This is an electronic reprint of the original article.
This reprint may differ from the original in pagination and typographic detail.

Author(s): Hencl, Stanislav; Onninen, Jani

Title: Jacobian of weak limits of Sobolev homeomorphisms

Year: 2018

Version:

Please cite the original version:

All material supplied via JYX is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.
Research Article

Stanislav Hencl and Jani Onninen*

**Jacobian of weak limits of Sobolev homeomorphisms**

DOI: 10.1515/acv-2016-0005

Received February 7, 2016; revised June 2, 2016; accepted June 3, 2016

**Abstract:** Let $\Omega$ be a domain in $\mathbb{R}^n$, where $n = 2, 3$. Suppose that a sequence of Sobolev homeomorphisms $f_k : \Omega \to \mathbb{R}^n$ with positive Jacobian determinants, $J(x, f_k) > 0$, converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, for some $p \geq 1$, to a mapping $f$. We show that $J(x, f) \geq 0$ a.e. in $\Omega$. Generalizations to higher dimensions are also given.

**Keywords:** Sobolev homeomorphism, weak limits, Jacobian

**MSC 2010:** Primary 26B10; secondary 46E35

Communicated by: Juha Kinnunen

1 Introduction

The main goal of this note is to establish when the sign of the Jacobian is preserved under $W^{1,p}$-weak convergence. Such a question pops out naturally in the variational approach to Geometric Function Theory (GFT) [2, 14, 22] and Nonlinear Elasticity (NE) [1, 4, 6, 19, 24, 25]. Both theories GFT and NE deal with minimizing sequences of Sobolev homeomorphisms. In the context of NE, one typically deals with two-dimensional or three-dimensional models and require that the deformation gradients belong to $M_{n,n}$, where $M_{m,n} = \{\text{real} \, m \times n \, \text{matrices}\}$, and $M_{n,n}^{+} = \{A \in M_{n,n} : \det A > 0\}$. The infimum of the energy is not attained, in general, at a homeomorphism; interpenetration of matter may occur. Even in a special case of Dirichlet energy injectivity is often lost when passing to the weak limit of the minimizing sequence, [3, 13, 15, 16]. Further examinations are needed to know the properties of such singular minimizers.

Throughout this text $\Omega$ will be a domain in $\mathbb{R}^n$. The class of Sobolev mappings $f : \Omega \to \mathbb{R}^n$ with nonnegative Jacobian determinant, $J(x, f) = \det Df(x) \geq 0$ almost everywhere, is closed under the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^n)$ provided $p > n$ (see [14, Theorem 8.4.2]). However, if $p \leq n$, passing to the weak $W^{1,p}$-limit of a sequence with nonnegative Jacobians one may lose the sign of the Jacobian. Indeed, there exists a sequence of Sobolev mappings $f_k : \Omega \to \mathbb{R}^n$ with $J(x, f_k) > 0$ almost everywhere such that the sequence converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p < n$, to the mapping $f(x) = (-x_1, x_2, \ldots, x_n)$, see [14, p. 181]. Moreover, following the construction in [18] such mappings $f_k$ can be made continuous. However, it is not obvious at all as to whether one can make a similar example with $f_k$ being homeomorphisms. This is the subject of our result here. Here $[\frac{q}{p}]$ denotes the integer part, i.e. $[\frac{2}{2}] = 1$, $[\frac{3}{2}] = 1$ and so on.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be a domain and let $p \geq 1$ for $n \in \{2, 3\}$ and $p > [\frac{q}{2}]$ for $n \geq 4$. Suppose that a sequence of Sobolev homeomorphisms $f_k : \Omega \to \mathbb{R}^n$ with $J(x, f_k) \geq 0$ converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$ to a mapping $f$ and further assume that $f(x, f_k)$ is not a.e. zero. Then $J(x, f) \geq 0$ a.e. in $\Omega$.

