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REGULARITY OF QUASILINEAR SUB-ELLIPTIC EQUATIONS IN THE HEISENBERG GROUP

SHIRSHO MUKHERJEE

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Jyväskylä, August 2018
Shirsho Mukherjee.

List of included articles

This dissertation comprises an introductory part and the following articles:

[A] Shirsho Mukherjee and Xiao Zhong, *$C^{1,\alpha}$-Regularity for variational problems in the Heisenberg Group*, preprint `https://arxiv.org/abs/1711.04671`, 2017.

[B] Shirsho Mukherjee, *On local Lipschitz regularity for Quasilinear equations in the Heisenberg Group*, preprint `https://arxiv.org/abs/1804.00751`, 2018.

[C] Shirsho Mukherjee, *$C^{1,\alpha}$-Regularity for Quasilinear equations in the Heisenberg Group*, to appear.

The articles are referred as [A],[B],[C] in the introduction, whereas other references are numbered [1],[2], etc. The author of this dissertation has actively taken part in the research of the paper [A].

INTRODUCTION

In this dissertation, we study the local interior regularity of solutions for a general class of certain quasilinear equations in divergence form. The considered setting is the Heisenberg Group, which has a structure that appears naturally in the quest of obtaining regularity for solutions of general second order equations. The techniques used for this purpose, involve methods from classical regularity theory with a longstanding history, along with some current results in contemporary literature.

1. Historical background

1.1. **Classical Elliptic Regularity.**

The main prototype model for quasilinear equations of divergence form arise from minimization of scalar variational integrals. To illustrate this, we consider a domain $\Omega \subset \mathbb{R}^n$ and a smooth function $f : \mathbb{R}^n \to \mathbb{R}$; the minimizer $u : \Omega \to \mathbb{R}$ of the functional

$$I(w) = \int_\Omega f(\nabla w)\, dx, \tag{1.1}$$

in the admissible class $K = \big\{w \in C^1(\bar{\Omega}) : w = u_0 \text{ in } \partial\Omega\big\}$ for a given function $u_0 \in C^1(\bar{\Omega})$, is a solution of the corresponding Euler-Lagrange equation

$$\operatorname{div}(\nabla f(\nabla u)) = 0. \tag{1.2}$$

This equation is said to be *uniformly elliptic*, if we have

$$\lambda|\xi|^2 \le \big\langle D^2 f(z)\,\xi, \xi\big\rangle \le \Lambda|\xi|^2 \quad \forall\ z, \xi \in \mathbb{R}^n, \tag{1.3}$$

for some positive constants $\Lambda \ge \lambda > 0$. The admissible class K needs to be extended to

$$\mathcal{K} = \Big\{w \in W^{1,2}(\Omega) : w - u_0 \in W_0^{1,2}(\Omega)\Big\}$$

in which the existence of minimizer can be shown from the so called *Direct methods in Calculus of Variations* and the minimizer is the *weak solution* of the equation (1.2), see [23]. It is also possible to show that a minimizer $u \in W^{1,2}(\Omega)$ is unique.

The question of smoothness of the minimizer u, first posed by Hilbert in 1900, remained unknown for few decades. By the *Caccioppoli inequalities* and difference-quotient arguments, one can show that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and that the weak partial derivative $u_{x_i} = \partial_{x_i} u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a local weak solution of the equation

$$\operatorname{div}(D^2 f(\nabla u)\nabla u_{x_i}) = 0,$$

for every $i \in \{1, \dots, n\}$. However, higher order Sobolev regularity of u could not be obtained by similar approach. The smoothness of minimizers was previously known by virtue of *Schauder estimates*, only for equations with continuous coefficients, i.e.

$$\operatorname{div}(A(x)\nabla u) = 0 \quad \text{in } \Omega, \tag{1.4}$$

where $A : \Omega \to \mathbb{R}^{n\times n}$ is bounded and continuous. Nevertheless, the problem was ultimately settled by Morrey [46] for the planar case $n = 2$ and by De Giorgi [7], Nash [50] independently, for $n \ge 3$. It was shown that, if $u \in W^{1,2}(\Omega)$ is a weak solution of equation (1.4) along with bare minimum hypothesis that A is only L^∞ and satisfies

$$\lambda|\xi|^2 \le \big\langle A(x)\,\xi, \xi\big\rangle \le \Lambda|\xi|^2,$$

then $u \in C^{0,\alpha}(\Omega)$ for some $\alpha = \alpha(n, \Lambda/\lambda) \in (0,1)$. Hence, for weak solutions $u \in W^{1,2}(\Omega)$ of the equation (1.2) with f satisfying (1.3), we can substitute $A(x) = D^2 f(\nabla u(x))$ and conclude $\nabla u \in C^{0,\alpha}(\Omega, \mathbb{R}^n)$. This, not only implies $u \in C^{1,\alpha}(\Omega)$, but also that it is smooth by the Schauder theory, thereby providing full affirmation to the Hilbert's problem. Later, a new proof of this theorem was provided by Moser [48] using *Harnack inequalities*, that follow from an iteration technique and the *John-Nirenberg lemma*, see [32].

The method of De Giorgi [7] is based on the fact that the weak solution u of (1.4) satisfy the following integral inequality

$$\int_{B_{r'}} |\nabla (u-k)^{\pm}|^2 \, dx \leq \frac{c}{(r-r')^2} \int_{B_r} |(u-k)^{\pm}|^2 \, dx, \tag{1.5}$$

for every $k \in \mathbb{R}$, $0 < r' < r$ and some $c = c(n, \Lambda/\lambda) > 0$, whenever $B_r \subset \Omega$ and $B_{r'}$ is concentric to B_r. Nowadays, the classes of all $W^{1,2}$-functions satisfying the inequality (1.5), are called De Giorgi classes $DG^{\pm}(\Omega)$ and $DG(\Omega) = DG^{+}(\Omega) \cap DG^{-}(\Omega)$. By application of Poincaré-Sobolev inequality, the integral inequality (1.5) and an iteration argument, it was shown that every $w \in DG(B_r)$ satisfies the oscillation estimate

$$\operatorname{osc}_{B_{r/2}} w \leq b_0 \operatorname{osc}_{B_r} w, \tag{1.6}$$

for some $b_0 \in (0,1)$. Consequently for every $0 < r' < r$, one has $\operatorname{osc}_{B_{r'}} w \leq c(r'/r)^{\alpha} \operatorname{osc}_{B_r} w$ by a standard iteration on (1.6), which thereby shows that all functions in the De Giorgi's class are locally Hölder continuous.

De Giorgi's ideas also shed light on the study of regularity for more general quasilinear elliptic equations. In the following decades, the techniques of [7] have been employed for investigating regularity for equations of the form $\operatorname{div} \mathcal{A}(\nabla u) = 0$ with

$$\lambda \, \mathrm{F}(|z|)|\xi|^2 \leq \big\langle D\mathcal{A}(z)\, \xi, \xi \big\rangle \leq \Lambda \, \mathrm{F}(|z|)|\xi|^2 \quad \forall \, z, \xi \in \mathbb{R}^n, \tag{1.7}$$

where $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathrm{F} : (0,\infty) \to (0,\infty)$; the equation is *degenerated* or *singular*, depending on the behavior of F. The main prototype is the *p-Laplace equation*

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad 1 < p < \infty, \tag{1.8}$$

which is degenerated for $p > 2$ and singular for $1 < p < 2$. It is easy to see that the solutions of equation (1.8) are also minimizers of (1.1) with $f(t) = t^p$ and existence can be shown in the class $\mathcal{K} = \{w \in W^{1,p}(\Omega) : w - u_0 \in W^{1,p}_0(\Omega)\}$ by the Direct methods. Before any inspection of local regularity, the *regularization* $\operatorname{div}\big((\varepsilon + |\nabla u|^2)^{(p-2)/2} \nabla u\big)$ is necessary and the limit $\varepsilon \to 0$ can be taken after obtaining uniform apriori estimates. Following this procedure, it can be shown easily by Moser's iteration that, a weak solution $u \in W^{1,p}(\Omega)$ of (1.8) satisfies

$$\sup_{B_{\sigma r}} |\nabla u| \;\leq\; \frac{c(n,p)}{(1-\sigma)^{n/p}} \Big(\fint_{B_r} |\nabla u|^p \, dx \Big)^{\frac{1}{p}} \tag{1.9}$$

whenver $B_r \subset \Omega$ and $0 < \sigma < 1$ and hence, $\nabla u \in L^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Higher regularity have been studied for $p > 2$ by Ladyzhenskaya-Ural'tseva [37] and Evans [12] (see also Uhlenbeck [62] for systems); it was shown that the weak solution u is locally $C^{1,\alpha}$, which is optimal in this case, see [38]. The partial derivatives u_{x_i} of a weak solution u satisfy a degenerated integral inequality, i.e. (1.5) with a weight $(\varepsilon + |\nabla u|^2)^{(p-2)/2}$. Nevertheless, the guiding principle in this respect also follows from [7], that if the degeneracy is locally confined in a set that is small in measure, then there is good control of it, thereby leading to the oscillation estimate (1.6) for u_{x_i}. The singular case $1 < p < 2$, is more difficult and the $C^{1,\alpha}$-regularity was ultimately established in the 80's independently by DiBenedetto [8], Lewis [39] and Tolksdorf [59]. In [8], it was shown that for the singular case, $u_{x_i}|u_{x_i}|^{(p-2)/2}$ satisfies an integral estimate slightly more general than (1.5), which is also good enough for the purpose. However, the proof by Tolksdorf [59], remarkably, provides a unitary treatment of both cases $1 < p < 2$ and $p \geq 2$, which is based on using the truncation

$$v = \min\big(m(r)/4,\, \max(m(r)/2 - u_{x_l}, 0)\big), \tag{1.10}$$

where $m(r) = \max_{1\leq i\leq n} \sup_{B_r} |u_{x_i}|$ for some fixed $B_r \subset \Omega$ and $l \in \{1, \ldots, n\}$. Noticeably, letting $E = \{x \in \Omega : m(r)/4 < u_{x_l} < m(r)/2\}$, there is no singularity of $|\nabla u|$ inside the set $E \cap B_r$; also $\nabla v = -\nabla u_{x_l}$ a.e. in E and ∇v vanishes almost everywhere in $\Omega \setminus E$. Thus, using v for the integral estimates, it is possible to obtain a Caccioppoli type inequality reminiscent of that for uniformly elliptic equation, devoid of the weight $(\varepsilon + |\nabla u|^2)^{(p-2)/2}$ and the proof follows from Moser's iteration, thereafter. The technique was followed up in [40] for equations with more general structure conditions.

We also remark that equation (1.8) is very singular for $p = 1$ and counterexamples can be found which show that, in this case the solutions have bounded but discontinuous gradient and hence, they are not $C^{1,\alpha}$. We refer to [45, 52, 8] for more details.

Alongside the development of the above topics, equations of the form

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0 \tag{1.11}$$

has also been a subject of deep scrutiny and exploration over a long period. The minimizers of general functionals of the form

$$I(w) = \int_\Omega f(x, w, \nabla w)\, dx,$$

are solutions of equations of the form (1.11) and this provides a perspective to study such equations from the variational point of view as well. For the equation (1.11), the existence of weak solutions are shown using *variational inequalities* corresponding to *monotone operators*. A comprehensive detail on this can be found in the book by Kinderlehrer-Stampacchia [33]. The local behavior of weak solutions have been investigated substantially by Serrin [57], with the structure conditions

$$\begin{aligned} &\big\langle A(x, u, z), z\big\rangle \geq |z|^\alpha - a_1|u|^\alpha - a_2;\\ &|A(x, u, z)| \leq a_3|z|^{\alpha-1} + a_4|u|^{\alpha-1} + a_5;\\ &|B(x, u, z)| \leq b_0|z|^{\alpha-1} + b_1|u|^{\alpha-1} + b_2, \end{aligned} \tag{1.12}$$

for $(x, u, z) \in \Omega\times\mathbb{R}\times\mathbb{R}^n$, $\alpha > 1$ and non-negative constants a_i, b_j. Following the iteration technique of Moser [48], it was shown that the Harnack inequality holds for solutions of equation (1.11) with A and B satisfying (1.12), which immediately implies that the solutions are Hölder continuous. Similar results can also be found in [37] and [61]. Now, similarly as in the aforementioned illustration, the standard technique was to differentiate the equation (1.11) and obtain the equation satisfied by the derivatives of the solution, in order to look for higher regularity. This requires the function $A(x, u, z)$ to be differentiable in all variables. However, it was shown by Giaquinta-Giusti [21], that this is superfluous and $C^{1,\alpha}$ regularity can be obtained just by assuming that $A(x, u, z)$ is uniformly differentiable with respect to z and Hölder continuous with respect to (x, u). Their technique involved a non-linear version of the *freezing argument* of Schauder estimates, followed by a perturbation argument of Campanato [2]. Although, their growth and ellipticity condition was uniform, but with the gradient bound as in (1.9), polynomial degeneracy similar to (1.12) can also be included in order to establish $C^{1,\alpha}$ regularity.

More detailed exposition on regularity theory and other related topics associated to quasilinear equations, can be found in the classical books by Gilbarg-Trudinger [22], Ladyzhenskaya-Ural'tseva [37] and Morrey [47].

1.2. **Hypoellipticity and Hörmander's condition.**
The development of *distribution theory* by Schwartz [55, 56] gave rise to the formation of a robust conceptual framework for solutions of linear partial differential equation of the form $\mathcal{L}u = f$, for distributions $u, f \in \mathscr{D}'(\Omega)$ and a generic linear operator of order m given by

$$\mathcal{L} = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \tag{1.13}$$

where $a_\alpha : \Omega \to \mathbb{C}$ for some $\Omega \subseteq \mathbb{R}^N$ and for every $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, $|\alpha| = \sum_{j=1}^N \alpha_j$ and $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}$. If f is compactly supported in Ω, then a particular solution is given by the convolution $u = \Gamma_0 * f$; here $\Gamma_x \in \mathscr{D}'(\mathbb{R}^N)$ is the *fundamental solution* of $\mathcal{L}$ (if it exists) satisfying

$$\mathcal{L}\,\Gamma_x = \delta_x,$$

where δ_x is the *Dirac distribution.* A general solution of the problem, can be obtained by combining this with solutions of $\mathcal{L}u = 0$. The question of existence of solutions lead to the notion of *local solvability*, that is, $\mathcal{L}$ is said to be locally solvable at $x_0 \in \Omega$ if there exists a neighborhood $U \subseteq \Omega$ containing x_0 such that for every $f \in C_0^\infty(U)$, there exists $u \in \mathscr{D}'(U)$ solving $\mathcal{L}u = f$. The *ellipticity* of $\mathcal{L}$ is defined via the principal symbol of (1.13), i.e.

$$p_0(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha,$$

as $\mathcal{L}$ is elliptic if $Z(p_0) = \left\{\xi \in \mathbb{R}^N : p_0(x, \xi) = 0\right\} = \{0\}$. It is known that if the coefficients $a_\alpha \in C^\infty(\Omega, \mathbb{C})$ and $\mathcal{L}$ as in (1.13) is elliptic, then $\mathcal{L}$ is locally solvable everywhere, see [36]. In case of constant coefficients $a_\alpha \in \mathbb{C}$, even if ellipticity is dropped, a fundamental solution for $\mathcal{L}$ always exists by the *Malgrange-Ehrenpreis theorem.*

The regularity of solutions of $\mathcal{L}u = f$ is addressed by the notion of *hypoellipticity*, where the operator $\mathcal{L}$ of (1.13) is said to be hypoelliptic in Ω if $a_\alpha \in C^\infty(\Omega, \mathbb{C})$ and the *singular support* of every distribution is $\mathcal{L}$-invariant i.e. for every $u \in \mathscr{D}'(\Omega)$

$$\text{sing supp } \mathcal{L}u = \text{sing supp } u;$$

in other words, for any open $U \subseteq \Omega$ and $u \in \mathscr{D}'(U)$, if $\mathcal{L}u \in C^\infty(U)$ then we have $u \in C^\infty(U)$. For the case of constant coefficients $a_\alpha \in \mathbb{C}$, a classical theorem (see [60]) states that $\mathcal{L}$ is hypoelliptic if and only if there exists a fundamental solution $\Gamma_0 \in C^\infty(\mathbb{R}^N \setminus \{0\})$. In fact, in this case, the hypoelliptic operators have been characterized by Hörmander as $\mathcal{L}$ is hypoelliptic if and only if

$$\lim_{|\xi| \to \infty} \frac{|\nabla p(\xi)|}{|p(\xi)|} = 0,$$

where $p(\xi)$ is the polynomial defined via Fourier transform $\widehat{\mathcal{L}u}(\xi) = p(\xi)\widehat{u}(\xi)$. Hence, the *Laplace equation*, *heat equation* and *Cauchy-Riemann equations* are hypoelliptic but the *wave equation* and *Schrödinger equation* are not. More generally, similarly as local solvability, if $a_\alpha \in C^\infty(\Omega, \mathbb{C})$ and $\mathcal{L}$ is elliptic, then it is also hypoelliptic, see [36]. However, if ellipticity is dropped, then there was no criterion to check hypoellipticity of linear operators just from the coefficients $a_\alpha(x)$. In connection with *Kolmogorov operator* (see [35]) and operators related to *Fokker-Planck equation*, numerous examples of operators have been found which were hypoelliptic. The following examples in $\mathbb{R}^2$,

(1) $\mathcal{L} = x_1 \partial^2_{x_2 x_2} + \partial_{x_1}$ hypoelliptic but not locally solvable at any $(0, x_2) \in \mathbb{R}^2$,
(2) $\mathcal{L} = x_1 \partial^2_{x_2 x_2} - \partial_{x_1}$ locally solvable everywhere but not hypoelliptic,

further illustrate the difficulty! The way hypoellipticity of most operators were checked, was by explicit computation of fundamental solutions, which in general can be a cumbersome task.

Thereafter, for the case of second order linear operators with smooth real coefficients, a major breakthrough was made by Hörmander [26] in 1967. First, it was shown that if

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=1}^{N} b_i(x)\partial_{x_i} + c(x) \tag{1.14}$$

is hypoelliptic and $a_{ij}, b_i, c \in C^\infty(\Omega)$, then the quadratic form is semidefinite in Ω. In other words $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq 0$ (or ≤ 0) for every $x \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$; hence in an open subset of $\{\mathrm{rank}(a_{ij}(x)) = r\}$, the operator $\mathcal{L}$ (or $-\mathcal{L}$) of (1.14) can be expressed as

$$\mathcal{L} = \sum_{i=1}^{r} X_i^2 + X_0 + c \quad \text{for some} \quad X_i = \sum_{k=1}^{N} b_{ik}(x)\partial_{x_k}, \tag{1.15}$$

where b_{ik}, c are real and C^∞ on the open set and $X_i^2 u = X_i X_i u$. Now, if the Lie algebra $\mathscr{L}(X_0, X_1, \ldots, X_r)$ generated by the vector fields of the above has dimension m in the open ball B_1, then its basis can be written as $\{\partial_{y_1}, \ldots, \partial_{y_m}\}$ for some different coordinate system $(y_1, \ldots, y_N)$ by *Frobenius theorem*. Hence in this coordinate, the operator $\mathcal{L}$ as in (1.15), is generated by $\partial^2_{y_i y_j}, \partial_{y_i}$ for $i, j \in \{1, \ldots, m\}$. If $m < N$, then notice that for any $w \in C^\infty(B_1)$ with $\mathcal{L}w \in C^\infty(B_1)$, we have that $\mathcal{L}\tilde{w} = \mathcal{L}w$ where $\tilde{w} = w\mathbb{1}_{\{y_N > 0\}} \notin C^\infty(B_1)$, which thereby violates hypoellipticity. Thus, a plausible natural condition sufficient for hypoellipticity would be $m = N$ everywhere. This is exactly the theorem proved by Hörmander in [26], which states that if

$$\dim\big(\mathscr{L}(X_0, X_1, \ldots, X_r)\big) = N \tag{1.16}$$

holds at every point in Ω, then the operator $\mathcal{L}$ as in (1.15) is hypoelliptic in Ω. Henceforth, the condition (1.16) is known as *Hörmander's condition.* It is noteworthy that although (1.16) has been effective in checking hypoellipticity, it is sufficient but not exactly a necessary condition unless the coefficients of (1.14) are constants i.e. $a_{ij}, b_i \in \mathbb{R}$. Also, in this case all commutators vanish and hence (1.16) holds if and only if either $\mathrm{span}\{X_1, \ldots, X_r\} = \mathbb{R}^N$ and $r = N$ ($\mathcal{L}$ is elliptic) or $\mathrm{span}\{X_0, X_1, \ldots, X_r\} = \mathbb{R}^N$ and $r = N - 1$ ($\mathcal{L}$ is parabolic). Thus all real constant coefficient hypoelliptic operators are either elliptic or parabolic. However, still no general assertion can be made on necessity of (1.16) for hypoelliptic operators of variable smooth coefficients, e.g. in $\mathbb{R}^2$ the operator $\mathcal{L} = \partial^2_{x_1 x_1} + e^{-2/{x_1}^2}\partial^2_{x_2 x_2}$ is hypoelliptic (see [6]), but $X_1 = \partial_{x_1}$ and $X_2 = e^{-1/{x_1}^2}\partial_{x_2}$ do not satisfy (1.16) on $\{x_1 = 0\}$. Further inspection on the nature failure of Hörmander's condition while hypoellipticity is preserved, was carried out by Oleĭnik-Radkevič [51] and Christ [6].

We refer to the books [27, 28, 29, 30] and references therein for more details on distribution theory and Hörmander's work on linear partial differential operators.

2. Sub-elliptic theory

A different proof of Hörmander's theorem was later found by Kohn [34], involving so called *pseudo-differential operators.* In both of the papers Kohn [34] and Hörmander [26], the proofs are based on apriori estimates, so called *sub-elliptic estimates*, of the type

$$\|u\|_{H^{m+\varepsilon}(\Omega')} \leq c\left(\|\mathcal{L}u\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\right) \tag{2.1}$$

for every $\Omega' \subset\subset \Omega$ and $m \geq 0$, where $\varepsilon > 0$ is small and $H^s(\Omega) = W^{s,2}(\Omega)$ are Sobolev spaces that include fractional order. Although sufficient for the purpose of hypoellipticity, this is somehow unsatisfactory. If the given data are only partially smooth, then the estimate (2.1) does not provide any reasonable improvement in regularity unlike the case of elliptic operators $\mathcal{L}$ as in (1.14) where, given $a_{ij}, b_i, c \in C^{m+1}(\Omega)$, we have $\|u\|_{H^{m+2}(\Omega')} \leq c\left(\|\mathcal{L}u\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\right)$. Obtaining improved sub-elliptic estimates was one of the main following quests undertaken during the mid 70's by Folland [16, 15], Folland-Stein [17] and Rothschild-Stein [53].

2.1. **Homogeneous groups.**

It is evident that constant coefficient operators are none other than translation-invariant operators on the Abelian Lie group $(\mathbb{R}^N,+)$. The models considered by Folland [16], are based on the following viewpoint : given any set of Hörmander vector fields, if there is a group operation making $\mathbb{R}^N$ into a (non-Abelian) Lie group such that the vector fields are translation-invariant with respect to it, then techniques from standard *harmonic analysis* could be applied to obtain better estimates for the corresponding operator.

To this end, consider a connected Lie group $\mathbb{G}=(\mathbb{R}^N,\cdot)$ with the origin as identity and with its corresponding Lie algebra $\mathfrak{g}$ being spanned by left invariant vector fields $X_1,\ldots,X_N$ that coincide with $\partial_{x_1},\ldots,\partial_{x_N}$ at the origin; also consider that for some $r\leq N$, the vector fields $X_1,\ldots,X_r$ satisfy the Hörmander's condition at every point. Notice that, this trivially holds for $r=N$ since linear independence is maintained everywhere in $\mathbb{G}$ starting from the origin, by virtue of left invariance and connectedness. But more intricate strucututures appear for $r<N$ since, in this case $\mathfrak{g}$ can be generated by taking the span of commutators $[X_i,X_j],[X_i,[X_j,X_k]]$, etc. among $X_1,\ldots,X_r$ until the full dimension N, is reached. In other words, the Lie algebra $\mathfrak{g}=\mathscr{V}_1\oplus\mathscr{V}_2\oplus\ldots\oplus\mathscr{V}_s$ is *graded* and *nilpotent* with some $s\in\mathbb{N}$ such that $N=\sum_{j=1}^s\dim(\mathscr{V}_j)$, $\mathscr{V}_1=\operatorname{span}\{X_1,\ldots,X_r\}$ and the other subspaces inductively defined as

$$
\begin{aligned}
&[\mathscr{V}_1,\mathscr{V}_j]=\mathscr{V}_{j+1}\quad\forall\, j\in\{1,\ldots,s-1\}\\
&[\mathscr{V}_1,\mathscr{V}_s]=\{0\}.
\end{aligned}
\tag{2.2}
$$

A graded algebra $\bigoplus_{j=1}^s\mathscr{V}_j$ with the structure (2.2), is said to be *stratified.* The Lie algebra $\mathfrak{g}$ admits a canonical map $\nu_\lambda:\mathfrak{g}\to\mathfrak{g}$ for every $\lambda\in(0,\infty)$, as $\nu_\lambda\big(\sum_{j=1}^s V_j\big)=\sum_{j=1}^s\lambda^jV_j$, where $V_j\in\mathscr{V}_j$ for every $j\in\{1,\ldots,s\}$. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp:\mathfrak{g}\to\mathbb{G}$ is a diffeomorphism (see [25]) and hence ν_λ give rise to a one parameter family of automorphisms called *dilations*, $\delta_\lambda:\mathbb{G}\to\mathbb{G}$ given by $\delta_\lambda(x_1,x_2,\ldots,x_N)=(\lambda^{\alpha_1}x_1,\lambda^{\alpha_2}x_2,\ldots,\lambda^{\alpha_N}x_N)$ where $\alpha_k=1$ for $1\leq k\leq r=\dim(\mathscr{V}_1)$ and $\alpha_k=j$ whenever $\dim(\mathscr{V}_{j-1})<k\leq\dim(\mathscr{V}_j)$, for every $k>r$. This makes $\mathbb{G}$ a *homogeneous group* with its *homogeneous dimension* defined by

$$
Q=\sum_{k=1}^N\alpha_k=\sum_{j=1}^s j\dim(\mathscr{V}_j),
\tag{2.3}
$$

and thus, if μ is a (bi-invariant) *Haar measure* on $\mathbb{G}$, then we have $\mu(\delta_\lambda(E))=\lambda^Q\mu(E)$ for every measurable $E\subset\mathbb{G}$. Although, there are many varities of smooth group operations on $\mathbb{R}^N$, the presence of homogeneous dilations δ_λ restrict them to be polynomials. Precisely, $x\cdot y=x+y+p(x,y)$ where $p(x,y)$ is a polynomial satisfying $p(x,0)=0=p(0,y)$ for every $x,y\in\mathbb{R}^N$; in other words, p does not contain monomials, see Stein [58]. In addition, one can take $x^{-1}=-x$ with an appropriate choice of coordinate system. These further imply that the Lebesgue measure of $\mathbb{R}^N$ is invariant under group translation and hence is a Haar measure of $\mathbb{G}$, unique upto multiplicative constant.

Simply connected homogeneous Lie groups with their Lie algebra stratified as in (2.2) are called *Carnot groups* of step s. There are several ways to define a *homogeneous norm* $\|.\|:\mathbb{G}\to[0,\infty)$ satisfying $\|\delta_\lambda x\|=\lambda\|x\|$ and $\|x\cdot y\|\leq c(\|x\|+\|y\|)$ for all $x,y\in\mathbb{G}$ and all are equivalent. One of the popular choices is

$$
\|x\|=\Big(\sum_{k=1}^N|x_k|^{Q/\alpha_k}\Big)^{1/Q}\qquad\forall\, x=(x_1,\ldots,x_N)\in\mathbb{R}^N.
\tag{2.4}
$$

Homogeneous norms give rise to left-invariant (quasi) distance functions as $d(x,y) = \|y^{-1}\cdot x\|$ satisfying $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x,y)$ and as a consequence, the *Hausdorff dimension* with respect to d, coincides with the homogeneous dimension Q. There are other ways to construct equivalent metrics on Carnot groups; the *Carnot-Carathèodory metric* is an example, which is constructed using the *horizontal curves*, i.e. absolutely continuous curves γ such that $\gamma'(t)$ is spanned by $X_1|_{\gamma(t)}, \dots, X_r|_{\gamma(t)}$. Since the Lebesgue measure is a Haar measure, we have $|B_r| = c(N)r^Q$ for any metric ball $B_r \subset \mathbb{G}$, regardless of the choice of metric.

A particular example of a Carnot group of step 2, is the *Heisenberg Group*, denoted as $\mathbb{H}^n$ for $n \geq 1$, where in this case $N = 2n+1$. The group operation is defined as

$$x \cdot y := \Big(x_1 + y_1, \dots, x_{2n} + y_{2n}, t + s + \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i)\Big) \tag{2.5}$$

for every $x = (x_1, \dots, x_{2n}, t)$, $y = (y_1, \dots, y_{2n}, s) \in \mathbb{H}^n$ and the Lie algebra $\mathfrak{g} \cong \mathbb{R}^{2n} \oplus \mathbb{R}$ is spanned by the left invariant vector fields

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t, \ X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t$$

for every $i \in \{1, \dots, n\}$ and the only non-zero commutator $T = \partial_t = [X_i, X_{n+i}]$. According to the above notions, here $r = 2n$ and the homogeneous dimension $Q = 2n+2$.

2.2. **Sub-elliptic linear operators.**
The operator $\mathcal{L} = \sum_{i=1}^r X_i^2$, so called the sub-elliptic Laplacian or *sub-Laplacian*, is hypoelliptic by Hörmander's theorem if the condition (1.16) holds for $X_1, \dots, X_r$, as illustrated before. Towards obtaining better sub-elliptic estimates, first it was shown by Folland [16] that, any operator $\mathcal{L}$ that is homogeneous of degree 2 in $\mathbb{G} = (\mathbb{R}^N, \cdot)$ with $Q > 2$, admits a unique fundamental solution $\Gamma_0 \in C_0^\infty(\mathbb{G} \setminus \{0\})$, which is homogeneous of degree $2 - Q$ i.e. $\Gamma_0(\delta_\lambda x) = \lambda^{2-Q}\Gamma_0(x)$. This implies that for any $f \in C_0^\infty(\mathbb{G})$ the convolution $u = f * \Gamma_0$ defined via group operation, solves $\mathcal{L}u = f$. The proof is not constructive and relies on some abstract results from distribution theory. However, for the case of sub-Laplacian in the Heisenberg group, the explicit fundamental solution $\Gamma_0(x) = c(n)\|x\|^{2-Q}$ was found earlier by Folland [15]. The behavior of fundamental solutions for subLaplacian of general Carnot groups, have been studied in extensive details in the monograph by Bonfiglioli-Lanconelli-Uguzzoni [1].

Upon obtaining fundamental solution on homogeneous groups, the estimates are usually carried out by obtaining representation formula of the type

$$X_i X_j u = \text{p.v.}\,(\mathcal{L}u * X_i X_j \Gamma_0) + c_{ij}\mathcal{L}u$$

and using *Calderòn-Zygmund theory* for the kernel $K(x) = X_i X_j \Gamma_0(x)$, leading to the apriori estimate $\|X_i X_j u\|_{L^p(\mathbb{G})} \leq c\|\mathcal{L}u\|_{L^p(\mathbb{G})}$ for $1 < p < \infty$. This was first carried out for Heisenberg groups in Folland-Stein [17]. Thereafter, by introduction of Sobolev spaces $HW^{k,p}$ (that is S^p_k or $S^{k,p}$ as the earlier notation) involving kth order L^p derivatives with respect to $X_1, \dots, X_r$, the results in [17] can be extended to obtain sharper sub-elliptic estimates

$$\|u\|_{HW^{k+2,p}(\mathbb{G})} \leq c\Big(\|\mathcal{L}u\|_{HW^{k,p}(\mathbb{G})} + \|u\|_{HW^{k,p}(\mathbb{G})}\Big);$$

subsequently these techniques for homogeneous groups have been used along with the so called, *lifting and approximation technique*, and apriori estimates like the above have been obtained for general Hörmander type operators by Rothschild-Stein [53].

In addition to this, Nagel-Stein-Wainger [49] have studied properties of certain metrics that appear naturally from family of Hörmander type vector fields and based on ideas from this, a general treatment of fundamental solutions for second order hypoelliptic operators was carried out later by Fefferman and Sánchez-Calle, see [54, 13].

2.3. **Quasilinear sub-elliptic equation.**
It is evident that the developments for sub-elliptic linear operators have remarkable structural similarity to that of Laplacian and linear elliptic operators. Hence, it is natural to seek a corresponding theory of quasilinear equations in the sub-elliptic setting that is reminiscent of the classical regularity theory for quasilinear elliptic equations.

Let $\mathfrak{X}u = (X_1u, \dots, X_ru)$. The following Poincaré inequality for $u \in C^{0,1}(B_r)$

$$\Big(\fint_{B_r} |u - \{u\}_{B_r}|^p \, dx \Big)^{1/p} \leq c\, r \Big(\fint_{B_r} |\mathfrak{X}u|^p \, dx \Big)^{1/p} \tag{2.6}$$

was proved by Jerison [31], first for homogeneous groups and then for general Hörmander vector fields following the lifting and approximation technique of [53] and exploiting certain results from [49]. The inequality (2.6), together with doubling condition for metric balls, implies a Sobolev embedding theorem which has been shown in different levels of generality by Garofalo-Nhieu [20], Franchi-Lu-Wheeden [19], Hajlasz-Koskela [24]. From Moser's iteration, this further implies the Harnack inequalities for weak solutions $u \in HW^{1,p}(\Omega)$ for the, so called the *sub-elliptic p-Laplacian* given by

$$\mathrm{div}_H(|\mathfrak{X}u|^{p-2}\mathfrak{X}u) = 0 \tag{2.7}$$

with div_H defined by the vector fields $X_1, \dots, X_r$, as shown by Capogna-Danielli-Garofalo [4]. This thereby leads to Hölder continuity of weak solutions of the equation (2.7). The class of quasilinear equations of the form $\mathrm{div}_H \, \mathcal{A}(\mathfrak{X}u) = 0$, which are uniformly sub-elliptic, i.e. $\nu^{-1}|\xi|^2 \leq \big\langle D\mathcal{A}(z)\xi, \xi \big\rangle \leq \nu|\xi|^2$, have been studied by Capogna [3] for the Heisenberg group. For weak solutions $u \in HW^{1,2}_{\mathrm{loc}}(\Omega)$, it was shown in [3] that $X_iu, Tu \in HW^{1,2}_{\mathrm{loc}}(\Omega)$ by using difference quotient arguments in exponential coordinates; thus equation differentiable and then following Sobolev inequality and Moser's iteration, one can conclude $X_iu, Tu \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ (by this notation, here we mean Hölder continuity in the sense of Folland-Stein [18] i.e. with respect to a homogeneous metric, which has been referred as Γ^{α} or $\Gamma^{0,\alpha}$ in [3] and others).

For the equation (2.7) with $p \neq 2$, there has not been any satisfactory result for quite some time regarding regularity of solutions higher than Hölder continuity even in the Heisenberg group, let alone Carnot groups or general Hörmander vector fields. The main source of difficulty was the non-commutativity of the vector fields $X_1, \dots, X_r$ unlike the equation (1.8) in the Euclidean setting. Regularity for $\mathfrak{X}u$ outside a measure-zero set, was obtained by Capogna-Garofalo [5] and for systems, by Föglein [14]. Hölder continuity of $\mathfrak{X}u$ was obtained for a small range of p by Domokos-Manfredi [10] using sub-elliptic Cordes perturbation technique of [11]. Then, it was shown by Domokos [9] that for weak solutions $u \in HW^{1,p}(\Omega)$ of (2.7), $Tu \in L^p_{\mathrm{loc}}(\Omega)$ for $1 < p < 4$, thereby extending an earlier result of Marchi [43]. Using this integrability result of [9], Lipschitz continuity of u was shown by Manfredi-Mingione [42] and later improved by Mingione-Zatorska-Zhong [44] to the range $2 \leq p < 4$. However, the imposed restriction $p < 4$ could not be removed.

This begs the natural question that, if there is any $C^{1,\alpha}$ regularity ($\Gamma^{1,\alpha}$ as in Folland-Stein [18]) for weak solutions of the equation (2.7) atleast in the Heisenberg group, reminiscent of the classical theory for the p-Laplace equation (1.8) in the Euclidean setting. To this end, the first result has been recently obtained by Zhong [63] where it was shown that the weak solution $u \in HW^{1,p}(\Omega)$ of (2.7), is locally Lipschitz and whenever $B_{2r} \subset \Omega$,

$$\sup_{B_r} |\mathfrak{X}u| \;\leq\; c(n,p) \Big(\fint_{B_{2r}} |\mathfrak{X}u|^p \, dx \Big)^{\frac{1}{p}} \tag{2.8}$$

holds for every $1 < p < \infty$, similarly as in (1.9). Furthermore, Hölder continuity of $\mathfrak{X}u$ was also shown in [63] for $p \geq 2$. However, the proof does not work for the singular case $1 < p < 2$ and the problem is significantly more difficult in this case. Finally, the problem is resolved in full strength in the paper [A], where we prove the following theorem.

Theorem 2.1 ([A, Theorem 1.3])**.** *Let $1 < p < \infty, \delta \geq 0$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of the equation*

$$\operatorname{div}_H \left((\delta + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} \mathfrak{X}u \right) = 0$$

in a domain $\Omega \subset \mathbb{H}^n$. Then $\mathfrak{X}u$ is locally Hölder continuous. Moreover, there exists a positive exponent $\alpha = \alpha(n,p) \leq 1$ such that for any ball $B_{r_0} \subset \Omega$ and any $0 < r \leq r_0$, we have

$$\max_{1 \leq l \leq 2n} \operatorname{osc}_{B_r} X_l u \, \leq \, c\Big(\frac{r}{r_0}\Big)^{\alpha} \Big(\fint_{B_{r_0}} (\delta + |\mathfrak{X}u|^2)^{\frac{p}{2}} \, dx \Big)^{\frac{1}{p}},$$

for some $c = c(n,p) > 0$.

This combined with (2.8), implies $\|\mathfrak{X}u\|_{C^{0,\alpha}(B_r;\mathbb{R}^{2n})} \leq c(n,p)\|\mathfrak{X}u\|_{L^p(B_{2r};\mathbb{R}^{2n})}$, when $B_{2r} \subset \Omega$, for every weak solution $u \in HW^{1,p}(\Omega)$ of (2.7). The theorem is proved first for $\delta > 0$ to ensure differentiablity of the equation by virtue of [3]; then the limit $\delta \to 0$ is taken upon the uniform estimate. The main idea behind the proof of Theorem 2.1 is to use the truncation $v = \min\big(\mu(r)/8, \max(\mu(r)/4 - X_l u, 0)\big)$ with $\mu(r) = \max_{1 \leq i \leq 2n} \sup_{B_r} |X_i u|$, similarly as (1.10) and use Moser's iteration and De Giorgi's arguments together with an integrability estimate of Tu, obtained in [63].

The ideas and techniques of [63] and [A] were applied for more general quasilinear equation of the form $\operatorname{div}_H \mathcal{A}(\mathfrak{X}u) = 0$, which has structure condition similar to (1.7) with $\mathrm{F}(t) = g(t)/t$ for a given C^1 function g satisfying

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \quad \text{for all } t > 0, \tag{2.9}$$

for some constants $g_0 \geq \delta > 0$ (this δ is a fixed constant and the usage is different from that in Theorem 2.1). The condition (2.9) have been previously introduced by Lieberman [41], in order to produce a natural extension of the structure conditions previously considered by Ladyzhenskaya and Ural'tseva [37]. Thus, the equation can be degenerated or singular, depending on the behavior of F; however (2.9) ensures that g is *doubling* and non-decreasing, which is necessary for the purpose. In this case, the natural domain for weak solutions is the *horizontal Orlicz-Sobolev space* $HW^{1,G}(\Omega)$ for $G(t) = \int_0^t g(s)\, ds$, see [B] for the definition.

Theorem 2.2 ([B, Theorem 1.1], [C, Theorem 1.3])**.** *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of the equation $\operatorname{div}_H \mathcal{A}(\mathfrak{X}u) = 0$ where the matrix $D\mathcal{A}$ is symmetric and satisfies the structure condition*

$$\begin{aligned} \frac{g(|z|)}{|z|}\, |\xi|^2 \leq \big\langle D\mathcal{A}(z)\, \xi, \xi \big\rangle \leq L\, \frac{g(|z|)}{|z|}\, |\xi|^2; \\ |\mathcal{A}(z)| \leq L\, g(|z|), \end{aligned} \tag{2.10}$$

for every $z, \xi \in \mathbb{R}^{2n}$, where $L \geq 1$, $G(t) = \int_0^t g(s)\, ds$ and g satisfies condition (2.9) with $g_0 \geq \delta > 0$. Then $\mathfrak{X}u$ is locally Hölder continuous and there exists positive constants $\sigma = \sigma(n, g_0, L) \in (0,1)$ and $c = c(n, \delta, g_0, L) > 0$ such that for any $B_{r_0} \subset \Omega$ and $0 < r < r_0/2$, we have

$$\begin{aligned} &(i) \quad \sup_{B_{\tau r}} G(|\mathfrak{X}u|) \leq \frac{c}{(1-\tau)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\, dx \quad \textit{for any } \tau \in (0,1); \\ &(ii) \quad \max_{1 \leq l \leq 2n} \fint_{B_r} G(|X_l u - \{X_l u\}_{B_r}|)\, dx \, \leq \, c\Big(\frac{r}{r_0}\Big)^{\sigma} \fint_{B_{r_0}} G(|\mathfrak{X}u|)\, dx. \end{aligned} \tag{2.11}$$

The proof of the first part of Theorem 2.2, requires an adaptation of the arguments in [63]; the whole of the paper [B] is devoted to this. The second part follows similarly as the proof of Theorem 2.1 in [A], which has been provided in [C].

Finally, we also consider equations of the form $\operatorname{div}_H A(x, u, \mathfrak{X}u) + B(x, u, \mathfrak{X}u) = 0$ in order to reproduce the classical regularity results of the equation (1.11), in the Heisenberg group. We show the Harnack inequalities and hence, the Hölder continuity of solutions, with structure conditions similar to (1.12) in [C], along the same lines of [61, 41]. In addition, assuming the following structure condition

$$
\begin{aligned}
&\frac{g(|z|)}{|z|}\,|\xi|^2 \le \big\langle D_z\, A(x,u,z)\,\xi,\xi\big\rangle \le L\,\frac{g(|z|)}{|z|}\,|\xi|^2;\\
&|A(x,u,z)-A(y,w,z)| \le L'\big(1+g(|z|)\big)\Big(|x-y|^\alpha+|u-w|^\alpha\Big);\\
&|B(x,u,z)| \le L'\big(1+g(|z|)\big)|z|,
\end{aligned}
\tag{2.12}
$$

for every $x, y \in \Omega$, $u, w \in [-M_0, M_0]$ and $z, \xi \in \mathbb{R}^{2n}$, for some $\alpha \in (0, 1], M_0 > 0$, g satisfying (2.9) and $L, L' \geq 1$, we also prove $C^{1,\alpha}$ regularity following arguments of [2, 21].

Theorem 2.3 ([C, Theorem 1.2])**.** *Let $u \in HW^{1,G}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of*

$$\operatorname{div}_H A(x, u, \mathfrak{X}u) + B(x, u, \mathfrak{X}u) = 0,$$

with $G(t) = \int_0^t g(s)ds$ and $|u| \leq M_0$ in Ω. Suppose the structure condition (2.12) *holds for some $L, L' \geq 1, \alpha \in (0, 1]$ and a function g satisfying* (2.9) *for some $g_0 \geq \delta > 0$, then there exists a constant $\beta = \beta(n, \delta, g_0, \alpha, L) \in (0, 1)$ such that $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ and for any open $\Omega' \subset\subset \Omega$, we have*

$$|\mathfrak{X}u|_{C^{0,\beta}(\Omega',\mathbb{R}^{2n})} \le C\big(n,\delta,g_0,\alpha,L,L',M_0,g(1),\operatorname{dist}(\Omega',\partial\Omega)\big). \tag{2.13}$$

Thus, from the progress in [A, B, C], now it is evident that the regularity theory for Heisenberg group is the same as that in the classical Euclidean setting. This should pave the way for further developments in sub-elliptic theory, in the setting of more general Carnot groups and general Hörmander vector fields.

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$C^{1,\alpha}$-REGULARITY FOR VARIATIONAL PROBLEMS IN THE HEISENBERG GROUP

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Abstract. We study the regularity of minima of scalar variational integrals of p-growth, $1 < p < \infty$, in the Heisenberg group and prove the Hölder continuity of horizontal gradient of minima.

1. Introduction

Following [40], we continue to study in this paper the regularity of minima of scalar variational integrals in the Heisenberg group $\mathbb{H}^n, n \geq 1$. Let Ω be a domain in $\mathbb{H}^n$ and $u : \Omega \to \mathbb{R}$ a function. We denote by $\mathfrak{X}u = (X_1u, X_2u, \ldots, X_{2n}u)$ the horizontal gradient of u. We study the following variational problem

$$I(u) = \int_\Omega f(\mathfrak{X}u)\, dx, \tag{1.1}$$

where the convex integrand function $f \in C^2(\mathbb{R}^{2n}; \mathbb{R})$ is of p-growth, $1 < p < \infty$. It satisfies the following growth and ellipticity conditions

$$\begin{aligned} (\delta + |z|^2)^{\frac{p-2}{2}} |\xi|^2 \leq \langle D^2 f(z)\xi, \xi\rangle &\leq L(\delta + |z|^2)^{\frac{p-2}{2}} |\xi|^2; \\ |Df(z)| &\leq L(\delta + |z|^2)^{\frac{p-2}{2}} |z| \end{aligned} \tag{1.2}$$

for all $z, \xi \in \mathbb{R}^{2n}$, where $\delta \geq 0, L \geq 1$ are constants.

It is easy to prove that a function in the horizontal Sobolev space $HW^{1,p}(\Omega)$ is a local minimizer of functional (1.1) if and only if it is a weak solution of the corresponding Euler-Lagrange equation of (1.1)

$$\mathrm{div}_H\big(Df(\mathfrak{X}u)\big) = \sum_{i=1}^{2n} X_i\big(D_i f(\mathfrak{X}u)\big) = 0. \tag{1.3}$$

where $Df = (D_1 f, D_2 f, \ldots, D_{2n} f)$ is the Euclidean gradient of f. See Section 2 for the definitions of horizontal Sobolev space $HW^{1,p}(\Omega)$, weak solutions and local minimizers.

A prototype example of integrand functions satisfying conditions (1.2) is

$$f(z) = \big(\delta + |z|^2\big)^{\frac{p}{2}}$$

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for a constant $\delta \geq 0$. Then the Euler-Lagrange equation (1.3) is reduced to the non-degenerate p-Laplacian equation

$$\operatorname{div}_H\Big(\big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}\mathfrak{X}u\Big) = 0, \tag{1.4}$$

when $\delta > 0$, and the p-Laplacian equation

$$\operatorname{div}_H\big(|\mathfrak{X}u|^{p-2}\mathfrak{X}u\big) = 0, \tag{1.5}$$

when $\delta = 0$. The weak solutions of equation (1.5) are called p-harmonic functions.

