Research Article

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Jacobian of weak limits of Sobolev homeomorphisms

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Abstract: Let $\Omega$ be a domain in $\mathbb{R}^n$, where $n = 2, 3$. Suppose that a sequence of Sobolev homeomorphisms $f_k: \Omega \to \mathbb{R}^n$ with positive Jacobian determinants, $J(x, f_k) > 0$, converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, for some $p \geq 1$, to a mapping $f$. We show that $J(x, f) \geq 0$ a.e. in $\Omega$. Generalizations to higher dimensions are also given.

Keywords: Sobolev homeomorphism, weak limits, Jacobian

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1 Introduction

The main goal of this note is to establish when the sign of the Jacobian is preserved under $W^{1,p}$-weak convergence. Such a question pops out naturally in the variational approach to Geometric Function Theory (GFT) [2, 14, 22] and Nonlinear Elasticity (NE) [1, 4, 6, 19, 24, 25]. Both theories GFT and NE deal with minimizing sequences of Sobolev homeomorphisms. In the context of NE, one typically deals with two-dimensional or three-dimensional models and require that the deformation gradients belong to $M^{m \times n}_{\text{rel}}$, where $M^{m \times n}_{\text{rel}} = \{\text{real } m \times n \text{ matrices}\}$, and $M^{m \times n}_{\text{rel}} = \{A \in M^{m \times n}_{\text{rel}} : \det A > 0\}$. The infimum of the energy is not attained, in general, at a homeomorphism; interpenetration of matter may occur. Even in a special case of Dirichlet energy injectivity is often lost when passing to the weak limit of the minimizing sequence, [3, 13, 15, 16]. Further examinations are needed to know the properties of such singular minimizers.

Throughout this text $\Omega$ will be a domain in $\mathbb{R}^n$. The class of Sobolev mappings $f: \Omega \to \mathbb{R}^n$ with nonnegative Jacobian determinant, $J(x, f) = \det Df(x) \geq 0$ almost everywhere, is closed under the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^n)$ provided $p \geq n$ (see [14, Theorem 8.4.2]). However, if $p < n$, passing to the weak $W^{1,p}$-limit of a sequence with nonnegative Jacobians one may lose the sign of the Jacobian. Indeed, there exists a sequence of Sobolev mappings $f_k: \Omega \to \mathbb{R}^n$ with $J(x, f_k) > 0$ almost everywhere such that the sequence converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p < n$, to the mapping $f(x) = (-x_1, x_2, \ldots, x_n)$, see [14, p. 181]. Moreover, following the construction in [18] such mappings $f_k$ can be made continuous. However, it is not obvious at all as to whether one can make a similar example with $f_k$ being homeomorphisms. This is the subject of our result here. Here $\lceil \frac{n}{2} \rceil$ denotes the integer part, i.e. $\lceil \frac{2}{2} \rceil = 1$, $\lceil \frac{3}{2} \rceil = 1$ and so on.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be a domain and let $p \geq 1$ for $n \in \{2, 3\}$ and $p > \lceil \frac{n}{2} \rceil$ for $n \geq 4$. Suppose that a sequence of Sobolev homeomorphisms $f_k: \Omega \to \mathbb{R}^n$ with $J(x, f_k) \geq 0$ converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$ to a mapping $f$ and further assume that $f(x, f_k)$ is not a.e. zero. Then $J(x, f) \geq 0$ a.e. in $\Omega$.

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It is worth noting that in Theorem 1 the Jacobian \( f(x, f) \) can have very different behavior than the Jacobians in the sequence without knowing that \( f(x, f_k) > 0 \) on a set of positive measure. Indeed, there exists a sequence of Sobolev homeomorphisms \( f_k \) with \( f(x, f_k) = 0 \) a.e., converging weakly in \( W^{1,p}(\Omega, \mathbb{R}^n) \), \( 1 \leq p < n \), to the mapping \( f(x) = x \). Let us briefly sketch this using the construction from [10]: we cover \( \Omega \) by cubes of diameter less than \( \frac{1}{k} \) and on each cube we follow the construction from [10] to obtain a homeomorphism with zero Jacobian a.e. It is possible to make the \( W^{1,p} \)-norm of the sequence uniformly bounded and hence find a weakly convergent subsequence. Furthermore, it follows from the construction that the sequence \( f_k \) converges uniformly to the identity. This also shows that there is a sequence with \( f(x, f_k) = 0 \) a.e. converging weakly in \( W^{1,p}(\Omega, \mathbb{R}^n) \), \( 1 \leq p < n \), to \( f(x) = (-x_1, x_2, \ldots, x_n) \).