**Stanislav Hencl:** Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ 186 00 Prague 8, Czech Republic, e-mail: hencl@karlin.mff.cuni.cz

**Corresponding author:** Jani Onninen: Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA;

and Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland, e-mail: jkonnine@syr.edu
It is worth noting that in Theorem 1 the Jacobian \( J(x, f) \) can have very different behavior than the Jacobians in the sequence without knowing that \( f(x, f_k) > 0 \) on a set of positive measure. Indeed, there exists a sequence of Sobolev homeomorphisms \( f_k \) with \( f(x, f_k) = 0 \) a.e., converging weakly in \( W^{1,p}(\Omega, \mathbb{R}^n) \), \( 1 \leq p < n \), to the mapping \( f(x) = x \). Let us briefly sketch this using the construction from [10]: we cover \( \Omega \) by cubes of diameter less than \( \frac{1}{k} \) and on each cube we follow the construction from [10] to obtain a homeomorphism with zero Jacobian a.e. It is possible to make the \( W^{1,p} \)-norm of the sequence uniformly bounded and hence find a weakly convergent subsequence. Furthermore, it follows from the construction that the sequence \( f_k \) converges uniformly to the identity. This also shows that there is a sequence with \( f(x, f_k) = 0 \) a.e. converging weakly in \( W^{1,p}(\Omega, \mathbb{R}^n) \), \( 1 \leq p < n \), to \( f(x) = (−x_1, x_2, \ldots , x_n) \).

Recently it was shown in [12] and [5] that a Jacobian of a Sobolev homeomorphism can change sign in dimension \( n \geq 4 \) for \( 1 \leq p < \left( \frac{n}{2} \right) \).

### 2 Preliminaries

#### 2.1 Degree and Jacobian

There are two basic approaches to the notion of local degree for a mapping, the algebraic (see e.g. Dold [7]) and the analytic (see e.g. Lloyd [17]). Both of these notions try to capture the idea of counting the preimages of a target point. For a continuous mapping \( f: \Omega \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \setminus \{ f(\Omega) \} \) the degree of \( f \) at \( y \) with respect to \( \Omega \) is denoted by \( \deg(f, \Omega, y) \). If \( f: \Omega \to \mathbb{R}^n \) is a homeomorphism, then \( \deg(f, \Omega, y) \) is either 1 or −1 for all \( y \in f(\Omega) \), see e.g. [17, Section IV.5] or [21, Section II.2.4, Theorem 3]. We say that a homeomorphism \( f \) is sense-preserving if \( \deg(f, \Omega, y) = 1 \). For a linear map \( A: \mathbb{R}^n \to \mathbb{R}^n \) with \( \det A \neq 0 \), it is easy to check from the definition that

\[
\deg(A, \Omega, y) = \text{sgn} \det A.
\]  (1)

We recall the following corollary [2, Corollary 2.8.2]. Given a homeomorphism \( f: \Omega \to \mathbb{R}^n \) suppose that \( f \) is differentiable at \( x \), with \( f(x, f) \neq 0 \). Then we have

\[
\deg(f, \Omega, f(x)) = \text{sgn} J(x, f).
\]  (2)

We will use the fact that the topological degree is stable under homotopy. That is for every continuous mapping \( H: \overline{\Omega} \times [0, 1] \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) such that \( y \notin H(\partial \Omega, t) \) for all \( t \in [0, 1] \) we have

\[
\deg(H(\cdot, 0), \Omega, y) = \deg(H(\cdot, 1), \Omega, y).
\]  (3)

#### 2.2 Differentiability of Sobolev mappings

A Sobolev homeomorphism \( f \in W^{1,p}(\Omega, \mathbb{R}^n) \) is differentiable almost everywhere if \( p > n−1, n \geq 3, \) and \( p \geq 1 \) for \( n = 2 \), see [9, 20, 26]. We will also need a generalization of the concept of differentiability, which is obtained by replacing the ordinary limit by an approximate limit, see e.g. [8, Section 6.1.3]. It is known that a Sobolev mapping \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \) is approximatively differentiable almost everywhere, see e.g. [8, Section 6.1.2, Theorem 2]. Moreover, such a mapping is \( L^1 \)-differentiable almost everywhere [27]; that is, for almost every \( x \in \Omega \) we have

\[
\lim_{r \to 0} \left\{ \int_{B(x, r)} \frac{|f(x) − f(x_*) − Df(x_*)(x − x_*)|}{r} \, dx \right\} = 0.
\]  (4)

Hereafter, the notation \( \int_{B(x, r)} \) means the integral average over the \( n \)-dimensional ball

\[
B(x_*, r) = \{ x \in \mathbb{R}^n : |x − x_*| < r \}.
\]

In order to illustrate our ideas and for reader’s comprehension, we first prove Theorem 1 in the simpler cases \( p \geq 1, n = 2; \) and \( p > n−1, n \geq 3, \) where we can avoid some technicalities.
3 Proof of Theorem 1 for $p > n - 1$, $n \geq 3$, and $p \geq 1$, $n = 2$

Each homeomorphism $f_j$ is either sense-preserving or sense-reversing. Under our assumptions there exists a point $x_j$ such that $f_j$ is differentiable at $x_j$, see Section 2.2, and $J(x_j, f_j) > 0$. By (2) we know that the degree of $f_j$ is one and hence each $f_j$ is sense-preserving.

