{(k_0^w)^2}]g_{00}^w = 0. \tag{134}$$

Using the fact that $\vec{k}_\|$ does not change from the wall frame to the plasma frame, (134) clearly simplifies to give us the wall frame dispersion relation:

$$[(k_0^w)^2 - (\vec{k}^w)^2 - m^2 + s\frac{\gamma_\| m^2 \theta_w'}{k_0^w}]g_{00}^w = 0. \tag{135}$$

4.3.4 Dispersion relation

We can now substitute for g_{00}^w in terms of g_{00}^{pl} from (124). However, we find that no exact spectral solution for g_{00}^{pl} emerges. Instead, using the fact that $k^2 - m^2$ is invariant, we get a differential equation.

$$\begin{aligned}
&[k_0^2 - \vec{k}^2 - m^2 + \gamma_\| \frac{sm^2\theta_w'}{k_0^w}](\frac{k_0^w}{k_0} g_{00}^{pl} \\
&\qquad - v_w \gamma_\| \gamma_w^2 \frac{sm^2\theta_w'}{2k_0}(\partial_{k_z} + v_w \partial_{k_0})(\frac{1}{k_0} g_{00}^{pl})) = 0 \qquad (136)\\
\Leftrightarrow\ &[k_0^2 - \vec{k}^2 - m^2]\frac{k_0^w}{k_0} g_{00}^{pl} + \gamma_\| \frac{sm^2\theta_w'}{k_0} g_{00}^{pl} \\
&\qquad -[k_0^2 - \vec{k}^2 - m^2] v_w \gamma_\| \gamma_w^2 \frac{sm^2\theta_w'}{2k_0^{pl}}(\partial_{k_z} + v_w \partial_{k_0})(\frac{1}{k_0} g_{00}^{pl}) = 0. \qquad (137)
\end{aligned}$$

To evaluate the last term, we apply the differential operator identity $OD(G) = D(OG) - (DO)G$. $D(OG)$ is a higher order term and can be neglected, which leaves us with the term

$$\begin{aligned}
(\partial_{k_z} + v_w \partial_{k_0})(k_0^2 - \vec{k}^2 - m^2) \cdot v_w \gamma_\| \gamma_w^2 \frac{sm^2\theta_w'}{2k_0^2} g_{00}^{pl} \\
= -2(k_z - v_w k_0) \cdot v_w \gamma_\| \gamma_w^2 \frac{sm^2\theta_w'}{2k_0^2} g_{00}^{pl} \\
= \gamma_\| \frac{sm^2\theta_w'}{k_0} \frac{\gamma_w v_w k_z^w}{k_0} g_{00}^{pl}.
\end{aligned} \tag{138}$$

Thus the constraint equation for g_{00}^{pl} becomes

$$[k_0^2 - \vec{k}^2 - m^2 + \gamma_{\|}\frac{sm^2\theta'_w}{k_0^w}(1 - \frac{\gamma_w v_w k_z^w}{k_0})]g_{00}^{pl} = 0$$
$$\Leftrightarrow [(k_0)^2 - \vec{k}^2 - m^2 + \gamma_{\|}\frac{sm^2\theta'_{pl}}{k_0}]g_{00}^{pl} = 0, \tag{139}$$

giving us the dispersion relation

$$k_0^2 = \vec{k}^2 + m^2 - \gamma_{\|}\frac{sm^2\theta'_{pl}}{k_0}.$$

Thus the frequency k_0 can be computed to be

$$\begin{aligned} k_0 &= \pm(\omega_0^2 - \gamma_{\|}\frac{sm^2\theta'}{k_0})^{\frac{1}{2}} \\ &= \pm\omega_0(1 \mp \gamma_{\|}\frac{sm^2\theta'}{|k_0|\omega_0^2})^{\frac{1}{2}} \\ &\approx \pm\omega_0(1 \mp \gamma_{\|}\frac{sm^2\theta'}{2|k_0|\omega_0^2}) \\ &= \pm\omega_0 - \gamma_{\|}\frac{sm^2\theta'}{2|k_0|\omega_0} \\ &= \pm(\omega_0 \mp \gamma_{\|}\frac{sm^2\theta'}{2|k_0|\omega_0}), \end{aligned} \tag{140}$$

where we define $\omega_0 = \vec{k}^2 - m^2$. According to the Feynman-Stückelberg interpretation of quantum mechanics, negative frequency solutions denote antiparticles. But in this case, the frequencies are not equal in magnitude: the frequency shift due to the second term is negative for particles and positive for antiparticles, i.e. the second term is CP-violating. Thus the magnitude of the on-shell energy is given by

$$\omega_{\pm} = \omega_0 \mp \gamma_{\|}\frac{sm^2\theta'}{2\omega_0^2}, \tag{141}$$

and the approximate spectral solution for g_{00}^{pl} is given by

$$g_{00}^{pl} = \sum_{\pm}\frac{\pi}{2Z_{s\pm}}n_s\delta(k_0^{pl} \mp \omega_{s\pm}), \tag{142}$$

where $Z_{s\pm} \equiv \frac{1}{\omega_{s\pm}}|\partial_{k_0}\Omega_s^2|_{k_0=\pm\omega_{s\pm}}$ is a normalisation factor, and n_s measures the particle density in phase space. [25]

Since it is no longer possible to find an exact algebraic dispersion relation, it is no longer entirely correct to think of the plasma as a collection of single-particle excitations, but we have shown that for the purposes of the problem in a slowly-varying background this is a good enough approximation. However, this partly explains the difficulty of the semiclassical derivation: in the Liouville equation, both the differential operator and the spinor state it acts on transform non-trivially, and it is not clear *a priori* exactly how.