Recently it was shown in [12] and [5] that a Jacobian of a Sobolev homeomorphism can change sign in dimension \( n \geq 4 \) for \( 1 \leq p < \frac{n}{2} \).

## 2 Preliminaries

### 2.1 Degree and Jacobian

There are two basic approaches to the notion of local degree for a mapping, the algebraic (see e.g. Dold [7]) and the analytic (see e.g. Lloyd [17]). Both of these notions try to capture the idea of counting the preimages of a target point. For a continuous mapping \( f : \Omega \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \setminus f(\partial \Omega) \) the degree of \( f \) at \( y \) with respect to \( \Omega \) is denoted by \( \deg(f, \Omega, y) \). If \( f : \Omega \to \mathbb{R}^n \) is a homeomorphism, then \( \deg(f, \Omega, y) \) is either 1 or \(-1\) for all \( y \in f(\Omega) \), see e.g. [17, Section IV.5] or [21, Section II.2.4, Theorem 3]. We say that a homeomorphism \( f \) is sense-preserving if \( \deg(f, \Omega, y) \equiv 1 \). For a linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) with \( \det A \neq 0 \), it is easy to check from the definition that

\[
\deg(A, \Omega, y) = \det A.
\]

We recall the following corollary [2, Corollary 2.8.2]. Given a homeomorphism \( f : \Omega \to \mathbb{R}^n \) suppose that \( f \) is differentiable at \( x \), with \( f(x, f) \neq 0 \). Then we have

\[
\deg(f, \Omega, f(x)) = \sgn J(x, f).
\]

We will use the fact that the topological degree is stable under homotopy. That is for every continuous mapping \( H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) such that \( y \notin H(\partial \Omega, t) \) for all \( t \in [0, 1] \) we have

\[
\deg(H(\cdot, 0), \Omega, y) = \deg(H(\cdot, 1), \Omega, y).
\]

### 2.2 Differentiability of Sobolev mappings

A Sobolev homeomorphism \( f \in W^{1,p}(\Omega, \mathbb{R}^n) \) is differentiable almost everywhere if \( p > n - 1 \) and \( p > 1 \) for \( n = 2 \), see [9, 20, 26]. We will also need a generalization of the concept of differentiability, which is obtained by replacing the ordinary limit by an approximate limit, see e.g. [8, Section 6.1.3]. It is known that a Sobolev mapping \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \) is \( \text{approximatively differentiable} \) almost everywhere, see e.g. [8, Section 6.1.2, Theorem 2]. Moreover, such a mapping is \( L^1 \)-differentiable almost everywhere [27]; that is, for almost every \( x \in \Omega \) we have

\[
\lim_{r \to 0} \frac{1}{\int_{B(x, r)} f(x) - f(x_0) - Df(x_0)(x-x_0)} \int_{B(x, r)} dx = 0.
\]

Hereafter, the notation \( \int_{B(x, r)} \) means the integral average over the \( n \)-dimensional ball

\[
B(x, r) = \{ x \in \mathbb{R}^n : |x-x_0| < r \}.
\]

In order to illustrate our ideas and for reader’s comprehension, we first prove Theorem 1 in the simpler cases \( p \geq 1, n = 2 \); and \( p > n - 1, n \geq 3 \), where we can avoid some technicalities.
Each homeomorphism $f_j$ is either sense-preserving or sense-reversing. Under our assumptions there exists a point $x_j$ such that $f_j$ is differentiable at $x_j$, see Section 2.2, and $J(x_j, f_j) > 0$. By (2) we know that the degree of $f_j$ is one and hence each $f_j$ is sense-preserving.