For the regularity of weak solutions of equation (1.3), the second author proved in [40] the following theorem, Theorem 1.1 of [40], from which follows the Lipschitz continuity of weak solutions for all $1 < p < \infty$. We remark that this result holds both for the non-degenerate case ($\delta > 0$) and for the degenerate one ($\delta = 0$). We also remark that it holds under a bit more general growth condition on the integrand function f than (1.2). Precisely, in [40] the integrand function f is assumed to satisfy

$$\begin{aligned} (\delta + |z|^2)^{\frac{p-2}{2}}|\xi|^2 \leq \langle D^2 f(z)\xi, \xi\rangle &\leq L(\delta + |z|^2)^{\frac{p-2}{2}}|\xi|^2; \\ |Df(z)| &\leq L(\delta + |z|^2)^{\frac{p-1}{2}} \end{aligned} \tag{1.6}$$

for all $z, \xi \in \mathbb{R}^{2n}$, where $\delta \geq 0, L \geq 1$ are constants.

Theorem 1.1. *Let $1 < p < \infty$, $\delta \geq 0$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (1.3) satisfying the structure condition (1.6). Then $\mathfrak{X}u \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^{2n})$. Moreover, for any ball $B_{2r} \subset \Omega$, we have that*

$$\sup_{B_r} |\mathfrak{X}u| \leq c\Big(\fint_{B_{2r}} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p}{2}}\, dx\Big)^{\frac{1}{p}}, \tag{1.7}$$

where $c > 0$ depends only on n, p, L.

Here and in the following, the ball B_r is defined with respect to the Carnot-Carathèodory metric (CC-metric) d; B_{2r} is the double size ball with the same center, see Section 2 for the definitions.

The second author also proved in [40] that the horizontal gradient of weak solutions of equation (1.3) is Hölder continuous when $p \geq 2$. We remark again that this result holds under the condition (1.6), and that it holds both for the non-degenerate case ($\delta > 0$) and for the degenerate one ($\delta = 0$).

Theorem 1.2. *Let $2 \leq p < \infty$, $\delta \geq 0$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (1.3) satisfying the structure condition (1.6). Then the horizontal gradient $\mathfrak{X}u$ is Hölder continuous. Moreover, there is a positive exponent $\alpha = \alpha(n,p,L) \leq 1$ such that for any ball $B_{r_0} \subset \Omega$ and any $0 < r \leq r_0$, we have*

$$\max_{1\leq l\leq 2n} \operatorname{osc}_{B_r} X_l u \leq c\Big(\frac{r}{r_0}\Big)^{\alpha}\Big(\fint_{B_{r_0}} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p}{2}}\, dx\Big)^{\frac{1}{p}}, \tag{1.8}$$

where $c > 0$ depends only on n, p, L.

We refer to the paper [40] and the references therein, e.g. [24, 19, 18, 26, 2, 3, 4, 8, 10, 17, 14, 15, 13, 33, 34, 32] for the earlier work on the regularity of weak solutions of equation (1.3).

The result in Theorem 1.2 leaves open the Hölder continuity of horizontal gradient of weak solutions for equation (1.3) in the case $1 < p < 2$. In this paper, we prove that the same result holds for this case, under the structure condition (1.2). This is the main result of this paper.

Theorem 1.3. *Let $1 < p < \infty$, $\delta \geq 0$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (1.3) satisfying the structure condition (1.2). Then the horizontal gradient $\mathfrak{X}u$ is Hölder continuous. Moreover, there is a positive exponent $\alpha = \alpha(n,p,L) \leq 1$ such that for any ball $B_{r_0} \subset \Omega$ and any $0 < r \leq r_0$, we have*

$$\max_{1\leq l\leq 2n} \operatorname{osc}_{B_r} X_l u \leq c\Big(\frac{r}{r_0}\Big)^{\alpha}\Big(\fint_{B_{r_0}} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p}{2}}\, dx\Big)^{\frac{1}{p}}, \tag{1.9}$$

where $c > 0$ depends only on n, p, L.

For $p \neq 2$, it is well known that weak solutions of equations of type (1.3) in the Euclidean spaces are of the class $C^{1,\alpha}$, that is, they have Hölder continuous derivatives, see [39, 29, 16, 12, 30, 37]. The $C^{1,\alpha}$-regularity is optimal when $p > 2$. This can been seen by examples. Theorem 1.3 shows that the regularity theory for equation (1.3) in the setting of Heisenberg group is similar to that in the setting of Euclidean spaces.

The proof of Theorem 1.3 is based on De Giorgi's method [11] and it works for all $1 < p < \infty$. The approach is similar to that of Tolksdorff [37] and Lieberman [31] in the setting of Euclidean spaces. The idea is to consider the double truncation of the horizontal derivative $X_l u, l = 1, 2, ..., 2n$, of the weak solution u to equation (1.3) satisfying the structure condition (1.2) with $\delta > 0$

$$v = \min\big(\mu(r)/8, \max(\mu(r)/4 - X_l u, 0)\big),$$

where

$$\mu(r) = \max_{1\leq i\leq 2n} \sup_{B_r} |X_i u|,$$

and $B_r \subset \Omega$ is a ball. The whole difficulties of this work lie in proving the following Caccioppoli type inequality for v. In the following lemma, $\eta \in C_0^\infty(B_r)$ is a non-negative cut-off function such that $0 \leq \eta \leq 1$ in B_r, $\eta = 1$ in $B_{r/2}$ and that $|\mathfrak{X}\eta| \leq 4/r$, $|\mathfrak{X}\mathfrak{X}\eta| \leq 16n/r^2$, $|T\eta| \leq 32n/r^2$ in B_r.

Lemma 1.1. *Let $\gamma > 1$ be a number. We have the following Caccioppoli type inequality*

$$\int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \leq c(\beta+2)^2 \frac{|B_r|^{1-1/\gamma}}{r^2} \mu(r)^4 \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{1/\gamma}$$

for all $\beta \geq 0$, where $c = c(n,p,L,\gamma) > 0$.

The proof of Lemma 1.1 is based on the integrability estimate for Tu, the vertical derivative of u, established in [40], see Lemma 2.4 and Lemma 2.5. To prove Lemma 1.1, we consider the equation for $X_l u$, see equations (2.3) and (2.4) of Lemma 2.1 in Section 2. We take the usual testing function

$$\varphi = \eta^{\beta+4} v^{\beta+3}$$

for equations (2.3) and (2.4), where $\beta \geq 0$. In the case $p \geq 2$, when equation (1.3) is degenerate, the proof of Lemma 1.1 is not difficult. On the contrary, in the case $1 < p < 2$, when equation (1.3) is singular, it is dedicated to prove the desired Caccioppoli inequality in Lemma 1.1. In order to prove Lemma 1.1, we prove two auxiliary lemmas, Lemma 3.1 and Lemma 3.2, where we establish the Caccioppoli type inequalities for $\mathfrak{X}u$ and Tu involving v. The essential feature of these inequalities is that we add weights such as the powers of $|\mathfrak{X}u|$, in order to deal with the singularity of equation (1.3) in the case $1 < p < 2$. The proof of Lemma 1.1 is given in Section 3.

Once Lemma 1.1 is established, the proof of Theorem 1.3 is similar to that in the setting of Euclidean spaces. We may follow the same line as that in [40]. The proof of Theorem 1.3 is given in Section 4. The proof of the auxiliary lemma, Lemma 3.1, is given in the Appendix.

2. Preliminaries

In this section, we fix our notation and introduce the Heisenberg group $\mathbb{H}^n$ and the known results on the sub-elliptic equation (1.3).

Throughout this paper, c is a positive constant, which may vary from line to line. Except explicitly being specified, it depends only on the dimension n of the Heisenberg group, and on the constants p and L in the structure condition (1.2). But, it does not depend on δ in (1.2).

2.1. **Heisenberg group $\mathbb{H}^n$.** The Heisenberg group $\mathbb{H}^n$ is identified with the Euclidean space $\mathbb{R}^{2n+1}, n \geq 1$. The group multiplication is given by

$$xy = (x_1 + y_1, \dots, x_{2n} + y_{2n}, t + s + \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i))$$

for points $x = (x_1, \dots, x_{2n}, t), y = (y_1, \dots, y_{2n}, s) \in \mathbb{H}^n$. The left invariant vector fields corresponding to the canonical basis of the Lie algebra are

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t,$$

and the only non-trivial commutator

$$T = \partial_t = [X_i, X_{n+i}] = X_i X_{n+i} - X_{n+i} X_i$$

for $1 \leq i \leq n$. We denote by $\mathfrak{X} = (X_1, X_2, \dots, X_{2n})$ the horizontal gradient. The second horizontal derivatives are given by the horizontal Hessian $\mathfrak{X}\mathfrak{X}u$ of a function u, with entries $X_i(X_j u), i, j = 1, \dots, 2n$. Note that it is not symmetric, in general. The standard Euclidean gradient of a function v in $\mathbb{R}^k$ is denoted by $Dv = (D_1 v, \dots, D_k v)$ and the Hessian matrix by $D^2 v$.

The Haar measure in $\mathbb{H}^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$. We denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{H}^n$ and by

$$\fint_E f\,dx = \frac{1}{|E|}\int_E f\,dx$$

the average of an integrable function f over set E.

A ball $B_\rho(x) = \{y \in \mathbb{H}^n : d(y, x) < \rho\}$ is defined with respect to the Carnot-Carathèodory metric (CC-metric) d. The CC-distance of two points in $\mathbb{H}^n$ is the length of the shortest horizontal curve joining them.

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{H}^n$ be an open set. The horizontal Sobolev space $HW^{1,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that the horizontal weak gradient $\mathfrak{X}u$ is also in $L^p(\Omega)$. $HW^{1,p}(\Omega)$, equipped with the norm

$$\| u \|_{HW^{1,p}(\Omega)} = \| u \|_{L^p(\Omega)} + \| \mathfrak{X}u \|_{L^p(\Omega)},$$

is a Banach space. $HW^{1,p}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ with this norm. We denote the local space by $HW^{1,p}_{\mathrm{loc}}(\Omega)$.

The following Sobolev inequality hold for functions $u \in HW_0^{1,q}(B_r)$, $1 \le q < Q = 2n+2$,

$$\Big(\fint_{B_r} |u|^{\frac{Qq}{Q-q}}\, dx \Big)^{\frac{Q-q}{Qq}} \le cr \Big(\fint_{B_r} |\mathfrak{X} u|^q \, dx \Big)^{\frac{1}{q}}, \tag{2.1}$$

where $B_r \subset \mathbb{H}^n$ is a ball and $c = c(n,q) > 0$.

2.2. **Known results on sub-elliptic equation** (1.3)**.** A function $u \in HW^{1,p}(\Omega)$ is a local minimizer of functional (1.1), that is,

$$\int_\Omega f(\mathfrak{X} u)\, dx \le \int_\Omega f(\mathfrak{X} u + \mathfrak{X}\varphi)\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$, if and only if it is a weak solution of equation (1.3), that is,

$$\int_\Omega \langle Df(\mathfrak{X} u), \mathfrak{X}\varphi \rangle \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$.

In the rest of this subsection, $u \in HW^{1,p}(\Omega)$ is a weak solution of equation (1.3) satisfying the structure condition (1.2) with $\delta > 0$. By Theorem 1.1, we have that

$$\mathfrak{X} u \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^{2n}).$$

Thanks to this and to the fact that we assume $\delta > 0$, equation (1.3) is uniformly elliptic. Then we can apply Capogna's results in [3]. Theorem 1.1 and Theorem 3.1 of [3] show that $\mathfrak{X} u$ and Tu are Hölder continuous in Ω, and that

$$\mathfrak{X} u \in HW^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^{2n}), \quad Tu \in HW^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega). \tag{2.2}$$

With the above regularity, we can easily prove the following three lemmas. They are Lemma 3.1, Lemma 3.2 and Lemma 3.3 of [40], respectively. We refer to [40] for the proofs.

Lemma 2.1. *Let $v_l = X_l u, l = 1, 2, \ldots, n$. Then v_l is a weak solution of*

$$\sum_{i,j=1}^{2n} X_i\big(D_jD_i f(\mathfrak{X} u) X_j v_l\big) + \sum_{i=1}^{2n} X_i\big(D_{n+l}D_i f(\mathfrak{X} u) Tu\big) + T\big(D_{n+l} f(\mathfrak{X} u)\big) = 0; \tag{2.3}$$

Let $v_{n+l} = X_{n+l} u, l = 1, 2, \ldots, n$. Then v_{n+l} is a weak solution of

$$\sum_{i,j=1}^{2n} X_i\big(D_jD_i f(\mathfrak{X} u) X_j v_{n+l}\big) - \sum_{i=1}^{2n} X_i\big(D_l D_i f(\mathfrak{X} u) Tu\big) - T\big(D_l f(\mathfrak{X} u)\big) = 0; \tag{2.4}$$

Lemma 2.2. *Tu is a weak solution of*

$$\sum_{i,j=1}^{2n} X_i\big(D_jD_i f(\mathfrak{X} u) X_j (Tu)\big) = 0. \tag{2.5}$$

Lemma 2.3. *For any $\beta \ge 0$ and all $\eta \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \eta^2 \big(\delta + |\mathfrak{X} u|^2\big)^{\frac{p-2}{2}} |Tu|^\beta |\mathfrak{X}(Tu)|^2 \, dx \le \frac{c}{(\beta+1)^2} \int_\Omega |\mathfrak{X}\eta|^2 \big(\delta + |\mathfrak{X} u|^2\big)^{\frac{p-2}{2}} |Tu|^{\beta+2}\, dx.$$

where $c = c(n,p,L) > 0$.

The following lemma is Corollary 3.2 of [40]. It shows the integrability of Tu. It is critical for the proof of the Hölder continuity of the horizontal gradient of solutions u in [40].

Lemma 2.4. *For any $\beta \geq 2$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$\int_\Omega \eta^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|Tu|^{\beta+2}\,dx \leq c(\beta)K^{\frac{\beta+2}{2}}\int_{spt(\eta)}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p+\beta}{2}}\,dx,$$

where $K=\|\mathfrak{X}\eta\|^2_{L^\infty}+\|\eta T\eta\|_{L^\infty}$ and $c(\beta)>0$ depends on n,p,L and β.

In this paper, we need the following version of Lemma 2.4, which is a bit stronger. The reason that this stronger version holds is that we have a stronger structure condition (1.2) than that one (1.6) in [40].

Lemma 2.5. *For any $\beta \geq 2$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$\int_\Omega \eta^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|Tu|^{\beta+2}\,dx \leq c(\beta)K^{\frac{\beta+2}{2}}\int_{spt(\eta)}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^{\beta+2}\,dx,$$

where $K=\|\mathfrak{X}\eta\|^2_{L^\infty}+\|\eta T\eta\|_{L^\infty}$ and $c(\beta)>0$ depends on n,p,L and β.

The following corollary follows easily from Lemma 2.3 and Lemma 2.5.

Corollary 2.1. *For any $q \geq 4$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \eta^{q+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|Tu|^{q-2}|\mathfrak{X}(Tu)|^2\,dx \leq c(q)K^{\frac{q+2}{2}}\int_{spt(\eta)}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^{q}\,dx,$$

where $K=\|\mathfrak{X}\eta\|^2_{L^\infty}+\|\eta T\eta\|_{L^\infty}$ and $c(q)=c(n,p,L,q)>0$.

In the rest of this subsection, we comment on the proof of Lemma 2.5. The proof of Lemma 2.5 is almost the same as that of Lemma 2.4 in [40]; it requires only minor modifications. Lemma 2.4 follows from two lemmas, that is, Lemma 3.4, Lemma 3.5 in [40]. To prove Lemma 2.5, we need stronger versions of Lemma 3.4 and Lemma 3.5 of [40], which we state here. The following lemma is a stronger version of Lemma 3.4 of [40].

Lemma 2.6. *For any $\beta \geq 0$ and all $\eta \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned}\int_\Omega \eta^2\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^\beta|\mathfrak{X}\mathfrak{X}u|^2\,dx \leq& c\int_\Omega\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^{\beta+2}\,dx\\ &+c(\beta+1)^4\int_\Omega \eta^2\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^\beta|Tu|^2\,dx,\end{aligned}$$

where $c=c(n,p,L)>0$.

The proof of Lemma 2.6 follows the same line as that of Lemma 3.4 of [40] with minor modifications. To prove Lemma 3.4 of [40], one uses $\varphi=\eta^2\big(\delta+|\mathfrak{X}u|^2\big)^{\beta/2}X_lu$ as a testing function for equations (2.3) when $l=1,2,...,n$ and for equation (2.4) when $l=n+1,n+2,...,2n$. Now, to prove Lemma 2.6, we use instead the testing function $\varphi=\eta^2|\mathfrak{X}u|^\beta X_lu$. The proof then is the same as that of Lemma 3.4 of [40] with obvious changes. To get through the proof, we remark that the structure condition (1.2) is essential. We omit the details of the proof of Lemma 2.6.

The following lemma is a stronger version of Lemma 3.5 of [40].

Lemma 2.7. *For any $\beta \geq 2$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \eta^{\beta+2}\big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|Tu|^\beta|\mathfrak{X}\mathfrak{X}u|^2\,dx$$
$$\leq c(\beta+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega \eta^\beta\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|Tu|^{\beta-2}|\mathfrak{X}\mathfrak{X}u|^2\,dx,$$

where $c = c(n, p, L) > 0$.

The proof of Lemma 2.7 is almost the same as that of Lemma 3.5, with obvious minor changes. The only difference is that we use the structure condition (1.2) whenever the structure condition (1.6) is used in the proof of Lemma 3.5 in [40]. We omit the details.

Once Lemma 2.6 and Lemma 2.7 are established, the proof of Lemma 2.5 is exactly the same as that of Lemma 2.4 in [40].

3. Proof of the main lemma, Lemma 1.1

Throughout this section, $u \in HW^{1,p}(\Omega)$ is a weak solution of equation (1.3) satisfying the structure condition (1.2) with $\delta > 0$. For any ball $B_r \subset \Omega$, we denote for $i = 1, 2, ..., 2n$,

$$\mu_i(r) = \sup_{B_r}|X_iu|, \quad \mu(r) = \max_{1\leq i\leq 2n}\mu_i(r). \tag{3.1}$$

Now fix $l \in \{1, 2, .., 2n\}$. We consider the following double truncation of X_lu

$$v = \min\big(\mu(r)/8, \max(\mu(r)/4 - X_lu, 0)\big). \tag{3.2}$$

We denote

$$E = \{x \in \Omega : \mu(r)/8 < X_lu < \mu(r)/4\}. \tag{3.3}$$

We note the following trivial inequality, which we use several times in this section

$$\mu(r)/8 \leq |\mathfrak{X}u| \leq (2n)^{1/2}\mu(r) \quad \text{in } E \cap B_r. \tag{3.4}$$

It follows from the regularity results (2.2) that

$$\mathfrak{X}v \in L^2_{\text{loc}}(\Omega; \mathbb{R}^{2n}), \quad Tv \in L^2_{\text{loc}}(\Omega) \tag{3.5}$$

and moreover

$$\mathfrak{X}v = \begin{cases} -\mathfrak{X}X_lu & \text{a.e. in } E; \\ 0 & \text{a.e. in } \Omega\setminus E, \end{cases} \qquad Tv = \begin{cases} -TX_lu & \text{a.e. in } E; \\ 0 & \text{a.e. in } \Omega\setminus E. \end{cases} \tag{3.6}$$

We note that the function

$$h(t) = \big(\delta + t^2\big)^{\frac{p-2}{2}}t^q$$

is non-decreasing on $[0, \infty)$ if $\delta \geq 0$ and $q \geq 0$ such that $p + q - 2 \geq 0$. Thus we have the following inequality, which is used several times in this section

$$\big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^q \leq c(n, p, q)\big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}}\mu(r)^q \quad \text{in } B_r, \tag{3.7}$$

where $c(n, p, q) = (2n)^{(q+p-2)/2}$ if $p \geq 2$ and $c(n, p, q) = (2n)^{q/2}$ if $1 < p < 2$.

To prove Lemma 1.1, we need the following two lemmas. The first lemma is similar to Lemma 3.3 of [40]. In this lemma, we prove a weighted Caccioppoli inequality for $\mathfrak{X}u$ involving v. It has an extra weight $|\mathfrak{X}u|^2$, comparing to that in Lemma 3.3 of [40]. This is essential for us to deal with the case $1 < p < 2$ when equation (1.3) is singular. The proof is also similar to that of Lemma 3.3 of [40]. It is standard, but lengthy. We give a detailed proof in the Appendix.

Lemma 3.1. *Let $1 < p < \infty$. For any $\beta \ge 0$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$
\begin{aligned}
&\int_\Omega \eta^{\beta+2} v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2 \, dx \\
&\qquad \le c(\beta+2)^2 \int_\Omega \eta^\beta \big(|\mathfrak{X}\eta|^2 + \eta|T\eta|\big) v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 \, dx \\
&\qquad\quad + c(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^\beta \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}v|^2 \, dx \\
&\qquad\quad + c \int_\Omega \eta^{\beta+2} v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |Tu|^2 \, dx,
\end{aligned} \tag{3.8}
$$

where $c = c(n,p,L) > 0$.

In the following is the second lemma that we need for the proof of Lemma 1.1, where we prove a weighted Caccioppoli inequality for Tu involving v. It has a weight $|\mathfrak{X}u|^4$, which is needed for us to deal with the case $1 < p < 2$. To state the lemma, we fix, throughout the rest of this section, a ball $B_r \subset \Omega$ and a cut-off function $\eta \in C_0^\infty(B_r)$ that satisfies

$$
0 \le \eta \le 1 \quad \text{in } B_r, \quad \eta = 1 \quad \text{in } B_{r/2} \tag{3.9}
$$

and

$$
|\mathfrak{X}\eta| \le 4/r, \quad |\mathfrak{X}\mathfrak{X}\eta| \le 16n/r^2, \quad |T\eta| \le 32n/r^2 \quad \text{in } B_r. \tag{3.10}
$$

Lemma 3.2. *Let $B_r \subset \Omega$ be a ball and $\eta \in C_0^\infty(B_r)$ be a cut-off function satisfying* (3.9) *and* (3.10). *Let $\tau \in (1/2, 1)$ and $\gamma \in (1,2)$ be two fixed numbers. Then, for any $\beta \ge 0$, we have*

$$
\begin{aligned}
&\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2 \, dx \\
&\qquad \le c(\beta+2)^{2\tau} \frac{|B_r|^{1-\tau}}{r^{2(2-\tau)}} \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^6 \, J^\tau,
\end{aligned} \tag{3.11}
$$

where $c = c(n,p,L,\tau,\gamma) > 0$ and

$$
J = \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2 \, dx \; + \; \mu(r)^4 \frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta} \, dx \Big)^{\frac{1}{\gamma}}. \tag{3.12}
$$

Proof. We denote by M the left hand side of (3.11)

$$
M = \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2 \, dx, \tag{3.13}
$$

where $1/2 < \tau < 1$. We use the following function

$$
\varphi = \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^4 \, Tu
$$

as a testing function for equation (2.5). We obtain that

$$
\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^4 D_jD_if(\mathfrak{X}u) X_jTu\, X_iTu\, dx \\
=&-(\tau(\beta+2)+4)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+3}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^4 TuD_jD_if(\mathfrak{X}u) X_jTu\, X_i\eta\, dx \\
&-\tau(\beta+4)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)-1} |\mathfrak{X}u|^4 TuD_jD_if(\mathfrak{X}u) X_jTu\, X_iv\, dx \\
&-4\int_\Omega \sum_{i,j,k=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} |\mathfrak{X}u|^2 X_kuTuD_jD_if(\mathfrak{X}u) X_jTu\, X_iX_ku\, dx \\
=&K_1+K_2+K_3,
\end{aligned}
\tag{3.14}
$$

where the integrals in the right hand side of (3.14) are denoted by K_1, K_2, K_3 in order, respectively. We estimate both sides of (3.14) as follows. For the left hand side, we have by the structure condition (1.2) that

$$
\text{left of (3.14)} \geq \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2\, dx = M. \tag{3.15}
$$

For the right hand side of (3.14), we estimate each item $K_i, i=1,2,3$, one by one. To this end, we denote

$$
\tilde{K} = \int_\Omega \eta^{(2\tau-1)(\beta+2)+6}\, v^{(2\tau-1)(\beta+4)} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |Tu|^2 |\mathfrak{X}(Tu)|^2\, dx. \tag{3.16}
$$

First, we estimate K_1 by the structure condition (1.2) and Hölder's inequality. We have

$$
\begin{aligned}
|K_1| \leq& c(\beta+2)\int_\Omega \eta^{\tau(\beta+2)+3}\, v^{\tau(\beta+4)} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |Tu||\mathfrak{X}(Tu)||\mathfrak{X}\eta|\, dx \\
\leq& c(\beta+2)\tilde{K}^{\frac12}\Big(\int_\Omega \eta^{\beta+2} v^{\beta+4} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}\eta|^2\, dx\Big)^{\frac12},
\end{aligned}
\tag{3.17}
$$

where $c = c(n,p,L,\tau) > 0$.

Second, we estimate K_2 also by the structure condition (1.2) and Hölder's inequality. We have

$$
\begin{aligned}
|K_2| \leq& c(\beta+2)\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)-1} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |Tu||\mathfrak{X}(Tu)||\mathfrak{X}v| dx \\
\leq& c(\beta+2)\tilde{K}^{\frac12}\Big(\int_\Omega \eta^{\beta+4} v^{\beta+2} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}v|^2\, dx\Big)^{\frac12}.
\end{aligned}
\tag{3.18}
$$

Finally, we estimate K_3. In the following, the first inequality follows from the structure condition (1.2), the second from Hölder's inequality and the third from

Lemma 3.1. We have

$$
\begin{aligned}
|K_3| &\leq c\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|Tu||\mathfrak{X}(Tu)||\mathfrak{X}\mathfrak{X}u|\,dx\\
&\leq c\tilde{K}^{\frac12}\Big(\int_\Omega \eta^{\beta+4}v^{\beta+4}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\Big)^{\frac12}\\
&\leq c\,\tilde{K}^{\frac12}\,I^{\frac12},
\end{aligned}
\tag{3.19}
$$

where I is the right hand side of (3.8) in Lemma 3.1

$$
\begin{aligned}
I =&c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta+4}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta+4}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\\
&+c\int_\Omega \eta^{\beta+4}v^{\beta+4}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|Tu|^2\,dx.
\end{aligned}
\tag{3.20}
$$

and $c=c(n,p,L)>0$. Notice that the integrals on the right hand side of (3.17) and (3.18) are both controlled from above by I. Hence, we can combine (3.17), (3.18) and (3.19) to obtain that

$$|K_1|+|K_2|+|K_3|\leq\ c\tilde{K}^{\frac12}I^{\frac12},$$

from which, together with the estimate (3.15) for the left hand side of (3.14), it follows that

$$M\leq c\tilde{K}^{\frac12}I^{\frac12}, \tag{3.21}$$

where $c=c(n,p,L,\tau)>0$. Now, we estimate $\tilde{K}$ by Hölder's inequality as follows.

$$
\begin{aligned}
\tilde{K}\ \leq&\Big(\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}(Tu)|^2\,dx\Big)^{\frac{2\tau-1}{\tau}}\\
&\times\Big(\int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|Tu|^{\frac{2\tau}{1-\tau}}|\mathfrak{X}(Tu)|^2\,dx\Big)^{\frac{1-\tau}{\tau}}\\
=&M^{\frac{2\tau-1}{\tau}}\,G^{\frac{1-\tau}{\tau}},
\end{aligned}
\tag{3.22}
$$

where M is as in (3.13) and we denote by G the second integral on the right hand side of (3.22)

$$G=\int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|Tu|^{\frac{2\tau}{1-\tau}}|\mathfrak{X}(Tu)|^2\,dx. \tag{3.23}$$

Now (3.22) and (3.21) yield that

$$M\leq cG^{1-\tau}I^\tau, \tag{3.24}$$

where $c=c(n,p,L,\tau)>0$. To estimate K, we estimate G and I from above. We estimate G by Corollary 2.1 with $q=2/(1-\tau)$, and we obtain that

$$
\begin{aligned}
G\leq&c\mu(r)^4\int_\Omega \eta^{q+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|Tu|^{q-2}|\mathfrak{X}(Tu)|^2\,dx\\
\leq&\frac{c}{r^{q+2}}\mu(r)^4\int_{B_r}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^q\,dx\\
\leq&\frac{c}{r^{q+2}}|B_r|\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}}\mu(r)^{q+4},
\end{aligned}
\tag{3.25}
$$

where $c = c(n, p, L, \tau) > 0$ and in the last inequality we used (3.7).

Now, we fix $1 < \gamma < 2$ and estimate each term of I in (3.20) as follows. For the first term of I, we have by Hölder's inequality and (3.7) that

$$\begin{aligned}&\int_\Omega \eta^{\beta+2} v^{\beta+4} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 \big(|\mathfrak{X}\eta|^2 + \eta|T\eta|\big)\, dx \\ &\qquad \le \frac{c}{r^2} \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^8 |B_r|^{1-\frac{1}{\gamma}} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{\frac{1}{\gamma}}.\end{aligned} \tag{3.26}$$

For the second term of I, we have by (3.7) that

$$\begin{aligned}&\int_\Omega \eta^{\beta+4} v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}v|^2\, dx \\ &\qquad \le c\big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^4 \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx\end{aligned} \tag{3.27}$$

For the third term of I, we have that

$$\begin{aligned}&\int_\Omega \eta^{\beta+4} v^{\beta+4} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |Tu|^2\, dx \\ &\quad \le \Big(\int_\Omega \eta^{\frac{2\gamma}{\gamma-1}} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |Tu|^{\frac{2\gamma}{\gamma-1}}\, dx\Big)^{1-\frac{1}{\gamma}} \\ &\quad\quad \times \Big(\int_\Omega \eta^{\gamma(\beta+2)} v^{\gamma(\beta+4)} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2\, dx\Big)^{\frac{1}{\gamma}} \\ &\quad \le \frac{c}{r^2} \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^8 |B_r|^{1-\frac{1}{\gamma}} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{\frac{1}{\gamma}}\end{aligned} \tag{3.28}$$

where $c = c(n, p, L, \gamma) > 0$. Here in the above inequalities, the first one follows from Hölder's inequality and the second from Lemma 2.5 and (3.7). Therefore, the estimates for three items of I above (3.26), (3.27) and (3.28) give us the following one for I

$$I \le c(\beta+2)^2 \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^4 J, \tag{3.29}$$

where J is defined as in (3.12)

$$J = \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \;+\; \mu(r)^4 \frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{\frac{1}{\gamma}}.$$

Now from the estimates (3.25) for G and (3.29) for I, we obtain the desired estimate for M by (3.24). Combing (3.25), (3.29) and (3.24), we end up with

$$M \le c(\beta+2)^{2\tau} \frac{|B_r|^{1-\tau}}{r^{2(2-\tau)}} \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^6 J^\tau, \tag{3.30}$$

where $c = c(n, p, L, \tau, \gamma) > 0$. This completes the proof. □

Now we prove the main lemma, Lemma 1.1. We restate Lemma 1.1 here.

Lemma 3.3. *Let $\gamma > 1$ be a number and for $B_r \subset \Omega$, $\eta \in C_0^\infty(B_r)$, be a cut-off function satisfying (3.9) and (3.10). We have the following Caccioppoli type inequality*

$$\int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \;\le\; c(\beta+2)^2 \mu(r)^4 \frac{|B_r|^{1-1/\gamma}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{1/\gamma} \tag{3.31}$$

for all $\beta \geq 0$*, where* $c = c(n, p, L, \gamma) > 0$*.*

Proof. We note that we may assume that $\gamma < 3/2$, since otherwise we can apply Hölder's inequality to the integral in the right hand side of the claimed inequality (3.31). So, we fix $1 < \gamma < 3/2$. Recall that

$$v = \min\big(\mu(r)/8, \max(\mu(r)/4 - X_l u, 0)\big),$$

where $l \in \{1, 2, ..., 2n\}$. We only prove the lemma for $l \in \{1, 2, ..., n\}$; we can prove the lemma similarly for $l \in \{n+1, n+2, ..., 2n\}$. Now fix $l \in \{1, 2, ..., n\}$. Let $\beta \geq 0$ and $\eta \in C_0^\infty(B_r)$ be a cut-off function satisfying (3.9) and (3.10). We use

$$\varphi = \eta^{\beta+4} v^{\beta+3}$$

as a test function for equation (2.3) to obtain that

$$\begin{aligned} -\int_\Omega \sum_{i,j=1}^{2n} D_j D_i f(\mathfrak{X}u) X_j X_l u X_i \varphi \, dx &= \int_\Omega \sum_{i=1}^{2n} D_{n+l} D_i f(\mathfrak{X}u) Tu X_i \varphi \, dx \\ &\quad - \int_\Omega T\big(D_{n+l} f(\mathfrak{X}u)\big) \varphi \, dx. \end{aligned} \tag{3.32}$$

Note that

$$X_i \varphi = (\beta+3)\eta^{\beta+4} v^{\beta+2} X_i v + (\beta+4)\eta^{\beta+3} v^{\beta+3} X_i \eta.$$

Thus (3.32) becomes

$$\begin{aligned} &-(\beta+3)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_j D_i f(\mathfrak{X}u) X_j X_l u X_i v \, dx \\ &\quad = (\beta+4)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+3} v^{\beta+3} D_j D_i f(\mathfrak{X}u) X_j X_l u X_i \eta \, dx \\ &\qquad + (\beta+4)\int_\Omega \sum_{i=1}^{2n} \eta^{\beta+3} v^{\beta+3} D_{n+l} D_i f(\mathfrak{X}u) Tu X_i \eta \, dx \\ &\qquad + (\beta+3)\int_\Omega \sum_{i=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_{n+l} D_i f(\mathfrak{X}u) X_i v \, Tu \, dx \\ &\qquad - \int_\Omega \eta^{\beta+4} v^{\beta+3} T\big(D_{n+l} f(\mathfrak{X}u)\big) \, dx. \end{aligned} \tag{3.33}$$

Note that

$$X_j X_l - X_l X_j = 0, \qquad \text{if } j \neq n+l,$$

and that

$$X_{n+l} X_l - X_l X_{n+l} = -T.$$

Therefore we have

$$\begin{aligned} &\sum_{i,j=1}^{2n} D_j D_i f(\mathfrak{X}u) X_j X_l u X_i \eta + \sum_{i=1}^{2n} D_{n+l} D_i f(\mathfrak{X}u) Tu X_i \eta \\ &= \sum_{i,j=1}^{2n} D_j D_i f(\mathfrak{X}u) X_l X_j u X_i \eta = \sum_{i=1}^{2n} X_l\big(D_i f(\mathfrak{X}u)\big) X_i \eta. \end{aligned}$$

Now we can combine the first two integrals in the right hand side of (3.33) by the above equality. Then (3.33) becomes

$$
\begin{aligned}
&-(\beta+3)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_j D_i f(\mathfrak{X}u) X_j X_l u X_i v\, dx \\
&\quad= (\beta+4)\int_\Omega \sum_{i=1}^{2n} \eta^{\beta+3} v^{\beta+3} X_l\big(D_i f(\mathfrak{X}u)\big) X_i\eta\, dx \\
&\qquad+ (\beta+3)\int_\Omega \sum_{i=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_{n+l} D_i f(\mathfrak{X}u) X_i v T u\, dx \\
&\qquad- \int_\Omega \eta^{\beta+4} v^{\beta+3} T\big(D_{n+l} f(\mathfrak{X}u)\big)\, dx \\
&\quad= I_1 + I_2 + I_3.
\end{aligned} \tag{3.34}
$$

Here we denote the terms in the right hand side of (3.34) by I_1, I_2, I_3, respectively.

We will estimate both sides of (3.34) as follows. For the left hand side, we have by the structure condition (1.2) that

$$
\begin{aligned}
\text{left of (3.34)} &\geq (\beta+3)\int_\Omega \eta^{\beta+4} v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}v|^2\, dx \\
&\geq c_0(\beta+2)\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx,
\end{aligned} \tag{3.35}
$$

where $c_0 = c_0(n,p,L) > 0$. Here we used (3.6) and (3.4).

For the right hand side of (3.34), we claim that each item I_1, I_2, I_3 satisfies the following estimate

$$
\begin{aligned}
|I_m| &\leq \frac{c_0}{6}(\beta+2)\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \\
&\quad+ c(\beta+2)^3 \frac{|B_r|^{1-1/\gamma}}{r^2}\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^4 \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{1/\gamma},
\end{aligned} \tag{3.36}
$$

where $m = 1,2,3$, $1 < \gamma < 3/2$ and c is a constant depending only on n, p, L and γ. Then the lemma follows from the estimate (3.35) for the left hand side of (3.34) and the above claim (3.36) for each item in the right. This completes the proof of the lemma, modulo the proof of the claim (3.36).

In the rest of the proof, we estimate I_1, I_2, I_3 one by one. First, for I_1, we have by integration by parts that

$$
I_1 = -(\beta+4)\int_\Omega \sum_{i=1}^{2n} D_i f(\mathfrak{X}u) X_l\big(\eta^{\beta+3} v^{\beta+3} X_i\eta\big)\, dx,
$$

from which it follows by the structure condition (1.2) that

$$
\begin{aligned}
|I_1| \le\,& c(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^{\beta+3} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u| \big(|\mathfrak{X}\eta|^2+\eta|\mathfrak{X}\mathfrak{X}\eta|\big)\, dx \\
&+ c(\beta+2)^2 \int_\Omega \eta^{\beta+3} v^{\beta+2} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u||\mathfrak{X}v||\mathfrak{X}\eta|\, dx \\
\le\,& \frac{c}{r^2}(\beta+2)^2 \big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^4 \int_{B_r} \eta^\beta v^\beta\, dx \\
&+ \frac{c}{r}(\beta+2)^2 \big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^2 \int_{B_r} \eta^{\beta+2} v^{\beta+1} |\mathfrak{X}v|\, dx,
\end{aligned} \tag{3.37}
$$

where $c = c(n,p,L) > 0$. Here the second inequality follows from (3.7), from the definitions of $\mu(r)$ and v, and from the factor that the support of η lies in B_r. Now we apply Young's inequality to the last term of inequality (3.37) to end up with the following estimate for I_1.

$$
\begin{aligned}
|I_1| \le\,& \frac{c_0}{6}(\beta+2)\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \\
&+ \frac{c}{r^2}(\beta+2)^3 \big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^4 \int_{B_r} \eta^\beta v^\beta\, dx,
\end{aligned} \tag{3.38}
$$

where $c = c(n,p,L) > 0$ and c_0 is the same constant as in (3.35). Now the claimed estimate (3.36) for I_1 follows from the above estimate (3.38) and Hölder's inequality.

Second, to estimate I_2, we have by the structure condition (1.2) that

$$
|I_2| \le c(\beta+2) \int_\Omega \eta^{\beta+4} v^{\beta+2} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}v||Tu|\, dx,
$$

from which it follows by Hölder's inequality that

$$
\begin{aligned}
|I_2| \le\,& c(\beta+2)\Big(\int_E \eta^{\beta+4} v^{\beta+2} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}v|^2\, dx\Big)^{\frac12} \\
&\times \Big(\int_E \eta^{\gamma(\beta+2)} v^{\gamma(\beta+2)} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}\, dx\Big)^{\frac{1}{2\gamma}} \\
&\times \Big(\int_\Omega \eta^q \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |Tu|^q\, dx\Big)^{\frac1q},
\end{aligned} \tag{3.39}
$$

where $q = 2\gamma/(\gamma-1)$. Here we used (3.6) so that in the second integral we can put the integration domain to be the set E, defined as in (3.3). This is critical, otherwise we would not have estimate for this integral and for the first integral in the case $1 < p < 2$. But now in set E we have (3.4), and we have the following estimates for these two integrals for the full range $1 < p < \infty$.

$$
\int_E \eta^{\beta+4} v^{\beta+2} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}v|^2\, dx \le c\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx, \tag{3.40}
$$

and

$$
\int_E \eta^{\gamma(\beta+2)} v^{\gamma(\beta+2)} \big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}\, dx \le c\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^{2\gamma} \int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx, \tag{3.41}
$$

where $c = c(n,p) > 0$. We estimate the last integral in the right hand side of (3.39) by Lemma 2.5. We have

$$\begin{aligned}\int_\Omega \eta^q \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |Tu|^q \, dx &\le \frac{c}{r^q} \int_{B_r} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^q \, dx \\ &\le \frac{c|B_r|}{r^q} \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^q,\end{aligned} \tag{3.42}$$

where $c = c(n,p,L,\gamma) > 0$. Here we used (3.7) again. Now combining the above three estimates (3.40), (3.41) and (3.42) for the three integrals in (3.39) respectively, we end up with the following estimate for I_2

$$|I_2| \le c(\beta+2) \frac{|B_r|^{\frac{1}{q}}}{r} \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^2 \Big(\int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2 \, dx \Big)^{\frac12} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta} \, dx \Big)^{\frac{1}{2\gamma}},$$

from which, together with Young's inequality, the claim (3.36) for I_2 follows.

Finally, we prove (3.36) for I_3. Recall that

$$I_3 = - \int_\Omega \eta^{\beta+4} v^{\beta+3} T\big(D_{n+l} f(\mathfrak{X}u)\big) \, dx.$$

Due to the regularity (3.5) for v, integration by parts yields

$$\begin{aligned} I_3 &= \int_\Omega D_{n+l} f(\mathfrak{X}u) T\big(\eta^{\beta+4} v^{\beta+3}\big) \, dx \\ &= (\beta+4) \int_\Omega \eta^{\beta+3} v^{\beta+3} D_{n+l} f(\mathfrak{X}u) T\eta \, dx \\ &\quad + (\beta+3) \int_\Omega \eta^{\beta+4} v^{\beta+2} D_{n+l} f(\mathfrak{X}u) Tv \, dx = I_3^1 + I_3^2, \end{aligned} \tag{3.43}$$

where we denote the last two integrals in the above equality by I_3^1 and I_3^2, respectively. The estimate for I_3^1 is easy. By the structure condition (1.2) and by (3.7), we have

$$\begin{aligned} |I_3^1| &\le c(\beta+2) \int_\Omega \eta^{\beta+3} v^{\beta+3} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u| |T\eta| \, dx \\ &\le \frac{c}{r^2} \big(\delta + \mu(r)^2\big)^{\frac{p-2}{2}} \mu(r)^4 \int_{B_r} \eta^\beta v^\beta \, dx. \end{aligned} \tag{3.44}$$

Thus by Hölder's inequality, I_3^1 satisfies estimate (3.36). Now we estimate I_3^2. We note that by (3.6) and the structure condition (1.2) we have

$$|I_3^2| \le c(\beta+2) \int_E \eta^{\beta+4} v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u| |\mathfrak{X}(Tu)| \, dx, \tag{3.45}$$

where the set E is

$$E = \{x \in \Omega : \mu(r)/8 < X_l u < \mu(r)/4\},$$

defined as in (3.3). We continue to estimate I_3^2 by Hölder's inequality

$$\begin{aligned} |I_3^2| \le c(\beta+2) &\Big(\int_E \eta^{(2-\gamma)(\beta+2)+4} v^{(2-\gamma)(\beta+4)} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |\mathfrak{X}(Tu)|^2 \, dx \Big)^{\frac12} \\ &\times \Big(\int_E \eta^{\gamma(\beta+2)} v^{\gamma\beta+4(\gamma-1)} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} \, dx \Big)^{\frac12}. \end{aligned}$$

We remark that in set E we have (3.4). Thus

$$|I_3^2| \le c(\beta+2)\big(\delta+\mu(r)^2\big)^{\frac{p-2}{4}}\mu(r)^{2(\gamma-1)-1}M^{\frac12}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac12}, \tag{3.46}$$

where

$$M=\int_\Omega \eta^{(2-\gamma)(\beta+2)+4}\,v^{(2-\gamma)(\beta+4)}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}(Tu)|^2\,dx. \tag{3.47}$$

Now we are in a position to apply Lemma 3.2 to estimate M from above. Lemma 3.2 with $\tau=2-\gamma$ gives us that

$$M\le c(\beta+2)^{2(2-\gamma)}\frac{|B_r|^{\gamma-1}}{r^{2\gamma}}\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}}\mu(r)^6\,J^{2-\gamma} \tag{3.48}$$

where $c=c(n,p,L,\gamma)>0$ and J is defined as in (3.12)

$$J=\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx+\mu(r)^4\frac{|B_r|^{1-\frac1\gamma}}{r^2}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac1\gamma}. \tag{3.49}$$

Now, it follows from (3.48) and (3.46) that

$$|I_3^2|\le c(\beta+2)^{3-\gamma}\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}}\mu(r)^{2\gamma}\,\frac{|B_r|^{\frac{\gamma-1}{2}}}{r^\gamma}\,J^{\frac{2-\gamma}{2}}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac12}.$$

By Young's inequality, we end up with

$$\begin{aligned}|I_3^2|\le&\frac{c_0}{12}(\beta+2)\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}}J\\&+c\,(\beta+2)^{\frac4\gamma-1}\big(\delta+\mu(r)^2\big)^{\frac{p-2}{2}}\mu(r)^4\frac{|B_r|^{1-\frac1\gamma}}{r^2}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac1\gamma},\end{aligned}$$

where $c_0>0$ is the same constant as in (3.36). Note that J is defined in (3.49). Thus I_3^2 satisfies a similar estimate to (3.36). Now the desired claim (3.36) for I_3 follows, since both I_3^1 and I_3^2 satisfy similar estimates. This concludes the proof of the claim (3.36), and hence the proof of the lemma. □

Remark 3.1. We can prove in the same way as that of Lemma 1.1 that the conclusion (3.31) holds for

$$v'=\min\big(\mu(r)/8,\max(\mu(r)/4+X_lu,0)\big).$$

The following corollary follows from Lemma 1.1 by Moser's iteration. It is proved for the case $p\ge2$ in [40], see Lemma 4.4 of [40]. Its proof is standard and is the same as in the Euclidean setting, see Proposition 4.1 of [12] or Lemma 2 of [37]. We include the proof here.

Corollary 3.1. *There exists a constant $\theta=\theta(n,p,L)>0$ such that the following statements hold. If we have*

$$|\{x\in B_r: X_lu<\mu(r)/4\}|\le\theta|B_r| \tag{3.50}$$

for an index $l\in\{1,\dots,2n\}$ and for a ball $B_r\subset\Omega$, then

$$\inf_{B_{r/2}}X_lu\ge 3\mu(r)/16;$$

Analogously, if we have

$$|\{x\in B_r: X_lu>-\mu(r)/4\}|\le\theta|B_r|, \tag{3.51}$$

for an index $l \in \{1, \dots, 2n\}$ and for a ball $B_r \subset \Omega$, then

$$\sup_{B_{r/2}} X_l u \leq \, -3\mu(r)/16.$$

Proof. Suppose that (3.50) holds for an index $l \in \{1, 2, ..., 2n\}$. We will apply Lemma 3.3 to prove Corollary 3.1. The case that (3.51) holds can be handled similarly by Lemma 3.3 for the function v', see Remark 3.1.

