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ON THE LOCAL AND GLOBAL REGULARITY OF TUG-OF-WAR GAMES

JOONAS HEINO

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JOONAS HEINO

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Jyväskylä, March 2018

Joonas Heino

List of included articles

This dissertation consists of an introductory part and the following articles:

[A] Á. Arroyo, J. Heino, and M. Parviainen. Tug-of-war games with varying probabilities and the normalized $p(x)$-Laplacian. *Commun. Pure Appl. Anal.*, 16(3):915–944, 2017.

[B] J. Heino. Uniform measure density condition and game regularity for tug-of-war games. *Bernoulli*, 24(1):408–432, 2018.

[C] J. Heino. A continuous time tug-of-war game for parabolic $p(x, t)$-Laplace type equations. Submitted.

In the introduction these articles are referred to as [A], [B], and [C], whereas the other references are numbered as [1], [2], etc.

The author of this dissertation has actively taken part in the research of the joint article [A].

Introduction

This thesis studies local and global regularity properties of a stochastic two-player zero-sum game called *tug-of-war*. In particular, we study value functions of the game locally as well as globally, that is, close to the boundaries of the game domains. Furthermore, we formulate a continuous time stochastic differential game and discuss, among other things, the equicontinuity of the families of value functions. The main motivation is to understand the properties of the games on their own right. As applications, we obtain an existence and a regularity result for a nonlinear elliptic p-Laplace type partial differential equation and a characterization of the solution to a parabolic p-Laplace type equation.

1. Backgrounds

Classically it is well known that linear partial differential equations arise in the probability theory, see for example [12]. The basic observation is that martingales and harmonic functions share a similar mean value property. This powerful connection attracted a lot of attention both in applications and pure mathematics. For example, Krylov and Safonov [22, 23] utilized the connection to establish regularity results for the elliptic and parabolic second order equations in a non-divergence form.

In 1950s, probabilistic interpretations for nonlinear partial differential equations started to arise from the optimal control theory and differential games, see for example [7]. These interpretations were based on dynamic programming principles, which heuristically speaking break a decision problem into smaller subproblems. However, probabilistic counterparts for nonlinear problems such as ∞-Laplace or p-Laplace equations remained still unknown.

In 2006, Peres, Schramm, Sheffield and Wilson discovered that a tug-of-war game is connected to the ∞-Laplacian. In the celebrated article [35], they proved that the tug-of-war game has a value u_ε, and the value u_ε satisfies a nonlinear mean value property. Moreover, they showed that u_ε approximates ∞-harmonic functions under certain general conditions. Later, Peres and Sheffield [36] proved similar results for p-harmonic functions for all $1 < p < \infty$. Furthermore, Manfredi, Parviainen and Rossi [31, 32] developed a tug-of-war game whose variant is also studied in this thesis. In [30], the authors formulated a time-dependent tug-of-war game that has a connection to p-parabolic functions.

To illustrate the probabilistic counterparts for p-Laplace type equations, let us start with the following examples. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and consider a particle inside the domain. Furthermore, let $F : \partial\Omega \to \mathbb{R}$ be a continuous function on the boundary of the domain. For given $\varepsilon > 0$, we extend F continuously to the ε-width strip outside the domain, and denote by τ the first time the particle hits this strip. Throughout, $B_\varepsilon(x_0)$ denotes the open ball centered at $x_0 \in \mathbb{R}^n$ and of the radius ε, and we use the notation

$$\fint_{B_\varepsilon(x_0)} u(y)\,dy := \frac{1}{|B_\varepsilon(x_0)|}\int_{B_\varepsilon(x_0)} u(y)\,dy$$

for the average integral with $|\cdot|$ denoting the n-dimensional Lebesgue measure.

1.1. **A random walk and the Laplacian.** First, we assume that the particle is moved randomly inside the domain Ω. To be more precise, let $\varepsilon > 0$ be a step size, and assume that the movement of the particle is started at a point $x_0 \in \Omega$. The position x_k of the particle at the round $k \in \{1, 2, \dots\}$ is selected according to the uniform distribution on the ball centered at the previous position x_{k-1} and of the radius ε. Then, we study the expectation of F at the first exit point x_τ,

$$u_\varepsilon(x_0) = \mathbb{E}^{x_0}\big[F(x_\tau)\big].$$

For simplicity, let us assume $u_\varepsilon \in C^2(\Omega)$, that is, the function u_ε is twice continuously differentiable in Ω. However, this condition is not necessary, and it could be relaxed by a notion of viscosity solutions, see for example [10]. By utilizing Taylor's formula, we can approximate

$$\begin{aligned} u_\varepsilon(y) = u_\varepsilon(x_0) + \big\langle Du_\varepsilon(x_0), y - x_0\big\rangle + \frac{1}{2}\big\langle D^2u_\varepsilon(x_0)(y-x_0), y - x_0\big\rangle \\ + o(|y-x_0|^2) \end{aligned} \tag{1}$$

as $|y - x_0| \to 0$. Furthermore, we can calculate

$$\int_{B_\varepsilon(x_0)} \langle \eta, y - x_0\rangle\,dy = 0$$

for all $\eta \in \mathbb{R}^n$ by using the symmetry. Therefore by combining this, (1), and a short computation, it holds

$$\begin{aligned} \fint_{B_\varepsilon(x_0)} u_\varepsilon(y)\,dy &= u_\varepsilon(x_0) + \sum_{i=1}^n \frac{\partial^2 u_\varepsilon}{\partial x_i^2}(x_0)\fint_{B_\varepsilon(0)} \frac{1}{2}z_i^2\,dz + o(\varepsilon^2) \\ &= u_\varepsilon(x_0) + \triangle u_\varepsilon(x_0)\frac{\varepsilon^2}{2(n+2)} + o(\varepsilon^2), \end{aligned} \tag{2}$$

where the standard *Laplace operator* $\triangle$ is defined by $\triangle u_\varepsilon := \sum_{i=1}^n \partial^2 u_\varepsilon/\partial x_i^2$. A function $u \in C^2(\Omega)$ is said to be *harmonic*, if it holds $\triangle u = 0$ in Ω.

Because each increment of the particle is chosen according to the uniform distribution on the corresponding ball, we can prove that u_ε satisfies the mean value property

$$u_\varepsilon(x_0) = \fint_{B_\varepsilon(x_0)} u_\varepsilon(y)\,dy$$

for all $x_0 \in \Omega$. Consequently, the stochastic process $u_\varepsilon(x_k)$, $k \in \{1, 2, \dots\}$, is a martingale with respect to the natural filtration of the particle. Moreover, the calculation (2) implies that

$$\triangle u_\varepsilon = 2(n+2)o(\varepsilon^2)\varepsilon^{-2} \to 0$$

as the step size $\varepsilon \to 0$. Actually, one can show that there exists a function u such that u_ε converges to u uniformly as $\varepsilon \to 0$ by considering a subsequence if necessary, and the function u solves the Dirichlet problem

$$\begin{cases} \triangle u(x) = 0 & \text{for } x \in \Omega, \\ u(x) = F(x) & \text{for } x \in \partial\Omega \end{cases}$$

assuming at this point that the limit satisfies $u \in C^2(\Omega)$. Therefore, we have described a connection between a random walk and harmonic functions.

1.2. **A pure tug-of-war and the ∞-Laplacian.** Next, assume that there are two players, *Player 1* and *Player 2*, who compete against each other. Let $x_0 \in \Omega$ be the starting point and $\varepsilon > 0$ the step size. At every round, a fair coin is flipped. The winner of the toss moves the particle to any point in the open ball centered at the current location and of the radius ε. The game is continued until the particle exits the domain. Before the game starts, the players have made an arrangement. Player 2 pays Player 1 the amount given by the function F at the first exit point x_τ outside the domain. Observe that Player 2 can receive money as well, because F can have negative values. This game described above is a tug-of-war game, usually called a *pure tug-of-war*. The particle is called a *token*, and F is a *pay-off* function.

We study the expected value of the pay-off function at the first exit point of the game domain, when Player 1 seeks to maximize and Player 2 to minimize, respectively, the pay-off function. This leads to the concept of value functions. A *history* in a game is the information of the past events before the corresponding round. The players move the token according to their strategies. A *strategy* is a sequence of functions that gives the next game position given the history of the game. Then, we define the *value function* for Player 1 and Player 2, respectively, by setting

$$u_1(x_0) = \sup_{S_1} \inf_{S_2} \begin{cases} \mathbb{E}^{x_0}_{S_1,S_2}\big[F(x_\tau)\big], & \text{if } \tau < \infty \text{ almost surely,} \\ -\infty, & \text{otherwise,} \end{cases}$$

$$u_2(x_0) = \inf_{S_2} \sup_{S_1} \begin{cases} \mathbb{E}^{x_0}_{S_1,S_2}\big[F(x_\tau)\big], & \text{if } \tau < \infty \text{ almost surely,} \\ \infty, & \text{otherwise.} \end{cases}$$

Here, the expectation is taken with respect to the measure $\mathbb{P}^{x_0}_{S_1,S_2}$ in a game that starts at x_0, and the players choose their moves according to the strategies S_1 and S_2, respectively. To make sure that the game ends, heuristically speaking, the value for a player is the worst possible at a point, if the player do not force the game to end. The game is said to have a *value*, if it holds $u_1 = u_2 =: u_\varepsilon$.

One can show that this game has a value u_ε, and the value u_ε satisfies the nonlinear mean value property

$$u_\varepsilon(x_0) = \frac{1}{2}\Bigg(\sup_{B_\varepsilon(x_0)} u_\varepsilon + \inf_{B_\varepsilon(x_0)} u_\varepsilon\Bigg), \tag{3}$$

see [35]. For simplicity, let us assume $u_\varepsilon \in C^2(\Omega)$ and $Du_\varepsilon \neq 0$ in Ω. By utilizing (1), we can calculate the Taylor expansion for $u_\varepsilon(x_0 + h)$ and $u_\varepsilon(x_0 - h)$ with $h = \varepsilon \frac{Du_\varepsilon(x_0)}{|Du_\varepsilon(x_0)|}$ to get

$$\frac{1}{2}\Big(u_\varepsilon(x_0 + h) + u_\varepsilon(x_0 - h)\Big) = u_\varepsilon(x_0) + \frac{1}{2}\varepsilon^2 \triangle_\infty^N u_\varepsilon(x_0) + o(\varepsilon^2)$$

as $\varepsilon \to 0$. Here, the *normalized* ∞*-Laplace operator* is defined by

$$\triangle_\infty^N u_\varepsilon := \frac{1}{|Du_\varepsilon|^2}\big\langle D^2 u_\varepsilon Du_\varepsilon, Du_\varepsilon\big\rangle = \frac{1}{|Du_\varepsilon|^2}\sum_{i,j=1}^{n} \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j},$$

and a viscosity solution to $\triangle_\infty^N u = 0$ is called ∞*-harmonic*. Thus, we deduce heuristically

$$\frac{1}{2}\Bigg(\sup_{B_\varepsilon(x_0)} u_\varepsilon + \inf_{B_\varepsilon(x_0)} u_\varepsilon\Bigg) = u_\varepsilon(x_0) + \frac{1}{2}\varepsilon^2 \triangle_\infty^N u_\varepsilon(x_0) + o(\varepsilon^2) \tag{4}$$

as $\varepsilon \to 0$. Consequently, because u_ε satisfies (3), it holds

$$\triangle_\infty^N u_\varepsilon(x_0) = 2o(\varepsilon^2)/\varepsilon^{-2} \to 0$$

as $\varepsilon \to 0$. This suggests the result in [35]. Indeed by considering a subsequence if necessary, the sequence of value functions (u_ε) converges uniformly to the unique viscosity solution to the Dirichlet problem

$$\begin{cases} \triangle_\infty^N u(x) = 0 & \text{for } x \in \Omega, \\ u(x) = F(x) & \text{for } x \in \partial\Omega \end{cases}$$

as $\varepsilon \to 0$. Jensen [17] proved the uniqueness of viscosity solutions to this Dirichlet problem. Therefore, we have described a connection between a pure tug-of-war game and ∞-harmonic functions.

The normalized ∞-Laplace operator is in a non-divergence form. It is related to the absolutely minimizing Lipschitz extensions, see [3]. Moreover, the ∞-Laplacian has applications in image processing and optimal mass transportation problems, see for example [13] and [14]. It also arises in a model for the sand-pile evolution, see [4].

1.3. **A tug-of-war with noise and the p-Laplacian.** Next, we assume that the token is influenced by the players and a random noise. Let $p:\Omega\to[p_{\min},p_{\max}]$ be a continuous function with $2<p_{\min}\le p_{\max}<\infty$, and let $\varepsilon>0$ be the step size and $x_0\in\Omega$ the starting point. We define probability functions $\alpha,\beta:\Omega\to(0,1)$ by

$$\alpha(x)=\frac{p(x)-2}{p(x)+n} \tag{5}$$

and $\beta(x)=1-\alpha(x)$ for all $x\in\Omega$. At every round k, the players toss a biased coin that gives heads with a probability $\alpha(x_{k-1})$ and tails with a probability $\beta(x_{k-1})$, where x_{k-1} denotes the current location of the token. If the outcome of the round is heads, a fair coin is flipped, and the winner of this toss moves the game token to any point in the open ball centered at x_{k-1} and of the radius ε. If the outcome in the first toss is tails, the game token is moved according to the uniform distribution on the ball centered at x_{k-1} and of the radius ε. The game is played until the token exits the game domain. Then, Player 2 pays Player 1 the amount given by F at the first exit point x_τ. The movements of the players are the tug-of-war parts of the game, and the random movements are the noise in the game. This game is a variant of the original tug-of-war game in [32].

Because Ω is bounded, and it holds $\inf_\Omega\beta>0$, the randomness in the game makes sure that the game ends almost surely. Thus, we can define the value functions for Player 1 and Player 2, respectively, by setting

$$\begin{aligned}u_1(x_0)&=\sup_{S_1}\inf_{S_2}\mathbb{E}^{x_0}_{S_1,S_2}\big[F(x_\tau)\big],\\ u_2(x_0)&=\inf_{S_2}\sup_{S_1}\mathbb{E}^{x_0}_{S_1,S_2}\big[F(x_\tau)\big].\end{aligned}$$

By minor modifications to the proofs in [29], it can be shown that this game has a value u_ε, and that the value u_ε satisfies the nonlinear mean value property

$$u_\varepsilon(x_0)=\frac{\alpha(x_0)}{2}\left(\sup_{B_\varepsilon(x_0)}u_\varepsilon+\inf_{B_\varepsilon(x_0)}u_\varepsilon\right)+\beta(x_0)\fint_{B_\varepsilon(x_0)}u_\varepsilon(y)\,dy. \tag{6}$$

For simplicity, let us assume $u_\varepsilon\in C^2(\Omega)$ and $Du_\varepsilon\neq0$ in Ω. By combining the calculations (2) and (4), and by multiplying the terms in a suitable way with α and β, we deduce

$$\begin{aligned}&\frac{\alpha(x_0)}{2}\left(\sup_{B_\varepsilon(x_0)}u_\varepsilon+\inf_{B_\varepsilon(x_0)}u_\varepsilon\right)+\beta(x_0)\fint_{B_\varepsilon(x_0)}u_\varepsilon(y)\,dy\\ &=u_\varepsilon(x_0)+\frac{\varepsilon^2}{2(p(x_0)+n)}\Big[\triangle u_\varepsilon(x_0)+\big(p(x_0)-2\big)\triangle^N_\infty u_\varepsilon(x_0)\Big]+o(\varepsilon^2).\end{aligned}$$

Thus, because u_ε satisfies (6), it holds

$$\triangle u_\varepsilon(x_0)+\big(p(x_0)-2\big)\triangle^N_\infty u_\varepsilon(x_0)\le 2(p_{\max}+n)\varepsilon^{-2}o(\varepsilon^2)\to0$$

as $\varepsilon \to 0$. A similar estimate also holds from below. Actually, one can show that the function u_ε converges uniformly to a function u as $\varepsilon \to 0$ by considering a subsequence if necessary, and the function u is a viscosity solution to the Dirichlet problem

$$\begin{cases} \triangle^N_{p(x)} u(x) = 0 & \text{for } x \in \Omega, \\ u(x) = F(x) & \text{for } x \in \partial\Omega. \end{cases} \tag{7}$$

The *normalized* $p(x)$*-Laplacian* is defined by

$$\triangle^N_{p(x)} u(x) := \triangle u(x) + (p(x) - 2)\triangle^N_\infty u(x) \tag{8}$$

for all $x \in \Omega$ and $u \in C^2(\Omega)$ such that $Du \neq 0$ in Ω. Here, we assumed the inequality $p(x) > 2$ for all $x \in \Omega$, because in this case $\alpha(x)$ in (5) is a positive probability for all $x \in \Omega$. However, one can prove in the whole range $1 < p(x) < \infty$ under suitable assumptions that there exists a viscosity solution to the Dirichlet problem (7). A game-theoretic proof of this is in [A, Theorem 6.2]. The *normalized* or *game-theoretic* p*-Laplacian* is defined by

$$\triangle^N_p u := \triangle u + (p - 2)\triangle^N_\infty u$$

for all constants $1 < p < \infty$ and $u \in C^2(\Omega)$ such that $Du \neq 0$ in Ω. On the other hand, the standard p*-Laplace operator* is defined by

$$\begin{aligned} \triangle_p u &:= \operatorname{div}\left(|Du|^{p-2} Du\right) \\ &= |Du|^{p-2}\Big[\triangle u + (p-2)\triangle^N_\infty u\Big]. \end{aligned}$$

A viscosity solution to $\triangle^N_p u = 0$ is called p*-harmonic*. Observe that a distributional weak solution to $\triangle_p u = 0$ coincides with a viscosity solution to $\triangle^N_p u = 0$, see [18].

In general, the Dirichlet problem (7) does not always have a solution that is continuous up to the boundary of a domain. In the classical case $p(x) = 2$ for all $x \in \Omega$, well-known examples such as the punctured disk by Zaremba [39] or the Lebesgue spine by Lebesgue [24] show that a solution may not exist.

Here, in all of our examples, we have assumed that the boundary of the domain is smooth. However, this is not always necessary. Peres and Sheffield [36] proved that in a game regular domain, there exists a p-harmonic function extending continuously to the boundary with the given continuous boundary values. A boundary point $y \in \partial\Omega$ is *game regular*, if a player has a strategy to end the game near y with a probability close to one whenever the game starts near y. The boundary point y is p*-regular*, if for any continuous boundary data F there exists a p-harmonic function u in Ω such that $\lim_{x\to y} u(x) = F(y)$. A sharp condition for a point to be p-regular is the celebrated Wiener's test, see for example [16]. However, it is not known whether a p-regular boundary point is necessarily game regular.

If p is a constant, a game regular boundary point is necessarily p-regular. This is true, because the game regularity implies that the value function u_ε is asymptotically uniformly continuous close to the boundary of the game domain. By copying the strategies and utilizing the translation invariance of the game, the value u_ε is also asymptotically uniformly continuous in the interior of the game domain. Consequently by applying a variant of the Arzelà-Ascoli theorem, we can deduce that there exists a continuous function u on the closure of the game domain with the given continuous boundary data F such that by considering a subsequence if necessary, the value u_ε converges uniformly to the function u as $\varepsilon \to 0$. Then by utilizing the stability principle for viscosity solutions, we can show that the function u is p-harmonic.

If p is not a constant function, this procedure is not possible, because we lose the translation invariance of the game. However, the value u_ε satisfies the dynamic programming principle related to the game. Thus, it is possible to prove the asymptotic continuity of u_ε in the interior of the game domain by studying the local regularity of functions satisfying the dynamic programming principle, see for example [A] or [27]. If we have the comparison principle at our disposal, an alternative approach not related to the game theory and developed by Barles and Souganidis in [6] is also available.

Finally, we point out that the version of tug-of-war games described in this subsection has a nice symmetry. In particular, the players do not affect the direction of the random noise. This allows us to prove sharp enough estimates for the density of the noise, see for example Lemma 5 below. However, the price of the symmetric noise is that the value of p cannot be strictly less than two. In contrast with the game described in this subsection, the players affect the direction of the noise in [A]. Consequently, we are able to let p get values in the whole range $1 < p(x) < \infty$, but the stochastic estimates developed in [B] are not directly applicable in the conditions of [A].

1.4. **Tug-of-war interpretations for other problems.** The probabilistic methods employed in [35] have given a new way to study the normalized ∞-Laplace operator. Recently, a lot of tug-of-war interpretations for different variants of ∞-Laplace type equations have also been discovered. Indeed, Antunović, Peres, Sheffield, and Somersille [1] studied a tug-of-war game for the normalized ∞-Laplacian with vanishing Neumann conditions, and Peres, Pete, and Somersille [34] studied a biased tug-of-war and its connection to the biased ∞-Laplacian. Furthermore, Bjorland, Caffarelli, and Figalli [8] formulated a nonlocal tug-of-war game for the infinity fractional Laplacian. Moreover, Armstrong and Smart [2] constructed a finite difference approach related to a tug-of-war game, and Del Pezzo and Rossi [11] used a tug-of-war game to obtain an existence result for a parabolic problem involving the ∞-Laplacian.

Interpretations for different variants of p-Laplace type equations with finite p have also been studied. For example, Lewicka and Manfredi [25] studied a tug-of-war game related to the obstacle problem for the normalized p-Laplacian. Moreover, Luiro, Parviainen, and Saksman [29] gave a game-theoretic proof of Harnack's inequality for p-harmonic functions. The case $p = 1$ is considered in [20], see also [19].

2. Local regularity of values and article [A]

In [A], we study a tug-of-war with noise, where the probabilities of the coin flips depend on the game location at every round. This game is a natural generalization of the original tug-of-war both from mathematical and application point of views.

Roughly speaking, we can describe the game as follows. Let $\Omega \subset \mathbb{R}^n$ be the bounded domain, where the game is played, and $\varepsilon > 0$. We extend the game domain to include the ε-width strip outside the domain, and denote the extended domain by Ω_ε. Furthermore, we define a continuous probability function $\alpha : \Omega_\varepsilon \to (0, 1)$ and a boundary correction function $\delta : \Omega_\varepsilon \to [0, 1]$, and denote by β the function $\beta := 1 - \alpha$. The game starts at a point $x_0 \in \Omega$. Then, for all game rounds $k \in \{1, 2, \dots\}$, both the players choose a direction, say ν_k^1 and ν_k^2, of the length ε. The game continues with a probability $1 - \delta(x_{k-1})$, where x_{k-1} denotes the game location at the round $k - 1$. In this case, for the next game location x_k, it holds $x_k = x_{k-1} + \nu_k^1$ or $x_k = x_{k-1} + \nu_k^2$ both with an equal probability $\alpha(x_{k-1})/2$. With an equal probability $\beta(x_{k-1})/2$, it holds $x_k = x_{k-1} + \nu_k^{1,\text{ort}}$ or $x_k = x_{k-1} + \nu_k^{2,\text{ort}}$, where $\nu_k^{i,\text{ort}}$ is chosen uniformly random from the $(n-1)$-dimensional ball of the radius ε and orthogonal to the vector ν_k^i for $i \in \{1, 2\}$. On the other hand, if the game stops at x_{k-1}, Player 2 pays Player 1 the amount given by a bounded Borel measurable function $F : \Omega_\varepsilon \to \mathbb{R}$ at a current point.

We show that the game in our setting has a value u_ε in Ω_ε [A, Theorem 3.7]. This is done by utilizing the dynamic programming principle

$$
\begin{aligned}
u(x) = \frac{1-\delta(x)}{2}\Bigg[&\sup_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu) + \beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h)\right)\\
&+ \inf_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu) + \beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h)\right)\Bigg]\\
&+ \delta(x)F(x)
\end{aligned}
\tag{9}
$$

related to the game. Above, we denote

$$
\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h) := \frac{1}{\mathcal{L}^{n-1}(B_\varepsilon^\nu)}\int_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h),
$$

where we denote by B_ε^ν the $(n-1)$-dimensional ball orthogonal to the vector ν and of the radius ε, and $\mathcal{L}^{n-1}$ is the $(n-1)$-dimensional Lebesgue measure. We first show by using an iteration argument that there exist a lower and an upper semicontinuous function satisfying (9), see [A, Proposition 3.3]. Here, we need the boundary correction function δ in our setting, because in such iterations, the measurability can be lost rather easily, see for example [29, Example 2.4]. Then, we can construct a strategy related to the lower semicontinuous solution to (9) and a strategy related to the upper semicontinuous solution to (9). Finally by utilizing these strategies, we show that the game has a value, and the value function u_ε is a solution to (9). Moreover, we prove that the solution to (9) is unique.

The main result of [A] is that the unique value of the game is locally Hölder continuous in an asymptotic way with respect to ε. Below, $\alpha_{\min}$ and $\alpha_{\max}$ denote the minimum and the maximum values of the function α, respectively.

Theorem 1. [A, Theorem 4.1] *Let $(x,z) \in B_R \times B_R$, $B_{2R} \subset \Omega$ and*

$$0 < \gamma < \frac{\alpha_{min}}{\alpha_{max}}.$$

Then, if u satisfies (9), it holds

$$|u(x) - u(z)| \le C\frac{|x-z|^\gamma}{R^\gamma} + C\frac{\varepsilon^\gamma}{R^\gamma}$$

with $C := C(\alpha_{min}, \alpha_{max}, n, R, \sup_{B_{2R}} u) < \infty$ and $0 < \varepsilon < 1$.

The heuristic idea of the proof is to consider two game sequences simultaneously. We can link these sequences to a single higher dimensional game by introducing a probability measure that has the measures of the original games as marginals through suitable couplings. However, we do not use stochastic arguments in the proof. Instead, we employ the method in [27]. In this method, we analyze functions satisfying the dynamic programming principle (9). The main difficulty is the loss of translation invariance so that the global or local regularity methods in [31, 36] or [28] are not directly applicable.

We consider the following application of Theorem 1. Let $p : \overline{\Omega} \to \mathbb{R}$ be a continuous function with values on a compact set $[p_{\min}, p_{\max}]$ for constants $1 < p_{\min} \le p_{\max} < \infty$. We define the function α by

$$\alpha(x) = \frac{p(x)-1}{p(x)+n}.$$

Then, we show in [A, Theorem 6.2] under suitable assumptions on $\partial\Omega$ that by passing to a subsequence if necessary, the value function u_ε converges uniformly to a viscosity solution to the Dirichlet problem

$$\begin{cases} \triangle_{p(x)}^N u(x) = 0 & \text{for } x \in \Omega, \\ u(x) = F(x) & \text{for } x \in \partial\Omega, \end{cases}$$

where $\triangle^N_{p(x)}$ is the normalized $p(x)$-Laplacian defined in (8), and F is a continuous pay-off of the game.

3. Global regularity of values and article [B]

In [B], we study the global regularity of the value in the tug-of-war game described in Section 1.3 in the constant p case, $2 < p < \infty$. In particular, we show that a uniform measure density condition implies the game regularity for the boundary points of the game domain. Observe that it is not known whether a regular boundary point for the p-Laplacian is necessarily game regular. Roughly speaking, a boundary point y satisfies the measure density condition, if the Lebesgue measure of the complement of the game domain in a ball centered at y is comparable to the Lebesgue measure of the whole ball. We recall that the boundary point y is game regular, if a player has a strategy to end the game near y with a probability close to one whenever the game starts near y.

Definition 2. [B, Definition 3.1] *A point $y \in \partial\Omega$ satisfies a measure density condition, if there is $c > 0$ such that*

$$|\Omega^c \cap B_r(y)| \geq c\,|B_r(y)|$$

for all $r > 0$.

Definition 3. [B, Definition 3.2] *A point $y \in \partial\Omega$ is game regular, if for all $\delta > 0$ and $\eta > 0$, there exist $\delta_0 > 0$ and $\epsilon_0 > 0$ such that for all fixed $\epsilon < \epsilon_0$ and $x_0 \in B_{\delta_0}(y)$, there is a strategy S_1^* for Player 1 such that*

$$\mathbb{P}^{x_0}_{S_1^*,S_2}(x_\tau \in B_\delta(y) \cap \Omega^c) \geq 1 - \eta,$$

where τ denotes the first time the token exits the game domain.

Theorem 4. [B, Theorem 3.7] *If $y \in \partial\Omega$ satisfies the measure density condition, then it is game regular for all $2 < p < \infty$.*

The idea of the proof is the following. Let $y \in \partial\Omega$ be the boundary point satisfying the measure density condition. First, to obtain game regularity for the boundary point y, we show in [B, Lemma 3.3] that it is enough to find a strategy S_1^* and a uniform and strictly positive lower bound for the probability of the event that the game ends before exiting a given ball, when Player 1 utilizes the strategy S_1^*, see also [36, Lemma 2.6]. Then, assume that the game starts at $x_0 \in \Omega$. We define S_1^* by the following procedure. Given a turn, the player always cancels the earliest move of the opponent which is not yet canceled. If all the moves of the opponent are canceled at the moment, the player moves the game token by the vector

$$\frac{\varepsilon}{2}\left(\frac{y - x_0}{|y - x_0|}\right).$$

Let $M := 2\lceil |y - x_0|/\varepsilon \rceil \in \mathbb{N}$. If Player 1 wins M more coin tosses than the opponent before the game ends, the player moves the game position to

$$y + \sum_{k \in I_3} v_k^3 \tag{10}$$

in the last turn. Here, I_3 denotes the indices of rounds a random movement has occurred in the game, and the random movements are denoted by v_k^3 for all $k \in I_3$.

We want to estimate the probability of the event that the point in (10) is on the complement of the game domain Ω. To this end, we apply the so called cylinder walk framework developed in [29]. We estimate the number of times a random movement has occurred with a high probability at the first time the player wins M more coin tosses than the opponent. This is done by utilizing Hoeffding's inequality and the Berry-Esseen theorem, see [B, Lemma 3.5]. Furthermore, we prove the following density estimate for the sum of independent and identically distributed random vectors.

Lemma 5. [B, Lemma A.4] *Let $\varepsilon > 0$, and let Z be distributed according to the uniform distribution on the ball $B_\epsilon(0) \subset \mathbb{R}^n$. For any $k \geq 2$, denote the density of the random variable $\sum_{i=1}^k Z_i$ by f_k, where the random variables Z_i, $i \in \{1, \dots, k\}$, are independent and distributed as Z. Then f_k is a decreasing radial function, and there exist universal constants $k_0 := k_0(n) > 2$ and $R := R(n) > 0$ such that for all $k \geq k_0$ and $R_* \in [0, R]$ we have*

$$f_k\big(R_*\sqrt{k}\epsilon\big) \geq C\left(\frac{1}{\sqrt{k}\epsilon}\right)^n$$

for a constant $C := C(n) > 0$.

This density estimate connects the measure density condition to our estimates. Consequently, we find a uniform and strictly positive lower bound for the probability of the event that the point in (10) is on the complement of the game domain Ω.

4. Continuous time tug-of-war games and article [C]

Motivated by the connection between the discrete time pure tug-of-war game and ∞-harmonic functions, Atar and Budhiraja [5] formulated a continuous time two-player zero-sum stochastic differential game (SDG) for the inhomogeneous ∞-Laplace equation. They proved that for a given bounded C^2 domain $\Omega \subset \mathbb{R}^n$, a continuous boundary data g, a uniformly continuous function h with $h > 0$ or $h < 0$, and the unique viscosity solution u to the equation $-2\triangle_\infty^N u = h$ in Ω with the boundary data g, the value of the game is equal to the solution u. The existence of the unique solution to this Dirichlet problem is proved in [35] by using the game theory. For a proof utilizing methods of partial differential equations, see for example [26].

In [C], we study a SDG that is defined in terms of an n-dimensional state process, and is driven by a $2n$-dimensional Brownian motion. The impacts of the players enter in both a diffusion and a drift coefficient of the state process. The game is played in the whole space $\mathbb{R}^n$ until a fixed time $T > 0$, and at that time Player 2 pays the opponent the amount given by a pay-off function g at a current point.

The SDG process is denoted by $X(t), t \in [0, T]$, and we model it by a stochastic differential equation

$$\begin{cases} dX(s) &= \rho\big(G(s)\big)\, ds + \sigma\big(X(s), G(s)\big)\, d\overline{W}(s) \\ X(0) &= x, \end{cases} \tag{11}$$

where $x \in \mathbb{R}^n$, and $\overline{W}$ is a $2n$-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ satisfying the standard assumptions. We let

$$G(s) = \big(a(s), b(s), c(s), d(s)\big),$$

where

$$a(s), b(s) \in \mathbb{S}^{n-1},\ c(s), d(s) \in [0, \infty),\ s \in [0, T]$$

are progressively measurable stochastic processes with respect to the filtration $\{\mathcal{F}_s\}$. Here, $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$. The pairs $\big(a(s), c(s)\big)$ and $\big(b(s), d(s)\big)$ are called *controls* of the players. Roughly speaking, $a(s)$ and $b(s)$ are the *directions*, and $c(s)$ and $d(s)$ are the *lengths* taken by the players at the time s. Furthermore, let $\mu \in \mathbb{R}^n$. Then, for $s \in [0, T]$, we define the function ρ in (11) by

$$\rho\big(G(s)\big) = \mu + \big(c(s) + d(s)\big)\big(a(s) + b(s)\big).$$

Let $p : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ be a Lipschitz continuous function with values on a compact set $[p_{\min}, p_{\max}]$ for constants $1 < p_{\min} \le p_{\max} < \infty$. Then, we define the $n \times 2n$ matrix σ in (11) to be

$$\begin{aligned} &\sigma\big(X(s), G(s)\big) \\ &= \Big[a(s)\sqrt{p\big(X(s), s\big) - 1};\quad P^{\perp}_{a(s)};\quad b(s)\sqrt{p\big(X(s), s\big) - 1};\quad P^{\perp}_{b(s)}\Big], \end{aligned}$$

where the $n \times (n-1)$ matrices $P^{\perp}_{a(s)}$ and $P^{\perp}_{b(s)}$ are defined such that the matrices

$$P^{\perp}_{a(s)}\big(P^{\perp}_{a(s)}\big)^T \text{ and } P^{\perp}_{b(s)}\big(P^{\perp}_{b(s)}\big)^T$$

are projections to the $(n-1)$-dimensional hyperspaces orthogonal to the vectors $a(s)$ and $b(s)$ at the time s, respectively.

The players can only use admissible controls. Roughly speaking, a player initially declares a bound $C < \infty$, and then plays as to keep $c(s) \le C$ for all $s \in [0, T]$, where $\big(a(s), c(s)\big)$ is the admissible control of the player. A *strategy* is a response to the control of the opponent. We only allow admissible strategies. The set of admissible controls is denoted by $\mathcal{AC}$, and the set of admissible strategies is denoted by $\mathcal{S}$, respectively.

We define the *lower* and *upper values* by

$$\begin{aligned} U^-(x,t) &= \inf_{S\in\mathcal{S}} \sup_{A\in\mathcal{AC}} \mathbb{E}\Big[e^{-r(T-t)} g\big(X(T)\big)\Big], \\ U^+(x,t) &= \sup_{S\in\mathcal{S}} \inf_{A\in\mathcal{AC}} \mathbb{E}\Big[e^{-r(T-t)} g\big(X(T)\big)\Big] \end{aligned} \tag{12}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $r \geq 0$, and g is the pay-off function. The game starts at a position x at a time t, and the expectation $\mathbb{E}$ is taken with respect to the measure $\mathbb{P}$. The game is said to have a *value* at (x,t), if it holds $U^-(x,t) = U^+(x,t)$.

In [C], the main result is to connect the value functions (12) to a parabolic terminal value problem

$$\begin{cases} \partial_t u(x,t) + \triangle^N_{p(x,t)} u(x,t) + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i}(x,t) = ru(x,t) & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{13}$$

in the whole range $1 < p(x,t) < \infty$. Here, the *normalized* $p(x,t)$*-Laplace operator* is defined as

$$\begin{aligned} &\triangle^N_{p(x,t)} u(x,t) \\ &:= \left(\frac{p(x,t)-2}{|Du(x,t)|^2}\right) \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) \frac{\partial u}{\partial x_i}(x,t) \frac{\partial u}{\partial x_j}(x,t) + \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x,t) \end{aligned}$$

for $x \in \mathbb{R}^n$ and $t \in (0,T)$, provided that $Du(x,t) \neq 0$. The vector $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)^T$ is the gradient with respect to x.

Theorem 6. [C, Theorem 1.3] *Let g be positive, bounded and Lipschitz continuous. Furthermore, let U^- and U^+ be the lower and upper values of the stochastic differential game defined in* (12)*, respectively. Then, the functions U^- and U^+ are viscosity solutions to* (13)*.*

For completeness, we also show that viscosity solutions to (13) are unique under suitable assumptions [C, Theorem 1.4]. Thus, the game has a value in our setting.

The idea in the proof of Theorem 6 is the following. First, we study the SDG with a uniform bound m on the action sets of the players. Then in this setup, we follow [37] and connect the value functions to terminal value problems of Bellman-Isaacs type equations. Viscosity solutions to these equations are unique, see for example [9, 15]. The existence of viscosity solutions follows by the construction of barriers [C, Lemma 2.2] and by utilizing Perron's method.

We show in [C, Lemma 3.3] that the unique viscosity solution to a Bellman-Isaacs type equation equals precisely the lower value function of the game under the uniform bound on the action sets. In the proof, we first regularize the viscosity solution by the sup- or inf-convolution procedures depending on which direction in the equality we aim to prove, and then by the standard mollification procedure. Based on the regularized solution, we formulate a discretized control and a strategy. Then, we apply the celebrated Itô's lemma, and finally pass to limits as the time discretization vanishes. Here, a key step is to utilize a fundamental estimate for diffusions hitting a set of positive measure, see [21, 22]. Finally, we pass the results to the original solution by taking uniform limits.

In [C, Lemma 4.6] we utilize the results of [23, 38] to show that the family of viscosity solutions to Bellman-Isaacs type equations is equicontinuous. Consequently, the Arzelà-Ascoli theorem allows us to find a converging subsequence of solutions to the Bellman-Isaacs type equation as the uniform bound m on the actions sets of the players tends to infinity. Moreover, the stability principle for viscosity solutions implies that the limit u is a viscosity solution to (13). The final part is to deduce that the subsequence of the corresponding lower value functions converges to the lower value function for the game without the uniform bound on the controls. Furthermore, the proofs for the upper value function are similar.

As an application, one could study the model described above in the context of the portfolio option pricing. This would be based on the idea that, in addition to a random noise, the prices of the underlying assets are influenced by the two competing players. The issuer and the holder try, respectively, to manipulate the drifts and the volatilities of the assets to minimize and maximize, respectively, the expected discounted reward at the time T. To a certain extent, we generalize the model developed in [33], see also the discrete time game in [30]. Indeed, the volatility of an asset may vary over the space and the time.

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Tug-of-war games with varying probabilities and the normalized $p(x)$-Laplacian

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TUG-OF-WAR GAMES WITH VARYING PROBABILITIES AND THE NORMALIZED $p(x)$-LAPLACIAN

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Abstract. We study a two player zero-sum tug-of-war game with varying probabilities that depend on the game location x. In particular, we show that the value of the game is locally asymptotically Hölder continuous. The main difficulty is the loss of translation invariance. We also show the existence and uniqueness of values of the game. As an application, we prove that the value function of the game converges to a viscosity solution of the normalized $p(x)$-Laplacian.

1. **Introduction.** The seminal works of Crandall, Evans, Ishii, Lions, Souganidis and others established a connection between the stochastic differential games and viscosity solution to Bellman-Isaacs equations in the early 80s. However, a similar connection between the p-Laplace or ∞-Laplace equations and the tug-of-war games with noise was discovered only rather recently in [19, 20].

In this paper we study a tug-of-war with noise with space dependent probabilities, which is a natural generalization of the original tug-of-war both from mathematical and application point of views. In particular, we prove that the value functions of the game in this setting are asymptotically Hölder continuous, Theorem 4.1. Here the main difficulty is the loss of translation invariance so that the global or local regularity methods in [19], [16] or [12] are not directly applicable. Instead, we employ the method in [11].

The main idea is to consider two game sequences simultaneously. Heuristically speaking, in a higher dimensional space, the sequences can be linked to a single higher dimensional game by introducing a probability measure that has the measures of the original game as marginals through suitable couplings. It is interesting to note that couplings of stochastic processes can be employed in the study of regularity for second order linear uniformly parabolic equations with continuous highest order coefficients, see for example [14], [21], and [10]. The method has also some similarities to the Ishii-Lions method [7], see also [18]. However, the method we use does not rely on the theorem of sums in the theory of viscosity solutions nor does

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it use stochastic tools. Indeed, it applies directly to functions satisfying a dynamic programming equation whether they arise from the stochastic games or numerical methods to PDEs.