As $f_j \rightharpoonup f$ in $L^p$, $p > 1$, we know that $\int_{\Omega} |Df_j|^p$ is uniformly bounded and hence we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$|Df_j|^p \rightharpoonup^w \mu$$

in measures.

Moreover, for $p = 1$ we can use De La Vale Poussin characterization of weak convergence in $L^1$ and we can find a continuous convex function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$\frac{\Phi(t)}{t} \text{ is increasing, } \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty \text{ and } \int_{\Omega} \Phi(|Df_j|) \leq 1. \quad (5)$$

It follows that we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$\Phi(|Df_j|) \rightharpoonup^w \mu$$

in measures.

It is well known that for almost every $x_0 \in \Omega$ we have

$$M_\mu(x_0) := \sup_{r > 0} \frac{\mu(B(x_0, r) \cap \Omega)}{|B(x_0, r)|} < \infty. \quad (6)$$

Let $\delta > 0$. For the contrary suppose that there is $x_0 \in \Omega$ such that (4) and (6) hold at $x_0$ and

$$J(x_0, f) < 0.$$
(ii) if \( n = 2 \) and \( p = 1 \), there exist a constant \( C \) and such that for all \( 0 < r < r_1 \) and \( j \geq j \), there is a set \( A \subset B(x_\cdot, r) \) such that

\[
|A| < C\delta|B(x_\cdot, r)| \quad \text{and} \quad r^2 \leq C \int_A |Df_j|.
\]

These would lead to a desired contradiction. Indeed, choose \( 0 < r < r_1 \) such that \( \mu(\partial B(x_\cdot, r)) = 0 \) and in case (i) we obtain after passing to a limit in \( j \) that

\[
\delta^{n-1-p} \leq C \lim_{j \to \infty} \int_{B(x_\cdot, r)} |Df_j|^p = C \frac{\mu(B(x_\cdot, r) \cap \Omega)}{|B(x_\cdot, r)|} \leq CM\mu(x_\cdot).
\]

After passing \( \delta \to 0^+ \) we obtain a contradiction with (6). In case (ii) we can use Jensen’s inequality and (5) to obtain

\[
\int_{B(x_\cdot, r)} \Phi(|Df_j|) \geq \frac{|A|}{r^2} \int_A \Phi(|Df_j|) \geq \frac{|A|}{r^2} \Phi \left( \int_A |Df_j| \right) \geq \frac{|A|}{r^2} \Phi \left( \frac{C^2}{|A|} \right) \geq C\delta \Phi \left( \frac{C}{\delta} \right).
\]

Similarly as above we obtain in the limit that

\[
C\delta \Phi \left( \frac{C}{\delta} \right) \leq CM\mu(x_\cdot)
\]

and now passing to a limit \( \delta \to 0^+ \) we obtain a contradiction using (5).

**Proof of (i).** We simplify the notation and write

\[
\varphi_j(x) = |f_j(x) - f(x_\cdot) - Df(x_\cdot)(x - x_\cdot)| \quad \text{and} \quad B_s = B(x_\cdot, s).
\]

In the following we use the notation \( \mathcal{H}^k(A) \) for the \( k \)-dimensional Hausdorff measure of the set \( A \). We claim that the set of radii

\[
I_G = \{ s \in [0, r] : \mathcal{H}^{n-1}(\{ x \in \partial B_s : \varphi_j(x) \geq \delta r \}) < 5^n\delta^{n-1}\mathcal{H}^{n-1}(\partial B_s) \}
\]

has measure at least \( \frac{3\pi}{2} \), i.e. \( |I_G| \geq \frac{3\pi}{2} \), otherwise

\[
\int_{B_s} \frac{\varphi_j(x)}{r} \, dx \geq \frac{1}{|B_s|} \int_0^\frac{r}{\delta} 5^n\delta^{n-1}\mathcal{H}^{n-1}(\partial B_s) \frac{\delta r}{r} \, ds = 5^n\delta^n \frac{|B_{\frac{r}{\delta}}|}{|B_s|}
\]

which contradicts (8).