Let $\beta \geq 0$ and

$$w = \eta^{\beta/2+2} v^{\beta/2+2},$$

where $\eta \in C_0^\infty(B_r)$ is a cut-off function satisfying (3.9) and (3.10) and v is defined as in (3.2). Then for any $\gamma > 1$, we have that

$$\begin{aligned} \int_{B_r} |\mathfrak{X} w|^2 \, dx \leq & c(\beta+2)^2 \Big(\int_{B_r} \eta^{\beta+2} v^{\beta+4} |\mathfrak{X}\eta|^2 \, dx + \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X} v|^2 \, dx \Big) \\ \leq & c(\beta+2)^4 \mu(r)^4 \frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta} \, dx \Big)^{\frac{1}{\gamma}}, \end{aligned} \tag{3.52}$$

where $c = c(n, p, L, \gamma) > 0$. Here the second inequality follows from Hölder's inequality and Lemma 3.3. By the Sobolev inequality (2.1), we also have that

$$\Big(\fint_{B_r} |w|^{2\chi} \, dx \Big)^{\frac{1}{\chi}} \; \leq \; c(n) \, r^2 \fint_{B_r} |\mathfrak{X} w|^2 \, dx, \tag{3.53}$$

where $\chi = Q/(Q-2) = (n+1)/n$. Combining (3.52) and (3.53), we obtain that

$$\Big(\fint_{B_r} (\eta v)^{\chi(\beta+4)} \, dx \Big)^{\frac{1}{\chi}} \; \leq \; c(\beta+2)^4 \, \mu(r)^4 \Big(\fint_{B_r} (\eta v)^{\gamma\beta} \, dx \Big)^{\frac{1}{\gamma}}, \tag{3.54}$$

where $c = c(n, p, L, \gamma) > 0$. Now, we choose $\gamma = (n+2)/(n+1)$. Thus $1 < \gamma < \chi$. We will iterate inequality (3.54). Let

$$\beta_i = \frac{4\chi}{\chi - \gamma} \Big(\big(\frac{\chi}{\gamma} \big)^{i+1} - 1 \Big), \quad i = 0, 1, 2, \dots.$$

Note that $\gamma \beta_{i+1} = \chi(\beta_i + 4)$. Thus (3.54) with $\beta = \beta_i$ becomes

$$M_{i+1} \leq c_i M_i^{\frac{\chi}{\gamma} \frac{\beta_i}{\beta_{i+1}}} \tag{3.55}$$

for every $i = 0, 1, 2, \dots$, where

$$c_i = c^{\frac{\chi}{\gamma} \frac{1}{\beta_{i+1}}} \beta_{i+1}^{\frac{4\chi}{\gamma} \frac{1}{\beta_{i+1}}},$$

and

$$M_i \; = \Big(\fint_{B_r} \big(\eta v / \mu(r) \big)^{\gamma \beta_i} \, dx \Big)^{\frac{1}{\gamma \beta_i}}.$$

Iterating (3.55), we obtain that

$$M_i \leq \; c M_0^{\left(\frac{\chi}{\gamma} \right)^i \frac{\beta_0}{\beta_i}}, \tag{3.56}$$

where $c = c(n, p, L) > 0$. Let $i \to \infty$, we end up with

$$\limsup_{i \to \infty} M_i \leq c \, M_0^{1-\gamma/\chi},$$

that is,

$$\sup_{B_r} \eta v/\mu(r) \le c\Big(\fint_{B_r} \big(\eta v/\mu(r)\big)^{4\chi}\, dx\Big)^{\frac{1}{4\chi}(1-\gamma/\chi)}, \tag{3.57}$$

where $c = c(n,p,L) > 0$. Now, since η satisfies (3.9) and (3.10), we derive from (3.57) by our assumption (3.50) that

$$\sup_{B_{r/2}} v \le c\mu(r)\, \theta^{\frac{1}{4\chi}(1-\gamma/\chi)} \le \mu(r)/16,$$

provided that θ is small enough. This implies that $X_l u \ge 3\mu(r)/16$ in $B_{r/2}$. The proof is finished. □

4. Hölder continuity of the horizontal gradient

In this section, we prove Theorem 1.3. This proof is divided into two cases, $\delta > 0$ and $\delta = 0$, in subsection 4.1 and subsection 4.2, respectively. The proof for the case $\delta > 0$ is the same as that of Theorem 1.2 of [40], with minor modifications. The proof for the case $\delta = 0$ follows from an approximation arguments, see [40]. We include the proof here.

4.1. **Proof of Theorem 1.3 for the case $\delta > 0$.** Let $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (1.3) satisfying the structure condition (1.2) with $\delta > 0$. We fix a ball $B_{r_0} \subset \Omega$. For all balls $B_r, 0 < r < r_0$, with the same center as B_{r_0}, we denote for $l = 1, 2, ..., 2n$,

$$\mu_l(r) = \sup_{B_r} |X_l u|, \quad \mu(r) = \max_{1\le l\le 2n} \mu_l(r),$$

and

$$\omega_l(r) = \operatorname{osc}_{B_r} X_l u, \quad \omega(r) = \max_{1\le l\le 2n} \omega_l(r).$$

Clearly, we have $\omega(r) \le 2\mu(r)$.

We define for any function w

$$A^+_{k,\rho}(w) = \{x \in B_\rho : (w(x)-k)^+ = \max(w(x)-k, 0) > 0\};$$

and we define $A^-_{k,\rho}(w)$ similarly. To prove Theorem 1.3, we need the following lemma.

Lemma 4.1. *Let $B_{r_0} \subset \Omega$ be a ball and $0 < r < r_0/2$. Suppose that there is $\tau > 0$ such that*

$$|\mathfrak{X} u| \ge \tau\mu(r) \quad \text{in } A^+_{k,r}(X_l u) \tag{4.1}$$

for an index $l \in \{1, 2, ..., 2n\}$ and for a constant $k \in \mathbb{R}$. Then for any $q \ge 4$ and any $0 < r'' < r' \le r$, we have

$$\begin{aligned} &\int_{B_{r''}} \big(\delta + |\mathfrak{X} u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}(X_l u - k)^+|^2\, dx \\ \le &\frac{c}{(r'-r'')^2} \int_{B_{r'}} \big(\delta + |\mathfrak{X} u|^2\big)^{\frac{p-2}{2}} |(X_l u - k)^+|^2\, dx + cK|A^+_{k,r'}(X_l u)|^{1-\frac{2}{q}} \end{aligned} \tag{4.2}$$

where $K = r_0^{-2}|B_{r_0}|^{2/q}\big(\delta + \mu(r_0)^2\big)^{p/2}$ and $c = c(n,p,L,q,\tau) > 0$.

Lemma 4.1 is similar to Lemma 4.3 of [40], which is valid for $p \geq 2$. Under our extra assumption (4.1), the proof of Lemma 4.1 is exactly the same as that of Lemma 4.3 of [40]. All of the steps go through in the same way. We remark here that there are two places in the proof of Lemma 4.3 of [40] where the assumption $p \geq 2$ is used. Now due to our assumption (4.1), we may get through the proof for $1 < p < \infty$. We omit the details of the proof of Lemma 4.1.

Remark 4.1. Similarly, we can obtain an inequality, corresponding to (4.2), with $(X_l u - k)^+$ replaced by $(X_l u - k)^-$ and $A^+_{k,r}(X_l u)$ replaced by $A^-_{k,r}(X_l u)$.

Theorem 1.3 follows easily from the following theorem by an interation argument.

Theorem 4.1. *There exists a constant* $s = s(n, p, L) \geq 1$ *such that for every* $0 < r \leq r_0/16$*, we have*

$$\omega(r) \leq (1 - 2^{-s})\omega(8r) + 2^s\big(\delta + \mu(r_0)^2\big)^{\frac{1}{2}}\left(\frac{r}{r_0}\right)^{\alpha}, \tag{4.3}$$

where $\alpha = 1/2$ *when* $1 < p < 2$ *and* $\alpha = 1/p$ *when* $p \geq 2$.

Proof. To prove Theorem 4.1, we fix a ball B_r, with the same center as B_{r_0}, such that $0 < r < r_0/16$. We may assume that

$$\omega(r) \geq \big(\delta + \mu(r_0)^2\big)^{\frac{1}{2}}\left(\frac{r}{r_0}\right)^{\alpha}, \tag{4.4}$$

since, otherwise, (4.3) is true with $s = 1$. In the following, we assume that (4.4) is true, and we prove Theorem 4.1. We divide the proof of Theorem 4.1 into two cases.

Case 1. For at least one index $l \in \{1, \ldots, 2n\}$, we have either

$$|\{x \in B_{4r} : X_l u < \mu(4r)/4\}| \leq \theta |B_{4r}| \tag{4.5}$$

or

$$|\{x \in B_{4r} : X_l u > -\mu(4r)/4\}| \leq \theta |B_{4r}|, \tag{4.6}$$

where $\theta = \theta(n, p, L) > 0$ is the constant in Corollary 3.1. Assume that (4.5) is true; the case (4.6) can be treated in the same way. We apply Corollary 3.1 and we obtain that

$$|X_l u| \geq 3\mu(4r)/16 \quad \text{in} \quad B_{2r}.$$

Thus we have

$$|\mathfrak{X} u| \geq 3\mu(2r)/16 \quad \text{in} \quad B_{2r}. \tag{4.7}$$

Due to (4.7), we can apply Lemma 4.1 with $q = 2Q$ to obtain that

$$\begin{aligned}\int_{B_{r''}} |\mathfrak{X}(X_i u - k)^+|^2 \, dx \leq & \frac{c}{(r' - r'')^2} \int_{B_{r'}} |(X_i u - k)^+|^2 \, dx \\ & + cK\big(\delta + \mu(2r)^2\big)^{\frac{2-p}{2}} |A^{\pm}_{k,r'}(X_i u)|^{1 - \frac{1}{Q}}\end{aligned} \tag{4.8}$$

where $K = r_0^{-2}|B_{r_0}|^{1/Q}\big(\delta + \mu(r_0)^2\big)^{p/2}$. The above inequality holds for all $0 < r'' < r' \leq 2r,\ i \in \{1, \ldots, 2n\}$ and all $k \in \mathbb{R}$. This means that for each i, $X_i u$ belongs to the De Giorgi class $DG^+(B_{2r})$, see Section 4.1 of [40] for the definition. The corresponding version of Lemma 4.1 for $(X_i u - k)^-$, see Remark 4.1, shows that $X_i u$ also belong to $DG^-(B_{2r})$. So, $X_i u$ belongs to $DG(B_{2r})$. Now we can apply

Theorem 4.1 of [40] to conclude that there is $s_0 = s_0(n,p,L) > 0$ such that for each $i \in \{1, 2, ..., 2n\}$

$$\operatorname{osc}_{B_r} X_i u \le (1 - 2^{-s_0}) \operatorname{osc}_{B_{2r}} X_i u + cK^{\frac{1}{2}} \big(\delta + \mu(2r)^2\big)^{\frac{2-p}{4}} r^{\frac{1}{2}}. \tag{4.9}$$

Now notice that when $1 < p < 2$, we have that

$$\big(\delta + \mu(2r)^2\big)^{\frac{2-p}{4}} \le \big(\delta + \mu(r_0)^2\big)^{\frac{2-p}{4}}.$$

When $p \ge 2$, our assumption (4.4) with $\alpha = 1/p$ gives

$$\big(\delta + \mu(2r)^2\big)^{\frac{2-p}{4}} \le 2^{\frac{p-2}{2}} \omega(r)^{\frac{2-p}{2}} \le 2^{\frac{p-2}{2}} \big(\delta + \mu(r_0)^2\big)^{\frac{2-p}{4}} \left(\frac{r}{r_0}\right)^{\frac{2-p}{2p}},$$

where in the first inequality we used that $\mu(2r) \ge \omega(2r)/2 \ge \omega(r)/2$. In both cases, (4.9) becomes

$$\operatorname{osc}_{B_r} X_i u \le (1 - 2^{-s_0}) \operatorname{osc}_{B_{2r}} X_i u + c\big(\delta + \mu(r_0)^2\big)^{\frac{1}{2}} \left(\frac{r}{r_0}\right)^{\alpha}, \tag{4.10}$$

where $c = c(n,p,L) > 0$, $\alpha = 1/2$ when $1 < p < 2$ and $\alpha = 1/p$ when $p \ge 2$. This shows that in this case Theorem 4.1 is true.

Case 2. If Case 1 does not happen, then for every $i \in \{1, \dots, 2n\}$, we have

$$|\{x \in B_{4r} : X_i u < \mu(4r)/4\}| > \theta |B_{4r}|, \tag{4.11}$$

and

$$|\{x \in B_{4r} : X_i u > -\mu(4r)/4\}| > \theta |B_{4r}|, \tag{4.12}$$

where $\theta = \theta(n,p,L) > 0$ is the constant in Corollary 3.1. Note that on the set $\{x \in B_{8r} : X_i u > \mu(8r)/4\}$, we have trivially

$$|\mathfrak{X} u| \ge \mu(8r)/4 \quad \text{in } A^+_{k,8r}(X_i u) \tag{4.13}$$

for all $k \ge \mu(8r)/4$. Thus, we can apply Lemma 4.1 with $q = 2Q$ to conclude that

$$\begin{aligned} \int_{B_{r''}} |\mathfrak{X}(X_i u - k)^+|^2 \, dx \le & \frac{c}{(r' - r'')^2} \int_{B_{r'}} |(X_i u - k)^+|^2 \, dx \\ & + cK\big(\delta + \mu(8r)^2\big)^{\frac{2-p}{2}} |A^+_{k,r'}(X_i u)|^{1 - \frac{1}{Q}} \end{aligned} \tag{4.14}$$

where $K = r_0^{-2} |B_{r_0}|^{1/Q} \big(\delta + \mu(r_0)^2\big)^{p/2}$, whenever $k \ge k_0 = \mu(8r)/4$ and $0 < r'' < r' \le 8r$. The above inequality is true all $i \in \{1, 2, ..., 2n\}$. We note that (4.11) implies trivially that

$$|\{x \in B_{4r} : X_i u < \mu(8r)/4\}| > \theta |B_{4r}|.$$

Now we can apply Lemma 4.2 of [40] to conclude that there exists $s_1 = s_1(n,p,L) > 0$ such that

$$\sup_{B_{2r}} X_i u \le \sup_{B_{8r}} X_i u - 2^{-s_1} \big(\sup_{B_{8r}} X_i u - \mu(8r)/4\big) + cK^{\frac{1}{2}} \big(\delta + \mu(8r)^2\big)^{\frac{2-p}{4}} r^{\frac{1}{2}}. \tag{4.15}$$

From (4.12), we can derive similarly, see Remark 4.1, that

$$\inf_{B_{2r}} X_i u \ge \inf_{B_{8r}} X_i u + 2^{-s_1} \big(-\inf_{B_{8r}} X_i u - \mu(8r)/4\big) - cK^{\frac{1}{2}} \big(\delta + \mu(8r)^2\big)^{\frac{2-p}{4}} r^{\frac{1}{2}}. \tag{4.16}$$

Note that the above two inequalities (4.15) and (4.16) yield

$$\operatorname{osc}_{B_{2r}} X_i u \le \ (1 - 2^{-s_1}) \operatorname{osc}_{B_{8r}} X_i u + 2^{-s_1 - 1} \mu(8r) + cK^{\frac{1}{2}} \big(\delta + \mu(8r)^2\big)^{\frac{2-p}{4}} r^{\frac{1}{2}},$$

and hence

$$\omega(2r) \le \big(1-2^{-s_1}\big)\omega(8r) + 2^{-s_1-1}\mu(8r) + cK^{\frac{1}{2}}\big(\delta+\mu(8r)^2\big)^{\frac{2-p}{4}} r^{\frac{1}{2}}. \tag{4.17}$$

Now notice that when $1<p<2$, we have that

$$\big(\delta+\mu(8r)^2\big)^{\frac{2-p}{4}} \le \big(\delta+\mu(r_0)^2\big)^{\frac{2-p}{4}}$$

When $p\ge 2$, our assumption (4.4) with $\alpha=1/p$ gives

$$\big(\delta+\mu(8r)^2\big)^{\frac{2-p}{4}} \le 2^{\frac{p-2}{2}}\mu(r)^{\frac{2-p}{2}} \le 2^{\frac{p-2}{2}}\big(\delta+\mu(r_0)^2\big)^{\frac{2-p}{4}}\left(\frac{r}{r_0}\right)^{\frac{2-p}{2p}},$$

where in the first inequality we used the fact that $\mu(8r)\ge\omega(8r)/2\ge\omega(r)/2$. In both cases, (4.17) becomes

$$\omega(2r) \le \big(1-2^{-s_1}\big)\omega(8r) + 2^{-s_1-1}\mu(8r) + c\big(\delta+\mu(r_0)^2\big)^{\frac{1}{2}}\left(\frac{r}{r_0}\right)^{\alpha}.$$

Now we notice from the conditions (4.11) and (4.12) that

$$\omega(8r)\ge\mu(8r)-\mu(4r)/4\ge 3\mu(8r)/4.$$

Then from the above two inequalities we arrive at

$$\omega(2r) \le \big(1-2^{-s_1-2}\big)\omega(8r) + c\big(\delta+\mu(r_0)^2\big)^{\frac{1}{2}}\left(\frac{r}{r_0}\right)^{\alpha},$$

where $c=c(n,p,L)>0$, $\alpha=1/2$ when $1<p<2$ and $\alpha=1/p$ when $p\ge 2$. This shows that also in this case Theorem 4.1 is true. Thus, Theorem 4.1 is true with the choice of $s=\max(1,s_0,s_1+2,\log_2 c)$. The proof of Theorem 4.1 is finished. □

4.2. **Proof of Theorem 1.3 for the case $\delta=0$.** The proof of Theorem 1.3 for this case follows from an approximation argument, exactly in the same way as that in Section 5.3 of [40]. Suppose that the integrand f of functional (1.1) satisfies the structure condition

$$\begin{aligned} |z|^{p-2}|\xi|^2 \le \langle D^2 f(z)\xi,\xi\rangle &\le L|z|^{p-2}|\xi|^2; \\ |Df(z)| &\le L|z|^{p-1} \end{aligned} \tag{4.18}$$

for all $z,\xi\in\mathbb{R}^{2n}$, where $L\ge 1$ is a constant. We may assume that $f(0)=0$. For $\delta>0$, we define

$$f_\delta(z) \;=\begin{cases} \big(\delta+f(z)^{\frac{2}{p}}\big)^{\frac{p}{2}}, & \text{if } 1<p<2; \\ \delta^{\frac{p-2}{2}}|z|^2+f(z), & \text{if } p\ge 2. \end{cases} \tag{4.19}$$

Then, it is easy to see that f_δ satisfies a structure condition similar to (1.2) for all $\delta>0$, that is,

$$\begin{aligned} \frac{1}{\tilde L}(\delta+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \le \langle D^2 f_\delta(z)\xi,\xi\rangle &\le \tilde L(\delta+|z|^2)^{\frac{p-2}{2}}|\xi|^2; \\ |Df_\delta(z)| &\le \tilde L(\delta+|z|^2)^{\frac{p-2}{2}}|z|, \end{aligned} \tag{4.20}$$

where $\tilde L=\tilde L(p,L)\ge 1$. Now let $u\in HW^{1,p}(\Omega)$ be a solution of (1.3) satisfying the structure condition (4.18). We denote by u_δ the unique weak solution of the following Dirichlet problem

$$
\begin{cases} \operatorname{div}_H\big(Df_\delta(\mathfrak{X}w)\big) = 0 & \text{in } \Omega; \\ w - u \in HW_0^{1,p}(\Omega). \end{cases} \tag{4.21}
$$

Then we may apply Theorem 1.3 for the case $\delta > 0$ to solution u_δ. We obtain the uniform estimate (1.9) for u_δ. Letting $\delta \to 0$, we conclude the proof of Theorem 1.3 for the case $\delta = 0$. The proof is finished.

5. Appendix

Proof of Lemma 3.1. Fix $l \in \{1, 2, ..., n\}$ and $\beta \geq 0$. Let $\eta \in C_0^\infty(\Omega)$ be a non-negative cut-off function. Set

$$
\varphi = \eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u. \tag{5.1}
$$

We use φ as a test-function in equation (2.3) to obtain that

$$
\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+2} D_j D_i f(\mathfrak{X}u) X_j X_i u X_i\big(|\mathfrak{X}u|^2 X_l u\big)\, dx \\
&= -(\beta+2) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+1} v^{\beta+2} |\mathfrak{X}u|^2 X_l u D_j D_i f(\mathfrak{X}u) X_j X_l u X_i \eta \, dx \\
&\quad - (\beta+2) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+1} |\mathfrak{X}u|^2 X_l u D_j D_i f(\mathfrak{X}u) X_i X_l u X_i v \, dx \\
&\quad - \int_\Omega \sum_{i=1}^{2n} D_{n+l} D_i f(\mathfrak{X}u) T u X_i\big(\eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u\big)\, dx \\
&\quad + \int_\Omega T\big(D_{n+l} f(\mathfrak{X}u)\big) \eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u \, dx \\
&= I_1^l + I_2^l + I_3^l + I_4^l.
\end{aligned} \tag{5.2}
$$

Here we denote the integrals in the right hand side of (5.2) by I_1^l, I_2^l, I_3^l and I_4^l in order respectively. Similarly, by equation (2.4) we have for all $l \in \{n+1, n+2, ..., 2n\}$

that

$$
\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+2} D_j D_i f(\mathfrak{X}u) X_j X_i u X_i\big(|\mathfrak{X}u|^2 X_l u\big)\, dx \\
&= -(\beta+2)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+1} v^{\beta+2} |\mathfrak{X}u|^2 X_l u D_j D_i f(\mathfrak{X}u) X_j X_l u X_i \eta\, dx \\
&\quad -(\beta+2)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+1} |\mathfrak{X}u|^2 X_l u D_j D_i f(\mathfrak{X}u) X_i X_l u X_i v\, dx \\
&\quad +\int_\Omega \sum_{i=1}^{2n} D_{l-n} D_i f(\mathfrak{X}u) T u X_i\big(\eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u\big)\, dx \\
&\quad -\int_\Omega T\big(D_{l-n} f(\mathfrak{X}u)\big) \eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u\, dx \\
&= I_1^l + I_2^l + I_3^l + I_4^l.
\end{aligned} \tag{5.3}
$$

Again we denote the integrals in the right hand side of (5.3) by I_1^l, I_2^l, I_3^l and I_4^l in order respectively. Summing up the above equation (5.2) and (5.3) for all l from 1 to $2n$, we end up with

$$
\int_\Omega \sum_{i,j,l} \eta^{\beta+2} v^{\beta+2} D_j D_i f(\mathfrak{X}u) X_j X_i u X_i\big(|\mathfrak{X}u|^2 X_l u\big)\, dx = \sum_l \sum_{m=1}^4 I_m^l. \tag{5.4}
$$

Here all sums for i, j, l are from 1 to $2n$.

In the following, we estimate both sides of (5.4). For the left hand of (5.4), note that

$$
X_i\big(|\mathfrak{X}u|^2 X_l u\big) = |\mathfrak{X}u|^2 X_i X_l u + X_i(|\mathfrak{X}u|^2) X_l u.
$$

Then by the structure condition (1.2), we have that

$$
\sum_{i,j,l} D_j D_i f(\mathfrak{X}u) X_j X_l u X_i\big(|\mathfrak{X}u|^2 X_l u\big) \geq \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2,
$$

which gives us the following estimate for the left hand side of (5.4)

$$
\text{left of (5.4)} \geq \int_\Omega \eta^{\beta+2} v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2\, dx. \tag{5.5}
$$

Then we estimate the right hand side of (5.4). We will show that I_m^l satisfies the following estimate for each $l = 1, 2, ..., 2n$ and each $m = 1, 2, 3, 4$

$$
\begin{aligned}
|I_m^l| \leq{}& \frac{1}{36n}\int_\Omega \eta^{\beta+2} v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2\, dx \\
&+ c(\beta+2)^2 \int_\Omega \eta^\beta \big(|\mathfrak{X}\eta|^2 + \eta|T\eta|\big) v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4\, dx \\
&+ c(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^\beta \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^4 |\mathfrak{X}v|^2\, dx \\
&+ c\int_\Omega \eta^{\beta+2} v^{\beta+2} \big(\delta + |\mathfrak{X}u|^2\big)^{\frac{p-2}{2}} |\mathfrak{X}u|^2 |Tu|^2\, dx,
\end{aligned} \tag{5.6}
$$

where $c = c(n,p,L) > 0$. Then the lemma follows from the above estimates (5.5) and (5.6) for both sides of (5.4). The proof of the lemma is finished, modulo the proof of (5.6). In the rest, we prove (5.6) in the order of $m = 1,2,3,4$.

First, when $m = 1$, we have for $I_1^l, l = 1,2,...,2n$, by the structure condition (1.2) that

$$|I_1^l| \le c(\beta+2)\int_\Omega \eta^{\beta+1}|\mathfrak{X}\eta| v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|\mathfrak{X}\mathfrak{X}u|\,dx,$$

from which it follows by Young's inequality that

$$\begin{aligned}|I_1^l| \le\, &\frac{1}{36n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &+c(\beta+2)^2\int_\Omega \eta^{\beta}|\mathfrak{X}\eta|^2 v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4\,dx.\end{aligned}\tag{5.7}$$

Thus (5.6) holds for $I_1^l, l = 1,2,...,2n$.

Second, when $m = 2$, we have for $I_1^l, l = 1,2,...,2n$, by the structure condition (1.2) that

$$|I_2^l| \le c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}v|\,dx,$$

from which it follows by Young's inequality that

$$\begin{aligned}|I_2^l| \le\, &\frac{1}{36n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &+c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx.\end{aligned}\tag{5.8}$$

This proves (5.6) for $I_2^l, l = 1,2,...,2n$.

Third, when $m = 3$, we note that

$$\begin{aligned}\big|X_i\big(\eta^{\beta+2}v^{\beta+2}|\mathfrak{X}u|^2X_lu\big)\big| \le\, &3\eta^{\beta+2}v^{\beta+2}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|\\ &+(\beta+2)\eta^{\beta+1}v^{\beta+2}|\mathfrak{X}u|^3|\mathfrak{X}\eta|+(\beta+2)\eta^{\beta+2}v^{\beta+1}|\mathfrak{X}u|^3|\mathfrak{X}v|.\end{aligned}$$

Thus by the structure condition (1.2), we have

$$\begin{aligned}|I_3^l| \le\, &c\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u||Tu|\,dx\\ &+c(\beta+2)\int_\Omega \eta^{\beta+1}|\mathfrak{X}\eta|v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|Tu|\,dx\\ &+c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|\mathfrak{X}v||Tu|\,dx,\end{aligned}$$

from which it follows by Young's inequality that

$$\begin{aligned}|I_3^l| \le\, &\frac{1}{36n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &+c\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|Tu|^2\,dx\\ &+c(\beta+2)^2\int_\Omega \eta^{\beta}|\mathfrak{X}\eta|^2v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4\,dx\\ &+c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx.\end{aligned}\tag{5.9}$$

This proves (5.6) for $I_3^l, l = 1, 2, ..., 2n$.

Finally, when $m = 4$, we prove (5.6) for I_4^l. We consider only the case $l = 1, 2, ..., n$. The case $l = n+1, n+2, ..., 2n$ can be treated similarly. Let

$$w = \eta^{\beta+2}|\mathfrak{X}u|^2 X_l u. \tag{5.10}$$

Then we can write test-function φ defined as in (5.1) as $\varphi = v^{\beta+2}w$. We rewrite T as $T = X_1X_{n+1} - X_{n+1}X_1$. Then integration by parts yields

$$\begin{aligned} I_4^l &= \int_\Omega T\big(D_{n+l}f(\mathfrak{X}u)\big)\varphi\, dx \\ &= \int_\Omega X_1\big(D_{n+l}f(\mathfrak{X}u)\big)X_{n+1}\varphi - X_{n+1}\big(D_{n+l}f(\mathfrak{X}u)\big)X_1\varphi\, dx. \end{aligned} \tag{5.11}$$

Note that

$$\mathfrak{X}\varphi = (\beta+2)v^{\beta+1}w\mathfrak{X}v + v^{\beta+2}\mathfrak{X}w.$$

Thus (5.11) becomes

$$\begin{aligned} I_4^l =& (\beta+2)\int_\Omega v^{\beta+1}w\Big(X_1\big(D_{n+l}f(\mathfrak{X}u)\big)X_{n+1}v - X_{n+1}\big(D_{n+l}f(\mathfrak{X}u)\big)X_1v\Big)\, dx \\ &+ \int_\Omega v^{\beta+2}\Big(X_1\big(D_{n+l}f(\mathfrak{X}u)\big)X_{n+1}w - X_{n+1}\big(D_{n+l}f(\mathfrak{X}u)\big)X_1w\Big)\, dx \\ =& J^l + K^l. \end{aligned} \tag{5.12}$$

Here we denote the first and the second integral in the right hand side of (5.11) by J^l and K^l, respectively. We estimate J^l as follows. By the structure condition (1.2) and the definition of w as in (5.10),

$$|J^l| \le c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}v|\, dx,$$

from which it follows by Young's inequality, that

$$\begin{aligned} |J^l| \le& \frac{1}{72n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\, dx \\ &+ c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v|^2\, dx. \end{aligned} \tag{5.13}$$

The above inequality shows that J^l satisfies similar estimate as (5.6) for all $l = 1, 2, ..., n$. Then we estimate K^l. Integration by parts again, yields

$$\begin{aligned} K^l =& (\beta+2)\int_\Omega v^{\beta+1}D_{n+l}f(\mathfrak{X}u)\Big(X_{n+1}vX_1w - X_1vX_{n+1}w\Big)\, dx \\ &- \int_\Omega v^{\beta+2}D_{n+l}f(\mathfrak{X}u)Tw\, dx \\ =& K_1^l + K_2^l. \end{aligned} \tag{5.14}$$

For K_1^l, we have by the structure condition (1.2) that

$$\begin{aligned} |K_1^l| \le& c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}v|\, dx \\ &+ c(\beta+2)^2\int_\Omega \eta^{\beta+1}v^{\beta+1}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v||\mathfrak{X}\eta|\, dx \end{aligned}$$

from which it follows by Young's inequality that

$$\begin{aligned}|K_1^l| \le \frac{1}{144n}&\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &+c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\\ &+c(\beta+2)^2\int_\Omega \eta^{\beta}|\mathfrak{X}\eta|^2v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4\,dx.\end{aligned} \tag{5.15}$$

The above inequality shows that K_1^l also satisfies similar estimate as (5.6) for all $l=1,2,...,n$. We continue to estimate K_2^l in (5.14). Note that

$$Tw = (\beta+2)\eta^{\beta+1}|\mathfrak{X}u|^2X_luT\eta+\eta^{\beta+2}|\mathfrak{X}u|^2X_lTu+\sum_{i=1}^{2n}2\eta^{\beta+2}X_luX_iuX_iTu.$$

Therefore we write K_2^l as

$$\begin{aligned}K_2^l = &-(\beta+2)\int_\Omega \eta^{\beta+1}v^{\beta+2}D_{n+l}f(\mathfrak{X}u)|\mathfrak{X}u|^2X_luT\eta\,dx\\ &-\int_\Omega \eta^{\beta+2}v^{\beta+2}D_{n+l}f(\mathfrak{X}u)|\mathfrak{X}u|^2X_lTu\,dx\\ &-2\sum_{i=1}^{2n}\int_\Omega \eta^{\beta+2}v^{\beta+2}D_{n+l}f(\mathfrak{X}u)X_luX_iuX_iTu\,dx.\end{aligned}$$

For the last two integrals in the above equality, we apply integration by parts. We obtain that

$$\begin{aligned}K_2^l = &-(\beta+2)\int_\Omega \eta^{\beta+1}v^{\beta+2}D_{n+l}f(\mathfrak{X}u)|\mathfrak{X}u|^2X_luT\eta\,dx\\ &+\int_\Omega X_l\Big(\eta^{\beta+2}v^{\beta+2}D_{n+l}f(\mathfrak{X}u)|\mathfrak{X}u|^2\Big)Tu\,dx\\ &+2\sum_{i=1}^{2n}\int_\Omega X_i\Big(\eta^{\beta+2}v^{\beta+2}D_{n+l}f(\mathfrak{X}u)X_luX_iu\Big)Tu\,dx.\end{aligned}$$

Now we may estimate the integrals in the above equality by the structure condition (1.2). We obtain the following estimate for K_2^l.

$$\begin{aligned}|K_2^l| \le c(\beta+2)&\int_\Omega \eta^{\beta+1}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|T\eta|\,dx\\ &+c\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u||Tu|\,dx\\ &+c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|\mathfrak{X}v||Tu|\,dx\\ &+c(\beta+2)\int_\Omega \eta^{\beta+1}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^3|\mathfrak{X}\eta||Tu|\,dx.\end{aligned}$$

By Young's inequality, we end up with the following estimate for K_2^l

$$
\begin{aligned}
|K_2^l| \le & \frac{1}{144n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^\beta\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\\
&+c\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|Tu|^2\,dx.
\end{aligned}
\tag{5.16}
$$

This shows that K_2^l also satisfies similar estimate as (5.6). Now we combine the estimates (5.15) for K_1^l and (5.16) for K_2^l. Recall that K^l is the sum of K_1^l and K_2^l as denoted in (5.14). We obtain that the following estimate for K^l.

$$
\begin{aligned}
|K^l| \le & \frac{1}{72n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^\beta\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\\
&+c\int_\Omega \eta^{\beta+2}v^{\beta+2}\big(\delta+|\mathfrak{X}u|^2\big)^{\frac{p-2}{2}}|\mathfrak{X}u|^2|Tu|^2\,dx.
\end{aligned}
\tag{5.17}
$$

Recall that I_4^l is the sum of J^l and K^l. We combine the estimates (5.13) for J^l and (5.17) for K^l, and we can see that the claimed estimate (5.6) holds for I_4^l for all $l=1,2,...,n$. We can prove (5.6) similarly for I_4^l for all $l=n+1,n+2,...,2n$. This finishes the proof of the claim (5.6) for I_m^l for all $l=1,2,...,2n$ and all $m=1,2,3,4$, and hence also the proof of the lemma. □

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[B]
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ON LOCAL LIPSCHITZ REGULARITY FOR QUASILINEAR EQUATIONS IN THE HEISENBERG GROUP

SHIRSHO MUKHERJEE

ABSTRACT. The goal of this article is to establish local Lipschitz continuity of weak solutions for a class of degenerated elliptic equations of divergence form, in the Heisenberg Group. The considered hypothesis for the growth and ellipticity condition, is a natural generalisation of the p-Laplace equation and more general quasilinear elliptic equations with polynomial or exponential type growth.

1. INTRODUCTION

Lipschitz continuity of weak solutions for variational problems in the Heisenberg Group $\mathbb{H}^n$, has been studied in [36], where equations with growth conditions of p-Laplacian type was considered. The purpose this paper is to reproduce this result, for a larger class of more general quasilinear equations.

In a domain $\Omega \subset \mathbb{H}^n$, for $n \geq 1$, we consider the equation

$$\sum_{i=1}^{2n} X_i(\mathcal{A}_i(\mathfrak{X}u)) = 0, \tag{1.1}$$

where $X_1, \ldots, X_{2n}$ are the horizontal vector fields, $\mathfrak{X}u = (X_1 u, \ldots, X_{2n}u)$ is the horizontal gradient of a function $u : \Omega \to \mathbb{R}$ and $\mathcal{A}_i : \mathbb{R}^{2n} \to \mathbb{R}$ are given C^1 functions. We denote $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as $\mathcal{A}(z) = (\mathcal{A}_1(z), \mathcal{A}_2(z), \ldots, \mathcal{A}_{2n}(z))$ for $z \in \mathbb{R}^{2n}$ and $D\mathcal{A}(z)$ as the $2n \times 2n$ Jacobian matrix $(\partial \mathcal{A}_i(z)/\partial z_j)_{ij}$. We assume that $D\mathcal{A}(z)$ is symmetric and satisfies

$$\begin{gathered} \frac{g(|z|)}{|z|}\,|\xi|^2 \;\leq\; \big\langle D\mathcal{A}(z)\,\xi, \xi \big\rangle \;\leq\; L\,\frac{g(|z|)}{|z|}\,|\xi|^2; \\ |\mathcal{A}(z)| \;\leq\; L\,g(|z|), \end{gathered} \tag{1.2}$$

for every $z, \xi \in \mathbb{R}^{2n}$, where $L \geq 1$ and $g : [0, \infty) \to [0, \infty)$ is a C^1 function, $g(0) = 0$ and there exists constants $g_0 \geq \delta > 0$, such that the following holds

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \quad \text{for all } t > 0. \tag{1.3}$$

In the Euclidean setting, conditions (1.2) and (1.3) have been introduced by Lieberman [23], in order to produce a natural extension of the structure conditions for elliptic operators in divergence form, previously considered by Ladyzhenskaya and Ural'tseva [21], which in his

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words is " in a sense, the best generalization ". The most prominent model case is produced from minimization of scalar variational integrals of the form

$$I(u) = \int_\Omega G(|\mathfrak{X}u|)\,dx,$$

where $G(t) = \int_0^t g(s)\,ds$. Clearly, the corresponding Euler-Lagrange equation is

$$\operatorname{div}_H\Big(g(|\mathfrak{X}u|)\,\frac{\mathfrak{X}u}{|\mathfrak{X}u|}\Big) = \sum_{i=1}^{2n} X_i\Big(g(|\mathfrak{X}u|)\,\frac{X_iu}{|\mathfrak{X}u|}\Big) = 0, \tag{1.4}$$

which forms a prototype example of the equation (1.1) with the structure condition (1.2). The condition (1.3) can appear naturally if one considers defining

$$\delta \;=\; \inf_{t>0}\;\frac{tg'(t)}{g(t)} \quad\text{ and }\quad g_0 \;=\; \sup_{t>0}\;\frac{tg'(t)}{g(t)}. \tag{1.5}$$

In view of this, one can check out the special case when g is a power-like function e.g. $g(t) = t(\varepsilon+t^2)^{\frac{p-2}{2}}$; in this case $tg'(t)/g(t) = 1+(p-2)t^2/(\varepsilon+t^2)$, which corresponds to $\delta = \min\{1, p-1\}$ and $g_0 = \max\{1, p-1\}$. Moreover, if $g(t) = t^{p-1}$ in particular, for $1 < p < \infty$, then it satisfies (1.3) with $\delta = p-1 = g_0$ and (1.4) becomes the sub-elliptic p-laplace equation $\operatorname{div}_H(|\mathfrak{X}u|^{p-2}\mathfrak{X}u) = 0$.

We refer to [32, 7, 33, 16, 15, 35, 12, 22] and references therein, for earlier works on regularity theory of elliptic equations in divergence form, including the p-laplace equations in the setting of the Euclidean spaces.

The conditions (1.2) and (1.3) encompass quasilinear equations for a wide class of structure function g. Some natural examples include functions having growth similar to that of power-like functions and there logarithmic perturbations. We enlist two particular examples:

(1) $g(t) = (e+t)^{a+b\sin(\log\log(e+t))} - e^a$ for $b > 0, a \geq 1 + b\sqrt{2}$

(2) $g(t) = t^\alpha(\log(a+t))^\beta$ for $\alpha, \beta > 0, a \geq 1,$

see [14, 26]. In addition, multiple candidates satisfying condition (1.3) can be glued together to form the function g. A suitable gluing of the monomials $t^{\alpha-\varepsilon}, t^\alpha$ and $t^{\beta+\varepsilon}$ for $\beta > \alpha > \varepsilon$ as shown in [23], can be constructed in such a way that certain non-standard growth conditions (so called (p,q)-growth condition) of Marcellini [25], can also be included in this setting. Lastly, we remark that the positivity of the constants in (1.5), is essential and the techniques do not apply to the borderline cases e.g. $\delta = 0$. Thus, the equations of the form (1.4) exclude the 1-laplace equation or minimal surface equation.

Regularity theory in the Heisenberg Group, begins from the seminal work of Hörmander [18], where linear equations have been considered. For the case of quasilinear equations in this setting, we refer to [2, 3, 5, 13, 10, 11, 27, 24, 9] etc. for earlier results on regularity of weak solutions. The local Lipschitz continuity of weak solutions for p-laplace equation in $\mathbb{H}^n$, has been shown in [36]. The techniques used in there, paves the way for this paper.

The natural domain for the weak solution of (1.1) is the Horizontal Orlicz-Sobolev space $HW^{1,G}(\Omega)$ (see Section 2 for details). This is defined similarly as the Horizontal Sobolev space $HW^{1,p}(\Omega)$ (see [24, 28, 36]). The following theorem is our main result.

Theorem 1.1. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of equation (1.1) with g is as in (1.2) and (1.3) and $G(t) = \int_0^t g(s)\, ds$. Then $\mathfrak{X}u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ and moreover for any $B_r \subset \Omega$, we have the estimate*

$$\sup_{B_{\sigma r}} G(|\mathfrak{X}u|) \le \frac{c}{(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\, dx \tag{1.6}$$

for any $0 < \sigma < 1$, where $c = c(n, \delta, g_0, L) > 0$ is a constant.

This paper is organised as follows. We provide some preliminary facts on Heisenberg group, Orlicz-Sobolev spaces and sub-elliptic equations in Section 2. Then we prove several Caccioppoli type inequalities of the horizontal and vertical derivatives in Section 3, followed by the proof of Theorem 1.1 in the end.

Finally, we remark that local $C^{1,\alpha}$-regularity of weak solutions of the p-laplace equation in $\mathbb{H}^n$, has been shown recently in [30]; the techniques can be adopted to show the same result for the equation (1.1), as well. Furthermore, $C^{1,\alpha}$-regularity can also be shown for general quasilinear equations of the form

$$\mathrm{div}_H\, A(x, u, \mathfrak{X}u) + B(x, u, \mathfrak{X}u) = 0,$$

with appropriate growth and ellipticity conditions. These topics shall be addressed in a follow up article [29], yet to appear.

2. Preliminaries

In this section, we fix the notations used and introduce the Heisenberg Group $\mathbb{H}^n$. Also, we provide some essential facts on Orlicz-Sobolev spaces and sub-elliptic equations.

Throughout this paper, we shall denote a postive constant by c which may vary from line to line. But c would depend only on the dimension n, the constant g_0 and L of (1.3) and (1.2), unless it is explicitly specified otherwise. The dependence on δ of (1.3) shall appear at the very end.

2.1. **Heisenberg Group.**

Here we provide the definition and properties of Heisenberg group that would be useful in this paper. For more details, we refer the reader to the books [1, 4].

Definition 2.1. For $n \ge 1$, the *Heisenberg Group* denoted by $\mathbb{H}^n$, is identified to the Euclidean space $\mathbb{R}^{2n+1}$ with the group operation

$$x \cdot y := \Big(x_1 + y_1,\ \ldots,\ x_{2n} + y_{2n},\ t + s + \frac{1}{2}\sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) \Big) \tag{2.1}$$

for every $x = (x_1, \ldots, x_{2n}, t)$, $y = (y_1, \ldots, y_{2n}, s) \in \mathbb{H}^n$.

Thus, $\mathbb{H}^n$ with the group operation (2.1) forms a non-Abelian Lie group, whose left invariant vector fields corresponding to the canonical basis of the Lie algebra, are

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2}\partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2}\partial_t,$$

for every $1 \le i \le n$ and the only non zero commutator is $T = \partial_t$. We have

$$[X_i\,, X_{n+i}] = T \quad \text{and} \quad [X_i\,, X_j] = 0 \ \ \forall\, j \ne n+i. \tag{2.2}$$

We call $X_1, \ldots, X_{2n}$ as *horizontal vector fields* and T as the *vertical vector field*. For a scalar function $f : \mathbb{H}^n \to \mathbb{R}$, we denote

$$\mathfrak{X} f := (X_1 f, \ldots, X_{2n} f) \quad \text{and} \quad \mathfrak{X}\mathfrak{X} f := (X_i(X_j f))_{i,j}$$

as the *Horizontal gradient* and *Horizontal Hessian*, respectively. From (2.2), we have the following trivial but nevertheless, an important inequality

$$|Tf| \leq 2|\mathfrak{X}\mathfrak{X} f|. \tag{2.3}$$

For a vector valued function $F = (f_1, \ldots, f_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$, the *Horizontal divergence* is defined as $\operatorname{div}_H(F) := \sum_{i=1}^{2n} X_i f_i$.

The Euclidean gradient of a function $g : \mathbb{R}^k \to \mathbb{R}$, shall be denoted by $\nabla g = (D_1 g, \ldots, D_k g)$ and the Hessian matrix by $D^2 g$.

A piecewise smooth rectifiable curve γ is called a *horizontal curve* if its tangent vectors are contained in the *horizontal sub-bundle* $\mathcal{H} = \operatorname{span}\{X_1, \ldots, X_{2n}\}$, that is $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for almost every t. For any $x, y \in \mathbb{H}^n$, if the set of all horizontal curves is denoted as

$$\Gamma(x, y) = \left\{\gamma : [0,1] \to \mathbb{H}^n : \gamma(0) = x, \gamma(1) = y,\ \gamma'(t) \in \mathcal{H}_{\gamma(t)}\right\},$$

then Chow's accessibility theorem (see [6]) gurantees $\Gamma(x, y) \neq \emptyset$. The *Carnot-Carathèodory metric* (CC-metric) is defined in terms of the length $\ell(\gamma)$ of horizontal curves, as

$$d(x, y) = \inf\left\{\ell(\gamma) : \gamma \in \Gamma(x, y)\right\}. \tag{2.4}$$

This is equivalent to the *Korànyi metric* $d_{\mathbb{H}^n}(x, y) = \|y^{-1} \cdot x\|_{\mathbb{H}^n}$, where the Korànyi norm for $x = (x_1, \ldots, x_{2n}, t) \in \mathbb{H}^n$ is given by

$$\|x\|_{\mathbb{H}^n} := \Big(\sum_{i=1}^{2n} x_i^2 + |t|\Big)^{\frac{1}{2}}. \tag{2.5}$$

Throughout this article we use CC-metric balls denoted by $B_r(x) = \{y \in \mathbb{H}^n : d(x, y) < r\}$ for $r > 0$ and $x \in \mathbb{H}^n$. However, by virtue of the equivalence of the metrics, all assertions for CC-balls can be restated to Korànyi balls.

The Haar measure of $\mathbb{H}^n$ is just the Lebesgue measure of $\mathbb{R}^{2n+1}$. The Hausdorff dimension with respect to the metric d is also the homogeneous dimension of the group $\mathbb{H}^n$, which shall be denoted as $Q = 2n+2$, throughout this paper. Thus, for any CC-metric ball B_r, we have that $|B_r| = c(n) r^Q$.

For $1 \leq p < \infty$, the *Horizontal Sobolev space* $HW^{1,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that the distributional horizontal gradient $\mathfrak{X} u$ is in $L^p(\Omega, \mathbb{R}^{2n})$. $HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\mathfrak{X} u\|_{L^p(\Omega,\mathbb{R}^{2n})}. \tag{2.6}$$

We define $HW^{1,p}_{\mathrm{loc}}(\Omega)$ as its local variant and $HW^{1,p}_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ with respect to the norm in (2.6). The Sobolev Embedding theorem has the following version in the setting of Heisenberg group (see [3, 4]).

Theorem 2.2 (Sobolev Embedding). *Let $B_r \subset \mathbb{H}^n$ and $1 < q < Q$. For all $u \in HW^{1,q}_0(B_r)$, there exists constant $c = c(n, q) > 0$ such that*

$$\Big(\fint_{B_r} |u|^{\frac{Qq}{Q-q}}\, dx\Big)^{\frac{Q-q}{Qq}} \leq c\, r \Big(\fint_{B_r} |\mathfrak{X} u|^q\, dx\Big)^{\frac{1}{q}}. \tag{2.7}$$

We remark that the Lipshcitz continuity that is considered, is implied in the sense of Folland-Stein i.e. the Lipshcitz continuity with respect to the CC-metric. It does not make any assertion on the regularity of the vertical derivative.