As $f_j \to f$ in $L^p$, $p > 1$, we know that $\int_{\Omega} |Df_j|^p$ is uniformly bounded and hence we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$|Df_j|^p \rightharpoonup^w \mu \text{ in measures.}$$

Moreover, for $p = 1$ we can use De La Vale Poussin characterization of weak convergence in $L^1$ and we can find an continuous convex function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$\frac{\Phi(t)}{t} \text{ is increasing, } \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty \text{ and } \int_{\Omega} \Phi(|Df_j|) \leq 1. \tag{5}$$

It follows that we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$\Phi(|Df_j|) \rightharpoonup^w \mu \text{ in measures.}$$

It is well known that for almost every $x_\epsilon \in \Omega$ we have

$$M\mu(x_\epsilon) := \sup_{r > 0} \frac{\mu(B(x_\epsilon, r) \cap \Omega)}{|B(x_\epsilon, r)|} < \infty. \tag{6}$$

Let $\delta > 0$. For the contrary we suppose that there is $x_\epsilon \in \Omega$ such that (4) and (6) hold at $x_\epsilon$ and $J(x_\epsilon, f) < 0$.

Without loss of generality we may and do assume that

$$Df(x_\epsilon) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}. \tag{7}$$

Using (4) we can find $0 < r_1$ small enough such that for all $0 < r < r_1$ we have

$$\int_{B(x_\epsilon, r)} \left| f(x) - f(x_\epsilon) - Df(x_\epsilon)(x - x_\epsilon) \right| r < \frac{\delta^n}{2}. \tag{8}$$

Since the sequence of mappings $f_j$ converges to $f$ weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, we have that the sequence of mappings $f_j$ converges to $f$ strongly in $L^{1}_{\operatorname{loc}}(\Omega, \mathbb{R}^n)$. Now, we may pick up an index $j$, large enough such that for all $j \geq j$,

$$\int_{\Omega} |f(x) - f_j(x)| dx < |B(0, 1)| \pi^{n+1} \frac{\delta^n}{2}.$$ 

The last two inequalities imply that for all $0 < r < r_1$ we have

$$\int_{B(x_\epsilon, r)} \left| f_j(x) - f_j(x_\epsilon) - Df_j(x_\epsilon)(x - x_\epsilon) \right| r < \delta^n. \tag{8}$$

Our next goal is to prove the following:

(i) if $p > n - 1$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j$, 

$$\delta^{n-1-p} r^{p} \leq C \int_{B(x_\epsilon, r)} |Df_j|^p.$$
(ii) if \( n = 2 \) and \( p = 1 \), there exist a constant \( C \) and such that for all \( 0 < r < r_1 \) and \( j \geq j_1 \), there is a set \( A \subset B(x, r) \) such that
\[
|A| < C \delta |B(x, r)| \quad \text{and} \quad r^2 \leq C \int_A |Df_j|.
\]

These would lead to a desired contradiction. Indeed, choose \( 0 < r < r_1 \) such that \( \mu(\partial B(x, r)) = 0 \) and in case (i) we obtain after passing to a limit in \( j \) that
\[
\delta^{n-1-p} \leq C \lim_{j \to \infty} \int_{B(x, r)} |Df_j|^p = C \frac{\mu(B(x, r) \cap \Omega)}{|B(x, r)|} \leq CM\mu(x).
\]

After passing \( \delta \to 0^+ \) we obtain a contradiction with (6). In case (ii) we can use Jensen’s inequality and (5) to obtain
\[
\int_{B(x, r)} \Phi(|Df_j|) \geq \frac{|A|}{r^2} \int_A \Phi(|Df_j|) \geq \frac{|A|}{r^2} \Phi \left( \frac{\int_A |Df_j|}{|A|} \right) \geq C \delta \Phi \left( \frac{C}{\delta} \right).
\]

Similarly as above we obtain in the limit that
\[
C \delta \Phi \left( \frac{C}{\delta} \right) \leq CM\mu(x)
\]
and now passing to a limit \( \delta \to 0^+ \) we obtain a contradiction using (5).