One of the key tools in studying the tug-of-war games is the dynamic programming principle. For the game in this paper, the dynamic programming principle (DPP) reads as

$$\begin{aligned} u(x) = \frac{1-\delta(x)}{2}\Bigg[&\sup_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu)+\beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\,d\mathcal{L}^{n-1}(h)\right) \\ &+\inf_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu)+\beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\,d\mathcal{L}^{n-1}(h)\right)\Bigg] + \delta(x)F(x) \end{aligned} \tag{1}$$

with a given boundary cut-off function δ, a boundary function F and probability functions $\alpha(x), \beta(x)$. Here, B_ε^ν denotes the $(n-1)$-dimensional ball orthogonal to ν. For more details, see Section 2. Heuristic idea behind the DPP is that the value at a point can be obtained by considering a single step in the game and summing up all the possible outcomes. At the point x, the game continues with a probability $1-\delta(x)$. In this case, the maximizer selects the direction $\nu_{\max}$ of fixed radius maximizing the expected payoff at the point. Similarly, the minimizer selects the direction $\nu_{\min}$ of the same radius minimizing the expectation. Then with a probability $\alpha(x)/2$, the game moves to $x+\nu_{\max}$ in the single step, and with the same probability, the game moves to $x+\nu_{\min}$. With a probability $\beta(x)/2$, the next game point is $x+\nu'_{\max}$, where $\nu'_{\max}$ is chosen according to the uniform distribution in a $(n-1)$-dimensional ball orthogonal to $\nu_{\max}$. Similarly with the same probability, the next game point is $x+\nu'_{\min}$, where $\nu'_{\min}$ is chosen uniformly random from a $(n-1)$-dimensional ball orthogonal to $\nu_{\min}$. If the game on the other hand stops at x, the payoff is given by the boundary function F at the point.

The first step in the paper is to show that a value function satisfies the dynamic programming principle above and that the value is unique. This is Theorem 3.7. We first prove existence of a measurable function satisfying the DPP by iterating the operator on the right hand side of (1). To this end, we guarantee the continuity and thus Borel measurability of the iterands by the boundary correction in the DPP above. Otherwise it is difficult to guarantee the measurability in such iterations. Then, the uniqueness and the continuity of the solution is obtained by using game theoretic arguments. In particular, we show that the solution coincides with the game value.

As an application, by using the regularity result, Arzelà-Ascoli's theorem and the DPP, we show in Theorem 6.2 that the values of the game converge to a continuous viscosity solution of the normalized $p(x)$-Laplace equation

$$\Delta_{p(x)}^N u(x) := \Delta u(x) + (p(x)-2)\Delta_\infty^N u(x) = 0,$$

where $\Delta_\infty^N u := |\nabla u|^{-2}\sum_{i,j=1}^n u_{x_ix_j}u_{x_i}u_{x_j}$ is the normalized infinity Laplacian, and $p:\overline{\Omega}\to(1,\infty)$ is a continuous function on the closure of the game domain Ω with $\inf_\Omega p > 1$ and $\sup_\Omega p < \infty$. Observe that we cover the range $1 < p(x) < \infty$. To guarantee that the limit takes the same boundary values, we need boundary estimates which are obtained in Theorem 5.2 by using barrier arguments.

2. **Preliminaries.**

Fix $n \geq 2$ and $\varepsilon > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For measurability reasons, we need the boundary correction function δ in the dynamic programming principle. Thus, we define the following open sets

$$I_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\},$$
$$O_\varepsilon = \{x \in \mathbb{R}^n \setminus \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$$

and the set $\Omega_\varepsilon := \overline{\Omega} \cup O_\varepsilon$. The function $\delta : \overline{\Omega}_\varepsilon \to [0, 1]$ is given by

$$\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus I_\varepsilon \\ 1 - \varepsilon^{-1} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in I_\varepsilon \\ 1 & \text{if } x \in \overline{O}_\varepsilon. \end{cases}$$

Let p be a continuous function on $\overline{\Omega}$ satisfying

$$1 < p_{\min} := \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) =: p_{\max} < \infty. \tag{2}$$

We require the finite upper bound $p_{\max}$ to make sure that the tug-of-war game defined below ends almost surely regardless of the strategies. Similarly, the upper bound comes into a play in the techniques we use in Section 3.2. On the other hand, the regularity and convergence results below require the lower bound in (2) for the function p. To prove existence and uniqueness of continuous solutions to (1) in Section 3, we utilize the uniform continuity of p. In Sections 4 and 5, the regularity techniques do not require the continuity of p, but in Section 6, we apply the continuity of p.

We define the functions $\alpha, \beta : \overline{\Omega} \to (0, 1)$ depending on $p(x)$ and the dimension n by

$$\alpha(x) = \frac{p(x) - 1}{p(x) + n} \quad \text{and} \quad \beta(x) = 1 - \alpha(x) = \frac{n + 1}{p(x) + n}.$$

By the assumptions on $p(x)$, the functions α and β are uniformly continuous. In addition, we have

$$\alpha_{\max} := \sup_{x\in\Omega} \alpha(x) < 1 \quad \text{and} \quad \alpha_{\min} := \inf_{x\in\Omega} \alpha(x) > 0. \tag{3}$$

We also denote $\beta_{\min} := 1 - \alpha_{\max} > 0$.

We consider averages of the form

$$\fint_{B_\varepsilon^\nu} u(x + h) d\mathcal{L}^{n-1}(h) := \frac{1}{\mathcal{L}^{n-1}(B_\varepsilon^\nu)} \int_{B_\varepsilon^\nu} u(x + h) d\mathcal{L}^{n-1}(h),$$

where $\mathcal{L}^{n-1}$ denotes the $(n-1)$-dimensional Lebesgue measure. The open ball of radius ε in the $(n-1)$-dimensional hyperplane $\nu^\perp$ orthogonal to $\nu \in \mathbb{R}^n$ is denoted by B_ε^ν, i.e.,

$$B_\varepsilon^\nu := B_\varepsilon(0) \cap \nu^\perp := \{z \in \mathbb{R}^n : |z| < \varepsilon \text{ and } \langle z, \nu\rangle = 0\}.$$

Throughout the paper, we denote open n-dimensional balls of radius $r > 0$ by $B_r(x)$ or by B_r, if the center point $x \in \mathbb{R}^n$ plays no role.

For brevity, the compact boundary strip of the game domain is denoted by

$$\Gamma_{\varepsilon,\varepsilon} := \overline{I}_\varepsilon \cup \overline{O}_\varepsilon.$$

Let F be a continuous boundary function $F : \Gamma_{\varepsilon,\varepsilon} \to \mathbb{R}$. In addition, we define an auxiliary function

$$W(x, \nu) := W(u; x, \nu) := \alpha(x) u(x + \nu) + \beta(x) \fint_{B_\varepsilon^\nu} u(x + h) d\mathcal{L}^{n-1}(h) \tag{4}$$

and an operator

$$T_\varepsilon u(x) := \frac{1-\delta(x)}{2}\left[\sup_{|\nu|=\varepsilon}\Big(W(u;x,\nu)\Big) + \inf_{|\nu|=\varepsilon}\Big(W(u;x,\nu)\Big)\right] + \delta(x)F(x) \qquad (5)$$

for all $x \in \overline{\Omega}_\varepsilon$ and continuous functions $u \in C(\overline{\Omega}_\varepsilon)$. By using this operator, we can identify the solutions to (1) with the fixed points of T_ε. Note that, despite the fact that $\alpha(x)$ and $\beta(x)$ are not defined in the outside strip $\overline{\Omega}_\varepsilon \setminus \overline{\Omega}$, (5) is well-defined by setting $T_\varepsilon u(x) = F(x)$ for all $x \in \overline{\Omega}_\varepsilon \setminus \overline{\Omega}$. Similarly, we set $\delta(x)F(x) = 0$ for all $x \in \Omega \setminus I_\varepsilon$.

The same boundary correction as above is also applied in [13, 6]. For an alternative approach, see [2]. Here, this correction is used in order to preserve measurability when iterating the operator. Indeed, in such iterations the measurability can rather easily be lost, see for example [13, Example 2.4]. In addition, an asymptotic expansion close to (1) is studied in [8].

2.1. **The two-player tug-of-war game.** In this subsection, we introduce the stochastic zero-sum tug-of-war game used in this work. Most of the methods of this paper arise from game theory, and some of the results are even directly proved by using game theory arguments (for example the uniqueness proof in Theorem 3.6).

Let us consider a game involving two players (say P_{I} and P_{II}). A token is placed at a starting point $x_0 \in \Omega$. Suppose that, after $j = 0, 1, 2, \ldots$ movements, the token is at a point $x_j \in \Omega$. Then,

- if $x_j \in \Omega \setminus I_\varepsilon$, P_{I} and P_{II} decide their possible movements ν^{I}_{j+1} and ν^{II}_{j+1}, respectively, with $\left|\nu^{\mathrm{I}}_{j+1}\right| = \left|\nu^{\mathrm{II}}_{j+1}\right| = \varepsilon$. A fair coin is tossed and if P_i wins the toss, we have two possibilities
  - with probability $\alpha(x_j)$, the token is moved to $x_{j+1} = x_j + \nu^i_{j+1}$, and
  - with probability $\beta(x_j)$, the token is moved to a point $x_{j+1} \in x_j + B^{\nu^i_{j+1}}_\varepsilon$ uniformly random

  with $i \in \{\mathrm{I}, \mathrm{II}\}$.
- If $x_j \in I_\varepsilon \cup O_\varepsilon$,
  - the game ends with probability $\delta(x_j)$ and then, P_{II} pays P_{I} the amount given by $F(x_j)$, and
  - with probability $1-\delta(x_j)$, the players play a game as in the previous case $x_j \in \Omega \setminus I_\varepsilon$.

Let τ denote the time when the game ends, and denote by $x_\tau \in \Gamma_{\varepsilon,\varepsilon}$ the position where the game ends. Then, P_{II} pays P_{I} the quantity $F(x_\tau)$. We define a *history* of the game as the vector $(x_0, x_1, \ldots, x_j)$ describing the positions of the token at each step after j repetitions. A *strategy* is a sequence of Borel measurable functions that gives the next game position given the history of the game. Therefore, we define $\mathcal{S}_i := (\mathcal{S}_i^j)_{j=1}^\infty$ with

$$\mathcal{S}_i^j : \{x_0\} \times \bigcup_{k=1}^{j-1} (\Omega_\varepsilon)^k \to \partial B_\varepsilon(0)$$

for all $j \geq 1$ and with both $i \in \{\mathrm{I}, \mathrm{II}\}$. For example, we have for P_{I} and for all $j \geq 1$ that

$$\mathcal{S}_{\mathrm{I}}^j\big((x_0, \ldots, x_{j-1})\big) = \nu^{\mathrm{I}}_j \in \partial B_\varepsilon(0).$$

Given a starting point $x_0 \in \Omega$ and strategies $\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}}$, we define a probability measure $\mathbb{P}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}_{\mathrm{II}}}$ on the natural product σ-algebra of the space of all game trajectories.

This measure is built by applying Kolmogorov's extension theorem to the family of transition densities

$$\pi_{\mathcal{S}_{\mathrm{I}},\mathcal{S}_{\mathrm{II}}}\big((x_0,\ldots,x_j),A\big)=\frac{1}{2}\Bigg[\alpha(x_j)\Big(\mathbb{I}_{x_j+\nu^{\mathrm{I}}_{j+1}}(A)+\mathbb{I}_{x_j+\nu^{\mathrm{II}}_{j+1}}(A)\Big)$$
$$+\frac{\beta(x_j)}{\omega_{n-1}\varepsilon^{n-1}}\Big(\mathcal{L}^{n-1}\Big(B_\varepsilon^{\nu^{\mathrm{I}}_{j+1}}(x_j)\cap A\Big)+\mathcal{L}^{n-1}\Big(B_\varepsilon^{\nu^{\mathrm{II}}_{j+1}}(x_j)\cap A\Big)\Big)\Bigg]$$

for all Borel subsets $A\subset\mathbb{R}^n$ as long as $x_j\in\Omega\setminus I_\varepsilon$ with the constant $\omega_{n-1}:=\mathcal{L}^{n-1}\big(B_1^z\big)$ for any $z\in\mathbb{R}^n\setminus\{0\}$. The measure $\mathbb{I}_z(A)$ is one if $z\in A$, and zero otherwise, for all $z\in\mathbb{R}^n$ and $A\subset\mathbb{R}^n$. In addition, we denote $B^z_\varepsilon(y):=y+B^z_\varepsilon$ for $z\in\mathbb{R}^n\setminus\{0\}$ and $y\in\mathbb{R}^n$. If $x_j\in I_\varepsilon$, the transition densities are multiplied with the boundary correction function δ similarly as in the DPP (1). If $x_j\notin\Omega$, the transition densities force $x_{j+1}=x_j$.

Here, we follow the ideas from [6], where the constant α case is covered. For the benefit of the reader and since the setting is slightly different, we give a self-contained proof.

Lemma 2.1. *The game ends almost surely in finite time regardless of the strategies $\mathcal{S}_I$ and $\mathcal{S}_{II}$.*

Proof. The idea of the proof is to consider solely random movements and to find a uniform lower bound for the probability of the event that the modulus of $|x_j|$ grows in a suitable fashion. In the proof, we need the fact $\beta_{\min}>0$.

Let $x_0\in\Omega$, $j\geq 0$ and let $x_{j+1}=x_j+h_j$, where h_j represents the displacement at each step of the game. By the vector calculus, we have

$$|x_{j+1}|^2=|x_j|^2+|h_j|^2+2\langle x_j,h_j\rangle.$$

In addition by the definition of the game, h_j is randomly chosen from B^ν_ε with a probability $\beta(x_j)/2$ for the vector $\nu:=\nu^{\mathrm{I}}_{j+1}$. Moreover, given that a random movement is chosen from B^ν_ε, we have $\langle x_j,h_j\rangle\geq 0$ with a probability of at least $\frac{1}{2}$ and the event $|h_j|\geq\frac{\varepsilon}{2}$ has a probability of

$$1-\frac{\mathcal{L}^{n-1}\big(B^\nu_{\varepsilon/2}\big)}{\mathcal{L}^{n-1}\big(B^\nu_\varepsilon\big)}=1-2^{1-n}.$$

Consequently, there is a positive probability of a random movement h_j such that $|h_j|\geq\varepsilon/2$ and $\langle x_j,h_j\rangle\geq 0$. In this case, we have

$$|x_{j+1}|^2\geq|x_j|^2+\frac{\varepsilon^2}{4}\tag{6}$$

with a probability of at least

$$\beta(x_j)\left(\frac{1}{4}-\frac{1}{2^{n+1}}\right)\geq\beta_{\min}\left(\frac{1}{4}-\frac{1}{2^{n+1}}\right)=:\theta>0.$$

Note that the universal constant θ does not depend on j and the fact $\beta_{\min}>0$ implies $\theta>0$. Now, let

$$j_0:=j_0(\varepsilon,\Omega)=4\left\lceil\operatorname{diam}(\Omega)\varepsilon^{-2}\right\rceil\in\mathbb{N}.$$

Then, after j_0 consecutive movements in the way (6) we have

$$|x_{j_0}|^2\geq|x_0|^2+j_0\frac{\varepsilon^2}{4}>|x_0|^2+\operatorname{diam}(\Omega).$$

Therefore, the token has exited the game domain after at most j_0 steps for any starting point x_0 with a probability of at least θ^{j_0}. Consequently, the probability of not exiting the game domain after j_0 steps is bounded above by $1-\theta^{j_0}$.

By repeating kj_0 times the game, the probability of not exiting Ω after kj_0 steps is bounded above by

$$\left(1-\theta^{j_0}\right)^k.$$

Thus, by letting $k\to\infty$, this probability goes to zero, and the proof is completed. □

For all starting points $x_0\in\Omega$, we define a *value function* for P_{I} and for P_{II} by

$$\begin{cases} u_{\mathrm{I}}(x_0)=\sup_{\mathcal{S}_{\mathrm{I}}}\inf_{\mathcal{S}_{\mathrm{II}}}\mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}_{\mathrm{II}}}[F(x_\tau)],\\ u_{\mathrm{II}}(x_0)=\inf_{\mathcal{S}_{\mathrm{II}}}\sup_{\mathcal{S}_{\mathrm{I}}}\mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}_{\mathrm{II}}}[F(x_\tau)]. \end{cases} \tag{7}$$

3. **Existence and uniqueness.**

In this section, the goal is to prove that there exists a unique continuous solution satisfying the dynamic programming principle (1). The proof is divided into two parts. In Section 3.1, by iterating the operator T_ε defined in (5), we show that there exist a lower and an upper semicontinuous solution to (1). Then in Section 3.2, we show that every measurable solution to (1) is bounded between the lower and the upper semicontinuous solutions. Further, we prove by using the tug-of-war game defined in Section 2.1 that, in fact, both semicontinuous solutions are the same.

3.1. **Existence of semicontinuous solutions to (1).**

In this subsection, by iterating the operator T_ε, we construct monotone sequences of bounded continuous functions. As a consequence, these sequences converge to semicontinuous functions which turn out to be solutions to (1). With that purpose, first, we need to show that T_ε maps continuous functions into continuous functions.

Lemma 3.1. *For any continuous function $u\in C(\overline{\Omega}_\varepsilon)$, the function $W(x,\nu)$ defined in (4) is continuous with respect to each variable on $\overline{\Omega}\times\partial B_\varepsilon(0)$.*

Proof. For fixed $|\nu|=\varepsilon$, we have for any $x,y\in\overline{\Omega}$ the estimate

$$\begin{aligned} &|\alpha(x)u(x+\nu)-\alpha(y)u(y+\nu)|\\ &\le|\alpha(x)u(x+\nu)-\alpha(x)u(y+\nu)|+|\alpha(x)u(y+\nu)-\alpha(y)u(y+\nu)|\\ &\le\alpha(x)\omega_u(|x-y|)+\|u\|_\infty\,\omega_\alpha(|x-y|), \end{aligned}$$

where ω_f is a modulus of continuity of the uniformly continuous function f. In a similar way, we have

$$\left|\beta(x)\fint_{B_\varepsilon^\nu}u(x+h)\,d\mathcal{L}^{n-1}(h)-\beta(y)\fint_{B_\varepsilon^\nu}u(y+h)\,d\mathcal{L}^{n-1}(h)\right|$$
$$\le\beta(x)\omega_u(|x-y|)+\|u\|_\infty\,\omega_\beta(|x-y|)$$

for $x,y\in\overline{\Omega}$. Thus, these inequalities imply that

$$|W(x,\nu)-W(y,\nu)|\le\omega_u(|x-y|)+\|u\|_\infty\left[\omega_\alpha(|x-y|)+\omega_\beta(|x-y|)\right] \tag{8}$$

for all $x,y\in\overline{\Omega}$. Hence, $W(\cdot,\nu)$ is a continuous function for fixed ν with modulus of continuity $\omega_u+\|u\|_\infty\left[\omega_\alpha+\omega_\beta\right]$.

For the continuity on ν, fix a point $x \in \overline{\Omega}$. Then, the modulus of continuity of α does not play any role. In addition, since the function u is continuous by the hypothesis, we only need to check the continuity of the function

$$\nu \mapsto \fint_{B_\varepsilon^\nu} u(x+h) d\mathcal{L}^{n-1}(h).$$

Let $|\nu| = |\chi| = \varepsilon$ and define a rotation $P : \nu^\perp \to \chi^\perp$ satisfying

$$|h - Ph| \leq C\,|h|\,|\nu - \chi| \tag{9}$$

for all $h \in \nu^\perp$, where $C > 0$ is a constant not depending on the choices of ν and χ. Therefore, we have

$$\begin{aligned}\fint_{B_\varepsilon^\nu} u(x+h)d\mathcal{L}^{n-1}(h) - \fint_{B_\varepsilon^\chi} u(x+h)d\mathcal{L}^{n-1}(h) \\ = \fint_{B_\varepsilon^\nu} \left[u(x+h) - u(x+Ph)\right] d\mathcal{L}^{n-1}(h).\end{aligned} \tag{10}$$

By recalling (9) together with the fact that we can choose ω_u to be increasing, we can estimate the expression in brackets in the equation (10) from above by

$$\omega_u\left(C\varepsilon\,|\nu - \chi|\right)$$

for $h \in B_\varepsilon^\nu$. Then, this same bound also holds for (10), and the continuity of $W(x,\cdot)$ for fixed $x \in \overline{\Omega}$ follows. □

Lemma 3.2. *For all $u \in C(\overline{\Omega}_\varepsilon)$, the operator T_ε defined in (5) satisfies $T_\varepsilon u \in C(\overline{\Omega}_\varepsilon)$. In addition, for all $u, v \in C(\overline{\Omega}_\varepsilon)$ such that $u \leq v$, we have*

$$T_\varepsilon u \leq T_\varepsilon v \ (\textit{monotonicity}).$$

Proof. The monotonicity of T_ε follows easily from the definition (5). Let $u \in C(\overline{\Omega}_\varepsilon)$ be a function with a modulus of continuity ω_u. By (5) and the fact that F is continuous on $\overline{O}_\varepsilon$, the function $T_\varepsilon u$ is continuous on the outside strip $\overline{O}_\varepsilon$. Thus, we have to check that $T_\varepsilon u$ is continuous on $\overline{\Omega}$.

First, let $x, y \in \Omega \setminus I_\varepsilon$ and recall the elementary inequalities

$$\left|\sup_{|\nu|=\varepsilon} W(x,\nu) - \sup_{|\nu|=\varepsilon} W(y,\nu)\right| \leq \sup_{|\nu|=\varepsilon} |W(x,\nu) - W(y,\nu)|,$$

$$\left|\inf_{|\nu|=\varepsilon} W(x,\nu) - \inf_{|\nu|=\varepsilon} W(y,\nu)\right| \leq \sup_{|\nu|=\varepsilon} |W(x,\nu) - W(y,\nu)|.$$

Then by the inequality (8) for any $|\nu| = \varepsilon$, we get that

$$\begin{aligned}&\frac{1}{2}\left|\left(\sup_{|\nu|=\varepsilon} + \inf_{|\nu|=\varepsilon}\right) W(x,\nu) - \left(\sup_{|\nu|=\varepsilon} + \inf_{|\nu|=\varepsilon}\right) W(y,\nu)\right| \\ &\leq \omega_u(|x-y|) + \|u\|_\infty \Big[\omega_\alpha(|x-y|) + \omega_\beta(|x-y|)\Big].\end{aligned}$$

Here, we use the shorthand notation

$$\left(\sup_{|\nu|=\varepsilon} + \inf_{|\nu|=\varepsilon}\right) W(x,\nu) := \sup_{|\nu|=\varepsilon} W(x,\nu) + \inf_{|\nu|=\varepsilon} W(x,\nu).$$

Therefore, since $\delta = 0$ on $\Omega \setminus I_\varepsilon$, we have shown that $T_\varepsilon u$ is continuous on $\Omega \setminus I_\varepsilon$.

Then, let $x, y \in I_\varepsilon$ and recall that $\sup_{x\in\Omega}\big(1-\delta(x)\big) = 1$ and $\omega_\delta(t) = t/\varepsilon$ for $t \geq 0$. Thus, we can estimate

$$\begin{aligned}&\left|\frac{1-\delta(x)}{2}\left(\sup_{|\nu|=\varepsilon} + \inf_{|\nu|=\varepsilon}\right) W(x,\nu) - \frac{1-\delta(y)}{2}\left(\sup_{|\nu|=\varepsilon} + \inf_{|\nu|=\varepsilon}\right) W(y,\nu)\right| \\ &\leq \omega_u(|x-y|) + \|u\|_\infty \left[\omega_\alpha(|x-y|) + \omega_\beta(|x-y|)\right] + \frac{\|u\|_\infty}{\varepsilon}|x-y|\end{aligned}$$

and

$$|\delta(x)F(x) - \delta(y)F(y)| \leq \delta(x)\omega_F(|x-y|) + \frac{\|F\|_\infty}{\varepsilon}|x-y|.$$

Consequently, $T_\varepsilon u$ is continuous in I_ε. Since the limiting values of the function $T_\varepsilon u$ coincide with the function values on the boundary ∂I_ε, there must exist a modulus of continuity for $T_\varepsilon u$, and hence $T_\varepsilon u \in C(\overline{\Omega}_\varepsilon)$. □

For the next result, let T_ε^k denote the k-th iteration of the operator T_ε for $k \in \mathbb{N}$, i.e.,

$$T_\varepsilon^k = T_\varepsilon(T_\varepsilon^{k-1}), \; T_\varepsilon^0 = \mathrm{Id},$$

with the identity operator $\mathrm{Id}(u) = u$ for all $u \in C(\overline{\Omega}_\varepsilon)$. By (5) and the monotonicity of T_ε, the sequence of iterates $\big\{T_\varepsilon^k(\inf F)\big\}_k$ is increasing and $\big\{T_\varepsilon^k(\sup F)\big\}_k$ is decreasing. Moreover,

$$\inf F \leq T_\varepsilon^k(\inf F) \leq T_\varepsilon^k(\sup F) \leq \sup F \tag{11}$$

for all $k \in \mathbb{N}$. Consequently, we can define the pointwise limit of both sequences

$$\begin{cases} \underline{u}(x) := \lim_{k\to\infty} T_\varepsilon^k(\inf F), \\ \overline{u}(x) := \lim_{k\to\infty} T_\varepsilon^k(\sup F) \end{cases} \tag{12}$$

for all $x \in \overline{\Omega}_\varepsilon$. In addition, since $\underline{u}$ and $\overline{u}$ are defined as the limit of monotone sequences of continuous functions, they are lower and upper semicontinuous functions, respectively.

Proposition 3.3. *The functions $\underline{u}$ and $\overline{u}$ defined in* (12) *are solutions to* (1) *and satisfy*

$$\underline{u} \leq \overline{u}. \tag{13}$$

Proof. The inequality (13) follows easily from (11). We only show that $\underline{u}$ is a solution to (1), since a similar argument can be applied to $\overline{u}$. To establish the result, we use Lemmas 3.1 and 3.2 and the fact that $\{T_\varepsilon^k(\inf F)\}$ is increasing to show that we can change the order of the limit and the infimum in the function $\underline{u}$.

Let $x \in \overline{\Omega}_\varepsilon$ and $u_k := T_\varepsilon^k(\inf F)$ for $k \in \mathbb{N}$. Then,

$$\begin{aligned}\underline{u}(x) &= \lim_{k\to\infty} u_{k+1}(x) = \lim_{k\to\infty} T_\varepsilon u_k(x) \\ &= \frac{1-\delta(x)}{2}\left[\lim_{k\to\infty}\sup_{|\nu|=\varepsilon} W(u_k; x,\nu) + \lim_{k\to\infty}\inf_{|\nu|=\varepsilon} W(u_k; x,\nu)\right] + \delta(x)F(x),\end{aligned}$$

where W denotes the auxiliary function defined in (4). Thus, we need to prove the equalities

$$\lim_{k\to\infty}\sup_{|\nu|=\varepsilon} W(u_k; x,\nu) = \sup_{|\nu|=\varepsilon} W(\underline{u}; x,\nu)$$

and

$$\lim_{k\to\infty}\inf_{|\nu|=\varepsilon} W(u_k; x,\nu) = \inf_{|\nu|=\varepsilon} W(\underline{u}; x,\nu).$$

The first equation follows from the fact that the sequence $\{u_k\}$ is pointwise increasing. For the second equation, we can assume $x \in \Omega$. Lemmas 3.1 and 3.2 imply that $W(u_k; x, \nu)$ is continuous with respect to ν for all $k \geq 1$. Therefore, we can define the compact set

$$C_k(\lambda) := \{\nu \in \mathbb{R}^n : |\nu| = \varepsilon \text{ and } W(u_k; x, \nu) \leq \lambda\}$$

for $\lambda \in \mathbb{R}$. Again, since $\{u_k\}$ is pointwise increasing, $C_{k+1}(\lambda) \subset C_k(\lambda)$ for all $k \geq 1$. Now, let

$$\lambda = \lim_{k\to\infty} \inf_{|\nu|=\varepsilon} W(u_k; x, \nu).$$

Because $W(u_k; x, \cdot)$ is continuous for all $k \geq 1$, there exists $\nu_k^* \in \partial B_\varepsilon(0)$ such that

$$\inf_{|\nu|=\varepsilon} W(u_k; x, \nu) = W(u_k; x, \nu_k^*).$$

This, together with the fact that $\{u_k\}$ is increasing, yields $C_k(\lambda) \neq \emptyset$ for all $k \geq 1$. Thus by Cantor's intersection theorem, we get

$$\bigcap_{k=1}^{\infty} C_k(\lambda) \neq \emptyset.$$

Choose $\tilde{\nu} \in \cap_{k=1}^{\infty} C_k(\lambda)$ so that we can estimate

$$\lambda \leq \inf_{|\nu|=\varepsilon} W(\underline{u}; x, \nu) \leq W(\underline{u}; x, \tilde{\nu}) = \lim_{k\to\infty} W(u_k; x, \tilde{\nu}) \leq \lambda.$$

The first inequality follows from the choice of λ and the fact that $\{u_k\}$ is increasing. In addition, we use the monotone convergence theorem in the first equality and the choice of $\tilde{\nu}$ in the last inequality. Therefore, the proof is complete. □

3.2. **Uniqueness of solutions to (1).** In this subsection, we prove the uniqueness of solutions to (1). To establish the result, we first show that any measurable solution of the equation (1) is between the solutions $\underline{u}$ and $\overline{u}$. Then, we show that, in fact, the functions $\underline{u}$ and $\overline{u}$ coincide. For the first result, we need the following technical lemma.

Lemma 3.4. *Let u be a measurable solution to (1). Assume that $\sup F < \sup_\Omega u$, and let $x \in \Omega$ be such that*

$$u(x) > \max\left\{\sup F, \sup_\Omega u - \lambda\right\} \tag{14}$$

for $\lambda > 0$. Then, there exist $|\nu_0| = \varepsilon$ and $h_0 \in B_\varepsilon^{\nu_0}$ satisfying the inequalities

$$|x + h_0|^2 \geq |x|^2 + 2^{\frac{2}{1-n}}\varepsilon^2 \tag{15}$$

and

$$u(x + h_0) \geq \sup_\Omega u - c(\alpha)\lambda \tag{16}$$

with a constant $c(\alpha) > 1$.

Proof. We obtain the inequalities (15) and (16) by analyzing the dynamic programming principle (1). The proof is similar to the proof of Lemma 2.1. Since u satisfies

(1), we have

$$\begin{aligned} u(x) &\leq (1-\delta(x)) \sup_{|\nu|=\varepsilon} \left(\alpha(x) u(x+\nu) + \beta(x) \fint_{B_\varepsilon^\nu} u(x+h) \, d\mathcal{L}^{n-1}(h) \right) + \delta(x) \sup F \\ &\leq (1-\delta(x))\alpha(x) \sup_{|\nu|=\varepsilon} u(x+\nu) + (1-\delta(x))\beta(x) \sup_{|\nu|=\varepsilon} \fint_{B_\varepsilon^\nu} u(x+h) \, d\mathcal{L}^{n-1}(h) \\ &\quad + \delta(x) \sup F. \end{aligned}$$

In addition, by utilizing the assumption (14) and $u = F$ on $\overline{O}_\varepsilon$, we get

$$\sup_{|\nu|=\varepsilon} u(x+\nu) \leq u(x) + \lambda.$$

Thus by (14), $0 < \delta(x) < 1$, $\alpha(x) + \beta(x) = 1$ and $\beta(x) \geq \beta_{\min} > 0$ for all $x \in \Omega$, we have

$$u(x) \leq \sup_{|\nu|=\varepsilon} \fint_{B_\varepsilon^\nu} u(x+h) \, d\mathcal{L}^{n-1}(h) + \frac{\alpha_{\max}}{\beta_{\min}} \lambda.$$

By the definition of supremum, there must exist $|\nu_0| = \varepsilon$ such that

$$u(x) - 2\frac{\alpha_{\max}}{\beta_{\min}} \lambda \leq \fint_{B_\varepsilon^{\nu_0}} u(x+h) \, d\mathcal{L}^{n-1}(h). \tag{17}$$

Next, we define a set $S \subset B_\varepsilon^{\nu_0}$ depending on x and ν_0. If $x \neq 0$ and $\nu_0 \in \operatorname{span}\{x\}$ or $x = 0$, we define

$$S := \left\{ h \in B_\varepsilon^{\nu_0} \ : \ |h| \geq (3/4)^{\frac{1}{n-1}} \varepsilon \right\}.$$

Otherwise, we set

$$S := \left\{ h \in B_\varepsilon^{\nu_0} \ : \ |h| \geq 2^{\frac{1}{1-n}} \varepsilon \text{ and } \langle x, h \rangle \geq 0 \right\}.$$

Observe that in both cases, the Lebesgue measure of the set S is the same. Indeed, it is clear that

$$\mathcal{L}^{n-1}(B_\varepsilon^{\nu_0}) - \mathcal{L}^{n-1}(B_{(3/4)^{1/(n-1)}\varepsilon}^{\nu_0}) = \frac{1}{4} \mathcal{L}^{n-1}(B_\varepsilon^{\nu_0}).$$

By symmetry, we get

$$\mathcal{L}^{n-1}\big(\{h \in B_\varepsilon^{\nu_0} : \langle x, h \rangle > 0\}\big) = \mathcal{L}^{n-1}\big(\{h \in B_\varepsilon^{\nu_0} : \langle x, h \rangle < 0\}\big),$$

and in the case $x \neq 0$ and $\nu_0 \notin \operatorname{span}\{x\}$, it holds

$$\mathcal{L}^{n-1}\big(\{h \in B_\varepsilon^{\nu_0} : \langle x, h \rangle = 0\}\big) = 0.$$

Thus, we have

$$\mathcal{L}^{n-1}(S) = \frac{1}{4} \mathcal{L}^{n-1}(B_\varepsilon^{\nu_0}). \tag{18}$$

In addition, because $(3/4)^{\frac{2}{n-1}} \geq 2^{\frac{2}{1-n}}$, the inequality (15) holds for each $h \in S$. The equality (18), together with (14) and (17), implies

$$\begin{aligned}\sup_{\Omega} u &\leq u(x) + \lambda \\ &\leq \fint_{B_\varepsilon^{\nu_0}} u(x+h)d\mathcal{L}^{n-1}(h) + \lambda \frac{1+\alpha_{\max}}{\beta_{\min}} \\ &= \frac{1}{4\mathcal{L}^{n-1}(S)} \left\{ \int_S u(x+h)d\mathcal{L}^{n-1}(h) + \int_{B_\varepsilon^{\nu_0}\setminus S} u(x+h)d\mathcal{L}^{n-1}(h) \right\} + \lambda \frac{1+\alpha_{\max}}{\beta_{\min}} \\ &\leq \frac{1}{4} \fint_S u(x+h)d\mathcal{L}^{n-1}(h) + \frac{3}{4}\sup_{\Omega} u + \frac{2\lambda}{\beta_{\min}}.\end{aligned}$$

By rearranging the terms and multiplying by 4, we have

$$\sup_{\Omega} u - \frac{8\lambda}{\beta_{\min}} \leq \fint_S u(x+h)d\mathcal{L}^{n-1}(h).$$

Hence, there must exist $h_0 \in S \subset B_\varepsilon^{\nu_0}$ satisfying (16). □

Proposition 3.5. *Any measurable solution u to (1) satisfies*

$$\underline{u} \leq u \leq \overline{u}$$

with $\underline{u}$ and $\overline{u}$ the semicontinuous functions defined in (12).

Proof. By the monotonicity of the operator T_ε and the definitions of $\underline{u}$ and $\overline{u}$, it is enough to show that

$$\inf F \leq u \leq \sup F. \tag{19}$$

Because u is a solution to (1), we have $u(x) = F(x)$ for $x \in \overline{O}_\varepsilon$. Hence, we need to show the estimate (19) for all $x \in \Omega$. We focus our attention on the second inequality, since the proof of the first inequality is analogous. We proceed by contradiction and assume that

$$\sup_{\Omega} u > \sup F.$$

By the assumption, for $\eta > 0$ there exists a point $x_1 \in \Omega$ such that

$$u(x_1) > \max\left\{\sup F, \sup_{\Omega} u - \eta\right\}.$$

The idea of the proof consists of finding a sequence of points $\{x_j\}$ satisfying $u(x_j) > \sup F$ for all j and $|x_{j_0}|$ is big enough for some large $j_0 \geq 1$. This is a contradiction, because $u = F$ on $\overline{O}_\varepsilon$. We obtain the sequence of points by using Lemma 3.4 iteratively.

Choose an integer $j_0 := j_0(\varepsilon, n, \Omega) \geq 1$ big enough such that

$$j 2^{\frac{2}{1-n}} \varepsilon^2 > \operatorname{diam}(\Omega) \tag{20}$$

for all $j \geq j_0$. Then, we fix the constant $\eta > 0$ small enough such that

$$0 < \eta < \frac{1}{c(\alpha)^{j_0}} \left(\sup_{\Omega} u - \sup F \right) \tag{21}$$

with the constant $c(\alpha) > 1$ from Lemma 3.4. We start from x_1 and choose x_2 such that $x_2 = x_1 + h_0$ with h_0 given by Lemma 3.4. Then, we have that $|x_2|^2 \geq |x_1|^2 + 2^{\frac{2}{1-n}}\varepsilon^2$ and

$$u(x_2) \geq \sup_{\Omega} u - c(\alpha)\eta > \sup F.$$

If $x_2 \in \overline{O}_\varepsilon$, we get a contradiction. Otherwise, we continue in the same way. We choose x_3 such that $x_3 = x_2 + h_1$ with h_1 given by Lemma 3.4. Then, we have that $|x_3|^2 \geq |x_1|^2 + 2 \cdot 2^{\frac{2}{1-n}} \varepsilon^2$ and

$$u(x_3) \geq \sup_\Omega u - c(\alpha)^2 \eta > \sup F.$$

After $j_0 - 1$ repetitions, assume that $x_{j_0} \in \Omega$. By the inequalities (20) and (21) it holds for the point x_{j_0+1} that

$$|x_{j_0+1}|^2 \geq |x_1|^2 + j_0 2^{\frac{2}{1-n}} \varepsilon^2 > |x_1|^2 + \operatorname{diam}(\Omega)$$

and

$$u(x_{j_0+1}) \geq \sup_\Omega u - c(\alpha)^{j_0} \eta > \sup F.$$

Since $x_{j_0+1} \notin \Omega$, the contradiction follows. □

The next theorem, together with (13), implies that the semicontinuous solutions to (1), $\underline{u}$ and $\overline{u}$, coincide.

Theorem 3.6. *Let $\underline{u}$ and $\overline{u}$ be the semicontinuous functions defined in (12). In addition, let u_I and u_{II} be the value functions defined in (7). Then, we have that*

$$\overline{u} \leq u_I \leq u_{II} \leq \underline{u}.$$

Proof. To establish the result, we show that under a suitable strategy for the other player, the process $\underline{u}(x_k)$ becomes a supermartingale and the process $\overline{u}(x_k)$ becomes a submartingale irregardless what the opponent does. Then, we are able to compare the functions $\underline{u}$ and $\overline{u}$ with the value functions of the game.

From the properties of inf and sup, it is clear that

$$u_{\mathrm{I}} \leq u_{\mathrm{II}}.$$

Thus, we need to prove that

$$u_{\mathrm{II}} \leq \underline{u} \quad \text{and} \quad \overline{u} \leq u_{\mathrm{I}}.$$

We only show $u_{\mathrm{II}} \leq \underline{u}$, since the argument in the other case is similar. Let $x_0 \in \Omega$ and denote a strategy $\mathcal{S}_{\mathrm{II}}^*$ for P_{II} such that

$$W(\underline{u}; x_j, \nu_j^{\mathrm{II}}) = \inf_{|\nu|=\varepsilon} W(\underline{u}; x_j, \nu)$$

for all $j \geq 0$, where W denotes the auxiliary function defined in (4). By a measure theoretical analysis, we can prove that this strategy is Borel measurable (for more details, see, for example [22, Theorem 5.3.1]).

Fix any strategy $\mathcal{S}_{\mathrm{I}}$ for P_{I}. Now, we can estimate

$$\begin{aligned}
&\mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}}, \mathcal{S}^*_{\mathrm{II}}}[\underline{u}(x_{j+1}) | x_0, \ldots, x_j] \\
&= \frac{1-\delta(x_j)}{2} \left[W(\underline{u}; x_j, \nu_{j+1}^{\mathrm{I}}) + W(\underline{u}; x_j, \nu_{j+1}^{\mathrm{II}}) \right] + \delta(x_j) F(x_j) \\
&\leq \frac{1-\delta(x_j)}{2} \left[\sup_{|\nu|=\varepsilon} W(\underline{u}; x_j, \nu) + \inf_{|\nu|=\varepsilon} W(\underline{u}; x_j, \nu) \right] + \delta(x_j) F(x_j).
\end{aligned}$$

Since $\underline{u}$ is a solution to (1), we have by the estimate above

$$\mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}}, \mathcal{S}^*_{\mathrm{II}}}[\underline{u}(x_{j+1}) | x_0, \ldots, x_j] \leq \underline{u}(x_j)$$

for all $j \geq 0$. Thus, the stochastic process $M_k := \underline{u}(x_k)$ is a supermartingale, when P_{II} uses the strategy $\mathcal{S}_{\mathrm{II}}^*$. By Lemma 2.1, the game ends almost surely, and since F is bounded, we get by using the optional stopping theorem that

$$u_{\mathrm{II}}(x_0) = \inf_{\mathcal{S}_{\mathrm{II}}} \sup_{\mathcal{S}_{\mathrm{I}}} \mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}_{\mathrm{II}}}[F(x_\tau)] \leq \sup_{\mathcal{S}_{\mathrm{I}}} \mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}^*_{\mathrm{II}}}[F(x_\tau)] \leq \underline{u}(x_0).$$

Therefore, the proof is finished. □

Now, Theorem 3.6 and Proposition 3.5 imply the uniqueness of solutions to (1). In addition, this unique function is continuous and the value function of the game.