4.3.5 Kinetic equation

The kinetic equation is given by the imaginary part of (134); we calculate it to first order in gradients:

$$\partial_t g_{00}^{pl} + \partial_i g_{i3}^{pl} - (m_R')_w \partial_{k_z}^w g_{01}^{pl} - (m_I')_w \partial_{k_z}^w g_{02}^{pl} = 0. \tag{143}$$

We evaluate the terms individually. We can use (122) to solve for g_{33}^{pl} in terms of g_{00}^{pl}:

$$\begin{aligned}
g_{33}^{pl} &= \gamma_w (g_{33}^w + v_w g_{00}^w) \\
&= \gamma_w (s\widetilde{g}_3^s + v_w g_{00}^w) \\
&= \gamma_w (\frac{k_z^w}{k_0^w} + s\frac{m^2\theta_w'}{2(k_0^w)^2}\partial_{k_z}^w + v_w) g_{00}^w \\
&= \gamma_w (\frac{k_z^w}{k_0^w} + s\frac{m^2\theta_w'}{2(k_0^w)^2}\partial_{k_z}^w + v_w)(\frac{k_0^w}{k_0} g_{00}^{pl} - v_w\gamma_\parallel \frac{sm^2\theta'}{2k_0^w k_0}\partial_{k_z}^w g_{00}^w) \\
&= \frac{k_z}{k_0} g_{00}^{pl} + \gamma_\parallel \frac{sm^2\theta_w'}{2k_0^w k_0}\partial_{k_z}^w (\frac{k_0^w}{k_0} g_{00}^{pl}),
\end{aligned} \tag{144}$$

and similarly

$$\begin{aligned}
&- (m_R')^w \partial_{k_z}^w g_{01}^{pl} - (m_I')^w \partial_{k_z}^w g_{02}^{pl} \\
=&- (m_R')^w \partial_{k_z}^w g_{01}^w - (m_I')^w \partial_{k_z}^w g_{02}^w \\
=&- (m_R')^w \frac{m_R}{\widetilde{k}_0}\partial_{k_z}^w \widetilde{g}_0^s - m_R' \partial_{k_z}^w (\frac{sm_I'}{2\widetilde{k}_0^2}\widetilde{g}_0^s - \frac{sm_I}{2\widetilde{k}_0^2}\partial_z^w \widetilde{g}_0^s) + \frac{(m_R')^w (m_I')^w}{2}\partial_{k_z}^w \partial_{k_z}^w (\frac{sk_z^w}{\widetilde{k}_0}\widetilde{g}_0^s) \\
&+ (m_I')^w \frac{m_I}{\widetilde{k}_0}\partial_{k_z}^w \widetilde{g}_0^s - m_I' \partial_{k_z}^w (\frac{sm_R'}{2\widetilde{k}_0^2}\widetilde{g}_0^s + \frac{sm_R}{2\widetilde{k}_0^2}\partial_z^w \widetilde{g}_0^s) - \frac{(m_R')^w (m_I')^w}{2}\partial_{k_z}^w \partial_{k_z}^w (\frac{sk_z^w}{\widetilde{k}_0}\widetilde{g}_0^s) \\
=&- \frac{(m^2)_w'}{\widetilde{k}_0}\partial_{k_z}^w \widetilde{g}_0^s - \frac{sm^2\theta_w'}{2\gamma_w \widetilde{k}_0^2}\partial_{k_z}^w \partial_z^{pl} \widetilde{g}_0^s.
\end{aligned} \tag{145}$$

Now if we substitute for g_{00}^w, we get

$$\partial_t g_{00}^{pl} + \partial_z(\frac{k_z}{k_0} g_{00}^{pl}) - \gamma_\| \gamma_w v_w \frac{sm^2\theta'_w k_z^w}{2(k_0^w)^2 k_0} \partial_z^w \partial_{k_z}^w (\frac{k_0^w}{k_0} g_{00}^{pl}) + \gamma_\| \gamma_w \frac{s(m^2\theta'_w)'_w}{2k_0} \partial_{k_z}^w (\frac{1}{k_0} g_{00}^{pl})$$
$$- \frac{(m^2)'_w}{k_0^w} \partial_{k_z}^w (\frac{k_0^w}{k_0} g_{00}^{pl} - \gamma_\| \gamma_w v_w \frac{sm^2\theta'_w}{2k_0^w k_0} \partial_{k_z}^w g_{00}^w) = 0. \quad (146)$$

We can write

$$\frac{k_z^w}{k_0^w} \partial_{k_z}^w \partial_z^w g_{00}^w = \partial_{k_z^w} (\frac{k_z^w}{k_0^w} \partial_z^w g_{00}^w) - \frac{\partial_z^w g_{00}^w}{k_0^w}.$$

Now the mixed derivative terms cancel out to the desired order, due to the wall frame kinetic equation:

$$\begin{aligned}
&\gamma_\| \gamma_w v_w \frac{sm^2\theta'_w}{2k_0^w k_0^{pl}} [\partial_{k_z}^w (-\frac{k_z^w}{k_0^w} \partial_z^w g_{00}^w + \frac{(m^2)'_w}{k_0^w} \partial_{k_z}^w g_{00}^w) + \frac{1}{k_0^w} \partial_z^w g_{00}^w - \gamma_w v_w \frac{1}{k_0^{pl}} \partial_{k_z}^w g_{00}^w] \\
\approx &\gamma_\| \gamma_w v_w \frac{sm^2\theta'_w}{2k_0^w k_0^{pl}} [\partial_{k_z}^w (-\partial_z g_{33}^w + \frac{(m^2)'}{k_0^w} \partial_{k_z}^w g_{00}^w) + \frac{1}{k_0^w} \partial_z^w g_{00}^w - \gamma_w v_w \frac{1}{k_0^{pl}} \partial_{k_z}^w g_{00}^w] \\
= &\gamma_\| \gamma_w v_w \frac{sm^2\theta'_w}{2k_0^w k_0^{pl}} [\frac{1}{k_0^w} \partial_z^w g_{00}^w - \gamma_w v_w \frac{1}{k_0^{pl}} \partial_{k_z}^w g_{00}^w]. \quad (147)
\end{aligned}$$

We are left with a z-derivative term and a k_z^w-derivative term. Putting these back in the kinetic equation, we get

$$\partial_t g_{00}^{pl} + (\frac{k_z}{k_0} + \gamma_\| \gamma_w v_w \frac{sm^2\theta'_w}{2k_0^w k_0^2}) \partial_z g_{00}^{pl}$$
$$- (\frac{(m^2)'_w}{k_0^w} - \gamma_\| \gamma_w \frac{s(m^2\theta'_w)'_w}{2k_0^w k_0} + \gamma_\| \gamma_w^2 v_w^2 \frac{sm^2\theta'_w}{k_0^w k_0^2} \frac{(m^2)'_w}{k_0^w}) \partial_{k_z}^w (\frac{k_0^w}{k_0} g_{00}^{pl}) = 0. \quad (148)$$

4.3.6 Projected kinetic equation

We will now insert the spectral solution for g_{00}^{pl} into the kinetic equation, and integrate separately over the positive and negative frequencies.