2.2. **Orlicz-Sobolev Spaces.**
In this subsection, we recall some facts on Orlicz-Sobolev functions, which shall be necessary later. Further details can be found in textbooks e.g. [20, 31].

Definition 2.3 (Young function)**.** If $\psi : [0,\infty) \to [0,\infty)$ is an non-decreasing, left continuous function with $\psi(0) = 0$ and $\psi(s) > 0$ for all $s > 0$, then any function $\Psi : [0,\infty) \to [0,\infty]$ of the form

$$\Psi(t) = \int_0^t \psi(s)\, ds \tag{2.8}$$

is called a *Young function*. A continuous Young function $\Psi : [0,\infty) \to [0,\infty)$ satisfying $\Psi(t) = 0$ iff $t = 0$, $\lim_{t\to\infty} \Psi(t)/t = \infty$ and $\lim_{t\to 0} \Psi(t)/t = 0$, is called *N-function*.

There are several different definitions available in various references. However, within a slightly restricted range of functions (as in our case), all of them are equivalent. We refer to the book by Rao-Ren [31], for a more general discussion.

Definition 2.4 (Conjugate)**.** The *generalised inverse* of a montone function ψ is defined as $\psi^{-1}(t) := \inf\{s \geq 0 \mid \psi(s) > t\}$. Given any Young function $\Psi(t) = \int_0^t \psi(s)ds$, its *conjugate* function $\Psi^* : [0,\infty) \to [0,\infty]$ is defined as

$$\Psi^*(s) := \int_0^s \psi^{-1}(t)\, dt \tag{2.9}$$

and (Ψ, Ψ^*) is called a *complementary pair*, which is *normalised* if $\Psi(1) + \Psi^*(1) = 1$.

A Young function Ψ is convex, increasing, left continuous and satisfies $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. The generalised inverse of Ψ is right continuous, increasing and coincides with the usual inverse when Ψ is continuous and strictly increasing. In general, the inequality

$$\Psi(\Psi^{-1}(t)) \leq t \leq \Psi^{-1}(\Psi(t)) \tag{2.10}$$

is satisfied for all $t \geq 0$ and equality holds when $\Psi(t)$ and $\Psi^{-1}(t) \in (0,\infty)$. It is also evident that that the conjugate function Ψ^* is also a Young function, $\Psi^{**} = \Psi$ and for any constant $c > 0$, we have $(c\,\Psi)^*(t) = c\,\Psi^*(t/c)$. Here are two standard examples of complementary pair of Young functions.

(1) $\Psi(t) = t^p/p$ and $\Psi^*(t) = t^{p^*}/p^*$ when $1 < p, p^* < \infty$ and $1/p + 1/p^* = 1$.
(2) $\Psi(t) = (1+t)\log(1+t) - t$ and $\Psi^*(t) = e^t - t - 1$.

Lemma 2.5. *If (Ψ, Ψ^*) is a complementary pair of N-functions, then for any $t > 0$ we have*

$$\Psi^*\left(\frac{\Psi(t)}{t}\right) \leq \Psi(t). \tag{2.11}$$

Proof. Let $\Psi(t) = \int_0^t \psi(s)ds$. From mean value theorem, there exists $s_0 \in (0,t]$ such that

$$\psi(s_0) = \frac{1}{t}\int_0^t \psi(s)\, ds = \frac{\Psi(t)}{t}$$

for every $t > 0$. Using definition (2.9) and mean value theorem again, we find that there exist $r_0 \in (0, \psi(s_0))$, such that we have

$$\Psi^*\left(\frac{\Psi(t)}{t}\right) = \int_0^{\Psi(t)/t} \psi^{-1}(r)\, dr = \frac{\Psi(t)}{t}\, \psi^{-1}(r_0).$$

Since ψ and ψ^{-1} are non-decreasing functions, hence $\psi^{-1}(r_0) \le \psi^{-1}(\psi(s_0)) = s_0 \le t$. Using this on the above, one easily gets (2.11), to complete the proof. $\square$

The following Young's inequality is well known. We refer to [31] for a proof.

Theorem 2.6 (Young's Inequality)**.** *Given a Young function $\Psi(t) = \int_0^t \psi(s) ds$, we have the following for all $s, t > 0$;*

(2.12) $$st \le \Psi(s) + \Psi^*(t)$$

and equality holds iff $t = \psi(s)$ or $s = \psi^{-1}(t)$.

Definition 2.7 (Doubling function)**.** The Young function Ψ is called *doubling* if there exists a constant $C_2 > 0$ such that for all $t \ge 0$, we have

$$\Psi(2t) \le C_2\, \Psi(t).$$

In the growth and ellipticity condition (1.2), the structure function g satisfying (1.3), is a doubling function. Its doubling constant $C_2 = 2^{g_0}$ (see Lemma 2.12 below). Henceforth, we restrict to Orlicz spaces of doubling functions, thereby avoiding unnecessary technicalities.

Definition 2.8. Let $\Omega \subset \mathbb{R}^m$ be open and μ be a σ-finite measure on Ω. For a doubling Young function Ψ, the *Orlicz space* $L^\Psi(\Omega, \mu)$ is defined as the vector space generated by the set $\{u : \Omega \to \mathbb{R} \mid u \text{ measurable}, \int_\Omega \Psi(|u|)\, d\mu < \infty\}$. The space is equipped with the following *Luxemburg norm*

(2.13) $$\|u\|_{L^\Psi(\Omega,\mu)} := \inf\left\{k > 0 : \int_\Omega \Psi\left(\frac{|u|}{k}\right) d\mu \le 1\right\}$$

If μ is the Lebesgue measure, the space is denoted by $L^\Psi(\Omega)$ and any $u \in L^\Psi(\Omega)$ is called a Ψ-integrable function.

The function $u \mapsto \|u\|_{L^\Psi(\Omega,\mu)}$ is lower semi continuous and $L^\Psi(\Omega, \mu)$ is a Banach space with the norm in (2.13). The following theorem is a generalised version of Hölder's inequality, which follows easily from the Young's inequality (2.12), see [31] or [34].

Theorem 2.9 (Hölder's Inequality)**.** *For every $u \in L^\Psi(\Omega, \mu)$ and $v \in L^{\Psi^*}(\Omega, \mu)$, we have*

(2.14) $$\int_\Omega |uv|\, d\mu \le 2\, \|u\|_{L^\Psi(\Omega,\mu)} \|v\|_{L^{\Psi^*}(\Omega,\mu)}$$

Remark 2.10. The factor 2 on the right hand side of the above, can be dropped if (Ψ, Ψ^*) is normalised and one is replaced by $\Psi(1)$ in the definition (2.13) of Luxemburg norm.

The *Orlicz-Sobolev space* $W^{1,\Psi}(\Omega)$ can be defined similarly by L^Ψ norms of the function and its gradient, see [31], that resembles $W^{1,p}(\Omega)$ for the special case of $\Psi(t) = t^p$. But here for $\Omega \subset \mathbb{H}^n$, we require the notion of *Horizontal Orlicz-Sobolev spaces*, analoguous to the horizontal Sobolev spaces defined in the previous subsection.

Definition 2.11. We define the space $HW^{1,\Psi}(\Omega) = \{u \in L^\Psi(\Omega) \mid \mathfrak{X}u \in L^\Psi(\Omega, \mathbb{R}^{2n})\}$ for an open set $\Omega \subset \mathbb{H}^n$ and a doubling Young function Ψ, along with the norm

$$\|u\|_{HW^{1,\Psi}(\Omega)} := \|u\|_{L^\Psi(\Omega)} + \|\mathfrak{X}u\|_{L^\Psi(\Omega,\mathbb{R}^{2n})};$$

the spaces $HW^{1,\Psi}_{\text{loc}}(\Omega)$, $HW^{1,\Psi}_0(\Omega)$ are defined, similarly as earlier.

We remark that, all these notions can be defined for a general metric space, equipped with a doubling measure and upper gradient. More details of these can be found in [34].

2.3. **Sub-elliptic equations.**
Here, we discuss the known results on existence and uniqueness of weak solutions of the equation (1.1). Using the notation of horizontal divergence, we rewrite (1.1) as

$$-\operatorname{div}_H(\mathcal{A}(\mathfrak{X}u)) = 0 \quad \text{in } \Omega, \tag{2.15}$$

where $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies (1.2) and the matrix $D\mathcal{A}(z)$ is symmetric. Now, we enlist some important properties of the structure function g, in the following lemma.

Lemma 2.12. *Let $g \in C^1([0,\infty))$ be a function that satisfies (1.3) for some constant $g_0 > 0$ and $g(0) = 0$. If $G(t) = \int_0^t g(s)ds$, then the following holds.*

$$(1)\ \ G \in C^2([0,\infty)) \textit{ is convex}; \tag{2.16}$$

$$(2)\ \ tg(t)/(1+g_0) \le G(t) \le tg(t) \quad \forall \ \ t \ge 0; \tag{2.17}$$

$$(3)\ \ g(s) \le g(t) \le (t/s)^{g_0} g(s) \quad \forall \ \ 0 \le s < t; \tag{2.18}$$

$$(4)\ \ G(t)/t \textit{ is an increasing function} \quad \forall \ \ t > 0; \tag{2.19}$$

$$(5)\ \ tg(s) \le tg(t) + sg(s) \quad \forall \ \ t, s \ge 0. \tag{2.20}$$

The proof of the above lemma is trivial (see Lemma 1.1 of [23]), so we omit it. Notice that (2.18) implies that g is increasing and doubling, with $g(2t) \le 2^{g_0} g(t)$. In fact, it is easy to see that, (1.3) implies $t \mapsto g(t)/t^{g_0}$ is decreasing and $t \mapsto g(t)/t^\delta$ is increasing. Thus,

$$\min\{\alpha^\delta, \alpha^{g_0}\} g(t) \le g(\alpha t) \le \max\{\alpha^\delta, \alpha^{g_0}\} g(t) \quad \text{for all } \alpha, t \ge 0. \tag{2.21}$$

Here onwards, we fix the following notations,

$$\mathrm{F}(t) := g(t)/t \quad \text{and} \quad G(t) := \int_0^t g(s)\, ds. \tag{2.22}$$

Thus, F and G are also doubling functions and G is a Young function. Now we restate the structure condition (1.2). For every $z, \xi \in \mathbb{R}^{2n}$, we have that

$$\begin{gathered} \mathrm{F}(|z|)|\xi|^2 \le \big\langle D\mathcal{A}(z)\,\xi, \xi \big\rangle \le L\, \mathrm{F}(|z|)|\xi|^2; \\ |\mathcal{A}(z)| \le L\,|z| \mathrm{F}(|z|). \end{gathered} \tag{2.23}$$

Definition 2.13. Any $u \in HW^{1,G}(\Omega)$ is called a weak solution of the equation (2.15) if for every $\varphi \in C_0^\infty(\Omega)$, we have that

$$\int_\Omega \big\langle \mathcal{A}(\mathfrak{X}u), \mathfrak{X}\varphi \big\rangle\, dx = 0. \tag{2.24}$$

In addition, for all non-negative $\varphi \in C_0^\infty(\Omega)$, if the integral above is positive (resp. negative) then u is called a weak supersolution (resp. subsolution) of the equation (2.15).

Monotonicity of the operator $\mathcal{A}$ is required for existence of weak solutions. This follows from the structure condition (2.23). First, notice that, from (2.23)

$$\begin{aligned}\big\langle \mathcal{A}(z)-\mathcal{A}(w), z-w\big\rangle &= \int_0^1 \big\langle D\mathcal{A}\big(w+t(z-w)\big)(z-w),(z-w)\big\rangle\,dt\\ &\geq |z-w|^2\int_0^1 \mathrm{F}(|w+t(z-w)|)\,dt,\end{aligned}$$

for any $z,w\in\mathbb{R}^{2n}$. Now, it is possible to show that

$$\begin{aligned} |z|/2\leq |tz+(1-t)w|\leq 3|z|/2 &\qquad \text{if } |z-w|\leq 2|z|,\ t\geq 3/4,\\ |z-w|/4\leq |tz+(1-t)w|\leq 3|z-w|/2 &\qquad \text{if } |z-w|>2|z|,\ t\leq 1/4,\end{aligned}$$

with appropriate use of triangle inequality. Combining the above inequalities and using the doubling property, we have the following monotonicity inequality

$$\big\langle \mathcal{A}(z)-\mathcal{A}(w), z-w\big\rangle \geq c(g_0)\begin{cases}|z-w|^2\,\mathrm{F}(|z|) & \text{if } |z-w|\leq 2|z|\\ |z-w|^2\,\mathrm{F}(|z-w|) & \text{if } |z-w|>2|z|\end{cases}\tag{2.25}$$

and therefore the following ellipticity condition

$$\big\langle \mathcal{A}(z), z\big\rangle \geq c(g_0)\,|z|^2\mathrm{F}(|z|)\geq c(g_0)G(|z|).\tag{2.26}$$

Remark 2.14. The inequality in (2.25) is reminiscent of the monotonicity inequality for the p-laplacian operator. Precisely, when $\mathcal{A}(z)=|z|^{p-2}z$ for $1<p<\infty$, we have

$$\big(|z|^{p-2}z-|w|^{p-2}w\big)\cdot(z-w)\geq c(p)\begin{cases}|z-w|^2(|z|+|w|)^{p-2} & \text{if } \ 1<p<2\\ |z-w|^p & \text{if } \ p\geq 2\end{cases}\tag{2.27}$$

and from this, one can also derive (2.25) for this special case.

Theorem 2.15 (Existence)**.** *If $u_0\in HW^{1,G}(\Omega)$ is a given function and the operator $\mathcal{A}$ has the structure condition* (2.23)*, then there exists a unique weak solution $u\in HW^{1,G}(\Omega)$ for the Dirichlet problem*

$$\begin{cases}-\operatorname{div}_H(\mathcal{A}(\mathfrak{X}u)) = \ 0 \ \ \textit{in } \Omega;\\ \quad u-u_0\in HW^{1,G}_0(\Omega).\end{cases}\tag{2.28}$$

The proof of this theorem is a standard variant of that for the Euclidean setting and relies on literature of variational inequalities for monotone operators by Kinderlehrer and Stampacchia [19]. Similarly as the proof of Theorem 17.1 in [17], it is possible to show that there exists $u\in\mathcal{K}$ satisfying the variational inequality

$$\int_\Omega \big\langle \mathcal{A}(\mathfrak{X}u), \mathfrak{X}w-\mathfrak{X}u\big\rangle\,dx\geq 0$$

for all $w\in\mathcal{K}$, where $\mathcal{K}=\{v\in HW^{1,G}(\Omega)\,|\,v-u_0\in HW^{1,G}_0(\Omega)\}$. Arguing with $w=u\pm\varphi$ for any $\varphi\in C^\infty_0(\Omega)$, it is easy to see that u satisfies (2.24) and hence, is a weak solution of (2.28). The conditions for existence of u, can be established from the monotonicity (2.25).

The uniqueness, follows from the following comparison principle, which can be easily proved by choosing an appropriate test function on (2.15) and using monotonicity.

Lemma 2.16 (Comparison Principle)**.** *Given $u, v \in HW^{1,G}(\Omega)$, if u and v respectively are weak super and subsolution of the equation* (2.15) *and $u \geq v$ on $\partial\Omega$ in the trace sense, then we have $u \geq v$ a.e. in Ω.*

We would also require that, the weak solution of the Dirichlet problem (2.28) is Lipschitz with respect to CC-metric, if it has smooth boundary value in strictly convex domain. The proof of this resembles the Hilbert-Haar theory in the Euclidean setting. Actually, this is the only place where we require that $D\mathcal{A}$ is symmetric.

We consider a bounded domain $D \subset \mathbb{R}^{2n+1}$ which is convex and there exists a constant $\varepsilon_0 > 0$ such that the following holds : for every $y \in \partial D$, there exists $b(y) \in \mathbb{R}^{2n+1}$ with $|b(y)| = 1$, such that

$$b(y) \cdot (x - y) \geq \varepsilon_0 |x - y|^2 \tag{2.29}$$

for all $x \in \bar{D}$. Here $(\cdot)$ is the Euclidean inner product and $|.|$ is the Euclidean norm of $\mathbb{R}^{2n+1}$. The following theorem shows existence of Lipschitz continuous solutions of (2.15). The statement and the proof of this theorem, are the same as those of Theorem 5.1 of [36]. For sake of completeness, we provide the proof here.

Theorem 2.17. *Let $D \subset \mathbb{H}^n$ be a bounded and convex domain satisfying* (2.29) *for some $\varepsilon_0 > 0$. Given $u_0 \in C^2(\bar{D})$, if $u \in HW^{1,G}(D)$ is the weak solution of the Dirichlet problem*

$$\begin{cases} \operatorname{div}_H(\mathcal{A}(\mathfrak{X}u)) = 0 & \text{in } D; \\ u - u_0 \in HW_0^{1,G}(D). \end{cases} \tag{2.30}$$

then there exists a constant $M = M\big(n, \varepsilon_0, \|\nabla u_0\|_{L^\infty(\bar{D})} + \|D^2 u_0\|_{L^\infty(\bar{D})}, \operatorname{diam}(D)\big) > 0$, such that we have

$$\|\mathfrak{X}u\|_{L^\infty(D)} \leq M$$

Proof. This proof is the same as that of Theorem 5.1 in [36], with minor changes. Here, we provide a brief outline for the reader's convenience. It is enough to show that

$$|u(x) - u(y)| \leq M d(x, y) \quad \forall\, x, y \in \bar{D} \tag{2.31}$$

for some constant $M = M\big(n, \varepsilon_0, \|\nabla u_0\|_{L^\infty(\bar{D})} + \|D^2 u_0\|_{L^\infty(\bar{D})}, \operatorname{diam}(D)\big) > 0$. To this end, first we fix $y \in \partial D$, then we consider the barrier functions

$$L^{\pm}(x) = u_0(y) + [\nabla u_0(y) \pm K\, b(y)] \cdot (x - y),$$

where $K = \frac{(2n+1)^2}{2\varepsilon_0} \|D^2 u_0\|_{L^\infty(\bar{D})}$. Taking ξ as an appropriate point between x and y and using Taylor's formula followed by the condition (2.29), we obtain

$$\begin{aligned} u_0(x) &= u_0(y) + \nabla u_0(y) \cdot (x - y) + \frac{1}{2} D^2 u_0(\xi)(x - y) \cdot (x - y) \\ &\leq u_0(y) + \nabla u_0(y) \cdot (x - y) + K\varepsilon_0 |x - y|^2 \leq L^+(x) \end{aligned}$$

and hence we get $L^-(x) \leq u_0(x) \leq L^+(x)$ for all $x \in \bar{D}$. Thus, if $u \in HW^{1,G}(D)$ is the weak solution of (2.30), since u_0 is continuous on the boundary, we have

$$L^-(x) \leq u(x) \leq L^+(x) \quad \forall\, x \in \partial D \tag{2.32}$$

upto a continuous representative of u.

Now, letting $b(y) = (\tilde{b}(y), b_t(y)) \in \mathbb{R}^{2n} \times \mathbb{R}$, it is easy to get

$$\mathfrak{X}\mathfrak{X}L^{\pm}(x) = \frac{1}{2}\big[\partial_t u_0(y) \pm K\, b_t(y)\big] \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

for every $x \in \bar{D}$. Thus $\mathrm{div}_H[\mathcal{A}(\mathfrak{X}L^{\pm})] = \mathrm{Tr}\left(D\mathcal{A}(\mathfrak{X}L^{\pm})^T \mathfrak{X}\mathfrak{X}L^{\pm}\right) = 0$, since the matrix $D\mathcal{A}(z)$ has been assumed to be symmetric. Thus, $L^{\pm}$ are solutions of the equation (2.30). Using (2.32) and comparison principle (Lemma 2.16), we get

$$L^-(x) \le u(x) \le L^+(x) \quad \forall\, x \in D.$$

Since $L^{\pm}$ are Lipschitz and $L^{\pm}(y) = u(y)$, it is evident that there exists $M > 0$ such that

$$-Md(x,y) \le u(x) - u(y) \le Md(x,y) \quad \forall\, x \in \bar{D}, y \in \partial D \tag{2.33}$$

Now, we need the fact that if u be a Lipschitz solution of (2.30), then the following holds

$$\sup_{x,y\in\bar{D}} \left(\frac{|u(x)-u(y)|}{d(x,y)} \right) = \sup_{x\in\bar{D},\, y\in\partial D} \left(\frac{|u(x)-u(y)|}{d(x,y)} \right). \tag{2.34}$$

We refer to [36] for a proof of (2.34). From (2.33) and (2.34), we immediately get (2.31) and the proof is finished. □

3. Local Boundedness of Horizontal gradient

We prove Theorem 1.1 in this section. In the following three subsections we prove some Caccioppoli type inequalities of the horizontal and vertical vector fields, under two supplementary assumptions (see (3.1) and (3.2) below). The proof of Theorem 1.1 is given at the end of this section, where we remove both assumptions one by one. Throughout this section, we denote $u \in HW^{1,G}(\Omega)$ as a weak solution of (2.15). We assume the growth and ellipticity conditions (2.23), retaining the notation (2.22).

Now we make two supplementary assumptions.

(3.1) (1) There exists $m_1, m_2 > 0$ such that $\lim_{t\to 0} \mathrm{F}(t) = m_1$ and $\lim_{t\to\infty} \mathrm{F}(t) = m_2$;

(3.2) (2) There exists $M > 0$ such that $\|\mathfrak{X}u\|_{L^\infty(\Omega)} \le M$.

The purpose of the assumptions, is to ensure the regularity of weak solutions of the equation (2.15). Since $\mathrm{F}(t) = g(t)/t$ and g is monotonic, F has possible singularities at $t \to 0$ or $t \to \infty$ (or both). The assmption (3.1) avoids this and consequently, the structure condition (2.23) along with (3.1) and (3.2), imply

$$\begin{aligned} \nu^{-1}|\xi|^2 \le \big\langle D\mathcal{A}(\mathfrak{X}u)\,\xi, \xi\big\rangle \le \nu\,|\xi|^2; \\ |\mathcal{A}(\mathfrak{X}u)| \le \nu\,|\mathfrak{X}u|, \end{aligned} \tag{3.3}$$

for some $\nu = \nu(g_0, L, M, m_1, m_2) > 0$. Thus, the equation (2.15) with (3.3), satisfies the conditions considered by Capogna in [2]. From Theorem 1.1 and Theorem 3.1 of [2], we get

$$\mathfrak{X}u \in HW^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}), \quad Tu \in HW^{1,2}_{\mathrm{loc}}(\Omega) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega). \tag{3.4}$$

However, every estimates in this section, are independent of the constants M, m_1, m_2. This enables us to remove both the assumptions (3.1) and (3.2), in the end.

3.1. **Caccioppoli type inequalities.**
By virtue of (3.4), we can differentiate the equation (2.15) and obtain the equations satisfied by $X_l u$ and Tu. This is shown in the following two lemmas.

Lemma 3.1. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of* (2.15)*, then Tu is a weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_j(Tu)) = 0. \tag{3.5}$$

The proof of the above lemma is quite easy and similar to Lemma 3.2 in [36]. So, we omit the proof. The following lemma is similar to Lemma 3.1 in [36].

Lemma 3.2. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of* (2.15)*, then for any $l \in \{1,\dots,n\}$, we have that $X_l u$ is weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_lu) + \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_{n+l}(\mathfrak{X}u)Tu) + T(\mathcal{A}_{n+l}(\mathfrak{X}u)) = 0 \tag{3.6}$$

and similarly, $X_{n+l}u$ is weak solution of

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_{n+l}u) - \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_l(\mathfrak{X}u)Tu) - T(\mathcal{A}_l(\mathfrak{X}u)) = 0. \tag{3.7}$$

Proof. We only prove (3.6), the proof of (3.7) is similar. Let $l \in \{1,2,\dots,n\}$ and $\varphi \in C_0^\infty(\Omega)$ be fixed. We choose test function $X_l\varphi$ in (2.15) to get

$$\int_\Omega \sum_{i=1}^{2n} \mathcal{A}_i(\mathfrak{X}u)X_iX_l\varphi\,dx = 0.$$

Recalling the commutation relation (2.2) and using integral by parts, we obtain

$$\begin{aligned} 0 &= \int_\Omega \sum_{i=1}^{2n} \mathcal{A}_i(\mathfrak{X}u)X_lX_i\varphi\,dx - \int_\Omega \mathcal{A}_{n+l}(\mathfrak{X}u)T\varphi\,dx \\ &= -\int_\Omega \sum_{i=1}^{2n} X_l(\mathcal{A}_i(\mathfrak{X}u)X_i\varphi)dx + \int_\Omega T(\mathcal{A}_{n+l}(\mathfrak{X}u))\varphi\,dx. \end{aligned} \tag{3.8}$$

From (2.2) again, notice that for every $i \in \{1,2,\dots,2n\}$,

$$X_l(\mathcal{A}_i(\mathfrak{X}u)) = \sum_{j=1}^{2n} D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_lu + D_i\mathcal{A}_{n+l}(\mathfrak{X}u)Tu. \tag{3.9}$$

Thus, (3.8) and (3.9) together completes the proof. □

The following Caccioppoli type inequality for Tu is quite standard and similar to that of Lemma 3.3 in [36]. We provide a proof for the reader's convenience.

Lemma 3.3. *For any $\gamma \geq 0$ and $\eta \in C_0^\infty(\Omega)$, there exists $c = c(n, g_0, L) > 0$ such that*

$$\int_\Omega \eta^2 G(|Tu|)^{\gamma+1}\mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}(Tu)|^2\,dx \leq \frac{c}{(\gamma+1)^2}\int_\Omega G(|Tu|)^{\gamma+1}\mathrm{F}(|\mathfrak{X}u|)|Tu|^2|\mathfrak{X}\eta|^2dx.$$

Proof. For some fixed $\eta \in C_0^\infty(\Omega)$ and $\gamma \geq 0$, we choose test funcion

$$\varphi = \eta^2 G(|Tu|)^{\gamma+1} Tu$$

in the equation (3.5) to get

$$\begin{aligned}
&\sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|Tu|)^{\gamma+1} D_j \mathcal{A}_i(\mathfrak{X}u) X_j(Tu) X_i(Tu)\, dx \\
&+ (\gamma+1) \sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|Tu|)^{\gamma} g(|Tu|)|Tu| D_j \mathcal{A}_i(\mathfrak{X}u) X_j(Tu) X_i(Tu)\, dx \\
&\qquad = -2 \sum_{i,j=1}^{2n} \int_\Omega \eta\, G(|Tu|)^{\gamma+1} Tu D_j \mathcal{A}_i(\mathfrak{X}u) X_j(Tu) X_i \eta\, dx.
\end{aligned}$$

We use the condition (2.17) on the first term and then use the structure condition (2.23), to estimate both sides of the above equality. We obtain

$$\begin{aligned}
&\int_\Omega \eta^2 G(|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}(Tu)|^2\, dx \\
&\qquad \leq \frac{c}{(\gamma+1)} \int_\Omega |\eta| G(|Tu|)^{\gamma+1} |Tu| \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}(Tu)||\mathfrak{X}\eta|\, dx \\
&\qquad \leq \; c\tau \int_\Omega \eta^2 G(|Tu|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}(Tu)|^2\, dx \\
&\qquad\qquad + \frac{c}{\tau(\gamma+1)^2} \int_\Omega G(|Tu|)^{\gamma+1} \mathrm{F}(|\mathfrak{X}u|) |Tu|^2 |\mathfrak{X}\eta|^2\, dx,
\end{aligned}$$

where we have used Young's inequality to obtain the latter inequality of the above. With the choice of a small enough $\tau > 0$, the proof is finished. $\square$

The following Caccioppoli type inequality for the horizontal vector fields is more involved than the above, due to the non-commutativity (2.2). For the case of p-laplace equations, similar inequalities have been proved before, using difference quotients for $2 \leq p \leq 4$ in [28] and directly, for $1 < p < \infty$ in [36].

Lemma 3.4. *For any $\gamma \geq 0$ and $\eta \in C_0^\infty(\Omega)$, there exists $c = c(n, g_0, L) > 0$ such that*

$$\begin{aligned}
\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2\, dx \leq c \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) \left(|\mathfrak{X}\eta|^2 + |\eta T\eta|\right) dx \\
+ c\,(\gamma+1)^4 \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^2\, dx.
\end{aligned}$$

Proof. We fix $l \in \{1, \dots, n\}$ and $\eta \in C_0^\infty(\Omega)$. Now, we choose $\varphi_l = \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} X_l u$ as a test function in (3.6) and obtain the following,

$$
\begin{aligned}
&\sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i X_l u \, dx \\
&\qquad + \; (\gamma+1) \sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i (G(|\mathfrak{X}u|)) \, dx \\
&\qquad = -2 \sum_{i,j=1}^{2n} \int_\Omega \eta \, G(|\mathfrak{X}u|)^{\gamma+1} X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i \eta \, dx \\
&\qquad\quad - \sum_{i=1}^{2n} \int_\Omega D_i \mathcal{A}_{n+l}(\mathfrak{X}u) X_i \varphi_l T u \; dx \\
&\qquad\quad + \int_\Omega T(\mathcal{A}_{n+l}(\mathfrak{X}u)) \, \varphi_l \, dx \\
&\qquad = J_{1,l} + J_{2,l} + J_{3,l}.
\end{aligned} \tag{3.10}
$$

Similarly, we choose $\varphi_{n+l} = \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} X_{n+l} u$ in (3.7) to get

$$
\begin{aligned}
&\sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_{n+l} u X_i X_{n+l} u \, dx \\
&\qquad + \; (\gamma+1) \sum_{i,j=1}^{2n} \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} X_{n+l} u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_{n+l} u X_i (G(|\mathfrak{X}u|)) \, dx \\
&\qquad = -2 \sum_{i,j=1}^{2n} \int_\Omega \eta \, G(|\mathfrak{X}u|)^{\gamma+1} X_{n+l} u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_{n+l} u X_i \eta \, dx \\
&\qquad\quad + \sum_{i=1}^{2n} \int_\Omega D_i \mathcal{A}_{l}(\mathfrak{X}u) X_i \varphi_{n+l} T u \; dx \\
&\qquad\quad - \int_\Omega T(\mathcal{A}_{l}(\mathfrak{X}u)) \, \varphi_{n+l} \, dx \\
&\qquad = J_{1,n+l} + J_{2,n+l} + J_{3,n+l}.
\end{aligned} \tag{3.11}
$$

We shall add (3.10) and (3.11) and estimate both sides. First, notice that

$$
X_i(G(|\mathfrak{X}u|)) = \frac{g(|\mathfrak{X}u|)}{|\mathfrak{X}u|} \sum_{k=1}^{2n} X_k u X_i X_k u.
$$

We shall use the above along with (2.17). Adding (3.10) and (3.11) and using the structure condition (2.23), we obtain that

$$
\sum_{l=1}^{2n} (J_{1,l} + J_{2,l} + J_{3,l}) \; \geq \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F} \, (|\mathfrak{X}u|) \, |\mathfrak{X}\mathfrak{X}u|^2 \, dx \tag{3.12}
$$

Now we claim the following, which combined with (3.12) concludes the proof of the lemma.
Claim : For every $k \in \{1,2,3\}, l \in \{1,\dots,2n\}$ and some $c = c(n, g_0, L) > 0$, we have

$$
\begin{aligned}
|J_{k,l}| \leq \frac{1}{12n} & \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}\mathfrak{X}u|^2 \, dx \\
& + c \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) \big(|\mathfrak{X}\eta|^2 + |\eta T\eta|\big) \, dx \\
& + c\,(\gamma+1)^4 \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^2 \, dx.
\end{aligned} \tag{3.13}
$$

We prove the claim by estimating each $J_{k,l}$ in (3.10) and (3.11), using (2.23).
For the first term, we obtain

$$
|J_{1,l}| \leq c \int_\Omega |\eta| G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u| \mathrm{F}(|\mathfrak{X}u|) |\mathfrak{X}\mathfrak{X}u| |\mathfrak{X}\eta| \, dx
$$

and the claim (3.13) for $J_{1,l}$, follows from Young's inequality.
We calculate $\mathfrak{X}\varphi_l$ and similary estimate the second term using (2.23), to get

$$
\begin{aligned}
|J_{2,l}| \leq c & \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu| |\mathfrak{X}\mathfrak{X}u| \, dx \\
& + c\,(\gamma+1) \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} g(|\mathfrak{X}u|) |\mathfrak{X}u| \mathrm{F}(|\mathfrak{X}u|) |Tu| |\mathfrak{X}\mathfrak{X}u| \, dx \\
& + c \int_\Omega |\eta| G(|\mathfrak{X}u|)^{\gamma+1} |\mathfrak{X}u| \mathrm{F}(|\mathfrak{X}u|) |\mathfrak{X}\eta| |Tu| \, dx.
\end{aligned} \tag{3.14}
$$

Recalling $tg(t) \leq (1+g_0)G(t)$ from (2.17), note that the second term of the right hand side of (3.14) can be replaced by the first term. Then the claim (3.13) for $J_{2,l}$, follows by applying Young's inequality on each terms of the above.

For the third term, we show the estimate only for (3.10) i.e. for $l \in \{1,\dots,n\}$, since the estimate for the other case is the same. We first use integral by parts, then we calculate $T\varphi_l$ and obtain the following;

$$
\begin{aligned}
J_{3,l} = - & \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathcal{A}_{n+l}(\mathfrak{X}u) X_l(Tu) \, dx \\
& - (\gamma+1) \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma} X_l u \, \mathcal{A}_{n+l}(\mathfrak{X}u) T(G(|\mathfrak{X}u|)) \, dx \\
& - \ 2 \int_\Omega \eta \, G(|\mathfrak{X}u|)^{\gamma+1} X_l u \, \mathcal{A}_{n+l}(\mathfrak{X}u) T\eta \, dx.
\end{aligned}
$$

Now, notice that

$$
T(G(|\mathfrak{X}u|)) = \frac{g(|\mathfrak{X}u|)}{|\mathfrak{X}u|} \sum_{k=1}^{2n} X_k u X_k(Tu) = \mathrm{F}\left(|\mathfrak{X}u|\right) \sum_{k=1}^{2n} X_k u X_k(Tu).
$$

Using this, we carry out integral by parts again, for the first two terms of $J_{3,l}$ and obtain

$$\begin{aligned}J_{3,l} = &\int_\Omega X_l\Big(\eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathcal{A}_{n+l}(\mathfrak{X}u)\Big)Tu\,dx\\ &-(\gamma+1)\int_\Omega\sum_{k=1}^{2n}X_k\Big(\eta^2 G(|\mathfrak{X}u|)^{\gamma}\mathrm{F}(|\mathfrak{X}u|)X_lu\,\mathcal{A}_{n+l}(\mathfrak{X}u)X_ku\Big)Tu\,dx\\ &-2\int_\Omega\eta\,G(|\mathfrak{X}u|)^{\gamma+1}X_lu\,\mathcal{A}_{n+l}(\mathfrak{X}u)T\eta\,dx.\end{aligned}$$

From standard calculations and structure condition (2.23), we get

$$\begin{aligned}|J_{3,l}| \le\ & c\,(\gamma+1)^2\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu||\mathfrak{X}\mathfrak{X}u|\,dx\\ &+\ c\,(\gamma+1)^2\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma}g(|\mathfrak{X}u|)|\mathfrak{X}u|\mathrm{F}(|\mathfrak{X}u|)|Tu||\mathfrak{X}\mathfrak{X}u|\,dx\\ &+\ c\,(\gamma+1)\int_\Omega |\eta|G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}\eta||Tu|\,dx\\ &+\ c\int_\Omega |\eta|G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\,(|\mathfrak{X}u|)\,|T\eta|\,dx.\end{aligned}\tag{3.15}$$

Similarly as the estimate of $J_{2,l}$ in (3.14), we use (2.17) to combine the first two terms of the right hand side of (3.15). Then, by applying Young's inequality on all terms except the last one, the claim (3.13) for $J_{3,l}$ follows. Thus, the proof is finished. □

3.2. **A Reverse type inequality.**
We follow the technique of Zhong [36] and obtain a reverse type inequality for Tu in the following lemma. This shall be crucial for obtaining estimates for horizontal and vertical derivatives, later. The following lemma is reminiscent to Lemma 3.5 in [36].

Lemma 3.5. *For any $\gamma \ge 1$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned}&\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\qquad\le c\,(\gamma+1)^2\|\mathfrak{X}\eta\|^2_{L^\infty}\int_\Omega G(\eta|Tu|)^{\gamma+1}|Tu|^{-2}|\mathfrak{X}u|^2\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx\end{aligned}\tag{3.16}$$

for some $c = c(n, g_0, L) > 0$.

Proof. First, notice that from (2.17), we have $G(\eta|Tu|)^{\gamma+1}|Tu|^{-2} \le \eta^2 G(\eta|Tu|)^{\gamma-1}g(\eta|Tu|)^2$ for every $\gamma \ge 1$. In other words, the integral in right hand side of (3.16), is not singular.

To prove the lemma, we fix $l \in \{1,\dots,n\}$ and invoke (3.8), i.e. for any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega\sum_{i=1}^{2n}X_l(\mathcal{A}_i(\mathfrak{X}u)X_i\varphi)dx = \int_\Omega T(\mathcal{A}_{n+l}(\mathfrak{X}u))\varphi\,dx.$$

We choose the test function $\varphi = \eta^2 G(\eta|Tu|)^{\gamma+1}X_lu$ in the above. Notice that

$$\begin{aligned}X_i\varphi = &\ \eta^2 G(\eta|Tu|)^{\gamma+1}X_iX_lu + (\gamma+1)\eta^3 G(\eta|Tu|)^{\gamma}g(\eta|Tu|)X_lu\,X_i(|Tu|)\\ &+\Big(2\eta G(\eta|Tu|)^{\gamma+1} + (\gamma+1)\eta^2 G(\eta|Tu|)^{\gamma}g(\eta|Tu|)|Tu|\Big)X_luX_i\eta\end{aligned}$$

and from (2.2), recall that $X_{n+l}X_l = X_lX_{n+l} - T$. Using these, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{2n}\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_l(\mathcal{A}_i(\mathfrak{X}u))X_lX_iu\,dx\\
&\qquad= \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_l(\mathcal{A}_{n+l}(\mathfrak{X}u))Tu\,dx\\
&\qquad\quad-(\gamma+1)\sum_{i=1}^{2n}\int_\Omega \eta^3 G(\eta|Tu|)^{\gamma} g(\eta|Tu|)X_luX_l(\mathcal{A}_i(\mathfrak{X}u))X_i(|Tu|)\,dx\\
&\qquad\quad-\sum_{i=1}^{2n}\int_\Omega \Big(2\eta G(\eta|Tu|)+(\gamma+1)\eta^2 g(\eta|Tu|)|Tu|\Big)G(\eta|Tu|)^{\gamma}X_luX_l(\mathcal{A}_i(\mathfrak{X}u))X_i\eta\,dx\\
&\qquad\quad+\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_lu\,T(\mathcal{A}_{n+l}(\mathfrak{X}u))\,dx\\
&\qquad= I_1+I_2+I_3+I_4.
\end{aligned}
$$

We shall estimate both sides of the above. To estimate the left hand side, we use the structure condition (2.23), to obtain

$$
\sum_{i=1}^{2n}\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1} X_l(\mathcal{A}_i(\mathfrak{X}u))X_lX_iu\,dx \geq \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|X_l(\mathfrak{X}u)|^2\,dx.
$$

For the right hand side, we claim the following for every $k\in\{1,2,3,4\}$,

$$
\begin{aligned}
|I_k| &\leq c\tau\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\
&\quad+\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega G(\eta|Tu|)^{\gamma+1}|Tu|^{-2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx
\end{aligned}
\tag{3.17}
$$

for some $c=c(n,g_0,L)>0$, where $\tau>0$ is any arbitrary constant. Assuming the claim and combining it with the previous estimate, we end up with

$$
\begin{aligned}
\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|X_l(\mathfrak{X}u)|^2\,dx &\leq \tau\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\
&\quad+\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega G(\eta|Tu|)^{\gamma+1}|Tu|^{-2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx
\end{aligned}
$$

for some $c=c(n,g_0,L)>0$ and every $l\in\{1,\dots,n\}$. Similarly, the above inequality can also be obtained when $l\in\{n,\dots,2n\}$. Then, by summing over the two inequalities and choosing $\tau>0$ small enough, it is easy to obtain (3.16), as required to complete the proof.

Thus, we are left with proving the claim (3.17), which we accomplish by estimating each I_k, one by one. For I_1, first we use integral by parts to get

$$\begin{aligned}
I_1 = & -\int_\Omega X_l\Big(\eta^2 G(\eta|Tu|)^{\gamma+1}Tu\Big)\mathcal{A}_{n+l}(\mathfrak{X}u)\,dx \\
= & -\int_\Omega \eta^2 G(\eta|Tu|)^\gamma\Big[G(\eta|Tu|)+(\gamma+1)\eta|Tu|g(\eta|Tu|)\Big]\mathcal{A}_{n+l}(\mathfrak{X}u)X_l(Tu)\,dx \\
& -\int_\Omega \eta\, G(\eta|Tu|)^\gamma\Big[2G(\eta|Tu|)+(\gamma+1)\eta|Tu|g(\eta|Tu|)\Big]Tu\,\mathcal{A}_{n+l}(\mathfrak{X}u)X_l\eta\,dx \\
= & \, I_{11}+I_{12}.
\end{aligned}$$

Recall that $tg(t)\le(1+g_0)G(t)$ for all $t>0$ from (2.17). Using this along with the structure condition (2.23), we will show that the claim (3.17) holds for both I_{11} and I_{12}.

For I_{11}, using (2.17),(2.23) and Young's inequality, we obtain

$$\begin{aligned}
|I_{11}| &\le c\,(\gamma+1)\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|\,dx \\
&\le \frac{\tau}{\|\mathfrak{X}\eta\|^2_{L^\infty}}\int_\Omega \eta^4 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\,dx \\
&\quad+\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|^2_{L^\infty}\int_\Omega G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx
\end{aligned} \tag{3.18}$$

Now, the following inequality can be proved in a way similar to that of the Caccioppoli type inequality of Tu in Lemma 3.3,

$$\int_\Omega \eta^4 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\,dx\le c\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^2|\mathfrak{X}\eta|^2\,dx,$$

for some $c=c(n,g_0,L)>0$. After using the above inequality for the first term of (3.18) and then using $|Tu|\le 2|\mathfrak{X}\mathfrak{X}u|$ for both terms, it is easy to see that (3.17) holds for I_{11}.

For I_{12}, using structure condition (2.23) and (2.17) again, we get

$$|I_{12}|\le c\,(\gamma+1)\int_\Omega |\eta|G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu||\mathfrak{X}\eta|\,dx \tag{3.19}$$

from which, (3.17) follows easily from Young's inequality and $|Tu|\le 2|\mathfrak{X}\mathfrak{X}u|$. Thus, combining the estimates (3.18) and (3.19), we conclude that the claim (3.17), holds for I_1.

The estimate of I_2 is similar. We use (2.23), (2.17) and Young's inequality, to get

$$\begin{aligned}
|I_2| &\le c\,(\gamma+1)\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}|Tu|^{-1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}(Tu)|\,dx \\
&\le \frac{\tau}{\|\mathfrak{X}\eta\|^2_{L^\infty}}\int_\Omega \eta^4 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|^2\,dx \\
&\quad+\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|^2_{L^\infty}\int_\Omega G(\eta|Tu|)^{\gamma+1}|Tu|^{-2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.
\end{aligned}$$

Notice that, the first term on the right hand side of the latter inequality of the above, is identical to that of (3.18). Hence, the claim (3.17) for I_2, follows similarly.

For I_3, using (2.17) and structure condition (2.23) again, we obtain

$$|I_3| \leq c\,(\gamma+1)\int_\Omega |\eta|\, G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}\eta|\,dx$$

which together with Young's inequality, is enough for claim (3.17). Finally, the fourth term has the following estimate.

$$\begin{aligned}
I_4 &= \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}X_l u\sum_{i=1}^{2n} D_i\mathcal{A}_{n+l}(\mathfrak{X}u)X_i(Tu)\,dx \\
&\leq \int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}|\mathfrak{X}u|\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}(Tu)|\,dx,
\end{aligned}$$

which is identical to the upper bound of I_{11} in (3.18). Hence, the claim (3.17) holds for I_4 as well and the proof is complete. $\square$

The inequality (3.16) of the above lemma yields the following intermediate inequality, which shall be essential for proving the final estimate for the horizontal gradient.

Corollary 3.6. *For any $\gamma \geq 1$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned}
&\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad\leq c^{\frac{\gamma+1}{2}}(\gamma+1)^{(\gamma+1)(1+g_0)}\int_\Omega \eta^2 G\big(\|\mathfrak{X}\eta\|_{L^\infty}|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx
\end{aligned} \tag{3.20}$$

where $c = c(n, g_0, L) > 0$.

Proof. Let us denote $\Psi(s) = \tau G(\sqrt{s})^{\gamma+1}$, where $\tau > 0$ is an arbitrary constant. Notice that Ψ is a N-function if $\gamma \geq 1$. Now we restate the inequality (3.16) of Lemma 3.5, as

$$\begin{aligned}
&\int_\Omega \eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad\leq \frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega \frac{\Psi(\eta^2|Tu|^2)}{|Tu|^2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.
\end{aligned} \tag{3.21}$$

Taking Ψ^* as the conjugate function of Ψ, we apply the Young's inequality (2.12) on the right hand side of the above to get

$$\begin{aligned}
&\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2\int_\Omega \frac{\Psi(\eta^2|Tu|^2)}{|Tu|^2}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad\leq \int_\Omega \eta^2\,\Psi^*\Big(\frac{\Psi(\eta^2|Tu|^2)}{\eta^2|Tu|^2}\Big)\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&\qquad\quad+\int_\Omega \eta^2\,\Psi\Big(\frac{c}{\tau}(\gamma+1)^2\|\mathfrak{X}\eta\|_{L^\infty}^2|\mathfrak{X}u|^2\Big)\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.
\end{aligned} \tag{3.22}$$

Recalling (2.11), notice that

$$\Psi^*\Big(\frac{\Psi(\eta^2|Tu|^2)}{\eta^2|Tu|^2}\Big) \leq \Psi(\eta^2|Tu|^2) = \tau G(\eta|Tu|)^{\gamma+1}$$

and using this together with (3.21) and (3.22), we end up with

$$\begin{aligned}\int_\Omega &\eta^2 G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\le \tau\int_\Omega \eta^2\, G(\eta|Tu|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\quad+\int_\Omega \eta^2\,\tau G\Big(\sqrt{c/\tau}\,(\gamma+1)\|\mathfrak{X}\eta\|_{L^\infty}|\mathfrak{X}u|\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.\end{aligned}$$

Thus, with a small enough $\tau>0$ and the doubling property of G, the proof is finished. $\square$

The inequality (3.20) is required in a slightly different form, which we state here in the following corollary. It is an easy consequence of Corollary 3.6, above.

Corollary 3.7. *For any $\gamma,\omega\ge 1$ and all non-negative $\eta\in C_0^\infty(\Omega)$, we have*

$$\begin{aligned}\int_\Omega &\eta^2 G\Big(\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\le \frac{c^{\frac{\gamma+1}{2}}(\gamma+1)^{(\gamma+1)(1+g_0)}}{\omega^{\frac{\gamma+1}{2}}}\int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\end{aligned}\tag{3.23}$$

where $K_\eta=\|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)}+\|\eta T\eta\|_{L^\infty(\Omega)}$ and $c=c(n,g_0,L)>0$ is a constant.