Theorem 3.7. *Let $\varepsilon > 0$ and let $F : \Gamma_{\varepsilon,\varepsilon} \to \mathbb{R}^n$ be a continuous function. Then, there exists a continuous function $u_\varepsilon : \overline{\Omega}_\varepsilon \to \mathbb{R}^n$ with the boundary data F such that it satisfies the dynamic programming principle (1). Moreover, this function is unique and it is the value function of the game, i.e., $u_\varepsilon = u_I = u_{II}$ with u_I and u_{II} defined in (7).*

4. **Local regularity.** In this section, we give a local regularity estimate for functions satisfying (1) in $\Omega \setminus I_\varepsilon$. The dynamic programming principle in $\Omega \setminus I_\varepsilon$ reduces to the equation

$$\begin{aligned} u(x) = \frac{1}{2}\Bigg[&\sup_{|\nu|=\varepsilon} \left(\alpha(x)u(x+\nu) + \beta(x) \fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h) \right) \\ &+ \inf_{|\nu|=\varepsilon} \left(\alpha(x)u(x+\nu) + \beta(x) \fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h) \right) \Bigg]. \end{aligned} \tag{22}$$

The regularity result is based on a method established by Luiro and Parviainen in [11]. The method consists of several steps. First, we choose a comparison function f having the desired regularity properties. Then, the idea is to analyze two different cases separately. At a small scale, we need to control the effects arising from the discretization. At a bigger scale, the key term of the comparison function is $C|x-z|^\gamma$ with $x, z \in \mathbb{R}^n, 0 < \gamma < 1$ and $C > 0$ big enough.

In the second step, we aim to prove that the error $u(x) - u(z) - f(x,z)$, where u is the solution to (22), is smaller in $(B_1 \times B_1) \setminus T$ than in $(B_2 \times B_2) \setminus (B_1 \times B_1 \setminus T)$ with both sets belonging to $\mathbb{R}^{2n}$. The set T is the set of points $(x,z) \in \mathbb{R}^{2n}$ such that $x = z$. Then, we thrive for a contradiction by assuming that the error is bigger in $(B_1 \times B_1) \setminus T$.

As a final step, we get a contradiction by using a multidimensional dynamic programming principle for the comparison function f. In the proof below, intuition based on suitable strategies is helpful even though we do not write down stochastic arguments.

Theorem 4.1. *Let $(x,z) \in B_R \times B_R$, $B_{2R} \subset \Omega$ and*

$$0 < \gamma < \frac{\alpha_{min}}{\alpha_{max}} - \kappa \tag{23}$$

for arbitrary small $\kappa \in \big(0, \alpha_{min}/\alpha_{max}\big)$ with $\alpha_{min}, \alpha_{max}$ defined in (3). Then, if u satisfies (22), we have

$$|u(x) - u(z)| \leq C\frac{|x-z|^\gamma}{R^\gamma} + C\frac{\varepsilon^\gamma}{R^\gamma} \tag{24}$$

with $C := C(p_{min}, p_{max}, n, R, \sup_{B_{2R}} u, \gamma)$, $0 < \varepsilon < 1$ and p_{min}, p_{max} defined in (2).

Proof. By using a scaling $x \mapsto Rx$, we can assume that $R = 1$. In addition by translation, it is enough to consider the claim (24) in the case $z = -x$. For simplicity, we assume $\sup_{B_2 \times B_2}(u(x) - u(z)) \leq 1$.

Given $C > 1$, let $N \in \mathbb{N}$ be such that

$$N \geq \frac{10^2 C}{\gamma}.$$

Then, we define the following functions in $\mathbb{R}^{2n}$

$$\begin{aligned} f_1(x,z) &= C\,|x-z|^\gamma + |x+z|^2\,, \\ f_2(x,z) &= \begin{cases} C^{2(N-i)}\varepsilon^\gamma & \text{if } (x,z) \in A_i, \\ 0 & \text{if } |x-z| > N\frac{\varepsilon}{10}, \end{cases} \\ f(x,z) &= f_1(x,z) - f_2(x,z) \end{aligned}$$

with $A_i = \left\{(x,z) \in \mathbb{R}^{2n} \;:\; (i-1)\frac{\varepsilon}{10} < |x-z| \leq i\frac{\varepsilon}{10}\right\}$ for $i = 0, 1, \ldots, N$. The function f_2 is called an *annular step function*, and it is needed to control the small scale jumps. Note that we have $\sup f_2 = C^{2N}\varepsilon^\gamma$ reached on

$$T := A_0 = \left\{(x,z) \in \mathbb{R}^{2n} \;:\; x = z\right\}.$$

It holds that $f_1 \geq 1$ in $(B_2 \times B_2) \setminus (B_1 \times B_1)$. Here, we need the term $|x+z|^2$ in the function f_1, because

$$|x+z|^2 = 2|x|^2 + 2|z|^2 - |x-z|^2 \geq 3$$

for all $x, z \in (B_2 \times B_2) \setminus (B_1 \times B_1)$ such that $|x - z| \leq 1$. Therefore, together with $u(x) - u(z) \leq 1$ in $B_2 \times B_2$ and $u(x) - u(z) = 0$ in T we have

$$u(x) - u(z) - f(x,z) \leq \sup f_2 = C^{2N}\varepsilon^\gamma, \tag{25}$$

if $(x,z) \in T$ or $(x,z) \in (B_2 \times B_2) \setminus (B_1 \times B_1)$. We have to show that this inequality is also true in $(B_1 \times B_1) \setminus T$. Thriving for a contradiction, write

$$M := \sup_{(x,z) \in B_1 \times B_1 \setminus T} (u(x) - u(z) - f(x,z))$$

and suppose that

$$M > C^{2N}\varepsilon^\gamma.$$

By (25), this is equivalent to

$$M = \sup_{(x,z) \in B_2 \times B_2} (u(x) - u(z) - f(x,z)). \tag{26}$$

For all $\eta > 0$, we choose a pair of points $(x,z) \in (B_1 \times B_1) \setminus T$ such that

$$M \leq u(x) - u(z) - f(x,z) + \frac{\eta}{2}. \tag{27}$$

Then by (22), we have

$$u(x) - u(z) \leq \frac{1}{2}\sup_{\nu_x,\nu_z}(W(x,\nu_x) - W(z,\nu_z)) + \frac{1}{2}\inf_{\nu_x,\nu_z}(W(x,\nu_x) - W(z,\nu_z)), \tag{28}$$

where W is the auxiliary function defined in (4).

Given $|\nu_x| = |\nu_z| = \varepsilon$, let $P_{\nu_z,-\nu_x}$ denote any rotation that sends ν_z to $-\nu_x$. By recalling $\alpha(x) + \beta(x) = 1$ for $x \in \Omega$, we can decompose the difference $W(x, \nu_x) - W(z, \nu_z)$. For simplicity, we may assume that $\alpha(x) \geq \alpha(z)$. Thus, we get

$$\begin{aligned} W(x,\nu_x) - W(z,\nu_z) &= \alpha(z)\left[u(x+\nu_x) - u(z+\nu_z)\right] \\ &\quad + \beta(x) \fint_{B_\varepsilon^{\nu_z}} \left[u(x + P_{\nu_z,-\nu_x} h) - u(z+h)\right] d\mathcal{L}^{n-1}(h) \\ &\quad + (\alpha(x) - \alpha(z)) \left[u(x+\nu_x) - \fint_{B_\varepsilon^{\nu_z}} u(z+h) d\mathcal{L}^{n-1}(h)\right]. \end{aligned} \tag{29}$$

Next, we use the counter assumption (26) to estimate each of the terms in (29) from above. Consequently, we can estimate

$$u(y) - u(\tilde{y}) \leq M + f(y, \tilde{y})$$

for all $y, \tilde{y} \in B_2$. Then, we define

$$\begin{aligned} G(f, x, z, \nu_x, \nu_z) &:= \alpha(z) f(x+\nu_x, z+\nu_z) \\ &\quad + \beta(x) \fint_{B_\varepsilon^{\nu_z}} f(x + P_{\nu_z,-\nu_x} h, z+h) d\mathcal{L}^{n-1}(h) \\ &\quad + (\alpha(x) - \alpha(z)) \fint_{B_\varepsilon^{\nu_z}} f(x+\nu_x, z+h) d\mathcal{L}^{n-1}(h). \end{aligned} \tag{30}$$

Thus, we have

$$W(x,\nu_x) - W(z,\nu_z) \leq M + G(f, x, z, \nu_x, \nu_z). \tag{31}$$

By taking the supremum, we obtain

$$\sup_{\nu_x,\nu_z} \left(W(x,\nu_x) - W(z,\nu_z)\right) \leq M + \sup_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z). \tag{32}$$

On the other hand, choose $|\varrho_x| = |\varrho_z| = \varepsilon$ such that

$$\inf_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z) \geq G(f, x, z, \varrho_x, \varrho_z) - \eta.$$

This together with (31) yields

$$\begin{aligned} \inf_{\nu_x,\nu_z} \left(W(x,\nu_x) - W(z,\nu_z)\right) &\leq W(x,\varrho_x) - W(z,\varrho_z) \\ &\leq M + G(f, x, z, \varrho_x, \varrho_z) \\ &\leq M + \inf_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z) + \eta. \end{aligned}$$

Therefore, by applying this inequality and (32) to (28) we get

$$u(x) - u(z) \leq M + \frac{1}{2}\left[\sup_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z) + \inf_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z)\right] + \frac{\eta}{2}.$$

Combining this with (27), we need to show

$$\sup_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z) + \inf_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z) < 2f(x,z).$$

This inequality follows from Proposition 4.2 below. Consequently, the equation (25) holds in $B_2 \times B_2$. □

Proposition 4.2. *Let f and T be as at the beginning of the proof of Theorem 4.1, and fix $x, z \in B_1 \times B_1 \setminus T$. In addition, let G be as in (30). Then, it holds that*

$$\sup_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z) + \inf_{\nu_x,\nu_z} G(f, x, z, \nu_x, \nu_z) < 2f(x,z).$$

The main part of the section is to show this estimate for G. This is done in several steps below.

4.1. **Proof of Proposition 4.2.** Let $V \subset \mathbb{R}^n$ be the space spanned by $x - z \neq 0$. We denote the orthogonal complement of V by $V^\perp$, i.e.,

$$V^\perp = \{y \in \mathbb{R}^n \ : \ \langle y, x - z\rangle = 0\}.$$

Given any $y \in \mathbb{R}^n$, we can decompose $y = y_V(x-z)/|x-z| + y_{V^\perp}$, where $y_V \in \mathbb{R}$ is the scalar projection of y onto V and $y_{V^\perp} \in V^\perp$, respectively. For the decomposed point it holds

$$\begin{aligned} y_V &= \langle y, \frac{x-z}{|x-z|}\rangle, \\ |y_{V^\perp}| &= \sqrt{|y|^2 - y_V^2}. \end{aligned}$$

By using this notation, the second order Taylor's expansion of f_1 is

$$\begin{aligned} &f_1(x + h_x, z + h_z) - f_1(x, z) \\ &\quad = C\gamma\, |x-z|^{\gamma-1} (h_x - h_z)_V + 2\langle x + z, h_x + h_z\rangle \\ &\qquad + \frac{1}{2} C\gamma\, |x-z|^{\gamma-2} \left\{ (\gamma-1)(h_x - h_z)_V^2 + |(h_x - h_z)_{V^\perp}|^2 \right\} \\ &\qquad + |h_x + h_z|^2 + \mathcal{E}_{x,z}(h_x, h_z), \end{aligned} \tag{33}$$

where $\mathcal{E}_{x,z}(h_x, h_z)$ is the error term. In the above, we used the calculations

$$\langle \nabla f_1(x,z), (h_x^T, h_z^T)\rangle = C\gamma |x-z|^{\gamma-2}\langle x - z, h_x - h_z\rangle + 2\langle x + z, h_x + h_z\rangle$$

and

$$D^2 f_1 = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix} + 2 \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{I} & \mathrm{I} \end{bmatrix}$$

with

$$A := C\gamma |x-z|^{\gamma-2} \left[(\gamma - 2) \frac{x-z}{|x-z|} \otimes \frac{x-z}{|x-z|} + \mathrm{I} \right].$$

The matrix I stands for the $n \times n$ identity matrix, and we denote the tensor product of two vectors by $\otimes$, i.e., $h \otimes s := hs^T$ for vectors $h, s \in \mathbb{R}^n$. By recalling the elementary formula $h^T(s \otimes s)h = \langle h, s\rangle^2$ for all $h, s \in \mathbb{R}^n$, we get (33).

By Taylor's theorem, the error term satisfies

$$|\mathcal{E}_{x,z}(h_x, h_z)| \leq C \left| (h_x^T, h_z^T) \right|^3 (|x-z| - 2\varepsilon)^{\gamma-3},$$

if $|x - z| > 2\varepsilon$. With the choice $N \geq \frac{100C}{\gamma}$ and if $|x-z| > \frac{N}{10}\varepsilon$, we can estimate

$$\begin{aligned} |\mathcal{E}_{x,z}(h_x, h_z)| &\leq C(2\varepsilon)^3 \left(\frac{|x-z|}{2} \right)^{\gamma-3} \\ &\leq 64 C \varepsilon^2 |x-z|^{\gamma-2} \frac{\varepsilon}{|x-z|} \\ &\leq 10\, |x-z|^{\gamma-2}\, \varepsilon^2, \end{aligned} \tag{34}$$

because $|h_x|, |h_z| \leq \varepsilon$. Therefore, to prove the result, we distinguish two separate cases. In the first case, we have $|x-z| \leq \frac{N}{10}\varepsilon$ and in the second case, we have $|x-z| > \frac{N}{10}\varepsilon$.

Proof of Proposition 4.2: Case $|x-z| \leq N\frac{\varepsilon}{10}$. In this case, we do not utilize the formula (33). We use concavity and convexity estimates for the terms in f_1 and the properties of the annular step function f_2. For $x, z \in B_1$ and $|h_x|, |h_z| < \varepsilon < 1$, it holds

$$|f_1(x+h_x, z+h_z) - f_1(x,z)| \leq 2C\varepsilon^\gamma + 16\varepsilon \leq 3C\varepsilon^\gamma$$

for $C > 16$. Consequently by (30), we have

$$\sup_{h_x,h_z} G(f_1, x, z, h_x, h_z) \leq f_1(x,z) + 3C\varepsilon^\gamma.$$

Together with $f_2 \geq 0$, these estimates yield

$$\sup_{h_x,h_z} G(f, x, z, h_x, h_z) \leq f_1(x,z) + 3C\varepsilon^\gamma. \tag{35}$$

Find $i \in \{1, 2, \ldots, N\}$ such that $(i-1)\frac{\varepsilon}{10} < |x-z| \leq i\frac{\varepsilon}{10}$ and choose $|\nu_x|, |\nu_z| < \varepsilon$ such that $(x+\nu_x, z+\nu_z) \in A_{i-1}$. Then for $C > 1$ large enough, we can estimate

$$\begin{aligned}
\sup_{h_x,h_z} G(f_2, x, z, h_x, h_z) &\geq G(f_2, x, z, \nu_x, \nu_z) \\
&\geq \alpha(z) f_2(x+\nu_x, z+\nu_z) \\
&= \alpha(z) C^{2(N-i+1)}\varepsilon^\gamma \\
&= \alpha(z)\left(C^2 - \frac{2}{\alpha(z)}\right) C^{2(N-i)}\varepsilon^\gamma + 2f_2(x,z) \\
&> 6C\varepsilon^\gamma + 2f_2(x,z),
\end{aligned}$$

where we use $f_2 \geq 0$ in the second inequality and $\alpha(z) > \alpha_{\min} > 0$ for all $z \in \Omega$ in the last inequality. Therefore, by $f = f_1 - f_2$ and (35) it holds

$$\begin{aligned}
\inf_{h_x,h_z} G(f, x, z, h_x, h_z) &\leq \sup_{h_x,h_z} G(f_1, x, z, h_x, h_z) - \sup_{h_x,h_z} G(f_2, x, z, h_x, h_z)) \\
&\leq f_1(x,z) - 2f_2(x,z) - 3C\varepsilon^\gamma.
\end{aligned}$$

Combining this inequality with (35), we get

$$\sup_{h_x,h_z} G(f, x, z, h_x, h_z) + \inf_{h_x,h_z} G(f, x, z, h_x, h_z) < 2f(x,z).$$

Hence, the proof of the case is complete.

Proof of Proposition 4.2: Case $|x-z| > N\frac{\varepsilon}{10}$. In this case, $f_2(x,z) = 0$ and hence $f \equiv f_1$. We apply (33) to get the result. For $\eta > 0$, let ν_x, ν_z be such that

$$\sup_{h_x,h_z} G(f, x, z, h_x, h_z) \leq G(f, x, z, \nu_x, \nu_z) + \eta.$$

Therefore for any $|\varrho_x|, |\varrho_z| \leq \varepsilon$, we get the following inequality

$$\begin{aligned}
&\sup_{h_x,h_z} G(f, x, z, h_x, h_z) + \inf_{h_x,h_z} G(f, x, z, h_x, h_z) \\
&\leq G(f, x, z, \nu_x, \nu_z) + G(f, x, z, \varrho_x, \varrho_z) + \eta.
\end{aligned} \tag{36}$$

By (34) and $|h_x|, |h_z| \leq \varepsilon$, the last two terms in (33) are bounded above by

$$(4 + 10|x-z|^{\gamma-2})\varepsilon^2.$$

We denote

$$E := E(f, x, z, \gamma, \varepsilon) := f(x,z) + (4 + 10|x-z|^{\gamma-2})\varepsilon^2,$$

and recall the notation $P_{h,s}$ denoting the rotation sending h to s for any vectors $|h| = |s|$ in $\mathbb{R}^n$. By (36) and (30), it suffices to study

$$\begin{aligned}
\textbf{[I]} &:= G(f,x,z,\nu_x,\nu_z) + G(f,x,z,\varrho_x,\varrho_z) - 2E \\
&= \alpha(z)\left[f(x+\nu_x,z+\nu_z) + f(x+\varrho_x,z+\varrho_z) - 2E\right] \\
&\quad + \beta(x)\Bigg[\fint_{B_\varepsilon^{\nu_z}} f(x+P_{\nu_z,-\nu_x}h, z+h)\,d\mathcal{L}^{n-1}(h) \\
&\quad + \fint_{B_\varepsilon^{\varrho_z}} f(x+P_{\varrho_z,-\varrho_x}h, z+h)\,d\mathcal{L}^{n-1}(h) - 2E\Bigg] \\
&\quad + (\alpha(x)-\alpha(z))\Bigg[\fint_{B_\varepsilon^{\nu_z}} f(x+\nu_x, z+h)\,d\mathcal{L}^{n-1}(h) \\
&\quad + \fint_{B_\varepsilon^{\varrho_z}} f(x+\varrho_x, z+h)\,d\mathcal{L}^{n-1}(h) - 2E\Bigg].
\end{aligned} \tag{37}$$

For simplicity, we decompose the previous expression into three terms to be examined separately, i.e.,

$$\textbf{[I]} = \alpha(z)\textbf{[II]} + \beta(x)\textbf{[III]} + (\alpha(x)-\alpha(z))\textbf{[IV]}. \tag{38}$$

Then by (33), we have

$$\begin{aligned}
\textbf{[II]} &\le C\gamma\,|x-z|^{\gamma-1}\left[(\nu_x-\nu_z)_V + (\varrho_x-\varrho_z)_V\right] \\
&\quad + 2\langle x+z, (\nu_x+\nu_z) + (\varrho_x+\varrho_z)\rangle \\
&\quad + \frac{1}{2}C\gamma\,|x-z|^{\gamma-2}\Big\{(\gamma-1)\left[(\nu_x-\nu_z)_V^2 + (\varrho_x-\varrho_z)_V^2\right] \\
&\quad + \left[|(\nu_x-\nu_z)_{V^\perp}|^2 + |(\varrho_x-\varrho_z)_{V^\perp}|^2\right]\Big\}.
\end{aligned} \tag{39}$$

Note that the first order terms in **[III]** vanishes when we integrate over the ball. Therefore, we can estimate

$$\begin{aligned}
\textbf{[III]} &\le \frac{1}{2}C\gamma\,|x-z|^{\gamma-2}\cdot \\
&\cdot\Bigg\{\fint_{B_\varepsilon^{\nu_z}} \left[(\gamma-1)(h-P_{\nu_z,-\nu_x}h)_V^2 + |(h-P_{\nu_z,-\nu_x}h)_{V^\perp}|^2\right] d\mathcal{L}^{n-1}(h) \\
&+ \fint_{B_\varepsilon^{\varrho_z}} \left[(\gamma-1)(h-P_{\varrho_z,-\varrho_x}h)_V^2 + |(h-P_{\varrho_z,-\varrho_x}h)_{V^\perp}|^2\right] d\mathcal{L}^{n-1}(h)\Bigg\}.
\end{aligned} \tag{40}$$

In addition, it holds

$$\begin{aligned}
\textbf{[IV]} &\le C\gamma\,|x-z|^{\gamma-1}(\nu_x+\varrho_x)_V + 2\langle x+z, \nu_x+\varrho_x\rangle \\
&\quad + \frac{1}{2}C\gamma\,|x-z|^{\gamma-2}\cdot \\
&\quad \cdot\Bigg\{\fint_{B_\varepsilon^{\nu_z}} \left[(\gamma-1)(\nu_x-h)_V^2 + |(\nu_x-h)_{V^\perp}|^2\right] d\mathcal{L}^{n-1}(h) \\
&\quad + \fint_{B_\varepsilon^{\varrho_z}} \left[(\gamma-1)(\varrho_x-h)_V^2 + |(\varrho_x-h)_{V^\perp}|^2\right] d\mathcal{L}^{n-1}(h)\Bigg\}.
\end{aligned} \tag{41}$$

We distinguish between two cases depending on the value of $(\nu_x - \nu_z)_V^2$ and fix $\tau_0 < \tau < 1$ with $0 < \tau_0 < 1$ defined later.

a) Case $|(\nu_x - \nu_z)_V| \geq (\tau+1)\varepsilon$: In this case, we choose $\varrho_x = -\nu_x$ and $\varrho_z = -\nu_z$. By replacing these vectors in the inequalities **[II]**, **[III]** and **[IV]** and using symmetry, we obtain

$$\textbf{[II]} \leq C\gamma\, |x-z|^{\gamma-2} \left[(\gamma-1)(\nu_x-\nu_z)_V^2 + |(\nu_x-\nu_z)_{V^\perp}|^2\right],$$

$$\textbf{[III]} \leq C\gamma\, |x-z|^{\gamma-2} \fint_{B_\varepsilon^{\nu_z}} |(h - P_{\nu_z,-\nu_x} h)_{V^\perp}|^2 \, d\mathcal{L}^{n-1}(h),$$

$$\textbf{[IV]} \leq C\gamma\, |x-z|^{\gamma-2} \left[(\gamma-1)\fint_{B_\varepsilon^{\nu_z}} (\nu_x-h)_V^2 d\mathcal{L}^{n-1}(h) + \fint_{B_\varepsilon^{\nu_z}} |(\nu_x-h)_{V^\perp}|^2 \, d\mathcal{L}^{n-1}(h)\right].$$

We used $\gamma - 1 < 0$ and the choice $P_{\nu_z,-\nu_x} = P_{-\nu_z,\nu_x}$ in the estimate for **[III]**. By assumption, it holds $(\nu_x - \nu_z)_V^2 \geq (\tau+1)^2\varepsilon^2$ implying

$$|(\nu_x - \nu_z)_{V^\perp}|^2 \leq \left[4 - (\tau+1)^2\right]\varepsilon^2.$$

Thus, we need to obtain uniform bounds for the terms $(\nu_x - h)_V^2$, $|(\nu_x - h)_{V^\perp}|^2$ and $|(h - P_{\nu_z,-\nu_x} h)_{V^\perp}|^2$ for $h \in B_\varepsilon^{\nu_z}$.

The assumption $|(\nu_x - \nu_z)_V| \geq (\tau+1)\varepsilon$, together with $|\nu_x|, |\nu_z| \leq \varepsilon$ and Pythagoras' theorem, implies

$$\begin{cases} \tau\varepsilon \leq |(\nu_x)_V| \leq \varepsilon, & 0 \leq |(\nu_x)_{V^\perp}| \leq \sqrt{1-\tau^2}\,\varepsilon, \\ \tau\varepsilon \leq |(\nu_z)_V| \leq \varepsilon, & 0 \leq |(\nu_z)_{V^\perp}| \leq \sqrt{1-\tau^2}\,\varepsilon. \end{cases} \tag{42}$$

Moreover, the same facts yield

$$|(\nu_x + \nu_z)_V| \leq (1-\tau)\varepsilon \quad \text{ and } \quad |(\nu_x + \nu_z)_{V^\perp}| \leq 2\sqrt{1-\tau^2}\,\varepsilon.$$

By combining these and using Pythagoras' theorem, we get

$$|\nu_x + \nu_z| < \sqrt{8}\,\sqrt{1-\tau}\,\varepsilon, \tag{43}$$

since $\tau < 1$. Let $h \in B_\varepsilon^{\nu_z}$. Then, we have

$$0 = \langle h, \nu_z\rangle = h_V(\nu_z)_V + \langle h_{V^\perp}, (\nu_z)_{V^\perp}\rangle$$

implying

$$h_V = -\frac{\langle h_{V^\perp}, (\nu_z)_{V^\perp}\rangle}{(\nu_z)_V}.$$

In addition by applying this equality together with (42) and $|h_{V^\perp}| \leq |h| \leq \varepsilon$, we obtain

$$|h_V| \leq \frac{\varepsilon}{\tau}\sqrt{1-\tau^2}.$$

Consequently, we get the estimates

$$(\nu_x - h)_V \geq \left(\tau - \frac{\sqrt{1-\tau^2}}{\tau}\right)\varepsilon \tag{44}$$

and

$$|(\nu_x - h)_{V^\perp}| \leq |(\nu_x)_{V^\perp}| + |h_{V^\perp}| \leq \left(1 + \sqrt{1-\tau^2}\right)\varepsilon. \tag{45}$$

We can assume that τ_0 is close enough to 1 guaranteeing the positivity of the quantity $\tau - \tau^{-1}\sqrt{1-\tau^2}$. In order to obtain the last estimate needed, we recall that $P_{\nu_z,-\nu_x}$ is any rotation sending the vector ν_z to $-\nu_x$. In particular, we choose a rotation satisfying

$$|h - P_{\nu_z,-\nu_x} h| \leq |\nu_z - P_{\nu_z,-\nu_x}\nu_z| = |\nu_z + \nu_x|$$

for every $|h| \leq \varepsilon$. Hence by recalling (43), we get

$$|(h - P_{\nu_z, -\nu_x} h)_{V^\perp}|^2 \leq 8(1-\tau)\varepsilon^2. \tag{46}$$

By replacing the estimates (44), (45) and (46) in **[II]**, **[III]** and **[IV]**, we can calculate

$$\begin{aligned}
\mathbf{[II]} &\leq C\gamma\, |x-z|^{\gamma-2}\, \varepsilon^2 \left[(\gamma-1)(\tau+1)^2 + 4 - (\tau+1)^2\right], \\
\mathbf{[III]} &\leq C\gamma\, |x-z|^{\gamma-2}\, \varepsilon^2 \left[8(1-\tau)\right], \\
\mathbf{[IV]} &\leq C\gamma\, |x-z|^{\gamma-2}\, \varepsilon^2 \left[(\gamma-1)\left(\tau - \frac{\sqrt{1-\tau^2}}{\tau}\right)^2 + \left(1+\sqrt{1-\tau^2}\right)^2\right].
\end{aligned}$$

In addition by (38), we get

$$\mathbf{[I]} \leq \mathbf{[V]} \cdot C\gamma\, |x-z|^{\gamma-2}\, \varepsilon^2,$$

where **[V]** is equal to

$$\begin{aligned}
&(\gamma-1)\left[\alpha(z)(\tau+1)^2 + (\alpha(x)-\alpha(z))\left(\tau - \frac{\sqrt{1-\tau^2}}{\tau}\right)^2\right] \\
&\qquad + (\alpha(x)-\alpha(z))(1+\sqrt{1-\tau^2}\ )^2 + \alpha(z)\left[(4-(\tau+1)^2)\right] + \beta(x)\ 8(1-\tau).
\end{aligned}$$

The assumption on γ in (23) implies that we can choose $\tau_0 := \tau_0(\kappa) < 1$ close enough to 1 such that the previous expression is negative, i.e.,

$$\begin{aligned}
\mathbf{[V]} &< (\gamma-1)\big(4\alpha(z) + \alpha(x) - \alpha(z)\big) + \alpha(x) - \alpha(z) + \kappa\alpha_{\max} \\
&< 4\big(\gamma\alpha_{\max} - \alpha_{\min}\big) + \kappa\alpha_{\max} \\
&< 0.
\end{aligned}$$

Now, by recalling (37), we have

$$\begin{aligned}
&G(f,x,z,\nu_x,\nu_z) + G(f,x,z,\omega_x,\omega_z) - 2f(x,z) \\
&\qquad\qquad \leq 8\varepsilon^2 + (20 + \mathbf{[V]}\cdot C\gamma)\, |x-z|^{\gamma-2}\, \varepsilon^2.
\end{aligned}$$

By choosing $C > 1$ large enough, we obtain

$$(20 + \mathbf{[V]}\cdot C\gamma)\, |x-z|^{\gamma-2}\, \varepsilon^2 < -10^8\, |x-z|^{\gamma-2}\, \varepsilon^2 < -10^7 \varepsilon^2.$$

This estimate yields

$$G(f,x,z,\nu_x,\nu_z) + G(f,x,z,\omega_x,\omega_z) - 2f(x,z) < 0.$$

b) Case $|(\nu_x - \nu_z)_V| \leq (\tau+1)\varepsilon$: In this case, the first order terms in (33) imply the result. By choosing $\varrho_x = -\varepsilon\frac{x-z}{|x-z|}$ and $\varrho_z = \varepsilon\frac{x-z}{|x-z|}$ in V and utilizing these in (39), (40) and (41), we get

$$\begin{aligned}
\mathbf{[II]} &\leq C\gamma\, |x-z|^{\gamma-1}\left[(\nu_x-\nu_z)_V - 2\varepsilon\right] + 2\langle x+z, \nu_x+\nu_z\rangle \\
&\quad + \frac{1}{2} C\gamma\, |x-z|^{\gamma-2}\left\{(\gamma-1)\left[(\nu_x-\nu_z)_V^2 + 4\varepsilon^2\right] + |(\nu_x-\nu_z)_{V^\perp}|^2\right\},
\end{aligned}$$

$$\begin{aligned}
\mathbf{[III]} &\leq \frac{1}{2} C\gamma\, |x-z|^{\gamma-2} \cdot \\
&\quad \cdot \left\{ \fint_{B_\varepsilon^{\nu_z}} \left[(\gamma-1)(h - P_{\nu_z,-\nu_x}h)_V^2 + |(h-P_{\nu_z,-\nu_x}h)_{V^\perp}|^2\right] d\mathcal{L}^{n-1}(h) \right\}
\end{aligned}$$

and

$$\begin{aligned}
\textbf{[IV]} \le{}& C\gamma\,|x-z|^{\gamma-1}\,[(\nu_x)_V-\varepsilon]+2\langle x+z,\nu_x-\varepsilon\frac{x-z}{|x-z|}\rangle\\
&+\frac{1}{2}C\gamma\,|x-z|^{\gamma-2}\cdot\left\{\fint_{B_\varepsilon^{\nu_z}}\left[(\gamma-1)(\nu_x-h)_V^2+|(\nu_x-h)_{V^\perp}|^2\right]d\mathcal{L}^{n-1}(h)\right.\\
&\left.+(\gamma-1)\varepsilon^2+\fint_{B_\varepsilon^{x-z}}|h|^2\,d\mathcal{L}^{n-1}(h)\right\}.
\end{aligned}$$

The second order terms in these inequalities can be estimated above by

$$3C\gamma\,|x-z|^{\gamma-2}\,\varepsilon^2.$$

In addition, we deduce that $(\nu_x-\nu_z)_V<\left[1+\left(\frac{\tau+1}{2}\right)^2\right]\varepsilon$. Therefore, we have

$$\begin{aligned}
\textbf{[II]} &\le C\gamma\,|x-z|^{\gamma-1}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\varepsilon+4\,|x+z|\,\varepsilon+3C\gamma\,|x-z|^{\gamma-2}\,\varepsilon^2,\\
\textbf{[III]} &\le 3C\gamma\,|x-z|^{\gamma-2}\,\varepsilon^2,\\
\textbf{[IV]} &\le 4\,|x+z|\,\varepsilon+3C\gamma\,|x-z|^{\gamma-2}\,\varepsilon^2.
\end{aligned}$$

By combining all these and recalling (37) and (38), we get

$$\begin{aligned}
&G(f,x,z,\nu_x,\nu_z)+G(f,x,z,\varrho_x,\varrho_z)-2f(x,z)\\
&\le C\alpha(z)\gamma\,|x-z|^{\gamma-1}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\varepsilon+4\alpha(x)\,|x+z|\,\varepsilon+8\varepsilon^2\\
&\quad+(20+3C\gamma)\,|x-z|^{\gamma-2}\,\varepsilon^2\\
&\le C\alpha(z)\gamma\,|x-z|^{\gamma-1}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\varepsilon+8\varepsilon^2+\left(\gamma+\frac{2}{C}\right)\gamma\,|x-z|^{\gamma-1}\,\varepsilon\\
&\le \gamma\,|x-z|^{\gamma-1}\,\varepsilon\left\{\gamma+1+C\alpha_{\min}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\right\}+8\varepsilon^2.
\end{aligned}$$

As in the previous case, we can choose the constant $C>1$ large enough to ensure the negativity of the previous equation. Thus, the proof is complete.

5. **Regularity near the boundary.** In this section, we show that the value function of the game is also asymptotically continuous near the boundary, if we assume some regularity on the boundary of the set. The proof is based on finding a suitable barrier function and a strategy for the other player so that the process under the barrier function is super- or submartingale depending on the form of the function. Then, the result follows by analyzing the barrier function and iterating the argument.

Fix $r>0$ and $z\in\mathbb{R}^n$ and define a barrier function

$$v(x)=a|x-z|^\sigma+b \tag{47}$$

for all $x\in\mathbb{R}^n\setminus\overline{B}_r(z)$ with some constants $\sigma<0$, $a<0$ and $b\ge 0$. Recall the auxiliary function W defined in (4). First, we prove the following properties of the function v.

Lemma 5.1. *Let $r > 0$ and $z \in \mathbb{R}^n$ and define the function v as in (47) with constants $a < 0$, $b \geq 0$ and $\sigma < 0$. Then, there is a constant $C > 0$ such that*

$$\sup_{|\nu|=\varepsilon} W(v; x, \nu) \leq W\left(v; x, \varepsilon \frac{x-z}{|x-z|}\right) + C\varepsilon^3 \tag{48}$$

and

$$\begin{aligned} &W\left(v; x, \varepsilon \frac{x-z}{|x-z|}\right) + W\left(v; x, -\varepsilon \frac{x-z}{|x-z|}\right) \\ &< 2v(x) + \varepsilon^2 a\sigma |x-z|^{\sigma-2}\Big(\beta(x) + \alpha(x)(\sigma-1)\Big) + C\varepsilon^3 \end{aligned} \tag{49}$$

for all $\varepsilon > 0$ and $x \in \mathbb{R}^n \setminus \overline{B}_r(z)$.

Proof. To establish the result, we apply Taylor's formula to the function v. The function v is real-analytic so we obtain by Taylor's formula

$$\begin{aligned} v(x+h) &= v(x) + \big\langle \nabla v(x), h \big\rangle + \frac{1}{2}\big\langle D^2 v(x)h, h\big\rangle + \mathcal{O}(|h|^3) \\ &= v(x) + \frac{1}{2} a\sigma |x-z|^{\sigma-2}\left(2\langle x-z, h\rangle + |h|^2 + (\sigma-2)\frac{\langle h, x-z\rangle^2}{|x-z|^2}\right) + \mathcal{O}(|h|^3) \end{aligned}$$

for all $h \in \mathbb{R}^n$. Let $C > 0$ be big enough so that $\big|\mathcal{O}(|h|^3)\big| \leq C\varepsilon^3$ for all $|h| \leq \varepsilon$. The function v is radially increasing, and the average integral over the first order term of v vanishes. In addition, we have

$$\int_{B_\varepsilon^{x-z}} \langle h, x-z\rangle^2 d\mathcal{L}^{n-1}(h) = 0.$$

Thus, Taylor's formula proves the equation (48).

Next, we prove the equation (49). Recall the notations $V = \operatorname{span}\{x-z\}$ and the orthogonal complement $V^\perp$. For a vector $h \in V$ such that $h = (x-z)\varepsilon/|x-z|$, we get

$$v(x+h) \leq v(x) + \frac{1}{2} a\sigma |x-z|^{\sigma-2}\Big(2|x-z|\varepsilon + \varepsilon^2(\sigma-1)\Big) + C\varepsilon^3.$$

Therefore, we obtain

$$v(x+h) + v(x-h) \leq 2v(x) + \varepsilon^2 a\sigma(\sigma-1)|x-z|^{\sigma-2} + C\varepsilon^3. \tag{50}$$

Also, for any vector $y \in B_\varepsilon^{z-x}$, we have $\langle y, x-z\rangle = 0$. Hence, by a short calculation, we have

$$\begin{aligned} \fint_{B_\varepsilon^{x-z}} v(x+y)\, d\mathcal{L}^{n-1}(y) &\leq v(x) + C\varepsilon^3 + \frac{1}{2} a\sigma |x-z|^{\sigma-2} \fint_{B_\varepsilon^{x-z}} |y|\ d\mathcal{L}^{n-1}(y) \\ &= v(x) + C\varepsilon^3 + \frac{1}{2} a\sigma |x-z|^{\sigma-2}\varepsilon^2\left(\frac{n-1}{n+1}\right). \end{aligned}$$

Thus, this inequality together with $B_\varepsilon^{z-x} = B_\varepsilon^{x-z}$ and (50) implies

$$\begin{aligned}
&W\left(v; x, \varepsilon \frac{x-z}{|x-z|}\right) + W\left(v; x, -\varepsilon \frac{x-z}{|x-z|}\right) \\
&= \alpha(x)\left(v\left(x - \varepsilon \frac{x-z}{|x-z|}\right) + v\left(x + \varepsilon \frac{x-z}{|x-z|}\right)\right) \\
&\quad + 2\beta(x) \fint_{B_\varepsilon^{z-x}} v(x+\tilde{h})\, d\mathcal{L}^{n-1}(\tilde{h}) \\
&\leq 2v(x) + \varepsilon^2 a\sigma |x-z|^{\sigma-2}\Big(\beta(x) + \alpha(x)(\sigma-1)\Big) + 2C\varepsilon^3. \qquad \square
\end{aligned}$$

Next, we prove the main theorem of this section. To get the result, we need to assume some regularity on the boundary of the set Ω.

Boundary Regularity Condition. There are universal constants $r_0, s \in (0,1)$ such that for all $r \in (0, r_0]$ and $y \in \partial\Omega$ there exists a ball

$$B_{sr}(z) \subset B_r(y) \setminus \Omega$$

for some $z \in B_r(y) \setminus \Omega$.

Assume that the set Ω satisfies this boundary condition. Then, the following theorem holds.

Theorem 5.2. *Let $y \in \partial\Omega$ and $r \in (0, r_0]$ with $r_0 \in (0,1)$ and the ball $B_{sr}(z) \subset B_r(y) \setminus \Omega$ given by the boundary regularity condition. Let u be the solution to (1) with continuous boundary data F. Then for all $\eta > 0$, there exist $\varepsilon_0 > 0$ and $k \geq 1$ such that*

$$u(x_0) - \sup_{B_{4r}(z)\cap\Gamma_{\varepsilon,\varepsilon}} F < \eta \tag{51}$$

for all $0 < \varepsilon < \varepsilon_0 < 1$ and $x_0 \in B_{4^{1-k}r}(y) \cap \overline{\Omega}_\varepsilon$.

Proof. The idea is to find a suitable barrier function so that by Lemma 5.1, if P_{I} pulls towards the point $z \in B_r(y) \setminus \Omega$, the game process inside the barrier function is a supermartingale. Then by utilizing the properties of the barrier function, we get the result by iteration.

Choose a constant $0 < \theta < 1$, independent of r, such that

$$\theta := \frac{s^\sigma - 2^\sigma}{s^\sigma - 4^\sigma}$$

with the parameter $s > 0$ from the boundary condition and a parameter $\sigma < 0$ that will be defined later. We extend the function F continuously to the set $\Gamma_{1,1}$ and use the same notation for the extension. Then, we choose $k \geq 1$ big enough such that

$$\theta^k \Big(\sup_{\Gamma_{1,1}} F - \inf_{\Gamma_{1,1}} F\Big) < \eta.$$

In addition, we denote the constants

$$b_U := \sup_{\Gamma_{\varepsilon,\varepsilon}} F$$

and

$$b_{4r} := \sup_{B_{4r}(z)\cap\Gamma_{\varepsilon,\varepsilon}} F.$$

Thus for the chosen k, independent of ε, it holds

$$\theta^k \big(b_U - b_{4r}\big) < \eta. \tag{52}$$

We define a function v_k such that

$$v_k(x) = a|x-z|^\sigma + b$$

in $B_{4^{2-k}r}(z) \setminus \overline{B}_{4^{1-k}sr}(z)$. The constants $a \leq 0$ and $b \geq 0$ can be calculated from the boundary values

$$v_k = \begin{cases} b_{4r} + \theta^{k-1}\big(b_U - b_{4r}\big) & \text{on } \partial B_{4^{2-k}r}(z) \\ b_{4r} & \text{on } \partial B_{4^{1-k}sr}(z). \end{cases} \tag{53}$$

If $b_U = b_{4r}$, it holds $a = 0$ and $b = b_U$. Otherwise, the values are $a < 0$ and $b \geq 0$. We consider the case with $a < 0$, since the proof of the other case is clear.