**Proof of (i).** We simplify the notation and write
\[
\varphi_j(x) = |f_j(x) - f(x, j - Df(x))(x - x_j)| \quad \text{and} \quad B_s = B(x, s).
\]

In the following we use the notation \( \mathcal{H}^k(A) \) for the \( k \)-dimensional Hausdorff measure of the set \( A \). We claim that the set of radii
\[
I_G = \left\{ s \in [0, r] : \mathcal{H}^{n-1}(\{ x \in \partial B_s : \varphi_j(x) \geq \delta r \}) < 5^n \delta^{n-1} \mathcal{H}^{n-1}(\partial B_s) \right\}
\]
has measure at least \( \frac{3r}{5} \), i.e. \( |I_G| \geq \frac{3r}{5} \), otherwise
\[
\int_{B_s} \frac{|\varphi_j(x)|}{r} \, dx \geq \frac{1}{|B_s|} \int_0^s 5^n \delta^{n-1} \mathcal{H}^{n-1}(\partial B_s) \frac{\delta r}{r} \, ds = 5^n \delta^n \frac{|B_{\delta r}|}{|B_r|}
\]
which contradicts (8).

On the other hand, the key point in our argument is that for \( x \in \Omega \), and for every \( s \in (0, r) \) we can find \( \beta = \beta(s) \in \partial B_s \) such that
\[
\varphi_j(\beta) \geq \frac{4}{5} s \quad \text{for every} \quad j = 1, 2, \ldots.
\]

Finding such a point \( \beta \) is the only place where we use the homeomorphism assumption of \( f_j \). Suppose on the contrary that (9) fails for every \( \beta \in \partial B_s \) and for some \( j \in \{1, 2, \ldots\} \). For \( x \in \partial B_s \) and \( t \in [0, 1] \) we consider the following homotopy:
\[
H(x, t) := (1 - t)(f_j(x) - f(x, t)) + tDf(x, j)(x - x_j).
\]

By (7) we know that \( Df(x, j) \) is an isometry and thus \( |Df(x, j)| = |z| \). Furthermore, if (9) does not hold, then for all \( x \in \partial B_s \) we have
\[
|H(x, t)| \geq |Df(x_0)(x - x_0)| - (1 - t)|f_j(x) - f(x_0) - Df(x_0)(x - x_0)| \geq s - (1 - t) \frac{4}{5}s > 0.
\]

It follows that \( H(x, t) \neq 0 \) for every \( x \in \partial B_s \) and all \( t \in [0, 1] \). Thus, by (3) and (1),
\[
\deg(f_j, B_s, f(x)) = \text{sgn det}(Df(x)) = -1.
\]

This contradicts the fact that \( f_j \) is sense-preserving.
We apply the Sobolev embedding theorem [8, Theorem 3 (i), p. 143] on the \((n - 1)\)-dimensional spheres. This way for almost every \(s \in (0, r)\) and for all \(z_1, z_2 \in \partial B(x, s)\) we have

\[
|f_j(z_1) - f_j(z_2)| \leq C(n, p)|z_1 - z_2|^{1 - \frac{n}{np}}\left(\int_{\partial B_s} |Df_j|^p\right)^{\frac{1}{p}}. \tag{10}
\]

Now let us fix \(s \in I_G\) so that (10) is satisfied on the sphere \(\partial B_s\). Since \(s \in I_G\), we find \(a = a(s) \in \partial B_s\) satisfying

\[
\varphi_j(a) < \delta r \quad \text{and} \quad |a - \beta| \leq C_0 \delta s,
\]

where \(C_0\) is some fixed constant (which depends only on \(n\)). Combining this with (9) we have found \(a, \beta \in \partial B_s\) such that

\[
\frac{4}{5}s - \delta r - 2C_0 \delta s \leq |\varphi_j(\beta)| - |\varphi_j(a)| - 2|a - \beta| \leq |f_j(\alpha) - f_j(\beta)|.
\]

This together with (10) implies that for \(s \in I_G \cap \left[\frac{r}{2}, r\right]\) and \(\delta\) small enough

\[
C\delta^p \leq \left(\frac{4}{5}s - \delta r - 2C_0 \delta s\right)^p \leq C(n, p)(\delta s)^{p-n+1} \int_{\partial B_s} |Df_j|^p. \tag{11}
\]

Integrating inequality (11) over the set \(I_G \cap \left[\frac{r}{2}, r\right]\) we obtain (i), finishing the proof of Theorem 1 in the case \(p > n - 1\).