Proof. Given any $\omega\ge 1$, note that from Lemma 2.12,

$$G\Big(\frac{t}{\sqrt{\omega}}\Big)\le\frac{t}{\sqrt{\omega}}g\Big(\frac{t}{\sqrt{\omega}}\Big)\le\frac{1+g_0}{\sqrt{\omega}}G(t).\tag{3.24}$$

Taking $K_\eta=\|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)}+\|\eta T\eta\|_{L^\infty(\Omega)}$, we use $\eta/\sqrt{\omega K_\eta}$ in place of η in (3.20), to get that

$$\begin{aligned}\int_\Omega &\frac{\eta^2}{\omega K_\eta} G\Big(\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\le c^{\frac{\gamma+1}{2}}(\gamma+1)^{(\gamma+1)(1+g_0)}\int_\Omega\frac{\eta^2}{\omega K_\eta}G\Big(\frac{\|\mathfrak{X}\eta\|_{L^\infty}|\mathfrak{X}u|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &\le\frac{c^{\frac{\gamma+1}{2}}(\gamma+1)^{(\gamma+1)(1+g_0)}}{\omega^{\frac{\gamma+1}{2}}}\int_\Omega\frac{\eta^2}{\omega K_\eta}G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx.\end{aligned}$$

In the latter inequality of the above, we have used $\|\mathfrak{X}\eta\|_{L^\infty}\le\sqrt{K_\eta}$, monotonicity of G and the inequality (3.24). After removing the factor $1/\omega K_\eta$ from both sides of the above, we end up with (3.23). This completes the proof. $\square$

3.3. **Horizontal and Vertical estimates.**

We first show that, the Caccioppoli type inequality of Lemma 3.4, can be improved using Corollary 3.7. This would be essential for the proof of Theorem 1.1.

Proposition 3.8. *If $u\in HW^{1,G}(\Omega)$ is a weak solution of equation (2.15), then for any $\gamma\ge 1$ and all non-negative $\eta\in C_0^\infty(\Omega)$, we have the following estimate*

$$\int_\Omega\eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\,dx\le c(\gamma+1)^{10(1+g_0)}K_\eta\int_{\mathrm{supp}(\eta)}G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx,$$

where $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$ *and* $c = c(n, g_0, L) > 0$ *is a constant.*

Proof. First, we recall the Caccioppoli type estimate of Lemma 3.4,

$$
\begin{aligned}
\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1} \mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx \le cK_\eta \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,dx \\
+ c\,(\gamma+1)^4 \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^2\,dx,
\end{aligned} \tag{3.25}
$$

where $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$ and $c = c(n, g_0, L) > 0$. Thus, to complete the proof, we require an estimate of the second integral of the right hand side of the above.

To this end, let us denote

$$
\Phi(s) = \omega K_\eta\, s\, G(\sqrt{s})^{\gamma+1} \tag{3.26}
$$

where $\omega \ge 1$ is a constant at our disposal, which shall be specified later. Let Φ^* be the conjugate of Φ. We estimate the last integral of (3.25) using the Young's inequality (2.12), as follows;

$$
\begin{aligned}
c(\gamma+1)^4 &\int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^2\,dx \\
&\le \int_\Omega \Phi\Big(c(\gamma+1)^4 \frac{\eta^2|Tu|^2}{\omega K_\eta}\Big)\mathrm{F}\,(|\mathfrak{X}u|)\;dx + \int_\Omega \Phi^*\Big(\omega K_\eta G(|\mathfrak{X}u|)^{\gamma+1}\Big)\mathrm{F}\,(|\mathfrak{X}u|)\;dx \\
&= Z_1 + Z_2
\end{aligned}
$$

where Z_1 and Z_2 are the respective terms of the right hand side. Now, we estimate Z_1 and Z_2, one by one. First, using doubling property for G and $|Tu| \le 2|\mathfrak{X}\mathfrak{X}u|$, notice that

$$
\begin{aligned}
Z_1 &= c(\gamma+1)^4 \int_\Omega \eta^2|Tu|^2 G\Big(\sqrt{c}(\gamma+1)^2\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\;dx \\
&\le c^{\frac{\gamma+1}{2}}(\gamma+1)^{4+2(\gamma+1)(1+g_0)} \int_\Omega \eta^2 G\Big(\frac{\eta|Tu|}{\sqrt{\omega K_\eta}}\Big)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx
\end{aligned} \tag{3.27}
$$

for some $c = c(n, g_0, L) > 0$. Now, we apply the estimate (3.23) from Corrollary 3.7 on the last term of (3.27), to get that

$$
\begin{aligned}
Z_1 &\le \frac{c^{\gamma+1}(\gamma+1)^{4+3(\gamma+1)(1+g_0)}}{\omega^{\frac{\gamma+1}{2}}} \int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&= \frac{1}{2}\int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}\mathfrak{X}u|^2\,dx,
\end{aligned} \tag{3.28}
$$

where ω is chosen as

$$
\omega = 2^{\frac{2}{\gamma+1}} c^2 (\gamma+1)^{6(1+g_0)+\frac{8}{\gamma+1}}. \tag{3.29}
$$

for an appropriate constant $c = c(n, g_0, L) > 0$.

To estimate Z_2, first notice that, from the inequality (2.11) and the definition (3.26)

$$
\Phi^*\Big(\omega K_\eta G(|\mathfrak{X}u|)^{\gamma+1}\Big) = \Phi^*\bigg(\frac{\Phi(|\mathfrak{X}u|^2)}{|\mathfrak{X}u|^2}\bigg) \le \Phi(|\mathfrak{X}u|^2) = \omega K_\eta|\mathfrak{X}u|^2\, G(|\mathfrak{X}u|)^{\gamma+1}. \tag{3.30}
$$

Using the above, we immediately have that

$$Z_2 \leq \omega K_\eta \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) dx. \tag{3.31}$$

Combining (3.28) and (3.31) with ω as in (3.29), we finally end up with

$$\begin{aligned} c(\gamma+1)^4 \int_\Omega \eta^2 G(|\mathfrak{X}u|)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^2\, dx &\leq \frac{1}{2}\int_\Omega \eta^2 G\big(|\mathfrak{X}u|\big)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2\, dx \\ &+ c(\gamma+1)^{6(1+g_0)+\frac{8}{\gamma+1}} K_\eta \int_\Omega G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right) dx \end{aligned}$$

for some $c = c(n, g_0, L) > 0$. This, together with (3.25), is enough to conclude the proof. $\square$

The following local estimate for the vertical derivative is an immediate consequence of the horizontal estimate of Proposition 3.8 and Corrollary 3.7, with the use of $|Tu| \leq 2|\mathfrak{X}\mathfrak{X}u|$.

Corollary 3.9. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of equation (2.15), then for any $\gamma \geq 1$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have the following estimate.*

$$\int_\Omega \eta^2 G\left(\frac{\eta|Tu|}{\sqrt{K_\eta}}\right)^{\gamma+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^2\, dx \leq c(\gamma) K_\eta \int_{\mathrm{supp}(\eta)} G(|\mathfrak{X}u|)^{\gamma+1}|\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)\, dx$$

where $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$ and $c(\gamma) = c(n, g_0, L, \gamma) > 0$ is a constant.

3.4. **Proof of Theorem 1.1.**
We recall that all the estimates above, rely on the apriori assumptions (3.1) and (3.2). We prove Theorem 1.1 here in three steps; first by assuming both (3.1) and (3.2), then by removing them one by one.

Proof of Theorem 1.1. First notice that, it is enough to establish the estimate (1.6) to finish the proof. If (1.6) holds apriori for a weak solution $u \in HW^{1,G}(\Omega)$ of (2.15), then monotonicity of g immediately implies $|\mathfrak{X}u| \in L^\infty(B_{\sigma r})$ along with the estimate

$$\sup_{B_{\sigma r}} |\mathfrak{X}u| \leq \max\left\{ 1\,,\ \frac{c}{g(1)(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\, dx \ \right\}.$$

Step 1 : We assume both (3.1) *and* (3.2).

The estimate (1.6) follows from Proposition 3.8 by standard Moser's iteration. Here, we provide a brief outline. Letting $w = G(|\mathfrak{X}u|)$, note that from (2.17)

$$|\mathfrak{X}w|^2 \leq |\mathfrak{X}u|^2\mathrm{F}\left(|\mathfrak{X}u|\right)^2 |\mathfrak{X}\mathfrak{X}u|^2 \leq (1+g_0)w\,\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}\mathfrak{X}u|^2,$$

and hence, from Proposition 3.8 we obtain

$$\int_\Omega \eta^2 w^\gamma |\mathfrak{X}w|^2\, dx \leq c\,(\gamma+1)^{10(1+g_0)}\, K_\eta \int_{\mathrm{supp}(\eta)} w^{\gamma+2}\, dx \tag{3.32}$$

for some $c = c(n, g_0, L) > 0$ and $K_\eta = \|\mathfrak{X}\eta\|^2_{L^\infty(\Omega)} + \|\eta T\eta\|_{L^\infty(\Omega)}$. Now we use a standard choice of test function $\eta \in C_0^\infty(B_r)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{r'}$ for $0 < r' < r$,

$$|\mathfrak{X}\eta| \leq 4/(r-r') \quad \text{and} \quad |\mathfrak{X}\mathfrak{X}\eta| \leq 16n/(r-r')^2.$$

Letting $\kappa = Q/(Q-2)$ and using Sobolev's inequality (2.7) for $q = 2$ on (3.32), we get that

$$\left(\int_{B_{r'}} w^{(\gamma+2)\kappa}\,dx\right)^{\frac{1}{\kappa}} \leq \frac{c(\gamma+2)^{12(1+g_0)}}{(r-r')^2}\int_{B_r} w^{\gamma+2}\,dx$$

for every $\gamma \geq 1$. Iterating this with $\gamma_i = 3\kappa^i - 2$ and $r_i = \sigma r + (1-\sigma)r/2^i$, we get

$$\sup_{B_{\sigma r}} w \leq \frac{c}{(1-\sigma)^{Q/3}}\left(\fint_{B_r} w^3\,dx\right)^{\frac{1}{3}}$$

for $c = c(n, g_0, L) > 0$ and this holds for every $B_r \subset \Omega$ and every $0 < \sigma < 1$. Then, a standard interpolation argument (see [8], p. 299–300) leads to

$$\sup_{B_{\sigma r}} w \leq \frac{c(q)}{(1-\sigma)^{Q/q}}\left(\fint_{B_r} w^q\,dx\right)^{\frac{1}{q}}$$

for every $q > 0$ and some $c(q) = c(n, g_0, L, q) > 0$. Taking $q = 1$, we get the estimate (1.6).

Step 2 : We assume (3.1) *but remove* (3.2).

Let $B_r = B_r(x_0) \subset \Omega$ be a fixed CC-ball. Given the weak solution $u \in HW^{1,G}(\Omega)$, there exists a smooth approximation $\phi_m \in C^\infty(B_r)$ such that $\phi_m \to u$ in $HW^{1,G}(B_r)$ as $m \to \infty$. By virtue of equivalence with the Korànyi metric, it is possible to find a concentric Korànyi ball $K_{\theta r} \subset\subset B_r$, for some constant $\theta = \theta(n) > 0$.

Now, let u_m be the weak solution of the following Dirichlet problem,

$$\begin{cases} \operatorname{div}_H(\mathcal{A}(\mathfrak{X}u_m)) = 0 & \text{in } K_{\theta r} \\ u_m - \phi_m \in HW_0^{1,G}(K_{\theta r}). \end{cases} \tag{3.33}$$

The choice of test function $u_m - \phi_m$, yields

$$\int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X}u_m), \mathfrak{X}u_m \big\rangle\,dx = \int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X}u_m), \mathfrak{X}\phi_m \big\rangle\,dx \tag{3.34}$$

Now, there exists $k = c(g_0, L) > 1$ such that combining ellipticity (2.26) and structure condition (2.23), one has $\big\langle \mathcal{A}(z), z\big\rangle \geq (2/k)\,|z||\mathcal{A}(z)|$. Using this along with (2.23) and doubling property of g, we estimate the right hand side of (3.34), as

$$\begin{aligned}
\int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X}u_m), \mathfrak{X}\phi_m \big\rangle\,dx &= \int_{|\mathfrak{X}u_m| \geq k|\mathfrak{X}\phi_m|} \big\langle \mathcal{A}(\mathfrak{X}u_m), \mathfrak{X}\phi_m \big\rangle\,dx + \int_{|\mathfrak{X}u_m| < k|\mathfrak{X}\phi_m|} \big\langle \mathcal{A}(\mathfrak{X}u_m), \mathfrak{X}\phi_m \big\rangle\,dx \\
&\leq \frac{1}{k}\int_{K_{\theta r}} |\mathcal{A}(\mathfrak{X}u_m)||\mathfrak{X}u_m|\,dx + \int_{|\mathfrak{X}u_m| < k|\mathfrak{X}\phi_m|} L\,g(|\mathfrak{X}u_m|)\,|\mathfrak{X}\phi_m|\,dx \\
&\leq \frac{1}{2}\int_{K_{\theta r}} \big\langle \mathcal{A}(\mathfrak{X}u_m), \mathfrak{X}u_m \big\rangle\,dx + k^{g_0} L\int_{K_{\theta r}} g(|\mathfrak{X}\phi_m|)\,|\mathfrak{X}\phi_m|\,dx.
\end{aligned}$$

Combining the above with (3.34) and using (2.26), we get

$$\int_{K_{\theta r}} G(|\mathfrak{X}u_m|)\,dx \leq c\int_{K_{\theta r}} G(|\mathfrak{X}\phi_m|)\,dx \leq c\int_{K_{\theta r}} G(|\mathfrak{X}u|)\,dx + o(1/m) \tag{3.35}$$

for $c = c(n, g_0, L) > 0$ and $o(1/m) \to 0$ as $m \to \infty$. Now, since ϕ_m is smooth and $K_{\theta r}$ satisfies the strong convexity condition (2.29), the equation (3.33) is an example of the Dirichlet problem (2.30). From Proposition 2.17, we have that

$$\|\mathfrak{X}u_m\|_{L^\infty(K_{\theta r})} \leq M$$

which is the assumption (3.2) for u_m. Now we can apply Step 1 and conclude

$$\sup_{B_{\sigma\tau r}} G(|\mathfrak{X}u_m|) \leq \frac{c}{(1-\sigma)^Q} \fint_{B_{\tau r}} G(|\mathfrak{X}u_m|)\, dx \tag{3.36}$$

for some $c = c(n, g_0, L) > 0$, $\sigma \in (0,1)$ and $\tau = \tau(n) > 0$ chosen such that $B_{\tau r} \subset K_{\theta r}$. This is followed up with standard argument, since (3.35) ensures that there exists $\tilde{u} \in HW^{1,G}(K_{\theta r})$ such that upto a subsequence $u_m \rightharpoonup \tilde{u}$. Since, $u_m - \phi_m \in HW_0^{1,G}(K_{\theta r})$, hence we have $\tilde{u} - u \in HW_0^{1,G}(K_{\theta r})$ and combined with the monotonicity (2.25), one can show $\tilde{u}$ is a weak solution of (2.15). From uniqueness, $\tilde{u} = u$. Taking $m \to \infty$ in (3.36) and (3.35), we conclude

$$\sup_{B_{\sigma\tau r}} G(|\mathfrak{X}u|) \leq \frac{c}{(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\, dx$$

and (1.6) follows from a simple covering argument.

Step 3: We remove both (3.2) *and* (3.1).

The assumption (3.1) is removed by a standard approximation argument. We use the regularization constructed in Lemma 5.2 of [23]. Here, we give a brief outline.

For any fixed $0 < \varepsilon < 1$ and some $\eta_\varepsilon \in C^{0,1}([0,\infty))$, we define

$$\mathrm{F}_\varepsilon(t) = \mathrm{F}\Big(\min\{\, t+\varepsilon\,,\, 1/\varepsilon\,\}\Big) \quad \text{and} \quad \mathcal{A}_\varepsilon(z) = \eta_\varepsilon(|z|)\mathrm{F}_\varepsilon(|z|)\, z + \Big(1 - \eta_\varepsilon(|z|)\Big)\mathcal{A}(z) \tag{3.37}$$

where $\mathcal{A}$ is given and $\mathrm{F}(t) = g(t)/t$. Thus, F_ε satisfies the assumption (3.1) with $m_1 = \mathrm{F}(\varepsilon)$ and $m_2 = \mathrm{F}(1/\varepsilon)$. Also, with the choice of η_ε as in [23](p. 343), it is possible to show that

$$\begin{gathered} \frac{1}{\tilde{L}}\mathrm{F}_\varepsilon(|z|)|\xi|^2 \leq \big\langle D\mathcal{A}_\varepsilon(z)\,\xi, \xi\big\rangle \leq \tilde{L}\,\mathrm{F}_\varepsilon(|z|)|\xi|^2; \\ |\mathcal{A}_\varepsilon(z)| \leq \tilde{L}|z|\mathrm{F}_\varepsilon(|z|), \end{gathered} \tag{3.38}$$

for some $\tilde{L} = \tilde{L}(\delta, g_0, L) > 0$. Reducing to a subsequence if necessary, it is easy to see that $\mathcal{A}_\varepsilon \to \mathcal{A}$ uniformly and $\mathrm{F}_\varepsilon \to \mathrm{F}$ uniformly on compact subsets of $(0,\infty)$, as $\varepsilon \to 0$.

Given weak solution $u \in HW^{1,G}(\Omega)$ of (2.15), we consider u_ε as the weak solution of the following regularized equation

$$\begin{cases} -\operatorname{div}_H(\mathcal{A}_\varepsilon(\mathfrak{X}u_\varepsilon)) = 0 & \text{in } \Omega'; \\ \quad u_\varepsilon - u \in HW_0^{1,G}(\Omega'), \end{cases} \tag{3.39}$$

for any $\Omega' \subset\subset \Omega$. Now, we are able to apply Step 2, to obtain uniform estimates for u_ε. Taking limit $\varepsilon \to 0$, we can obtain (1.6). This concludes the proof. $\square$

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$C^{1,\alpha}$-REGULARITY OF QUASILINEAR EQUATIONS ON THE HEISENBERG GROUP

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Abstract. In this article, we reproduce results of classical regularity theory of quasilinear elliptic equations in the divergence form, in the setting of Heisenberg Group. The conditions encompass a very wide class of equations with isotropic growth conditions, which are a generalization of the p-Laplace type equations in this respect; these also include all equations with polynomial or exponential type growth. In addition, some even more general conditions have also been explored.

Contents

1. Introduction

Regularity theory for weak solutions of second order quasilinear elliptic equations in the Euclidean spaces, has been well-developed over a long period of time since the pioneering work of De Giorgi [9] and has involved significant contributions of many authors. For more details on this topic, we refer to [38, 10, 39, 20, 18, 42, 15, 27], etc. and references therein. A comprehensive study of the subject can be found in the nowadays classical books by Gilbarg-Trudinger [22], Ladyzhenskaya-Ural'tseva [26] and Morrey [33].

The goal of this paper is to obtain regularity results in the setting of Heisenberg Group $\mathbb{H}^n$, that are previously known in the Euclidean setting. We consider the equation

$$\mathrm{Q}u = \mathrm{div}_H\, A(x, u, \mathfrak{X}u) + B(x, u, \mathfrak{X}u) = 0 \tag{1.1}$$

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in a domain $\Omega \subset \mathbb{H}^n$ for any $n \geq 1$, where $\mathfrak{X}u = (X_1 u, \dots, X_{2n} u)$ is the horizontal gradient of a function $u : \Omega \to \mathbb{R}$ and div_H is the horizontal divergence of a vector field (see Section 2 for details). Here $A : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ are given locally integrable functions. We also assume that A is differentiable and the $(2n \times 2n)$ matrix $D_p A(x, z, p) = (\partial A_i(x, z, p)/\partial p_j)_{ij}$ is symmetric for every $x \in \Omega, z \in \mathbb{R}$ and $p = (p_1, \dots, p_{2n}) \in \mathbb{R}^{2n}$. Thus, the results of this setting can also be applied to minimizers of a variational integral

$$I(u) = \int_\Omega f(x, u, \mathfrak{X}u)\, dx$$

for a smooth scalar function $f : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$; the Euler-Lagrange equation corresponding to the functional I, would be an equation of the form (1.1). The equations in settings similar to ours, are often referred as sub-elliptic equations.

In addition to A and B, we consider a C^1-function $g : [0, \infty) \to [0, \infty)$ also as given data, which satisfies $g(0) = 0$ and there exists constants $g_0 \geq \delta \geq 0$ such that the following holds,

$$\delta \leq \frac{t g'(t)}{g(t)} \leq g_0 \quad \text{for all } t > 0. \tag{1.2}$$

The function g shall be used in the hypothesis of growth and ellipticity conditions satisfied by A and B, as given below. The condition (1.2) appears in the work of Lieberman [29], in the Euclidean setting. In the case of Heisenberg Groups, a special class of quasilinear equations with growth conditions involving (1.2), has been recently studied in [35]. We remark that the special case $g(t) = t^{p-1}$ for $1 < p < \infty$, would correspond to equations with p-laplacian type growth. For a more detailed discussion on the relevance of the condition (1.2) and more examples of such function g, we refer to [29, 31, 1, 35] etc.

The study of regularity theory for sub-elliptic equations goes back to the fundamental work of Hörmander [24]. We refer to [5, 6, 8, 16, 13, 14, 32, 30, 12] and references therein, for earlier results on regularity of weak solutions of quasilinear equations.

The structure conditions for the equation (1.1) used in this paper, have been introduced in [29], which are generalizations of the so called natural conditions for elliptic equations in divergence form; these have been extensively studied by Ladyzhenskaya-Ural'tseva in [26] for equations in the Euclidean setting. The first structure condition is as follows.

Given some non-negative constants a_1, a_2, a_3, b_0, b_1 and χ, we assume that A and B satisfies

$$\begin{aligned}
&\big\langle A(x, z, p), p \big\rangle \geq |p| g(|p|) - a_1\, g\left(\frac{|z|}{R}\right) \frac{|z|}{R} - g(\chi)\chi; \\
&|A(x, z, p)| \leq a_2\, g(|p|) + a_3\, g\left(\frac{|z|}{R}\right) + g(\chi); \\
&|B(x, z, p)| \leq \frac{1}{R}\left[b_0\, g(|p|) + b_1\, g\left(\frac{|z|}{R}\right) + g(\chi) \right],
\end{aligned} \tag{1.3}$$

where $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^{2n}$ and $0 < R < \frac{1}{2} \operatorname{diam}(\Omega)$. Similar growth conditions have been considered previously in [22],[26] and [40] for the special case $g(t) = t^{\alpha - 1}$ for $\alpha > 1$.

For weak solutions of equation (1.1) with the above structure conditions, the appropriate domain is the Horizontal Orlicz-Sobolev space $HW^{1,G}(\Omega)$ (see Section 2 for the definition), where $G(t) = \int_0^t g(s) ds$. The following is the first result of this paper.

Theorem 1.1. *Let $u \in HW^{1,G}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of the equation* (1.1)*, with $G(t) = \int_0^t g(s)ds$ and $|u| \le M$ in Ω. Suppose the structure condition* (1.3) *holds for some $\chi \ge 0$, $0 < R \le R_0$ and a function g satisfying* (1.2) *with $\delta > 0$, then there exists $c > 0$ and $\alpha \in (0,1)$ dependent on $n, \delta, g_0, a_1, a_2, a_3, b_0 M, b_1$ such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ and*

$$\operatorname{osc}_{B_r} u \le c\left(\frac{r}{R}\right)^\alpha \Big(\operatorname{osc}_{B_R} u + \chi R\Big), \tag{1.4}$$

whenver $B_{R_0} \subset\subset \Omega$ and B_r, B_R are concentric to B_{R_0} with $0 < r < R \le R_0$.

The above theorem follows as a consequence of Harnack inequalities, Theorem 3.4 and Theorem 3.5 in Section 3. Similar Harnack inequalities in the sub-elliptic setting, has also been shown in [6] for the special case of polynomial type growth. The proof of these are standard imitations of the corresponding classical results due to Serrin [37], see also [40, 29].

Theorem 1.1 is necessary for our second result, the $C^{1,\alpha}$-regularity of weak solutions. This is new and relies on some recent development in [35], which in turn is based on the work of Zhong [43]. The structure conditions considered for this, are as follows.

Given the constants $L, L' \ge 1$ and $\alpha \in (0,1]$, we assume that the following holds,

$$\begin{aligned}
&\frac{g(|p|)}{|p|}\,|\xi|^2 \le \big\langle D_p\, A(x,z,p)\,\xi,\xi\big\rangle \le L\,\frac{g(|p|)}{|p|}\,|\xi|^2;\\
&|A(x,z,p) - A(y,w,p)| \le L'\big(1+g(|p|)\big)\Big(|x-y|^\alpha + |z-w|^\alpha\Big);\\
&|B(x,z,p)| \le L'\big(1+g(|p|)\big)|p|,
\end{aligned} \tag{1.5}$$

for every $x, y \in \Omega$, $z, w \in [-M_0, M_0]$ and $p, \xi \in \mathbb{R}^{2n}$, where $M_0 > 0$ is another given constant. The following theorem is the second result of this paper.

Theorem 1.2. *Let $u \in HW^{1,G}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of the equation* (1.1)*, with $G(t) = \int_0^t g(s)ds$ and $|u| \le M_0$ in Ω. Suppose the structure condition* (1.5) *holds for some $L, L' \ge 1, \alpha \in (0,1]$ and a function g satisfying* (1.2) *with $\delta > 0$, then there exists a constant $\beta = \beta(n, \delta, g_0, \alpha, L) \in (0,1)$ such that $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ and for any open $\Omega' \subset\subset \Omega$, we have*

$$|\mathfrak{X}u|_{C^{0,\beta}(\Omega',\mathbb{R}^{2n})} \le C\big(n, \delta, g_0, \alpha, L, L', M_0, g(1), \operatorname{dist}(\Omega', \partial\Omega)\big). \tag{1.6}$$

Pertaining to the growth conditions involving (1.2), local Lipshcitz continuity for the class of equations of the form $\operatorname{div}_H \mathcal{A}(\mathfrak{X}u) = 0$, has been shown in [35]. As a follow up, here we show the $C^{1,\alpha}$-regularity for this case as well, with a robust gradient estimate unlike (1.6).

Theorem 1.3. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of the equation $\operatorname{div}_H \mathcal{A}(\mathfrak{X}u) = 0$, where $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, the matrix $D\mathcal{A}$ is symmetric and the following structure condition holds,*

$$\begin{gathered}
\frac{g(|p|)}{|p|}\,|\xi|^2 \le \big\langle D\mathcal{A}(p)\,\xi,\xi\big\rangle \le L\,\frac{g(|p|)}{|p|}\,|\xi|^2;\\
|\mathcal{A}(p)| \le L\, g(|p|).
\end{gathered} \tag{1.7}$$

for every $p, \xi \in \mathbb{R}^{2n}$, $L \ge 1$ is a given constant and g satisfies (1.2) *with $\delta > 0$. Then $\mathfrak{X}u$ is locally Hölder continuous and there exists $\sigma = \sigma(n, g_0, L) \in (0,1)$ and $c = c(n, \delta, g_0, L) > 0$ such that for any $B_{r_0} \subset \Omega$ and $0 < r < r_0/2$, we have*

$$\max_{1\le l\le 2n} \fint_{B_r} G(|X_l u - \{X_l u\}_{B_r}|)\,dx \le c\Big(\frac{r}{r_0}\Big)^\sigma \fint_{B_{r_0}} G(|\mathfrak{X}u|)\,dx. \tag{1.8}$$

The proof of the above theorem, follows similarly along the line of that in [34]. It involves Caccioppoli type estimates of the horizontal and vertical vector fields along with the use of an integrability estimate of [43] and a double truncation of [39] and [28].

We remark that the spaces $C^{0,\alpha}$ and $C^{1,\alpha}$ considered in this paper, are in the sense of Folland-Stein [17]. In other words, the spaces are defined with respect to the homogeneous metric of the Heisenberg Group, see Section 2 for details. No assertions are made concerning the regularity of the vertical derivative.

This paper is organised as follows. In Section 2, we provide a brief review on Heisenberg Group and Orlicz spaces. Then in Section 3, first we prove a global maximum principle exploring some generalised growth conditions along the lines of [29]; then we prove the Harnack inequalities, thereby leading to the proof of Theorem 1.1. The whole of Section 4 is devoted to the proof of Theorem 1.3. Finally in Section 5, the proof of Theorem 1.2 is provided and some possible extensions of the structure conditions are discussed.

2. Preliminaries

In this section, we fix the notations used and provide a brief introduction of the Heisenberg Group $\mathbb{H}^n$. Also, we provide some essential facts on Orlicz spaces and the Horizontal Sobolev spaces, which are required for the purpose of this setting.

2.1. **Heisenberg Group.**

Here we provide the definition and properties of Heisenberg group that would be useful in this paper. For more details, we refer the reader to [2],[7], etc.

Definition 2.1. For $n \geq 1$, the *Heisenberg Group* denoted by $\mathbb{H}^n$, is identified to the Euclidean space $\mathbb{R}^{2n+1}$ with the group operation

$$x \cdot y := \Big(x_1 + y_1,\ \ldots,\ x_{2n} + y_{2n},\ t + s + \frac{1}{2} \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) \Big) \tag{2.1}$$

for every $x = (x_1, \ldots, x_{2n}, t)$, $y = (y_1, \ldots, y_{2n}, s) \in \mathbb{H}^n$.

Thus, $\mathbb{H}^n$ with the group operation (2.1) forms a non-Abelian Lie group, whose left invariant vector fields corresponding to the canonical basis of the Lie algebra, are

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2} \partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2} \partial_t,$$

for every $1 \leq i \leq n$ and the only non zero commutator $T = \partial_t$. We have

$$[X_i\,, X_{n+i}] = T \quad \text{and} \quad [X_i\,, X_j] = 0 \ \ \forall\ j \neq n+i. \tag{2.2}$$

We call $X_1, \ldots, X_{2n}$ as *horizontal vector fields* and T as the *vertical vector field.* For a scalar function $f : \mathbb{H}^n \to \mathbb{R}$, we denote $\mathfrak{X} f = (X_1 f, \ldots, X_{2n} f)$ and $\mathfrak{X}\mathfrak{X} f = (X_i(X_j f))_{i,j}$ as the *Horizontal gradient* and *Horizontal Hessian*, respectively. From (2.2), we have the following trivial but nevertheless, an important inequality $|Tf| \leq 2|\mathfrak{X}\mathfrak{X} f|$. For a vector valued function $F = (f_1, \ldots, f_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$, the *Horizontal divergence* is defined as

$$\mathrm{div}_H(F) = \sum_{i=1}^{2n} X_i f_i.$$

The Euclidean gradient of a function $g : \mathbb{R}^k \to \mathbb{R}$, shall be denoted by $\nabla g = (D_1 g, \ldots, D_k g)$ and the Hessian matrix by $D^2 g$.

The *Carnot-Carathèodory metric* (CC-metric) is defined as the length of the shortest horizontal curves, connecting two points. This is equivalent to the *Korànyi metric*, denoted as $d_{\mathbb{H}^n}(x,y) = \|y^{-1}\cdot x\|_{\mathbb{H}^n}$, where the Korànyi norm for $x=(x_1,\ldots,x_{2n},t)\in\mathbb{H}^n$ is

$$\|x\|_{\mathbb{H}^n} := \Big(\sum_{i=1}^{2n} x_i^2 + |t|\Big)^{\frac{1}{2}}. \tag{2.3}$$

Throughout this article we use CC-metric balls denoted by $B_r(x) = \{y\in\mathbb{H}^n : d(x,y)<r\}$ for $r>0$ and $x\in\mathbb{H}^n$. However, by virtue of the equivalence of the metrics, all assertions for CC-balls can be restated to Korànyi balls.

The Haar measure of $\mathbb{H}^n$ is just the Lebesgue measure of $\mathbb{R}^{2n+1}$. For a measurable set $E\subset\mathbb{H}^n$, we denote the Lebesgue measure as $|E|$. For an integrable function f, we denote

$$\{f\}_E = \fint_E f\,dx = \frac{1}{|E|}\int_E f\,dx.$$

The Hausdorff dimension with respect to the metric d is also the homogeneous dimension of the group $\mathbb{H}^n$, which shall be denoted as $Q=2n+2$, throughout this paper. Thus, for any CC-metric ball B_r, we have that $|B_r| = c(n)r^Q$.

For $1\le p<\infty$, the *Horizontal Sobolev space* $HW^{1,p}(\Omega)$ consists of functions $u\in L^p(\Omega)$ such that the distributional horizontal gradient $\mathfrak{X}u$ is in $L^p(\Omega,\mathbb{R}^{2n})$. $HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\mathfrak{X}u\|_{L^p(\Omega,\mathbb{R}^{2n})}. \tag{2.4}$$

We define $HW^{1,p}_{\mathrm{loc}}(\Omega)$ as its local variant and $HW^{1,p}_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ with respect to the norm in (2.4). The Sobolev Embedding theorem has the following version in the setting of Heisenberg group (see [6],[7]).

Theorem 2.2 (Sobolev Embedding)**.** *Let $B_r\subset\mathbb{H}^n$ and $1<q<Q$. For all $u\in HW_0^{1,q}(B_r)$, there exists constant $c=c(n,q)>0$ such that*

$$\Big(\fint_{B_r} |u|^{\frac{Qq}{Q-q}}\,dx\Big)^{\frac{Q-q}{Qq}} \le c\,r\Big(\fint_{B_r}|\mathfrak{X}u|^q\,dx\Big)^{\frac{1}{q}}. \tag{2.5}$$

Hölder spaces with respect to homogeneous metrics have appeared in Folland-Stein [17] and therefore, are sometimes called are known as Folland-Stein classes and denoted by Γ^α or $\Gamma^{0,\alpha}$ in some literature. However, here we maintain the classical notation and define

$$C^{0,\alpha}(\Omega) = \{u\in L^\infty(\Omega) : |u(x)-u(y)| \le c\,d(x,y)^\alpha\ \forall\ x,y\in\Omega\} \tag{2.6}$$

for $0<\alpha\le 1$, which are Banach spaces with the norm

$$\|u\|_{C^{0,\alpha}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega}\frac{|u(x)-u(y)|}{d(x,y)^\alpha}. \tag{2.7}$$

These have standard extensions to classes $C^{k,\alpha}(\Omega)$ for $k\in\mathbb{N}$, which consists of functions having horizontal derivatives up to order k in $C^{0,\alpha}(\Omega)$. The local counterparts are denoted as $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$. Now, the definition of Morrey and Campanato spaces in sub-elliptic setting differs in different texts. Here, we adopt the definition similar to the classical one.

For any domain $\Omega \subset \mathbb{H}^n$ and $\lambda > 0$, we define the *Morrey space* as

$$\mathcal{M}^{1,\lambda}(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_{B_r} |u| \, dx < c\, r^{\lambda} \ \forall \ B_r \subset \Omega, r > 0 \right\} \tag{2.8}$$

and the *Campanato space* as

$$\mathcal{L}^{1,\lambda}(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_{B_r} \left| u - \{u\}_{B_r} \right| dx < c\, r^{\lambda} \ \forall \ B_r \subset \Omega, r > 0 \right\}, \tag{2.9}$$

where in both definitions B_r represents balls with metric d. These spaces are Banach spaces and have properties similar to the classical spaces in the Euclidean setting. We shall use the fact that for every $0 < \alpha < 1$ and $Q = 2n + 2$, we have

$$\mathcal{L}^{1,Q+\alpha}(\Omega) \subset C^{0,\alpha}(\Omega), \tag{2.10}$$

where the inclusion is to be understood as taking continuous representatives. For details on classical Morrey and Campanato spaces, we refer to [25] and for the sub-elliptic setting we refer to [7].

2.2. **Orlicz-Sobolev Spaces.**

In this subsection, we recall some basic facts on Orlicz-Sobolev functions, which shall be necessary later. Further details can be found in textbooks e.g. [25],[36].

Definition 2.3 (Young function)**.** If $\psi : [0, \infty) \to [0, \infty)$ is an non-decreasing, left continuous function with $\psi(0) = 0$ and $\psi(s) > 0$ for all $s > 0$, then any function $\Psi : [0, \infty) \to [0, \infty]$ of the form

$$\Psi(t) = \int_0^t \psi(s) \, ds \tag{2.11}$$

is called a *Young function.* A continuous Young function $\Psi : [0, \infty) \to [0, \infty)$ satisfying $\Psi(t) = 0$ iff $t = 0$, $\lim_{t\to\infty} \Psi(t)/t = \infty$ and $\lim_{t\to 0} \Psi(t)/t = 0$, is called *N-function.*

There are several different definitions available in various references. However, within a slightly restricted range of functions (as in our case), all of them are equivalent. We refer to the book of Rao-Ren [36], for a more general discussion.

Definition 2.4 (Conjugate)**.** The *generalised inverse* of a montone function ψ is defined as $\psi^{-1}(t) := \inf\{s \geq 0 \mid \psi(s) > t\}$. Given any Young function $\Psi(t) = \int_0^t \psi(s) ds$, its *conjugate* function $\Psi^* : [0, \infty) \to [0, \infty]$ is defined as

$$\Psi^*(s) := \int_0^s \psi^{-1}(t) \, dt \tag{2.12}$$

and (Ψ, Ψ^*) is called a *complementary pair*, which is *normalised* if $\Psi(1) + \Psi^*(1) = 1$.

A Young function Ψ is convex, increasing, left continuous and satisfies $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. The generalised inverse of Ψ is right continuous, increasing and coincides with the usual inverse when Ψ is continuous and strictly increasing. In general, the inequality

$$\Psi(\Psi^{-1}(t)) \leq t \leq \Psi^{-1}(\Psi(t)) \tag{2.13}$$

is satisfied for all $t \geq 0$ and equality holds when $\Psi(t)$ and $\Psi^{-1}(t) \in (0, \infty)$. It is also evident that that the conjugate function Ψ^* is also a Young function, $\Psi^{**} = \Psi$ and for any constant $c > 0$, we have $(c\,\Psi)^*(t) = c\,\Psi^*(t/c)$.

Here are two standard examples of complementary pair of Young functions.

(1) $\Psi(t) = t^p/p$ and $\Psi^*(t) = t^{p^*}/p^*$ when $1 < p, p^* < \infty$ and $1/p + 1/p^* = 1$.

(2) $\Psi(t) = (1+t)\log(1+t) - t$ and $\Psi^*(t) = e^t - t - 1$.

The following Young's inequality is well known. We refer to [36] for a proof.

Theorem 2.5 (Young's Inequality). *Given a Young function $\Psi(t) = \int_0^t \psi(s)ds$, we have*

$$st \le \Psi(s) + \Psi^*(t) \tag{2.14}$$

for all $s, t > 0$ and equality holds if and only if $t = \psi(s)$ or $s = \psi^{-1}(t)$.

A Young function Ψ is called *doubling* if there exists a constant $C_2 > 0$ such that for all $t \ge 0$, we have $\Psi(2t) \le C_2\,\Psi(t)$. By virtue of (1.2), the structure function g is doubling with the doubling constant $C_2 = 2^{g_0}$ and hence, we restrict to Orlicz spaces of doubling functions.

Definition 2.6. Let $\Omega \subset \mathbb{R}^m$ be Borel and ν be a σ-finite measure on Ω. For a doubling Young function Ψ, the *Orlicz space* $L^\Psi(\Omega,\nu)$ is defined as the vector space generated by the set $\{u : \Omega \to \mathbb{R} \mid u \text{ measurable}, \int_\Omega \Psi(|u|)\,d\nu < \infty\}$. The space is equipped with the following *Luxemburg norm*

$$\|u\|_{L^\Psi(\Omega,\nu)} := \inf\left\{k > 0 : \int_\Omega \Psi\left(\frac{|u|}{k}\right) d\nu \le 1\right\} \tag{2.15}$$

If ν is the Lebesgue measure, the space is denoted by $L^\Psi(\Omega)$ and any $u \in L^\Psi(\Omega)$ is called a Ψ-integrable function.

The function $u \mapsto \|u\|_{L^\Psi(\Omega,\nu)}$ is lower semi continuous and $L^\Psi(\Omega,\nu)$ is a Banach space with the norm in (2.15). The following theorem is a generalised version of Hölder's inequality, which follows easily from the Young's inequality (2.14), see [36] or [41].

Theorem 2.7 (Hölder's Inequality). *For every $u \in L^\Psi(\Omega,\nu)$ and $v \in L^{\Psi^*}(\Omega,\nu)$, we have*

$$\int_\Omega |uv|\,d\nu \le 2\,\|u\|_{L^\Psi(\Omega,\nu)}\|v\|_{L^{\Psi^*}(\Omega,\nu)} \tag{2.16}$$

Remark 2.8. The factor 2 on the right hand side of the above, can be dropped if (Ψ, Ψ^*) is normalised and one is replaced by $\Psi(1)$ in the definition (2.15) of Luxemburg norm.

The *Orlicz-Sobolev space* $W^{1,\Psi}(\Omega)$ can be defined similarly by L^Ψ norms of the function and its gradient, see [36], that resembles $W^{1,p}(\Omega)$. But here for $\Omega \subset \mathbb{H}^n$, we require the notion of *Horizontal Orlicz-Sobolev spaces*, analoguous to the horizontal Sobolev spaces defined in the previous subsection.

Definition 2.9. We define the space $HW^{1,\Psi}(\Omega) = \{u \in L^\Psi(\Omega) \mid \mathfrak{X}u \in L^\Psi(\Omega,\mathbb{R}^{2n})\}$ for an open set $\Omega \subset \mathbb{H}^n$ and a doubling Young function Ψ, along with the norm

$$\|u\|_{HW^{1,\Psi}(\Omega)} := \|u\|_{L^\Psi(\Omega)} + \|\mathfrak{X}u\|_{L^\Psi(\Omega,\mathbb{R}^{2n})}.$$

The spaces $HW^{1,\Psi}_{\text{loc}}(\Omega)$, $HW^{1,\Psi}_0(\Omega)$ are defined, similarly as earlier.

We remark that, all these notions can be defined for a general metric space, equipped with a doubling measure. We refer to [41] for the details.

The following theorem, so called (Ψ,Ψ)-Poincaré inequality, has been proved (see Proposition 6.23 in [41]) in the setting of a general metric space with a doubling measure and metric upper gradient. We provide the statement in the setting of Heisenberg Group.

Theorem 2.10. *Given any doubling N-function Ψ with doubling constant $c_2 > 0$, every $u \in HW^{1,\Psi}(\Omega)$ satisfies the following inequality for every $B_r \subset \Omega$ and some $c = c(n, c_2) > 0$,*

$$\fint_{B_r} \Psi\left(\frac{|u - \{u\}_{B_r}|}{r}\right) dx \leq c \fint_{B_r} \Psi(|\mathfrak{X}u|)\, dx. \tag{2.17}$$

In case of $\Psi(t) = t^p$, the inequality is referred as (p,p)-Poincaré inequality. The following corrollary follows easily from (2.17) and the $(1,1)$-Poincaré inequality on $\mathbb{H}^n$.

Corollary 2.11. *Given a convex doubling N-function Ψ with doubling constant $c_2 > 0$, there exists $c = c(n, c_2)$ such that for every $B_r \subset \Omega$ and $u \in HW^{1,\Psi}(\Omega) \cap HW_0^{1,1}(\Omega)$, we have*

$$\fint_{B_r} \Psi\left(\frac{|u|}{r}\right) dx \leq c \fint_{B_r} \Psi(|\mathfrak{X}u|)\, dx. \tag{2.18}$$

Given a domain $\Omega \subset \mathbb{H}^n$, using (2.18) and arguments with chaining method (see [23]), it is also possible to show that for u, Ψ and $c = c(n, c_2) > 0$ as in Corrollary 2.11, we have

$$\fint_{\Omega} \Psi\left(\frac{|u|}{\operatorname{diam}(\Omega)}\right) dx \leq c \fint_{\Omega} \Psi(|\mathfrak{X}u|)\, dx. \tag{2.19}$$

Now we enlist some important properties of the function g that satisfies (1.2).

Lemma 2.12. *Let $g \in C^1([0,\infty))$ be a function that satisfies (1.2) for some constant $g_0 > 0$ and $g(0) = 0$. If $G(t) = \int_0^t g(s)ds$, then the following holds.*

$$(1)\ \ G \in C^2([0,\infty)) \text{ is convex}; \tag{2.20}$$

$$(2)\ \ tg(t)/(1+g_0) \leq G(t) \leq tg(t) \quad \forall\ \ t \geq 0; \tag{2.21}$$

$$(3)\ \ g(s) \leq g(t) \leq (t/s)^{g_0} g(s) \quad \forall\ \ 0 \leq s < t; \tag{2.22}$$

$$(4)\ \ G(t)/t \text{ is an increasing function} \quad \forall\ \ t > 0; \tag{2.23}$$

$$(5)\ \ tg(s) \leq tg(t) + sg(s) \quad \forall\ \ t, s \geq 0. \tag{2.24}$$

The proof is trivial (see Lemma 1.1 of [29]), so we omit it. Notice that (2.22) implies that g is increasing and doubling, with $g(2t) \leq 2^{g_0} g(t)$ and

$$\min\{1, \alpha^{g_0}\} g(t) \leq g(\alpha t) \leq \max\{1, \alpha^{g_0}\} g(t) \quad \text{for all } \alpha, t \geq 0. \tag{2.25}$$

Since G is convex, an easy application of Jensen's inequality yields

$$\fint_{\Omega} G(|w - \{w\}_{\Omega}|)\, dx \ \leq\ c(g_0) \min_{k \in \mathbb{R}} \fint_{\Omega} G(|w - k|)\, dx \quad \forall\, w \in L^G(\Omega) \tag{2.26}$$

All the above properties hold even if $\delta = 0$ in (1.2) and they are purposefully kept that way. However, the properties corresponding to $\delta > 0$, shall be required in some situations. For this case, (2.21) and (2.22) becomes

$$tg(t)/(1+g_0) \leq G(t) \leq tg(t)/(1+\delta) \quad \forall\ \ t \geq 0; \tag{2.27}$$

$$(t/s)^{\delta} g(s) \leq g(t) \leq (t/s)^{g_0} g(s) \quad \forall\ \ 0 \leq s < t, \tag{2.28}$$

and hence $t \mapsto g(t)/t^{g_0}$ is decreasing and $t \mapsto g(t)/t^{\delta}$ is increasing.

3. Hölder continuity of weak solutions

In this section, we show that weak solutions of quasilinear equations in the Heisenberg Group satisfy the Harnack inequalities, which leads to the Hölder continuity, thereby proving Theorem 1.1. The techniques are standard, based on appropriate modifications of similar results in the Euclidean setting, by Trudinger [40] and Lieberman [29].

On a domain $\Omega \subset \mathbb{H}^n$, we consider the prototype quasilinear operator in divergence form

$$\mathrm{Q}u = \mathrm{div}_H\, A(x, u, \mathfrak{X}u) + B(x, u, \mathfrak{X}u) \tag{3.1}$$

throughout this paper, where $A : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ are given functions. Appropriate additional hypothesis on structure conditions satisfied by A and B, shall be assumed in the following subsections, accordingly as required.