We extend the function v_k to the set $\mathbb{R}^n \setminus \overline{B}_{4^{1-k}sr-2\varepsilon}(z)$ and use the same notation for the extension. We may assume that $x_0 \in \Omega$, and observe that

$$x_0 \in B_{4^{1-k}r}(y) \cap \Omega \subset \Omega \cap B_{2\cdot 4^{1-k}r}(z) \setminus \overline{B}_{4^{1-k}sr}(z). \tag{54}$$

Assume that P_{II} plays the game by pulling towards the point z given a turn, i.e., he moves the game token by the vector $-\varepsilon(x_m - z)/|x_m - z|$, if he wins the mth toss. This strategy is denoted by $\mathcal{S}_{\text{II}}^*$. Also, fix a strategy for P_{I} and denote it by $\mathcal{S}_{\text{I}}$.

By using Lemma 5.1 for all $m \geq 1$, we can estimate

$$\begin{aligned}
&\mathbb{E}^{x_0}_{\mathcal{S}_{\text{I}},\mathcal{S}^*_{\text{II}}}\big[v_k(x_{m+1})|x_0,\dots,x_m\big] \\
&\leq \frac{1-\delta(x_m)}{2}\left(\sup_{|\nu|=\varepsilon} W(v_k;x_m,\nu) + W\left(v_k;x_m,-\varepsilon\frac{x_m-z}{|x_m-z|}\right)\right) + \delta(x_m)F(x_m) \\
&\leq \frac{1-\delta(x_m)}{2}\bigg(2v_k(x_m) + \varepsilon^2 a\sigma|x_m-z|^{\sigma-2}\cdot \\
&\qquad\qquad \cdot\big(\beta(x_m)+\alpha(x_m)(\sigma-1)\big) + 2C\varepsilon^3\bigg) + \delta(x_m)F(x_m)
\end{aligned}$$

for some $C > 0$. Next, we need to choose the constant $\sigma < 0$ small enough. Recall that $\alpha_{\min} > 0$. Let us fix the value

$$\sigma := \frac{2}{\alpha_{\min}}(\alpha_{\min} - 1)$$

implying

$$\beta(x_m) + \alpha(x_m)(\sigma - 1) < -1$$

for all $x_m \in \Omega$. In addition, we have

$$a\sigma|x_m - z|^{\sigma-2} > a\sigma\big(\operatorname{diam}(\Omega)+1\big)^{\sigma-2} > 0$$

for all $x_m \in \Omega$. Thus by choosing $\varepsilon_0 := \varepsilon_0(\alpha_{\min}, r, \Omega, k) > 0$ small enough, we can ensure that

$$\mathbb{E}^{x_0}_{\mathcal{S}_{\text{I}},\mathcal{S}^*_{\text{II}}}[v_k(x_{m+1})|x_0,\dots,x_m] \leq \big(1-\delta(x_m)\big)v_k(x_m) + \delta(x_m)F(x_m) \leq v_k(x_m)$$

for all $\varepsilon < \varepsilon_0$. We have shown that the process

$$M_m := v_k(x_m)$$

is a supermartingale, when P_{II} uses the strategy $\mathcal{S}_{\text{II}}^*$ and P_{I} uses any strategy $\mathcal{S}_{\text{I}}$.

Define a boundary function $F_{v_k} : \Gamma_{\varepsilon,\varepsilon} \to \mathbb{R}$ such that

$$F_{v_k} = v_k|_{\Gamma_{\varepsilon,\varepsilon}}.$$

By Theorem 3.7, we have $u = u_{\mathrm{I}}$ with u_{I} the value function for P_{I} defined in (7). Since $F \le F_{v_k}$, $(M_m)_{m=1}^{\infty}$ is a supermartingale, F_{v_k} is bounded and $\tau < \infty$ almost surely, we can estimate with the help of the optimal stopping theorem

$$\begin{aligned} u(x_0) &= \sup_{\mathcal{S}_{\mathrm{I}}} \inf_{\mathcal{S}_{\mathrm{II}}} \mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}_{\mathrm{II}}}\big[F(x_\tau)\big] \le \sup_{\mathcal{S}_{\mathrm{I}}} \mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}^*_{\mathrm{II}}}\big[F(x_\tau)\big] \\ &\le \sup_{\mathcal{S}_{\mathrm{I}}} \mathbb{E}^{x_0}_{\mathcal{S}_{\mathrm{I}},\mathcal{S}^*_{\mathrm{II}}}\big[F_{v_k}(x_\tau)\big] \le v_k(x_0). \end{aligned}$$

By using the boundary values (53), we can calculate the constants a and b in the function v_k and deduce by (54) that

$$v_k(x_0) \le b_{4r} + \theta^k \big(b_U - b_{4r}\big).$$

Hence by (52), we have shown the estimate (51). □

Corollary 5.3. *Let $\eta > 0$ and let u be the solution to (1) with continuous boundary data F. Then, there is a constant $\overline{r} \in (0, r_0]$ such that for all $r \in (0, \overline{r}]$ there exist constants $k \ge 1$ and $\varepsilon_0 > 0$ such that for any $y \in \Gamma_{\varepsilon,\varepsilon}$ it holds*

$$u(x_0) - F(y) < \eta$$

for all $\varepsilon < \varepsilon_0$ and $x_0 \in B_{4^{1-k}r}(y) \cap \overline{\Omega}_\varepsilon$.

Proof. First, assume that $y \in \partial\Omega$. Theorem 5.2 implies that for any $r \in (0, r_0]$, there are constants $k \ge 1$ and $\varepsilon_0 > 0$ such that

$$u(x_0) - \sup_{B_{4r}(z)\cap\Gamma_{\varepsilon,\varepsilon}} F < \frac{\eta}{10}$$

for all $\varepsilon < \varepsilon_0$ and $x_0 \in B_{4^{1-k}r}(y) \cap \overline{\Omega}_\varepsilon$. Let $y^* \in \overline{B}_{4r}(z) \cap \Gamma_{\varepsilon,\varepsilon}$ be such that

$$\sup_{B_{4r}(z)\cap\Gamma_{\varepsilon,\varepsilon}} F < F(y^*) + \frac{\eta}{10}.$$

The boundary function F is continuous on the compact set $\Gamma_{\varepsilon,\varepsilon}$, so there is a modulus of continuity ω_F for the function F. Thus, we can estimate

$$\begin{aligned} u(x_0) - F(y) &= u_\varepsilon(x_0) - F(y^*) + F(y^*) - F(y) \\ &< u - \sup_{B_{4r}(z)\cap\Gamma_{\varepsilon,\varepsilon}} F + \frac{\eta}{10} + \omega_F\big(|y^* - y|\big) \\ &< \frac{\eta}{5} + \omega_F\big(|y^* - y|\big). \end{aligned}$$

It holds $|z - y| < r$ implying the estimate $|y^* - y| \le |y^* - z| + |z - y| < 5r$. We choose $\overline{r} > 0$ so small that

$$\omega_F(5r) < \frac{\eta}{10}$$

for all $r < \overline{r}$. This yields that for any $r < \overline{r}$, we have

$$u(x_0) - F(y) < \frac{\eta}{2}$$

for all $\varepsilon < \varepsilon_0$ and $x_0 \in B_{4^{1-k}r}(y) \cap \overline{\Omega}_\varepsilon$.

Next, assume that $y \notin \partial\Omega$. Pick a point $y_b \in \partial\Omega$ such that $y \in B_\varepsilon(y_b)$. We choose $\varepsilon_0 > 0$ so small that

$$\omega_F(\varepsilon_0) < \frac{\eta}{2}.$$

This implies that

$$\big|F(y) - F(y_b)\big| \le \omega_F\big(|y - y_b|\big) < \frac{\eta}{2}$$

for all $\varepsilon < \varepsilon_0$. Since $y_b \in \partial\Omega$, we can use the estimates above to get the result. □

Remark 5.4. By using a similar argument to Theorem 5.2, it holds for all $y \in \partial\Omega$ and $\eta > 0$ that there exist $\varepsilon_0 > 0$ and $k \geq 1$ such that

$$u(x_0) - \inf_{B_{4r}(z)\cap\Gamma_{\varepsilon,\varepsilon}} F > -\eta$$

for all $0 < \varepsilon < \varepsilon_0 < 1$ and $x_0 \in B_{4^{1-k}r}(y) \cap \overline{\Omega}_\varepsilon$. Hence, we have for all $r > 0$ small enough, $k \geq 1$ big enough and $\varepsilon > 0$ small enough the lower bound

$$u(x_0) - F(y) > -\eta$$

for $y \in \Gamma_{\varepsilon,\varepsilon}$ and $x_0 \in B_{4^{1-k}r}(y) \cap \overline{\Omega}_\varepsilon$.

6. Application.

In this section, we prove that the uniform limit of functions satisfying (1) as $\varepsilon \to 0$ is a weak solution to the normalized homogeneous $p(x)$-Laplace equation

$$\begin{aligned} \Delta^N_{p(x)} u(x) &:= \Delta u(x) + \big(p(x) - 2\big)\Delta^N_\infty u(x) \\ &= \Delta u(x) - \Delta^N_\infty u(x) + \big(p(x) - 1\big)\Delta^N_\infty u(x) \\ &= 0. \end{aligned} \tag{55}$$

This equation is in a non divergence form so we define weak solutions via viscosity theory. There is a related version of the equation (55), called a strong $p(x)$-Laplacian, in a divergence form, which has recently received attention and studied using distributional weak theory (see for example [1, 23, 17]). For some questions, the viscosity point of view is very natural in the sense that the equation (55) has the Pucci operator bounds used for example in Section 4 in [3].

We define for all vectors $x, h \in \mathbb{R}^n$ and symmetric $n \times n$ matrices X

$$\mathbb{F}_{p(x)}(x, h, X) := \operatorname{trace}(X) - \sum_{i,j}^n \frac{h_i h_j}{|h|^2} X_{ij} + \big(p(x) - 1\big) \sum_{i,j}^n \frac{h_i h_j}{|h|^2} X_{ij}.$$

These functions are discontinuous, when $h = 0$. Therefore, we define viscosity solutions via semicontinuous extensions. For more details about the extensions, see for example [5, 4]. We denote by $\lambda_{\min}(X)$ and by $\lambda_{\max}(X)$ the smallest and the largest eigenvalues of a symmetric matrix X.

Definition 6.1. A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity solution to the equation (55), if for all $x \in \Omega$ and $\phi \in C^2$ such that $u(x) = \phi(x)$ and $u(y) > \phi(y)$ for $y \neq x$ we have

$$\begin{cases} 0 \;\geq\; \mathbb{F}_{p(x)}(x, \nabla\phi(x), D^2\phi(x)), & \text{if} \quad \nabla\phi(x) \neq 0, \\ 0 \;\geq\; \lambda_{\min}((p(x) - 2)D^2\phi(x)) + \operatorname{trace}(D^2\phi(x)), & \text{if} \quad \nabla\phi(x) = 0. \end{cases} \tag{56}$$

We also require that for all $x \in \Omega$ and $\phi \in C^2$ such that $u(x) = \phi(x)$ and $u(y) < \phi(y)$ for $y \neq x$ all the inequalities are reversed, and we use $\lambda_{\max}$ in the role of $\lambda_{\min}$.

It is equivalent to require that $u - \phi$ has a local strict minimum at x instead of $u(x) = \phi(x)$ and $u(y) > \phi(y)$ for $y \neq x$ (see for example [9]). Next, we prove that by passing to a subsequence if necessary, the value function of the game converges uniformly to a solution of the equation (55). To prove that the limiting function u is a viscosity solution to (55), we use an argument similar to the stability principle for viscosity solutions. We apply the DPP (1) for a test function $\phi \in C^2$ and deduce the connection by utilizing the uniform convergence.

Theorem 6.2. *Let u_ε denote the unique continuous solution to (1) with $\varepsilon > 0$ and with a continuous boundary function $F : \Gamma_{\varepsilon,\varepsilon} \to \mathbb{R}$. Then, there are a function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence $\{\varepsilon_i\}$ such that u_{ε_i} converges uniformly to u on $\overline{\Omega}$ and the function u is a viscosity solution to (55) with the boundary data F.*

Proof. To find the function u, we use a variant of the Arzelà-Ascoli's theorem (see for example [16, p. 15-16]). By Theorems 4.1 and 5.2 together with Remark 5.4, the assumptions for Arzelà-Ascoli's theorem are satisfied and hence, there exist a continuous function u on $\overline{\Omega}$ with the boundary values F and a subsequence $\{\varepsilon_i\}$ such that $u_{\varepsilon_i} \to u$ uniformly on $\overline{\Omega}$ as $i \to \infty$. Thus, it is enough to show that u is a viscosity solution to (55).

Let $x \in \Omega$ and $\phi \in C^2$ such that $u - \phi$ has a strict local minimum at x. Then, we have

$$\inf_{B_r(x)} (u - \phi) = u(x) - \phi(x) < u(z) - \phi(z)$$

for some $r > 0$ and for all $z \in B_r(x) \setminus \{x\}$. The uniform convergence yields

$$\inf_{B_r(x)} (u_\varepsilon - \phi) < u_\varepsilon(z) - \phi(z)$$

for all $z \in B_r(x)\setminus\{x\}$ and for all $\varepsilon > 0$ small enough. Thus, we can use the definition of the infimum and deduce that for all $\eta_\varepsilon > 0$, there exists a point $x_\varepsilon \in B_r(x) \subset \Omega$ such that

$$u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) \le u_\varepsilon(z) - \phi(z) + \eta_\varepsilon$$

for all $z \in B_r(x)$ and $\varepsilon > 0$ small enough with $x_\varepsilon \to x$ as $\varepsilon \to 0$. We define $\varphi := \phi + u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon)$ so that

$$\varphi(x_\varepsilon) = u_\varepsilon(x_\varepsilon) \text{ and } u_\varepsilon(z) \ge \varphi(z) - \eta_\varepsilon$$

for all $z \in B_r(x)$. Therefore, these together with the fact that u_ε is a solution to (1) imply

$$\begin{aligned} u_\varepsilon(x_\varepsilon) = T_\varepsilon u_\varepsilon(x_\varepsilon) &\ge T_\varepsilon \varphi(x_\varepsilon) - (1 - \delta(x_\varepsilon))\eta_\varepsilon \\ &= T_\varepsilon \phi(x_\varepsilon) - \eta_\varepsilon + u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) + \delta(x_\varepsilon)\Lambda_\varepsilon, \end{aligned}$$

where we use the monotonicity of T_ε and denote $\Lambda_\varepsilon := \eta_\varepsilon + \phi(x_\varepsilon) - u_\varepsilon(x_\varepsilon)$. This inequality yields

$$\eta_\varepsilon \ge T_\varepsilon \phi(x_\varepsilon) - \phi(x_\varepsilon) + \delta(x_\varepsilon)\Lambda_\varepsilon. \tag{57}$$

By the Taylor's expansion of ϕ at x_ε with $|\nu| = 1$, we get

$$\frac{1}{2}\phi(x_\varepsilon + \varepsilon\nu) + \frac{1}{2}\phi(x_\varepsilon - \varepsilon\nu) = \phi(x_\varepsilon) + \frac{\varepsilon^2}{2}\langle D^2\phi(x_\varepsilon)\nu, \nu\rangle + o(\varepsilon^2), \tag{58}$$

$$\fint_{B_\varepsilon^\nu} \phi(x_\varepsilon + h)\, d\mathcal{L}^{n-1}(h) = \phi(x_\varepsilon) + \frac{\varepsilon^2}{2(n+1)}\Delta_{\nu^\perp}\phi(x_\varepsilon) + o(\varepsilon^2). \tag{59}$$

In (59), we utilize the orthonormal basis $\mathcal{V}$ including ν and an orthonormal basis for $\nu^\perp$ to obtain

$$\frac{1}{2}\fint_{B_\varepsilon^\nu} \langle D^2\phi(x_\varepsilon)h, h\rangle\, d\mathcal{L}^{n-1}(h) = \frac{\varepsilon^2}{2(n+1)}\Delta_{\nu^\perp}\phi(x_\varepsilon)$$

in a similar way to [15]. Here, the operator $\Delta_{\nu^\perp}$ denotes the Laplacian on the plane $\nu^\perp$, i.e.,

$$\Delta_{\nu^\perp}\phi(x_\varepsilon) = \sum_{j=2}^{n} \langle D^2\phi(x_\varepsilon)\nu_j, \nu_j\rangle$$

with $\nu_2, \dots, \nu_n$ the orthonormal basis vectors for $\nu^\perp$. Observe that

$$\Delta\phi(z) = \operatorname{trace}(D^2\phi(z)) = \Delta_{\nu^\perp}\phi(z) + \langle D^2\phi(z)\nu, \nu\rangle \tag{60}$$

for any $|\nu| = 1$ and $z \in \Omega$. To see this, we apply the orthonormal basis $\mathcal{V}$ and a change of variables $x_1 = \nu, x_2 = \nu_2, \dots, x_n = \nu_n$. Then, we deduce the equation (60) by the chain rule.

There exists a vector $\nu_{\min} := \nu_{\min}(\varepsilon)$ minimizing

$$\alpha(x_\varepsilon)\phi(x_\varepsilon + \varepsilon\nu) + \beta(x_\varepsilon)\fint_{B_\varepsilon^\nu} \phi(x_\varepsilon + h)\,d\mathcal{L}^{n-1}(h) \tag{61}$$

with $|\nu| = 1$. Thus by (58) and (59) with $\nu = \nu_{\min}$ and the fact that $-\nu^\perp \equiv \nu^\perp$, we obtain

$$\begin{aligned}
T_\varepsilon\phi(x_\varepsilon) =& \frac{1-\delta(x_\varepsilon)}{2}\sup_{|\nu|=1}\left(\alpha(x_\varepsilon)\phi(x_\varepsilon + \varepsilon\nu) + \beta(x_\varepsilon)\fint_{B_\varepsilon^\nu} \phi(x_\varepsilon + h)\,d\mathcal{L}^{n-1}(h)\right) \\
&+ \frac{1-\delta(x_\varepsilon)}{2}\inf_{|\nu|=1}\left(\alpha(x_\varepsilon)\phi(x_\varepsilon + \varepsilon\nu) + \beta(x_\varepsilon)\fint_{B_\varepsilon^\nu} \phi(x_\varepsilon + h)\,d\mathcal{L}^{n-1}(h)\right) \\
&+ \delta(x_\varepsilon)F(x_\varepsilon) \\
\geq& (1-\delta(x_\varepsilon))\frac{\alpha(x_\varepsilon)}{2}\left\{\phi(x_\varepsilon + \varepsilon\nu_{\min}) + \phi(x_\varepsilon - \varepsilon\nu_{\min})\right\} \\
&+ (1-\delta(x_\varepsilon))\beta(x_\varepsilon)\fint_{B_\varepsilon^{\nu_{\min}}} \phi(x_\varepsilon + h)\,d\mathcal{L}^{n-1}(h) + \delta(x_\varepsilon)F(x_\varepsilon) + o(\varepsilon^2) \\
=& (1-\delta(x_\varepsilon))\phi(x_\varepsilon) + (1-\delta(x_\varepsilon))\frac{\beta(x_\varepsilon)\varepsilon^2}{2(n+1)}\Big\{\Delta_{\nu_{\min}^\perp}\phi(x_\varepsilon) \\
&\qquad\qquad + (p(x_\varepsilon)-1)\langle D^2\phi(x_\varepsilon)\nu_{\min}, \nu_{\min}\rangle\Big\} \\
&+ \delta(x_\varepsilon)F(x_\varepsilon) + o(\varepsilon^2).
\end{aligned}$$

By this estimate and (57), we have

$$\begin{aligned}
\eta_\varepsilon \geq& -\phi(x_\varepsilon) + (1-\delta(x_\varepsilon))\bigg\{\phi(x_\varepsilon) + \frac{\beta(x_\varepsilon)\varepsilon^2}{2(n+1)}\Big[\Delta_{\nu_{\min}^\perp}\phi(x_\varepsilon) \\
&\qquad + (p(x_\varepsilon)-1)\langle D^2\phi(x_\varepsilon)\nu_{\min}, \nu_{\min}\rangle\Big]\bigg\} \\
&+ \delta(x_\varepsilon)\big(F(x_\varepsilon) + \Lambda_\varepsilon\big) + o(\varepsilon^2).
\end{aligned} \tag{62}$$

First, assume that $|\nabla\phi(x)| \neq 0$. Then by $x_\varepsilon \to x$ as $\varepsilon \to 0$, it turns out that

$$\nu_{\min} \to -\frac{\nabla\phi(x)}{|\nabla\phi(x)|} =: \nu_{\min}^* \tag{63}$$

as $\varepsilon \to 0$. Here, we need the fact that $\alpha_{\min} > 0$, i.e., $p_{\min} > 1$. In addition by (60), we have

$$\Delta_{(\nu_{\min}^*)^\perp}\phi(x) + (p(x)-1)\langle D^2\phi(x)\nu_{\min}^*, \nu_{\min}^*\rangle = \Delta_{p(x)}^N\phi(x). \tag{64}$$

Choose $\eta_\varepsilon = o(\varepsilon^2)$, divide both sides in (62) by ε^2 and let $\varepsilon \to 0$. Therefore by (62), (63), (64) and the facts that $x_\varepsilon \to x$ and $\delta(x_\varepsilon)\varepsilon^{-2} \to 0$ as $\varepsilon \to 0$, we obtain

$$0 \geq \frac{\beta(x)}{2(n+1)}\Delta_{p(x)}^N\phi(x).$$

By $\beta(x) \geq \beta_{\min} > 0$, we get the inequality $\Delta^N_{p(x)}\phi(x) \leq 0$.

Next, assume that $|\nabla\phi(x)| = 0$. By (60), we have

$$\begin{aligned} &\Delta_{\nu_{\min}^\perp}\phi(x_\varepsilon) + (p(x_\varepsilon) - 1)\langle D^2\phi(x_\varepsilon)\nu_{\min}, \nu_{\min}\rangle \\ &= \Delta\phi(x) + (p(x_\varepsilon) - 2)\langle D^2\phi(x_\varepsilon)\nu_{\min}, \nu_{\min}\rangle. \end{aligned} \tag{65}$$

Assume that $p(x) > 2$. Then by the continuity of p and $x_\varepsilon \to x$ as $\varepsilon \to 0$, we have $p(x_\varepsilon) > 2$ for all ε small enough. Thus, we can estimate

$$(p(x_\varepsilon) - 2)\langle D^2\phi(x_\varepsilon)\nu_{\min}, \nu_{\min}\rangle \geq (p(x_\varepsilon) - 2)\lambda_{\min}(D^2\phi(x_\varepsilon))$$

for all ε small enough. As above, this estimate together with (62), (65) and the continuity of $z \mapsto \lambda_{\min}(D^2\phi(z))$ imply

$$0 \geq \Delta\phi(x) + \lambda_{\min}((p(x) - 2)D^2\phi(x)).$$

The cases $p(x) = 2$ and $p(x) < 2$ can be treated in a similar fashion.

To show the required reverse inequality, we choose $|\nu_{\max}| = 1$ such that it maximizes (61), and we consider the reverse inequality with $\phi \in C^2$ such that $u - \phi$ has a strict maximum at x. The proof is analogous to the above. □

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[B]

Uniform measure density condition and game regularity for tug-of-war games

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Uniform measure density condition and game regularity for tug-of-war games

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We show that a uniform measure density condition implies game regularity for all $2 < p < \infty$ in a stochastic game called "tug-of-war with noise". The proof utilizes suitable choices of strategies combined with estimates for the associated stopping times and density estimates for the sum of independent and identically distributed random vectors.

Keywords: density estimate for the sum of i.i.d. random vectors; game regularity; hitting probability; p-harmonic functions; p-regularity; stochastic games; uniform distribution in a ball; uniform measure density condition

1. Introduction

The profound connection between stochastic processes and classical linear partial differential equations has been pivotal. For example, this connection was made use of in [8,9] to establish regularity result for the second order equations in a non-divergence form. Recently, a connection between nonlinear infinity harmonic functions and tug-of-war games was discovered in [15]. Later in [16], the authors found a stochastic game related to p-harmonic functions. They proved among other things by using a game approach that in a game regular domain there exists a p-harmonic function extending continuously to the boundary with the given continuous boundary values. However, a problem asking if a regular boundary point for the p-Laplacian is necessarily game regular was left open.

We study a modified version of a "tug-of-war with noise" developed in [13] and also related to p-harmonic functions. First, the players choose a step length $\epsilon > 0$ and a starting point x_0. Then, they toss a biased coin, and if they get heads (probability α), the players play a "tug-of-war", that is, they toss a fair coin and the winner of the toss can move the game position to any point of the open ball centered at x_0 and of the radius ϵ. If in the first toss, they get tails (probability β), the game point moves according to the uniform distribution in the open ball centered at x_0 and of the radius ϵ. After the first move, the players play the same game from the new game position. The game ends, when the game position exits the game domain for the first time. In the end, Player 2 pays to Player 1 the amount given by the payoff function at the first point outside the domain. We consider this version of the game because the players do not affect the direction of the noise and hence, we can prove sharp enough estimates for the density of the noise.

We give a stochastic proof that a uniform measure density condition implies game regularity (Theorem 3.7). Roughly, a boundary point y is game regular, if Player 1 has a strategy to end the game near y with a probability close to one whenever the game starts near y as well. A boundary

1350-7265

point y satisfies a measure density condition, if the Lebesgue measure of the complement of the game domain in the ball centered at y is comparable to the Lebesgue measure of the whole ball. The proof of Theorem 3.7 utilizes a stochastic density estimate for the sum of independent and identically distributed random vectors (Lemma A.4). In addition, we use a "cylinder walk" framework together with a cancellation strategy for Player 1 to connect the stochastic estimates to the setting. We omit the case $p = 2$, because in that case the process is merely a random walk and the result follows from the classical invariance principle.

Game theory has already given new insights to partial differential equations. For instance, the ideas emerging from nonlinear game theory have led to simpler as well as completely different proofs for PDEs (see, for example, [1] and [10]). In addition, a dynamic programming principle related to the game also arises from discretization schemes (see, for instance, [14]).

We expect the techniques developed in this paper to be useful for a larger class of partial differential equations as well. In addition, stochastic estimates on where the game spends time under cancellation strategies are likely to be important for further results.

This work is organized as follows. In Section 2, we describe the preliminaries needed in the paper. Then in Section 3, we show that the uniform measure density condition implies game regularity for all $2 < p < \infty$. For brevity, we do not write down all the stochastic calculations needed in the section, but the calculations are in the Appendix.

2. Preliminaries

First, let us start by introducing the notation. We denote the standard Euclidean open ball by $B_r(x_0) \subset \mathbb{R}^n$,

$$B_r(x_0) = \left\{ z \in \mathbb{R}^n : |z - x_0| < r \right\}.$$

Lebesgue measure is denoted by $|\cdot|$, and in addition, the notation $C_{n,p}$ means that the universal constant depends only on n and p. Throughout the paper, we use the asymptotic notation $\mathcal{O}(\epsilon)$. For example, if a real-valued function f satisfies the inequality $f(\epsilon) \leq \mathcal{O}(\epsilon)$, it means that there exists a constant $C > 0$ such that $|f(\epsilon)| \leq C\epsilon$ for all $\epsilon > 0$ small enough.

Let $2 < p < \infty$, $\epsilon > 0$ and dimension $n \geq 1$. Fix a bounded, non-empty and open set $\Omega \subset \mathbb{R}^n$. Next, we recall the two-player zero-sum-game called "tug-of-war with noise". First, choose a starting point $x_0 \in \Omega$ for the game, and then, the players toss a biased coin with probabilities α and β. The probabilities depend on n and p by

$$\alpha = \frac{p-2}{p+n}, \qquad \beta = \frac{n+2}{p+n}. \tag{2.1}$$

The players get heads with the probability α, and in this case, they will toss a fair coin and the winner of the toss can move the game position to any point of the open ball $B_\epsilon(x_0)$. Tossing of a fair coin and the movement after the toss are the "tug-of-war" parts of the game. On the other hand, if they get tails, the next game position will be decided by the uniform distribution in the ball $B_\epsilon(x_0)$. A random movement is the "noise" part of the game. After the first move is decided, the players continue playing the same game from the new position.

The game procedure yields a sequence of game positions $x_0, x_1, x_2, \ldots$, where every x_k is a random variable. A history of a game up to step k is a vector of the first $k+1$ game positions $x_0, \ldots, x_k$ and k coin tosses $c_1, \ldots, c_k$, that is,

$$h_k := \big(x_0, (c_1, x_1), \ldots, (c_k, x_k)\big).$$

In the above, $c_j \in \mathcal{C} := \{0, 1, 2\}$, where 0 denotes that Player 1 wins, 1 that Player 2 wins and 2 that a random movement occurs.

To prescribe boundary values, let us denote a compact boundary strip of width ϵ by

$$\Gamma_\epsilon := \Big\{z \in \mathbb{R}^n \setminus \Omega : \inf_{y \in \partial\Omega} |z - y| \le \epsilon\Big\}.$$

The reason to use the boundary strip instead of just the boundary is that $B_\epsilon(x) \subset \Omega_\epsilon := \Omega \cup \Gamma_\epsilon$ for all $x \in \Omega$. After the first time the game position is in Γ_ϵ, the players do not move it anymore. For all $k \ge 0$, the history h_k belongs to the space $H^k := x_0 \times (\mathcal{C}, \Omega_\epsilon)^k$ with $H^0 := x_0$. We denote the space of all game sequences by

$$H^\infty := \bigcup_{k \ge 0} H^k = x_0 \times (\mathcal{C}, \Omega_\epsilon) \times (\mathcal{C}, \Omega_\epsilon) \times \cdots .$$

A *strategy* for Player 1 is a sequence of Borel measurable functions that give the next game position given the history of the game. To be more precise, a strategy for Player 1 is $S_1 := (S_{1,k})_{k=0}^{\infty}$ with

$$S_{1,k} : H^k \to \mathbb{R}^n$$

for all $k \ge 0$. For example, if Player 1 wins the $(k+1)$th toss,

$$S_{1,k}\big(x_0, (c_1, x_1), \ldots, (c_k, x_k)\big) = x_{k+1} \in B_\epsilon(x_k)$$

for all $h_k \in H^k$. Similarly Player 2 deploys a strategy S_2.

We denote the first hitting time to the set Γ_ϵ by

$$\tau := \tau(\omega) = \inf\{k : x_k \in \Gamma_\epsilon, k = 0, 1, 2, \ldots\}.$$

The game process is a discrete time adapted process with respect to the filtration $\mathcal{F}_0 := \sigma(x_0)$ and

$$\mathcal{F}_k := \sigma\big(x_0, (c_1, x_1), \ldots, (c_k, x_k)\big) \qquad \text{for } k \ge 1,$$

so τ is a stopping time. The game ends at the random time τ, and the payoff is $F(x_\tau)$, where $F : \Gamma_\epsilon \to \mathbb{R}$ is a fixed, bounded and Borel measurable *payoff function*. In the end, Player 2 pays the amount $F(x_\tau)$ to Player 1.

To establish a unique probability measure, we need to know a starting point x_0 and strategies S_1 and S_2. Then, the probability measure $\mathbb{P}^{x_0}_{S_1,S_2}$ on the natural product σ-algebra is built by applying Kolmogorov's extension theorem to the family of transition densities

$$
\begin{aligned}
&\pi_{S_1,S_2}\big(x_0,(c_1,x_1),\ldots,(c_k,x_k),(C,A)\big)\\
&\quad=\frac{\alpha}{2}\delta_0(C)\delta_{S_1(x_0,(c_1,x_1),\ldots,(c_k,x_k))}(A)+\frac{\alpha}{2}\delta_1(C)\delta_{S_2(x_0,(c_1,x_1),\ldots,(c_k,x_k))}(A)\\
&\qquad+\beta\delta_2(C)\frac{|A\cap B_\epsilon(x_k)|}{|B_\epsilon(x_k)|}
\end{aligned}
$$

for any subset $C\subset\mathcal{C}$ and Borel subset $A\subset\Omega_\epsilon$ as long as $x_k\in\Omega$. If $x_k\notin\Omega$, the transition probability forces $x_{k+1}=x_k$.

The expected payoff is

$$
\mathbb{E}^{x_0}_{S_1,S_2}\big[F(x_\tau)\big]=\int_{H^\infty}F\big(x_\tau(\omega)\big)\,d\mathbb{P}^{x_0}_{S_1,S_2},
$$

when the game starts from x_0 and the players use strategies S_1 and S_2. The *value of the game for Player* 1 is given by

$$
u^1_\epsilon(x_0)=\sup_{S_1}\inf_{S_2}\mathbb{E}^{x_0}_{S_1,S_2}\big[F(x_\tau)\big]
$$

and the *value of the game for Player* 2 is given by

$$
u^2_\epsilon(x_0)=\inf_{S_2}\sup_{S_1}\mathbb{E}^{x_0}_{S_1,S_2}\big[F(x_\tau)\big],
$$

respectively. The game has a value that is, there exists a unique value function $u_\epsilon:=u^1_\epsilon=u^2_\epsilon$ (see [13] and [11]).

Since Ω is bounded, the game ends almost surely for any choice of strategies. This is true due to the fact that for $n_0\geq 1$ large enough, we have $n_0\epsilon>\operatorname{diam}(\Omega)$, and almost surely there will be infinitely many blocks of length n_0 consisting of solely random moves in the game.

Observe that the history h_k contains all the information at the moment k, and since the strategies are a collection of Borel measurable functions from all possible histories, it is clear that the game process will not be a *Markov process* in general.

This version of the tug-of-war game has good symmetry properties, which we will utilize in the proofs. Other versions of tug-of-war games have been studied for example in [16] and [6] and a continuous time game in [2].

A rough outline of the connection between the version of the game considered in this paper and p-harmonic functions is the following. First, assume that we have a p-harmonic function in an open set $\Omega'\supset\Omega$ with a nonvanishing gradient. Then, the p-harmonic function is real analytic, and Theorem 4.1 in [13] states that the game with probabilities (2.1) and with the values of the p-harmonic function on the boundary approximates the p-harmonic function in the game domain. The proof is based on the gradient strategy for the p-harmonic function and on the optional stopping theorem as well as on the asymptotic expansion in [12].

The general case requires game regularity of the boundary of the game domain. Then, it is possible to use a barrier argument to get estimates close to the boundary. By copying the strategies and utilizing the translation invariance of the game, the same estimates also holds in the interior of the game domain. Finally, a variant of the classical Arzelà–Ascoli's theorem provides a convergent subsequence. To prove that the limit is a viscosity solution to the homogeneous p-Laplace equation, a dynamic programming principle related to the game is applied (for more details about the principle, see, for example, [11]).

3. Measure density condition implies game regularity

We show in Theorem 3.7 that a uniform measure density condition implies game regularity for all $p > 2$. To establish this, we first show in Lemma 3.3 a more attainable criterion for game regularity. Then in Theorem 3.6, we use a "cylinder walk" framework, introduced in [10], to obtain some important hitting probability estimates.

Definition 3.1. *A point $y \in \partial\Omega$ satisfies a measure density condition if there is $c > 0$ such that*

$$\left|\Omega^c \cap B_r(y)\right| \geq c\left|B_r(y)\right|$$

for all $r > 0$.

Definition 3.2. *A point $y \in \partial\Omega$ is game regular, if for all $\delta > 0$ and $\eta > 0$, there exist $\delta_0 > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ and $x_0 \in B_{\delta_0}(y)$, there is a strategy S_1^* for Player 1 such that*

$$\mathbb{P}^{x_0}_{S_1^*,S_2}\left(x_\tau \in B_\delta(y) \cap \Omega^c\right) \geq 1 - \eta.$$

If every boundary point of Ω is game regular, we say that Ω is game regular.

Roughly speaking, game regularity means that whenever the game starts near a boundary point y, Player 1 has a strategy to end the game near y with a high probability. Next, we give a more attainable criterion to obtain game regularity. We modify the idea from [16], page 13.

Lemma 3.3. *A boundary point $y \in \partial\Omega$ is game regular if there exists a constant $\theta > 0$ such that for all $\delta > 0$, there are parameters $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for all $\epsilon < \epsilon_0$ and $x_0 \in B_{\delta_0}(y)$, there is a strategy S_1^* for Player 1 such that*

$$\mathbb{P}^{x_0}_{S_1^*,S_2}\big(\textit{the game ends before exiting the ball } B_\delta(y)\big) \geq \theta.$$

Proof. The idea of the proof is the following. By choosing $\delta_0 > 0$ small enough, we can start the game as near the point y as we want, and in order to exit the ball $B_\delta(y)$, the game sequence has to exit all the concentric smaller balls inside $B_\delta(y)$ as well. The probability to exit all the concentric balls inside $B_\delta(y)$ can be estimated above via the uniform probability θ; it is less than $(1-\theta)^k$, where k is the amount of concentric balls inside $B_\delta(y)$. Thus, the probability to end the game near y is close to one, when k is big enough.

To be more precise, let $\delta > 0$ and $\eta > 0$. Now, there are $\theta > 0$, $\epsilon_{0,1} > 0$ and $0 < \delta_{0,1} < \delta$ such that for all $\epsilon < \epsilon_{0,1}$ and for all $x_0 \in B_{\delta_{0,1}}(y)$, we have a strategy S_1^1 for Player 1 such that

$$\mathbb{P}^{x_0}_{S_1^1, S_2}\big(\text{the game ends before exiting the ball } B_\delta(y)\big) \geq \theta.$$

We can assume that $\epsilon_{0,1} < \delta_{0,1}/2$. Again similarly as above, for the constant $\delta_{0,1} - \epsilon_{0,1}$, there are $\epsilon_{0,2} > 0$ and $0 < \delta_{0,2} < \delta_{0,1}/2$ such that for all $\epsilon < \epsilon_{0,2}$ and for all $x_0 \in B_{\delta_{0,2}}(y)$, we have a strategy S_1^2 for Player 1 such that the probability to end the game before exiting the ball $B_{\delta_{0,1}-\epsilon_{0,1}}(y)$ is at least θ. We can do this as many times we want. Let us do this $k \in \mathbb{N}$ times, where k is such that

$$(1-\theta)^k \leq \eta.$$

Define $\delta_0 := \delta_{0,k}$ and $\epsilon_0 := \min\{\epsilon_{0,1}, \dots, \epsilon_{0,k}\}$, and fix any $x_0 \in B_{\delta_0}(y)$ and $\epsilon < \epsilon_0$. We can assume that $\epsilon < \frac{1}{2}\min\{\delta_0, \delta_{0,k-1} - \delta_{0,k}, \dots, \delta_{0,1} - \delta_{0,2}\}$ so that the game position cannot jump over many concentric balls during one turn. Denote the first time the game sequence exits $B_{\delta_{0,i-1}-\epsilon_{0,i-1}}(y)$ by $\tau^i := \tau^i(\omega)$ for all $i \in \{1, \dots, k\}$ with $\delta_{0,0} := \delta$ and $\epsilon_{0,0} := 0$. Also, denote the set

$$A_i := \big\{\text{exits the ball } B_{\delta_{0,i-1}-\epsilon_{0,i-1}}(y) \text{ before the game ends}\big\}$$

for all $i \in \{1, \dots, k\}$.

Recall that the game ends at the random time τ. Define a strategy S_1^* for Player 1 such that first, Player 1 uses the strategy S_1^k. If $\tau^k < \tau$, Player 1 starts to use the strategy S_1^{k-1} after the stopping time τ^k. Similarly, if $\tau^{k-1} < \tau$, Player 1 starts to use the strategy S_1^{k-2} after the stopping time τ^{k-1}. Thus, if it holds $0 < \tau^k < \tau^{k-1} < \cdots < \tau^1 < \tau$, after every stopping time τ^i, Player 1 starts to use the strategy S_1^{i-1} for all $i \in \{2, \dots, k\}$ and for all game sequences $\omega \in H^\infty$. After the stopping time τ^1, Player 1 does not change her strategy anymore. Observe that the earlier strategy S_1^i does not affect the game after the first time the game sequence exits $B_{\delta_{0,i-1}-\epsilon_{0,i-1}}(y)$ for every $i \in \{2, \dots, k\}$. Roughly this means that for every $i \in \{2, \dots, k\}$, after the stopping time τ^i, Player 1 forgets everything that has happened prior the time τ^i.

Let S_2 be any strategy for Player 2. The strategy S_2 can depend heavily on the past, so it could well be that our game process does not have any Markovian structure at any game round. However, the uniform θ is independent of the information available, so roughly, Player 2 cannot gain too much from the information of the past.