We have

$$g_{00}^{pl} = \sum_{\pm} \frac{\pi}{2Z_{s\pm}} n_s \delta(k_0 \mp \omega_{s\pm}),$$

where n_s measures the particle number density in phase space. The particle distribution functions are defined by $f_{s+} \equiv n_s(\omega_{s+}, k_z, z)$ and $f_{s-} \equiv 1 - n_s(-\omega_{s-}, -k_z, z)$. [25]

So far, we have taken k_0 and k_z to be independent variables. However, inserting the spectral solution and integrating over it forces the particles on-shell, so that $k_0 = \pm\omega_\pm$. We define the generating function of the dispersion relation,

$$\psi = k_0^2 - \vec{k}^2 - m^2 + \gamma_\| \frac{sm^2\theta'}{k_0}.$$

The integration constrains us to the curve along which $\psi = 0$. Integrating along this curve corresponds to making a quasiparticle approximation, where the two-point function can be approximated with a δ-function. We do not take into account the self-energy corrections, which would lead to further thermal corrections to the quasiparticle dispersion relation.

Henceforth, we take k_z to be the independent variable, while the on-shell energy, $\omega_\pm$, is a function of k_z.

We now need to integrate over derivatives of g_{00}:

$$\int dk_0 \partial_{k_z}(f(k_0, z)\delta(\psi(k_0, z))) = \int dk_0(\partial_{k_z}(f(k_0, z))\delta\psi + f(k_0, z)\partial_{k_z}(\delta\psi)). \tag{149}$$

The second term can be rewritten as $\int dk_0 f(\frac{\partial k_0}{\partial k_z})_{k_0=\omega} \partial_{k_0}\delta(\psi)$, which gives us a surface term and $\int dk_0((\frac{\partial k_0}{\partial k_z})_{k_0=\omega}\partial_{k_0} f)\delta(\psi)$. Here we have used the fact that $\frac{\partial k_0}{\partial k_z}$ is purely a function of k_z. The surface term vanishes, leaving us with

$$\int dk_0(\partial_{k_z} f + \frac{\partial k_0}{\partial k_z}\partial_{k_0} f)\delta(\psi) = \sum_{\pm\omega_\pm} \frac{1}{|\partial_{k_0}\psi|_{\psi=0}}(\partial_{k_z} + \frac{\partial k_0}{\partial k_z}\partial_{k_0} f). \tag{150}$$

This sum is over the on-shell frequencies, $k_0 = \pm\omega_\pm$. Henceforth we denote the on-shell frequency by $k_0 = \omega$.

We now compute the derivatives from the generating function. Here it is necessary to take into account the fact that $\gamma_\|$ is a function of k_0^w, i.e. $\gamma_\| = \frac{k_0^w}{\sqrt{(k_0^w)^2 - k_\|^2}}$; k_0^w is in turn a function of k_z and k_0. But since k_0 is now the on-shell energy, it depends on k_z as well. (We note that when k_0 is on-shell, k_0^w is on-shell as well, since the wall frame dispersion relation is a solution to the plasma frame constraint; cf. equation (135).)

$$2\omega\frac{\partial\omega}{\partial k_z} = 2k_z + \gamma_\| \frac{sm^2\theta'}{\omega^2}\frac{\partial\omega}{\partial k_z} - \frac{sm^2\theta'}{\omega}\partial_\omega^w \gamma_\| (\frac{\partial\omega^w}{\partial k_z} + \frac{\partial\omega^w}{\partial\omega}\frac{\partial\omega}{\partial k_z}).$$

This is an equation for $\frac{\partial\omega}{\partial k_z}$, which approximately gives

$$\frac{\partial\omega}{\partial k_z} \approx \frac{k_z}{\omega} + \gamma_\parallel \frac{sm^2\theta'}{\omega^3}\frac{k_z}{\omega} + v_w \frac{sm^2\theta'}{2\omega^2\omega^w}\gamma_w(\gamma_\parallel - \gamma_\parallel^3) + \frac{sm^2\theta'}{2\omega^2\omega^w}\gamma_w(\gamma_\parallel - \gamma_\parallel^3)\frac{k_z}{\omega}. \tag{151}$$

Similarly

$$2\omega\frac{\partial\omega}{\partial z} = (m^2)' - \gamma_\parallel \frac{s(m^2\theta')'}{\omega} - sm^2\theta'\partial_z(\frac{\gamma_\parallel}{\omega}),$$

from which we get

$$\frac{\partial\omega}{\partial z} \approx \frac{(m^2)'}{2\omega} - \gamma_\parallel\frac{s(m^2\theta')'}{2\omega^2} - \frac{sm^2\theta'}{2\omega}(\frac{1}{\omega}\frac{\partial\omega^w}{\partial z}\partial_\omega^w\gamma_\parallel + \gamma_\parallel\frac{\partial\omega}{\partial z}\partial_\omega\frac{1}{\omega}). \tag{152}$$

Note that these agree in magnitude with those computed from the generating function.