Here onwards, throughout this paper, we fix the notations

$$\mathrm{F}(t) := g(t)/t \quad \text{and} \quad G(t) := \int_0^t g(s)\, ds. \tag{3.2}$$

We remark that the conditions chosen for A, always ensure some sort of ellipticity for the operator (3.1) and the existence of weak solutions $u \in HW^{1,G}(\Omega)$ for $\mathrm{Q}u = 0$ is always assured. Any pathological situation, where this does not hold, is avoided.

3.1. **Global Maximum principle.**

Given weak solution $u \in HW^{1,G}(\Omega)$ for $\mathrm{Q}u = 0$, here we show global L^∞ estimates of u under appropriate boundary conditions. The method and techniques are adaptations of similar classical results in [29] for quasilinear equations in the Euclidean setting.

Here, we assume that u satisfies the boundary condition $u - u_0 \in HW_0^{1,G}(\Omega)$ for some $u_0 \in L^\infty(\bar{\Omega})$. In addition, we assume that there exists $b_0 > 0$ and $M \geq \|u_0\|_{L^\infty}$ such that

$$\big\langle A(x, z, p), p\big\rangle \geq |p| g(|p|) - f_1(|z|); \tag{3.3}$$

$$zB(x, z, p) \leq b_0 \big\langle A(x, z, p), p\big\rangle + f_2(|z|), \tag{3.4}$$

holds for all $x \in \Omega$, $|z| \geq M$ and $p \in \mathbb{R}^{2n}$, where f_1, f_2 and g are non-negative increasing functions. Also, we require $\big\langle A(x, u, \mathfrak{X}u), \mathfrak{X}u\big\rangle \in L^1(\Omega)$ and $u \in L^\infty(\Omega)$. The first condition (3.3), can be viewed as a weak ellipticity condition.

Additional conditions on f_1 and f_2, yields apriori integral estimates as in the following lemma. Similar results in Euclidean setting, can be found in [22] and [26].

Lemma 3.1. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of $\mathrm{Q}u = 0$ in Ω along with the conditions* (3.3) *and* (3.4) *and $u - u_0 \in HW_0^{1,G}(\Omega)$. If the functions f_1, f_2 and g satisfy*

$$(1)\ tg(t) \leq a_1 G(t); \tag{3.5}$$

$$(2)\ tg(t) f_1(Rt) + G(t) f_2(Rt) \leq a_1 G(t)^2, \tag{3.6}$$

for some $a_1 \geq 1, R > 0$ and every $t > M/R$, then there exists $c(n) > 0$ such that for $Q = 2n + 2$ and $c = c(n)[(1 + a_1)(1 + 2b_0)]^Q$, we have

$$\sup_\Omega G(|u|/R) \leq \max\Big\{\frac{c}{R^Q} \int_\Omega G(|u|/R)\, dx\,,\ (1 + a_1) G(M/R)\Big\}. \tag{3.7}$$

Proof. The proof is similar to that of Lemma 2.1 in [29] (see also Lemma 10.8 in [22]) and follows from standard Moser's iteration. We provide a brief outline.

Note that, we can assume $|u| \geq M$ without loss of generality, as otherwise we are done; we provide the proof for $u \geq M$, the proof for $u \leq -M$ is similar. The test function $\varphi = h(u)$ is used for the equation $\mathrm{Q}u = 0$, where letting $G = G(|u|/R)$ and $\tau = G(M/R)$, we choose

$$h(u) = uG^\beta \big|(1-\tau/G)^+\big|^{Q\beta+1},$$

for $\beta \geq 2b_0$ and $Q = 2n+2$. Thus $\varphi/u \geq 0$ and $\varphi = 0$ on $\partial\Omega$, since $M \geq \|u_0\|_{L^\infty}$. Hence, applying φ as a test function and using (3.4), we get

$$\begin{aligned}\int_\Omega \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\varphi\big\rangle\,dx &= \int_\Omega B(x,u,\mathfrak{X}u)\varphi\,dx \\ &\leq \int_\Omega \Big[b_0\big\langle A(x,u,\mathfrak{X}u),\mathfrak{X}u\big\rangle + f_2(|u|)\Big]\frac{\varphi}{u}\,dx.\end{aligned} \tag{3.8}$$

Note that $\mathfrak{X}\varphi = h'(u)\mathfrak{X}u$ and we have

$$h'(u) = \frac{\varphi}{u} + \bigg[\beta\Big(1-\frac{\tau}{G}\Big) + (Q\beta+1)\frac{\tau}{G}\bigg]G^{\beta-1}\big|(1-\tau/G)^+\big|^{Q\beta} g\Big(\frac{|u|}{R}\Big)\frac{u}{R},$$

which implies $h'(u) \geq (\beta+1)\varphi/u$ and $h'(u) \leq a_1(Q+2)(\beta+1)|(1-\tau/G)^+|^{Q\beta}G^\beta$ from (3.5). For every $\beta \geq 2b_0$, we obtain that

$$\begin{aligned}\frac{1}{2}\int_\Omega h'(u)g(|\mathfrak{X}u|)|\mathfrak{X}u|\,dx &\leq \int_\Omega \Big(h'(u) - b_0\varphi/u\Big)\Big[\big\langle A(x,u,\mathfrak{X}u),\mathfrak{X}u\big\rangle + f_1(|u|)\Big]\,dx \\ &\leq \int_\Omega \Big[f_2(|u|)\varphi/u + \Big(h'(u)\ - b_0\varphi/u\Big)f_1(|u|)\Big]\,dx,\end{aligned} \tag{3.9}$$

where we have used $h'(u) \geq 2b_0\,\varphi/u$ and (3.3) for the first inequality and (3.8) for the second inequality of the above. From (3.9) and (3.6), we obtain

$$\frac{1}{2}\int_\Omega h'(u)g(|\mathfrak{X}u|)|\mathfrak{X}u|\,dx \leq a_1(\beta+1)(2n+4)\int_\Omega \big|(1-\tau/G)^+\big|^{Q\beta}G^{\beta+1}\,dx. \tag{3.10}$$

Now, leting $w = \psi(G) = \frac{1}{2}G^{\beta+1}|(1-\tau/G)^+|^{Q\beta+1}$, note that $|\psi'(G)| \leq h'(u)g(|u|/R)|\mathfrak{X}u|/R$. Then, we use (2.24) of Lemma 2.12 with $t = |\mathfrak{X}u|$ and $s = |u|/R$, to obtain

$$\begin{aligned}\int_\Omega |\mathfrak{X}w|\,dx \leq \int_\Omega h'(u)g\Big(\frac{|u|}{R}\Big)\frac{|\mathfrak{X}u|}{R}\,dx &\leq \int_\Omega h'(u)\bigg[g\Big(\frac{|u|}{R}\Big)\frac{|u|}{R^2} + g(|\mathfrak{X}u|)\frac{|\mathfrak{X}u|}{R}\bigg]\,dx \\ &\leq \frac{c(n)}{R}a_1(\beta+1)\int_\Omega \big|(1-\tau/G)^+\big|^{Q\beta}G^{\beta+1}\,dx\end{aligned} \tag{3.11}$$

for some $c(n) > 0$, where for the last inequality of the above, we have used (3.10) and (3.5). Recalling Sobolev's inequality (2.5) with $q = 1$, we have

$$\Big(\int_\Omega w^\kappa\,dx\Big)^{1/\kappa} \leq c(n)\int_\Omega |\mathfrak{X}w|\,dx$$

for $\kappa = Q/(Q-1) = (2n+2)/(2n+1)$. Combining this with (3.11), we obtain

$$\Big(\int_\Omega \big|(1-\tau/G)^+\big|^{\kappa(Q\beta+1)}G^{\kappa(\beta+1)}\,dx\Big)^{1/\kappa} \leq \frac{c(n)}{R}a_1(\beta+1)\int_\Omega \big|(1-\tau/G)^+\big|^{Q\beta}G^{\beta+1}\,dx \tag{3.12}$$

which can be reduced to $\|v\|_{L^{\kappa\gamma}(\Omega,\mu)} \leq (\gamma/\gamma_0)^{1/\gamma}\|v\|_{L^{\gamma}(\Omega,\mu)}$, where $v = G|(1-\tau/G)^+|^Q$, $\gamma = \beta+1, \gamma_0 = 2b_0+1$ and the measure μ satisfying $d\mu = (\frac{c(n)}{R}a_1\gamma_0)^Q(1-\tau/G)^{-Q}dx$. Iterating with $\gamma_m = \kappa^m\gamma_0$ for $m = 0,1,2,\dots$ and taking $m\to\infty$, we finally obtain

$$\sup_\Omega G|(1-\tau/G)^+|^Q \leq c(n)\bigg(\frac{a_1(2b_0+1)}{R}\bigg)^Q\int_\Omega G\,dx$$

for some $c(n)>0$. It is easy to see that this yields (3.7), since $\sup_\Omega G > (1+a_1)\tau$ implies $\sup_\Omega G|(1-\tau/G)^+|^Q \geq \big(\frac{a_1}{1+a_1}\big)^Q \sup_\Omega G$. Thus, the proof is finished. □

Now, we are ready to prove the global maximum principle. For the Euclidean setting, similar theorems have been proved before, see e.g. Theorem 10.10 in [22].

Theorem 3.2. *Let $u\in HW^{1,G}(\Omega)$ be a weak solution of $\mathrm{Q}u=0$ in Ω with $\sup_{\partial\Omega}|u|<\infty$. We assume that there exists non-negative increasing functions f_1, f_2 and g such that the conditions* (3.3) *and* (3.4) *hold for $R = \operatorname{diam}(\Omega)$ and $0<b_0<1$; furthermore we assume $\Psi(t) = tg(t)$ is convex and g satisfies* (3.5) *for some $a_1\geq 1$. Then there exists $c_0 = c_0(n,a_1)$ sufficiently small such that, if f_1 and f_2 satisfy*

$$f_1(|z|) + \frac{f_2(|z|)}{1-b_0} \leq c_0\Psi\bigg(\frac{|z|}{R}\bigg) \tag{3.13}$$

for all $|z|\geq\sup_{\partial\Omega}|u|$, then for some $c(n,b_0,a_1)>0$, we have

$$\sup_\Omega G(|u|/R) \leq c(n,b_0,a_1)\sup_{\partial\Omega} G(|u|/R) \tag{3.14}$$

Proof. First notice that, since $\Psi(t) = tg(t)$ and g is increasing, we have $G(t)\leq\Psi(t)$ and from (3.5), we have $\Psi(t)\leq a_1G(t)$. These together imply that G is convex and doubling and so is Ψ, with 2^{a_1} as their doubling constant.

Let us denote $M = \sup_{\partial\Omega}|u|$ and $\Omega^+ = \{u>M\}$. We choose $\varphi = (u-M)^+$ as a test function for $\mathrm{Q}u=0$ and use (3.4) to get

$$\begin{aligned}\int_{\Omega^+}\big\langle A(x,u,\mathfrak{X}u),\mathfrak{X}u\big\rangle\,dx &= \int_{\Omega^+}(u-M)B(x,u,\mathfrak{X}u)\,dx\\ &\leq \int_{\Omega^+}\big(1-M/u\big)\Big[b_0\big\langle A(x,u,\mathfrak{X}u),\mathfrak{X}u\big\rangle + f_2(|u|)\Big]\,dx\\ &\leq \int_{\Omega^+} b_0\big\langle A(x,u,\mathfrak{X}u),\mathfrak{X}u\big\rangle\,dx + \int_{\Omega^+}f_2(|u|)\,dx,\end{aligned} \tag{3.15}$$

and then we use (3.15) together with (3.3) and (3.13) to obtain

$$\int_{\Omega^+}\Psi(|\mathfrak{X}u|)\,dx \leq \int_{\Omega^+}\bigg[f_1(|u|)+\frac{f_2(|u|)}{1-b_0}\bigg]\,dx \leq c_0\int_{\Omega^+}\Psi\bigg(\frac{|u|}{R}\bigg)\,dx. \tag{3.16}$$

Now, from the Poincaré inequality (2.19), we have

$$\int_\Omega\Psi\bigg(\frac{\varphi}{R}\bigg)\,dx \leq c(n,a_1)\int_\Omega\Psi(|\mathfrak{X}\varphi|)\,dx = c(n,a_1)\int_{\Omega^+}\Psi(|\mathfrak{X}u|)\,dx. \tag{3.17}$$

We have $\Psi(2\varphi/R) \leq 2^{a_1}\Psi(\varphi/R)$ from the doubling condition and letting $\Omega^* = \{u > 2M\}$, notice that $\Psi(u/R) \leq \Psi(2\varphi/R)$ on Ω^*. Using these together with (3.17) and (3.16), we get

$$\int_{\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx \leq \tau_0 \int_{\Omega^+} \Psi\left(\frac{|u|}{R}\right) dx = \tau_0 \left[\int_{\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx + \int_{\Omega^+\setminus\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx\right] \tag{3.18}$$

where $\tau_0 = 2^{a_1}c(n, a_1)c_0 < 1$ for small enough c_0. Hence, from (3.18), we arrive at

$$(1-\tau_0)\int_{\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx \leq \tau_0 \int_{\Omega^+\setminus\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx,$$

which, after adding $(1-\tau_0)\int_{\Omega^+\setminus\Omega^*}\Psi(|u|/R)dx$ on both sides, imply

$$(1-\tau_0)\int_{\Omega^+} \Psi\left(\frac{|u|}{R}\right) dx \leq \int_{\Omega^+\setminus\Omega^*} \Psi\left(\frac{|u|}{R}\right) dx \leq |\Omega^+|\Psi(2M/R). \tag{3.19}$$

From a similar argument with $\Omega^- = \{u < -M\}$, we can obtain

$$(1-\tau_0)\int_{\Omega^-} \Psi\left(\frac{|u|}{R}\right) dx \leq |\Omega^-|\Psi(2M/R). \tag{3.20}$$

Now for $\Omega^0 = \{|u| \leq M\}$, we directly have

$$(1-\tau_0)\int_{\Omega^0} \Psi\left(\frac{|u|}{R}\right) dx \leq |\Omega^0|\Psi(2M/R) \tag{3.21}$$

since Ψ is increasing. Thus, adding (3.19),(3.20) and (3.21), we obtain

$$(1-\tau_0)\int_{\Omega} \Psi\left(\frac{|u|}{R}\right) dx \leq |\Omega|\Psi(2M/R). \tag{3.22}$$

Now, if $c_0 < 1/a_1$, notice that multiplying $\Psi(|z|/R)$ on both sides of (3.13) and using inequality $G(t) \leq \Psi(t) \leq a_1 G(t)$, we can obtain

$$\Psi(|z|/R)f_1(|z|) + G(|z|/R)\frac{f_2(|z|)}{1-b_0} \leq a_1 G(|z|/R)^2$$

which is similar to (3.6). Hence, we can combine (3.7) of Lemma 3.1 with (3.22) and conclude $\sup_\Omega G(|u|/R) \leq c(n, b_0, a_1)G(M/R)$, which completes the proof. □

Remark 3.3. With minor modifications of the above arguments, the global bound can also be shown corresponding to u^+ for weak supersolutions u i.e. for $\mathrm{Q}u \geq 0$.

3.2. **Harnack Inequality.**

Here we show that weak solutions of $\mathrm{Q}u = 0$, satisfy Harnack inequality. The proofs are standard modifications of those in [40] and [29] for the Euclidean setting. We also refer to [6] for the Harnack inequalities on special cases, in the sub-elliptic setting.

In this subsection, we consider

$$\big\langle A(x,z,p), p\big\rangle \geq |p|g(|p|) - a_1\, g\left(\frac{|z|}{R}\right)\frac{|z|}{R} - g(\chi)\chi \tag{3.23}$$

$$|A(x,z,p)| \leq a_2\, g(|p|) + a_3\, g\left(\frac{|z|}{R}\right) + g(\chi) \tag{3.24}$$

for given non-negative constants a_1, a_2, a_3, and $\chi, R > 0$.

Theorem 3.4. *In $B_R \subset \Omega$, let $u \in HW^{1,G}(B_R) \cap L^\infty(B_R)$ be a weak supersolution, $\mathrm{Q}u \geq 0$ with $|u| \leq M$ in B_R and with the structure conditions* (3.23),(3.24) *and*

$$\operatorname{sign}(z) B(x,z,p) \leq \frac{1}{R}\Big[b_0\, g(|p|) + b_1\, g\Big(\frac{|z|}{R}\Big) + g(\chi)\Big] \tag{3.25}$$

for given non-negative constants a_1, a_2, a_3, b_0, b_1 and $g \in C^1([0,\infty))$ that satisfies (1.2) *with $\delta \geq 0$. Then for any $q > 0$ and $0 < \sigma < 1$, there exists $c = c(n, g_0, a_1, a_2, a_3, b_0 M, b_1, q) > 0$ such that, letting $Q = 2n+2$, we have*

$$\sup_{B_{\sigma R}} u^+ \leq \frac{c}{(1-\sigma)^{(1+g_0)Q/q}} \Big[\Big(⨍_{B_R} |u^+|^q \, dx \Big)^{\frac{1}{q}} + \chi R. \Big] \tag{3.26}$$

Proof. The proof is based on Moser's iteration, similar to that of Theorem 1.2 in [29]. We provide an outline. First notice that, using $\bar z = z + \chi R$, the structure conditions (3.23),(3.24) and (3.25) can be reduced to

$$\big\langle A(x,z,p), p \big\rangle \geq |p| g(|p|) - (1+a_1) g\big(|\bar z|/R\big) |\bar z|/R; \tag{3.27}$$

$$|A(x,z,p)| \leq a_2\, g(|p|) + (1+a_3) g\big(|\bar z|/R\big); \tag{3.28}$$

$$\bar z B(x,z,p) \leq b_0 |p| g(|p|) + (1+b_0+b_1) g\big(|\bar z|/R\big) |\bar z|/R. \tag{3.29}$$

To obtain (3.29), we multiply $\bar z$ on (3.25) and use (2.24) of Lemma 2.12 with $t = |\bar z|/R$ and $s = |p|$.

Hence, we use $\bar u = u^+ + \chi R$ for the proof. Given any $\sigma \in (0,1)$, we choose a standard cutoff function $\eta \in C_0^\infty(B_R)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\sigma R}$ and $|\mathfrak{X}\eta| \leq 2/(1-\sigma)R$. Then, for some $\gamma \in \mathbb{R}$ and $\beta \geq 1 + |\gamma|$ which are chosen later, we use

$$\varphi = \eta^\gamma \bar u\, G(\eta \bar u/R)^{\beta-1} e^{b_0 \bar u}$$

as a test function for $\mathrm{Q}u \geq 0$, to get

$$\begin{aligned}
&(1+b_0) \int_{B_R} \eta^\gamma G(\eta\bar u/R)^{\beta-1} e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\bar u \big\rangle \, dx \\
&\quad + \frac{\beta-1}{R} \int_{B_R} \eta^\gamma \bar u\, G(\eta\bar u/R)^{\beta-2} g(\eta\bar u/R) e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\bar u \big\rangle \, dx \\
&\leq -\frac{\beta-1}{R} \int_{B_R} \eta^\gamma |\bar u|^2 G(\eta\bar u/R)^{\beta-2} g(\eta\bar u/R) e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\eta \big\rangle \, dx \\
&\qquad - \gamma \int_{B_R} \eta^{\gamma-1} \bar u\, G(\eta\bar u/R)^{\beta-1} e^{b_0\bar u} \big\langle A(x,u,\mathfrak{X}u), \mathfrak{X}\eta \big\rangle \, dx \\
&\qquad + \int_{B_R} \eta^\gamma G(\eta\bar u/R)^{\beta-1} e^{b_0\bar u} \bar u\, B(x,u,\mathfrak{X}u) \, dx.
\end{aligned} \tag{3.30}$$

Now we use the structure condition (3.27) for the left hand side and (3.28),(3.29) for the right hand side of the above inequality. Then, we use (2.21) and (2.22) of Lemma 2.12 and also the fact that $e^{b_0\chi R} \le e^{b_0\bar{u}} \le e^{b_0(M+\chi R)}$, since $|u| \le M$ in B_R. We obtain

$$\begin{aligned}
\beta \int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|\,dx& \\
\le \frac{a_2\beta\, e^{b_0 M}}{(1-\sigma)} \int_{B_R} \eta^{\gamma-1} G(\eta\bar{u}/R)^{\beta-1} \frac{\bar{u}}{R}\, g(|\mathfrak{X}\bar{u}|)\,dx& \\
+ \beta(1+g_0)C_1 e^{b_0 M} \int_{B_R} \eta^{\gamma-1} G(\eta\bar{u}/R)^{\beta-1} g\Big(\frac{\bar{u}}{R}\Big)\frac{\bar{u}}{R}\,dx& \\
= I_1 + I_2&
\end{aligned} \tag{3.31}$$

where $C_1 = (1+a_1)(1+b_0) + (1+b_0+b_1) + (1+a_3)/(1-\sigma)$. Here onwards, we use $c = c(n, g_0, a_1, a_2, a_3, b_0 M, b_1) > 0$ as a large enough constant, throughout the rest of the proof. Now we estimate both I_1 and I_2 as follows.

For I_1, we use (2.24) with $t = \frac{2}{(1-\sigma)} a_2 e^{b_0 M}\bar{u}/\eta R$ and $s = |\mathfrak{X}\bar{u}|$, to obtain

$$\begin{aligned}
I_1 &\le \frac{\beta}{2}\int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|\,dx \\
&\quad + \frac{c\beta}{(1-\sigma)}\int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} \frac{\bar{u}}{\eta R}\, g\Big(\frac{\bar{u}}{(1-\sigma)\eta R}\Big)\,dx \\
&\le \frac{\beta}{2}\int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|\,dx \\
&\quad + \frac{c\beta}{(1-\sigma)^{1+g_0}}\int_{B_R} \eta^{\gamma-(2+2g_0)} G(\eta\bar{u}/R)^{\beta}\,dx,
\end{aligned} \tag{3.32}$$

where we have used $g(\bar{u}/\eta R) \le \eta^{-2g_0} g(\eta\bar{u}/R)$ for the latter inequality of the above.

For I_2, we trivially have

$$I_2 \le \frac{c\beta}{(1-\sigma)}\int_{B_R} \eta^{\gamma-1} G(\eta\bar{u}/R)^{\beta}\,dx. \tag{3.33}$$

Letting $\theta = 2+2g_0$ and combining (3.31) with (3.32) and (3.33), we obtain

$$\frac{\beta}{2}\int_{B_R} \eta^\gamma G(\eta\bar{u}/R)^{\beta-1} g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|\,dx \le \frac{c\beta}{(1-\sigma)^{\theta/2}}\int_{B_R} \eta^{\gamma-\theta} G(\eta\bar{u}/R)^{\beta}\,dx. \tag{3.34}$$

Now, we use Sobolev inequality

$$\Big(\int_{B_R} |w|^\kappa\,dx\Big)^{\frac{1}{\kappa}} \le c(n)\int_{B_R} |\mathfrak{X}w|\,dx$$

for $\kappa = Q/(Q-1) = (2n+2)/(2n+1)$ and $w = \eta^\gamma G(\eta\bar{u}/R)^\beta$ with the choice of $\gamma = -(Q-1)\theta$, so that $\kappa\gamma = -Q\theta = \gamma-\theta$. Combining with (3.34), we obtain

$$\Big(\int_{B_R} \eta^{-Q\theta} G(\eta\bar{u}/R)^{\kappa\beta}\,dx\Big)^{\frac{1}{\kappa}} \le \frac{c\beta}{(1-\sigma)^{\theta/2}}\int_{B_R} \eta^{-Q\theta} G(\eta\bar{u}/R)^{\beta}\,dx.$$

Iterating the above with $\beta_0 = q \ge Q\theta$ and $\beta_m = \kappa^m\beta_0$ and letting $m \to \infty$, we get

$$\sup_{B_R} G(\eta\bar{u}/R) \le \frac{c(q)}{(1-\sigma)^{Q\theta/2q}}\Big(\fint_{B_R} \eta^{-Q\theta} G(\eta\bar{u}/R)^{q}\,dx\Big)^{\frac{1}{q}}. \tag{3.35}$$

Hence, using (2.21), we get

$$\sup_{B_{\sigma R}} \bar{u} \leq \frac{c(q)}{(1-\sigma)^{Q\theta/2q}} \Big(\fint_{B_R} |\bar{u}|^q \, dx \Big)^{\frac{1}{q}}$$

for all $q \geq Q\theta$ and $c(q) = c(n, g_0, a_1, a_2, a_3, b_0 M, b_1, q) > 0$. Then from the interpolation argument in [11], we get the above for all $q > 0$. This concludes the proof. □

Theorem 3.5. *In $B_R \subset \Omega$, let $u \in HW^{1,G}(B_R) \cap L^\infty(B_R)$ be a weak subsolution,* $\mathrm{Q}u \leq 0$ *with $0 \leq u \leq M$ in B_R and with the structure conditions* (3.23),(3.24) *and*

$$\operatorname{sign}(z) B(x,z,p) \geq -\frac{1}{R}\Big[b_0\, g(|p|) + b_1\, g\Big(\frac{|z|}{R}\Big) + g(\chi) \Big] \tag{3.36}$$

for given non-negative constants a_1, a_2, a_3, b_0, b_1 and $g \in C^1([0,\infty))$ that satisfies (1.2) *with $\delta > 0$. Then there exists positive constants q_0 and c depending on $n, \delta, g_0, a_1, a_2, a_3, b_0 M, b_1$ such that, letting $Q = 2n+2$, we have*

$$\Big(\fint_{B_{R/2}} u^{q_0} \, dx \Big)^{\frac{1}{q_0}} \leq c \Big(\inf_{B_{R/4}} u + \chi R \Big) \tag{3.37}$$

Proof. Taking $\bar{u} = u + \chi R$ and $\eta \in C_0^\infty(B_{R/2})$ similarly as in the proof of Theorem 3.4, we can use the test function $\varphi = \eta^\gamma \bar{u} G(\bar{u}/\eta R) e^{-b_0 \bar{u}}$ on $\mathrm{Q}u \leq 0$ and obtain

$$\Big(\fint_{B_{R/2}} \bar{u}^{-q} \, dx \Big)^{-\frac{1}{q}} \leq c(q) \inf_{B_{R/4}} \bar{u} \tag{3.38}$$

for any $q > 0$. Now for any $0 < r \leq R$, we choose $\eta \in C_0^\infty(B_r)$ such that $0_{\mathbb{H}} \leq \eta \leq 1, \eta = 1$ in $B_{r/2}$ and $|\mathfrak{X}\eta| \leq 2/r$. Then we choose test function $\varphi = \eta^{g_0} \bar{u} G(\bar{u}/r)^{-1}$ in $\mathrm{Q}u \leq 0$. Here we use the fact that g satisfies (1.2) with $\delta > 0$, so that from (2.27) and (2.28), we have

$$G(\bar{u}/r)^{-1} - G(\bar{u}/r)^{-2} g(\bar{u}/r) \bar{u}/r \leq -G(\bar{u}/r)^{-1} \delta/(1+\delta).$$

Thus, using test function φ and structure conditions (3.27),(3.28) and (3.36), we obtain

$$\int_{B_r} \eta^{g_0} \frac{g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|}{G(\bar{u}/r)} \, dx \leq c \int_{B_r} \Big[(a_1 + a_3 + b_0 + b_1) \frac{g(\bar{u}/r)\bar{u}/r}{G(\bar{u}/r)} \Big] \, dx \leq c r^Q$$

where we suppress the dependence of a_i, b_j, g_0, δ and denote constant as c. Now, recalling (2.24), we use $t \leq t g(t)/g(s) + s$, with $t = |\mathfrak{X}u|$ and $s = \bar{u}/r$, to obtain

$$\begin{aligned} \int_{B_{r/2}} \frac{|\mathfrak{X}\bar{u}|}{\bar{u}} \, dx &\leq \int_{B_{r/2}} \Big[\frac{g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|}{\bar{u} g(\bar{u}/r)} + \frac{1}{r} \Big] \, dx \\ &\leq \frac{c}{r} \int_{B_r} \Big[\eta^{g_0} \frac{g(|\mathfrak{X}\bar{u}|)|\mathfrak{X}\bar{u}|}{G(\bar{u}/r)} + 1 \Big] \, dx \leq c\, r^{Q-1} \end{aligned} \tag{3.39}$$

Taking $w = \log(\bar{u})$, we use Poincaré inequality and (3.39) to get

$$\fint_{B_{r/2}} |w - \{w\}_{B_{r/2}}| \, dx \leq c\, r \fint_{B_{r/2}} |\mathfrak{X}w| \, dx = \frac{c}{r^{Q-1}} \int_{B_{r/2}} \frac{|\mathfrak{X}\bar{u}|}{\bar{u}} \, dx \leq c,$$

which shows that $w \in \mathrm{BMO}(B_{r/2})$. John-Nirenberg type inequalities in the setting of metric spaces with doubling measures, is known; we refer to [3]. This is applicable in our setting and the above inequality imples exponential inetegrability for $w = \log(\bar{u})$. Thus there exists $q_0 > 0$ and $c_0 > 0$ such that

$$\left(\fint_{B_{r/2}} \bar{u}^{-q_0}\, dx\right)\left(\fint_{B_{r/2}} \bar{u}^{q_0}\, dx\right) \leq \left(\fint_{B_{r/2}} e^{q_0|w-\{w\}_{B_{r/2}}|}\, dx\right)^2 \leq c_0^2. \tag{3.40}$$

for any $r \leq R$. Thus, (3.38) with $q = q_0$ and (3.40), concludes the proof. ☐

From Theorem 3.4 and Theorem 3.5, the following corrollary is immediate.

Corollary 3.6. *In $B_R \subset \Omega$, let $u \in HW^{1,G}(B_R) \cap L^\infty(B_R)$ be a weak solution of* $\mathrm{Q}u = 0$ *with $0 \leq u \leq M$ in B_R and with the structure conditions* (3.23),(3.24) *and*

$$|B(x, z, p)| \leq \frac{1}{R}\left[b_0\, g(|p|) + b_1\, g\left(\frac{|z|}{R}\right) + g(\chi)\right] \tag{3.41}$$

for given non-negative constants a_1, a_2, a_3, b_0, b_1 and $g \in C^1([0, \infty))$ that satisfies (1.2) *with $\delta > 0$. Then there exists $c = c(n, \delta, g_0, a_1, a_2, a_3, b_0M, b_1) > 0$ such that we have*

$$\sup_{B_{R/4}} u \leq c\left(\inf_{B_{R/4}} u + \chi R\right) \tag{3.42}$$

Thus, bounded weak solutions satisfy the Harnack inequality (3.42), which implies the Hölder continuity of weak solutions. By standard arguments, it is possible to show that there exists $\alpha = \alpha(n, \delta, g_0, a_1, a_2, a_3, b_0M, b_1) \in (0, 1)$ and $c = c(n, \delta, g_0, a_1, a_2, a_3, b_0M, b_1) > 0$ such that, we have

$$\mathrm{osc}_{B_r}\, u \leq c\left(\frac{r}{R}\right)^\alpha \left(\mathrm{osc}_{B_R}\, u + \chi R\right). \tag{3.43}$$

for every $0 < r < R$ and $B_R \subset \Omega$. This is enough to prove Theorem 1.1.

Remark 3.7. The growth and ellipticity conditions (3.23),(3.24) and (3.41) are special cases of the more general conditions in (3.3) and (3.4). When g satisfies (1.2), it is easy to see that (3.5) holds with $a_1 = 1 + g_0$ and (3.3), (3.4) and (3.6) holds if $f_1(|z|), f_2(|z|) \sim g(|z|/R)|z|/R + g(\chi)\chi$. Therefore, it is not restrictive to assume $|u| \leq M$ since we have Theorem 3.2 for the above cases. Furthermore, (3.41) can be relaxed to

$$|zB(x, z, p)| \leq b_0|p|g(|p|) + b_1 g\left(\frac{|z|}{R}\right)\frac{|z|}{R} + g(\chi)\chi \tag{3.44}$$

so that, in this case (3.29) can be obtained immediately.

4. Hölder continuity of Horizontal gradient

In this section, we consider a homogenous quasilinear equation where the operator does not depend on x and u. Estimates for this equation shall be necessary in Section 5. However, all results in this section are obtained independently, without any reference to the rest of this paper, apart from the usage of the structure function g in (1.2).

We warn the reader that in this section z is used as a variable in $\mathbb{R}^{2n}$, unlike the other sections. This is done to maintain continuity with [35].

In a domain $\Omega \subset \mathbb{H}^n$, we consider

$$\mathrm{div}_H(\mathcal{A}(\mathfrak{X}u)) = 0 \quad \text{in } \Omega, \tag{4.1}$$

where $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a given C^1 function. We denote $\mathcal{A}(z) = (\mathcal{A}_1(z), \mathcal{A}_2(z), \dots, \mathcal{A}_{2n}(z))$ for all $z \in \mathbb{R}^{2n}$ and $D\mathcal{A}(z)$ as the $2n \times 2n$ Jacobian matrix $(\partial \mathcal{A}_i(z)/\partial z_j)_{ij}$. We assume that $D\mathcal{A}(z)$ is symmetric and satisfies

$$\begin{gathered}\mathrm{F}(|z|)|\xi|^2 \leq \big\langle D\mathcal{A}(z)\,\xi, \xi\big\rangle \leq L\,\mathrm{F}(|z|)|\xi|^2; \\ |\mathcal{A}(z)| \leq L\,|z|\mathrm{F}(|z|).\end{gathered} \tag{4.2}$$

for every $z, \xi \in \mathbb{R}^{2n}$ and $L \geq 1$, where we denote $\mathrm{F}(t) = g(t)/t$ maintaining the notation (3.2). Here $g : [0, \infty) \to [0, \infty)$ is a given C^1 function satisfying (1.2) and $g(0) = 0$.

The above equation has been considered previously in [35] where local boundedness of $\mathfrak{X}u$ for a weak solution u of (4.1), has been established. The goal of this section is to prove the local Hölder continuity of $\mathfrak{X}u$. We restate Theorem 1.3 here, which is the main result of this section.

Theorem 4.1. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of the equation (4.1) with structure condition (4.2) and g satisfies (1.2) with $\delta > 0$. Then $\mathfrak{X}u$ is locally Hölder continuous and there exists $\sigma = \sigma(n, g_0, L) \in (0, 1)$ such that for any $B_{r_0} \subset \Omega$ and $0 < r < r_0/2$, we have*

$$\max_{1\leq l\leq 2n} \fint_{B_r} G(|X_l u - \{X_l u\}_{B_r}|)\,dx \leq c\Big(\frac{r}{r_0}\Big)^{\sigma} \fint_{B_{r_0}} G(|\mathfrak{X}u|)\,dx \tag{4.3}$$

where $c > 0$ depends on n, δ, g_0, L.

4.1. **Previous Results.**

Here we provide some results that are known and previously obtained, which would be essential for our purpose. For more details, we refer to [35] and references therein.

The following monotonicity and ellipticity inequalities follow easily from (4.2).

$$(1)\ \big\langle \mathcal{A}(z) - \mathcal{A}(w), z - w\big\rangle \geq c(g_0)\begin{cases}|z-w|^2\,\mathrm{F}(|z|) & \text{if } |z-w| \leq 2|z| \\ |z-w|^2\,\mathrm{F}(|z-w|) & \text{if } |z-w| > 2|z|\end{cases} \tag{4.4}$$

$$(2)\ \big\langle \mathcal{A}(z), z\big\rangle \geq c(g_0)\,|z|^2\mathrm{F}(|z|) \geq c(g_0)\,G(|z|) \tag{4.5}$$

for all $z, w \in \mathbb{R}^{2n}$ and some constant $c(g_0) > 0$. These are essential to show the existence of a weak solution $u \in HW^{1,G}(\Omega)$ of the equation (4.1). We refer to [35] for a brief discussion on existence and uniqueness for (4.1). The following theorem is Theorem 1.1 of [35], which shows the local Lipschitz continuity of the weak solutions.

Theorem 4.2. *Let $u \in HW^{1,G}(\Omega)$ be a weak solution of equation (4.1) satisfying structure condition (4.2) and g satisfies (1.2) with $\delta > 0$. Then $\mathfrak{X}u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$; moreover for any $B_r \subset \Omega$, we have*

$$\sup_{B_{\sigma r}} G(|\mathfrak{X}u|) \leq \frac{c}{(1-\sigma)^Q} \fint_{B_r} G(|\mathfrak{X}u|)\,dx \tag{4.6}$$

for any $0 < \sigma < 1$, where $c = c(n, g_0, \delta, L) > 0$ is a constant.

Now, we also require the following apriori assumption as considered in [35], in order to temporarily remove possible singularities of the function F. Here onwards, this shall be assumed until the end of this section.

$$(A):\ \text{There exists } m_1, m_2 > 0 \text{ such that } \lim_{t\to 0}\mathrm{F}(t) = m_1 \text{ and } \lim_{t\to\infty}\mathrm{F}(t) = m_2. \tag{4.7}$$

This combined with the local boundedness of $\mathfrak{X}u$ from Theorem 4.2, makes the equation (4.1) to be uniformly elliptic and enables us to conlcude

$$\mathfrak{X}u \in HW^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n}), \quad Tu \in HW^{1,2}_{\mathrm{loc}}(\Omega) \cap C^{0,\alpha}_{\mathrm{loc}}(\Omega) \tag{4.8}$$

from Theorem 1.1 and Theorem 3.1 of Capogna [5]. However, every estimates in this section, are independent of the constants m_1 and m_2 and (4.7) shall be ultimately removed.

The regularity (4.8) is necessary to differentiate the equation (4.1) and obtain the equations satisfied by $X_l u$ and Tu, as shown in the following two lemmas. The proofs are simple and omitted here, we refer to [35] and [43] for details.

Lemma 4.3. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of* (4.1)*, then Tu is a weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_j(Tu)) = 0. \tag{4.9}$$

Lemma 4.4. *If $u \in HW^{1,G}(\Omega)$ is a weak solution of* (4.1)*, then for any $l \in \{1, \ldots, n\}$, we have that $X_l u$ is weak solution of*

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_lu) + \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_{n+l}(\mathfrak{X}u)Tu) + T(\mathcal{A}_{n+l}(\mathfrak{X}u)) = 0 \tag{4.10}$$

and similarly, $X_{n+l}u$ is weak solution of

$$\sum_{i,j=1}^{2n} X_i(D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_{n+l}u) - \sum_{i=1}^{2n} X_i(D_i\mathcal{A}_l(\mathfrak{X}u)Tu) - T(\mathcal{A}_l(\mathfrak{X}u)) = 0. \tag{4.11}$$

We enlist some Caccioppoli type inequalitites, that are very similar to those in [43] and [34]. They will be essential for the estimates in the next subsection.

The following lemma is similar to Lemma 3.3 in [43], the proof is trivial and omitted here.

Lemma 4.5. *For any $\beta \geq 0$ and all $\eta \in C^\infty_0(\Omega)$, we have, for some $c = c(n, g_0, L) > 0$, that*

$$\int_\Omega \eta^2\, \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^\beta |\mathfrak{X}(Tu)|^2\, dx \leq \frac{c}{(\beta+1)^2} \int_\Omega |\mathfrak{X}\eta|^2 \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^{\beta+2}\, dx.$$

The following lemma is similar to Corollary 3.2 of [43] and Lemma 2.5 of [34]. This is crucial for the proof of the Hölder continuity of the horizontal gradient. The proof of the lemma is similar to that in [43] and involves few other Caccioppoli type estimates. An outline is provided in Appendix II, for the reader's convenience.

Lemma 4.6. *For any $q \geq 4$ and all non-negative $\eta \in C^\infty_0(\Omega)$, we have that*

$$\int_\Omega \eta^q\, \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^q\, dx \leq c(q) K^{q/2} \int_{\mathrm{supp}(\eta)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^q\, dx, \tag{4.12}$$

where $K = \|\mathfrak{X}\eta\|^2_{L^\infty} + \|\eta T\eta\|_{L^\infty}$ and $c(q) = c(n, g_0, L, q) > 0$.

The following corollary follows immediately from Lemma 4.5 and Lemma 4.6.

Corollary 4.7. *For any $q \geq 4$ and all non-negative $\eta \in C^\infty_0(\Omega)$, we have*

$$\int_\Omega \eta^{q+2}\, \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^{q-2} |\mathfrak{X}(Tu)|^2\, dx \leq c(q) K^{\frac{q+2}{2}} \int_{spt(\eta)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^q\, dx,$$

where $K = \|\mathfrak{X}\eta\|^2_{L^\infty} + \|\eta T\eta\|_{L^\infty}$ and $c(q) = c(n, g_0, L, q) > 0$.

4.2. **The truncation argument.**
In this subsection, we follow the technique of [34] and prove Caccioppoli type inequalities invovling a double truncation of horizontal derivatives. In the setting of Euclidean spaces, similar ideas have been implemented previously by Tolksdorff [39] and Lieberman [28].

Here onwards, throughout this section, we shall denote $u \in HW^{1,G}(\Omega)$ as a weak solution of (4.1) and equipped with local Lipschitz continuity from Theorem 4.2, we denote

$$\mu_i(r) = \sup_{B_r} |X_i u|, \quad \mu(r) = \max_{1\leq i\leq 2n} \mu_i(r). \tag{4.13}$$

for a fixed ball $B_r \subset \Omega$.

We fix any $l \in \{1, 2, .., 2n\}$ and consider the following double truncation

$$v := \min\big(\mu(r)/8\,,\, \max\left(\mu(r)/4 - X_l u, 0\right)\big). \tag{4.14}$$

It is important to note that, from the regularity (4.8), we have

$$\mathfrak{X} v \in L^2_{\text{loc}}(\Omega; \mathbb{R}^{2n}), \quad T v \in L^2_{\text{loc}}(\Omega) \tag{4.15}$$

and moreover, letting

$$E = \{x \in \Omega : \mu(r)/8 < X_l u < \mu(r)/4\}, \tag{4.16}$$

we have that

$$\mathfrak{X} v = \begin{cases} -\mathfrak{X} X_l u & \text{a.e. in } E; \\ 0 & \text{a.e. in } \Omega \setminus E, \end{cases} \quad \text{and} \quad T v = \begin{cases} -T X_l u & \text{a.e. in } E; \\ 0 & \text{a.e. in } \Omega \setminus E. \end{cases} \tag{4.17}$$

The properties of this truncation shall be exploited for proving all the following Caccioppoli type estimates. In particular, notice that

$$\mu(r)/8 \leq |\mathfrak{X} u| \leq (2n)^{1/2} \mu(r) \quad \text{in } E \cap B_r; \tag{4.18}$$

since $\mathrm{F}(t) = g(t)/t$, (4.18) combined with (2.25) implies

$$\frac{1}{8^{g_0}(2n)^{1/2}} \mathrm{F}(\mu(r)) \leq \mathrm{F}(|\mathfrak{X} u|) \leq 8(2n)^{g_0/2} \mathrm{F}(\mu(r)) \quad \text{in } E \cap B_r, \tag{4.19}$$

which shall be used several times during the estimates that follow in this subsection. The main lemma required to prove Theorem 4.1, is the following.

Lemma 4.8. *Let v be the truncation (4.14) and $\eta \in C_0^\infty(B_r)$ be a non-negative cut-off function such that $0 \leq \eta \leq 1$ in B_r, $\eta = 1$ in $B_{r/2}$ and that $|\mathfrak{X}\eta| \leq 4/r$, $|\mathfrak{X}\mathfrak{X}\eta| \leq 16n/r^2$. Then we have the following Caccioppoli type inequality*

$$\int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X} v|^2\, dx \leq c(\beta+2)^2 \frac{|B_r|^{1-1/\gamma}}{r^2} \mu(r)^4 \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{1/\gamma} \tag{4.20}$$

for all $\beta \geq 0$ and $\gamma > 1$, where $c = c(n, g_0, L, \gamma) > 0$ is a constant.

In the setting of equations with p-laplace type growth, the above lemma has been shown previously in [34](see Lemma 1.1). The proof is going to be similar. Hence, we would require two auxillary lemmas, similarly as in [34].

We also remark that the inequality (4.20) also holds corresponding to the truncation

$$v' = \min\big(\mu(r)/8, \max(\mu(r)/4 + X_l u, 0)\big),$$

and the proof can be carried out in the same way as that of Lemma 4.8.

The following lemma is the analogue of Lemma 3.1 of [34]. The proof is similar and lengthy, which we provide in the Appendix I.

Lemma 4.9. *For any $\beta \geq 0$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$
\begin{aligned}
&\int_\Omega \eta^{\beta+2} v^{\beta+2} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2\, dx \\
&\qquad \leq c(\beta+2)^2 \int_\Omega \eta^\beta \big(|\mathfrak{X}\eta|^2 + \eta|T\eta|\big) v^{\beta+2} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4\, dx \\
&\qquad\quad + c(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^\beta \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4 |\mathfrak{X}v|^2\, dx \\
&\qquad\quad + c \int_\Omega \eta^{\beta+2} v^{\beta+2} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2 |Tu|^2\, dx,
\end{aligned}
\tag{4.21}
$$

where v is as in (4.14) *and $c = c(n, g_0, L) > 0$.*

Throughout the rest of this subsection, we fix a ball $B_r \subset \Omega$ and a cut-off function $\eta \in C_0^\infty(B_r)$ that satisfies

$$
0 \leq \eta \leq 1 \quad \text{in } B_r, \quad \eta = 1 \quad \text{in } B_{r/2}
\tag{4.22}
$$

and

$$
|\mathfrak{X}\eta| \leq 4/r, \quad |\mathfrak{X}\mathfrak{X}\eta| \leq 16n/r^2, \quad |T\eta| \leq 32n/r^2 \quad \text{in } B_r.
\tag{4.23}
$$

The following technical lemma, that is required for the proof of Lemma 4.8, is a weighted Caccioppoli inequality for Tu involving v similar to that in Lemma 3.2 of [34]. We provide the proof here for sake of completeness.

Lemma 4.10. *Let $B_r \subset \Omega$ be a ball and $\eta \in C_0^\infty(B_r)$ be a cut-off function satisfying* (4.22) *and* (4.23)*. Let $\tau \in (1/2, 1)$ and $\gamma \in (1,2)$ be two fixed numbers. Then, for any $\beta \geq 0$, we have the following estimate,*

$$
\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2\, dx \leq c(\beta+2)^{2\tau} \frac{|B_r|^{1-\tau}}{r^{2(2-\tau)}} \mathrm{F}(\mu(r))\mu(r)^6\, J^\tau,
\tag{4.24}
$$

where $c = c(n, g_0, L, \tau, \gamma) > 0$ and

$$
J = \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \; + \; \mu(r)^4 \frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx \Big)^{\frac{1}{\gamma}}.
\tag{4.25}
$$

Proof. We denote the left hand side of (4.24) by M,

$$
M = \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4 |\mathfrak{X}(Tu)|^2\, dx,
\tag{4.26}
$$

where $1/2 < \tau < 1$. Now we use $\varphi = \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)}|\mathfrak{X}u|^4\, Tu$ as a test function for the equation (4.9). We obtain that

$$
\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)}|\mathfrak{X}u|^4 D_j\mathcal{A}_i(\mathfrak{X}u)X_jTu\, X_iTu\, dx\\
=&-(\tau(\beta+2)+4)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+3}\, v^{\tau(\beta+4)}|\mathfrak{X}u|^4 TuD_j\mathcal{A}_i(\mathfrak{X}u)X_jTu\, X_i\eta\, dx\\
&-\tau(\beta+4)\int_\Omega \sum_{i,j=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)-1}|\mathfrak{X}u|^4 TuD_j\mathcal{A}_i(\mathfrak{X}u)X_jTu\, X_iv\, dx\\
&-4\int_\Omega \sum_{i,j,k=1}^{2n} \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)}|\mathfrak{X}u|^2 X_kuTuD_j\mathcal{A}_i(\mathfrak{X}u)X_jTu\, X_iX_ku\, dx\\
=&K_1+K_2+K_3,
\end{aligned}
\tag{4.27}
$$

where the integrals in the right hand side of (4.27) are denoted by K_1, K_2, K_3 in order. To prove the lemma, we estimate both sides of (4.27) as follows.