By the reasoning above, we can estimate iteratively

$$\begin{aligned}
&\mathbb{P}^{x_0}_{S_1^*, S_2}\big(\text{exits the ball } B_\delta(y) \text{ before the game ends}\big)\\
&= \mathbb{E}^{x_0}_{S_1^k, S_2}\left[\chi_{A_k}\mathbb{E}^{x_0}_{S_1^{k-1}, S_2}\left[\prod_{l=1}^{k-1}\chi_{A_l}\middle|\mathcal{F}_{\tau^k}\right]\right]\\
&= \mathbb{E}^{x_0}_{S_1^k, S_2}\big[\chi_{A_k}\mathbb{E}^{x_0}_{S_1^{k-1}, S_2}\big[\chi_{A_{k-1}}\cdots\mathbb{E}^{x_0}_{S_1^1, S_2}[\chi_{A_1}|\mathcal{F}_{\tau^2}]\cdots|\mathcal{F}_{\tau^k}\big]\big]\\
&\leq (1-\theta)^k \leq \eta.
\end{aligned}$$

This implies that

$$\mathbb{P}^{x_0}_{S_1^*,S_2}\big(x_\tau \in B_\delta(y)\cap\Omega^c\big) \geq \mathbb{P}^{x_0}_{S_1^*,S_2}\big(\text{the game ends before exiting } B_\delta(y)\big)$$
$$\geq 1-\eta.$$

Thus, we have shown the game regularity. □

To see that the uniform measure density condition implies game regularity, we need a "cylinder walk" framework.

Cylinder walk. Set the constants α, $\beta > 0$ with $\alpha + \beta = 1$ as before in (2.1), and fix the cylinder size $r > 0$. Consider the following random walk (called the "cylinder walk") in a $(n+1)$-dimensional cylinder $B_r(0)\times[0,r]$. Suppose that we are at a point $(x_j,t_j)\in B_r(0)\times[0,r]$. Next, we move to the point $(x_j,t_j-\epsilon)$ with the probability $\alpha/2$ and to the point $(x_j,t_j+\epsilon)$ with the probability $\alpha/2$. With the probability β we move to the point (x_{j+1},t_j), where x_{j+1} is chosen from the ball $B_\epsilon(x_j)$ according to the uniform distribution.

We have the following estimate for the probability that the cylinder walk exits the cylinder through its bottom; the proof is in the Appendix of the paper [10].

Lemma 3.4. *Let us start the cylinder walk from the point* $(0,t)$ *with* $0<t<r$. *Then, the probability that the walk exits the cylinder through its bottom is at least*

$$1 - C_{n,p}(t+\epsilon)/r$$

for all $\epsilon > 0$ *small enough.*

Assume that the origin $0\in\mathbb{R}^{n+1}$ at the bottom of the cylinder belongs to the set $\partial\Omega\times\{0\}$ and that this boundary point satisfies the measure density condition. The set $\Omega\cap B_r(0)\times\{0\}\subset B_r(0)\times[0,r]$. We are interested in the probability that the cylinder walk exits through the bottom and in addition, at the first time the walk hits the bottom, the process is in the complement of the set Ω. Since the origin satisfies the measure density condition, the complement has some positive Lebesgue measure. This suggests that the event we are interested in could have some positive probability measure.

The cylinder walk can be constructed by combining three independent random constructions. There is a "horizontal" random walk with the initial position $\tilde{x}_0 = x\in B_r(0)$. The point $\tilde{x}_{j+1}$ is chosen according to the uniform distribution in the ball $B_\epsilon(\tilde{x}_j)\subset\mathbb{R}^n$ for all $j\geq 0$. Further, there is a "vertical" random walk in the real axis with steps $+\epsilon$ or $-\epsilon$ and with the initial position $\tilde{t}_0 = t\in\,]0,r[$. For all $j\geq 0$, the next positions are $\tilde{t}_{j+1}=\tilde{t}_j+\epsilon$ or $\tilde{t}_{j+1}=\tilde{t}_j-\epsilon$ both with probability $\frac{1}{2}$. In addition, there is the increasing sequence

$$U_j = \sum_{m=1}^{j}\mathrm{Ber}_m,$$

where the Ber_m's are independent Bernoulli variables with $\mathrm{Ber}_m(\omega) \in \{0, 1\}$ and $\mathbb{P}(\mathrm{Ber}_m = 1) = \alpha$. Therefore, a copy of the cylinder walk is obtained by letting for $j \geq 0$

$$t_j = \tilde{t}_{U_j}, \qquad x_j = \tilde{x}_{j-U_j}.$$

Let τ_g stand for the first moment t_j exits the cylinder through its bottom or top, that is, the first j such that $t_j \in \mathbb{R}\backslash]0, r[$. Also, let $\tilde{\tau}_g$ stand for the first moment $\tilde{t}_j$ exits the cylinder through its bottom or top. Here, the subindex g refers to a "good exit".

We assume that $x = 0$. First, let us study the properties of the function $\tau_g - U_{\tau_g} = \tau_g - \tilde{\tau}_g$. The random variable $\tau_g - \tilde{\tau}_g$ is the number of times a random horizontal movement has occurred at the first moment the cylinder walk hits the bottom or top. The proof of the lemma below is in the Appendix for completeness.

Lemma 3.5. *Let $\tau_g, \tilde{\tau}_g, \alpha, \beta$ and r be as above, and let $n_0 \geq 1$ and $\gamma \in]0, 1[$. Then, there is a universal constant $C := C_{n_0,n,p,\gamma} > 1$ such that for all $a > 0$, $C\epsilon \leq t < r/2$ and ϵ small enough it holds*

$$\mathbb{P}(\tau_g - \tilde{\tau}_g \geq n_0) \geq 1 - \gamma \quad \textit{and} \tag{3.1}$$

$$\mathbb{P}\big(\tau_g - \tilde{\tau}_g \geq a\epsilon^{-2}\big) \leq 1 - \frac{2}{\sqrt{2\pi}} \int_{\frac{t}{\sqrt{a}}\nu_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds + \gamma + \mathcal{O}(\epsilon) \tag{3.2}$$

with the constant

$$\nu_{n,p} := 2\sqrt{\frac{\beta + 0.01\alpha}{0.99\alpha}}.$$

For any $a > 0$, $n_0 \geq 1$ and $\gamma \in]0, 1[$ the inequalities (3.1) and (3.2) yield

$$\begin{aligned} \mathbb{P}\big(n_0 \leq \tau_g - \tilde{\tau}_g < a\epsilon^{-2}\big) &\geq \mathbb{P}(\tau_g - \tilde{\tau}_g \geq n_0) - \mathbb{P}\big(\tau_g - \tilde{\tau}_g \geq a\epsilon^{-2}\big) \\ &\geq \frac{2}{\sqrt{2\pi}} \int_{\frac{t}{\sqrt{a}}\nu_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds - 2\gamma - \mathcal{O}(\epsilon) \end{aligned} \tag{3.3}$$

for all $C\epsilon \leq t < r/2$ and ϵ small enough with a large $C > 1$ independent of ϵ. Observe that

$$\frac{2}{\sqrt{2\pi}} \int_{\frac{t}{\sqrt{a}}\nu_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds \to 1$$

as $t \to 0$. Thus, the inequality (3.3) points out that for the cylinder walk started from the height $C\epsilon \leq t < r_0$, the random variable $\tau_g - \tilde{\tau}_g$ is very likely between the times n_0 and $a\epsilon^{-2}$ for all r_0, γ and ϵ small enough and fixed $n_0 \geq 1$ and $a > 0$.

Next, we concentrate on the distribution of the random variable $\tilde{x}_k$. Assume that Z is a random vector with the uniform distribution in the ball $B_\epsilon(0) \subset \mathbb{R}^n$. The density of the random vector Z is

$$f_Z(x) = \frac{1}{|B_\epsilon(0)|} \chi_{B_\epsilon(0)}(x).$$

We denote the measure of the unit ball by $\omega_n := |B_1(0)|$. Let $k_0 := k_{0,n} > 2$ denote the constant in Lemma A.4 and fix any $k \geq k_0$. For the density of the random variable $\tilde{x}_k = \sum_{i=1}^{k} Z_i$, where the random vectors Z_i are independent and distributed as Z, we use the notation $f_k := f_{\sum_{i=1}^{k} Z_i}$. The density f_k is a decreasing radial function. In the Appendix, we have derived in (A.8) and (A.10) the following estimates: There are constants $C_n > 0$ and $C_1 > 0$ such that

$$f_k(0) \leq C_n \left(\frac{1}{\sqrt{k}\epsilon} \right)^n,$$

and

$$f_k(C_* \sqrt{k}\epsilon) \geq \left(\frac{1}{C_1} \right)^n \left(\frac{0.99}{\omega_n} - C_n (C_*)^n \right) \left(\frac{1}{\sqrt{k}\epsilon} \right)^n \tag{3.4}$$

for all $C_* \in]0, C_1[$. By the comment after the statement of Lemma A.4 in the Appendix, we have

$$f_k(C_* \sqrt{k}\epsilon) \geq \zeta \left(\frac{1}{\sqrt{k}\epsilon} \right)^n$$

for some $\zeta := \zeta_n > 0$, if we choose $C_* > 0$ so small that

$$C_* < \left(\frac{0.99}{\omega_n C_n} \right)^{1/n}. \tag{3.5}$$

Let τ_b stand for the first j when $|x_j|$ reaches $[r, \infty[$. Here, the subindex b refers to a "bad exit". Recall that the origin at the bottom of the cylinder satisfies the measure density condition. Let $C_{n,p} > 0$ denote the constant in Lemma 3.4, and for all $\delta > 0$, denote

$$A_\delta := B_\delta(0) \cap \Omega^c.$$

Theorem 3.6. *Consider the cylinder $B_{\delta/3}(0) \times [0, \delta/3]$ for any fixed $\delta > 0$. Then, there exist constants $\theta := \theta_{n,p} > 0$, $\epsilon_0 := \epsilon_{0,n,p,\delta} > 0$ and $\delta_0 := \delta_{0,n,p,\delta} > 0$ such that*

$$\mathbb{P}(\tau_b \leq \tau_g \text{ or } t_{\tau_g} \geq \delta/3 \text{ or } x_{\tau_g} \notin A_{\delta/3}) \leq 1 - \theta$$

for all $\epsilon < \epsilon_0$ whenever the cylinder walk starts from the point $(0, t)$ for some $0 < t \leq \delta_0$.

Proof. To establish the result, we use the inequality (3.3) to estimate how many times it is likely that a random horizontal movement has occurred at the first time the cylinder walk hits the bottom. Then, we use the estimate (3.4) and the fact that vertical and horizontal movements are independent to estimate the probability that we are in the complement of the set Ω at the first time the walk exits the cylinder through its bottom.

Let $0 < \lambda < 1$, where the exact value of λ will be fixed later. Define

$$\delta_0 := \frac{\delta\lambda}{3C_{n,p}}, \tag{3.6}$$

and start the cylinder walk from the point $(0,t)$ for some $0 < t \leq \delta_0$ in the cylinder $B_{\delta/3}(0) \times [0, \delta/3]$.

We recall the constant $k_0 := k_{0,n} > 2$ from Lemma A.4. In addition, let $C := C_{n_0,n,p,\gamma} > 1$ be the constant from Lemma 3.5 with $n_0 = k_0$ and $0 < \gamma < 1$ defined later. First, assume that $t < C\epsilon$. Then, the number of ϵ-steps required to reach the bottom from t is less than the universal constant C. Hence, the probability that the cylinder walk exits the cylinder through its bottom and in addition, it holds $x_{\tau_g} = 0$, is greater than or equal to $(\alpha/2)^C$. From this the statement immediately follows in the case $t < C\epsilon$.

Next, assume that $t \geq C\epsilon$. Lemma 3.4 states that

$$\mathbb{P}(\tau_b \leq \tau_g \text{ or } t_{\tau_g} \geq \delta/3) \leq 3C_{n,p}\delta^{-1}(t+\epsilon) \leq 3C_{n,p}\delta^{-1}(\delta_0+\epsilon).$$

Therefore, we have by (3.6) that

$$\mathbb{P}(\tau_b \leq \tau_g \text{ or } t_{\tau_g} \geq \delta/3 \text{ or } x_{\tau_g} \notin A_{\delta/3}) \leq \mathcal{O}(\epsilon) + \lambda + 1 - \mathbb{P}(x_{\tau_g} \in A_{\delta/3}).$$

The inequality (3.3) and the remark after suggest the estimate

$$\begin{aligned}\mathbb{P}(x_{\tau_g} \in A_{\delta/3}) &= \mathbb{P}(\tilde{x}_{\tau_g - \tilde{\tau}_g} \in A_{\delta/3}) \\ &\geq \mathbb{P}\big(\tilde{x}_{\tau_g - \tilde{\tau}_g} \in A_{\delta/3} \text{ and } k_0 \leq \tau_g - \tilde{\tau}_g < \delta^2\epsilon^{-2}\big) \\ &= \sum_{k=k_0}^{\lfloor \delta^2\epsilon^{-2} \rfloor} \mathbb{P}(\tilde{x}_{\tau_g - \tilde{\tau}_g} \in A_{\delta/3} \text{ and } \tau_g - \tilde{\tau}_g = k).\end{aligned}$$

Denote the index set

$$I := \big\{k_0, k_0+1, \dots, \lfloor \delta^2\epsilon^{-2} \rfloor\big\}.$$

Since the random variables $\tilde{x}_k$ and $\tau_g - \tilde{\tau}_g$ are independent for all $k \in I$, we have

$$\sum_{k\in I} \mathbb{P}(\tilde{x}_{\tau_g - \tilde{\tau}_g} \in A_{\delta/3} \text{ and } \tau_g - \tilde{\tau}_g = k) = \sum_{k\in I} \mathbb{P}(\tilde{x}_k \in A_{\delta/3})\mathbb{P}(\tau_g - \tilde{\tau}_g = k).$$

Let $k \in I$ and choose the constant $C_* > 0$ as in (3.5). We may assume that $C_* < 1/3$. Because $\sqrt{k}\epsilon < \delta$ and the density f_k is a decreasing radial function, we can calculate

$$\mathbb{P}(\tilde{x}_k \in A_{\delta/3}) \geq \mathbb{P}\big(\tilde{x}_k \in B_{C_*\sqrt{k}\epsilon}(0) \cap \Omega^c\big) \geq f_k(C_*\sqrt{k}\epsilon)\big|B_{C_*\sqrt{k}\epsilon}(0) \cap \Omega^c\big|.$$

By using the estimate (3.4) and the uniform measure density condition, we obtain

$$\begin{aligned}&f_k(C_*\sqrt{k}\epsilon)\big|B_{C_*\sqrt{k}\epsilon}(0) \cap \Omega^c\big| \\ &\geq \left(\frac{1}{C_1}\right)^n \left(\frac{0.99}{\omega_n} - C_n(C_*)^n\right)\left(\frac{1}{\sqrt{k}\epsilon}\right)^n c\big|B_{C_*\sqrt{k}\epsilon}(0)\big| \\ &= \omega_n c\left(\frac{C_*}{C_1}\right)^n \left(\frac{0.99}{\omega_n} - C_n(C_*)^n\right) =: \hat{C}_n,\end{aligned}$$

where the constant $c > 0$ comes from the uniform measure density condition. This together with the inequality (3.3) yield

$$\begin{aligned}
&\sum_{k\in I} \mathbb{P}(\tilde{x}_k \in A_{\delta/3})\mathbb{P}(\tau_g - \tilde{\tau}_g = k) \\
&\quad \geq \omega_n c \left(\frac{C_*}{C_1}\right)^n \left(\frac{0.99}{\omega_n} - C_n (C_*)^n\right) \mathbb{P}\big(k_0 \leq \tau_g - \tilde{\tau}_g < \delta^2 \epsilon^{-2}\big) \\
&\quad \geq \hat{C}_n \frac{2}{\sqrt{2\pi}} \int_{\frac{\lambda}{3}\tilde{C}_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds - 2\gamma \hat{C}_n - \mathcal{O}(\epsilon)
\end{aligned}$$

with the constant

$$\tilde{C}_{n,p} := \frac{2}{C_{n,p}} \sqrt{\frac{\beta + 0.01\alpha}{0.99\alpha}}.$$

Therefore, we have shown

$$\begin{aligned}
&\mathbb{P}(\tau_b \leq \tau_g \text{ or } t_{\tau_g} \geq \delta/3 \text{ or } x_{\tau_g} \notin A_{\delta/3}) \\
&\quad \leq 1 - \hat{C}_n \frac{2}{\sqrt{2\pi}} \int_{\frac{1}{3}\tilde{C}_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds + \lambda + 2\gamma \hat{C}_n + \mathcal{O}(\epsilon) \\
&\quad \leq 1 - \frac{1}{2}\hat{C}_n \frac{2}{\sqrt{2\pi}} \int_{\frac{1}{3}\tilde{C}_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds
\end{aligned}$$

for all $C\epsilon \leq t \leq \delta_0$ and ϵ, λ and γ small enough. Thus, this concludes the case $t \geq C\epsilon$. To combine the cases, we define

$$\theta := \min\left\{\frac{1}{2}\hat{C}_n \frac{2}{\sqrt{2\pi}} \int_{\frac{1}{3}\tilde{C}_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds, (\alpha/2)^C\right\}.$$

Consequently, the proof is complete. □

If Player 1 plays by canceling the moves of the other player, we obtain Theorem 3.7. Observe that this strategy is not optimal for Player 1 in the sense that Player 1 also tries to cancel the moves that might benefit her.

The cancellation strategy was introduced in the paper [10] to prove Harnack's inequality for p-harmonic functions via tug-of-war games. In addition, the cancellation strategy can be used to prove regularity properties for viscosity solutions of the inhomogeneous p-Laplace equation (see [18]).

Theorem 3.7. *If $y \in \partial\Omega$ satisfies the measure density condition, then it is game regular for $p > 2$.*

Proof. To establish the result, our aim is to use Lemma 3.3 and therefore, to find a uniform lower bound for the probability that the game ends before exiting a given ball. If Player 1 plays by canceling the moves of the other player, the lower bound $\theta > 0$ for the probability is obtained by using Theorem 3.6.

We can clearly assume that $y = 0$. Let $\delta > 0$, and consider the cylinder $B_{\delta/3}(0) \times [0, \delta/3]$. Define a constant δ_0 as in (3.6), and find $\epsilon_0 > 0$ and $\lambda > 0$ small enough such that we can apply Theorem 3.6. Let $x_0 \in B_{\delta_0}(0)$ and $\epsilon < \epsilon_0$. At every moment, we can divide the game position as a sum of vectors

$$x_0 + \sum_{k \in I_1} v_k^1 + \sum_{k \in I_2} v_k^2 + \sum_{k \in I_3} v_k^3.$$

Here, I_1 denotes the indices of rounds when Player 1 has moved with the vectors v_k^1 as her moves. Similarly, Player 2 has moved in the indices of rounds I_2 with the moves v_k^2 as his moves. The random movements have occurred in the indices of rounds I_3, and these random movements are denoted by v_k^3.

Let

$$M := 2\left\lceil \frac{|x_0|}{\epsilon} \right\rceil,$$

where the factor 2 is due to the fact that the players cannot step to the boundary of $B_\epsilon(x_j)$ for any j. Define the following strategy S_1^* for Player 1 for the game that starts from x_0. She always tries to cancel the earliest move of Player 2 which she has not yet been able to cancel. If all the moves at that moment are cancelled and she wins the coin toss, she moves the game point by the vector

$$-\epsilon/2 \frac{x_0}{|x_0|}.$$

She does this until she has won $M - 1$ more coin tosses than Player 2. If she wins her Mth more coin toss, her move will be such that the game position is

$$\sum_{k \in I_3} v_k^3$$

after the move. Observe that the game, with the strategy S_1^*, is related to the cylinder walk, when we start the cylinder walk from the point $(0, M\epsilon/2)$ with $M\epsilon/2 \to |x_0| < \delta_0$ as $\epsilon \to 0$.

Let us define three conditions for the game sequences of the game:

(**A**) Player 1 has won the coin toss M more times than Player 2, and at the moment this happens, the game sequence is in the set Ω^c.

(**B**) Player 2 has won the coin toss at least $\frac{\delta}{3\epsilon}$ more times than Player 1.

(**C**) $|\sum_{k \in I_3} v_k^3| \geq \frac{\delta}{3}$.

We are interested in the following event

$$\mathbf{X} := \{\text{the condition } (\mathbf{A}) \text{ happens before conditions } (\mathbf{B}) \text{ and } (\mathbf{C})\},$$

and Theorem 3.6 states that there is a constant $\theta := \theta_{n,p} > 0$ such that

$$\mathbb{P}_{S_1^*, S_2}(\mathbf{X}) \geq \theta.$$

Now, we can estimate

$$\mathbb{P}_{S_1^*,S_2}\big(\text{the game ends before exiting the ball } B_\delta(0)\big) \geq \mathbb{P}_{S_1^*,S_2}(\mathbf{X}).$$

Above, we also used the fact that the game sequences for which the game has ended before Player 1 has won M more coin tosses than Player 2 are good for our purposes. To finish the proof, we can use Lemma 3.3, and thus the proof is complete. □

It is worth mentioning that in the case $p > n$, every point becomes game regular. This is proved in [16], and the same also holds for the version of the game considered in this paper. Roughly, as p increases, the probability for the player to end the game before exiting a given ball increases.

Appendix: Hitting probabilities for a cylinder walk

Fix the cylinder size $r > 0$. The cylinder walk in a cylinder $B_r(0) \times [0, r] \subset \mathbb{R}^{n+1}$ can be constructed by combining three independent random constructions. There is a "horizontal" random walk with the initial position $\tilde{x}_0 = x \in B_r(0)$. The point $\tilde{x}_{j+1}$ is chosen according to the uniform distribution in the ball $B_\epsilon(\tilde{x}_j) \subset \mathbb{R}^n$ for all $j \geq 0$. Further, there is a "vertical" random walk in the real axis with steps $+\epsilon$ or $-\epsilon$ and with the initial position $\tilde{t}_0 = t \in]0, r[$. The next positions are $\tilde{t}_{j+1} = \tilde{t}_j + \epsilon$ or $\tilde{t}_{j+1} = \tilde{t}_j - \epsilon$ both with probability $\frac{1}{2}$ for all $j \geq 0$. In addition, there is the increasing sequence

$$U_j = \sum_{m=1}^{j} \mathrm{Ber}_m,$$

where the Ber_m's are independent Bernoulli variables with $\mathrm{Ber}_m(\omega) \in \{0, 1\}$ and $\mathbb{P}(\mathrm{Ber}_m = 1) = \alpha \in]0, 1[$. Thus, a copy of the cylinder walk is obtained by letting for $j \geq 0$

$$t_j = \tilde{t}_{U_j}, \qquad x_j = \tilde{x}_{j-U_j}.$$

Let τ_g stand for the first moment t_j exits the cylinder through its bottom or top, and let $\tilde{\tau}_g$ stand for the first moment $\tilde{t}_j$ exits the cylinder through its bottom or top.

Recall Hoeffding's (or Azuma's or Bernstein's) inequality for a sum of independent and identically distributed random variables (see, for example, [7], page 198).

Theorem A.1. *Let Y_m be independent and identically distributed symmetric $\mathbb{R}^n$-valued random variables, $m \in \{1, 2, \dots, N\}$, that are uniformly bounded: $|Y_m| \leq b$ almost surely for all m. Then,*

$$\mathbb{P}\left(\max_{1\leq m\leq N}\left|\sum_{i=1}^{m} Y_i\right| \geq \lambda\right) \leq 4n \exp\left(-\frac{\lambda^2}{2Nb^2 n}\right).$$

In the theorem above, the factor 4 instead of 2 comes from the use of Levy–Kolmogorov's inequality (see, for example, [19], page 397)

$$\mathbb{P}\left(\max_{1\le m\le N}\left|\sum_{i=1}^{m} Y_i\right| \ge \lambda\right) \le 2\mathbb{P}\left(\left|\sum_{i=1}^{N} Y_i\right| \ge \lambda\right).$$

We assume that $x = 0$, and denote $\beta = 1 - \alpha$.

Lemma A.2. *Let* τ_g *and* $\tilde{\tau}_g$ *be as above*, $n_0 \ge 1$ *and* $\gamma \in]0, 1[$. *Then*, *there is a constant* $C := C_{n_0,n,p,\gamma} > 0$ *such that it holds*

$$\mathbb{P}(\tilde{\tau}_g \ge n_0) \ge 1 - \gamma \quad \textit{and} \tag{A.1}$$

$$\mathbb{P}(\tau_g - \tilde{\tau}_g \ge n_0) \ge 1 - \gamma \tag{A.2}$$

for all $C\epsilon \le t < r/2$ *and* $\epsilon < r/(4C)$.

Proof. The vertical movement consists of the moves $+\epsilon$ or $-\epsilon$ in the real axis. Let Y_i be independent and identically distributed random variables with $Y_i(\omega) \in \{-\epsilon, \epsilon\}$ and $\mathbb{P}(Y_i = \epsilon) = \mathbb{P}(Y_i = -\epsilon) = \frac{1}{2}$ for all i. Assume $t < r/2$, and recall the cylinder size $B_r(0) \times [0, r]$. Now, it holds

$$\begin{aligned}\mathbb{P}(\tilde{\tau}_g \ge n_0) &= \mathbb{P}\left(\max_{k<n_0}\sum_{i=1}^{k} Y_i < \min\{t, r-t\}\right) \\ &= 1 - \mathbb{P}\left(\max_{k<n_0}\sum_{i=1}^{k} Y_i \ge t\right).\end{aligned}$$

Random variables Y_i are bounded, $|Y_i| \le \epsilon$ for all $i \ge 1$. By using Hoeffding's inequality that is, Theorem A.1, we can deduce that for $C > 0$ and $t \ge C\epsilon$ it holds

$$\mathbb{P}\left(\max_{k<n_0}\sum_{i=1}^{k} Y_i \ge t\right) \le 4\exp\left(-\frac{t^2}{2n_0\epsilon^2}\right) \le 4\exp\left(-\frac{C^2}{2n_0}\right).$$

Consequently, there is $\overline{C} := \overline{C}_{n_0,\gamma} > 1$ large enough such that (A.1) holds for all $\overline{C}\epsilon \le t < r/2$ and $\epsilon < r/(4\overline{C})$.

For the second part, let us consider the event

$$B := \{0.99\alpha\tau_g < \tilde{\tau}_g < (\alpha + \beta/2)\tau_g\}. \tag{A.3}$$

For any $j_0 \ge 1$, denote the sets

$$\begin{aligned}B^* &:= \{U_j < (\alpha + \beta/2)j \text{ for all } j \ge j_0\} \quad \text{and} \\ B_* &:= \{U_j > 0.99\alpha j \text{ for all } j \ge j_0\}.\end{aligned}$$

Again, apply Hoeffding's inequality with $Y_m = \mathrm{Ber}_m - \alpha$, $\lambda = j\beta/2$, $b = 1$ and $N = j$ to get

$$\mathbb{P}\big(U_j \geq (\alpha + \beta/2)j\big) = \mathbb{P}(U_j - \alpha j \geq j\beta/2) \leq \mathbb{P}\big(|U_j - \alpha j| \geq j\beta/2\big)$$
$$\leq 4\exp\Big(-\frac{1}{8}\beta^2 j\Big).$$

In a similar fashion, we can calculate

$$\mathbb{P}(U_j \leq 0.99\alpha j) \leq 4\exp\Big(-\frac{\alpha^2 j}{2 \cdot 10^4}\Big).$$

Thus by choosing j_0 large enough and summing over all indices, we get

$$\mathbb{P}\big((B^*)^c\big) = \mathbb{P}\big(U_j \geq (\alpha + \beta/2)j \text{ for some } j \geq j_0\big)$$
$$\leq \sum_{j \geq j_0} \mathbb{P}\big(U_j \geq (\alpha + \beta/2)j\big)$$
$$\leq \sum_{j \geq j_0} 4\exp\Big(-\frac{1}{8}\beta^2 j\Big) \leq \frac{\gamma}{8}.$$

By a similar argument, it holds

$$\mathbb{P}(B_*) \geq 1 - \gamma/8$$

for j_0 large enough. Hence, we choose a large index $j_0 := j_{0,n,p,\gamma}$ such that

$$\mathbb{P}\big(B_* \text{ and } B^*\big) \geq \mathbb{P}(B_*) - \mathbb{P}\big((B^*)^c\big) \geq 1 - \gamma/4.$$

Observe that

$$\big\{B_* \text{ and } B^* \text{ and } \tau_g \geq j_0\big\} \subset B.$$

Therefore, we get

$$\mathbb{P}(B) \geq \mathbb{P}(\tau_g \geq j_0) - \gamma/4.$$

Since $\tau_g \geq \tilde{\tau}_g$ always, we have $\{\tilde{\tau}_g \geq j_0\} \subset \{\tau_g \geq j_0\}$. Combining this with a similar argument to (A.1), we can deduce that there is $\tilde{C} := \tilde{C}_{j_0,\gamma} > 1$ large enough such that for all $\tilde{C}\epsilon \leq t < r/2$ and $\epsilon < r/(4\tilde{C})$ it holds

$$\mathbb{P}(B) \geq 1 - \gamma/2. \tag{A.4}$$

By a direct calculation, we have

$$B \subset \Big\{\frac{\beta}{2\alpha + \beta}\tilde{\tau}_g < \tau_g - \tilde{\tau}_g < \frac{\beta + 0.01\alpha}{0.99\alpha}\tilde{\tau}_g\Big\}.$$

Therefore, we obtain by using (A.4)

$$\begin{aligned}\mathbb{P}(\tau_g - \tilde{\tau}_g \ge n_0) &\ge \mathbb{P}\left(\frac{\beta}{2\alpha+\beta}\tilde{\tau}_g \ge n_0 \text{ and } B\right)\\ &= \mathbb{P}\big(\tilde{\tau}_g \ge (2\alpha+\beta)n_0(\beta)^{-1} \text{ and } B\big)\\ &\ge \mathbb{P}\big(\tilde{\tau}_g \ge (2\alpha+\beta)n_0(\beta)^{-1}\big) - \gamma/2\end{aligned}$$

for all $\tilde{C}\epsilon \le t < r/2$ and $\epsilon < r/(4\tilde{C})$. Thus, this estimate and a similar argument to (A.1) imply that there is $C := C_{n_0,n,p,\gamma} > \max\{\overline{C}, \tilde{C}\} > 1$ large enough such that (A.2) holds for all $C\epsilon \le t < r/2$ and $\epsilon < r/(4C)$. □

Lemma A.3. *Let $\tau_g, \tilde{\tau}_g, r$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ be as at the beginning of the Appendix. In addition, let $C := C_{n_0,n,p,\gamma} > 1$ be the constant from Lemma A.2 for $\gamma \in]0, 1[$ and $n_0 \ge 1$. Then for all $a > 0$, $C\epsilon \le t < r/2$ and ϵ small enough, we have*

$$\mathbb{P}\big(\tau_g - \tilde{\tau}_g \ge a\epsilon^{-2}\big) \le 1 - \frac{2}{\sqrt{2\pi}}\int_{\frac{t}{\sqrt{a}}\nu_{n,p}}^{\infty} e^{-\frac{s^2}{2}}\, ds + \gamma + \mathcal{O}(\epsilon)$$

with the constant

$$\nu_{n,p} := 2\sqrt{\frac{\beta + 0.01\alpha}{0.99\alpha}}.$$

Proof. By using the inequality (A.4) and the inclusion after it, we can deduce

$$\begin{aligned}\mathbb{P}\big(\tau_g - \tilde{\tau}_g \ge a\epsilon^{-2}\big) &\le \mathbb{P}\big(\tau_g - \tilde{\tau}_g \ge a\epsilon^{-2} \text{ and } B\big) + \mathbb{P}\big(B^c\big)\\ &\le \mathbb{P}\left(\tilde{\tau}_g \ge \frac{0.99\alpha a}{(\beta+0.01\alpha)\epsilon^2}\right) + \gamma\end{aligned}$$

for all $C\epsilon \le t < r/2$ and $\epsilon < r/(4C)$ with the set B defined in (A.3).

We estimate the probability of the event $\{\tilde{\tau}_g \ge d\epsilon^{-2}\}$ for all $d > 0$. Consider the following independent and identically distributed random variables: $Z_i(\omega) \in \{1, -1\}$, $\mathbb{P}(Z_i = -1) = \mathbb{P}(Z_i = 1) = \frac{1}{2}$ and $\mathbb{E}[Z_i]^2 = 1$ for all $i \ge 1$. For these random variables, we have the following equality (see, for example, [7], page 351)

$$\mathbb{P}\left(\max_{1\le m\le N}\sum_{i=1}^{m} Z_i \ge l\right) = 2\mathbb{P}\left(\sum_{i=1}^{N} Z_i \ge l\right) - \mathbb{P}\left(\sum_{i=1}^{N} Z_i = l\right)$$

for all integers $N \ge 1$ and $l \ge 1$. Further, since $\mathbb{E}|Z_i|^3 = 1 < \infty$ for all $i \ge 1$, we can use the Berry–Esseen theorem to determine the speed in the central limit theorem (see, for example, [19], page 63), and thus

$$2\mathbb{P}\left(\sum_{i=1}^{N} Z_i \ge l\sqrt{N}\right) - \mathbb{P}\left(\sum_{i=1}^{N} Z_i = l\sqrt{N}\right) \ge \frac{2}{\sqrt{2\pi}}\int_{l}^{\infty} e^{-\frac{s^2}{2}}\, ds - \mathcal{O}\big(N^{-1/2}\big).$$

Observe that for all $\epsilon > 0$ small enough

$$\frac{\lceil t\epsilon^{-1}\rceil}{\sqrt{\lfloor d\epsilon^{-2}\rfloor}} \leq \frac{2t}{\sqrt{d}}.$$

Therefore, for all $t < r/2$ and $\epsilon > 0$ small enough, we have

$$\begin{aligned} \mathbb{P}\big(\tilde{\tau}_g \geq d\epsilon^{-2}\big) &\leq \mathbb{P}\left(\max_{1\leq m\leq \lfloor d\epsilon^{-2}\rfloor} \sum_{i=1}^{m} Z_i < \lceil t\epsilon^{-1}\rceil\right) \\ &\leq 1 - \frac{2}{\sqrt{2\pi}} \int_{\frac{2t}{\sqrt{d}}}^{\infty} e^{-\frac{s^2}{2}}\,ds + \mathcal{O}(\epsilon). \end{aligned}$$

□

Lemma 3.5 is now an immediate consequence of Lemmas A.2 and A.3.

Next, we prove a technical result (Lemma A.4 below) that we use in Section 3 above. First, in order to keep the calculations simple, let the dimension n be one for now. Assume that Z is distributed according to the uniform distribution in $]-\epsilon, \epsilon[$ for some $\epsilon > 0$. Then for two independent Z_1 and Z_2 both distributed as Z, the density of the random variable $Z_1 + Z_2$ can be computed via convolution. Thus, since $f_Z(x) = 1/(2\epsilon)\chi_{]-\epsilon,\epsilon[}(x)$, we have

$$f_{Z_1+Z_2}(x) = \int_{-\infty}^{\infty} f_Z(x-y) f_Z(y)\,dy = \left(\frac{1}{2\epsilon}\right)^2 \big(2\epsilon - |x|\big)\chi_{]-2\epsilon,2\epsilon[}(x).$$

For any $k \geq 1$, denote the density $f_k := f_{\sum_{i=1}^{k} Z_i}$, where Z_i are independent random variables distributed as Z. Similarly as in the case $k = 2$, we can deduce and prove by induction (see, for example, [17], page 197) that for any $k \geq 1$

$$f_k(x) = \begin{cases} \dfrac{1}{(k-1)!(2\epsilon)^k} \displaystyle\sum_{j=0}^{\lfloor \frac{x+k\epsilon}{2\epsilon} \rfloor} (-1)^j \binom{k}{j} (x + k\epsilon - 2j\epsilon)^{k-1}, & \text{if } x \in]-k\epsilon, k\epsilon[, \\ 0, & \text{otherwise.} \end{cases}$$

Unfortunately, it is hard to get quantitative estimates from it.

There have been a lot of studies on the concentration function of a sum of independent random variables (see, for example, [4]). However, we are interested in the pointwise value of the function f_k at the origin, and we will estimate the value by hand for the reader in a rather accessible way.

The characteristic function of the random variable Z can be easily calculated,

$$\varphi_Z(t) = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{itx}\,dx = \frac{\sin(\epsilon t)}{\epsilon t}.$$

Let $k \geq 2$. Because of the independence,

$$\varphi_{\sum_{i=1}^{k} Z_i}(t) = \left(\frac{\sin(\epsilon t)}{\epsilon t}\right)^k.$$

Now, we have $\int_{-\infty}^{\infty} |\varphi_{\sum_{i=1}^k Z_i}(t)|\, dt < \infty$, so we can use the well-known inversion formula

$$f_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi_{\sum_{i=1}^k Z_i}(t)\, dt.$$

This inversion formula yields

$$f_k(0) = \frac{1}{\pi} \int_0^{\epsilon^{-1}} \left(\frac{\sin(\epsilon t)}{\epsilon t} \right)^k dt + \frac{1}{\pi} \int_{\epsilon^{-1}}^{\infty} \left(\frac{\sin(\epsilon t)}{\epsilon t} \right)^k dt.$$

Define

$$h(z) := 2\frac{1 - \cos z}{z^2}$$

so that we have for any $0 \le m \le 2\pi$

$$\frac{\sin z}{z} \le 1 - h(m)\frac{z^2}{6} \tag{A.5}$$

for all $|z| \le m$. This inequality is true since the function $\sin z/z$ decreases for $0 < z \le \pi$ implying

$$\left(\frac{\sin(m/2)}{m/2} \right)^2 \le \left(\frac{\sin(z/2)}{z/2} \right)^2$$

for all $0 < z \le 2\pi$. This inequality yields

$$1 - \cos z - h(m)\frac{z^2}{2} \ge 0$$

for all $0 < z \le 2\pi$ so we have the inequality (A.5), since both sides of the inequality (A.5) are even functions. By using the inequality (A.5), a change of variables formula and the inequality $1 - z \le e^{-z}$ for all $z \in \mathbb{R}$, we have

$$\begin{aligned} \frac{1}{\pi} \int_0^{\epsilon^{-1}} \left(\frac{\sin(\epsilon t)}{\epsilon t} \right)^k dt &= \frac{1}{\pi\epsilon} \int_0^1 \left(\frac{\sin z}{z} \right)^k dz \\ &\le \frac{1}{\pi\epsilon} \int_0^1 \left(1 - h(1)\frac{z^2}{6} \right)^k dz \\ &\le \frac{1}{\pi\epsilon} \int_0^1 e^{-\frac{z^2 kh(1)}{6}}\, dz. \end{aligned}$$

Again, via changing the variables we derive

$$\frac{1}{\pi\epsilon} \int_0^1 e^{-\frac{z^2 kh(1)}{6}}\, dz \le \frac{1}{\epsilon}\sqrt{\frac{6}{k\pi h(1)}} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{u^2}{2}}\, du = \sqrt{\frac{3}{2\pi h(1)}} \frac{1}{\sqrt{k}\epsilon}.$$

Thus, we have estimated

$$\frac{1}{\pi}\int_0^{\epsilon^{-1}}\left(\frac{\sin(\epsilon t)}{\epsilon t}\right)^k dt \le \sqrt{\frac{3}{2\pi h(1)}}\frac{1}{\sqrt{k}\epsilon}.$$

Because $\sin z \le 1$ for all $z \in \mathbb{R}$, we can estimate the second integral directly, and hence

$$\frac{1}{\pi}\int_{\epsilon^{-1}}^{\infty}\left(\frac{\sin(\epsilon t)}{\epsilon t}\right)^k dt \le \frac{1}{\pi\epsilon}\int_1^{\infty}\frac{1}{z^k}\,dz = \frac{1}{\pi\epsilon(k-1)}.$$

Therefore, we have derived the estimate

$$f_k(0) \le \sqrt{\frac{3}{2\pi h(1)}}\frac{1}{\sqrt{k}\epsilon} + \frac{1}{\pi\epsilon(k-1)}.$$

Next, we extend the argument to the higher dimensions as well. Assume that Z is a random vector with the uniform distribution in the n-ball $B_\epsilon(0)$, $n \ge 1$. The density of the random vector Z is

$$f_Z(x) = \frac{1}{|B_\epsilon(0)|}\chi_{B_\epsilon(0)}(x).$$

Using the same approach as in dimension one, we first need the characteristic function of the random vector Z. Denote the measure of the unit ball by $\omega_n := |B_1(0)| = \pi^{n/2}/\,\Gamma(\frac{n}{2}+1)$, where the function Γ is the usual gamma function. The random variable Z is invariant under rotation, that is, the density function is a constant on every sphere $S_r^{n-1}(0) := \{x \in \mathbb{R}^n : |x| = r\}$ for all $r > 0$. Hence, by rotating the ball $B_\epsilon(0)$, we see that $\varphi_Z(u) = \varphi_Z((r, 0, \ldots, 0))$ for all $u \in \mathbb{R}^n$ such that $|u| = r$. Let $r > 0$, and direct computation with a change of variables $x = \epsilon y$ yields

$$\begin{aligned}
\varphi_Z\big((r,0,\ldots,0)\big) &= \int_{\mathbb{R}^n} e^{irx_1} f_Z(x)\,dx \\
&= \frac{1}{\omega_n}\int_{B_1(0)} e^{i\varepsilon r y_1}\,dy_1\cdots dy_n \\
&= \frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\big(1-y_1^2\big)^{(n-1)/2} e^{i\varepsilon r y_1}\,dy_1 \\
&= \frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\big(1-y_1^2\big)^{(n-1)/2}\cos(\varepsilon r y_1)\,dy_1.
\end{aligned}$$

A spherical Bessel function of order $n/2$, often denoted by $J_{n/2}(z)$, has an integral representation

$$J_{n/2}(z) = \left(\frac{z}{2}\right)^{n/2}\frac{1}{\Gamma(\frac{n+1}{2})\sqrt{\pi}}\int_{-1}^{1}\big(1-t^2\big)^{\frac{n-1}{2}}\cos(zt)\,dt$$

(see, for example, [20]). We can use this integral formula to obtain

$$\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\big(1-y_1^2\big)^{(n-1)/2}\cos(\varepsilon r y_1)\,dy_1 \\ = (\varepsilon r/2)^{-n/2}\Gamma\left(\frac{n}{2}+1\right)J_{n/2}(\varepsilon r).$$

Thus, we have derived the characteristic function

$$\varphi_Z(u) = \left(\frac{2}{\varepsilon|u|}\right)^{n/2}\Gamma\left(\frac{n}{2}+1\right)J_{n/2}\big(\varepsilon|u|\big) \tag{A.6}$$

for all $u \in \mathbb{R}^n$. Spherical Bessel functions have a connection to our calculations in dimension $n = 1$, since one could show that

$$\frac{\sin z}{z} = \sqrt{\frac{\pi}{2z}}J_{\frac{1}{2}}(z)$$

holds for all $z \in \mathbb{R}$.