On the other hand, the key point in our argument is that for \( x_\cdot \in \Omega \), and for every \( s \in (0, r) \) we can find \( \beta = \beta(s) \in \partial B_s \) such that

\[
\varphi_j(\beta) \geq \frac{4}{5}s \quad \text{for every} \quad j = 1, 2, \ldots .
\]

Finding such a point \( \beta \) is the only place where we use the homeomorphism assumption of \( f_j \). Suppose on the contrary that (9) fails for every \( \beta \in \partial B_s \) and for some \( j \in \{ 1, 2, \ldots \} \). For \( x \in \partial B_s \) and \( t \in [0, 1] \) we consider the following homotopy:

\[
H(x, t) := (1-t)(f_j(x) - f(x_\cdot)) + tDf(x_\cdot)(x - x_\cdot).
\]

By (7) we know that \( Df(x_\cdot) \) is an isometry and thus \( |Df(x_\cdot)z| = |z| \). Furthermore, if (9) does not hold, then for all \( x \in \partial B_s \) we have

\[
|H(x, t)| \geq |Df(x_\cdot)(x - x_\cdot)| - (1-t)|f_j(x) - f(x_\cdot) - Df(x_\cdot)(x - x_\cdot)| \geq s - (1-t)\frac{4}{5}s > 0.
\]

It follows that \( H(x, t) \neq 0 \) for every \( x \in \partial B_s \) and all \( t \in [0, 1] \). Thus, by (3) and (1),

\[
\deg(f_j, B_s, f(x_\cdot)) = \text{sgn} \det(Df(x_\cdot)) = -1.
\]

This contradicts the fact that \( f_j \) is sense-preserving.
Integrating inequality (11) over the set identifying finising the proof of (ii).

Let $p$ case. This together with (10) implies that for all $z_1, z_2 \in \partial B(x, s)$ we have
\[
|f_j(z_1) - f_j(z_2)| \leq C(n, p)|z_1 - z_2|^{1 - \frac{p}{n}} \left( \int_{\partial B_s} |Df_j|^p \right)^{\frac{1}{p}}.
\] (10)

Now let us fix $s \in I_G$ so that (10) is satisfied on the sphere $\partial B_s$. Since $s \in I_G$, we find $a = a(s) \in \partial B_s$ satisfying
\[
\varphi_j(a) < \delta r \quad \text{and} \quad |a - \beta| < C_0 \delta s,
\]
where $C_0$ is some fixed constant (which depends only on $n$). Combining this with (9) we have found $a, \beta \in \partial B_s$ such that
\[
\frac{4}{5} s - \delta r - 2 C_0 \delta s \leq |\varphi_j(\beta) - |\varphi_j(a)| - 2|a - \beta| \leq |f_j(\alpha) - f_j(\beta)|.
\]
This together with (10) implies that for $s \in I_G \cap [\frac{r}{2}, r]$ and $\delta$ small enough
\[
C \delta^p \leq \left( \frac{4}{5} s - \delta r - 2 C_0 \delta s \right)^p \leq C(n, p)(\delta s)^{p-1} \int_{\partial B_s} |Df_j|^p.
\] (11)

Integrating inequality (11) over the set $I_G \cap [\frac{r}{2}, r]$ we obtain (i), finishing the proof of Theorem 1 in the case $p > n - 1$. \hfill \Box

**Proof of (ii).** We proceed as above. For $s \in I_G$ we can find $\beta = \beta(s) \in \partial B_s$ so that (9) holds. In fact we consider the measurable set
\[
A := \{x \in B_r : \varphi_j(x) > \delta r\}.
\]
By Chebyshev’s inequality and (8) we obtain
\[
|A| \leq \frac{1}{\delta r} \int_{\partial B_s} |\varphi_j(x)| \, dx \leq \frac{1}{\delta r} \delta^2 r^2 r = C \delta |B_r|.
\]