**Proof of (ii).** We proceed as above. For \(s \in I_G\) we can find \(\beta = \beta(s) \in \partial B_s\) so that (9) holds. In fact we consider the measurable set

\[
A := \{x \in B_r : \varphi_j(x) > \delta r\}.
\]

By Chebyshev’s inequality and (8) we obtain

\[
|A| \leq \frac{1}{\delta r} \int_{B_s} |\varphi_j(x)| \, dx \leq \frac{1}{\delta r} \delta^2 r^2 = C\delta |B_r|.
\]

Let \(s \in I_G \cap \left[\frac{r}{2}, r\right]\). The point \(\beta \in \partial B_s\) with (9) clearly belongs to \(A \cap \partial B_s\) and the closest point \(a\) on the relative boundary of \(\partial B_s \cap A\) satisfies

\[
|\varphi_j(a)| = \delta r
\]

by the definition of \(A\). It follows that for every \(s \in I_G \cap \left[\frac{r}{2}, r\right]\) we have

\[
s \leq C \int_{\partial B_s \cap A} |Df_j|.
\]

Integrating this over \(I_G \cap \left[\frac{r}{2}, r\right]\) we obtain

\[
r^2 \leq C \int_A |Df_j|
\]

finishing the proof of (ii).

The above proof was based on the Sobolev embedding theorem on spheres and therefore does not work for \(p < n - 1\). To overcome these difficulties we follow Hencl and Malý [11] and use the theory of linking numbers and its topological invariance. For the convenience of the reader we recall the needed properties of linking numbers here.

## 4 Linking number

We use the notation \(B_d\) for the unit ball in \(\mathbb{R}^d\) and \(S_{d-1}\) for the unit sphere. By \(\overline{B}_d(c, r)\) we denote the closed ball with center \(c\) and radius \(r > 0\).
Let \( n, t, q \) be positive integers with \( t + q = n - 1 \). Let us consider the mapping \( \Phi(\xi, \eta) : \mathbb{B}_{t-1} \times \mathbb{B}_{q+1} \to \mathbb{R}^n \) defined coordinatewise as \( \Phi(\xi, \eta) = x \), where
\[
\begin{align*}
x_1 &= (2 + \eta_1)\xi_1, \\
\vdots \\
x_{t+1} &= (2 + \eta_1)\xi_{t+1}, \\
x_{t+2} &= \eta_2, \\
\vdots \\
x_{t+q+1} &= \eta_{q+1}.
\end{align*}
\]
Denote by \( \mathbb{A} \) the annulus
\[
\Phi(\mathbb{S}_t \times \mathbb{B}_{q+1}) = \{ x \in \mathbb{R}^n : \left( \sqrt{x_1^2 + \cdots + x_{t+1}^2 - 2} \right)^2 + x_{t+2}^2 + \cdots + x_n^2 < 1 \}.
\]
Of course, given \( x \in \mathbb{A} \) we can find a unique \( \xi \in \mathbb{S}_t \) and \( \eta \in \mathbb{B}_{q+1} \) such that \( \Phi(\xi, \eta) = x \). We will denote these as \( \xi(x) \) and \( \eta(x) \).

A **link** is a pair \((\varphi, \psi)\) of parametrized surfaces \( \varphi : \mathbb{S}_t \to \mathbb{R}^n \), \( \psi : \mathbb{S}_q \to \mathbb{R}^n \). The **linking number** of the link \((\varphi, \psi)\) is defined as the topological degree
\[
\mathcal{L}(\varphi, \psi) = \text{deg}(L, \mathbb{A}, 0),
\]
where the mapping \( L = L_{\varphi, \psi} : \mathbb{A} \to \mathbb{R}^n \) is defined as
\[
L(x) = \varphi(\xi(x)) - \psi(-\eta(x)),
\]
or equivalently
\[
L(\Phi(\xi, \eta)) = \varphi(\xi) - \psi(-\eta), \quad \xi \in \mathbb{S}_t, \: \eta \in \mathbb{B}_{q+1},
\]
where \( \psi \) is an arbitrary continuous extension of \( \psi \) to \( \mathbb{B}_{q+1} \) (of course, the degree does not depend on the way how we extend \( \psi \), it depends only on the values on the boundary \( \partial \mathbb{A} = \Phi(\mathbb{S}_t \times \mathbb{S}_q) \)). Geometrically speaking, for \( t = q = 1 \), the linking number is the number of loops of a curve \( \varphi \) around a curve \( \psi \) counting orientation into account as \(+1\) or \(-1\). For the introductions to the linking number in \( \mathbb{R}^3 \) and its application to the theory of knots see [23].