It is possible to express $J_{n/2}(z)$ as a product of factors such that each factor vanishes at one of the zeros of $z^{-n/2}J_{n/2}(z)$. Denote the zeros of the function $z^{-n/2}J_{n/2}(z)$ by $\pm j_{n/2,1}, \pm j_{n/2,2}, \pm j_{n/2,3}, \dots$ with $j_{n/2,l} > 0$ for all $l = 1, 2, \dots$ and $j_{n/2,1} \le j_{n/2,2} \le j_{n/2,3} \le \cdots$. Then, we have the infinite product formula of the Bessel function

$$J_{n/2}(z) = \left(\frac{z}{2}\right)^{n/2}\frac{1}{\Gamma(\frac{n}{2}+1)}\prod_{l=1}^{\infty}\left(1-\frac{z^2}{j_{n/2,l}^2}\right) \tag{A.7}$$

(see [20], pages 497–498). The number of zeros of $z^{-n/2}J_{n/2}(z)$ between the origin and the point

$$l_m := m\pi + \frac{\pi}{4}(n+1)$$

is exactly m for all m big enough (see [20], pages 495–497). Consequently, the infinite sum $\sum_{l=1}^{\infty} j_{n/2,l}^{-2}$ converges, since

$$\sum_{l=p}^{\infty} j_{n/2,l}^{-2} \le \sum_{l=p}^{\infty}\left(\frac{1}{(l-1)\pi + \pi/4(n+1)}\right)^2 < \infty$$

for some p big enough. Therefore, the infinite product in the formula (A.7) is well defined for all $z \in \mathbb{R}$.

Via independence we have

$$\varphi_{\sum_{i=1}^{k} Z_i}(u) = \big(\varphi_Z(u)\big)^k,$$

and the inversion formula together with the characteristic function (A.6), the infinite product formula (A.7) and a change of variables $z = \epsilon u$ yield

$$\begin{aligned} f_k(0) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \big(\varphi_Z(u)\big)^k \, du \\ &= \frac{1}{(2\pi)^n \epsilon^n} \int_{B_s(0)} \left[\prod_{l=1}^{\infty} \left(1 - \frac{|z|^2}{j_{n/2,l}^2} \right) \right]^k dz \\ &\quad + \frac{2^{kn/2} \Gamma(\frac{n}{2}+1)^k}{(2\pi)^n \epsilon^n} \int_{\mathbb{R}^n \setminus B_s(0)} \left(\frac{1}{|z|} \right)^{kn/2} \big(J_{n/2}(|z|)\big)^k \, dz \end{aligned}$$

for all $s > 0$.

Now, the function

$$1 - \frac{|z|^2}{j_{n/2,l}^2} \geq 0$$

for all $l \geq 1$, if $0 \leq |z| \leq j_{n/2,1}$. In addition, since $1 - z \leq e^{-z}$ for all $z \in \mathbb{R}$, we have

$$\begin{aligned} &\frac{1}{(2\pi)^n \epsilon^n} \int_{B_{j_{n/2,1}}(0)} \left[\prod_{l=1}^{\infty} \left(1 - \frac{|z|^2}{j_{n/2,l}^2} \right) \right]^k dz \\ &\quad \leq \frac{|S_1^{n-1}|}{(2\pi)^n \epsilon^n} \int_0^{j_{n/2,1}} e^{-r^2 k \sum_{l=1}^{\infty} j_{n/2,l}^{-2}} r^{n-1} \, dr \\ &\quad \leq \frac{|S_1^{n-1}|}{(2\pi)^n \epsilon^n} \int_0^{\infty} e^{-r^2 k \sum_{l=1}^{\infty} j_{n/2,l}^{-2}} r^{n-1} \, dr. \end{aligned}$$

Hence, we can integrate with a change of variables $r = (k \sum_{l=1}^{\infty} j_{n/2,l}^{-2})^{-1/2} t$ to obtain

$$\begin{aligned} &\frac{|S_1^{n-1}|}{(2\pi)^n \epsilon^n} \int_0^{\infty} e^{-r^2 k \sum_{l=1}^{\infty} j_{n/2,l}^{-2}} r^{n-1} \, dr \\ &\quad = \frac{n \int_0^{\infty} e^{-t^2} t^{n-1} \, dt}{\Gamma(\frac{n}{2}+1) \pi^{n/2} 2^n (\sum_{l=1}^{\infty} j_{n/2,l}^{-2})^{n/2}} \left(\frac{1}{\sqrt{k}\epsilon} \right)^n. \end{aligned}$$

Thus, there is a constant

$$c_n^1 := \frac{n \int_0^{\infty} e^{-t^2} t^{n-1} \, dt}{\Gamma(\frac{n}{2}+1) \pi^{n/2} 2^n (\sum_{l=1}^{\infty} j_{n/2,l}^{-2})^{n/2}} > 0$$

such that

$$\frac{1}{(2\pi)^n\epsilon^n}\int_{B_{j_{n/2,1}}(0)}\left[\prod_{l=1}^{\infty}\left(1-\frac{|z|^2}{j_{n/2,l}^2}\right)\right]^k dz \le c_n^1\left(\frac{1}{\sqrt{k}\epsilon}\right)^n.$$

It holds $J_{n/2}(|z|) \le 1$ for all $z \in \mathbb{R}^n$ (see, for example, [20]). Therefore, we get by a direct calculus

$$\begin{aligned}
&\frac{2^{kn/2}\Gamma(\frac{n}{2}+1)^k}{(2\pi)^n\epsilon^n}\int_{\mathbb{R}^n\setminus B_{j_{n/2,1}}(0)}\left(\frac{1}{|z|}\right)^{kn/2}\big(J_{n/2}(|z|)\big)^k\,dz\\
&\le \frac{|S_1^{n-1}|2^{kn/2}\Gamma(\frac{n}{2}+1)^k}{(2\pi)^n\epsilon^n}\int_{j_{n/2,1}}^{\infty} r^{n-1-kn/2}\,dr\\
&= \frac{(j_{n/2,1})^n}{2^{n-1}\Gamma(\frac{n}{2}+1)\pi^{n/2}\epsilon^n}\left(\frac{1}{k-2}\right)\left(\left(\frac{2}{j_{n/2,1}}\right)^{n/2}\Gamma\left(\frac{n}{2}+1\right)\right)^k
\end{aligned}$$

for all $k > 2$. There exists the following lower bound for the first zero $j_{v,1}$ (see [5] and for example [3])

$$j_{v,1} > v + \frac{\pi+1}{2} > v+2$$

for all $v > -\frac{1}{2}$. Thus, if n is even, $n = 2h$ for some $h \ge 1$, we get

$$\frac{\Gamma(h+1)2^h}{(j_{h,1})^h} < \frac{h!2^h}{(h+2)^h} < 1.$$

Similarly, if n is odd, $n = 2h+1$ for some $h \ge 0$, we get

$$\frac{\Gamma(h+\frac{3}{2})2^{h+1/2}}{(j_{h+1/2,1})^{h+1/2}} < \frac{(2h+2)!2^h\sqrt{2\pi}}{4^{h+1}(h+1)!(h+2.5)^{h+1/2}} < 1.$$

Hence, there exists a constant $k_0 := k_{0,n} > 2$ such that

$$\left(\frac{1}{k-2}\right)\left(\left(\frac{2}{j_{n/2,1}}\right)^{n/2}\Gamma\left(\frac{n}{2}+1\right)\right)^k \le \left(\frac{1}{\sqrt{k}}\right)^n$$

for all $k \ge k_0$. Denote

$$c_n^2 := \frac{(j_{n/2,1})^n}{2^{n-1}\Gamma(\frac{n}{2}+1)\pi^{n/2}} > 0$$

and

$$C_n := 2\max\{c_n^1, c_n^2\}.$$

Thus, we have derived the estimate

$$f_k(0) \le C_n \left(\frac{1}{\sqrt{k}\epsilon} \right)^n \tag{A.8}$$

for all $k \ge k_0$.

Let $k \ge k_0$. Theorem A.1 implies that there is a constant $C_1 := C_{1,n} > 0$ big enough such that for all $\epsilon > 0$

$$\mathbb{P}\left(\left| \sum_{i=1}^{k} Z_i \right| < C_1 \sqrt{k}\epsilon \right) \ge 0.99.$$

By using the convolution formula, we have that

$$\begin{aligned} f_k(x) &= \int_{\mathbb{R}^n} f_{k-1}(x-y)\chi_{B_\epsilon(0)}(y)\,dy = \int_{B_\epsilon(0)} f_{k-1}(x-y)\,dy \\ &= \int_{B_\epsilon(x)} f_{k-1}(y)\,dy \end{aligned} \tag{A.9}$$

holds for all $x \in \mathbb{R}^n$. The function f_1 is a decreasing radial function. Thus, we can deduce by using the formula (A.9) that f_2 is also a decreasing radial function, and by induction f_k as well. Therefore, we can denote the density f_k as a function of the radius $|u|$ for all $u \in \mathbb{R}^n$, and we have for any $C_* \in]0, C_1[$

$$f_k(0)\left|B_{C_*\sqrt{k}\epsilon}(0)\right| + f_k(C_*\sqrt{k}\epsilon)\left(\left|B_{C_1\sqrt{k}\epsilon}(0)\right| - \left|B_{C_*\sqrt{k}\epsilon}(0)\right|\right) \ge 0.99.$$

This inequality yields

$$\begin{aligned} f_k(C_*\sqrt{k}\epsilon) &\ge \frac{0.99 - f_k(0)|B_{C_*\sqrt{k}\epsilon}(0)|}{|B_{C_1\sqrt{k}\epsilon}(0)| - |B_{C_*\sqrt{k}\epsilon}(0)|} \\ &\ge \frac{0.99}{|B_{C_1\sqrt{k}\epsilon}(0)|} - f_k(0)\left(\frac{C_*}{C_1}\right)^n. \end{aligned}$$

Now, we use the estimate (A.8) to obtain

$$\begin{aligned} f_k(C_*\sqrt{k}\epsilon) &\ge \frac{0.99}{|B_{C_1\sqrt{k}\epsilon}(0)|} - C_n\left(\frac{C_*}{C_1\sqrt{k}\epsilon}\right)^n \\ &= \left(\frac{1}{C_1}\right)^n \left(\frac{0.99}{\omega_n} - C_n(C_*)^n\right)\left(\frac{1}{\sqrt{k}\epsilon}\right)^n. \end{aligned}$$

Thus, we have proven the following lemma.

Lemma A.4. *Let $\epsilon > 0$ and let Z be distributed according to the uniform distribution in the ball $B_\epsilon(0) \subset \mathbb{R}^n$. For any $k \ge 2$, denote the density of the random variable $\sum_{i=1}^{k} Z_i$ by f_k, where* *the random variables* Z_i, $i \in \{1, \dots, k\}$, *are independent and distributed as* Z. *Then* f_k *is a decreasing radial function*, *and there exist universal constants* $k_0 := k_{0,n} > 2$, $C_1 := C_{1,n} > 0$ *and* $C_n > 0$ *such that for all* $k \geq k_0$ *and* $C_* \in]0, C_1[$ *we have*

$$f_k(C_*\sqrt{k}\epsilon) \geq \left(\frac{1}{C_1}\right)^n \left(\frac{0.99}{\omega_n} - C_n(C_*)^n\right)\left(\frac{1}{\sqrt{k}\epsilon}\right)^n. \tag{A.10}$$

Observe that

$$f_k(C_*\sqrt{k}\epsilon) \geq \zeta \left(\frac{1}{\sqrt{k}\epsilon}\right)^n$$

for some $\zeta := \zeta_n > 0$, if we choose $C_* > 0$ so small that

$$C_* < \left(\frac{0.99}{\omega_n C_n}\right)^{1/n}.$$

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A continuous time tug-of-war game for parabolic $p(x,t)$-Laplace type equations

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A CONTINUOUS TIME TUG-OF-WAR GAME FOR PARABOLIC $p(x,t)$-LAPLACE TYPE EQUATIONS

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ABSTRACT. We formulate a stochastic differential game in continuous time that represents the unique viscosity solution to a terminal value problem for a parabolic partial differential equation involving the normalized $p(x,t)$-Laplace operator. Our game is formulated in a way that covers the full range $1 < p(x,t) < \infty$. Furthermore, we prove the uniqueness of viscosity solutions to our equation in the whole space under suitable assumptions.

1. Introduction

In this paper, we study a two-player zero-sum stochastic differential game (SDG) that is defined in terms of an n-dimensional state process, and is driven by a $2n$-dimensional Brownian motion for $n \geq 2$. The players' impacts on the game enter in both a diffusion and a drift coefficient of the state process. The game is played in $\mathbb{R}^n$ until a fixed time $T > 0$, and at that time a player pays the other player the amount given by a pay-off function g at a current point. We show that the game has a value, and characterize the value function of the game as a viscosity solution u to a parabolic terminal value problem

$$\begin{cases} \partial_t u(x,t) + \triangle^N_{p(x,t)} u(x,t) + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i}(x,t) = ru(x,t) & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

for $\mu \in \mathbb{R}^n$ and $r \geq 0$. Moreover, we show that the viscosity solution u is unique under suitable assumptions. Here, the normalized $p(x,t)$-Laplacian is defined as

$$\triangle^N_{p(x,t)} u(x,t) := \left(\frac{p(x,t)-2}{|Du(x,t)|^2} \right) \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) \frac{\partial u}{\partial x_i}(x,t) \frac{\partial u}{\partial x_j}(x,t) + \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x,t)$$

for $x \in \mathbb{R}^n$ and $t \in (0,T)$, provided that $Du(x,t) \neq 0$. The vector $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)^T$ is the gradient with respect to x, and the function

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$p : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is Lipschitz continuous with values on a compact set $[p_{\min}, p_{\max}]$ for constants $1 < p_{\min} \le p_{\max} < \infty$.

This work is motivated by a connection between p-harmonic functions and a stochastic game called tug-of-war, see the seminal papers [PSSW09, PS08, MPR12] in the elliptic case and [MPR10] in the parabolic case. Furthermore, Atar and Budhiraja [AB10] formulated a game in continuous time representing the unique viscosity solution to a certain elliptic inhomogeneous problem with the normalized ∞-Laplacian. The contribution of our work is the identification of a game in continuous time that corresponds to the parabolic normalized $p(x,t)$-Laplace operator. Moreover, our game covers the full range $1 < p(x,t) < \infty$. In the game formulation, we increased the dimension of the Brownian motion that drives our state process to let p also get values below two. This approach is new even for constant p.

In this work, the main difficulties arise from the variable dependence in p and from the unboundedness of the game domain. It is simpler to approximate viscosity solutions and to prove comparison principles to our equations without the variable dependence in p. Furthermore, we overcome the loss of translation invariance on the SDG by utilizing the Hölder continuity of solutions to Bellman-Isaacs type equations. Because the game domain is unbounded, we need to eliminate solutions growing too fast when $|x| \to \infty$. We show that under a linear growth bound a viscosity solution to our equation is unique.

1.1. **SDG formulation.** We fix a time $T > 0$, and model $X(t), t \in [0, T]$ by a stochastic differential equation

$$\begin{cases} dX(s) &= \rho\big(G(s)\big)\, ds + \sigma\big(X(s), G(s)\big)\, d\overline{W}(s) \\ X(0) &= x, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$, and $\overline{W}$ is a $2n$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ satisfying the standard assumptions. In our model, there are two competing players. We let

$$G(s) = \big(a(s), b(s), c(s), d(s)\big),$$

where

$$a(s), b(s) \in \mathbb{S}^{n-1},\ c(s), d(s) \in [0, \infty),\ s \in [0, T]$$

are progressively measurable stochastic processes with respect to the filtration $\{\mathcal{F}_s\}$. Throughout the paper, $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$. The pairs $\big(a(s), c(s)\big)$ and $\big(b(s), d(s)\big)$ are called controls of the players. Roughly speaking, $a(s)$ and $b(s)$ are the directions, and $c(s)$ and $d(s)$ are the lengths taken by the players at the time s. Furthermore, let $\mu \in \mathbb{R}^n$. Then, for $s \in [0, T]$, we define the function ρ in (1.1) by

$$\rho\big(G(s)\big) = \mu + \big(c(s) + d(s)\big)\big(a(s) + b(s)\big).$$

Recall that $p : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a Lipschitz continuous function taking values on the compact set $[p_{\min}, p_{\max}]$. We define the $n \times 2n$ matrix σ in (1.1) to be

$$\begin{aligned}&\sigma\big(X(s), G(s)\big)\\ &= \Big[a(s)\sqrt{p\big(X(s),s\big)-1};\quad P^{\perp}_{a(s)};\quad b(s)\sqrt{p\big(X(s),s\big)-1};\quad P^{\perp}_{b(s)}\Big],\end{aligned}$$

where the $n \times (n-1)$ matrices $P^{\perp}_{a(s)}$ and $P^{\perp}_{b(s)}$ are defined such that the matrices

$$P^{\perp}_{a(s)}\big(P^{\perp}_{a(s)}\big)^T \text{ and } P^{\perp}_{b(s)}\big(P^{\perp}_{b(s)}\big)^T$$

are projections to the $(n-1)$-dimensional hyperspaces orthogonal to the vectors $a(s)$ and $b(s)$ at the time s, respectively. For more details on σ, see Section 2 below.

We only allow players to use admissible controls. Roughly speaking, a player initially declares a bound $C < \infty$, and then plays as to keep $c(s) \le C$ for all s, where $\big(a(s), c(s)\big)$ is the admissible control of the player.

Definition 1.1. *Given a control $A := \big(a(s), c(s)\big)$, that is, a progressively measurable process with respect to the Brownian filtration $\{\mathcal{F}_s\}$ with $a(s) \in \mathbb{S}^{n-1}$, $c(s) \in [0,\infty)$, and $s \in [0,T]$, we set*

$$\Lambda(A) = \operatorname*{ess\,sup}_{\omega\in\Omega}\ \sup_{s\in[0,T]} c(s) \in [0,\infty]. \tag{1.2}$$

Then, we define the set of admissible controls by

$$\mathcal{AC} = \{A \text{ control} : \Lambda(A) < \infty\}.$$

Given an admissible control A, we say that the compact set $\mathbb{S}^{n-1} \times [0,\Lambda(A)]$ is an action set. A strategy is a response to the control of the opponent.

Definition 1.2. *A strategy is a function*

$$S : \mathcal{AC} \to \mathcal{AC}$$

such that for all $t \in [0,T]$, if

$$\mathbb{P}\big(A(s) = \tilde{A}(s) \text{ for a.e. } s \in [0,t]\big) = 1 \text{ and } \Lambda(A) = \Lambda(\tilde{A}),$$

then

$$\mathbb{P}\big(S(A)(s) = S(\tilde{A})(s) \text{ for a.e. } s \in [0,t]\big) = 1 \text{ and } \Lambda(S(A)) = \Lambda(S(\tilde{A})).$$

Given a strategy S, we set

$$\Lambda(S) := \sup_{A\in\mathcal{AC}} \Lambda(S(A)) \in [0,\infty]. \tag{1.3}$$

Then, we define the set of admissible strategies by

$$\mathcal{S} = \{S \text{ strategy} : \Lambda(S) < \infty\}.$$

We define the lower and upper values of the game with the dynamics (1.1) by

$$\begin{aligned} U^-(x,t) &= \inf_{S\in\mathcal{S}} \sup_{A\in\mathcal{AC}} \mathbb{E}\Big[e^{-r(T-t)}g\big(X(T)\big)\Big], \\ U^+(x,t) &= \sup_{S\in\mathcal{S}} \inf_{A\in\mathcal{AC}} \mathbb{E}\Big[e^{-r(T-t)}g\big(X(T)\big)\Big] \end{aligned} \tag{1.4}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where $r \geq 0$, and g is the pay-off function. The game starts at a position x at a time t, and the expectation $\mathbb{E}$ is taken with respect to the measure $\mathbb{P}$. The game is said to have a value at (x,t), if it holds $U^-(x,t) = U^+(x,t)$.

1.2. **Statement of the main results.** Let us denote

$$\begin{aligned} &F\big((x,t), u(x,t), Du(x,t), D^2u(x,t)\big) \\ &:= \triangle^N_{p(x,t)} u(x,t) + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i}(x,t) - ru(x,t) \end{aligned}$$

for all $(x,t) \in \mathbb{R}^n \times (0,T)$, where D^2u is the matrix consisting of the second order derivatives with respect to x. We consider the terminal value problem

$$\begin{cases} \partial_t u + F\big((x,t), u, Du, D^2u\big) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{1.5}$$

where g is a positive, bounded and Lipschitz continuous function. A common notion of a weak solution to this equation is a viscosity solution. In this paper, we prove the following main result.

Theorem 1.3. *Let g be positive, bounded and Lipschitz continuous. Furthermore, let U^- and U^+ be the lower and upper values of the stochastic differential game defined in (1.4), respectively. Then, the functions U^- and U^+ are viscosity solutions to (1.5).*

For completeness, we show that a viscosity solution to (1.5) is unique under suitable assumptions.

Theorem 1.4. *Let g be positive, bounded and Lipschitz continuous. Then, a viscosity solution u to the equation (1.5) is unique, if u satisfies a linear growth bound*

$$|u(x,t)| \leq c(1+|x|) \tag{1.6}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$ and for $c < \infty$ independent of x,t.

Because g is bounded, the functions U^- and U^+ satisfy (1.6). Thus, Theorems 1.3 and 1.4 imply the following.

Corollary 1.5. *The game has a value at every $(x,t) \in \mathbb{R}^n \times [0,T]$.*

As an application, one could study our model in the context of the portfolio option pricing. This would be based on the idea that, in addition to a random noise, the prices of the underlying assets are influenced by the two competing players. Roughly speaking, one can see the players as the issuer and the holder of the corresponding option. The issuer and the holder try, respectively, to manipulate the drifts and the volatilities of the assets to minimize and maximize, respectively, the expected discounted reward at the time T. The time T can be interpreted as a maturity; it is the time on which the corresponding financial instrument must either be renewed or it will cease to exist. To some extent, we generalize the model developed by Nyström and Parviainen in [NP17]. Indeed, our contribution is the introduction of a local volatility p. The volatility of an asset may vary over the space and the time.

1.3. **An outline of the proofs of Theorems 1.3 and 1.4.** Our approach is influenced by the papers [Swi96, AB10, NP17]. First, we examine games with uniformly bounded action sets, and in the end, let the uniform bound tend toward infinity. Here, the important step is to connect the value functions under uniformly bounded action sets to the terminal value problems of Bellman-Isaacs type equations

$$\begin{cases} \partial_t u - F_m^-\big((x,t), u, Du, D^2u\big) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{1.7}$$

and

$$\begin{cases} \partial_t u - F_m^+\big((x,t), u, Du, D^2u\big) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.8}$$

The exact definitions of F_m^- and F_m^+ are given in Section 2 below. Here, m denotes the uniform bound on the controls. The uniqueness of viscosity solutions to (1.7) and (1.8) follows, for example, from [GGIS91, BL08]. Furthermore, the existence of viscosity solutions to the equations (1.7) and (1.8) follows by the construction of suitable barriers (Lemma 2.2) and by the use of Perron's method.

In Section 3, the main result is Lemma 3.3 in which we show that a lower value function with uniformly bounded action sets equals to the unique solution u_m to (1.7). In the proof, we first regularize the solution u_m by sup- or inf-convolutions depending on which direction in the equality we aim to prove, and then we mollify u_m by the standard mollifier. Finally, we deduce the result by utilizing Itô's formula to the regularized solution and passing to limits.

In Section 4, we examine the problem (1.5). First, we prove Theorem 1.4. To prove a comparison principle, we double the variables and apply the celebrated theorem of sums, see [CI90]. Because we only consider solutions satisfying a linear growth bound in the whole space, we utilize a quadratic barrier function for all large x. Furthermore, we use the Lipschitz continuity of p to estimate the error coming from a penalty function. To continue, in Lemma 4.5 we show that

$$F_m^- \to F$$

as $m \to \infty$. Furthermore, in Lemma 4.6 we utilize the results of [KS80, Wan92] to show that the family

$$\{u_m : m \geq 1\}$$

is equicontinuous. Finally by the reduction of test functions (Lemma 4.4) and the stability principle for viscosity solutions, we can utilize the Arzelà-Ascoli theorem to find a solution u to (1.5) and a subsequence (u_{m_j}) converging uniformly to u as $j \to \infty$. To complete the proof of Theorem 1.3, we also need the fact that the subsequence of the corresponding lower value functions converges to the lower value function for the game without the uniform bound on the controls. In addition, all the proofs in the context of the equation (1.8) are analogous.

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2. Preliminaries

Let $\overline{W} = (W^1, W^2)^T$ be a $2n$-dimensional Brownian motion such that $W^1 = (W_1^1, \dots, W_n^1)$ and $W^2 = (W_1^2, \dots, W_n^2)$ are n-dimensional Brownian motions. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ denote a complete filtered probability space with a right-continuous filtration supporting the process $\overline{W}$. As mentioned above, we consider the following stochastic differential equation

$$\begin{cases} dX(s) &= \rho\big(G(s)\big)\, ds + \sigma\big(X(s), G(s)\big)\, d\overline{W}(s) \\ X(0) &= x \end{cases} \tag{2.9}$$

for $s \in [0, T]$, $T > 0$ and $x \in \mathbb{R}^n$ with $G : [0, T] \to \mathcal{CS}$, $\rho : \mathcal{CS} \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathcal{CS} \to M^{n\times 2n}$. Here, we define $\mathcal{CS} := \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times [0, \infty) \times [0, \infty)$, where $\mathcal{CS}$ refers to control space. Furthermore, $M^{n\times 2n}$ is the set of $n \times 2n$ matrices.

We are interested in the following form of the functions G, ρ and σ. Let $A_1 := \big(a(s), c(s)\big)$ and $A_2 := \big(b(s), d(s)\big)$ be admissible controls of the players in the sense of Definition 1.1, respectively. Furthermore, let $\mu \in \mathbb{R}^n$. Then, for $s \in [0, T]$, we define

$$G(s) = \big(a(s), b(s), c(s), d(s)\big),$$

and

$$\rho\big(G(s)\big) = \mu + \big(c(s) + d(s)\big)\big(a(s) + b(s)\big).$$

Let $\nu \in \mathbb{S}^{n-1}$, and denote the orthogonal complement of ν by

$$\nu^{\perp} := \{z \in \mathbb{R}^n : \langle z, \nu \rangle = 0\}.$$

We set $P_\nu^\perp$ to be a $n\times(n-1)$ matrix such that the columns are $p_\nu^1,\dots,p_\nu^{n-1}$, where $\{p_\nu^1,\dots,p_\nu^{n-1}\}$ is a fixed orthonormal basis of $\nu^\perp$,

$$P_\nu^\perp = \begin{bmatrix} p_\nu^1 & \cdots & p_\nu^{n-1}\end{bmatrix}.$$

We can define the basis of $\nu^\perp$ in a way that the function $\nu\mapsto P_\nu^\perp$ is continuous. In addition, let $p:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ be a Lipschitz continuous function such that

$$p_{\min} = \inf_{y\in\mathbb{R}^n\times[0,T]} p(y) > 1 \text{ and } p_{\max} = \sup_{y\in\mathbb{R}^n\times[0,T]} p(y) < \infty. \tag{2.10}$$

With respect to the time variable t, we only need that p is Hölder continuous for all fixed x, but we avoid further technical difficulties. Now, we define the $n\times 2n$ matrix σ to be

$$\begin{aligned}&\sigma\big(X(s),G(s)\big)\\ &= \Big[a(s)\sqrt{p\big(X(s),s\big)-1};\quad P_{a(s)}^\perp;\quad b(s)\sqrt{p\big(X(s),s\big)-1};\quad P_{b(s)}^\perp\Big].\end{aligned}$$

By the game dynamics (2.9), we get

$$\begin{aligned} dX_i(s) = &\Big[\mu_i + \big(c(s)+d(s)\big)\big(a_i(s)+b_i(s)\big)\Big]\,ds\\ &+\sqrt{p\big(X(s),s\big)-1}\Big(a_i(s)\,dW_1^1(s)+b_i(s)\,dW_1^2(s)\Big)\\ &+\sum_{k=2}^n\big(\vec{p}^{\,i}_{a(s)}\big)_{k-1}\,dW_k^1(s)+\sum_{k=2}^n\big(\vec{p}^{\,i}_{b(s)}\big)_{k-1}\,dW_k^2(s)\end{aligned}\tag{2.11}$$

for all $i\in\{1,\dots,n\}$. Here, $\big(\vec{p}^{\,i}_\nu\big)$ denotes the i-th row vector of $P_\nu^\perp$.

By a strong solution to the stochastic differential equation (2.9), we mean a progressively measurable process $(X(l))$ with respect to the Brownian filtration $\{\mathcal{F}_l\}$ such that $X(l)$ coincides with the right-hand side of (2.9) for all $l\in[0,T]$ almost surely. Moreover, a solution is pathwise unique, if any two given solutions $\big(X(l),Y(l)\big)$ satisfy

$$\mathbb{P}\big(\sup_{l\in[0,T]}|X(l)-Y(l)|>0\big)=0.$$

Let us denote by $|\cdot|_F$ the Frobenius norm

$$||\sigma||_F := \sqrt{\operatorname{trace}(\sigma\sigma^T)}$$

for all $\sigma\in M^{n\times 2n}$. Then by (2.10), it holds

$$\mathbb{E}\int_0^T ||\sigma\big(X(l),G(l)\big)||_F^2\,dl \le 2T(p_{\max}-2+n)<\infty. \tag{2.12}$$

Hence, the stochastic integral in the right-hand side of (2.9) is well defined. Because the controls of the players are admissible, it holds

$$\mathbb{E}\int_0^T \Big|\rho\big(G(s)\big)\Big|^2\,ds \le \big(|\mu|+2(\Lambda(A_1)+\Lambda(A_2))\big)^2 T<\infty \tag{2.13}$$

for $\Lambda(A_1), \Lambda(A_2) < \infty$, where $\Lambda(\cdot)$ is defined in (1.2). Furthermore, the functions ρ and σ are continuous with respect to the controls. Moreover, we can estimate

$$\begin{aligned}||\sigma\big(x, G(t)\big) - \sigma\big(y, G(t)\big)||_F &\leq \sqrt{2}\big|\sqrt{p(x,t)-1} - \sqrt{p(y,t)-1}\big| \\ &\leq \frac{|p(x,t)-p(y,t)|}{\sqrt{2p_{\min}-2}} \\ &\leq \frac{L_p}{\sqrt{2p_{\min}-2}}|x-y|\end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and $t \in [0,T]$ with L_p denoting the Lipschitz constant of p. Therefore by combining this, (2.12), (2.13) and [Kry09, Theorem 2.5.7], the SDE (2.9) admits a pathwise unique strong solution.

Throughout, we denote by $||\cdot||$ the matrix norm

$$||M|| := \sup_{|x|=1} \big|\langle Mx, x\rangle\big|$$

for all $n\times n$ matrices M. Furthermore, $S(n)$ denotes the set of all symmetric $n \times n$ matrices, I is the $n \times n$ identity matrix, and for $\xi \in \mathbb{R}^n$, we denote by $\xi\otimes\xi$ the $n\times n$ matrix for which $(\xi\otimes\xi)_{ij} = \xi_i\xi_j$. A function $\zeta : [0,\infty) \to [0,\infty)$ is said to be a modulus, if it is continuous, nondecreasing, and satisfies $\zeta(0) = 0$.

2.1. **Viscosity solutions to Bellman-Isaacs equations with uniformly bounded action sets.** We define $\Phi : \mathcal{CS} \times \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ through

$$\Phi\big(a, b, c, d; (x,t), \nu, M\big) = -\operatorname{trace}\left(\mathcal{A}_{a,b}^{(x,t)}M\right) - (c+d)\langle a+b, \nu\rangle - \langle \mu, \nu\rangle,$$

where

$$\mathcal{A}_{a,b}^{(x,t)} := \frac{1}{2}\big(p(x,t)-2\big)(a\otimes a + b\otimes b) + I. \tag{2.14}$$

Observe that the matrix $\mathcal{A}_{a,b}^{(x,t)}$ is symmetric with eigenvalues between the values

$$\lambda := \min\{1, p_{\min}-1\} \text{ and } \Lambda := \max\{1, p_{\max}-1\}. \tag{2.15}$$

Given $m \in \{1, 2, \dots\}$, we let

$$\mathcal{H}_m := \mathbb{S}^{n-1} \times [0, m],$$

and define $F_m^-, F_m^+ : \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ through

$$\begin{aligned}F_m^-\big((x,t), \xi, \nu, M\big) &= \inf_{(a,c)\in\mathcal{H}_m} \sup_{(b,d)\in\mathcal{H}_m} \Phi\big(a,b,c,d;(x,t),\nu,M\big) + r\xi, \\ F_m^+\big((x,t), \xi, \nu, M\big) &= \sup_{(b,d)\in\mathcal{H}_m} \inf_{(a,c)\in\mathcal{H}_m} \Phi\big(a,b,c,d;(x,t),\nu,M\big) + r\xi\end{aligned}$$

for $r \geq 0$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive bounded Lipschitz function such that

$$\sup_{x \in \mathbb{R}^n} g(x) + \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|g(x) - g(y)|}{|x - y|} < L_g \tag{2.16}$$

for some $L_g < \infty$. We study terminal value problems

$$\begin{cases} \partial_t u - F_m^-\big((x,t), u, Du, D^2u\big) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{2.17}$$

and

$$\begin{cases} \partial_t u - F_m^+\big((x,t), u, Du, D^2u\big) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{2.18}$$

A common notion of weak solutions to these equations is viscosity solutions. We only consider solutions u which satisfy a linear growth condition

$$|u(x,t)| \leq c(1 + |x|) \tag{2.19}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$ and for some $c < \infty$ independent of x, t. We prove that there exists a unique viscosity solution to the equation (2.17) satisfying the condition (2.19). We omit the proof for (2.18), because it is analogous. The proofs are based on the comparison principle and Perron's method.

Definition 2.1. *(i) A lower semicontinuous function $\overline{u}_m : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a viscosity supersolution to (2.17), if it satisfies (2.19),*

$$\overline{u}_m(x,T) \geq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for all $\phi \in C^{2,1}\big(\mathbb{R}^n \times (0,T)\big)$ such that

- $\overline{u}_m(x_0,t_0) = \phi(x_0,t_0)$
- $\overline{u}_m(x,t) > \phi(x,t)$ *for all* $(x,t) \neq (x_0,t_0)$

it holds

$$\partial_t \phi(x_0,t_0) - F_m^-\big((x_0,t_0), \phi(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)\big) \leq 0.$$

(ii) An upper semicontinuous function $\underline{u}_m : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a viscosity subsolution to (2.17), if it satisfies (2.19),

$$\underline{u}_m(x,T) \leq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for all $\phi \in C^{2,1}\big(\mathbb{R}^n \times (0,T)\big)$ such that

- $\underline{u}_m(x_0,t_0) = \phi(x_0,t_0)$
- $\underline{u}_m(x,t) < \phi(x,t)$ *for all* $(x,t) \neq (x_0,t_0)$

it holds

$$\partial_t\phi(x_0,t_0) - F_m^-\big((x_0,t_0),\phi(x_0,t_0),D\phi(x_0,t_0),D^2\phi(x_0,t_0)\big) \geq 0.$$

(iii) If a function $u_m : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a viscosity supersolution and a subsolution to (2.17), then u_m is a viscosity solution to (2.17).

Observe that we require the growth condition (2.19) as a standing assumption for viscosity super- and subsolutions. We start with the following lemma.

Lemma 2.2. *Let $y \in \mathbb{R}^n$, $0 < \varepsilon < 1$, and let L_g be the constant in (2.16) for g. Then, the functions*

$$\overline{a}(x,t) = g(y) + \frac{A}{\varepsilon^{1/2}}(T-t) + 2L_g\big(|x-y|^2+\varepsilon\big)^{1/2},$$

$$\underline{a}(x,t) = g(y) - \frac{A}{\varepsilon^{1/2}}(T-t) - 2L_g\big(|x-y|^2+\varepsilon\big)^{1/2}$$

are viscosity super- and subsolutions to (2.17), respectively, if we choose A, independent of y,ε and m, large enough.

Proof. Because g is Lipschitz continuous with (2.16), we get

$$\underline{a}(x,T) \leq g(x) \leq \overline{a}(x,T)$$

for all $x \in \mathbb{R}^n$. Furthermore, $\underline{a}$ and $\overline{a}$ satisfy (2.19). First, we prove that $\overline{a}$ is a supersolution. To establish this, since $\overline{a}$ is a smooth function, we need to show that

$$\partial_t\overline{a}(x,t) - F_m^-\big((x,t),\overline{a}(x,t),D\overline{a}(x,t),D^2\overline{a}(x,t)\big) \leq 0$$

for all $(x,t) \in \mathbb{R}^n \times (0,T)$. Let $(x,t) \in \mathbb{R}^n \times (0,T)$. By a direct calculation, it holds

$$D\overline{a}(x,t) = 2L_g\big(|x-y|^2+\varepsilon\big)^{-1/2}(x-y)$$

and

$$D^2\overline{a}(x,t) = 2L_g\big(|x-y|^2+\varepsilon\big)^{-1/2}\left(I - \frac{(x-y)\otimes(x-y)}{|x-y|^2+\varepsilon}\right).$$

Thus, we can estimate

$$\begin{aligned}
&-\operatorname{trace}\left(\mathcal{A}_{a,b}^{(x,t)}D^2\overline{a}(x,t)\right)\\
&= 2L_g\big(|x-y|^2+\varepsilon\big)^{-1/2}\Big\{\operatorname{trace}\left(\mathcal{A}_{a,b}^{(x,t)}\big(|x-y|^2+\varepsilon\big)^{-1}(x-y)\otimes(x-y)\right)\\
&\qquad\qquad - \operatorname{trace}\left(\mathcal{A}_{a,b}^{(x,t)}\right)\Big\}\\
&\geq -2n\Lambda L_g\big(|x-y|^2+\varepsilon\big)^{-1/2}
\end{aligned}$$

for all $a,b \in \mathbb{S}^{n-1}$. Furthermore, we have $\partial_t\overline{a}(x,t) = -A\varepsilon^{-1/2}$.

We can assume $x \neq y$, because otherwise the next term below is zero. It holds

$$
\begin{aligned}
&\inf_{(a,c)\in\mathcal{H}_m} \sup_{(b,d)\in\mathcal{H}_m} -(c+d)\big\langle a+b, D\overline{a}(x,t)\big\rangle \\
&\geq 2L_g\big(|x-y|^2+\varepsilon\big)^{-1/2} \inf_{(a,c)\in\mathcal{H}_m} -c\big\langle a-(x-y)/|x-y|, x-y\big\rangle \\
&\geq 0.
\end{aligned}
$$

In addition, we can estimate

$$
\Big|\big\langle \mu, D\overline{a}(x,t)\big\rangle\Big| \leq 2L_g|\mu||x-y|\big(|x-y|^2+\varepsilon\big)^{-1/2} \leq 2L_g|\mu|.
$$

By combining our estimates above, we have

$$
\begin{aligned}
&\partial_t\overline{a}(x,t) - F_m^-\big((x,t),\overline{a}(x,t),D\overline{a}(x,t),D^2\overline{a}(x,t)\big) \\
&\leq -A\varepsilon^{-1/2} + 2n\Lambda L_g\big(|x-y|^2+\varepsilon\big)^{-1/2} + 2L_g|\mu| - r\overline{a}(x,t) \\
&\leq \varepsilon^{-1/2}\big(-A+2n\Lambda L_g\big) + 2L_g|\mu|.
\end{aligned}
$$

Hence, if we choose

$$
A = 4L_g\big(n\Lambda + |\mu|\big),
$$

we can conclude that $\overline{a}$ is a supersolution to (2.17).