For the left hand side, we have by the structure condition (4.2) that

$$
\text{left of (4.27)} \geq \int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}(Tu)|^2\, dx = M,
\tag{4.28}
$$

and for the right hand side of (4.27), we estimate each item $K_i, i = 1, 2, 3$, one by one.

To this end, we denote

$$
\tilde{K} = \int_\Omega \eta^{(2\tau-1)(\beta+2)+6}\, v^{(2\tau-1)(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|Tu|^2|\mathfrak{X}(Tu)|^2\, dx.
\tag{4.29}
$$

First, we estimate K_1 by the structure condition (4.2) and Hölder's inequality, to get

$$
\begin{aligned}
|K_1| \leq& c(\beta+2)\int_\Omega \eta^{\tau(\beta+2)+3}\, v^{\tau(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|Tu||\mathfrak{X}(Tu)||\mathfrak{X}\eta|\, dx\\
\leq& c(\beta+2)\tilde{K}^{\frac{1}{2}}\Big(\int_\Omega \eta^{\beta+2}v^{\beta+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}\eta|^2\, dx\Big)^{\frac{1}{2}},
\end{aligned}
\tag{4.30}
$$

where $c = c(n, g_0, L, \tau) > 0$.

Second, we estimate K_2 also by the structure condition (4.2) and Hölder's inequality,

$$
\begin{aligned}
|K_2| \leq& c(\beta+2)\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)-1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|Tu||\mathfrak{X}(Tu)||\mathfrak{X}v|dx\\
\leq& c(\beta+2)\tilde{K}^{\frac{1}{2}}\Big(\int_\Omega \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\, dx\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{4.31}
$$

Finally, we estimate K_3. In the following, the first inequality follows from the structure condition (4.2), the second from Hölder's inequality and the third from Lemma 4.9. We have

$$
\begin{aligned}
|K_3| &\le c\int_\Omega \eta^{\tau(\beta+2)+4}\, v^{\tau(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^3|Tu||\mathfrak{X}(Tu)||\mathfrak{X}\mathfrak{X}u|\,dx\\
&\le c\tilde{K}^{\frac12}\Big(\int_\Omega \eta^{\beta+4}v^{\beta+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\Big)^{\frac12}\\
&\le c\,\tilde{K}^{\frac12}\, I^{\frac12},
\end{aligned}
\tag{4.32}
$$

where I is the right hand side of (4.21) in Lemma 4.9

$$
\begin{aligned}
I = {} & c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\\
&+c\int_\Omega \eta^{\beta+4}v^{\beta+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|Tu|^2\,dx.
\end{aligned}
\tag{4.33}
$$

where $c=c(n,g_0,L)>0$. Notice that the integrals on the right hand side of (4.30) and (4.31) are both controlled from above by I. Hence, we can combine (4.30), (4.31) and (4.32) to obtain

$$|K_1|+|K_2|+|K_3|\le\; c\tilde{K}^{\frac12}I^{\frac12},$$

from which, together with the estimate (4.28) for the left hand side of (4.27), it follows that

$$M\le c\tilde{K}^{\frac12}I^{\frac12}, \tag{4.34}$$

where $c=c(n,g_0,L,\tau)>0$. Now, we estimate $\tilde{K}$ by Hölder's inequality as follows.

$$
\begin{aligned}
\tilde{K}\;&\le\Big(\int_\Omega \eta^{\tau(\beta+2)+4}\,v^{\tau(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}(Tu)|^2\,dx\Big)^{\frac{2\tau-1}{\tau}}\\
&\quad\times\Big(\int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|Tu|^{\frac{2\tau}{1-\tau}}|\mathfrak{X}(Tu)|^2\,dx\Big)^{\frac{1-\tau}{\tau}}\\
&=M^{\frac{2\tau-1}{\tau}}H^{\frac{1-\tau}{\tau}},
\end{aligned}
\tag{4.35}
$$

where M is as in (4.26) and we denote by H the second integral on the right hand side of (4.35)

$$H=\int_\Omega \eta^{\frac{2\tau}{1-\tau}+4}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|Tu|^{\frac{2\tau}{1-\tau}}|\mathfrak{X}(Tu)|^2\,dx. \tag{4.36}$$

Combining (4.35) and (4.34), we get

$$M\le cH^{1-\tau}I^\tau, \tag{4.37}$$

for some $c=c(n,g_0,L,\tau)>0$. To estimate M, we estimate H and I from above. We estimate H by Corollary 4.7 with $q=2/(1-\tau)$ and monotonicity of g, to obtain

$$
\begin{aligned}
H&\le c\mu(r)^4\int_\Omega \eta^{q+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^{q-2}|\mathfrak{X}(Tu)|^2\,dx\\
&\le\frac{c}{r^{q+2}}\mu(r)^4\int_{B_r}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^q\,dx\le\frac{c}{r^{q+2}}|B_r|\mathrm{F}(\mu(r))\mu(r)^{q+4},
\end{aligned}
\tag{4.38}
$$

where $c = c(n, g_0, L, \tau) > 0$.

Now, we fix $1 < \gamma < 2$ and estimate each term of I in (4.33) as follows. For the first term of I, we have by Hölder's inequality and monotonicity of g that

$$\begin{aligned}(4.39)\qquad &\int_\Omega \eta^{\beta+2} v^{\beta+4} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 \big(|\mathfrak{X}\eta|^2 + \eta|T\eta|\big)\, dx \\ &\qquad\qquad \le \frac{c}{r^2} \mathrm{F}(\mu(r))\mu(r)^8 |B_r|^{1-\frac{1}{\gamma}} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{\frac{1}{\gamma}}.\end{aligned}$$

For the second term of I, we similarly have

$$(4.40)\qquad \int_\Omega \eta^{\beta+4} v^{\beta+2} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^4 |\mathfrak{X}v|^2\, dx \le c\mathrm{F}(\mu(r))\mu(r)^4 \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx.$$

For the third term of I, we have that

$$\begin{aligned}(4.41)\qquad &\int_\Omega \eta^{\beta+4} v^{\beta+4} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^2 |Tu|^2\, dx \\ &\qquad \le \Big(\int_\Omega \eta^{\frac{2\gamma}{\gamma-1}} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^2 |Tu|^{\frac{2\gamma}{\gamma-1}}\, dx\Big)^{1-\frac{1}{\gamma}} \\ &\qquad\quad \times \Big(\int_\Omega \eta^{\gamma(\beta+2)} v^{\gamma(\beta+4)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^2\, dx\Big)^{\frac{1}{\gamma}} \\ &\qquad \le \frac{c}{r^2} \mathrm{F}(\mu(r))\mu(r)^8 |B_r|^{1-\frac{1}{\gamma}} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{\frac{1}{\gamma}}\end{aligned}$$

where $c = c(n, g_0, L, \gamma) > 0$. Here in the above inequalities, the first one follows from Hölder's inequality and the second from Lemma 4.6 and monotonicity of g. Combining the estimates for three items of I above (4.39), (4.40) and (4.41), we get the following estimate for I,

$$(4.42)\qquad I \le c(\beta+2)^2 \mathrm{F}(\mu(r))\mu(r)^4 J,$$

where J is defined as in (4.25)

$$J = \int_{B_r} \eta^{\beta+4} v^{\beta+2} |\mathfrak{X}v|^2\, dx \ + \ \mu(r)^4 \frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2} \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta}\, dx\Big)^{\frac{1}{\gamma}}.$$

Now from the estimates (4.38) for G and (4.42) for I, we obtain the desired estimate for M by (4.37). Combing (4.38), (4.42) and (4.37), we end up with

$$(4.43)\qquad M \le c(\beta+2)^{2\tau} \frac{|B_r|^{1-\tau}}{r^{2(2-\tau)}} \mathrm{F}(\mu(r))\mu(r)^6 J^\tau,$$

where $c = c(n, g_0, L, \tau, \gamma) > 0$. This completes the proof. □

Now we provide the proof of Lemma 4.8, for completeness.

Proof of Lemma 4.8. First, notice that we may assume $\gamma < 3/2$, since otherwise we can apply Hölder's inequality to the integral in the right hand side of the claimed inequality (4.20). Also, we recall from (4.14), that for some $l \in \{1, \ldots, 2n\}$,

$$v = \min\big(\mu(r)/8\,,\, \max\left(\mu(r)/4 - X_l u, 0\right)\big).$$

We prove the lemma assuming $l \in \{1, \ldots, n\}$; the case for $l \in \{n+1, \ldots, 2n\}$ can be proven similarly. Henceforth, we fix $1 < \gamma < 3/2$ and $l \in \{1, \ldots, n\}$ throughout the rest of the proof. Let $\beta \geq 0$ and $\eta \in C_0^\infty(B_r)$ be a cut-off function satisfying (4.22) and (4.23). Using test function $\varphi = \eta^{\beta+4} v^{\beta+3}$ for the equation (4.10), we obtain

$$
\begin{aligned}
-(\beta+3) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i v \, dx \\
= (\beta+4) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+3} v^{\beta+3} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i \eta \, dx \\
+ (\beta+4) \int_\Omega \sum_{i=1}^{2n} \eta^{\beta+3} v^{\beta+3} D_i \mathcal{A}_{n+l}(\mathfrak{X}u) T u X_i \eta \, dx \\
+ (\beta+3) \int_\Omega \sum_{i=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_i \mathcal{A}_{n+l}(\mathfrak{X}u) X_i v \, T u \, dx \\
- \int_\Omega \eta^{\beta+4} v^{\beta+3} \, T\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big) \, dx.
\end{aligned}
\tag{4.44}
$$

Now notice that from (2.2), we have

$$
\begin{aligned}
&\sum_{i,j=1}^{2n} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i \eta + \sum_{i=1}^{2n} D_i \mathcal{A}_{n+l}(\mathfrak{X}u) T u X_i \eta \\
&\quad = \sum_{i,j=1}^{2n} D_j \mathcal{A}_i(\mathfrak{X}u) X_l X_j u X_i \eta = \sum_{i=1}^{2n} X_l\big(\mathcal{A}_i(\mathfrak{X}u)\big) X_i \eta.
\end{aligned}
$$

Thus, we can combine the first two integrals in the right hand side of (4.44) by the above equality. Then (4.44) becomes

$$
\begin{aligned}
-(\beta+3) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i v \, dx \\
= (\beta+4) \int_\Omega \sum_{i=1}^{2n} \eta^{\beta+3} v^{\beta+3} X_l\big(\mathcal{A}_i(\mathfrak{X}u)\big) X_i \eta \, dx \\
+ (\beta+3) \int_\Omega \sum_{i=1}^{2n} \eta^{\beta+4} v^{\beta+2} D_i \mathcal{A}_{n+l}(\mathfrak{X}u) X_i v T u \, dx \\
- \int_\Omega \eta^{\beta+4} v^{\beta+3} T\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big) \, dx \\
= I_1 + I_2 + I_3,
\end{aligned}
\tag{4.45}
$$

where we denote the terms in the right hand side of (4.45) by I_1, I_2, I_3, respectively.

We will estimate both sides of (4.45) as follows. For the left hand side, denoting E as in (4.16) and using structure condition (4.2), we have

$$
\begin{aligned}
\text{left of (4.45)} &\geq (\beta+3)\int_E \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v|^2\,dx \\
&\geq c_0(\beta+2)\mathrm{F}(\mu(r))\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx,
\end{aligned}
\tag{4.46}
$$

for a constant $c_0 = c_0(n, g_0, L) > 0$. Here we have used (4.17) and (4.19).

For the right hand side of (4.45), we claim that each item I_1, I_2, I_3 satisfies

$$
\begin{aligned}
|I_m| \;&\leq \frac{c_0}{6}(\beta+2)\mathrm{F}(\mu(r))\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx \\
&\quad + c(\beta+2)^3\frac{|B_r|^{1-1/\gamma}}{r^2}\mathrm{F}(\mu(r))\mu(r)^4\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{1/\gamma},
\end{aligned}
\tag{4.47}
$$

where $m = 1, 2, 3$, $1 < \gamma < 3/2$ and c is a constant depending only on n, g_0, L and γ. Then the lemma follows from the estimate (4.46) for the left hand side of (4.45) and the above claim (4.47) for each item in the right. Thus, we are only left with proving the claim (4.47).

In the rest of the proof, we estimate I_1, I_2, I_3 one by one. First for I_1, using integration by parts, we have that

$$
I_1 = -(\beta+4)\int_\Omega \sum_{i=1}^{2n}\mathcal{A}_i(\mathfrak{X}u)X_l\big(\eta^{\beta+3}v^{\beta+3}X_i\eta\big)\,dx,
$$

from which it follows by the structure condition (4.2), that

$$
\begin{aligned}
|I_1| \leq\, & c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta+3}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|\big(|\mathfrak{X}\eta|^2+\eta|\mathfrak{X}\mathfrak{X}\eta|\big)\,dx \\
&+ c(\beta+2)^2\int_\Omega \eta^{\beta+3}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||\mathfrak{X}v||\mathfrak{X}\eta|\,dx \\
\leq\, & \frac{c}{r^2}(\beta+2)^2\mathrm{F}(\mu(r))\mu(r)^4\int_{B_r}\eta^\beta v^\beta\,dx \\
&+ \frac{c}{r}(\beta+2)^2\mathrm{F}(\mu(r))\mu(r)^2\int_{B_r}\eta^{\beta+2}v^{\beta+1}|\mathfrak{X}v|\,dx,
\end{aligned}
\tag{4.48}
$$

where $c = c(n, g_0, L) > 0$. For the latter inequality of (4.48), we have used the fact that $g(t) = t\mathrm{F}(t)$ is monotonically increasing. Now we apply Young's inequality to the last term of (4.48) to end up with

$$
\begin{aligned}
|I_1| \leq\, & \frac{c_0}{6}(\beta+2)\mathrm{F}(\mu(r))\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx \\
&+ \frac{c}{r^2}(\beta+2)^3\mathrm{F}(\mu(r))\mu(r)^4\int_{B_r}\eta^\beta v^\beta\,dx,
\end{aligned}
\tag{4.49}
$$

where $c = c(n, g_0, L) > 0$ and c_0 is the same constant as in (4.46). The claimed estimate (4.47) for I_1, follows from the above estimate (4.49) and Hölder's inequality.

To estimate I_2, we have by the structure condition (4.2) that

$$|I_2| \leq c(\beta+2)\int_\Omega \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v||Tu|\,dx,$$

from which it follows by Hölder's inequality that

$$\begin{aligned} |I_2| \leq & c(\beta+2)\Big(\int_E \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v|^2\,dx\Big)^{\frac{1}{2}} \\ & \times\Big(\int_E \eta^{\gamma(\beta+2)}v^{\gamma(\beta+2)}\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx\Big)^{\frac{1}{2\gamma}} \\ & \times\Big(\int_\Omega \eta^q\,\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^q\,dx\Big)^{\frac{1}{q}}, \end{aligned} \tag{4.50}$$

where $q = 2\gamma/(\gamma-1)$. The fact that the integrals are on the set E, is crucial since we can use (4.19) and the following estimates can not be carried out unless the function F is increasing. We have the following estimates for the first two integrals of the above, using (4.19).

$$\int_E \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}v|^2\,dx \leq c\mathrm{F}(\mu(r))\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx, \tag{4.51}$$

and

$$\int_E \eta^{\gamma(\beta+2)}v^{\gamma(\beta+2)}\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx \leq c\mathrm{F}(\mu(r))\mu(r)^{2\gamma}\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx, \tag{4.52}$$

where $c = c(n, g_0, L) > 0$. We estimate the last integral in the right hand side of (4.50) by (4.12) of Lemma 4.6 and monotonicity of g, to obtain

$$\int_\Omega \eta^q\,\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^q\,dx \leq \frac{c}{r^q}\int_{B_r}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^q\,dx \leq \frac{c|B_r|}{r^q}\mathrm{F}(\mu(r))\mu(r)^q, \tag{4.53}$$

where $c = c(n, g_0, L, \gamma) > 0$. Now combining the above three estimates (4.51), (4.52) and (4.53) for the three integrals in (4.50) respectively, we end up with the following estimate for I_2

$$|I_2| \leq c(\beta+2)\frac{|B_r|^{\frac{\gamma-1}{2\gamma}}}{r}\mathrm{F}(\mu(r))\mu(r)^2\Big(\int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{2\gamma}},$$

from which, together with Young's inequality, the claim (4.47) for I_2 follows.

Finally, we prove the claim (4.47) for I_3. Recall that

$$I_3 = -\int_\Omega \eta^{\beta+4}v^{\beta+3}T\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)\,dx.$$

By virtue of the regularity (4.15) for v, integration by parts yields

$$\begin{aligned} I_3 = & \int_\Omega \mathcal{A}_{n+l}(\mathfrak{X}u)T\big(\eta^{\beta+4}v^{\beta+3}\big)\,dx \\ = & (\beta+4)\int_\Omega \eta^{\beta+3}v^{\beta+3}\mathcal{A}_{n+l}(\mathfrak{X}u)T\eta\,dx \\ & + (\beta+3)\int_\Omega \eta^{\beta+4}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)Tv\,dx = I_3^1 + I_3^2, \end{aligned} \tag{4.54}$$

where we denote the last two integrals in the above equality by I_3^1 and I_3^2, respectively. The estimate for I_3^1 easily follows from the structure condition (4.2) and monotonicity of g, as

$$
\begin{aligned}
|I_3^1| &\leq c(\beta+2)\int_\Omega \eta^{\beta+3}v^{\beta+3}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||T\eta|\,dx \\
&\leq \frac{c}{r^2}(\beta+2)\mathrm{F}(\mu(r))\mu(r)^4\int_{B_r}\eta^\beta v^\beta\,dx.
\end{aligned}
\tag{4.55}
$$

Thus by Hölder's inequality, I_3^1 satisfies estimate (4.47). To estimate I_3^2, note that by (4.17) and the structure condition (4.2) we have

$$
|I_3^2| \leq c(\beta+2)\int_E \eta^{\beta+4}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||\mathfrak{X}(Tu)|\,dx, \tag{4.56}
$$

where the set E is as in (4.16). For $1<\gamma<3/2$, we continue to estimate I_3^2 by Hölder's inequality as follows,

$$
\begin{aligned}
|I_3^2| \leq c(\beta+2)&\Big(\int_E \eta^{(2-\gamma)(\beta+2)+4}v^{(2-\gamma)(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}(Tu)|^2\,dx\Big)^{\frac{1}{2}} \\
&\times\Big(\int_E \eta^{\gamma(\beta+2)}v^{\gamma\beta+4(\gamma-1)}\mathrm{F}\left(|\mathfrak{X}u|\right)\,dx\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Since, we have (4.19) on the set E, hence

$$
|I_3^2| \leq c(\beta+2)\mathrm{F}(\mu(r))^{\frac{1}{2}}\mu(r)^{2(\gamma-1)-1}M^{\frac{1}{2}}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{2}}, \tag{4.57}
$$

where

$$
M = \int_\Omega \eta^{(2-\gamma)(\beta+2)+4}\,v^{(2-\gamma)(\beta+4)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}(Tu)|^2\,dx. \tag{4.58}
$$

Now we can apply Lemma 4.10 to estimate M from above. Note that Lemma 4.10 with $\tau = 2-\gamma$, gives us that

$$
M \leq c(\beta+2)^{2(2-\gamma)}\frac{|B_r|^{\gamma-1}}{r^{2\gamma}}\mathrm{F}(\mu(r))\mu(r)^6\,J^{2-\gamma} \tag{4.59}
$$

where $c = c(n, g_0, L, \gamma) > 0$ and J is defined as in (4.25)

$$
J = \int_{B_r}\eta^{\beta+4}v^{\beta+2}|\mathfrak{X}v|^2\,dx + \mu(r)^4\frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{\gamma}}. \tag{4.60}
$$

Now, it follows from (4.59) and (4.57) that

$$
|I_3^2| \leq c(\beta+2)^{3-\gamma}\mathrm{F}(\mu(r))\mu(r)^{2\gamma}\,\frac{|B_r|^{\frac{\gamma-1}{2}}}{r^\gamma}\,J^{\frac{2-\gamma}{2}}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{2}}.
$$

By Young's inequality, we end up with

$$
\begin{aligned}
|I_3^2| \leq{}& \frac{c_0}{12}(\beta+2)\mathrm{F}(\mu(r))J \\
&+ c(\beta+2)^{\frac{4}{\gamma}-1}\mathrm{F}(\mu(r))\mu(r)^4\frac{|B_r|^{1-\frac{1}{\gamma}}}{r^2}\Big(\int_{B_r}\eta^{\gamma\beta}v^{\gamma\beta}\,dx\Big)^{\frac{1}{\gamma}},
\end{aligned}
$$

where $c_0 > 0$ is the same constant as in (4.47). Note that, with J as in (4.60), I_3^2 satisfies an estimate similar to (4.47). Now the desired claim (4.47) for I_3 follows, since both I_3^1 and I_3^2 satisfy similar estimates. This concludes the proof of the claim (4.47), and hence the proof of the lemma. □

The following corollary follows from Lemma 4.8 by Moser's iteration. We refer to [34] for the proof.

Corollary 4.11. *There exists a constant $\theta = \theta(n, g_0, L) > 0$ such that the following statements hold. If we have*

$$|\{x \in B_r : X_l u < \mu(r)/4\}| \le \theta |B_r| \tag{4.61}$$

for an index $l \in \{1, \dots, 2n\}$ and for a ball $B_r \subset \Omega$, then

$$\inf_{B_{r/2}} X_l u \ge 3\mu(r)/16;$$

Analogously, if we have

$$|\{x \in B_r : X_l u > -\mu(r)/4\}| \le \theta |B_r|, \tag{4.62}$$

for an index $l \in \{1, \dots, 2n\}$ and for a ball $B_r \subset \Omega$, then

$$\sup_{B_{r/2}} X_l u \le -3\mu(r)/16.$$

4.3. **Proof of Theorem 4.1.**

At the end of this subsection, we provide the proof of Theorem 4.1. As before, we denote $u \in HW^{1,G}(\Omega)$ as a weak solution of equation (4.1) We fix a ball $B_{r_0} \subset \Omega$. For all balls $B_r, 0 < r < r_0$, concentric to B_{r_0}, we denote for $l = 1, 2, ..., 2n$,

$$\mu_l(r) = \sup_{B_r} |X_l u|, \quad \mu(r) = \max_{1 \le l \le 2n} \mu_l(r),$$

and

$$\omega_l(r) = \operatorname{osc}_{B_r} X_l u, \quad \omega(r) = \max_{1 \le l \le 2n} \omega_l(r).$$

We clearly have $\omega(r) \le 2\mu(r)$. For any function w, we define

$$A^+_{k,\rho}(w) = \{x \in B_\rho : (w(x) - k)^+ = \max(w(x) - k, 0) > 0\};$$

and $A^-_{k,\rho}(w)$ is similarly defined.

The following lemma is similar to Lemma 4.1 of [34] and Lemma 4.3 of [43]. For sake of completeness, we provide a proof in Appendix I.

Lemma 4.12. *Let $B_{r_0} \subset \Omega$ be a ball and $0 < r < r_0/2$. Suppose that there is $\tau > 0$ such that*

$$|\mathfrak{X} u| \ge \tau \mu(r) \quad \text{in } A^+_{k,r}(X_l u) \tag{4.63}$$

for an index $l \in \{1, 2, ..., 2n\}$ and for a constant $k \in \mathbb{R}$. Then for any $q \ge 4$ and any $0 < r'' < r' \le r$, we have

$$\begin{aligned} &\int_{B_{r''}} \mathrm{F}(|\mathfrak{X} u|) |\mathfrak{X}(X_l u - k)^+|^2 \, dx \\ &\le \frac{c}{(r' - r'')^2} \int_{B_{r'}} \mathrm{F}(|\mathfrak{X} u|) |(X_l u - k)^+|^2 \, dx + cK |A^+_{k,r'}(X_l u)|^{1 - \frac{2}{q}} \end{aligned} \tag{4.64}$$

where $K = r_0^{-2}|B_{r_0}|^{2/q}\mu(r_0)^2\mathrm{F}(\mu(r_0))$ *and* $c = c(n,p,L,q,\tau) > 0$.

Remark 4.13. Similarly, we can obtain an inequality, corresponding to (4.64), with $(X_l u-k)^+$ replaced by $(X_l u - k)^-$ and $A^+_{k,r}(X_l u)$ replaced by $A^-_{k,r}(X_l u)$.

Lemma 4.14. *There exists a constant* $s = s(n, g_0, L) \geq 0$ *such that for every* $0 < r \leq r_0/16$, *we have the following,*

$$\omega(r) \leq (1 - 2^{-s})\omega(8r) + 2^s\mu(r_0)\left(\frac{r}{r_0}\right)^{\alpha}, \tag{4.65}$$

where $\alpha = 1/2$ *when* $0 < g_0 < 1$ *and* $\alpha = 1/(1 + g_0)$ *when* $g_0 \geq 1$.

Proof. To prove the lemma, we fix a ball B_r concentric to B_{r_0}, such that $0 < r < r_0/16$.
Letting $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1 + g_0)$ when $g_0 \geq 1$, we may assume that

$$\omega(r) \geq \mu(r_0)\left(\frac{r}{r_0}\right)^{\alpha}, \tag{4.66}$$

since, otherwise, (4.65) is true with $s = 0$. In the following, we assume that (4.66) is true and we divide the proof into two cases.

Case 1. For at least one index $l \in \{1, \ldots, 2n\}$, we have either

$$|\{x \in B_{4r} : X_l u < \mu(4r)/4\}| \leq \theta|B_{4r}| \tag{4.67}$$

or

$$|\{x \in B_{4r} : X_l u > -\mu(4r)/4\}| \leq \theta|B_{4r}|, \tag{4.68}$$

where $\theta = \theta(n, g_0, L) > 0$ is the constant in Corollary 4.11. Assume that (4.67) is true; the case (4.68) can be treated in the same way. We apply Corollary 4.11 to obtain that

$$|X_l u| \geq 3\mu(4r)/16 \quad \text{in} \quad B_{2r}.$$

Thus we have

$$|\mathfrak{X}u| \geq 3\mu(2r)/16 \quad \text{in} \quad B_{2r}. \tag{4.69}$$

Due to (4.69), we can apply Lemma 4.12 with $q = 2Q$ to obtain

$$\begin{aligned}\int_{B_{r''}} |\mathfrak{X}(X_i u - k)^+|^2\, dx &\leq \frac{c}{(r' - r'')^2}\int_{B_{r'}} |(X_i u - k)^+|^2\, dx \\ &\quad + cK\mathrm{F}(\mu(2r))^{-1}|A^+_{k,r'}(X_i u)|^{1-\frac{1}{Q}}\end{aligned} \tag{4.70}$$

where $K = r_0^{-2}|B_{r_0}|^{1/Q}\mu(r_0)^2\mathrm{F}(\mu(r_0))$. The above inequality holds for all $0 < r'' < r' \leq 2r$, $i \in \{1, \ldots, 2n\}$ and all $k \in \mathbb{R}$, which means that for each i, $X_i u$ belongs to the De Giorgi class $DG^+(B_{2r})$, see [43] for details. The corresponding version of Lemma 4.12 for $(X_i u - k)^-$, see Remark 4.13, shows that $X_i u$ also belong to $DG^-(B_{2r})$ and hence $X_i u$ belongs to $DG(B_{2r})$. Now we can apply Theorem 4.1 of [43] to conclude that there is $s_0 = s_0(n, p, L) > 0$ such that for each $i \in \{1, 2, ..., 2n\}$

$$\operatorname{osc}_{B_r} X_i u \leq (1 - 2^{-s_0}) \operatorname{osc}_{B_{2r}} X_i u + cK^{\frac{1}{2}}\mathrm{F}(\mu(2r))^{-\frac{1}{2}} r^{\frac{1}{2}}. \tag{4.71}$$

Now, from doubling property of g, see (2.22) of Lemma 2.12, we have $g(\mu(r_0)) \leq \left(\frac{\mu(r_0)}{\mu(2r)}\right)^{g_0} g(\mu(2r))$ whenever $2r \leq r_0$ and hence

$$\mathrm{F}(\mu(r_0))/\mathrm{F}(\mu(2r)) \leq \big(\mu(r_0)/\mu(2r)\big)^{g_0-1}.$$

Thus, notice that when $0 < g_0 < 1$, we have

$$\mathrm{F}(\mu(2r))^{-1} \leq \mathrm{F}(\mu(r_0))^{-1}$$

and when $g_0 \geq 1$, our assumption (4.66) with $\alpha = 1/(1+g_0)$ gives

$$\begin{aligned}\mathrm{F}(\mu(2r))^{-1} \leq \left(\frac{\mu(r_0)}{\mu(2r)}\right)^{g_0-1} \mathrm{F}(\mu(r_0))^{-1} &\leq 2^{g_0-1}\mathrm{F}(\mu(r_0))^{-1}\left(\frac{\mu(r_0)}{\omega(r)}\right)^{g_0-1} \\ &\leq 2^{g_0-1}\mathrm{F}(\mu(r_0))^{-1}\left(\frac{r}{r_0}\right)^{\frac{1-g_0}{1+g_0}}\end{aligned}$$

where in the second inequality we used that $\mu(2r) \geq \omega(2r)/2 \geq \omega(r)/2$. In both cases, we find that (4.71) becomes

$$\operatorname{osc}_{B_r} X_i u \leq (1 - 2^{-s_0}) \operatorname{osc}_{B_{2r}} X_i u + c\mu(r_0)\left(\frac{r}{r_0}\right)^{\alpha}, \tag{4.72}$$

where $c = c(n, g_0, L) > 0$, $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1+g_0)$ when $g_0 \geq 1$. This shows that the lemma holds in this case.

Case 2. If Case 1 does not happen, then for every $i \in \{1, \dots, 2n\}$, we have

$$|\{x \in B_{4r} : X_i u < \mu(4r)/4\}| > \theta|B_{4r}|, \tag{4.73}$$

and

$$|\{x \in B_{4r} : X_i u > -\mu(4r)/4\}| > \theta|B_{4r}|, \tag{4.74}$$

where $\theta = \theta(n, g_0, L) > 0$ is the constant in Corollary 4.11.

Note that on the set $\{x \in B_{8r} : X_i u > \mu(8r)/4\}$, we trivially have

$$|\mathfrak{X}u| \geq \mu(8r)/4 \quad \text{in } A^+_{k,8r}(X_i u) \tag{4.75}$$

for all $k \geq \mu(8r)/4$. Thus, we can apply Lemma 4.12 with $q = 2Q$ to conclude that

$$\begin{aligned}\int_{B_{r''}} |\mathfrak{X}(X_i u - k)^+|^2\, dx \ &\leq \frac{c}{(r' - r'')^2}\int_{B_{r'}} |(X_i u - k)^+|^2\, dx \\ &\quad + cK\,\mathrm{F}(\mu(8r))^{-1}|A^+_{k,r'}(X_i u)|^{1-\frac{1}{Q}}\end{aligned} \tag{4.76}$$

where $K = r_0^{-2}|B_{r_0}|^{1/Q}\mu(r_0)^2\mathrm{F}(\mu(r_0))$, whenever $k \geq k_0 = \mu(8r)/4$ and $0 < r'' < r' \leq 8r$. The above inequality is true all $i \in \{1, 2, ..., 2n\}$. We note that (4.73) trivially implies

$$|\{x \in B_{4r} : X_i u < \mu(8r)/4\}| > \theta|B_{4r}|.$$

Now we can apply Lemma 4.2 of [43] to conclude that there exists $s_1 = s_1(n, p, L) > 0$ such that the following holds,

$$\sup_{B_{2r}} X_i u \leq \sup_{B_{8r}} X_i u - 2^{-s_1}\big(\sup_{B_{8r}} X_i u - \mu(8r)/4\big) + cK^{\frac{1}{2}}\mathrm{F}(\mu(8r))^{-1/2}r^{\frac{1}{2}}. \tag{4.77}$$

From (4.74), we can derive similarly, see Remark 4.13, that

$$\inf_{B_{2r}} X_i u \geq \inf_{B_{8r}} X_i u + 2^{-s_1}\big(-\inf_{B_{8r}} X_i u - \mu(8r)/4\big) - cK^{\frac{1}{2}}\mathrm{F}(\mu(8r))^{-1/2}r^{\frac{1}{2}}. \tag{4.78}$$

The above two inequalities (4.77) and (4.78) yield

$$\operatorname{osc}_{B_{2r}} X_i u \leq \ (1 - 2^{-s_1}) \operatorname{osc}_{B_{8r}} X_i u + 2^{-s_1-1}\mu(8r) + cK^{\frac{1}{2}}\mathrm{F}(\mu(8r))^{-1/2}r^{\frac{1}{2}},$$

and hence

$$\omega(2r) \leq \big(1 - 2^{-s_1}\big)\omega(8r) + 2^{-s_1-1}\mu(8r) + cK^{\frac{1}{2}}\mathrm{F}(\mu(8r))^{-1/2}r^{\frac{1}{2}}. \tag{4.79}$$

By using doubling condition of g and the inequality $\mu(8r) \geq \omega(8r)/2 \geq \omega(r)/2$ along with the assumption (4.66), we proceed by the same argument as in the preceeding case, to conclude

$$\omega(2r) \leq \big(1 - 2^{-s_1}\big)\omega(8r) + 2^{-s_1-1}\mu(8r) + c\mu(r_0)\left(\frac{r}{r_0}\right)^{\alpha}$$

for $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1+g_0)$ when $g_0 \geq 1$.

Now we notice that (4.73) implies that $\inf_{B_{4r}} X_i u \leq \mu(4r)/4$ and (4.74) implies that $\sup_{B_{4r}} X_i u \geq -\mu(4r)/4$ for every $i \in \{1, \ldots, 2n\}$. Hence

$$\omega(8r) \geq \mu(8r) - \mu(4r)/4 \geq 3\mu(8r)/4.$$

Then from the above two inequalities we arrive at

$$\omega(2r) \leq \big(1 - 2^{-s_1-2}\big)\omega(8r) + c\mu(r_0)\left(\frac{r}{r_0}\right)^{\alpha},$$

where $c = c(n, g_0, L) > 0$, $\alpha = 1/2$ when $0 < g_0 < 1$ and $\alpha = 1/(1+g_0)$ when $g_0 \geq 1$. This shows that also in this case the lemma is true. Thus, the proof of the lemma follows from choice of $s = \max(0, s_0, s_1 + 2, \log_2 c)$. □

Proof of Theorem 4.1.
We first consider the apriori assumption (4.7) so that, equipped with this assumption, we have the above lemma, Lemma 4.14. Now, by an iteration on (4.65), it is easy to see that

$$\omega(r) \leq c\Big(\frac{r}{r_0}\Big)^{\sigma}\big[\omega(r_0/2) + \mu(r_0/2)\big] \tag{4.80}$$

for some $\sigma = \sigma(n, g_0, L) \in (0, 1)$, $r \leq r_0/2$ and $c = c(n, g_0, L) > 0$. Using (4.80), observe that

$$\begin{aligned} \fint_{B_r} G(|X_l u - \{X_l u\}_{B_r}|)\,dx &\leq c\,G(\omega_l(r)) \leq c\,G\bigg(\Big(\frac{r}{r_0}\Big)^{\sigma}\big[\omega(r_0/2) + \mu(r_0/2)\big]\bigg) \\ &\leq c\Big(\frac{r}{r_0}\Big)^{\sigma}\sup_{B_{r_0/2}} G(|\mathfrak{X}u|) \end{aligned} \tag{4.81}$$

where we have used (2.26) for the first inequality and (2.21) for the last inequality of the above. Hence from (4.6), we end up with

$$\fint_{B_r} G(|X_l u - \{X_l u\}_{B_r}|)\,dx \leq c\Big(\frac{r}{r_0}\Big)^{\sigma}\fint_{B_{r_0}} G(|\mathfrak{X}u|)\,dx \tag{4.82}$$

which gives us the estimate (4.3).

Now, to complete the proof, first we need to show that the estimate (4.82) is uniform, without the assumption (4.7). This involves a standard approximation argument, using the following regularization, as constructed [29];

$$\mathrm{F}_\varepsilon(t) = \mathrm{F}\Big(\min\{\,t+\varepsilon\,,\,1/\varepsilon\,\}\Big) \quad\text{and}\quad \mathcal{A}_\varepsilon(z) = \eta_\varepsilon(|z|)\mathrm{F}_\varepsilon(|z|)\,z + \Big(1 - \eta_\varepsilon(|z|)\Big)\mathcal{A}(z) \tag{4.83}$$

where $0 < \varepsilon < 1$, $\eta_\varepsilon \in C^{0,1}([0,\infty))$ as in [29] and $\mathrm{F}(t) = g(t)/t$ for g satisfying (1.2) with $\delta > 0$. Then, given $u \in HW^{1,G}(B_r)$ we consider u_ε that solves $\mathrm{div}_H(\mathcal{A}_\varepsilon(\mathfrak{X}u_\varepsilon)) = 0$ and

$u_\varepsilon - u \in HW_0^{1,G}(B_r)$. We have $\mathcal{A}_\varepsilon \to \mathcal{A}$ and $\mathrm{F}_\varepsilon \to \mathrm{F}$ uniformly on compact subsets and F_ε satisfies the assumption (4.7) with $m_1 = \mathrm{F}(\varepsilon)$ and $m_2 = \mathrm{F}(1/\varepsilon)$. Since the estimate (4.82) are independent of m_1 and m_2, hence the limit $\varepsilon \to 0$ can be taken to obtain the uniform estimate, where the constant depends on n, δ, g_0, L.

Now, we show that the uniform estimate (4.82) implies that $X_l u$ is Hölder continuous for every $l \in \{1, \dots, 2n\}$. Using (2.21) and Jensen's inequality on (4.82), notice that

$$
\begin{aligned}
&\Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big) g\Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big) \\
&\qquad \leq (1+g_0) G\Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big) \leq c\Big(\frac{r}{r_0}\Big)^\sigma \fint_{B_{r_0}} G(|\mathfrak{X} u|)\, dx
\end{aligned}
\tag{4.84}
$$

for some $c = c(n, \delta, g_0, L) > 0$. Now, observe that if $\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx \geq 1$ then,

$$
\Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big) g\Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big) \geq g(1) \fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx;
$$

otherwise if $\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx \leq 1$, then from doubling condition

$$
\Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big) g\Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big) \geq g(1) \Big(\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx\Big)^{1+g_0}.
$$

Notice that, both cases of the above when combined with (4.84), yield

$$
\fint_{B_r} |X_l u - \{X_l u\}_{B_r}|\, dx \leq C\Big(n, \delta, g_0, L, g(1), \|u\|_{HW^{1,G}(\Omega)}\Big) \Big(\frac{r}{r_0}\Big)^{\frac{\sigma}{1+g_0}}
\tag{4.85}
$$

which implies that $X_l u \in \mathcal{L}^{1, Q+\sigma'}(B_r)$ and hence, recalling (2.10), $X_l u \in C^{0,\sigma'}(B_r)$ with $\sigma' = \sigma/(1+g_0)$ for some $\sigma = \sigma(n, g_0, L) \in (0,1)$. This completes the proof. $\square$

Remark 4.15. Let $B_R \subset B_{R_0} \subset\subset \Omega$ be concentric balls for $0 < R < R_0$. As illustrated in the above proof, if $w \in HW^{1,G}(\Omega)$ with $\|u\|_{HW^{1,G}(\Omega)} \leq M$, satisfies the inequality

$$
\int_{B_R} G(|\mathfrak{X} w - \{\mathfrak{X} w\}_{B_R}|)\, dx \leq C(R/R_0)^\lambda
$$

for some positive constants $C = C(n, \delta, g_0, R_0, M) > 0$ and $\lambda \in (0, Q+1)$ with $Q = 2n+2$, then we have $\mathfrak{X} w \in \mathcal{L}^{1,\lambda'}(B_R, \mathbb{R}^{2n})$; where if $\lambda \in (0, Q)$ then $\lambda' = \lambda$ and if $\lambda \in (Q, Q+1)$ then $\lambda' = Q + (\lambda - Q)/(1+g_0)$. This shall be used in the next section.

5. $C^{1,\alpha}$-regularity of weak solutions

In this section, we prove Theorem 1.2. In a fixed subdomain Ω' compactly contained in Ω, we show that the weak solutions are locally $C^{1,\beta}$ in Ω'. The proof is standard, based on the results of the preceeding section and a Campanato type perturbation technique. Similar arguments in the Euclidean setting, can be found in [10, 18, 29], etc.

5.1. **The perturbation argument.**

Given $\Omega' \subset\subset \Omega$, we fix $x_0 \in \Omega'$ and a ball $B_R = B_R(x_0) \subset \Omega'$ for $R \leq R_0 = \frac{1}{2}\operatorname{dist}(\Omega', \partial\Omega)$ and consider $u \in HW^{1,G}(B_R) \cap L^\infty(B_R)$ as weak solution of $\mathrm{Q}u = 0$ in B_R, where Q is defined as in (3.1). We recall the structure conditions for Theorem 1.2, as follows;

$$\frac{g(|p|)}{|p|}\,|\xi|^2 \leq \big\langle D_p A(x,z,p)\,\xi, \xi\big\rangle \leq L\,\frac{g(|p|)}{|p|}\,|\xi|^2; \tag{5.1}$$

$$|A(x,z,p) - A(y,w,p)| \;\leq\; L'\big(1 + g(|p|)\big)\Big(|x-y|^\alpha + |z-w|^\alpha\Big); \tag{5.2}$$

$$|B(x,z,p)| \;\leq\; L'\big(1 + g(|p|)\big)|p| \tag{5.3}$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^{2n}$ and the matrix $D_p A(x,z,p)$ is symmetric. In addition, we recall the hypothesis of Theorem 1.2 that, there exists $M_0 > 0$ such that $|u| \leq M_0$ in Ω'.

From structure condition (5.1), it is not difficult to check that $A(x,z,p)$ satisfies conditions reminiscent of (3.23) and (3.24); the condition on variable z for (3.23) and (3.24) are absolved in the constants L and L', since the solution u is bounded. However, the condition (5.3) on B is more relaxed than (3.41) and (3.44), which is necessary for $C^{1,\beta}$-regularity.

Thus, this allows us to apply Theorem 1.1 and conclude u is Hölder continuous with

$$\operatorname{osc}_{B_R} u \leq \theta(R) = \gamma R^\tau \tag{5.4}$$

for some $\gamma = \gamma(M_0, \operatorname{dist}(\Omega', \partial\Omega)) > 0$ and $\tau \in (0,1)$ can be chosen to be as small as required. Here onwards, we suppress the dependence of the data $n, \delta, g_0, \alpha, L, L', M_0, \operatorname{dist}(\Omega', \partial\Omega)$; all positive constants depending on these shall be denoted as c, throughout this subsection, until the end of the proof of theorem 1.2.

Let us denote $\mathcal{A} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as

$$\mathcal{A}(p) = A(x_0, u(x_0), p), \tag{5.5}$$

so that from (5.1), $\mathcal{A}$ satisfies the structure condition (4.2) and hence also the monotonicity and ellipticity conditions (4.4) and (4.5) (with possible dependence on g_0 and δ). Hence, for the problem

$$\begin{cases} \operatorname{div}_H(\mathcal{A}(\mathfrak{X}\tilde{u})) = 0 \ \text{ in } B_R; \\ \ \tilde{u} - u \in HW_0^{1,G}(B_R). \end{cases} \tag{5.6}$$

we can use the monotonicity inequalities and uniform estimates from Section 4.

Lemma 5.1. *If $u \in HW^{1,G}(B_R) \cap C(\bar{B}_R)$ is given, then there exists a unique weak solution $\tilde{u} \in HW^{1,G}(B_R) \cap C(\bar{B}_R)$ for the problem* (5.6)*, which satisfies the following:*

$$(i)\ \sup_{B_R} |u - \tilde{u}| \leq \operatorname{osc}_{B_R} u\ ; \tag{5.7}$$

$$(ii)\ \int_{B_R} G(|\mathfrak{X}\tilde{u}|)\,dx \leq c \int_{B_R} G(|\mathfrak{X}u|)\,dx. \tag{5.8}$$

Proof. Existence and uniqueness is standard from monotonicity of $\mathcal{A}$, we refer to [35] for more details. Also, (5.7) follows easily from Comparison principle and the fact that

$$\inf_{\partial B_R} u \leq \tilde{u} \leq \sup_{\partial B_R} u \quad \text{in } B_R,$$

which is easy to show by considering $\varphi = (\tilde{u} - \sup_{\partial B_R} u)^+$ (and similarly the other case) as a test function for (5.6), see Lemma 5.1 in [10].

The proof of (5.8) is also standard. Using test function $\varphi = \tilde{u} - u$ on (5.6), we get

$$\int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}\tilde{u}\big\rangle\, dx = \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}u\big\rangle\, dx. \tag{5.9}$$

Now we choose $k = k(\delta, g_0, L) > 0$ such that combining ellipticity (4.5) and boundedness of $\mathcal{A}$, we have $\big\langle \mathcal{A}(p), p\big\rangle \geq (2/k)|p||\mathcal{A}(p)|$. Hence, we obtain

$$\begin{aligned}\int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}u\big\rangle\, dx &\leq \frac{1}{k}\int_{|\mathfrak{X}\tilde{u}|\geq k|\mathfrak{X}u|} |\mathcal{A}(\mathfrak{X}\tilde{u})||\mathfrak{X}\tilde{u}|\, dx \;+\; \int_{|\mathfrak{X}\tilde{u}|< k|\mathfrak{X}u|} |\mathcal{A}(\mathfrak{X}\tilde{u})||\mathfrak{X}u|\, dx\\ &\leq \frac{1}{2}\int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}\tilde{u}), \mathfrak{X}\tilde{u}\big\rangle\, dx \;+\; k^{g_0} c\int_{B_R} g(|\mathfrak{X}u|)\,|\mathfrak{X}u|\, dx.\end{aligned}$$

which combined with (5.9) and the ellipticity (4.5), concludes the proof. □

To proceed with the proof of Theorem 1.2, we shall need the following technical lemma which is a variant of a lemma of Campanato [4]. This is elementary but a fundamental lemma. We refer to [21] or [19, Lemma 2.1] for a proof.

Lemma 5.2. *Let $\phi : (0,\infty) \to [0,\infty)$ be a non-decreasing function and $A, B > 1, \alpha > 0$ be fixed constants. Suppose that for any $\rho < r \leq R_0$ and $\epsilon > 0$, we have*

$$\phi(\rho) \leq A\left[\left(\frac{\rho}{r}\right)^{\alpha} + \kappa\right]\phi(r) + Br^{\alpha-\epsilon};$$

then there exists a constant $\kappa_0 = \kappa_0(\alpha, A, B) > 0$ such that if $\kappa < \kappa_0$, we have

$$\phi(\rho) \leq c\left(\frac{\rho}{r}\right)^{\alpha-\epsilon}\big[\phi(r) + Br^{\alpha-\epsilon}\big]$$

for all $\rho < r \leq R_0$, where $c = c(\alpha, \epsilon, A) > 0$ is a constant.