Let $s \in I_G \cap [\frac{r}{2}, r]$. The point $\beta \in \partial B_s$ with (9) clearly belongs to $A \cap \partial B_s$ and the closest point $a$ on the relative boundary of $\partial B_s \cap A$ satisfies
\[
|\varphi_j(a)| = \delta r
\]
by the definition of $A$. It follows that for every $s \in I_G \cap [\frac{r}{2}, r]$ we have
\[
s \leq C \int_{\partial B_s \cap A} |Df_j|.
\]

Integrating this over $I_G \cap [\frac{r}{2}, r]$ we obtain
\[
r^3 \leq C \int_A |Df_j|
\]
finishing the proof of (ii). \hfill \Box

The above proof was based on the Sobolev embedding theorem on spheres and therefore does not work for $p < n - 1$. To overcome these difficulties we follow Hencl and Malý [11] and use the theory of linking numbers and its topological invariance. For the convenience of the reader we recall the needed properties of linking numbers here.

## 4 Linking number

We use the notation $B_d$ for the unit ball in $\mathbb{R}^d$ and $S_{d-1}$ for the unit sphere. By $\overline{B}_d(c, r)$ we denote the closed ball with center $c$ and radius $r > 0$. 

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Let \( n, t, q \) be positive integers with \( t + q = n - 1 \). Let us consider the mapping \( \Phi(\xi, \eta): \mathbb{B}_{t+1} \times \mathbb{B}_{q+1} \rightarrow \mathbb{R}^n \) defined coordinatewise as \( \Phi(\xi, \eta) = x \), where
\[
\begin{align*}
    x_1 &= (2 + \eta_1)\xi_1, \\
    \vdots \\
    x_{t+1} &= (2 + \eta_1)\xi_{t+1}, \\
    x_{t+2} &= \eta_2, \\
    \vdots \\
    x_{t+q+1} &= \eta_{q+1}.
\end{align*}
\]
Denote by \( A \) the anuloid
\[
\Phi(S_t \times B_{q+1}) = \left\{ x \in \mathbb{R}^n : \left( \sqrt{x_1^2 + \cdots + x_{t+1}^2} - 2 \right)^2 + x_{t+2}^2 + \cdots + x_n^2 < 1 \right\}.
\]
Of course, given \( x \in A \) we can find a unique \( \xi \in S_t \) and \( \eta \in B_{q+1} \) such that \( \Phi(\xi, \eta) = x \). We will denote these as \( \xi(x) \) and \( \eta(x) \).

A link is a pair \((\varphi, \psi)\) of parametrized surfaces \( \varphi: S_t \rightarrow \mathbb{R}^n, \psi: S_q \rightarrow \mathbb{R}^n \). The linking number of the link \((\varphi, \psi)\) is defined as the topological degree
\[
L(\varphi, \psi) = \text{deg}(L, A, 0),
\]
where the mapping \( L = L_{\varphi, \psi}: A \rightarrow \mathbb{R}^n \) is defined as
\[
L(x) = \varphi(\xi(x)) - \psi(-\eta(x)),
\]
or equivalently
\[
L(\Phi(\xi, \eta)) = \varphi(\xi) - \psi(-\eta), \quad \xi \in S_t, \quad \eta \in B_{q+1},
\]
where \( \psi \) is an arbitrary continuous extension of \( \psi \) to \( B_{q+1} \) (of course, the degree does not depend on the way how we extend \( \psi \), it depends only on the values on the boundary \( \partial A = \Phi(S_t \times S_q) \)). Geometrically speaking, for \( t = q = 1 \), the linking number is the number of loops of a curve \( \varphi \) around a curve \( \psi \) counting orientation into account as +1 or −1. For the introductions to the linking number in \( \mathbb{R}^3 \) and its application to the theory of knots see [23].