The proof that $\underline{a}$ is a subsolution to (2.17) is very similar to the above. We need to show that

$$
\partial_t\underline{a}(x,t) - F_m^-\big((x,t),\underline{a}(x,t),D\underline{a}(x,t),D^2\underline{a}(x,t)\big) \geq 0.
$$

Observe that for $x \neq y$, we have this time

$$
\begin{aligned}
&\inf_{(a,c)\in\mathcal{H}_m} \sup_{(b,d)\in\mathcal{H}_m} -(c+d)\big\langle a+b, D\overline{a}(x,t)\big\rangle \\
&\leq 2L_g\big(|x-y|^2+\varepsilon\big)^{-1/2} \sup_{(b,d)\in\mathcal{H}_m} -d\big\langle (x-y)/|x-y|+b, x-y\big\rangle \\
&\leq 0
\end{aligned}
$$

by estimating the infimum instead of the supremum. Thus, by repeating the argument above, we have

$$
\begin{aligned}
&\partial_t\underline{a}(x,t) - F_m^-\big((x,t),\underline{a}(x,t),D\underline{a}(x,t),D^2\underline{a}(x,t)\big) \\
&\geq \varepsilon^{-1/2}\big(A-2n\Lambda L_g\big) - 2L_g|\mu| - r\underline{a}(x,t).
\end{aligned}
$$

Recall the assumption (2.16) implying $-r\underline{a}(x,t) \geq -rL_g$. Therefore by adjusting the constant A large enough, we can conclude that $\underline{a}$ is a subsolution to (2.17). □

A useful tool for us is the comparison principle.

Lemma 2.3. *Let $\underline{u}_m$ and $\overline{u}_m$ be continuous viscosity sub- and supersolutions to (2.17) in the sense of Definition 2.1, respectively. Then, it holds*

$$
\underline{u}_m(x,t) \leq \overline{u}_m(x,t)
$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$.

The proof of the comparison principle can be found from [BL08], see also [GGIS91]. Now, Lemmas 2.2 and 2.3 applied to Perron's method yield the following result.

Proposition 2.4. *There exists a unique viscosity solution* u_m *to* (2.17) *in the sense of Definition 2.1.*

Observe that by comparison with a sufficiently large constant, the unique solution u_m to (2.17) is not merely of linear growth (2.19). It is even bounded.

3. The SDG with uniformly bounded action sets

In this section, we examine the game dynamics under uniform bounds on the action sets of the players. In particular, we prove that the unique solution to (2.17) equals the lower value function of the game under the uniform bound. For the upper value function, the proof is similar.

Definition 3.1. *Let* $\mathcal{AC}$ *be the set of admissible controls, and let* $\mathcal{S}$ *be the set of admissible strategies in the sense of Definitions 1.1 and 1.2, respectively. For* $m \in \{1,2,\dots\}$*, we set*

$$\mathcal{AC}_m := \{A \in \mathcal{AC} : \Lambda(A) \le m\},$$
$$\mathcal{S}_m := \{S \in \mathcal{S} : \Lambda(S) \le m\},$$

where $\Lambda(\cdot)$ *is defined in* (1.2) *and* (1.3).

Let $m \in \{1,2,\dots\}$, and assume that the players choose their controls and strategies from the sets $\mathcal{AC}_m$ and $\mathcal{S}_m$, respectively. As before, the SDE (2.9) admits a pathwise unique strong solution. We define the lower and upper value functions of the game with controls in $\mathcal{AC}_m$ and strategies in $\mathcal{S}_m$ by setting

$$\begin{aligned} U_m^-(x,t) &= \inf_{S\in\mathcal{S}_m} \sup_{A\in\mathcal{AC}_m} \mathbb{E}\Big[e^{-r(T-t)} g\big(X(T)\big)\Big], \\ U_m^+(x,t) &= \sup_{S\in\mathcal{S}_m} \inf_{A\in\mathcal{AC}_m} \mathbb{E}\Big[e^{-r(T-t)} g\big(X(T)\big)\Big] \end{aligned} \tag{3.20}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, where g is the pay-off (2.16). The game starts at x at a time t, and the expectation $\mathbb{E}$ is taken with respect to the measure $\mathbb{P}$.

In Lemma 3.5 below, we assume that the solution u_m to (2.17) is twice differentiable and that the solution and its derivatives of first and second order are Lipschitz continuous. Hence, we first study the so called sup- and inf-convolutions of the function u_m. In particular, for a large $j \in \mathbb{N}$, let us denote $T_j := T - j^{-1}$ and $R_j^n := \mathbb{R}^n \times [j^{-1}, T_j]$. Then for j fixed and $\varepsilon > 0$ small, we define

$$u_\varepsilon(x,t) = \sup_{(z,s)\in\mathbb{R}^n\times[0,T]} \Big(u_m(z,s) - \frac{1}{2\varepsilon}\big((t-s)^2 + |x-z|^2\big)\Big)$$

whenever $(x,t) \in R_j^n$. The sup-convolution u_ε has well-known properties. Indeed, u_ε is locally Lipschitz continuous, semiconvex and $u_\varepsilon \searrow u_m$ as $\varepsilon \to 0$, see for example [CIL92]. Moreover, u_ε yields a good approximation of u_m in the viscosity sense. The proof of the following lemma follows [Ish95], where they consider an elliptic case. For the benefit of the reader, we give the proof in our parabolic setting.

Lemma 3.2. *Let u_m be a viscosity solution to (2.17), and let u_ε be the sup-convolution of u_m. Then for ε small enough, it holds*

$$F_m^-\big((x,t), u_\varepsilon(x,t), Du_\varepsilon(x,t), D^2u_\varepsilon(x,t)\big) \le \partial_t u_\varepsilon(x,t) + \zeta(\varepsilon)$$

for a.e. $(x,t) \in R_j^n$ with a bounded modulus of continuity $\zeta(\varepsilon)$.

Proof. By the comparison principle and the assumption (2.16) on g, it holds $0 \le u_m \le L_g$. Therefore for all $(x,t) \in R_j^n$ and $\varepsilon > 0$ small enough, there exists a point $(x^*, t^*) \in \mathbb{R}^n \times]0,T[$, where the supremum used in the definition of u_ε is obtained. In particular, it holds

$$0 \le u_m(x,t) \le u_\varepsilon(x,t) \le L_g - \frac{1}{2\varepsilon}\big((t-t^*)^2 + |x-x^*|^2\big).$$

Hence, this yields $|t - t^*| < j^{-1}$, if $\varepsilon < 1/(2L_g j^2)$.

By the Lipschitz continuity and the semiconvexity of u_ε, it holds

$$\begin{aligned} u_\varepsilon(z,s) \le{}& u_\varepsilon(x,t) + \partial_t u_\varepsilon(x,t)(s-t) + \langle Du_\varepsilon(x,t), z-x\rangle \\ &+ \frac{1}{2}\big\langle D^2u_\varepsilon(x,t)(z-x), z-x\big\rangle + o\big(|s-t| + |z-x|^2\big) \end{aligned} \tag{3.21}$$

for a.e. $(x,t) \in R_j^n$ as $(z,s) \to (x,t)$, see [Jen88, Lemmas 3.3 and 3.15]. Here, we also applied the fundamental Aleksandrov's theorem for convex functions, see for example [EG92, Theorem 6.4.1]. Moreover, the estimate (3.21) implies that we can choose (x^*, t^*) such that

$$\begin{aligned} x^* &= x + \varepsilon Du_\varepsilon(x,t), \\ t^* &= t + \varepsilon\partial_t u_\varepsilon(x,t) \end{aligned} \tag{3.22}$$

for a.e. $(x,t) \in R_j^n$, see [CIL92, Lemma A.5] or [Kat15, Theorem 4.7]. Let $(x,t) \in R_j^n$ such that (3.21) holds. We define $v : R_j^n \to \mathbb{R}$ through

$$\begin{aligned} v(z,s) ={}& \partial_t u_\varepsilon(x,t)(s-t) + \langle Du_\varepsilon(x,t), z-x\rangle \\ &+ \frac{1}{2}\big\langle D^2u_\varepsilon(x,t)(z-x), z-x\big\rangle \end{aligned}$$

for $(z,s) \in R_j^n$. We want to find a local maximum of a function at (x^*,t^*,x,t) up to an error in order to use the parabolic theorem of sums. Because it holds $v(x,t)=0$ and

$$u_m(y,l) - \frac{1}{2\varepsilon}\big((l-s)^2 + |y-z|^2\big) \le u_\varepsilon(z,s)$$

for all $(z,s),(y,l) \in R_j^n$, we can estimate by (3.21)

$$\begin{aligned} &u_m(y,l) - v(z,s) - \frac{1}{2\varepsilon}\big((l-s)^2 + |y-z|^2\big) \\ &\le u_m(x^*,t^*) - v(x,t) - \frac{1}{2\varepsilon}\big((t-t^*)^2 + |x-x^*|^2\big) \\ &\quad + o\big(|s-t| + |z-x|^2\big) \end{aligned}$$

for any $(y,l) \in R_j^n$ as $(z,s) \to (x,t)$. By using this inequality, we can deduce

$$\begin{aligned} &u_m(y,l) - v(z,s) \\ &\le u_m(x^*,t^*) - v(x,t) + \frac{1}{\varepsilon}\langle x^*-x, y-x^*\rangle + \frac{1}{\varepsilon}(t^*-t)(l-t^*) \\ &\quad + \frac{1}{\varepsilon}\langle x-x^*, z-x\rangle + \frac{1}{\varepsilon}(t-t^*)(s-t) + \frac{1}{2\varepsilon}\big(|y-x^*|^2 + |z-x|^2\big) \\ &\quad - \frac{1}{\varepsilon}\langle y-x^*, z-x\rangle + o\big(|s-t| + |l-t^*| + |z-x|^2\big) \end{aligned} \tag{3.23}$$

for all $y \in \mathbb{R}^n$ as $(z,s,l) \to (x,t,t^*)$. This is true, because by direct calculations it holds

$$\begin{aligned} &\frac{1}{2\varepsilon}\big((l-s)^2 - (t-t^*)^2\big) \\ &= \frac{1}{2\varepsilon}\Big(\big(t-s+l-t^*\big)^2 - 2(t^*-t)^2 + 2(t^*-t)(l-s)\Big) \\ &\le \frac{1}{\varepsilon}(t^*-t)(l-t^*) + \frac{1}{\varepsilon}(t-t^*)(s-t) + o\big(|s-t| + |l-t^*|\big) \end{aligned}$$

as $(s,l) \to (t,t^*)$ and

$$\begin{aligned} &\langle x^*-x, y-x^*\rangle + \langle x-x^*, z-x\rangle + \frac{1}{2}\big(|y-x^*|^2 + |z-x|^2\big) - \langle y-x^*, z-x\rangle \\ &= \frac{1}{2}\big(|y-z|^2 + |x-x^*|^2\big) \end{aligned}$$

for all $y,z \in \mathbb{R}^n$.

For the following notation and the parabolic theorem of sums, we refer the reader to [CIL92], see also [Kat15]. By the estimate (3.23), it holds

$$\begin{aligned} &\left(\frac{1}{\varepsilon}(x^*-x), \frac{1}{\varepsilon}(t^*-t), \frac{1}{\varepsilon}(x-x^*), \frac{1}{\varepsilon}(t-t^*), \frac{1}{\varepsilon}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}\right) \\ &\in \mathcal{P}^{2,+}\Big(u_m(x^*,t^*) - v(x,t)\Big). \end{aligned}$$

Thus by [Kat15, Theorem 6.7], there exist symmetric matrices $Y := Y(\varepsilon)$ and $Z := Z(\varepsilon)$ such that

$$\left(\frac{1}{\varepsilon}(t^*-t), \frac{1}{\varepsilon}(x^*-x), Y\right) \in \overline{\mathcal{P}}^{2,+} u_m(x^*,t^*)$$
$$\left(\frac{1}{\varepsilon}(t^*-t), \frac{1}{\varepsilon}(x^*-x), Z\right) \in \overline{\mathcal{P}}^{2,-} v(x,t)$$

and

$$\begin{bmatrix} Y & 0 \\ 0 & -Z \end{bmatrix} \le \frac{3}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}. \tag{3.24}$$

Therefore, because u_m is a subsolution, this and (3.22) yield

$$F_m^-\big((x^*,t^*), u_m(x^*,t^*), Du_\varepsilon(x,t), Y\big) \le \partial_t u_\varepsilon(x,t). \tag{3.25}$$

Furthermore, since $D^2 v(x,t) = D^2 u_\varepsilon(x,t)$, the degenerate ellipticity of F_m^- implies

$$\begin{aligned} &F_m^-\big((x,t), u_\varepsilon(x,t), Du_\varepsilon(x,t), D^2u_\varepsilon(x,t)\big) \\ &\le F_m^-\big((x,t), u_\varepsilon(x,t), Du_\varepsilon(x,t), Z\big). \end{aligned}$$

By combining this and (3.25), the proof is complete, if we can show that there exists a modulus ζ such that

$$\begin{aligned} &F_m^-\big((x,t), u_\varepsilon(x,t), Du_\varepsilon(x,t), Z\big) \\ &\le F_m^-\big((x^*,t^*), u_m(x^*,t^*), Du_\varepsilon(x,t), Y\big) + \zeta(\varepsilon). \end{aligned} \tag{3.26}$$

We prove this inequality by utilizing (3.24).

Let $a, b \in \mathbb{S}^{n-1}$. We multiply from the left both sides in (3.24) by

$$\begin{bmatrix} \mathcal{A}_{a,b}^{(x^*,t^*)} & \mathcal{A}_{a,b}^{(x,t),(x^*,t^*)} \\ \mathcal{A}_{a,b}^{(x,t),(x^*,t^*)} & \mathcal{A}_{a,b}^{(x,t)} \end{bmatrix},$$

where

$$\mathcal{A}_{a,b}^{(x,t),(x^*,t^*)} := \frac{1}{2}\Big(\sqrt{p(x^*,t^*)-1}\sqrt{p(x,t)-1} - 1\Big)(a\otimes a + b\otimes b) + I,$$

and the matrices $\mathcal{A}_{a,b}^{(x,t)}$ and $\mathcal{A}_{a,b}^{(x^*,t^*)}$ are defined in (2.14). Then by taking traces and observing

$$\operatorname{trace}(a\otimes a + b\otimes b) = 2,$$

we get

$$\begin{aligned} &-\operatorname{trace}\big(\mathcal{A}_{a,b}^{(x,t)} Z\big) + \operatorname{trace}\big(\mathcal{A}_{a,b}^{(x^*,t^*)} Y\big) \\ &\le \frac{3}{\varepsilon}\Big(\operatorname{trace}\big(\mathcal{A}_{a,b}^{(x^*,t^*)} + \mathcal{A}_{a,b}^{(x,t)}\big) - 2\operatorname{trace}\big(\mathcal{A}_{a,b}^{(x,t),(x^*,t^*)}\big)\Big) \\ &= \frac{3}{\varepsilon}\Big(\sqrt{p(x,t)-1} - \sqrt{p(x^*,t^*)-1}\Big)^2. \end{aligned} \tag{3.27}$$

Because it holds $p_{\min} > 1$ and

$$\sqrt{f} - \sqrt{h} = \frac{(\sqrt{f} + \sqrt{h})(\sqrt{f} - \sqrt{h})}{\sqrt{f} + \sqrt{h}} = \frac{f - h}{\sqrt{f} + \sqrt{h}}$$

for any $f, h > 0$, we can estimate

$$\frac{3}{\varepsilon}\Big(\sqrt{p(x,t) - 1} - \sqrt{p(x^*,t^*) - 1}\Big)^2 \leq \frac{3L_p^2}{2(p_{\min} - 1)} \cdot \frac{1}{2\varepsilon}\big((t - t^*)^2 + |x - x^*|^2\big)$$

with L_p denoting the Lipschitz constant of p. Therefore, because $\mathcal{H}_m$ is compact, Φ is continuous with respect to the variables in $\mathcal{CS}$ and a, b are arbitrary, this and (3.27) imply

$$\begin{aligned} &F_m^-\big((x,t), u_\varepsilon(x,t), Du_\varepsilon(x,t), Z\big) - F_m^-\big((x^*,t^*), u_m(x^*,t^*), Du_\varepsilon(x,t), Y\big) \\ &\leq \frac{3L_p^2}{2(p_{\min} - 1)} \cdot \frac{1}{2\varepsilon}\big((t - t^*)^2 + |x - x^*|^2\big). \end{aligned}$$

The solution u_m is Hölder continuous, see Lemma 4.6 below. In particular, there exists a modulus ζ_u, independent of m, such that

$$\frac{1}{2\varepsilon}\big((t - t^*)^2 + |x - x^*|^2\big) \leq u_m(x^*,t^*) - u_m(x,t) \leq \zeta_u\big(\sqrt{2L_g\varepsilon}\big).$$

Thus by denoting

$$\zeta(\varepsilon) := \frac{3L_p^2}{2(p_{\min} - 1)}\zeta_u\big(\sqrt{2L_g\varepsilon}\big)$$

and recalling (3.26), the proof is complete. □

We prove the following main lemma of this section.

Lemma 3.3. *Let u_m be the unique viscosity solution to the equation (2.17). Furthermore, let U_m^- be the lower value function of the game defined in (3.20). Then, it holds*

$$u_m(x,t) = U_m^-(x,t)$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$.

Proof. To establish the result, we regularize the solution u_m first by the sup-convolution and then by the standard mollification. Then, we apply Lemma 3.5 below to the regularized function and finally pass to the limits.

Let $j \in \mathbb{N}$ be large and $\varepsilon > 0$ small. By Lemma 3.2, it holds

$$F_m^-\big((x,t), u_\varepsilon(x,t), Du_\varepsilon(x,t), D^2u_\varepsilon(x,t)\big) \leq \partial_t u_\varepsilon(x,t) + \zeta(\varepsilon) \tag{3.28}$$

for a.e. $(x,t) \in R_j^n$ with a bounded modulus of continuity $\zeta(\varepsilon)$. Let $\delta > 0$, and denote by ϕ_δ the standard mollifier in $\mathbb{R}^{n+1}$. Then for δ small enough, the function $u_\varepsilon^\delta := \phi_\delta * u_\varepsilon$ is well defined on R_{j-1}^n. Because u_ε is bounded, the mollification ensures that u_ε^δ is bounded uniformly in δ, and u_ε^δ is Lipschitz continuous. Moreover, u_ε^δ is smooth, and $Du_\varepsilon^\delta, \partial_t u_\varepsilon^\delta$ and $D^2u_\varepsilon^\delta$ are bounded and Lipschitz continuous on R_{j-1}^n. In addition, because u_ε is continuous on

R_j^n, it holds that $u_\varepsilon^\delta \to u_\varepsilon$ uniformly as $\delta \to 0$ on R_{j-1}^n. We can also show that it holds

$$\begin{aligned} Du_\varepsilon^\delta(x,t) &\to Du_\varepsilon(x,t),\\ \partial_t u_\varepsilon^\delta(x,t) &\to \partial_t u_\varepsilon(x,t),\\ D^2 u_\varepsilon^\delta(x,t) &\to D^2 u_\varepsilon(x,t) \end{aligned}$$

as $\delta \to 0$ for a.e. $(x,t) \in R_{j-1}^n$, see for example [EG92]. Furthermore, we have

$$F_m^-\big((x,t), u_\varepsilon^\delta(x,t), Du_\varepsilon^\delta(x,t), D^2u_\varepsilon^\delta(x,t)\big) \le \partial_t u_\varepsilon^\delta(x,t) + \zeta(\varepsilon) + \gamma_\delta(x,t)$$

for all $(x,t) \in R_{j-1}^n$, where it holds

$$\begin{aligned}\gamma_\delta(x,t) := \max\Big\{&F_m^-\big((x,t), u_\varepsilon^\delta(x,t), Du_\varepsilon^\delta(x,t), D^2u_\varepsilon^\delta(x,t)\big) - \partial_t u_\varepsilon^\delta(x,t), \zeta(\varepsilon)\Big\}\\ &- \zeta(\varepsilon).\end{aligned}$$

By using the convergences above and (3.28), we see $\gamma_\delta \to 0$ as $\delta \to 0$ for a.e. $(x,t) \in R_{j-1}^n$. It also holds that γ_δ is uniformly continuous on R_{j-1}^n and bounded from above uniformly with respect to δ. This is true, because the operator F_m^- and the variables are uniformly continuous, and u_ε^δ is uniformly Lipschitz and semiconvex with respect to δ. Now by doing minor adjustments to the proof of Lemma 3.5 below, we can argue that

$$\begin{aligned} u_\varepsilon^\delta(x,t) \le \inf_{S\in\mathcal{S}_m} \sup_{A\in\mathcal{AC}_m} \mathbb{E}\bigg[&\int_t^{T_{j-1}} e^{-r(l-t)} h_\varepsilon^\delta\big(X(l),l\big)\,dl\\ &+ e^{-r(T_{j-1}-t)} u_\varepsilon^\delta\big(X(T_{j-1}),T_{j-1}\big)\bigg] \end{aligned} \tag{3.29}$$

for all $(x,t) \in R_{j-1}^n$ with $h_\varepsilon^\delta := \zeta(\varepsilon) + \gamma_\delta$ and ε small enough. This is true, because h_ε^δ is uniformly continuous.

Next, for j fixed, we let $\delta \to 0$ and $\varepsilon \to 0$. First, we make a rough estimate for the drift part and apply Doob's martingale inequality for the diffusion part of the process $\big(X(l)\big)$ to get the following. For all $\theta > 0$, we choose $R := R(\theta, m, \mu, n, p_{\max}, T) > 0$, independent of controls and strategies, large enough such that

$$\mathbb{P}\Big(\sup_{t\le l\le T} \big|X(l) - x\big| \ge R\Big) \le \theta,$$

see for example [Eva13, Theorem 2.7.2.2]. Then by Egorov's theorem, we find a set $U_\theta \subset B_R(x) \times [0,T]$ such that $|U_\theta| \le \theta$ and

$$\gamma_\delta \to 0 \text{ uniformly as } \delta \to 0 \text{ on } \big(B_R(x) \times [(j-1)^{-1}, T_{j-1}]\big) \setminus U_\theta. \tag{3.30}$$

Now, we estimate

$$\begin{aligned} \mathbb{E}\int_t^{T_{j-1}} e^{-r(l-t)} h_\varepsilon^\delta\big(X(l),l\big)\,dl \le\ & I_1^{\varepsilon,\delta}(\theta) + I_2^{\varepsilon,\delta}(\theta)\\ &+ \big(C_\gamma + \zeta(\varepsilon)\big)(T_{j-1}-t)\theta, \end{aligned} \tag{3.31}$$

where we denoted by $C_\gamma < \infty$ a constant such that $\sup_{R^n_{j-1}} \gamma_\delta < C_\gamma$ and

$$I_1^{\varepsilon,\delta}(\theta) := \mathbb{E}\int_t^{T_{j-1}} e^{-r(l-t)} h_\varepsilon^\delta\big(X(l),l\big)\chi_{U_\theta}\big(X(l),l\big)\,dl,$$
$$I_2^{\varepsilon,\delta}(\theta) := \mathbb{E}\int_t^{T_{j-1}} e^{-r(l-t)} h_\varepsilon^\delta\big(X(l),l\big)\chi_{\left(B_R(x)\times[t,T_{j-1}]\right)\setminus U_\theta}\big(X(l),l\big)\,dl.$$

By a fundamental estimate in [Kry09, Theorem 3.4], see also [KS79], it holds

$$\mathbb{E}\int_t^{T_{j-1}} \Big[e^{-r(l-t)}\chi_{U_\theta}\big(X(l),l\big)\Big]\,dl \le C(T_{j-1}-t)|U_\theta|$$

for a constant $C := C(n, p_{\min}, p_{\max}, m, \mu, r) < \infty$. Hence, we have

$$I_1^{\varepsilon,\delta}(\theta) \le C(T_{j-1}-t)\theta\big(C_\gamma + \zeta(\varepsilon)\big). \tag{3.32}$$

Furthermore, because we have (3.30) and $\zeta(\varepsilon) \to 0$ as $\varepsilon \to 0$, it holds $I_2^{\varepsilon,\delta}(\theta) \to 0$ by first letting $\delta \to 0$ and then $\varepsilon \to 0$.

Combining this together with the estimates (3.29), (3.31) and (3.32), and letting $\delta, \theta, \varepsilon \to 0$, we have proven

$$u_m(x,t) \le \inf_{S\in\mathcal{S}_m}\sup_{A\in\mathcal{AC}_m} \mathbb{E}\Big[e^{-r(T_{j-1}-t)}u_m\big(X(T_{j-1}),T_{j-1}\big)\Big]$$

for all $(x,t) \in R^n_{j-1}$. Finally by recalling $T_{j-1} = T-(j-1)^{-1}$ and letting $j \to \infty$, we see by utilizing the barrier constructed in Lemma 2.2 that

$$u_m(x,t) \le \inf_{S\in\mathcal{S}_m}\sup_{A\in\mathcal{AC}_m} \mathbb{E}\Big[e^{-r(T-t)}g\big(X(T)\big)\Big]. \tag{3.33}$$

Here, we also applied Jensen's inequality, Itô's isometry and (2.11) to get

$$\mathbb{E}\Big(|X(T_{j-1})-X(T)|^2 + j^{-1}\Big)^{1/2} \le \Big(\mathbb{E}|X(T_{j-1})-X(T)|^2 + j^{-1}\Big)^{1/2}$$
$$\le \Big(\tilde{C}(j-1)^{-1} + j^{-1}\Big)^{1/2}$$

with a constant $\tilde{C} := \tilde{C}(m,\mu,n,p_{\max}) < \infty$ to estimate the terms in the barrier.

The proof of the opposite inequality in (3.33) is analogous. In particular, we first apply the inf-convolution

$$\tilde{u}_\varepsilon(x,t) = \inf_{(z,s)\in\mathbb{R}^n\times[0,T]}\Big(u_m(z,s) + \frac{1}{2\varepsilon}\big((t-s)^2 + |x-z|^2\big)\Big)$$

whenever $(x,t) \in R^n_j$, and deduce an opposite type of inequality similar to (3.28) with the same modulus of continuity ζ. Then, we make the standard mollification, and deduce the result by passing to the limits as before. Therefore, the proof is complete. □

In the result above, we utilized the following two lemmas.

Lemma 3.4. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable, and let $a, b \in \mathbb{S}^{n-1}$ and $c, d \in [0, m]$ with $m \in \mathbb{N}$. Furthermore, assume that Du and D^2u are Lipschitz continuous, and D^2u is bounded. Then, the function*

$$(x,t) \mapsto \Phi\big(a, b, c, d; (x,t), Du(x,t), D^2u(x,t)\big)$$

is also Lipschitz continuous.

Proof. By a direct computation, it holds

$$\begin{aligned} &\big\langle (c+d)(a+b) + \mu, Du(x,t) - Du(z,s)\big\rangle \\ &\quad + \operatorname{trace}\Big[D^2u(x,t) - D^2u(z,s)\Big] \\ &\le L\big(|x-z|^2 + (t-s)^2\big)^{1/2} \end{aligned} \tag{3.34}$$

for all $(x,t), (z,s) \in \mathbb{R}^n \times [0,T]$ and for a constant $L := L(m, \mu, n, L_1, L_2)$ with L_1 denoting the Lipschitz constant of Du and L_2 denoting the Lipschitz constant of D^2u, respectively. Furthermore, because D^2u is bounded, we have

$$C_0 := \sup_{(z,l)\in\mathbb{R}^n\times[0,T]} \big|\big|D^2u(z,l)\big|\big| < \infty.$$

Therefore, we can estimate

$$\begin{aligned} &\big(p(x,t) - 2\big) \operatorname{trace}\Big((a\otimes a + b\otimes b)D^2u(x,t)\Big) \\ &\quad - \big(p(z,l) - 2\big) \operatorname{trace}\Big((a\otimes a + b\otimes b)D^2u(z,l)\Big) \\ &= \big(p(x,t) - 2\big) \operatorname{trace}\Big((a\otimes a + b\otimes b)\big(D^2u(x,t) - D^2u(z,l)\big)\Big) \\ &\quad + \big(p(x,t) - p(z,l)\big) \operatorname{trace}\Big((a\otimes a + b\otimes b)D^2u(z,l)\Big) \\ &\le \tilde{L}\big(|x-z|^2 + (t-s)^2\big)^{1/2} \end{aligned}$$

for all $(x,t), (z,s) \in \mathbb{R}^n \times [0,T]$ and for a constant $\tilde{L} := \tilde{L}(p_{\max}, n, L_2, L_p, C_0)$ with L_p denoting the Lipschitz constant of p. Thus, this estimate, together with the estimate (3.34), completes the proof.

□

Lemma 3.5. *Let u_m be the unique viscosity solution to the equation (2.17), and let U_m^- be the lower value function of the game defined in (3.20). Furthermore, assume that u_m is twice differentiable such that u_m, $\partial_t u_m$, Du_m, D^2u_m are Lipschitz continuous, and Du_m, D^2u_m are bounded in $\mathbb{R}^n \times [0,T)$. Then, it holds*

$$u_m(x,t) = U_m^-(x,t)$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$.

Proof. The idea of the proof is to apply Itô's formula to connect the solution u_m and the lower value function U_m with uniformly bounded action sets.

We construct a discretized control and a strategy based on the solution u_m, and in the end, pass to a limit with the discretization parameter.

Let $k \geq 1$ be an integer, $(x,t) \in \mathbb{R}^n \times [0,T)$ and denote $\triangle t := (T-t)/k$ and $t_i := t + i\triangle t$ for all $i \in \{0,\dots,k\}$. Note that $t_0 = t$ and $t_k = T$, and set $E_i := [t_{i-1}, t_i)$ for all $i \in \{1,\dots,k\}$. For the time interval E_1, we can choose a constant control $(a^1, c_1) \in \mathcal{H}_m$ such that

$$\begin{aligned}\sup_{(b,d)\in\mathcal{H}_m} &\Phi\big(a^1, b, c_1, d; (x,t), Du_m(x,t), D^2u_m(x,t)\big) + ru_m(x,t)\\ &\leq \partial_t u_m(x,t) + \frac{1}{k},\end{aligned} \tag{3.35}$$

since u_m is a solution to (2.18). Let $s \in E_1$, and let $\big(b(l), d(l)\big) \in \mathcal{AC}_m$ be an arbitrary control. We define $X(s)$ as in (2.9) with $X(t) = x$ and controls (a^1, c_1) and $\big(b(l), d(l)\big)$, $l \in [t,s]$. By the assumptions, u_m is regular enough to utilize Itô's formula. Thus, it holds

$$\begin{aligned}&u_m\big(X(s),s\big) - u_m(x,t)\\ &= \int_t^s \partial_t u_m\big(X(l),l\big)\,dl + \sum_{i=1}^n \int_t^s \frac{\partial u_m}{\partial x_i}\big(X(l),l\big)\,dX_i(l)\\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \int_t^s \frac{\partial^2 u_m}{\partial x_i \partial x_j}(X(l),l)\,d\langle X_i, X_j\rangle(l).\end{aligned} \tag{3.36}$$

For brevity, we denote

$$\begin{aligned}\Phi_1^X(s) &:= \Phi\big(a^1, b(s), c_1, d(s); (X(s),s), Du_m(X(s),s), D^2u_m(X(s),s)\big),\\ \Phi_1^x(s) &:= \Phi\big(a^1, b(s), c_1, d(s); (x,s), Du_m(x,s), D^2u_m(x,s)\big).\end{aligned}$$

Therefore by utilizing (2.11) and (3.36), we get

$$\begin{aligned}u_m\big(X(s),s\big) = u_m(x,t) &+ \int_t^s \big(\partial_t u_m\big(X(l),l\big) - \Phi_1^X(l)\big)\,dl\\ &+ N\big(X(s),s\big).\end{aligned} \tag{3.37}$$

Here, it holds

$$\begin{aligned}N\big(X(s),s\big) = \sum_{i=2}^n &\bigg(\int_t^s \big\langle Du_m(X(l),l), p_{a^1}^{i-1}\big\rangle\,dW_i^1(l)\\ &\quad + \int_t^s \big\langle Du_m(X(l),l), p_{b(l)}^{i-1}\big\rangle\,dW_i^2(l)\bigg)\\ &+ \sqrt{p\big(X(l),l\big)-1}\bigg(\int_t^s \big\langle Du_m(X(l),l), a^1\big\rangle\,dW_1^1(l)\\ &\quad + \int_t^s \big\langle Du_m(X(l),l), b(l)\big\rangle\,dW_2^1(l)\bigg),\end{aligned}$$

where we recall that p_ν^i denotes the i-th column vector of the matrix $P_\nu^\perp$ for all $\nu \in \mathbb{S}^{n-1}$.

We note that for any adapted one dimensional process $\{\theta(l)\}_{l\in[0,T]}$ with $\mathbb{E}\int_0^T \theta^2(l)\,dl < \infty$, it holds

$$\mathbb{E}\int_0^h \theta(l)\,dW(l) = 0$$

for all $h \in [0,T]$, where W is a one dimensional Brownian motion starting from the origin. Thus, because Du_m and p are assumed to be bounded, it holds

$$\mathbb{E}N\big(X(s),s\big) = 0.$$

Therefore by estimating the function $(z,l) \mapsto e^{-rl}u_m(z,l)$ instead of $(z,l) \mapsto u_m(z,l)$ in a similar way to (3.37), it holds

$$\begin{aligned}
&\mathbb{E}\big[e^{-rs}u_m\big(X(s),s\big) - e^{-rt}u_m(x,t)\big] \\
&= \mathbb{E}\int_t^s e^{-rl}\big(\partial_t u_m\big(X(l),l\big) - \Phi_1^X(l) - ru_m\big(X(l),l\big)\big)\,dl.
\end{aligned}$$

This implies

$$\begin{aligned}
u_m(x,t) = \mathbb{E}\Big[&e^{-r(s-t)}u_m\big(X(s),s\big) \\
&- \int_t^s e^{-r(l-t)}\big(\partial_t u_m\big(X(l),l\big) - \Phi_1^X(l) - ru_m\big(X(l),l\big)\big)\,dl\Big].
\end{aligned}$$

Next, we add and subtract terms so that we can utilize (3.35). In particular, it holds

$$\begin{aligned}
u_m(x,t) = \mathbb{E}\Big[&e^{-r(s-t)}u_m\big(X(s),s\big) + K_1 + K_2 + K_3 \\
&+ \int_t^s e^{-r(l-t)}\big(-\partial_t u_m(x,t) + \Phi_1^x(t) + ru_m(x,t)\big)\,dl\Big],
\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
K_1 &= \int_t^s e^{-r(l-t)}\big(\partial_t u_m(x,t) - \partial_t u_m\big(X(l),l\big)\big)\,dl, \\
K_2 &= \int_t^s e^{-r(l-t)}\big(\Phi_1^X(l) - \Phi_1^x(t)\big)\,dl, \\
K_3 &= \int_t^s e^{-r(l-t)}\big(ru_m\big(X(l),l\big) - ru_m(x,t)\big)\,dl.
\end{aligned}$$

Hence by using (3.35) to estimate the last term in (3.38), we get

$$u_m(x,t) \le \mathbb{E}\Big[e^{-r(s-t)}u_m\big(X(s),s\big) + K_1 + K_2 + K_3\Big] + \frac{s-t}{k}. \tag{3.39}$$

We recall that u_m, $\partial_t u_m$, Du_m, and D^2u_m are Lipschitz continuous, and we denote the largest Lipschitz constant of these functions by L_m. Then, we can estimate

$$\mathbb{E}|K_1| + \mathbb{E}|K_3| \le (1+r)L_m\Big[(s-t)^2 + \mathbb{E}\int_t^s |X(l)-x|\,dl\Big].$$

Furthermore, let us denote

$$C_{0,m} := \sup_{(z,l)\in\mathbb{R}^n\times[0,T]} \big|\big|D^2 u_m(z,l)\big|\big|,$$

which is assumed to be bounded. Then, Lemma 3.4 yields

$$\big|\Phi_1^X(l) - \Phi_1^x(t)\big| \le L\big(|X(l)-x|^2 + (s-t)^2\big)^{1/2}$$

for all $l \in [t,s]$ and for a constant $L := L(m,\mu,p_{\max},n,L_m,C_{0,m},L_p)$. Here, recall that the constant L_p is the Lipschitz constant of p. Therefore by combining these estimates with (3.39), we get

$$\begin{aligned} u_m(x,t) \le& \mathbb{E}\Big[e^{-r(s-t)}u_m\big(X(s),s\big)\Big] + C\mathbb{E}\int_t^s |X(l)-x|\,dl \\ &+ C(s-t)^2 + \frac{s-t}{k} \end{aligned} \tag{3.40}$$

for a constant $C := C(m,\mu,p_{\max},n,L_m,C_{0,m},L_p,r)$. By recalling (2.11) and utilizing Jensen's inequality and Itô's isometry, we see

$$\int_t^s \mathbb{E}|X(l)-x|\,dl \le \tilde{C}\big((s-t)^2 + (s-t)^{3/2}\big)$$

for a constant $\tilde{C} := \tilde{C}(m,\mu,p_{\max},n)$. Thus, combining this with (3.40) and letting $s \to t_1$, we have

$$u_m(x,t) \le \mathbb{E}\Big[e^{-r\Delta t}u_m\big(X(t_1),t_1\big)\Big] + C(\Delta t)^2 + C(\Delta t)^{3/2} + \frac{\Delta t}{k} \tag{3.41}$$

for some generic constant C.

Next, we replicate the same argument as above in the time interval E_2. By Lemma 3.4, it follows that there are a covering $U_2 := \big(B(y^{2,i}, r_{2,i})\big)_{i=1}^\infty$ of $\mathbb{R}^n$ and a sequence of controls $\mathcal{C}_2 := (a^{2,i}, c_{2,i})_{i=1}^\infty$ depending on the covering U_2 such that

$$\begin{aligned} \sup_{(b,d)\in\mathcal{H}_m} &\Big(\Phi\big(a^{2,i}, b, c_{2,i}, d; (y,t_1), Du_m(y,t_1), D^2u_m(y,t_1)\big) + ru_m(y,t_1)\Big) \\ &\le \partial_t u_m(y,t_1) + \frac{1}{k} \end{aligned} \tag{3.42}$$

if $y \in B(y^{2,i}, r_{2,i})$. For $y \in \mathbb{R}^n$, let $I_2(y)$ be the smallest index i for which it holds $y \in B(y^{2,i}, r_{2,i})$ in the covering U_2 of $\mathbb{R}^n$. Then, we define a function $z^2 : \mathbb{R}^n \to \mathcal{H}_m$ by

$$z^2(y) = \big(a^{2,I_2(y)}, c_{2,I_2(y)}\big)$$

for all $y \in \mathbb{R}^n$. Observe that we can construct z^2 in such a way that it is Borel measurable. Furthermore, we define a control $\big(a^2(l), c_2(l)\big)$ such that

$$\big(a^2(l), c_2(l)\big) = \begin{cases} (a^1, c_1), & \text{if } l \in E_1, \\ z^2\big(X(t_1)\big), & \text{if } l \in E_2. \end{cases}$$

By the inequality (3.42), we can now repeat the argument above to get

$$u_m\big(X(t_1),t_1\big) \le \mathbb{E}\Big[e^{-r\Delta t}u_m\big(X(t_2),t_2\big)\Big] + C(\Delta t)^2 + C(\Delta t)^{3/2} + \frac{\Delta t}{k}.$$

Thus, combining this estimate with (3.41), it holds

$$u_m(x,t) \le \mathbb{E}\Big[e^{-r2\Delta t}u_m\big(X(t_2),t_2\big)\Big] + 2C(\Delta t)^2 + 2C(\Delta t)^{3/2} + \frac{2\Delta t}{k}.$$

The idea is to replicate the argument in all time intervals $E_1,\dots,E_k$. Indeed, after the k-th iteration, we get a control $\big(a^k(l),c_k(l)\big)$ such that

$$\big(a^k(l),c_k(l)\big) = \begin{cases} (a^{k-1}(l),c_{k-1}(l)), & \text{if } l\in\cup_{i=1}^{k-1}E_i \\ z^k\big(X(t_{k-1})\big), & \text{if } l\in E_k. \end{cases}$$

Here, z^k corresponds to the triplet $\big(\mathcal{C}_k,U_k,I_k(\cdot)\big)$ in the same way as above. In particular, we have

$$u_m(x,t) \le \mathbb{E}\Big[e^{-r(T-t)}g\big(X(T)\big)\Big] + (T-t)\big(C\Delta t + C(\Delta t)^{1/2}\big) + \Delta t, \quad (3.43)$$

because it holds $k=(T-t)/\triangle t$ and $u_m(z,T)=g(z)$ for all $z\in\mathbb{R}^n$.