The kinetic equation is given by (148). However, the momentum derivative is with respect to the wall frame momentum; we need to boost to the plasma frame to get the Boltzmann equation. Using the inverse Lorentz boost,

$$\begin{aligned}\partial_{k_z}^w(\frac{k_0^w}{k_0}g_{00}^{pl}) &= -\frac{k_0^w}{k_0^2}\gamma_w v_v g_{00}^{pl} + \frac{k_0^w}{k_0}\gamma_w(\partial_{k_z} + v_w\partial_{k_0})g_{00}^{pl} \\ &= -\frac{k_0^w}{k_0^2}\gamma_w v_v g_{00}^{pl} + \gamma_w v_w\frac{k_0^w}{k_0}\partial_{k_0}g_{00}^{pl} + \gamma_w\frac{k_0^w}{k_0}\partial_{k_z}g_{00}^{pl}.\end{aligned} \tag{153}$$

Thus it turns out that the Boltzmann equation has some extra terms. Combining these and integrating by parts, we get:

$$\begin{aligned}\int dk_0 v_w\widetilde{F}_2(\partial_{k_0} - \frac{1}{k_0})n_s\delta(\psi) &= -v_w\int dk_0(\partial_{k_0}\widetilde{F}_2 + \frac{\widetilde{F}_2}{k_0})n_s\delta(\psi) \\ &= \sum_\pm \frac{-1}{|\partial_{k_0}\psi|}v_w(\partial_{k_0}\widetilde{F}_2 + \frac{\widetilde{F}_2}{k_0})n_s,\end{aligned} \tag{154}$$

where $\widetilde{F}_2 = \gamma_w\frac{k_0^w}{k_0}F_2$.

Integrating over the on-shell projections, we get:

$$\sum_\pm \frac{1}{|\partial_{k_0}\psi|}[F_1\partial_z n_s + \widetilde{F}_2\partial_{k_z}n_s - v_w(\partial_{k_0}\widetilde{F}_2 + \frac{\widetilde{F}_2}{k_0})n_s + [F_1(\frac{\partial k_0}{\partial z}) + \widetilde{F}_2(\frac{\partial k_0}{\partial k_z})]\partial_{k_0}n_s] = 0. \tag{155}$$

This leaves us with the equation

$$\sum_{\pm\omega_\pm} \frac{1}{|\partial_{k_0}\psi|}[F_1\partial_z n_s + \widetilde{F}_2\partial_{k_z} n_s - v_w \frac{\widetilde{F}_2}{k_0} n_s] = 0. \tag{156}$$

We separate out the particle and antiparticle parts. For particles, the distribution function is given by $f = n_s(\omega_+, k_z, z)$ and the on-shell frequency is given by $k_0 = \omega_+$. Thus we get

$$\begin{aligned} &F_1\partial_z n_s + \widetilde{F}_2\partial_{k_z} n_s - v_w \frac{\widetilde{F}_2}{k_0} n_s \\ =&(-v_w + \frac{k_z}{\omega_+} + v_w\gamma_\parallel \frac{sm^2\theta'}{2\gamma_w\omega_+^w\omega_+^2})\partial_z f - (\frac{(m^2)'}{2\omega_+} - \gamma_\parallel \frac{s(m^2\theta')'}{2\omega_+^2} \\ &+ v_w^2\gamma_\parallel \frac{sm^2\theta'}{2\omega_+^2\omega_+^w}\frac{(m^2)'}{2\omega_+})(\partial_{k_z} f - \frac{v_w}{\omega_+} f). \end{aligned} \tag{157}$$

For antiparticles, the distribution function is given by $f = 1 - n_s(-\omega_-, -k_z, z)$ and the on-shell frequency is given by $k_0 = -\omega_-$. This gives us

$$\begin{aligned} &\bar{F}_1\partial_z(1 - f(-\omega_-, -k_z, z)) + \bar{\widetilde{F}}_2\partial_{k_z}(1 - f(-\omega_-, -k_z, z)) \\ &- v_w \frac{\bar{\widetilde{F}}_2}{k_0}(1 - f(-\omega_-, -k_z, z)) \\ =&(-v_w + \frac{-k_z}{-k_0} + v_w\gamma_\parallel \frac{sm^2\theta'}{2\gamma_w(-\omega_-^w)\omega_-^2})(-\partial_z f) - (\frac{(m^2)'}{-2\omega_-} - \gamma_\parallel \frac{s(m^2\theta')'}{2\gamma_w\omega_-^2} \\ &+ v_w^2\gamma_\parallel \frac{sm^2\theta'}{2\omega_-^2(-\omega_-^w)}\frac{(m^2)'}{2(-\omega_-)})(\partial_{k_z} f - \frac{v_w}{-\omega_-}(1 - f)) \\ =&-[(-v_w + \frac{k_z}{\omega_-} - v_w\gamma_\parallel \frac{sm^2\theta'}{2\gamma_w\omega_-^w\omega_-^2})\partial_z f - (\frac{(m^2)'}{2\omega_-} + \gamma_\parallel \frac{s(m^2\theta')'}{2\omega_-^2} \\ &- v_w^2\gamma_\parallel \frac{sm^2\theta'}{2\omega_-^2\omega_-^w}\frac{(m^2)'}{2\omega_-})(\partial_{k_z} f + \frac{v_w}{\omega_-}(1 - f))]. \end{aligned} \tag{158}$$

We can identify this equation with the Liouville equation for the time-evolution of the distribution function of a fluid:

$$[\partial_t + v_g\partial_z + F_z\partial_{k_z}]f = \frac{df}{dt},$$

where $\frac{df}{dt} = \frac{v_w}{\omega_\pm} n_{s\pm}$ is the source term, and hence identify the terms in the equation with the group velocity and semiclassical force:

$$v_g = \frac{k_z}{\omega_\pm} \pm v_w \gamma_\parallel \frac{s m^2 \theta'}{2\gamma_w \omega_\pm^w \omega_\pm^2} \tag{159}$$

$$F_z = -\frac{(m^2)'}{2\omega_\pm} \pm \gamma_\parallel \frac{s(m^2\theta')'}{2\omega_\pm^2} \mp v_w^2 \gamma_\parallel \frac{s m^2 \theta'}{2\omega_\pm^2 \omega_\pm^w} \frac{(m^2)'}{2\omega_\pm}. \tag{160}$$

Equation (160) is the main result of this thesis; it is new in the literature. Clearly, it has a CP-even term to first order, and CP-odd terms to second order in θ'.

We note that this expression for the force incorporates the exact velocity dependence; so far, we have not made any assumptions about the velocity, so this result is valid for extremely fast as well as slow walls. However, the subsequent calculation of the transport equations with the exact velocity dependence is quite complicated, so this has so far only been performed for low wall velocities in the literature. We will be doing this in Chapter 5. Our eventual goal in the forthcoming publication is to calculate these exactly, for any wall velocity; it will be interesting to compare these results with the simplified ones existing in the literature.