The canonical link is the pair \((\mu, \nu)\), where
\[
\begin{align*}
    \mu(\xi) &= \Phi(\xi, 0), \quad \xi \in S_t, \\
    \nu(\eta) &= \Phi(e_1, \eta), \quad \eta \in S_q.
\end{align*}
\]
For example in dimension \( n = 3 \) we get that
\[
\begin{align*}
    \mu(S_1) &= \{ x \in \mathbb{R}^3 : x_3 = 0, \ x_1^2 + x_2^2 = 4 \}, \\
    \nu(S_1) &= \{ x_2 = 0, (x_1 - 2)^2 + x_2^2 = 1 \}.
\end{align*}
\]
It is well known that the linking number is a topological invariant. The simple proof of the following proposition can be found in [11].

**Proposition 2.** Let \( n, t, q \) be positive integers with \( t + q = n - 1 \). Let \( f: B_n(4) \rightarrow \mathbb{R}^n \) be a homeomorphism. Then \( L(f \circ \mu, f \circ \nu) \) is 1 if \( f \) is sense preserving and −1 if \( f \) is sense reversing.

Analogously, we can pick \( a \in B_{q+1}(0, \frac{1}{10}) \) and \( b \in B_{t+1}(e_1, \frac{1}{10}) \cap B_{t+1} \) and consider the pair
\[
\begin{align*}
    \mu_a(\xi) &= \Phi(\xi, a), \quad \xi \in S_t, \\
    \nu_b(\eta) &= \Phi(b, \eta), \quad \eta \in S_q.
\end{align*}
\]
Similarly to the previous proposition we have:

**Proposition 3.** Let \( n, t, q \) be positive integers with \( t + q = n - 1 \), \( a \in B_{q+1}(0, \frac{1}{10}) \) and \( b \in B_{t+1}(e_1, \frac{1}{10}) \cap B_{t+1} \). Let \( f: B_n(4) \rightarrow \mathbb{R}^n \) be a homeomorphism. Then \( L(f \circ \mu_a, f \circ \nu_b) \) is 1 if \( f \) is sense preserving and −1 if \( f \) is sense reversing.
5 Proof of Theorem 1 for $p > \frac{q}{2}$, $n \geq 3$, and $p \geq 1$, $n = 3$

Our argument is similar to the proof given in Section 3 and therefore some details are only sketched. By $\mu$ we again denote the $w^*$ limit of (some subsequence) $\int |Df|^p$ for $p > \frac{q}{2}$ and of $\int \Phi(|Df|)$ for $p = 1$ and $n = 3$.

By $C_1$ and $C_2$ we denote a fixed constants whose exact value will be determined later. We fix $\delta > 0$ and we choose a point $x_0$ such that (4) and (6) hold and without loss of generality we assume that the derivative of $f$ at $x_0$ is given by (7).

We fix $r_1 > 0$ such that for all $0 < r < r_1$ we have

$$\int_{B(x_0, 4r)} \left| \frac{f(x) - f(x_0)}{r} - Df(x_0)(x - x_0) \right| \, dx < C_1 \frac{\delta^n}{2}$$

and again for all $j \geq j$, we obtain

$$\int_{B(x_0, 4r)} \left| \frac{f_j(x) - f(x_0)}{r} - Df(x_0)(x - x_0) \right| \, dx < C_1 \delta^n.$$

(12)

We fix $t$, $q \leq \frac{q}{2}$ such that $t + q = n - 1$ (e.g. $t = q = \frac{n-1}{2}$ for $n$ odd and $t = \frac{n-2}{2}$, $q = \frac{n}{2}$ for $n$ even). Our goal is to prove the following:

(i) if $p > \frac{q}{2}$ and $n \geq 3$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j$,

$$\gamma^{\min(t, q) - p} r^n \leq C \int_{B(x_0, 4r)} |Df|^p,$$

(ii) if $p = 1$ and $n = 3$, we have $A \subset B(x_0, 4r)$ such that

$$|A| < C_2 \delta |B(x_0, 4r)| \quad \text{and} \quad r^3 \leq C \int_A |Df|.$$

Analogously to reasoning in Section 3 we obtain a contradiction using $\min(t, q) - p < 0$ for $p > \frac{q}{2}$ and (5) for $p = 1$ and $n = 3$.