The **canonical link** is the pair \((\mu, \nu)\), where
\[
\begin{align*}
\mu(\xi) &= \Phi(\xi, 0), \quad \xi \in \mathbb{S}_t, \\
\nu(\eta) &= \Phi(e_1, \eta), \quad \eta \in \mathbb{S}_q.
\end{align*}
\]
For example in dimension \( n = 3 \) we get that
\[
\begin{align*}
\mu(\mathbb{S}_1) &= \{ x \in \mathbb{R}^3 : x_3 = 0, \: x_1^2 + x_2^2 = 4 \}, \\
\nu(\mathbb{S}_1) &= \{ x_2 = 0, \: (x_1 - 2)^2 + x_3^2 = 1 \}.
\end{align*}
\]
It is well known that the linking number is a topological invariant. The simple proof of the following proposition can be found in [11].

**Proposition 2.** Let \( n, t, q \) be positive integers with \( t + q = n - 1 \). Let \( f : \mathbb{B}_n(4) \to \mathbb{R}^n \) be a homeomorphism. Then \( \mathcal{L}(f \circ \mu, f \circ \nu) \) is 1 if \( f \) is sense preserving and \(-1\) if \( f \) is sense reversing.

Analogously, we can pick \( a \in \mathbb{B}_{q+1}(0, \frac{1}{10}) \) and \( b \in \mathbb{B}_{t+1}(e_1, \frac{1}{10}) \cap \mathbb{B}_{t+1} \) and consider the pair
\[
\begin{align*}
\mu_a(\xi) &= \Phi(\xi, a), \quad \xi \in \mathbb{S}_t, \\
\nu_b(\eta) &= \Phi(b, \eta), \quad \eta \in \mathbb{S}_q.
\end{align*}
\]
Similarly to the previous proposition we have:

**Proposition 3.** Let \( n, t, q \) be positive integers with \( t + q = n - 1 \), \( a \in \mathbb{B}_{q+1}(0, \frac{1}{10}) \) and \( b \in \mathbb{B}_{t+1}(e_1, \frac{1}{10}) \cap \mathbb{B}_{t+1} \). Let \( f : \mathbb{B}_n(4) \to \mathbb{R}^n \) be a homeomorphism. Then \( \mathcal{L}(f \circ \mu_a, f \circ \nu_b) \) is 1 if \( f \) is sense preserving and \(-1\) if \( f \) is sense reversing.
5 Proof of Theorem 1 for \( p > \left[ \frac{q}{2} \right] \), \( n \geq 3 \), and \( p \geq 1 \), \( n = 3 \)

Our argument is similar to the proof given in Section 3 and therefore some details are only sketched. By \( \mu \) we again denote the \( w^* \) limit of some subsequence \( \int |Df|^p \) for \( p > \left[ \frac{q}{2} \right] \) and of \( \int \Phi(|Df|) \) for \( p = 1 \) and \( n = 3 \).

By \( C_1 \) and \( C_2 \) we denote a fixed constants whose exact value will be determined later. We fix \( \delta > 0 \) and we choose a point \( x_0 \) such that (4) and (6) hold and without loss of generality we assume that the derivative of \( f \) at \( x_0 \) is given by (7).