4.3.7 Helicity states

Equation (160) gives us the force acting on the s-states (the eigenvalues of $\widetilde{S}_z^w$, in the double-boosted frame).

Our ultimate goal is to find the force on the plasma frame helicity states:

$$F_{pl}^h = \sum_s \langle u(p, h^{pl}) | F_{pl} | u(p, h^{pl}) \rangle$$

Once again, the force depends linearly on spin, so this is proportionate to the projections of the plasma frame helicity states on to the double-boosted spin states

$$s_h^{pl} = \langle u(p, h^{pl}) | \widetilde{S}_z | u(p, h^{pl}) \rangle$$

The spin operator, $\widetilde{S}_z$, transforms as

$$S_z^{pl} = S_w^{-1} S_\parallel^{-1} \widetilde{S}_z S_\parallel S_w \tag{161}$$

$$= \gamma_\parallel [\widetilde{S}_z - i\gamma_w (1 + v_w \alpha_z)(\vec{v}_\parallel \times \vec{\alpha})_z]. \tag{162}$$

The force acts on the eigenstates of $\widetilde{S}_z$. These are the $k_\parallel = 0$ frame spin states, defined by

$$u^{pl}(\vec{k}^{pl}, s) = S_w^{-1} S_\parallel^{-1} \widetilde{u}(\widetilde{k}_z, s) \quad (163)$$
$$\widetilde{S}_z \widetilde{u}(\widetilde{k}_z, s) = s \widetilde{u}(\widetilde{k}_z, s) \quad (164)$$

We can write the plasma frame helicity states as a combination of plasma frame spin states:

$$u(\vec{k}^{pl}, h) = \sum c_s^{pl} u(\vec{k}^{pl}, s) \quad (165)$$

so the expectation value of the plasma frame spin can be calculated as before:

$$\begin{aligned}
s_h^{pl} &= \sum_s s |c_s^{pl}|^2 \\
&= \sum_s \frac{\bar{u}(\vec{k}^{pl}, h) S_w^{-1} S_\parallel^{-1} \widetilde{S}_z \widetilde{u}(\widetilde{k}_z, s) \bar{\widetilde{u}}(\widetilde{k}_z, s) S_\parallel S_w u(\vec{k}^{pl}, h)}{4m^2} \\
&= \frac{\bar{u}(\vec{k}^{pl}, h) S_w^{-1} S_\parallel^{-1} \widetilde{S}_z S_\parallel S_w [S_w^{-1} S_\parallel^{-1} \sum_s \widetilde{u}(\widetilde{k}_z, s) \bar{\widetilde{u}}(\widetilde{k}_z, s) S_\parallel S_w] u(\vec{k}^{pl}, h)}{4m^2} \\
&= \frac{\bar{u}(\vec{k}^{pl}, h) (S_w^{-1} S_\parallel^{-1} \widetilde{S}_z S_\parallel S_w)(\not{k}_z^{pl} + m) u(\vec{k}^{pl}, h)}{4m^2} \\
&= \frac{\bar{u}(\vec{k}^{pl}, h) (S_w^{-1} S_\parallel^{-1} \widetilde{S}_z S_\parallel S_w) u(\vec{k}^{pl}, h)}{2m} \\
&= \langle S_w^{-1} S_\parallel^{-1} \widetilde{S}_z S_\parallel S_w \rangle_h \\
&= \gamma_w h \frac{k_z^{pl}}{|\vec{k}^{pl}|} + v_w v_\parallel^2 \gamma_w \gamma_\parallel \frac{h \omega^{pl}}{|\vec{k}^{pl}|} \quad (166)
\end{aligned}$$

where we have used the fact that the expectation value of α_i is $h \frac{k_i^{pl}}{|\vec{k}^{pl}|}$. Thus the force on the helicity states is given by

$$F_z = -\frac{(m^2)'}{2\omega_\pm} \pm (\gamma_\parallel \frac{(m^2\theta')'}{2\omega_\pm^2} - v_w^2 \gamma_\parallel \frac{m^2\theta'}{2\omega_\pm^2 \omega_\pm^w} \frac{(m^2)'}{2\omega_\pm})(\gamma_w h \frac{k_z^{pl}}{|\vec{k}^{pl}|} + v_w v_\parallel^2 \gamma_w \gamma_\parallel \frac{h\omega^{pl}}{|\vec{k}^{pl}|}) \quad (167)$$

This is also a new result.

5 Transport equations in the wall frame

5.1 Another look at the semiclassical dynamics in the wall frame

Following [21], we can rewrite the dispersion relation (135) as

$$\omega^w = \omega_0 \mp s\frac{\theta'}{2}\frac{\omega_{0z}}{\omega_0}, \tag{168}$$

where $\omega_0 = \sqrt{(p_{cz} - \alpha_{CP})^2 + p_x^2 + p_y^2 + m^2}$ and $\omega_{0z} = \sqrt{(p_{cz} - \alpha_{CP})^2 + m^2}$. Clearly, the limit $\omega_0 = \omega_{0z}$ corresponds to the $p_\parallel = 0$ case.

The group velocity of the WKB wavepacket along the z-axis is calculated as before to be

$$v_{gz} = (\frac{\partial \omega}{\partial p_{cz}})_z = \frac{p_{cz} - \alpha_{CP}}{\omega_0}(1 \mp s\frac{\theta'}{2}\frac{\omega_0^2 - \omega_{0z}^2}{\omega_0^2 \omega_{0z}}), \tag{169}$$

and hence the kinetic momentum is given by

$$p_z = \omega v_{gz} = (p_{cz} - \alpha_{CP})(1 \mp s\frac{\theta'}{2\omega_{0z}}). \tag{170}$$

Now, we want to integrate over physical three-momentum, so it is most convenient to make a change of variables. Following [20], we assume that it is the kinetic momentum that is conserved in collisions.

In this frame, the energy is a good quantum number. The energy can be expressed in terms of the physical momentum as

$$E_w = E_0 \pm \Delta E = E_0 \mp s\frac{\theta' m^2}{2E_0 E_{0z}}, \tag{171}$$

where we define $E_0 = \sqrt{p_z^2 + p_x^2 + p_y^2 + m^2}$ and $E_{0z} = \sqrt{p_z^2 + m^2}$.