Proof of Theorem 1.2.
Let $u \in HW^{1,G}(\Omega)$ be a weak solution of $\mathrm{Q}u = 0$. For $B_R \subset \Omega' \subset\subset \Omega$, we have $|u| \leq M_0$ in $\bar{B}_R$ and we can regard that $u \in HW^{1,G}(B_R) \cap C(\bar{B}_R)$. Let us denote

$$I = \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}u), (\mathfrak{X}u - \mathfrak{X}\tilde{u})\big\rangle\, dx, \tag{5.10}$$

where $\mathcal{A}$ is as in (5.5) and $\tilde{u} \in HW^{1,G}(B_R) \cap C(\bar{B}_R)$ is the weak solution of (5.6). Since $u = \tilde{u}$ in ∂B_R, the function $u - \tilde{u}$ can be used to test the equations satisfied by u and $\tilde{u}$, which shall be used to estimate I to obtain both lower and upper bounds.

First, using $u - \tilde{u}$ as test function for $\mathrm{Q}u = 0$, we obtain

$$\begin{aligned} I &= \int_{B_R} \big\langle A(x_0, u(x_0), \mathfrak{X}u) - A(x, u, \mathfrak{X}u), (\mathfrak{X}u - \mathfrak{X}\tilde{u})\big\rangle\, dx\\ &\quad + \int_{B_R} B(x, u, \mathfrak{X}u)(u - \tilde{u})\, dx\\ &\leq c\big(R^{\alpha} + \theta(R)^{\alpha}\big)\int_{B_R} g(1 + |\mathfrak{X}u|)|\mathfrak{X}u - \mathfrak{X}\tilde{u}|\, dx\\ &\quad + c\,\theta(R)\int_{B_R} g(1 + |\mathfrak{X}u|)|\mathfrak{X}u|\, dx\end{aligned} \tag{5.11}$$

with $\theta(R)$ as in (5.4), where we have used structure condition (5.2) and (5.3) for the first term and (5.7) for the second term of the right hand side of (5.11). Now we use (2.24) of Lemma 2.12 and (5.8) of Lemma 5.1 to estimate the first term of the above and obtain that

$$I \le c\,\theta(R)^{\alpha} \int_{B_R} G(1+|\mathfrak{X}u|)\,dx. \tag{5.12}$$

Secondly, to obtain the upper bound for I, we shall use the monotonicity inequality (4.4). Let us denote $S_1 = \{x \in B_R : |\mathfrak{X}u - \mathfrak{X}\tilde{u}| \le 2|\mathfrak{X}u|\}$ and $S_2 = \{x \in B_R : |\mathfrak{X}u - \mathfrak{X}\tilde{u}| > 2|\mathfrak{X}u|\}$. Taking $u - \tilde{u}$ as test function for (5.6) and using (4.4), we obtain

$$\begin{aligned} I &= \int_{B_R} \big\langle \mathcal{A}(\mathfrak{X}u) - \mathcal{A}(\mathfrak{X}\tilde{u}), (\mathfrak{X}u - \mathfrak{X}\tilde{u})\big\rangle\,dx \\ &\ge c\int_{S_1} \mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}u - \mathfrak{X}\tilde{u}|^2\,dx + c\int_{S_2} \mathrm{F}(|\mathfrak{X}u - \mathfrak{X}\tilde{u}|)|\mathfrak{X}u - \mathfrak{X}\tilde{u}|^2\,dx \end{aligned} \tag{5.13}$$

Recalling $G(t) \le t^2\mathrm{F}(t)$ from (2.21), we have from (5.12) and (5.13), that

$$\int_{S_2} G(|\mathfrak{X}u - \mathfrak{X}\tilde{u}|)\,dx \le c\,\theta(R)^{\alpha} \int_{B_R} G(1+|\mathfrak{X}u|)\,dx. \tag{5.14}$$

Now since $|\mathfrak{X}u - \mathfrak{X}\tilde{u}| \le 2|\mathfrak{X}u|$ in S_1 by definition, we obtain the following from (2.21), monotonicity of g and Hölder's inequality;

$$\begin{aligned} \int_{S_1} G(|\mathfrak{X}u - \mathfrak{X}\tilde{u}|)\,dx &\le c\Big(\int_{S_1} \mathrm{F}(|\mathfrak{X}u|)|\mathfrak{X}u - \mathfrak{X}\tilde{u}|^2\,dx\Big)^{\frac12}\Big(\int_{S_1} G(|\mathfrak{X}u|)\,dx\Big)^{\frac12} \\ &\le c\,\theta(R)^{\alpha/2}\int_{B_R} G(1+|\mathfrak{X}u|)\,dx \end{aligned} \tag{5.15}$$

where the latter inequality of the above follows from (5.12) and (5.13). Now, we add (5.14) and (5.15) to obtain the estimate of the integral over whole of B_R,

$$\int_{B_R} G(|\mathfrak{X}u - \mathfrak{X}\tilde{u}|)\,dx \le c\,\theta(R)^{\alpha/2} \int_{B_R} G(1+|\mathfrak{X}u|)\,dx. \tag{5.16}$$

Recalling (4.6) and (5.8), note that for any $0 < r \le R/2$, we have

$$\int_{B_r} G(|\mathfrak{X}\tilde{u}|)\,dx \le r^Q \sup_{B_{R/2}} G(|\mathfrak{X}\tilde{u}|) \le c\Big(\frac{r}{R}\Big)^Q \int_{B_R} G(|\mathfrak{X}\tilde{u}|)\,dx \le c\Big(\frac{r}{R}\Big)^Q \int_{B_R} G(|\mathfrak{X}u|)\,dx.$$

where $Q = 2n+2$. Combining the above with (5.16), we obtain

$$\int_{B_r} G(|\mathfrak{X}u|)\,dx \;\le\; c\left(\frac{r}{R}\right)^Q \int_{B_R} G(|\mathfrak{X}u|)\,dx + c\,\theta(R)^{\alpha/2}\int_{B_R} G(1+|\mathfrak{X}u|)\,dx. \tag{5.17}$$

Now, we follow the bootstrap technique of Giaquinta-Giusti [18]. Here onwards the constants dependent on $g(1)$ in addition to the aforementioned data, shall be denoted as C.

For $0 < \rho \le R_0$, let us denote $\Phi(\rho) = \int_{B_\rho} G(|\mathfrak{X}u|)\,dx$, so that we rewrite (5.17) as

$$\Phi(r) \le c\left(\frac{r}{R}\right)^Q \Phi(R) + cR^{\vartheta}\int_{B_R} G(1+|\mathfrak{X}u|)\,dx \tag{5.18}$$

where $\vartheta = \tau\alpha/2$ with $\tau \in (0,1)$ as in (5.4). We proceed by induction, with the hyposthesis

$$\int_{B_R} G(1+|\mathfrak{X}u|)\,dx \le CR^{(k-1)\vartheta} \quad \text{for some } k\in\mathbb{N},\ k\vartheta < Q. \tag{5.19}$$

The hypothesis clearly holds for $k = 0$. Assuming the hypothesis (5.19) holds for some $k \in \mathbb{N}$, first notice that by virtue of (2.26), we have

$$\int_{B_R} G(|\mathfrak{X}u - \{\mathfrak{X}u\}_{B_R}|)\,dx \le CR^{(k-1)\vartheta}$$

which further implies that $\mathfrak{X}u \in \mathcal{L}^{1,(k-1)\vartheta}(\Omega')$, see Remark 4.15. Now using (5.19) in (5.18), we apply Lemma 5.2 to obtain that

$$\Phi(R) \le c\left(\frac{R}{R_0}\right)^{k\vartheta}\big[\Phi(R_0) + C\big],$$

which, from definition of Φ, implies the hypothesis (5.19) for $k+1$ and $\mathfrak{X}u \in \mathcal{L}^{1,k\vartheta}(\Omega')$. We choose can choose ϑ small enough and carry on a finite induction for $k = 0, 1, \dots (m-1)$ where m is chosen such that $(m-1)\vartheta < Q < m\vartheta < Q+1$. Thus, after the last induction step, we conclude that $\Phi(R) \le CR^{m\vartheta}$ and we have

$$\int_{B_R} G(|\mathfrak{X}u - \{\mathfrak{X}u\}_{B_R}|)\,dx \le CR^{m\vartheta}.$$

Hence from Remark 4.15, $\mathfrak{X}u \in \mathcal{L}^{1,\lambda}(\Omega')$ where $\lambda = Q + (m\vartheta - Q)/(1+g_0)$. Recalling (2.10), this further implies $\mathfrak{X}u \in C^{0,\beta}(\Omega')$ with $\beta = \frac{m\vartheta - Q}{1+g_0}$ and the proof is finished. □

5.2. **Concluding Remarks.**
Here we discuss some possible extensions of the structure conditions that can be included and results similar to the above can be obtained with minor modifications of the arguments.

(1) Any dependence of x in structure conditions for $A(x,z,p)$ and $B(x,z,p)$ has been suppressed so far, for sake of simplicity. However, we remark that for some given non-negative measurable functions $a_1, a_2, a_4, a_5, b_1, b_2$, the structure condition

$$\begin{aligned}
&\big\langle A(x,z,p), p\big\rangle \ge |p|g(|p|) - a_1(x)\, g\left(\frac{|z|}{R}\right)\frac{|z|}{R} - a_2(x);\\
&|A(x,z,p)| \le a_3\, g(|p|) + a_4(x)\, g\left(\frac{|z|}{R}\right) + a_5(x);\\
&|B(x,z,p)| \le \frac{1}{R}\left[b_0\, g(|p|) + b_1(x)\, g\left(\frac{|z|}{R}\right) + b_2(x)\right],
\end{aligned}$$

can also be considered for obtaining the Harnack inequalities. In this case, we would require $a_1, a_2, a_4, a_5, b_1, b_2 \in L^q_{\text{loc}}(\Omega)$ for some $q > Q$. Similar arguments can be carried out with a choice of $\chi > 0$, such that $\|a_5\|_{L^q(B_R)} + \|b_2\|_{L^q(B_R)} \le g(\chi)$ and $\|a_2\|_{L^q(B_R)} \le g(\chi)\chi$. We refer to [29] and [6] for more details of such cases.

(2) The function $g(t)/t$ in the growth conditions can be replaced by $f(t)$, where f is a continuous doubling positive function on $(0,\infty)$ and $t \mapsto f(t)t^{1-\delta}$ is non-decreasing. A C^1-function $\tilde{g}$ can be found satisfying (1.2) and $\tilde{g}(t) \sim tf(t)$(see [29, Lemma 1.6]), which is sufficient to carry out all of the above arguments.

Appendix I

Proof of Lemma 4.9.
Fix $l \in \{1, 2, ..., n\}$ and $\beta \geq 0$. Let $\eta \in C_0^\infty(\Omega)$ be a non-negative cut-off function. Using

$$\varphi = \eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u \tag{5.20}$$

as a test-function in equation (4.10), we get

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+2} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_i u X_i\big(|\mathfrak{X}u|^2 X_l u\big)\, dx \\
&= -(\beta+2) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+1} v^{\beta+2} |\mathfrak{X}u|^2 X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i \eta\, dx \\
&\quad - (\beta+2) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+1} |\mathfrak{X}u|^2 X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_i X_l u X_i v\, dx \\
&\quad - \int_\Omega \sum_{i=1}^{2n} D_i \mathcal{A}_{n+l}(\mathfrak{X}u) T u X_i\big(\eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u\big)\, dx \\
&\quad + \int_\Omega T\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big) \eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u\, dx \\
&= I_1^l + I_2^l + I_3^l + I_4^l.
\end{aligned} \tag{5.21}$$

Here we denote the integrals in the right hand side of (4.10) by I_1^l, I_2^l, I_3^l and I_4^l in order respectively. Similarly for all $l \in \{n+1, n+2, ..., 2n\}$, from equation (4.11), we have

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+2} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_i u X_i\big(|\mathfrak{X}u|^2 X_l u\big)\, dx \\
&= -(\beta+2) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+1} v^{\beta+2} |\mathfrak{X}u|^2 X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i \eta\, dx \\
&\quad - (\beta+2) \int_\Omega \sum_{i,j=1}^{2n} \eta^{\beta+2} v^{\beta+1} |\mathfrak{X}u|^2 X_l u D_j \mathcal{A}_i(\mathfrak{X}u) X_i X_l u X_i v\, dx \\
&\quad + \int_\Omega \sum_{i=1}^{2n} D_i \mathcal{A}_{l-n}(\mathfrak{X}u) T u X_i\big(\eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u\big)\, dx \\
&\quad - \int_\Omega T\big(\mathcal{A}_{l-n}(\mathfrak{X}u)\big) \eta^{\beta+2} v^{\beta+2} |\mathfrak{X}u|^2 X_l u\, dx \\
&= I_1^l + I_2^l + I_3^l + I_4^l.
\end{aligned} \tag{5.22}$$

Again we denote the integrals in the right hand side of (5.22) by I_1^l, I_2^l, I_3^l and I_4^l in order respectively. Summing up the above equation (5.21) and (5.22) for all l from 1 to $2n$, we end up with

$$\int_\Omega \sum_{i,j,l} \eta^{\beta+2} v^{\beta+2} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_i u X_i\big(|\mathfrak{X}u|^2 X_l u\big)\, dx = \sum_l \sum_{m=1}^4 I_m^l, \tag{5.23}$$

where all sums for i, j, l are from 1 to $2n$.

In the following, we estimate both sides of (5.23). For the left hand of (5.23), note that

$$X_i\big(|\mathfrak{X}u|^2 X_l u\big) = |\mathfrak{X}u|^2 X_i X_l u + X_i(|\mathfrak{X}u|^2) X_l u.$$

Then by the structure condition (4.2), we have that

$$\sum_{i,j,l} D_j \mathcal{A}_i(\mathfrak{X}u) X_j X_l u X_i\big(|\mathfrak{X}u|^2 X_l u\big) \geq \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2,$$

which gives us the following estimate for the left hand side of (5.23)

$$\text{left of (5.23)} \geq \int_\Omega \eta^{\beta+2} v^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2\, dx. \tag{5.24}$$

Now we estimate the right hand side of (5.23). We will show that I_m^l satisfies the following estimate for each $l = 1, 2, ..., 2n$ and each $m = 1, 2, 3, 4$

$$\begin{aligned} |I_m^l| \leq{}& \frac{1}{36n} \int_\Omega \eta^{\beta+2} v^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2\, dx \\ &+ c(\beta+2)^2 \int_\Omega \eta^\beta \big(|\mathfrak{X}\eta|^2 + \eta|T\eta|\big) v^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^4\, dx \\ &+ c(\beta+2)^2 \int_\Omega \eta^{\beta+2} v^\beta \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^4 |\mathfrak{X}v|^2\, dx \\ &+ c \int_\Omega \eta^{\beta+2} v^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^2 |Tu|^2\, dx, \end{aligned} \tag{5.25}$$

where $c = c(n, g_0, L) > 0$. Then the lemma follows from the above estimates (5.24) and (5.25) for both sides of (5.23). The proof of the lemma is finished, modulo the proof of (5.25). In the rest, we prove (5.25) in the order of $m = 1, 2, 3, 4$.

First, when $m = 1$, we have for $I_1^l, l = 1, 2, ..., 2n$, by the structure condition (4.2) that

$$|I_1^l| \leq c(\beta+2) \int_\Omega \eta^{\beta+1} |\mathfrak{X}\eta| v^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^3 |\mathfrak{X}\mathfrak{X}u|\, dx,$$

from which it follows by Young's inequality that

$$\begin{aligned} |I_1^l| \leq{}& \frac{1}{36n} \int_\Omega \eta^{\beta+2} v^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^2 |\mathfrak{X}\mathfrak{X}u|^2\, dx \\ &+ c(\beta+2)^2 \int_\Omega \eta^\beta |\mathfrak{X}\eta|^2 v^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^4\, dx. \end{aligned} \tag{5.26}$$

Thus (5.25) holds for $I_1^l, l = 1, 2, ..., 2n$.

Second, when $m = 2$, we have for $I_1^l, l = 1, 2, ..., 2n$, by the structure condition (4.2) that

$$|I_2^l| \leq c(\beta+2) \int_\Omega \eta^{\beta+2} v^{\beta+1} \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^3 |\mathfrak{X}\mathfrak{X}u| |\mathfrak{X}v|\, dx,$$

from which it follows by Young's inequality that

$$
\begin{aligned}
|I_2^l| \le & \frac{1}{36n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
& + c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx.
\end{aligned}
\tag{5.27}
$$

This proves (5.25) for $I_2^l, l=1,2,...,2n$.

Third, when $m=3$, we use

$$
\begin{aligned}
\left|X_i\big(\eta^{\beta+2}v^{\beta+2}|\mathfrak{X}u|^2X_lu\big)\right| \le & 3\eta^{\beta+2}v^{\beta+2}|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u| \\
& + (\beta+2)\eta^{\beta+1}v^{\beta+2}|\mathfrak{X}u|^3|\mathfrak{X}\eta| + (\beta+2)\eta^{\beta+2}v^{\beta+1}|\mathfrak{X}u|^3|\mathfrak{X}v|.
\end{aligned}
$$

and the structure condition (4.2), to obtain

$$
\begin{aligned}
|I_3^l| \le & c\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u||Tu|\,dx \\
& + c(\beta+2)\int_\Omega \eta^{\beta+1}|\mathfrak{X}\eta|v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^3|Tu|\,dx \\
& + c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^3|\mathfrak{X}v||Tu|\,dx,
\end{aligned}
$$

from which it follows by Young's inequality that

$$
\begin{aligned}
|I_3^l| \le & \frac{1}{36n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
& + c\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|Tu|^2\,dx \\
& + c(\beta+2)^2\int_\Omega \eta^{\beta}|\mathfrak{X}\eta|^2v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4\,dx \\
& + c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx.
\end{aligned}
\tag{5.28}
$$

This proves (5.25) for $I_3^l, l=1,2,...,2n$.

Finally, when $m=4$, we prove (5.25) for I_4^l. We consider only the case $l=1,2,...,n$. The case $l=n+1,n+2,...,2n$ can be treated similarly. Let us denote

$$
w=\eta^{\beta+2}|\mathfrak{X}u|^2X_lu.
\tag{5.29}
$$

so that we can write test-function φ defined as in (5.20) as $\varphi=v^{\beta+2}w$. Then, for I_4^l in (5.21), we rewrite $T=X_1X_{n+1}-X_{n+1}X_1$ and use integration by parts to obtain

$$
I_4^l=\int_\Omega T\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)\varphi\,dx=\int_\Omega X_1\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)X_{n+1}\varphi-X_{n+1}\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)X_1\varphi\,dx.
\tag{5.30}
$$

Using $\mathfrak{X}\varphi = (\beta+2)v^{\beta+1}w\mathfrak{X}v + v^{\beta+2}\mathfrak{X}w$ in (5.30), we get

$$
\begin{aligned}
I_4^l = &(\beta+2)\int_\Omega v^{\beta+1}w\Big(X_1\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)X_{n+1}v - X_{n+1}\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)X_1v\Big)\,dx \\
&+ \int_\Omega v^{\beta+2}\Big(X_1\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)X_{n+1}w - X_{n+1}\big(\mathcal{A}_{n+l}(\mathfrak{X}u)\big)X_1w\Big)\,dx \\
= &J^l + K^l.
\end{aligned} \tag{5.31}
$$

Here we denote the first and the second integral in the right hand side of (5.30) by J^l and K^l, respectively. Now we estimate J^l as follows. From structure condition (4.2) and (5.29)

$$
|J^l| \le c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^3|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}v|\,dx,
$$

from which it follows by Young's inequality, that

$$
\begin{aligned}
|J^l| \le &\frac{1}{72n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&+ c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx.
\end{aligned} \tag{5.32}
$$

The above inequality shows that J^l satisfies similar estimate as (5.25) for all $l = 1, 2, ..., n$. Now we estimate K^l. Integration by parts again, yields

$$
\begin{aligned}
K^l = &(\beta+2)\int_\Omega v^{\beta+1}\mathcal{A}_{n+l}(\mathfrak{X}u)\Big(X_{n+1}vX_1w - X_1vX_{n+1}w\Big)\,dx \\
&- \int_\Omega v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)Tw\,dx \\
= &K_1^l + K_2^l.
\end{aligned} \tag{5.33}
$$

For K_1^l, we have by the structure condition (4.2) that

$$
\begin{aligned}
|K_1^l| \le &c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^3|\mathfrak{X}\mathfrak{X}u||\mathfrak{X}v|\,dx \\
&+ c(\beta+2)^2\int_\Omega \eta^{\beta+1}v^{\beta+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v||\mathfrak{X}\eta|\,dx
\end{aligned}
$$

from which it follows by Young's inequality that

$$
\begin{aligned}
|K_1^l| \le &\frac{1}{144n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx \\
&+ c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx \\
&+ c(\beta+2)^2\int_\Omega \eta^{\beta}|\mathfrak{X}\eta|^2v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4\,dx.
\end{aligned} \tag{5.34}
$$

The above inequality shows that K_1^l also satisfies similar estimate as (5.25) for all $l = 1, 2, ..., n$. We continue to estimate K_2^l in (5.33). Note that

$$
Tw = (\beta+2)\eta^{\beta+1}|\mathfrak{X}u|^2X_luT\eta + \eta^{\beta+2}|\mathfrak{X}u|^2X_lTu + \sum_{i=1}^{2n}2\eta^{\beta+2}X_luX_iuX_iTu.
$$

Therefore we write K_2^l as

$$
\begin{aligned}
K_2^l = &-(\beta+2)\int_\Omega \eta^{\beta+1}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)|\mathfrak{X}u|^2X_luT\eta\,dx\\
&-\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)|\mathfrak{X}u|^2X_lTu\,dx\\
&-2\sum_{i=1}^{2n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)X_luX_iuX_iTu\,dx.
\end{aligned}
$$

For the last two integrals in the above equality, we apply integration by parts to get

$$
\begin{aligned}
K_2^l = &-(\beta+2)\int_\Omega \eta^{\beta+1}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)|\mathfrak{X}u|^2X_luT\eta\,dx\\
&+\int_\Omega X_l\Big(\eta^{\beta+2}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)|\mathfrak{X}u|^2\Big)Tu\,dx\\
&+2\sum_{i=1}^{2n}\int_\Omega X_i\Big(\eta^{\beta+2}v^{\beta+2}\mathcal{A}_{n+l}(\mathfrak{X}u)X_luX_iu\Big)Tu\,dx.
\end{aligned}
$$

Now we may estimate the integrals in the above equality by the structure condition (4.2), to obtain the following estimate for K_2^l.

$$
\begin{aligned}
|K_2^l| \le &\, c(\beta+2)\int_\Omega \eta^{\beta+1}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|T\eta|\,dx\\
&+c\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u||Tu|\,dx\\
&+c(\beta+2)\int_\Omega \eta^{\beta+2}v^{\beta+1}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^3|\mathfrak{X}v||Tu|\,dx\\
&+c(\beta+2)\int_\Omega \eta^{\beta+1}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^3|\mathfrak{X}\eta||Tu|\,dx.
\end{aligned}
$$

By Young's inequality, we end up with the following estimate for K_2^l

$$
\begin{aligned}
|K_2^l| \le &\,\frac{1}{144n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta}\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\\
&+c\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|Tu|^2\,dx.
\end{aligned}
\tag{5.35}
$$

This shows that K_2^l also satisfies similar estimate as (5.25). Now we combine the estimates (5.34) for K_1^l and (5.35) for K_2^l. Recall that $K^l = K_1^l + K_2^l$ as denoted in (5.33). We obtain that the following estimate for K^l.

$$
(5.36)\qquad \begin{aligned}
|K^l| \leq & \frac{1}{72n}\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|\mathfrak{X}\mathfrak{X}u|^2\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta}\big(|\mathfrak{X}\eta|^2+\eta|T\eta|\big)v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4\,dx\\
&+c(\beta+2)^2\int_\Omega \eta^{\beta+2}v^{\beta}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^4|\mathfrak{X}v|^2\,dx\\
&+c\int_\Omega \eta^{\beta+2}v^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^2|Tu|^2\,dx.
\end{aligned}
$$

Recall that $I_4^l = J^l + K^l$. We combine the estimates (5.32) for J^l and (5.36) for K^l, and we can see that the claimed estimate (5.25) holds for I_4^l for all $l = 1, 2, ..., n$. We can prove (5.25) similarly for I_4^l for all $l = n+1, n+2, ..., 2n$. This finishes the proof of the claim (5.25) for I_m^l for all $l = 1, 2, ..., 2n$ and all $m = 1, 2, 3, 4$, and hence also the proof of the lemma. □

Proof of Lemma 4.12.
Recalling (4.63), notice that $\tau\mu(r) \leq |\mathfrak{X}u| \leq (2n)^{\frac{1}{2}}\mu(r)$ in $A^+_{k,r}(X_lu)$. Then this combined with doubling condition of g, implies that

$$
(5.37)\qquad \frac{\tau^{g_0}}{(2n)^{1/2}}\mathrm{F}(\mu(r)) \leq \mathrm{F}(|\mathfrak{X}u|) \leq \frac{(2n)^{g_0/2}}{\tau}\mathrm{F}(\mu(r)) \quad \text{in } A^+_{k,r}(X_lu).
$$

In the proof, we only consider $l \in \{1, \dots, n\}$; the proof is similar for $l \in \{n, \dots, 2n\}$. In addition, note that we can also assume $|k| \leq \mu(r_0)$ without loss of generality, to prove (4.64). This proof is very similar to that of Lemma 4.3 in [43].

Let $\eta \in C_0^\infty(B_{r'})$ is a standard cutoff function such that $\eta = 1$ in $B_{r''}$ and $|\mathfrak{X}\eta| \leq 2/(r'-r'')$, we choose $\varphi = \eta^2(X_lu-k)^+$ as a test function in equation (4.10) to get

$$
\begin{aligned}
&\int_{B_r}\sum_{i,j}\eta^2 D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_luX_i((X_lu-k)^+)\,dx\\
&\qquad = -2\int_{B_r}\sum_{i,j}\eta(X_lu-k)^+D_j\mathcal{A}_i(\mathfrak{X}u)X_jX_luX_i\eta\,dx\\
&\qquad\quad -\int_{B_r}\sum_{i}D_i\mathcal{A}_{n+l}(\mathfrak{X}u)TuX_i(\eta^2(X_lu-k)^+)\,dx\\
&\qquad\quad +\int_{B_r}\eta^2(X_lu-k)^+T(\mathcal{A}_{n+l}(\mathfrak{X}u))\,dx
\end{aligned}
$$

Using structure condition (4.2) and Young's inequality, we obtain

$$
\begin{aligned}
&\int_{B_r} \eta^2\, \mathrm{F}\,(|\mathfrak{X}u|)|\mathfrak{X}(X_l u-k)^+|^2\, dx \\
&\qquad\le c\int_{B_r} |\mathfrak{X}\eta|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|(X_l u-k)^+|^2\, dx \\
&\qquad\quad +\; c\int_{A^+_{k,r}} \eta^2\, \mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^2\, dx \\
&\qquad\quad +\; c\int_{B_r} \eta^2 (X_l u-k)^+ \mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}(Tu)|\, dx \\
&\qquad= \;J_1+J_2+J_3.
\end{aligned}
\tag{5.38}
$$

Notice that to show (4.64) from (5.38), we need to estimate J_2 and J_3. First, we estimate J_2 using Hölder's inequality, (4.12) and (5.37) as follows.

$$
\begin{aligned}
J_2 &\le \Big(\int_{B_{r_0/2}} \mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^q\, dx\Big)^{\frac{2}{q}} \Big(\int_{A^+_{k,r}} \mathrm{F}\,(|\mathfrak{X}u|)\; dx\Big)^{1-\frac{2}{q}} \\
&\le c\, r_0^{-2}\mu(r_0)^2 \mathrm{F}(\mu(r_0))|B_{r_0}|^{\frac{2}{q}}|A^+_{k,r}(X_l u)|^{1-\frac{2}{q}}
\end{aligned}
\tag{5.39}
$$

for $c=c(n,g_0,L,q,\tau)>0$.

The estimate of J_3 is more involved. We wish to show the following, which combined with (5.38) and (5.39), completes the proof of this lemma.

$$
J_3 \le \mathcal{M}/2 + c\, r_0^{-2}\mu(r_0)^2 \mathrm{F}(\mu(r_0))|B_{r_0}|^{\frac{2}{q}}|A^+_{k,r}(X_l u)|^{1-\frac{2}{q}} \tag{5.40}
$$

for some $c=c(n,p,L,q,\tau)>0$, where

$$
\mathcal{M} := \int_{B_r} \eta^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}(X_l u-k)^+|^2\, dx + \int_{B_r} |\mathfrak{X}\eta|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|(X_l u-k)^+|^2\, dx. \tag{5.41}
$$

In order to prove the claim (5.40), we follow the iteration argument of Zhong [43].

For any $\kappa\ge 0$, we take $\eta^2|(X_l u-k)^+|^2|Tu|^\kappa Tu$ as a test function in (4.9) and use structure condition (4.2), to obtain

$$
\begin{aligned}
&(\kappa+1)\int_{B_r} \eta^2|(X_i u - k)^+|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^\kappa|\mathfrak{X}(Tu)|^2 dx \\
&\qquad\le\; c\int_{B_r} \eta|(X_i u-k)^+|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^{\kappa+1}|\mathfrak{X}(Tu)||\mathfrak{X}\eta|\, dx \\
&\qquad\quad + c\int_{B_r} \eta^2|(X_l u-k)^+|\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^{\kappa+1}|\mathfrak{X}(Tu)||\mathfrak{X}(X_l u-k)^+|\, dx
\end{aligned}
$$

Using Cauchy-Schwartz inequality on the above, we obtain

$$
\begin{aligned}
&\int_{B_r} \eta^2|(X_i u - k)^+|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^\kappa|\mathfrak{X}(Tu)|^2 dx \\
&\qquad\le c\,\mathcal{M}^{\frac{1}{2}}\Big(\int_{B_r} \eta^2|(X_i u-k)^+|^2 \mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^{2\kappa+2}|\mathfrak{X}(Tu)|^2 dx\Big)^{\frac{1}{2}}
\end{aligned}
\tag{5.42}
$$

for $c = c(n, g_0, L) > 0$ and $\mathcal{M}$ as defined in (5.41). Now we iterate (5.42), choosing the sequence $\kappa_m = 2^m - 2$ for $m \in \mathbb{N}$. For any $m \geq 1$, we set

$$a_m = \int_{B_r} \eta^2 \mathrm{F}\left(|\mathfrak{X}u|\right) |(X_l u - k)^+|^2 |Tu|^{\kappa_m} |\mathfrak{X}(Tu)|^2 dx$$

and obtain $a_1 \leq (c\mathcal{M})^{\frac{1}{2}} a_2^{\frac{1}{2}} \leq \ldots \leq (c\mathcal{M})^{(1-\frac{1}{2^m})} a_{m+1}^{\frac{1}{2^m}}$, for every $m \in \mathbb{N}$. Now, for some large enough m to be chosen later, we estimate a_{m+1}. Recalling, $|k| \leq \mu(r_0)$ and using Corrolary 4.7, we obtain

$$\begin{aligned} a_{m+1} &\leq c\,\mu(r_0)^2 \int_{B_{r_0/2}} \mathrm{F}\left(|\mathfrak{X}u|\right) |Tu|^{\kappa_{m+1}} |\mathfrak{X}(Tu)|^2\, dx \\ &\leq c\, r_0^{-(\kappa_{m+1}+4)} \int_{B_{r_0}} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^{\kappa_{m+1}+2}\, dx \end{aligned} \tag{5.43}$$

for some $c = c(n, g_0, L, m) > 0$. Hence, we get

$$a_{m+1} \leq c\, r_0^{-(\kappa_{m+1}+4)} \mathrm{F}(\mu(r_0)) \mu(r_0)^{\kappa_{m+1}+4}\, |B_{r_0}|. \tag{5.44}$$

Now we go back to the estimate of J_3. From Hölder's inequality and (5.37),

$$\begin{aligned} J_3 &\leq c\Big(\int_{B_r} \eta^2 \mathrm{F}\left(|\mathfrak{X}u|\right) |(X_l u - k)^+|^2 |\mathfrak{X}(Tu)|^2\, dx \Big)^{\frac{1}{2}} \Big(\int_{A^+_{k,r}} \mathrm{F}\left(|\mathfrak{X}u|\right) dx \Big)^{\frac{1}{2}} \\ &\leq c\, a_1^{1/2} \mathrm{F}(\mu(r_0))^{1/2} |A^+_{k,r}(X_l u)|^{1/2}. \end{aligned}$$

for $c = c(n, g_0, L, \tau) > 0$. We continue further, using the iteration to estimate $a_1^{1/2}$ in terms of a_{m+1} and $\mathcal{M}$. Then we use (5.44) and obtain

$$\begin{aligned} J_3 &\leq c\,\mathcal{M}^{\frac{1}{2}(1-\frac{1}{2^m})}\, a_{m+1}^{\frac{1}{2^{m+1}}}\, \mathrm{F}(\mu(r_0))^{1/2} |A^+_{k,r}(X_l u)|^{\frac{1}{2}} \\ &\leq \frac{c}{r_0^{(1+\frac{1}{2^m})}} \mathcal{M}^{\frac{1}{2}(1-\frac{1}{2^m})} \mathrm{F}(\mu(r_0))^{\frac{1}{2}(1+\frac{1}{2^m})} \mu(r_0)^{(1+\frac{1}{2^m})} |B_{r_0}|^{\frac{1}{2^{m+1}}} |A^+_{k,r}(X_l u)|^{\frac{1}{2}} \end{aligned}$$

Using Young's inequality on the above, we finally obtain

$$J_3 \leq \mathcal{M}/2 + c\, r_0^{-2} \mathrm{F}(\mu(r_0)) \mu(r_0)^2 |B_{r_0}|^{\frac{1}{2^m+1}} |A^+_{k,r}(X_l u)|^{\frac{2^m}{2^m+1}} \tag{5.45}$$

for some $c = c(n, g_0, L, \tau, m) > 0$. The claim (5.40) follows immediately from (5.45), with the choice of $m = m(q) \in \mathbb{N}$ such that $2^m/(2^m+1) \geq 1 - 2/q$. This completes the proof. $\square$

Appendix II

Here we provide an outline of the proof of Lemma 4.6 for the reader's convenience. It requires some Caccioppoli type estimates of horizontal and vertical derivatives, similar to those in [43]. The proof of Lemma 4.6 shall follow in the end.

The following Lemma is similar to Lemma 3.4 in [43] and Lemma 2.6 in [34]. The proof is similar and easier than the proof of Lemma 4.9 in Appendix I, so we omit it. We refer the reader to [34] for some remarks on the proof of Lemma 2.6 in it.

Lemma 5.3. *For any $\beta \geq 0$ and $\eta \in C_0^\infty(\Omega)$, there exists $c = c(n, g_0, L) > 0$ such that*

$$\int_\Omega \eta^2 \,\mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^\beta |\mathfrak{X}\mathfrak{X}u|^2 \, dx \leq c \int_\Omega (|\mathfrak{X}\eta|^2 + \eta |T\eta|) \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^{\beta+2} \, dx$$
$$+ c(\beta+1)^4 \int_\Omega \eta^2 \,\mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^\beta |Tu|^2 \, dx.$$

The following lemma is similar to Lemma 3.5 of [43].

Lemma 5.4. *For any $\beta \geq 2$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \eta^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |Tu|^\beta |\mathfrak{X}\mathfrak{X}u|^2 \, dx \leq c(\beta+1)^2 \|\mathfrak{X}\eta\|_{L^\infty}^2 \int_\Omega \eta^\beta \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^2 |Tu|^{\beta-2} |\mathfrak{X}\mathfrak{X}u|^2 \, dx,$$

for some constant $c = c(n, g_0, L) > 0$.

Proof. Note that have the following identity for any $\varphi \in C_0^\infty(\Omega)$, which can be easily obtained using $X_l\varphi$ as a test function in equation (4.1) (see the proof of Lemma 3.5 in [35]).

$$\int_\Omega \sum_{i=1}^{2n} X_l(\mathcal{A}_i(\mathfrak{X}u) X_i\varphi) \, dx = \int_\Omega T(\mathcal{A}_{n+l}(\mathfrak{X}u)) \varphi \, dx \tag{5.46}$$

Let $\eta \in C_0^\infty(\Omega)$ be a non-negative cut-off function. Fix any $l \in \{1, 2, \ldots, n\}$ and $\beta \geq 2$, let $\varphi = \eta^{\beta+2} |Tu|^\beta X_l u$. We use φ as a test function in (5.46). Note that

$$X_i\varphi = \eta^{\beta+2} |Tu|^\beta X_i X_l u + \beta \eta^{\beta+2} |Tu|^{\beta-2} Tu X_l u X_i(Tu) + (\beta+2) \eta^{\beta+1} X_i\eta |Tu|^\beta X_l u$$

and that $X_{n+l} X_l = X_l X_{n+l} - T$. Using these, we obtain

$$\begin{aligned}
\int_\Omega \sum_i \eta^{\beta+2} |Tu|^\beta X_l(\mathcal{A}_i(\mathfrak{X}u)) X_l X_i u \, dx &= \int_\Omega \eta^{\beta+2} X_l(\mathcal{A}_{n+l}(\mathfrak{X}u)) |Tu|^\beta Tu \, dx \\
&\quad - (\beta+2) \int_\Omega \sum_i \eta^{\beta+1} |Tu|^\beta X_l(\mathcal{A}_i(\mathfrak{X}u)) X_l u X_i\eta \, dx \\
&\quad + \int_\Omega \eta^{\beta+2} T\left(\mathcal{A}_{n+l}(\mathfrak{X}u)\right) |Tu|^\beta X_l u \, dx. \\
&\quad - \beta \int_\Omega \sum_i \eta^{\beta+2} |Tu|^{\beta-2} Tu X_l u X_l(\mathcal{A}_i(\mathfrak{X}u)) X_i Tu \, dx \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{5.47}$$

We will estimate both sides of (5.47) as follows. For the left hand side, the structure condition (4.2) implies that

$$\int_\Omega \sum_i \eta^{\beta+2} |Tu|^\beta X_l(\mathcal{A}_i(\mathfrak{X}u)) X_l X_i u \, dx \geq \int_\Omega \eta^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |Tu|^\beta |X_l \mathfrak{X}u|^2 \, dx.$$

For the right hand side, we will show that for each item, the following estimate is true.

$$\begin{aligned}
|I_k| &\leq c\tau \int_\Omega \eta^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)\, |Tu|^\beta |\mathfrak{X}\mathfrak{X}u|^2 \, dx \\
&\quad + \frac{c(\beta+1)^2 \|\mathfrak{X}\eta\|_{L^\infty}^2}{\tau} \int_\Omega \eta^\beta \mathrm{F}\,(|\mathfrak{X}u|)\, |\mathfrak{X}u|^2 |Tu|^{\beta-2} |\mathfrak{X}\mathfrak{X}u|^2 \, dx,
\end{aligned} \tag{5.48}$$

for $k = 1, 2, 3, 4$, where $c = c(n, p, L) > 0$ and $\tau > 0$ is a constant. By the above estimates for both sides of (5.47), we end up with

$$\begin{aligned}\int_\Omega \eta^{\beta+2} \mathrm{F}\,(|\mathfrak{X}u|)|Tu|^\beta |X_l\mathfrak{X}u|^2\,dx \;&\le\; c\tau \int_\Omega \eta^{\beta+2}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^\beta |\mathfrak{X}\mathfrak{X}u|^2\,dx\\ &+\;\frac{c(\beta+1)^2\|\mathfrak{X}\eta\|^2_{L^\infty}}{\tau}\int_\Omega \eta^\beta \mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u|^2|Tu|^{\beta-2}|\mathfrak{X}\mathfrak{X}u|^2\,dx.\end{aligned}$$

The above inequality is true for all $l = 1, 2, \dots, n$. Similarly, we can prove that it is true also for all $l = n+1, \dots, 2n$. Now, by choosing $\tau > 0$ small enough, we complete the proof of the lemma, assuming the proof of (5.48).

To prove (5.48), we start with I_4. By structure condition (4.2) and Young's inequality

$$\begin{aligned}|I_4| &\le c\beta \int_\Omega \eta^{\beta+2}\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u||Tu|^{\beta-1}|X_l\mathfrak{X}u||\mathfrak{X}(Tu)|\,dx\\ &\le \frac{\tau}{\|\mathfrak{X}\eta\|^2_{L^\infty}}\int_\Omega \eta^{\beta+4}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^\beta|\mathfrak{X}(Tu)|^2\,dx\\ &\quad+\frac{c\beta^2\|\mathfrak{X}\eta\|^2_{L^\infty}}{\tau}\int_\Omega \eta^\beta\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u|^2|Tu|^{\beta-2}|X_l\mathfrak{X}u|^2\,dx.\end{aligned}$$

We then apply Lemma 4.5 to estimate the first integral in the right hand side.

$$\int_\Omega \eta^{\beta+4}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^\beta|\mathfrak{X}(Tu)|^2\,dx \le\; c\int_\Omega \eta^{\beta+2}|\mathfrak{X}\eta|^2\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^{\beta+2}\,dx. \tag{5.49}$$

Using this, we obtain

$$\begin{aligned}|I_4| &\le c\tau\int_\Omega \eta^{\beta+2}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^{\beta+2}\,dx\\ &\quad+\frac{c\beta^2\|\mathfrak{X}\eta\|^2_{L^\infty}}{\tau}\int_\Omega \eta^\beta\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u|^2|Tu|^{\beta-2}|X_l\mathfrak{X}u|^2\,dx.\end{aligned}\tag{5.50}$$

Since $|Tu| \le 2|\mathfrak{X}\mathfrak{X}u|$, (5.50) implies that I_4 satisfies (5.48).

To prove that (5.48) holds for I_1, integration by parts yields

$$\begin{aligned}I_1 &= -\int_\Omega \mathcal{A}_{n+l}(\mathfrak{X}u)X_l(\eta^{\beta+2}|Tu|^\beta Tu)\,dx\\ &= -\,(\beta+1)\int_\Omega \eta^{\beta+2}|Tu|^\beta\mathcal{A}_{n+l}(\mathfrak{X}u)X_l(Tu)\,dx\\ &\quad-(\beta+2)\int_\Omega \eta^{\beta+1}\mathcal{A}_{n+l}(\mathfrak{X}u)X_l\eta|Tu|^\beta Tu\,dx = I_{11}+I_{12}.\end{aligned}$$

We will show that (5.48) holds for both I_{11} and I_{12}. For I_{11}, by structure condition (4.2) and Young's inequality,

$$\begin{aligned}|I_{11}| \;&\le c(\beta+1)\int_\Omega \eta^{\beta+2}\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u||Tu|^\beta|\mathfrak{X}(Tu)|\,dx\\ &\le \frac{\tau}{\|\mathfrak{X}\eta\|^2_{L^\infty}}\int_\Omega \eta^{\beta+4}\mathrm{F}\,(|\mathfrak{X}u|)\,|Tu|^\beta|\mathfrak{X}(Tu)|^2\,dx\\ &\quad+\frac{c(\beta+1)^2\|\mathfrak{X}\eta\|^2_{L^\infty}}{\tau}\int_\Omega \eta^\beta\mathrm{F}\,(|\mathfrak{X}u|)\,|\mathfrak{X}u|^2|Tu|^\beta\,dx,\end{aligned}$$

which, together with (5.49) and the fact $|Tu| \leq 2|\mathfrak{X}\mathfrak{X}u|$, implies that (5.48) holds for I_{11}. For I_{12}, (5.48) follows from

$$|I_{12}| \leq c(\beta+2)\int_\Omega \eta^{\beta+1}|\mathfrak{X}\eta| \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||Tu|^{\beta+1}\,dx,$$

and Young's inequality. This proves that I_1 satisfies (5.48).

For I_2, we have by structure condition (4.2), that

$$|I_2| \leq c(\beta+2)\int_\Omega \eta^{\beta+1}|\mathfrak{X}\eta| \mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||Tu|^{\beta}|X_l\mathfrak{X}u|\,dx,$$

from which, together with Young's inequality and $|Tu| \leq 2|\mathfrak{X}\mathfrak{X}u|$, (5.48) for I_2 follows.

Finally, I_3 has the same bound as that of I_{11}. We have

$$|I_3| \leq c\int_\Omega \eta^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u||Tu|^{\beta}|\mathfrak{X}(Tu)|\,dx,$$

thus I_3 satisfies (5.48), too. This completes the proof of (5.48), and hence that of the lemma. □

The following corollary is easy to prove, by using Hölder's inequality on Lemma 5.4.

Corollary 5.5. *For any $\beta \geq 2$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \eta^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^{\beta}|\mathfrak{X}\mathfrak{X}u|^2\,dx \leq c^{\frac{\beta}{2}}(\beta+1)^{\beta}\|\mathfrak{X}\eta\|_{L^\infty}^{\beta}\int_\Omega \eta^2\,\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^{\beta}|\mathfrak{X}\mathfrak{X}u|^2\,dx,$$

where $c = c(n, g_0, L) > 0$.

Now, we resate Lemma 4.6 as follows.

Lemma 5.6. *For any $\beta \geq 2$ and all non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$\int_\Omega \eta^{\beta+2}\,\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^{\beta+2}\,dx \leq c(\beta)K^{\frac{\beta+2}{2}}\int_{\mathrm{supp}(\eta)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^{\beta+2}\,dx, \tag{5.51}$$

where $K = \|\mathfrak{X}\eta\|_{L^\infty}^2 + \|\eta T\eta\|_{L^\infty}$ and $c(\beta) = c(n, g_0, L, \beta) > 0$.

Proof. First, we show the following claim. For all non-negative $\eta \in C_0^\infty(\Omega)$, we show that

$$\int_\Omega \eta^2\,\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^{\beta}|\mathfrak{X}\mathfrak{X}u|^2\,dx \leq c(\beta+1)^{10}K\int_{\mathrm{supp}(\eta)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^{\beta+2}\,dx, \tag{5.52}$$

where $K = \|\mathfrak{X}\eta\|_{L^\infty}^2 + \|\eta T\eta\|_{L^\infty}$ and $c = c(n, g_0, L) > 0$. Then, (5.51) follows easily from Corollary 5.5, the estimate (5.52) and the fact that $|Tu| \leq 2|\mathfrak{X}\mathfrak{X}u|$. Thus, we are only left with the proof of the claimed estimate (5.52).

To prove (5.52), notice that by Lemma 5.3, we only need to estimate the integral

$$\int_\Omega \eta^2\,\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^{\beta}|Tu|^2\,dx.$$

From Hölder's inequality, we have

$$\int_\Omega \eta^2\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^{\beta}|Tu|^2\,dx \leq \Big(\int_\Omega \eta^{\beta+2}\mathrm{F}\left(|\mathfrak{X}u|\right)|Tu|^{\beta+2}\,dx\Big)^{\frac{2}{\beta+2}}\Big(\int_{\mathrm{supp}(\eta)}\mathrm{F}\left(|\mathfrak{X}u|\right)|\mathfrak{X}u|^{\beta+2}\,dx\Big)^{\frac{\beta}{\beta+2}}.$$

Then, using $|Tu| \leq |\mathfrak{X}\mathfrak{X}u|$ on the above, we obtain the following from Lemma 5.3, Corollary 5.5 and Young's inequality,

$$\int_\Omega \eta^2 \,\mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^\beta |\mathfrak{X}\mathfrak{X}u|^2 \, dx \leq c(\beta+1)^{\frac{4(\beta+2)}{\beta}+2} K \int_{\mathrm{supp}(\eta)} \mathrm{F}\left(|\mathfrak{X}u|\right) |\mathfrak{X}u|^{\beta+2} \, dx,$$

which proves the claim (5.52) and hence completes the proof. $\square$

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