We fix \( r_1 > 0 \) such that for all \( 0 < r < r_1 \) we have
\[
\int_{B(x_i, 4r)} \left| \frac{f(x) - f(x_0) - Df(x)(x-x_0)}{r} \right| dx < C_1 \delta^n \frac{\delta}{2}
\]

and again for all \( j \geq j \), we obtain
\[
\int_{B(x_i, 4r)} \left| \frac{f_j(x) - f(x_0) - Df(x)(x-x_0)}{r} \right| dx < C_1 \delta^n. \tag{12}
\]

We fix \( t, q \leq \left[ \frac{q}{2} \right] \) such that \( t + q = n - 1 \) (e.g. \( t = q = \frac{n-1}{2} \) for \( n \) odd and \( t = \frac{n-2}{2}, q = \frac{n}{2} \) for \( n \) even). Our goal is to prove the following:

(i) if \( p > \left[ \frac{q}{2} \right] \) and \( n \geq 3 \), then there exists a constant \( C \) (depending only on \( p \) and \( n \)) such that for all \( 0 < r < r_1 \) and \( j \geq j \),
\[
\delta^{\min(t, q) - p} r^n \leq C \int_{B(x_i, 4r)} |Df|^p,
\]

(ii) if \( p = 1 \) and \( n = 3 \), we have \( A \subset B(x_i, 4r) \) such that
\[
|A| < C_2 \delta |B(x_i, 4r)| \quad \text{and} \quad r^3 \leq C \int_{A} |Df|.
\]

Analogously to reasoning in Section 3 we obtain a contradiction using \( \min(t, q) - p < 0 \) for \( p > \left[ \frac{q}{2} \right] \) and (5) for \( p = 1 \) and \( n = 3 \).

**Proof of (i).** Without loss of generality we will assume that \( x_0 = 0 \). We write
\[
\varphi(x) = |f_j(rx) - f(0) - Df(0)rx|.
\]

Let us fix \( y \in \mu_d(S_t) \) and denote
\[
B_{\mu_d(S_t)}(y, \delta) = \{ x \in \mu_d(S_t) : |x-y| < \delta \},
\]
the ball of radius \( \delta \) on the link \( \mu_d(S_t) \). We can clearly choose a constant \( C_1 \) small enough at the beginning of the proof so that (12) implies that the set of good links
\[
I_a = \left\{ a \in B_{q+1} \left( 0, \frac{1}{10} \right) : \mathcal{H}^q(x \in \mu_d(S_t) : \varphi_a(x) \geq \delta r) < \mathcal{H}^q(\mu_d(S_t)(y, \delta)) \right\},
\]
\[
I_b = \left\{ b \in B_{t+1} \left( e_1, \frac{1}{10} \right) \cap B_{t+1} : \mathcal{H}^q(x \in \nu_b(S_q) : \varphi_b(x) \geq \delta r) < \mathcal{H}^q(\nu_b(S_q)(y, \delta)) \right\}
\]

has measure at least
\[
\mathcal{H}^q(I_a) > \frac{1}{2} \mathcal{H}_{q+1} \left( 0, \frac{1}{10} \right) \quad \text{and} \quad \mathcal{H}^{t+1}(I_b) > \frac{1}{2} \mathcal{H}_{t+1} \left( e_1, \frac{1}{10} \right) \cap B_{t+1}.
\]

The key point of our argument is that for every \( a \in B_{q+1}(0, \frac{1}{10}) \) and every \( b \in B_{t+1}(e_1, \frac{1}{10}) \cap B_{t+1} \) we can find \( \xi \in S_t \) and \( \eta \in S_q \) such that
\[
\varphi_a(\mu_d(\xi)) = |f_j(r\mu_d(\xi)) - f(0) - Df(0)r\mu_d(\xi)| > \frac{r}{10} \quad \text{or}
\]
\[
\varphi_b(\nu_b(\eta)) = |f_j(r\nu_b(\eta)) - f(0) - Df(0)r\nu_b(\eta)| > \frac{r}{10}.
\]