Substituting for the energy in (52), we get the force in the wall frame in terms of our kinetic variables:

$$F_w = -\frac{(m^2)'}{2E_0} \pm s\frac{(m^2\theta')'}{2E_0E_{0z}} \mp s\frac{\theta' m^2 (m^2)'}{4E_0^3 E_{0z}}, \tag{172}$$

which contains a CP-even term that is first order in derivatives and a CP-odd term that is second order in derivatives. This force generates CP-even and CP-odd perturbations in the phase space distribution of the particles due to the passage of the bubble wall.

5.2 Semiclassical Boltzmann equations

We recall that since we are at the semiclassical limit, the distribution of a species of fermions in the plasma evolves according to the Boltzmann equation:

$$(\partial_t + v_g\partial_z + F_z\partial_{k_z})f_i = C[f_i],$$

where $C[f_i]$ is the collision term, which is most easily evaluated in the plasma frame. Thus our eventual goal in the forthcoming publication is to evaluate the transport equations in the plasma frame; this would be a new result in the literature. However, in this thesis, we will restrict ourselves to deriving the left-hand side of the transport equations in the wall frame, and compare them to the earlier results in the literature.

We are following the calculation performed in [21], and using the same notation to facilitate comparison. Our intermediate results differ slightly because we have paid more attention to the exact v_w dependence in the γ_w-factors; however, in the low-v_w regime, which is where calculations have been performed so far, this dependence has no effect. In a forthcoming publication we intend to evaluate the transport equations while keeping the full velocity dependence; at this point the γ_w factors could have an effect.

As we have seen, f_i measures the number density of particles of species i in phase space. From this, we can get the particle density in ordinary space by integrating over the momentum space:

$$n(\vec{x}) = \int d^3p f(\vec{x}, \vec{p}). \tag{173}$$

The equilibrium phase space distribution of particles of a given species in the plasma is given by

$$f_i^{eq}(\vec{x}, \vec{p}) = \frac{1}{\exp\{\beta\gamma_w(E_w + v_w p_{zw})\} \pm 1} \tag{174}$$

for fermions (bosons), in terms of the wall frame energy and momentum. This describes a distribution that is in *thermodynamic equilibrium*, i.e. both *kinetic equilibrium* and *chemical equilibrium*.

We will be dealing here with fermions, that obey the Fermi-Dirac statistics. The distribution gets perturbations from equilibrium due to the passage of the bubble wall:

$$f_i(\vec{x}, \vec{p}) = \frac{1}{\exp\{\beta(\gamma_w(E_w + v_w p_{zw}) - \mu_i)\} + 1} + \delta f_i(\vec{x}, \vec{p}), \tag{175}$$

where by definition, $\int d^3p \delta f_i = 0$.

Thus the first term describes a distribution that is in local kinetic equilibrium, but not necessarily in chemical equilibrium; the second term describes perturbations away from kinetic equilibrium. If a particle species is in kinetic equilibrium, the particles obey the Fermi-Dirac distribution, which resembles the first term for some μ_i and T; this equilibrium is enforced by elastic scattering processes.

The terms δf model the perturbations away from kinetic equilibrium; they are non-trivially momentum-dependent and anisotropic in momentum space.

For a species in chemical equilibrium, the rate of any reaction is equal to the rate of its reverse reaction. This is parametrised by chemical potentials, μ_i, that correspond to conserved particle numbers - in this case, the baryon number. Mathematically, μ_i parametrises the change in the free energy of the system in response to a change in the conserved particle number. Thus in chemical equilibrium, if the chemical potential of a particle species is μ_i, then the potential of the corresponding antiparticle species is $-\mu_i$. [26]

However, since we are not necessarily in chemical equilibrium here, the μ_i in our equation is a *pseudo*-chemical potential due to the force, with a CP-even and a CP-odd part (in contrast to the true chemical potential, which is purely CP-odd). [20]

We assume that the rate of the elastic scattering processes that enforce kinetic equilibrium is faster than that of processes enforcing chemical equilibrium, as well as fast compared to the time scale of the wall. Thus it is the kinetic momentum that is conserved in the scatterings of WKB particles, and thus it is convenient to express our ansatz in terms of kinetic (rather than canonical) variables. [20]

Since we are in the WKB regime with a slowly-varying potential, our perturbations are also suppressed by θ'. Thus we can expand them in orders of $|m^2|'$ and θ'. We recall that the force had a CP-even term to first order in $|m^2|'$, that would generate an even first order perturbation, and higher order CP-odd terms, that would give rise to corresponding higher order perturbations.

We can therefore expand the perturbations to first and second order in gradients as

$$\mu_i = \mu_{1e} + \mu_{2o} + \mu 2e$$
$$\delta f_i = \delta f_{1e} + \delta f_{2o} + \delta f_{2e},$$

where $\mu_{1e} \sim |m^2|'$, $\mu_{20} \sim \theta''$, $\mu_{2e} \sim |m^2|''$, etc.

Since these perturbations are small, we are close to equilibrium, and can therefore Taylor expand $f_i(\vec{x}, \vec{p})$ around $E = E_0$. Expanding up to the second order and taking the exact v_w dependence, we get

$$f_i(\vec{x}, \vec{p}) = f_{0,v_w} + \frac{1}{\gamma_w}\frac{\partial f}{\partial E_0}(\pm\gamma_w \Delta E - \mu_{1e} - \mu_{2o} - \mu_{2e}) + \frac{1}{2\gamma_w^2}\frac{\partial^2 f}{\partial E_0^2}(\gamma_w^2(\Delta E)^2 + \mu_{1e}^2 \mp 2\gamma_w \Delta E \mu_{1e}) + \delta f_{1e} + \delta f_{2o} + \delta f_{2e}. \quad (176)$$

Our results differ slightly from those of [21], since we have accounted for the γ_w-dependence of the plasma frame energy.