Let $S\in\mathcal{S}_m$, and recall that the control $\big(b(l),d(l)\big)$ is arbitrary. We set

$$\big(b(l),d(l)\big) := S\big(a^k(l),c_k(l)\big)$$

for all $l\in[0,T]$. Then by (3.43), it holds

$$\begin{aligned} u_m(x,t) &\le \mathbb{E}\Big[e^{-r(T-t)}g\big(X(T)\big)\Big] + (T-t)\big(C\Delta t + C(\Delta t)^{1/2}\big) + \Delta t \\ &\le \sup_{A\in\mathcal{AC}_m}\mathbb{E}\Big[e^{-r(T-t)}g\big(X(T)\big)\Big] + (T-t)\big(C\Delta t + C(\Delta t)^{1/2}\big) + \Delta t. \end{aligned}$$

Because $S\in\mathcal{S}_m$ is arbitrary, by letting $k\to\infty$, this yields

$$u_m(x,t) \le \inf_{S\in\mathcal{S}_m}\sup_{A\in\mathcal{AC}_m}\mathbb{E}\Big[e^{-r(T-t)}g\big(X(T)\big)\Big].$$

Next, we prove the opposite inequality. Observe that

$$(a,b,c,d;y)\mapsto\Phi\big(a,b,c,d;(y,l),Du_m(y,l),D^2u_m(y,l)\big)$$

is uniformly continuous in $\mathcal{CS}\times\mathbb{R}^n$ for all fixed $l\in[0,T]$. Thus, we deduce that for given $k\ge1$ and $j\in\{1,\dots,k\}$, there are a covering

$$\tilde U_j := \Big(B(\tilde y^{j,i},\tilde r_{j,i})\times B\big((\tilde a^{j,i},\tilde c_{j,i}),\hat r_{j,i}\big)\Big)_{i=1}^{\infty}$$

of $\mathbb{R}^n\times\mathcal{H}_m$ and a sequence of controls $\tilde{\mathcal{C}}_j := (b^{j,i},d_{j,i})_{i=1}^{\infty}\subset\mathcal{H}_m$ depending on the covering $\tilde U_j$ such that for all indices $i\ge1$, it holds

$$\begin{aligned} &\Phi\big(a,b^{j,i},c,d_{j,i};Du_m(y,t_{j-1}),D^2u_m(y,t_{j-1})\big)+ru_m(y,t_{j-1}) \\ &\ge \partial_t u_m(y,t_{j-1}) - \frac{1}{k} \end{aligned}$$

for all $y \in B(\tilde{y}^{j,i}, \tilde{r}_{j,i})$ and $(a,c) \in B\big((\tilde{a}^{j,i}, \tilde{c}_{j,i}), \hat{r}_{j,i}\big)$, because u_m is a solution to (2.18). For $\big(y,(a,c)\big) \in \mathbb{R}^n \times \mathcal{H}_m$, let $\tilde{I}_j := \tilde{I}_j(y,a,c)$ be the smallest index i for which it holds

$$\big(y,(a,c)\big) \in B(\tilde{y}^{j,i}, \tilde{r}_{j,i}) \times B\big((\tilde{a}^{j,i}, \tilde{c}_{j,i}), \hat{r}_{j,i}\big).$$

Then, we define a Borel measurable map $\tilde{z}^j : \mathbb{R}^n \times \mathcal{H}_m \to \mathcal{H}_m$ by

$$\tilde{z}^j\big(y,(a,c)\big) = \Big(b^{j,\tilde{I}_j}, d_{j,\tilde{I}_j}\Big)$$

for all $\big(y,(a,c)\big) \in \mathbb{R}^n \times \mathcal{H}_m$. Let $A = \big(a(l), c(l)\big) \in \mathcal{AC}_m$, and let us define an admissible strategy $\bar{S} \in \mathcal{S}_m$ by

$$\bar{S} = \tilde{z}^j\Big(X(t_{j-1}), \big(a(l), c(l)\big)\Big),$$

if $l \in E_j$. We define $X(l), l \in [t,T]$, in (2.9) with controls A, $\bar{S}$ and $X(t) = x$. Therefore by a similar reasoning to the above, it holds

$$\begin{aligned} u_m(x,t) &\geq \mathbb{E}\Big[e^{-r(T-t)} g\big(X(T)\big)\Big] - (T-t)\big(C\Delta t + C(\Delta t)^{1/2}\big) - \Delta t \\ &\geq \inf_{S \in \mathcal{S}_m} \mathbb{E}\Big[e^{-r(T-t)} g\big(X(T)\big)\Big] - (T-t)\big(C\Delta t + C(\Delta t)^{1/2}\big) - \Delta t. \end{aligned}$$

Hence by letting $k \to \infty$, we get

$$u_m(x,t) \geq \inf_{S \in \mathcal{S}_m} \sup_{A \in \mathcal{AC}_m} \mathbb{E}\Big[e^{-r(T-t)} g\big(X(T)\big)\Big].$$

Thus, the proof is complete. □

4. Going to the limit: action sets without a uniform bound

In this section, we let bounds on the controls increase. To this end, we first show that a viscosity solution to the limiting equation is unique under suitable assumptions. Then by utilizing the stability principle and the equicontinuity of the families of viscosity solutions to the terminal value problems (2.17) and (2.18), we see that there exist subsequences of solutions to (2.17) and (2.18) converging uniformly to the unique solution to the limiting equation. The final part is to show that the subsequences of the corresponding lower and upper value functions converge to the lower and upper value functions for the original game without the uniform bound on the controls.

Let $J_0 := \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \big(\mathbb{R}^n \setminus \{0\}\big) \times S(n)$, and define $F : J_0 \to \mathbb{R}$ through

$$F\big((x,t), \xi, \nu, M\big) = \big(p(x,t) - 2\big)\frac{\langle M\nu, \nu\rangle}{|\nu|^2} + \operatorname{trace}(M) + \langle \mu, \nu\rangle - r\xi.$$

Then, the limiting terminal value problem for (2.17) and (2.18) as $m \to \infty$ is

$$\begin{cases} \partial_t u + F\big((x,t), u, Du, D^2 u\big) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.44}$$

As before, this equation is understood in the viscosity sense. We take care of the points, where the gradient of the underlying function in the operator F vanishes, via semicontinuous envelopes. Let us denote

$$F_*\big((x,t), \xi, \nu, M\big) := \liminf_{\tilde{\nu} \to \nu} F\big((x,t), \xi, \tilde{\nu}, M\big)$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, $\xi \in \mathbb{R}$, $\nu \in \mathbb{R}^n$ and $M \in S(n)$, and $F^* := -(-F)_*$. The following definition parallels Definition 2.1.

Definition 4.1. *(i) A lower semicontinuous function $\overline{u} : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a viscosity supersolution to (4.44), if it satisfies the growth bound (2.19),*

$$\overline{u}(x,T) \geq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for all $\phi \in C^{2,1}\big(\mathbb{R}^n \times (0,T)\big)$ such that

- $\overline{u}(x_0,t_0) = \phi(x_0,t_0)$
- $\overline{u}(x,t) > \phi(x,t)$ *for all* $(x,t) \neq (x_0,t_0)$

it holds

$$\partial_t \phi(x_0,t_0) + F\big((x_0,t_0), \phi(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)\big) \leq 0$$

whenever $D\phi(x_0,t_0) \neq 0$, and

$$\partial_t \phi(x_0,t_0) + F_*\big((x_0,t_0), \phi(x_0,t_0), 0, D^2\phi(x_0,t_0)\big) \leq 0,$$

whenever $D\phi(x_0,t_0) = 0$.

(ii) An upper semicontinuous function $\underline{u} : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a viscosity subsolution to (4.44), if it satisfies the growth bound (2.19),

$$\underline{u}(x,T) \leq g(x)$$

for all $x \in \mathbb{R}^n$, and if the following holds. For all $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for all $\phi \in C^{2,1}\big(\mathbb{R}^n \times (0,T)\big)$ such that

- $\underline{u}(x_0,t_0) = \phi(x_0,t_0)$
- $\underline{u}(x,t) < \phi(x,t)$ *for all* $(x,t) \neq (x_0,t_0)$

it holds

$$\partial_t \phi(x_0,t_0) + F\big((x_0,t_0), \phi(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)\big) \geq 0,$$

whenever $D\phi(x_0,t_0) \neq 0$, and

$$\partial_t \phi(x_0,t_0) + F^*\big((x_0,t_0), \phi(x_0,t_0), 0, D^2\phi(x_0,t_0)\big) \geq 0,$$

whenever $D\phi(x_0,t_0) = 0$.

(iii) If a function $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a viscosity supersolution and a subsolution to (4.44), then u is a viscosity solution to (4.44).

Remark 4.2. *Observe that for any test function $\phi \in C^{2,1}\big(\mathbb{R}^n \times (0,T)\big)$ such that $D\phi(x_0,t_0) \neq 0$ or $D^2\phi(x_0,t_0) = \mathbf{0}$ in the Definition 4.1, it holds*

$$\begin{aligned} &F_*\big((x_0,t_0), \phi(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)\big) \\ &= F^*\big((x_0,t_0), \phi(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)\big) \end{aligned}$$

for all $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$.

To prove a comparison principle for the equation (4.44), we follow the path developed in [GGIS91], see also [CGG91, JLM01, KMP12]. Here, the main difficulties arise from the (x,t) dependence in F as well as from the unboundedness of the domain.

Theorem 4.3. *Let $\underline{u}$ and $\overline{u}$ be continuous viscosity sub- and supersolutions to (4.44) in the sense of Definition 4.1, respectively. Then, it holds*

$$\underline{u}(x,t) \leq \overline{u}(x,t)$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$.

Proof. The proof is by contradiction. We assume that

$$\alpha := \sup_{\mathbb{R}^n \times [0,T]} \big(\underline{u} - \overline{u}\big) > 0. \tag{4.45}$$

Let $\varepsilon, \delta, \gamma > 0$, and define

$$w_{\varepsilon,\delta,\gamma}(x,y,t) = \underline{u}(x,t) - \overline{u}(y,t) - \frac{1}{4\varepsilon}|x-y|^4 - B_{\delta,\gamma}(x,y,t)$$

for all $x, y \in \mathbb{R}^n$ and $t \in (0,T]$, where

$$B_{\delta,\gamma}(x,y,t) := \delta(|x|^2 + |y|^2) + \gamma t^{-1}. \tag{4.46}$$

The function $B_{\delta,\gamma}$ plays the role of a barrier for all large x, y and $t = 0$.

We can show, see [GGIS91, Proposition 2.3], that there are constants $K, K' > 0$ independent of x, y, t such that

$$\underline{u}(x,t) - \overline{u}(y,t) \leq K|x-y| + K'(1+t) \tag{4.47}$$

for all $x, y \in \mathbb{R}^n$ and $t \in [0,T]$. Indeed, because for $R' > 0$ it holds

$$\Big|F\big((x,t), \xi, p, M\big)\Big| \leq (p_{\max} - 2 + n + |\mu|)R' + r|\xi| < \infty$$

for all $(x,t,\xi,p,M) \in J_0$ such that $|p| \leq R'$ and $||M|| \leq R'$, we can utilize the same arguments as in [GGIS91, Proposition 2.3]. Therefore by the estimate (4.47), it holds $\alpha < \infty$ in (4.45).

We denote by $(\hat{x}, \hat{y}, \hat{t})$ a maximum point of $w_{\varepsilon,\delta,\gamma}$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,T]$. The growth condition (2.19) and the barrier (4.46) ensure that $w_{\varepsilon,\delta,\gamma}(x,y,t) < 0$, when x, y are outside a compact set $E \subset \mathbb{R}^n \times \mathbb{R}^n$ depending on δ, and $t \in (0,T]$. Therefore, because $w_{\varepsilon,\delta,\gamma}$ is continuous and (4.45) holds with

$\alpha < \infty$, the maximum point exists for all δ, γ small enough and any ε. Furthermore by (4.45), we can find $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$ such that

$$\underline{u}(x_0, t_0) - \overline{u}(x_0, t_0) > \alpha - \varepsilon/3.$$

Because $\underline{u} - \overline{u}$ is continuous, we may assume that $t_0 > 0$. Consequently, for $\varepsilon < \alpha$ there are $\delta_0 := \delta_0(\varepsilon) > 0$ and $\gamma_0 := \gamma_0(\varepsilon) > 0$ such that

$$w_{\varepsilon,\delta,\gamma}(\hat{x}, \hat{y}, \hat{t}) \geq \underline{u}(x_0, t_0) - \overline{u}(x_0, t_0) - 2\delta|x_0| - \gamma t_0^{-1} > \alpha - \varepsilon \tag{4.48}$$

for all $\delta < \delta_0$ and $\gamma < \gamma_0$. Let $\varepsilon < \alpha/2, \delta < \delta_0$ and $\gamma < \gamma_0$. Then by (4.48) we can estimate

$$\underline{u}(\hat{x}, \hat{t}) - \overline{u}(\hat{y}, \hat{t}) > \frac{1}{4\varepsilon}|\hat{x} - \hat{y}|^4 + B_{\delta,\gamma}(\hat{x}, \hat{y}, \hat{t}) \geq \frac{1}{4\varepsilon}|\hat{x} - \hat{y}|^4.$$

This and (4.47) imply

$$|\hat{x} - \hat{y}| \leq 4\varepsilon\big(K|\hat{x} - \hat{y}|^{-3} + K'(1 + T)|\hat{x} - \hat{y}|^{-4}\big).$$

Therefore, we have $|\hat{x} - \hat{y}| < C$ for some $C < \infty$ independent of ε, δ and γ. Moreover, it holds

$$|\hat{x} - \hat{y}| \leq \max\big\{\varepsilon^{1/8}, 4K\varepsilon^{5/8} + 4K'(1 + T)\sqrt{\varepsilon}\big\} =: \zeta(\varepsilon). \tag{4.49}$$

By an analogous argument, we can deduce $\hat{t} > 0$. Because it holds $\underline{u}(z, T) \leq \overline{u}(z, T)$ for all $z \in \mathbb{R}^n$ by the assumptions, the inequality (4.48) yields $\hat{t} < T$. In addition, because $|\hat{x} - \hat{y}|$ is bounded, the estimate (4.47) implies that $w_{\varepsilon,\delta,\gamma}(\hat{x}, \hat{y}, \hat{t})$ is uniformly bounded from above with respect to δ. Hence, because $w_{\varepsilon,\delta,\gamma}(\hat{x}, \hat{y}, \hat{t})$ increases as $\delta \to 0$, the quantity $\lim_{\delta\to 0} w_{\varepsilon,\delta,\gamma}(\hat{x}, \hat{y}, \hat{t})$ exists. Therefore by denoting $(\tilde{x}, \tilde{y}, \tilde{t})$ a global maximum point of $w_{\varepsilon,\delta/2,\gamma}$, we have

$$w_{\varepsilon,\delta/2,\gamma}(\tilde{x}, \tilde{y}, \tilde{t}) \geq w_{\varepsilon,\delta,\gamma}(\hat{x}, \hat{y}, \hat{t}) + \delta/2\big(|\hat{x}|^2 + |\hat{y}|^2\big)$$

implying

$$\delta\big(|\hat{x}|^2 + |\hat{y}|^2\big) \to 0 \tag{4.50}$$

as $\delta \to 0$.

By theorem of sums, see [CIL92, Theorem 8.3], there exist symmetric matrices $X := X(\varepsilon, \delta)$ and $Y := Y(\varepsilon, \delta)$, and real numbers $\tau_{\underline{u}}$ and $\tau_{\overline{u}}$, such that $\tau_{\underline{u}} - \tau_{\overline{u}} = \partial_t B_{\delta,\gamma}(\hat{x}, \hat{y}, \hat{t}) = -\gamma\hat{t}^{-2}$ and

$$\begin{aligned}
&\Big(\tau_{\underline{u}}, \varepsilon^{-1}\,|\hat{x} - \hat{y}|^2\,(\hat{x} - \hat{y}) + 2\delta\hat{x},\, X\Big) \in \overline{\mathcal{P}}^{2,+}\underline{u}(\hat{x}, \hat{t}),\\
&\Big(\tau_{\overline{u}}, \varepsilon^{-1}\,|\hat{x} - \hat{y}|^2\,(\hat{x} - \hat{y}) - 2\delta\hat{y},\, Y\Big) \in \overline{\mathcal{P}}^{2,-}\overline{u}(\hat{y}, \hat{t}).
\end{aligned} \tag{4.51}$$

Furthermore by computing the second derivatives of the function $B_{\delta,\gamma}(x, y, t) + \frac{1}{4\varepsilon}|x - y|^4$, it holds

$$\begin{aligned}
\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq (1 + 4\varepsilon\delta)\begin{bmatrix} M & -M \\ -M & M \end{bmatrix} + 2\varepsilon\begin{bmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{bmatrix}\\
+ 2\delta(1 + 2\delta)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\end{aligned} \tag{4.52}$$

with

$$M := \varepsilon^{-1}\Big(2(\hat{x}-\hat{y})\otimes(\hat{x}-\hat{y}) + |\hat{x}-\hat{y}|^2 I\Big),$$

and

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \geq -(\varepsilon^{-1} + 3\varepsilon^{-1}|\hat{x}-\hat{y}|^2 + 2\delta)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.53}$$

Thus, because $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution, it holds by (4.51)

$$\begin{aligned} \tau_{\underline{u}} + F^*\big((\hat{x},\hat{t}), \underline{u}(\hat{x},\hat{t}), \varepsilon^{-1}|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y}) + 2\delta\hat{x}, X\big) &\geq 0, \\ \tau_{\overline{u}} + F_*\big((\hat{y},\hat{t}), \overline{u}(\hat{y},\hat{t}), \varepsilon^{-1}|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y}) - 2\delta\hat{y}, Y\big) &\leq 0, \end{aligned} \tag{4.54}$$

see also Remark 4.2.

We consider two different cases depending on the behavior of $\hat{x}-\hat{y}$ as $\delta \to 0$. First, assume that $\hat{x}-\hat{y} \to 0$ as $\delta \to 0$. Then by the estimate (4.52), it holds

$$\limsup_{\delta\to 0}\langle Xz, z\rangle \leq 0 \text{ and } \liminf_{\delta\to 0}\langle Yz, z\rangle \geq 0$$

for all $z \in \mathbb{R}^n$. Thus by combining this with (4.54), and recalling (4.48), the degenerate ellipticity of F and $\delta\hat{x}, \delta\hat{y} \to 0$ as $\delta \to 0$ by (4.50), we can estimate

$$\begin{aligned} \gamma T^{-2} &\leq \limsup_{\delta\to 0} F^*\big((\hat{x},\hat{t}), \underline{u}(\hat{x},\hat{t}), 0, \mathbf{0}\big) - \liminf_{\delta\to 0} F_*\big((\hat{y},\hat{t}), \overline{u}(\hat{y},\hat{t}), 0, \mathbf{0}\big) \\ &\leq 0. \end{aligned}$$

Hence, because it holds $\gamma > 0$, we have found a contradiction.

Next, we assume $\hat{x}-\hat{y} \to \eta \neq 0$ for some subsequence still denoted by (δ). For brevity, let us denote

$$\begin{aligned} \tilde{\xi}_x &:= \varepsilon^{-1}|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y}) + 2\delta\hat{x}, \\ \tilde{\xi}_y &:= \varepsilon^{-1}|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y}) - 2\delta\hat{y}, \end{aligned}$$

$\xi_x := \tilde{\xi}_x/|\tilde{\xi}_x|$ and $\xi_y := \tilde{\xi}_y/|\tilde{\xi}_y|$ assuming $\tilde{\xi}_x, \tilde{\xi}_y \neq 0$. Then, because of (4.48) and (4.54), we can estimate

$$\begin{aligned} 0 < \big(p(\hat{x},\hat{t}) - 2\big)\big\langle X\xi_x, \xi_x\big\rangle - \big(p(\hat{y},\hat{t}) - 2\big)\big\langle Y\xi_y, \xi_y\big\rangle \\ + \sum_{i=1}^{n}\lambda_i\big(X-Y\big) + 2\langle\mu, \delta\hat{x} + \delta\hat{y}\rangle - r\alpha/2, \end{aligned} \tag{4.55}$$

where λ_i denotes the i-th eigenvalue of the corresponding matrix. Because the first two matrices in the right-hand side of (4.52) annihilate, we have

$$X - Y \leq 4\delta(1+2\delta)I. \tag{4.56}$$

Thus to complete the proof, we need to estimate the first two terms in the right-hand side of (4.55).

Let us define $\xi_\delta := (\hat{x}-\hat{y})/|\hat{x}-\hat{y}| \in \mathbb{S}^{n-1}$ for all δ small enough. Then, it holds

$$\xi_\delta \to \eta/|\eta| \tag{4.57}$$

as $\delta \to 0$. Observe that by the convergence (4.50), it also holds

$$\xi_x, \xi_y \to \eta/|\eta| \tag{4.58}$$

as $\delta \to 0$. Furthermore by (4.52) and (4.53), X and Y are uniformly bounded with respect to δ, see also [Ish89, Lemma 5.3]. Thus, because the function p is bounded, the convergences (4.57) and (4.58) imply

$$\begin{aligned} &\big(p(\hat{x},\hat{t})-2\big)\big\langle X\xi_x,\xi_x\big\rangle-\big(p(\hat{y},\hat{t})-2\big)\big\langle Y\xi_y,\xi_y\big\rangle \\ &=\big(p(\hat{x},\hat{t})-1\big)\big\langle X\xi_\delta,\xi_\delta\big\rangle-\big(p(\hat{y},\hat{t})-1\big)\big\langle Y\xi_\delta,\xi_\delta\big\rangle \\ &\quad-\big\langle (X-Y)\xi_\delta,\xi_\delta\big\rangle+E_\delta(\hat{x},\hat{y},\hat{t}) \end{aligned} \tag{4.59}$$

for some error $E_\delta(\hat{x},\hat{y},\hat{t})$ such that

$$E_\delta(\hat{x},\hat{y},\hat{t}) \to 0$$

as $\delta \to 0$. For the vector

$$\big(\xi_\delta^T\sqrt{p(\hat{x},\hat{t})-1},\xi_\delta^T\sqrt{p(\hat{y},\hat{t})-1}\big)\in\mathbb{R}^{2n}$$

in the estimate (4.52), it holds

$$\begin{aligned} &(p(\hat{x},\hat{t})-1)\langle X\xi_\delta,\xi_\delta\rangle-(p(\hat{y},\hat{t})-1)\langle Y\xi_\delta,\xi_\delta\rangle \\ &\leq\Big(\sqrt{p(\hat{x},\hat{t})-1}-\sqrt{p(\hat{y},\hat{t})-1}\Big)^2\Big((1+4\varepsilon\delta)\big\langle M\xi_\delta,\xi_\delta\big\rangle \\ &\qquad\qquad+2\varepsilon\big\langle M^2\xi_\delta,\xi_\delta\big\rangle\Big)+4(p_{\max}-1)\delta(1+2\delta) \\ &\leq\frac{L_p^2}{4(p_{\min}-1)}|\hat{x}-\hat{y}|^2\Big((1+4\varepsilon\delta)3\varepsilon^{-1}|\hat{x}-\hat{y}|^2+18\varepsilon^{-1}|\hat{x}-\hat{y}|^4\Big) \\ &\quad+4(p_{\max}-1)\delta(1+2\delta), \end{aligned} \tag{4.60}$$

where L_p is the Lipschitz constant of p. Moreover by the estimates (4.48) and (4.49), it holds

$$\begin{aligned} \frac{|\hat{x}-\hat{y}|^4}{4\varepsilon}&<\underline{u}(\hat{x},\hat{t})-\overline{u}(\hat{y},\hat{t})-\alpha+\varepsilon \\ &\leq\sup_{|x-y|<\zeta(\varepsilon),t\in[0,T]}\big(\underline{u}(x,t)-\overline{u}(y,t)\big)-\alpha+\varepsilon. \end{aligned}$$

This estimate, together with (4.45), implies

$$\lim_{\varepsilon\to0}\limsup_{\delta,\gamma\to0}\frac{|\hat{x}-\hat{y}|^4}{\varepsilon}=0.$$

Therefore by combining this, (4.50), (4.56), (4.59) and (4.60) with the estimate (4.55), we have found a contradiction by first letting $\delta,\gamma\to0$ and then $\varepsilon\to0$. Hence, the proof is complete. □

A typical phenomenon for equations of p-Laplacian type is that the set of test functions used in their definition can be reduced.

Lemma 4.4. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be continuous. Then, to test whether or not u is a viscosity super- or subsolution at (x_0,t_0) in the sense of Definition 4.1, it is enough to consider test functions $\phi \in C^{2,1}\big(\mathbb{R}^n \times (0,T)\big)$ such that either*

- $D\phi(x_0,t_0) \neq 0$ *or*
- $D\phi(x_0,t_0) = 0$ *and* $D^2\phi(x_0,t_0) = 0$.

Proof. We only provide the proof in the context of supersolutions. Let $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$. Assume that there exist $\delta > 0$ and a test function $\phi \in C^{2,1}\big(\mathbb{R}^n \times (0,T)\big)$ such that $u(x_0,t_0) = \phi(x_0,t_0)$, $u(x,t) > \phi(x,t)$ for $(x,t) \neq (x_0,t_0)$, $D\phi(x_0,t_0) = 0$, $D^2\phi(x_0,t_0) \neq \mathbf{0}$ and

$$0 < \partial_t\phi(x_0,t_0) + F_*\big((x_0,t_0), \phi(x_0,t_0), 0, D^2\phi(x_0,t_0)\big) - \delta. \tag{4.61}$$

Observe that $u - \phi$ has a strict global minimum at (x_0,t_0). We define a function

$$w_j(x,t,y,s) := u(x,t) - \phi(y,s) + \frac{j}{4}|x-y|^4 + \frac{j}{2}(t-s)^2$$

for $x,y \in \mathbb{R}^n, t,s \in [0,T]$. Let $R := \max\{2|x_0|,1\} > 0$, and denote by (x_j,t_j,y_j,s_j) a minimum point of w_j on a compact set $K := \overline{B}_R(0) \times [0,T] \times \overline{B}_R(0) \times [0,T]$. Because $w_j(x_j,t_j,y_j,s_j)$ increases as j increases, and it is bounded from above by $w_j(x_0,t_0,x_0,t_0) = 0$ for all j, the limit

$$\lim_{j\to\infty} w_j(x_j,t_j,y_j,s_j) < \infty$$

exists. Consequently, the estimate

$$w_{j/2}(x_{j/2},t_{j/2},y_{j/2},s_{j/2}) \leq w_j(x_j,t_j,y_j,s_j) - \frac{j}{8}|x_j-y_j|^4 - \frac{j}{4}(t_j-s_j)^2$$

implies

$$j|x_j-y_j|^4 + j(t_j-s_j)^2 \to 0 \tag{4.62}$$

as $j \to \infty$. Furthermore, because the global minimum of $u - \phi$ is strict, it holds

$$(x_j,t_j,y_j,s_j) \to (x_0,t_0,x_0,t_0) \tag{4.63}$$

as $j \to \infty$. In particular, the point (x_j,t_j,y_j,s_j) is not on the boundary of the set K for all j large enough, because it holds $(x_0,t_0) \in B_R(0) \times (0,T)$.

We prove the case $x_j = y_j$ for an infinite sequence of j:s, and consider only such indices j. The proof in the case $x_j \neq y_j$ for all j large enough is similar to the proof of Theorem 4.3, see also [CGG91, JLM01]. By denoting $\varphi(x,y) := \frac{j}{4}|x-y|^4$, it holds

$$D_x\varphi(x_j,y_j) = -D_y\varphi(x_j,y_j) = 0 \text{ and } D^2_{xx}\varphi(x_j,y_j) = D^2_{yy}\varphi(x_j,y_j) = \mathbf{0}.$$

Furthermore, the function

$$(y,s) \mapsto \phi(y,s) - \varphi(x_j,y) - \frac{j}{2}(t_j-s)^2$$

has a local maximum at (y_j, s_j). These imply $D\phi(y_j, s_j) = -D_y\varphi(x_j, y_j) = 0$, $\partial_t\phi(y_j, s_j) = -j(t_j - s_j)$ and $D^2\phi(y_j, s_j) \le -D^2_{yy}\varphi(x_j, y_j) = \mathbf{0}$. Thus, because p and $(y, s) \mapsto \lambda_i\big(D^2\phi(y, s)\big)$ for any i are continuous with λ_i denoting the i-th eigenvalue of the corresponding matrix, the assumption (4.61) and the convergence (4.63) yield

$$\begin{aligned} 0 &< \partial_t\phi(y_j, s_j) + \lambda_{\max}\Big(\big(p(y_j, s_j) - 1\big)D^2\phi(y_j, s_j)\Big) \\ &\quad + \sum_{i \ne i_{\min}} \lambda_i\Big(D^2\phi(y_j, s_j)\Big) - r\phi(y_j, s_j) - \frac{\delta}{2} \\ &\le -j(t_j - s_j) - r\phi(y_j, s_j) - \frac{\delta}{2} \end{aligned} \tag{4.64}$$

for all j large enough. Furthermore, because the function

$$\begin{aligned} (x, t) \mapsto \Psi(x, t) := &- \varphi(x, y_j) - \frac{j}{2}(t - s_j)^2 + \varphi(x_j, y_j) + \frac{j}{2}(t_j - s_j)^2 \\ &+ u(x_j, t_j) \end{aligned}$$

tests u from below at (x_j, t_j), and it holds $D_x\Psi(x_j, t_j) = 0$, we have

$$0 \ge \Psi_t(x_j, t_j) + F_*\big((x_j, t_j), u(x_j, t_j), 0, D^2_{xx}\Psi(x_j, t_j)\big).$$

Thus, because it holds $\Psi_t(x_j, t_j) = -j(t - s_j)$ and $D^2_{xx}\Psi(x_j, t_j) = \mathbf{0}$, by combining this and (4.64), we get

$$0 < r\big(u(x_j, t_j) - \phi(y_j, s_j)\big) - \delta/2.$$

Hence, because u is continuous and (4.63) holds, we find a contradiction for all j large enough. □

The following lemma suggests that F is the correct limiting equation in our setting. The proof for the equation F^+_m is analogous.

Lemma 4.5. *Let $(x_m, t_m), (x, t) \in \mathbb{R}^n \times [0, \infty)$, $\xi_m, \xi \in \mathbb{R}$, $\nu_m, \nu \in \mathbb{R}^n \setminus \{0\}$ and $M_m, M \in S(n)$ be such that*

$$(x_m, t_m) \to (x, t),\ \ \xi_m \to \xi,\ \ \nu_m \to \nu \text{ and } M_m \to M$$

as $m \to \infty$. Then, it holds

$$F^-_m\big((x_m, t_m), \xi_m, \nu_m, M_m\big) \to -F\big((x, t), \xi, \nu, M\big)$$

as $m \to \infty$.

Proof. It is clear that $\langle \mu, \nu_m\rangle \to \langle \mu, \nu\rangle$ and $r\xi_m \to r\xi$ as $m \to \infty$. To complete the proof, we utilize the key inequality

$$\big\langle \nu_m/|\nu_m| + \xi, \nu_m\big\rangle \ge 0 \tag{4.65}$$

whenever $\xi \in \mathbb{S}^{n-1}$.

We set

$$\tilde{\Phi}_m := \inf_{(a,c)\in\mathcal{H}_m} \sup_{(b,d)\in\mathcal{H}_m} \Big[- \operatorname{trace}\Big(\mathcal{A}^{(x_m,t_m)}_{a,b} M_m\Big) - (c + d)\langle a + b, \nu_m\rangle \Big].$$

Because $\big(\nu_m/|\nu_m|, 0\big) \in \mathcal{H}_m$, it holds

$$\tilde{\Phi}_m \le \sup_{(b,d)\in\mathcal{H}_m} \Big[-\operatorname{trace}\Big(\mathcal{A}^{(x_m,t_m)}_{\frac{\nu_m}{|\nu_m|},b} M_m\Big) - d\langle \nu_m/|\nu_m| + b, \nu_m\rangle\Big].$$

Therefore, this estimate and (4.65) imply

$$\tilde{\Phi}_m \le n\Lambda\big|\big|M_m\big|\big|,$$

where Λ is defined in (2.15). Hence, $\tilde{\Phi}_m$ is bounded from above as $m \to \infty$.

Because $\big(-\nu_m/|\nu_m|, m\big) \in \mathcal{H}_m$, we can estimate

$$\tilde{\Phi}_m \ge \inf_{(a,c)\in\mathcal{H}_m} \Big[-\operatorname{trace}\Big(\mathcal{A}^{(x_m,t_m)}_{a,-\frac{\nu_m}{|\nu_m|}} M_m\Big) - (c+m)\langle a - \nu_m/|\nu_m|, \nu_m\rangle\Big].$$

Now, (4.65) implies that the second term after the infimum is bounded from below as $m \to \infty$. Hence by the definition of the infimum, there exists $(a_m, c_m) \in \mathcal{H}_m$ such that

$$\begin{aligned}\tilde{\Phi}_m \ge &-\operatorname{trace}\Big(\mathcal{A}^{(x_m,t_m)}_{a_m,-\frac{\nu_m}{|\nu_m|}} M_m\Big)\\ &- (c_m+m)\langle a_m - \nu_m/|\nu_m|, \nu_m\rangle - \frac{1}{m}.\end{aligned} \tag{4.66}$$

Next, we prove that

$$a_m \to \frac{\nu}{|\nu|} \tag{4.67}$$

as $m \to \infty$. To establish this, it suffices to show that for given $\eta > 0$, there is $m_0 := m_0(\eta)$ such that

$$\langle a_m, \nu_m\rangle \ge |\nu_m| - \eta$$

for all $m \ge m_0$. We assume, on the contrary, that there is $\eta > 0$ such that for all $m \ge 0$

$$\langle a_m, \nu_m\rangle < |\nu_m| - \eta.$$

Thus in this case, (4.66) implies

$$\tilde{\Phi}_m \ge -n\Lambda\big|\big|M_m\big|\big| + \eta(c_m + m) - \frac{1}{m}.$$

This contradicts the boundedness of $\tilde{\Phi}_m$ as $m \to \infty$, and hence, (4.67) holds.

Recall that the function p is continuous which implies $p(x_m, t_m) \to p(x,t)$ as $m \to \infty$. Therefore by combining the assumptions, (4.65) and (4.67) with (4.66), we get

$$\begin{aligned}\liminf_{m\to\infty} \tilde{\Phi}_m &\ge -\operatorname{trace}\Big(\mathcal{A}^{(x,t)}_{\frac{\nu}{|\nu|},-\frac{\nu}{|\nu|}} M\Big)\\ &= -\big(p(x,t) - 2\big)\frac{\langle M\nu, \nu\rangle}{|\nu|^2} - \operatorname{trace}(M).\end{aligned}$$

Thus, we have proven

$$\liminf_{m\to\infty} F^-_m\big((x_m,t_m), \xi_m, \nu_m, M_m\big) \ge -F\big((x,t), \xi, \nu, M\big).$$

Next, we prove that

$$\limsup_{m\to\infty} F_m^-\big((x_m,t_m),\xi_m,\nu_m,M_m\big) \le -F\big((x,t),\xi,\nu,M\big). \tag{4.68}$$

Again, as $\big(\nu_m/|\nu_m|,m\big)\in\mathcal{H}_m$, we have

$$\tilde\Phi_m \le \sup_{(b,d)\in\mathcal{H}_m}\Big[-\operatorname{trace}\Big(\mathcal{A}^{(x_m,t_m)}_{\frac{\nu_m}{|\nu_m|},b}M_m\Big)-(m+d)\langle\nu_m/|\nu_m|+b,\nu_m\rangle\Big].$$

Because the second term after the supremum is bounded from above by (4.65), we find $\big(b_m,d_m\big)\in\mathcal{H}_m$ such that

$$\tilde\Phi_m \le -\operatorname{trace}\Big(\mathcal{A}^{(x_m,t_m)}_{\frac{\nu_m}{|\nu_m|},b_m}M_m\Big)-(m+d_m)\langle\nu_m/|\nu_m|+b_m,\nu_m\rangle+\frac{1}{m} \tag{4.69}$$

by the definition of the supremum. Moreover, $\tilde\Phi_m$ is bounded also from below, because we can use (4.65) and estimate the supremum in $\tilde\Phi_m$ with the choice $(-\nu_m/|\nu_m|,0)\in\mathcal{H}_m$. This and the estimate (4.69) imply $b_m\to -\nu/|\nu|$ as $m\to\infty$ in a similar way to the above. Therefore, this, together with the estimate (4.65) in the inequality (4.69), by taking $\limsup_{m\to\infty}$, completes the proof of (4.68). □

For all $M\in S(n)$, we utilize the Pucci operators

$$P^+(M):=\sup_{A\in\mathcal{A}_{\lambda,\Lambda}}\operatorname{trace}(AM)$$

and

$$P^-(M):=\inf_{A\in\mathcal{A}_{\lambda,\Lambda}}\operatorname{trace}(AM),$$

where $\mathcal{A}_{\lambda,\Lambda}\subset S(n)$ is the set of symmetric $n\times n$ matrices whose eigenvalues belong to $[\lambda,\Lambda]$.

Lemma 4.6. *Let u_m be the unique solution to (2.17) ensured by Proposition 2.4. Then, the function u_m is Hölder continuous on $\mathbb{R}^n\times[0,T]$ with a Hölder constant independent of m. In particular, the sequence*

$$\{u_m : m\ge 1\}$$

is equicontinuous on $\mathbb{R}^n\times[0,T]$.

Proof. Let $m\ge 1$ and $(x,t)\in\mathbb{R}^n\times(0,T)$. Furthermore, let $\varphi\in C^2\big(\mathbb{R}^n\times(0,T)\big)$ test u_m from below at (x,t). First, we assume $D\varphi(x,t)\neq 0$. Because u_m is a supersolution to (2.17), we can find a vector b_m on a compact set $\mathbb{S}^{n-1}$ such that

$$\begin{aligned}0&\ge \partial_t\varphi(x,t)+\operatorname{trace}\Big(\mathcal{A}^{(x,t)}_{\frac{D\varphi(x,t)}{|D\varphi(x,t)|},b_m}D^2\varphi(x,t)\Big)+\big\langle\mu,D\varphi(x,t)\big\rangle-r\varphi(x,t)\\&\ge \partial_t\varphi(x,t)+P^-(D^2\varphi(x,t))+\big\langle\mu,D\varphi(x,t)\big\rangle-r\varphi(x,t).\end{aligned}$$

Next, we assume $D\varphi(x,t) = 0$. Now, since there is no more gradient dependence in Φ, the term inside inf sup in Φ is always bounded, and hence for any $\nu \in \mathbb{S}^{n-1}$, there is $b_m \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned} 0 &\geq \partial_t \varphi(x,t) + \operatorname{trace}\left(\mathcal{A}_{\nu,b_m}^{(x,t)} D^2\varphi(x,t)\right) + \langle \mu, D\varphi(x,t)\rangle - r\varphi(x,t) \\ &\geq \partial_t \varphi(x,t) + P^-(D^2\varphi(x,t)) + \langle \mu, D\varphi(x,t)\rangle - r\varphi(x,t). \end{aligned}$$

Let $\phi \in C^2\big(\mathbb{R}^n \times (0,T)\big)$ test u_m from above at (x,t). In a similar way to the above, if $D\phi(x,t) \neq 0$, we can find $a_m \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned} 0 &\leq \partial_t \phi(x,t) + \operatorname{trace}\left(\mathcal{A}_{a_m,-\frac{D\phi(x,t)}{|D\phi(x,t)|}}^{(x,t)} D^2\phi(x,t)\right) + \langle \mu, D\phi(x,t)\rangle - r\phi(x,t) \\ &\leq \partial_t \phi(x,t) + P^+\big(D^2\phi(x,t)\big) + \langle \mu, D\phi(x,t)\rangle - r\phi(x,t), \end{aligned}$$

because u_m is a subsolution to (2.17). Furthermore, if $D\phi(x,t) = 0$, for any $\nu \in \mathbb{S}^{n-1}$, there is $a_m \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned} 0 &\leq \partial_t \phi(x,t) + \operatorname{trace}\left(\mathcal{A}_{a_m,\nu}^{(x,t)} D^2\phi(x,t)\right) + \langle \mu, D\phi(x,t)\rangle - r\phi(x,t) \\ &\leq \partial_t \phi(x,t) + P^+(D^2\phi(x,t)) + \langle \mu, D\phi(x,t)\rangle - r\phi(x,t). \end{aligned}$$

Thus, we have shown that u_m is a super- and a subsolution to the equations

$$\begin{cases} \partial_t u_m(x,t) + P^-\big(D^2 u_m(x,t)\big) + \langle \mu, Du_m(x,t)\rangle - ru_m(x,t) = 0, \\ \partial_t u_m(x,t) + P^+\big(D^2 u_m(x,t)\big) + \langle \mu, Du_m(x,t)\rangle - ru_m(x,t) = 0, \end{cases}$$

respectively. Therefore, the classical result of [Wan92, Theorem 4.19], see also [KS80], implies that the function u_m is Hölder continuous with a Hölder constant independent of m. □

We are now in a position to prove the main theorem of the paper.

Proof of Theorem 1.3. By the comparison principle Lemma 2.3 and (2.16), we see that the sequence (u_m) of solutions to (2.17) is uniformly bounded with respect to m. Hence, because Lemma 4.6 holds, by the Arzelà-Ascoli theorem, there exist u, continuous on $\mathbb{R}^n \times [0,T]$, and a subsequence (m_j) such that it holds

$$u_{m_j} \to u$$

uniformly on $\mathbb{R}^n \times [0,T]$ as $j \to \infty$. By Lemmas 4.4 and 4.5, the stability principle for viscosity solutions yields that u is a viscosity solution to (4.44). Therefore by Lemma 3.3, the final part is to show that the value function U_m^- with uniformly bounded controls converges to the value function U^- as $m \to \infty$. This follows from the properties of the infimum and the supremum, because the boundary values g are bounded, and for the set of admissible strategies, it holds $\mathcal{S} = \bigcup_m \mathcal{S}_m$. For more details, see for example [NP17, the proof of Theorem 1.2].

The corresponding proofs in the context of U^+, U_m^+ and the equation (2.18) are analogous to the above. In particular, let u_m^+ be the unique viscosity solution to (2.18). The proof of Lemma 3.2 for u_m^+ and F_m^+ is essentially the same as before. Then by minor adjustments to the proofs of Lemmas 3.3 and 3.5, we can show that $u_m^+ = U_m^+$ on $\mathbb{R}^n \times [0, T]$. Finally, the uniform boundedness and the equicontinuity of the family (u_m^+), together with the convergence of U_m^+ to U^+ as $m \to \infty$, follows as before. Therefore, the proof is complete.

□

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