\[
\Rightarrow r < \frac{10}{\mathcal{H}^q(\mu_d(\xi))} \quad \text{or} \quad r < \frac{10}{\mathcal{H}^q(\nu_b(\eta))}
\]
We prove the observation by contradiction and we suppose that (13) does not hold. We define
\[
f_s(x) = (1 - s)(f(0) + Df(0)rx) + sf_j(rx)
\]
and we consider the homotopy \( H(\mathbb{R}^n) \rightarrow \mathbb{R}^n \) defined as
\[
H(\Phi(\xi, \eta), s) = (f_s \circ \mu_a)(\xi) - (f_s \circ v_b)(-\eta),
\]
where \((f_s \circ v_b)\) denotes a continuous extension of \( f_s \circ v_b \to \mathbb{R}^n \) as in the definition of the linking number, which in addition depends continuously on \( s \). From [11] we know that the mapping \( f_j \in W^{1,p}, p > \frac{q}{2} \), with nonnegative and nonzero Jacobian is sense preserving. By Proposition 3 we get that
\[
\text{deg}(H(x, 1), A, 0) = 1.
\]
On the other hand
\[
\text{deg}(H(x, 0), A, 0) = -1
\]
since the affine mapping \( f(0) + Df(0)rx \) is sense reversing. To obtain a contradiction (with the preservation of the degree under homotopy) it is now enough to show that for every \( \xi \in S_t \), for every \( \eta \in S_q \) and for every \( s \in [0, 1] \) we have \( H(\Phi(\xi, \eta), s) \neq 0 \). It is easy to see that
\[
\text{dist}(f_0 \circ \mu_a(S_t), (f_0 \circ v_b)(S_q)) \geq \text{dist}(f_0 \circ \mu(S_t), (f_0 \circ v)(S_q)) - \frac{6r}{10} \geq \frac{3r}{10}.
\]
Since (13) does not hold, we obtain from the definition of \( f_s \) that
\[
\text{dist}(f_s \circ \mu_a(S_t), (f_s \circ v_b)(S_q)) \geq \frac{3r}{10} - \frac{r}{10} - \frac{r}{10}
\]
which implies \( H(\Phi(\xi, \eta), s) \neq 0 \).

By (13) and the symmetry we may assume without loss of generality that
\[
\tilde{I}_a = \{ a \in I_a : \text{there exists } \xi \in S_t \text{ such that } \varphi_j(\mu_a(\xi)) > \frac{r}{10} \}
\]
satisfies \( 3t^d \tilde{I}_a \geq \frac{1}{2} |B_{q+1}(0, \frac{1}{10})| \). Since \( p > \frac{q}{2} \geq t \), we can use the Sobolev embedding theorem on the \( t \)-dimensional space \( r\mu_a(S_t) \) and we have for almost every \( a \in \tilde{I}_a \) and for all \( z_1, z_2 \in r\mu_a(S_t) \),
\[
|f_j(z_1) - f_j(z_2)| \leq C|z_1 - z_2|^{1 - \frac{q}{p}} \left( \int_{r\mu_a(S_t)} |Df_j|^p \right)^{\frac{1}{p}}.
\]
(14)

Now let us fix \( a \in \tilde{I}_a \) so that (14) is satisfied and find \( \xi \in S_t \) so that for \( \beta = \mu_a(\xi) \) we have \( \varphi_j(\beta) > \frac{r}{10} \) as in the definition of \( \tilde{I}_a \). Using \( a \in I_a \) we find \( a \in \mu_a(S_t) \) satisfying
\[
\varphi_j(\alpha) < \delta r \quad \text{and} \quad |\alpha - \beta| \leq \delta.
\]
Thus we have found \( \alpha, \beta \in \mu_a(S_t) \) such that
\[
\frac{r}{10} - 3\delta r \leq |\varphi_j(\beta)| - |\varphi_j(\alpha)| - 2r|\alpha - \beta| \leq |f_j(\alpha) - f_j(\beta)|.
\]
This together with (14) implies that for almost every \( a \in \tilde{I}_a \) and \( \delta \) small enough we have
\[
r^t \leq C\delta^{t-1} \int_{r\mu_a(S_t)} |Df_j|^p.
\]
(15)

Integrating inequality (15) over the set \( \tilde{I}_a \) we obtain (i).

Proof of (ii). If \( p = 1 \) and \( n = 3 \), then for each \( a \in \tilde{I}_a \) we can find \( \xi \in S(t) \) so that (13) holds for \( \beta = \mu_a(\xi) \). The measurable set
\[
A := \{ x \in B(x, 4r) : \varphi_j(x) > \delta r \}
\]
satisfies \( |A| \leq C\delta |B(x, 4r)| \) by inequality (12) and Chebyshev’s inequality. Now clearly \( \xi \) with (13) satisfies
\[
r\beta = r\mu_a(\xi) \in A.
\]
In (14) and (15) instead of integrating over the entire \( r\mu_a(S_t) \) we integrate only over the set \( r\mu_a(S_t) \cap A \). Integrating over \( \tilde{I}_a \) we obtain the desired conclusion.
Acknowledgment: The authors would like to thank the referee for carefully reading the manuscript and for his comments that helped to improve it.

Funding: Stanislav Hencl was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education. Jani Onninen was supported by the NSF grant DMS-1301570.

References