Henceforth we omit the wall frame and i subscripts to simplify the notation. The evolution of f is governed by the wall frame Boltzmann equation,

$$L[f] \equiv (\dot{z}\partial_z + \dot{p}_z \partial_{p_z})f = C[f], \quad (177)$$

where

$$\dot{p}_z = F_w = -\frac{(m^2)'}{2E_0} \pm s\frac{(m^2\theta')'}{2E_0 E_{0z}} \mp s\frac{\theta' m^2 (m^2)'}{4E_0^3 E_{0z}}$$

$$\dot{z} = v_g = \frac{p_{zw}}{E_0}(1 \pm \frac{s\theta'}{2}\frac{m^2}{E_0^2 E_{0z}}).$$

Substituting these into equation (177), we find that

$$L[f] = (v_g \partial_z + F_w \partial_{p_z})f.$$

Denoting $\frac{\partial f}{\partial E_0}$ by f', we get

$$L[f]_{particles} = ((\frac{p_z}{E_0} + \frac{p_z}{E_0}\frac{s\theta' m^2}{2E_0 E_{0z}})\partial_z + (-\frac{(m^2)'}{2E_0} + s\frac{(m^2\theta')'}{2E_0 E_{0z}} - s\frac{\theta' m^2 (m^2)'}{4E_0^3 E_{0z}})\partial_{p_z})(f_{0,v_w} + \frac{1}{\gamma_w} f'(\gamma_w \Delta E - \mu_{1e} - \mu_{2o} - \mu_{2e}) + \frac{1}{2\gamma_w^2} f''(\gamma_w^2 (\Delta E)^2 + \mu_{1e}^2 - 2\gamma_w \Delta E \mu_{1e}) + \delta f_{1e} + \delta f_{2o} + \delta f_{2e}). \quad (178)$$

$$L[f]_{antiparticles} = ((\frac{p_z}{E_0} - \frac{p_z}{E_0}\frac{s\theta' m^2}{2E_0 E_{0z}})\partial_z + (-\frac{(m^2)'}{2E_0} - s\frac{(m^2\theta')'}{2E_0 E_{0z}} + s\frac{\theta' m^2 (m^2)'}{4E_0^3 E_{0z}})\partial_{p_z})(f_{0,v_w}$$
$$+ \frac{1}{\gamma_w} f'(-\gamma_w \Delta E - \bar{\mu}_{1e} - \bar{\mu}_{2o} - \bar{\mu}_{2e}) + \frac{1}{2\gamma_w^2} f''(\gamma_w^2 (\Delta E)^2 + \bar{\mu}_{1e}^2 + 2\gamma_w \Delta E \bar{\mu}_{1e})$$
$$+ \bar{\delta f}_{1e} + \bar{\delta f}_{2o} + \bar{\delta f}_{2e}). \quad (179)$$

We get the CP-even part of the Boltzmann equations by adding the particle and antiparticle equations, and the CP-odd part by subtracting them. We define

$$\mu_1 = \mu_{1e} + \bar{\mu}_{1e}, \quad (180)$$
$$\delta f_1 = \delta f_{1e} + \bar{\delta f}_{1e} \quad (181)$$

for the even terms, and

$$\mu_2 = \mu_{2o} - \bar{\mu}_{2o}, \quad (182)$$
$$\delta f_2 = \delta f_{2o} - \bar{\delta f}_{2o} \quad (183)$$

for the odd terms. The second-order even terms drop off. We note that since the chemical potentials measure local departure from chemical equilibrium, they are spatially dependent, but constant with respect to momentum. However, the δf_i terms are non-trivially dependent on momentum.

For the even part we get

$$L[f]_{even} = -\frac{p_z}{\gamma_w E_0} f'_{0,v_w} \partial_z \mu_1 + \frac{p_z}{E_0}\partial_z \delta f_1 - \frac{(m^2)'}{E_0}\partial_{p_z}\delta f_1,$$

while the Liouville term for the CP-odd equation is

$$\begin{aligned} L[f]_{odd} = &-\frac{p_z}{\gamma_w E_0} f'_{0,v_w}\partial_z \mu_2 + \frac{v_w}{\gamma_w}\frac{(m^2)'}{2E_0} f''_{0,v_w}\mu_2 + v_w \frac{s(m^2\theta')'}{E_0 E_{0z}} f'_{0,v_w} \\ &+ v_w \frac{s\theta' m^2 (m^2)'}{2E_0^2 E_{0z}}(f''_{0,v_w} - \frac{f'_{0,v_w}}{E_0}) \\ &+ \frac{\theta' m^2 |p_z|}{2E_0^2 E_{0z}}(f''_{0,v_w} - \frac{f'_{0,v_w}}{E_0})\partial_z \mu_1 - \frac{v_w}{\gamma_w}\frac{s(m^2\theta')'}{2E_0 E_{0z}} f''_{0,v_w}\mu_1 \\ &- \frac{v_w}{\gamma_w}\frac{s\theta' m^2 (m^2)'}{4E_0^2 E_{0z}}(f'''_{0,v_w} - \frac{f''_{0,v_w}}{E_0})\mu_1 + \frac{p_z}{E_0}\partial_z \delta f_2 - \frac{(m^2)'}{2E_0}\partial_{p_z}\delta f_2 \\ &+ \frac{\theta' m^2 |p_z|}{2E_0^3 E_{0z}}\partial_z \delta f_1 + (\frac{s(m^2\theta')'}{2E_0 E_{0z}} - \frac{s\theta' m^2 (m^2)'}{4E_0^3 E_{0z}})\partial_{p_z}\delta f_1. \end{aligned} \quad (184)$$

We note that our results differ slightly from those of [21]. The additional γ_w factors can be attributed to the difference in the Taylor expansion. However, for small velocities, these have no effect. The additional factors of 2 in the first two lines come from the subtraction of the equations for particles and antiparticles (note that these vanish if we absorb the factor of 2 into the definitions of the potentials: e.g. $\mu_1 = \frac{1}{2}(\mu_{1e} + \bar{\mu}_{1e})$, $\mu_2 = \frac{1}{2}(\mu_{2o} - \bar{\mu}_{2o})$.)

5.3 Fluid equations

These equations look rather complicated. To make them easier to deal with, we assume that the velocity is small. This allows us to expand perturbatively in powers of the velocity, and truncate the expansion to the first order:

$$f_{0,v_w} \approx f_0 + v_w p_z f_0' \tag{185}$$
$$f_{0,v_w}' \approx f_0' + v_w p_z f_0'', \tag{186}$$

and so on. Clearly, in this low-v_w regime, $\gamma_w = 1$.

We average these equations over momentum, weighting them by 1 and $\frac{p_z}{E_0}$. These averages are defined by

$$\langle X \rangle = \frac{\int d^3p X(p)}{\int d^3p f_0'(m=0)}.$$

We make use of the fact that f_0 and its derivatives are even functions of momentum, while terms like $v_w p_z f_0'$ are odd (they change sign under the transformation $p_z \to -p_z$). The integrals of odd terms (for instance, terms proportional to s or p_z) vanish. We define the velocity of the plasma as

$$u_2 = \langle \frac{p_z}{E_0} \delta f_2 \rangle.$$

This gives us two transport equations for the second-order perturbations:

$$v_w \langle \frac{p_z^2}{E_0} f_0'' \rangle \mu_2' + v_w (m^2)' \langle \frac{f_0''}{2E_0} \rangle \mu_2 - m^2 \theta' \langle \frac{|p_z|}{2E_0^2 E_{0z}} (\frac{f_0'}{E_0} - f_0'') \rangle \mu_1' - u_2' = \langle C[f] \rangle \tag{187}$$

$$-\langle \frac{p_z^2}{E_0^2} f_0' \rangle \mu_2' + \theta' m^2 \langle \frac{p_z}{2E_0^3 E_{0z}} f_0 \rangle u_1' + \langle \frac{p_z^2}{E_0^2} \partial_z \delta f_2 \rangle - (m^2)' \langle \frac{p_z}{2E_0^2} \partial_{p_z} \delta f_2 \rangle$$
$$= \langle \frac{p_z}{E_0} C[f] \rangle - 2v_w (m^2 \theta')' \langle \frac{|p_z|}{E_0^2 E_{0z}} f_0' \rangle + 2v_w m^2 \theta' (m^2)' \langle \frac{|p_z|}{2E_0^3 E_{0z}} (f_0'' - \frac{f_0'}{E_0}) \rangle, \tag{188}$$

Following the notation in [21] and [15], we can write the momentum-averaged transport equations of the second-order perturbations in the form

$$\begin{pmatrix} v_w K_1 & 1 \\ -K_4 & v_w K_5 \end{pmatrix} \begin{pmatrix} \mu_2' \\ u_2' \end{pmatrix} = \begin{pmatrix} \langle C[f] \rangle \\ \langle \frac{p_z}{E_0} C[f] \rangle \end{pmatrix} - v_w (m^2)' \begin{pmatrix} K_2 \mu_2 \\ K_6 u_2 \end{pmatrix} + \begin{pmatrix} S_\mu \\ 2S_\theta + S_u \end{pmatrix}, \tag{189}$$

where the source terms are given by

$$S_\mu = K_7 \theta' m^2 \mu_1' \tag{190}$$

$$S_\theta = -v_w K_8 (m^2 \theta')' + v_w K_9 \theta' m^2 (m^2)') \tag{191}$$

$$S_u = -\widetilde{K}_{10} m^2 \theta' u_1', \tag{192}$$

and the moment functions are defined as [21]

$$K_1 = -\langle \frac{p_z^2}{E_0} f_0'' \rangle, \qquad \widetilde{K}_6 = [\frac{E_0^2 - p_z^2}{2E_0^3} f_0'] \tag{193}$$

$$K_2 = \langle \frac{f_0''}{2E_0} \rangle, \qquad K_7 = \langle \frac{|p_z|}{2E_0^2 E_{0z}} (\frac{f_0'}{E_0} - f_0'') \rangle \tag{194}$$

$$K_3 = \langle \frac{f_0'}{2E_0} \rangle, \qquad K_8 = \langle \frac{|p_z| f_0'}{2E_0^2 E_{0z}} \rangle \tag{195}$$

$$K_4 = \langle \frac{p_z^2}{E_0^2} f_0' \rangle, \qquad K_9 = \langle \frac{|p_z|}{4E_0^3 E_{0z}} (\frac{f_0'}{E_0} - f_0'') \rangle \tag{196}$$

$$\widetilde{K}_5 = [\frac{p_z^2}{E_0} f_0'], \qquad \widetilde{K}_{10} = [\frac{|p_z| f_0}{2E_0^3 E_{0z}}], \tag{197}$$

where the moments $\widetilde{K}_i$ denote averages involving δf_2.

Note, however, that our source term differs from that in [21] by a factor of 2; this is due to the difference in equation (184).

These are existing results from the literature, that hold in the low-v_w regime; however, they have not been calculated for a general value of v_w, and we do not know if the low-v_w assumption is valid. Thus it would be interesting to calculate the results for a general v_w. This is one of the objectives of our forthcoming publication.

6 Conclusions and Outlook

We have obtained the semiclassical force on the helicity states in the plasma frame, which is a new result in the literature.

We have verified that the semiclassical and Schwinger-Keldysh formalisms agree with one another in the regime of slowly-varying backgrounds in the wall frame and the double-boosted frame. However, we have not yet obtained directly commensurable results in the plasma frame, since we have yet to derive the force from the semiclassical formalism. This would therefore be one direction of future interest. It would also be interesting to directly derive the force on helicity states from the Schwinger-Keldysh formalism.

We have obtained the CP-even and CP-odd parts of the Boltzmann equation for the plasma in the wall frame. This has been done in [21], but our results differ slightly since we have kept the exact γ_w dependence in the perturbative expansion. The next step would be to calculate them in the plasma frame, where it is easiest to evaluate the self-energies and collision terms, and subsequently integrate over them, keeping the full velocity dependence.

The CP-odd terms are the ones that directly give rise to baryogenesis, while the CP-even equation governs the friction between the plasma and the wall and hence the wall velocity, having an indirect impact on baryogenesis. Thus both these equations are important.

Our calculation was considerably simplified by the assumption of a small velocity. However, we do not yet know whether this assumption holds true or not; calculation of the wall velocity requires computing the friction between the wall and the plasma from the CP-even equation. This is a non-trivial and model-dependent calculation. Results in the literature so far have been primarily based on the low-v_w assumption, but in principle it would be interesting to solve the equations with a full momentum code. A direction of future interest would be to compute these results while keeping the full velocity dependence; we intend to do this in a forthcoming publication.

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