Proof of (i). Without loss of generality we will assume that $x_0 = 0$. We write

$$\varphi_j(x) = |f_j(rx) - f(0) - Df(0)rx|.$$

Let us fix $y \in \mu_\delta(S_t)$ and denote

$$B_{\mu_\delta(S_t)}(y, \delta) = \{x \in \mu_\delta(S_t) : |x - y| < \delta\},$$

the ball of radius $\delta$ on the link $\mu_\delta(S_t)$. We can clearly choose a constant $C_1$ small enough at the beginning of the proof so that (12) implies that the set of good links

$$I_a = \left\{ a \in B_{q+1}\left(0, \frac{1}{4}\right) : \mathcal{H}^q(x \in \mu_\delta(S_t) : \varphi_j(x) \geq \delta r) < \mathcal{H}^q(B_{\mu_\delta(S_t)}(y, \delta)) \right\},$$

$$I_b = \left\{ b \in B_{q+1}\left(e_1, \frac{1}{4}\right) \cap B_{q+1} : \mathcal{H}^q(x \in \nu_\delta(S_q) : \varphi_j(x) \geq \delta r) < \mathcal{H}^q(B_{\nu_\delta(S_q)}(y, \delta)) \right\}$$

has measure at least

$$\mathcal{H}^{q+1}(I_a) > \frac{1}{2}|B_{q+1}\left(0, \frac{1}{4}\right)| \quad \text{and} \quad \mathcal{H}^{q+1}(I_b) > \frac{1}{2}|B_{q+1}\left(e_1, \frac{1}{4}\right) \cap B_{q+1}|.$$

The key point of our argument is that for every $a \in B_{q+1}(0, \frac{1}{10})$ and every $b \in B_{q+1}(e_1, \frac{1}{10}) \cap B_{q+1}$ we can find $x \in S_q$ and $\eta \in S_q$ such that

$$\varphi_j(x) > \frac{r}{10} \quad \text{or} \quad \varphi_j(\mu_\delta(S_q)) > \frac{r}{10},$$

$$\varphi_j(\nu_\delta(\eta)) > \frac{r}{10}.$$

(13)
We prove the observation by contradiction and we suppose that \((13)\) does not hold. We define

\[ f_s(x) = (1 - s)(f(0) + Df(0)rx) + sf_j(rx) \]

and we consider the homotopy \(H(\Phi, s) = (f_s \circ \mu_a)(\xi) - (f_s \circ \nu_b)(-\eta),\)

where \((f_s \circ \nu_b)\) denotes a continuous extension of \(f_s \circ \nu_b\) to \(B_{q+1}^p\) as in the definition of the linking number, which in addition depends continuously on \(s\). From [11] we know that the mapping \(f_j \in W^{1,p}, p > \left[\frac{q}{2}\right]\), with nonnegative and nonzero Jacobian is sense preserving. By Proposition 3 we get that

\[ \text{deg}(H(x, 1), A, 0) = 1. \]

On the other hand

\[ \text{deg}(H(x, 0), A, 0) = -1 \]

since the affine mapping \(f(0) + Df(0)rx\) is sense reversing. To obtain a contradiction (with the preservation of the degree under homotopy) it is now enough to show that for every \(\xi \in S_t\), for every \(\eta \in S_q\) and for every \(s \in [0, 1]\) we have \(H(\Phi(\xi, \eta), s) \neq 0\). It is easy to see that

\[ \text{dist}((f_0 \circ \mu_a)(s_t), (f_0 \circ \nu_b)(s_q)) \geq \text{dist}((f_0 \circ \mu)(s_t), (f_0 \circ \nu)(s_q)) - \frac{6r}{10} \geq \frac{3r}{10}. \]

Since \((13)\) does not hold, we obtain from the definition of \(f_s\) that

\[ \text{dist}((f_s \circ \mu_a)(s_t), (f_s \circ \nu_b)(s_q)) \geq \frac{3r}{10} - \frac{r}{10} - \frac{r}{10} \]

which implies \(H(\phi(\xi, \eta), s) = 0\).

By \((13)\) and the symmetry we may assume without loss of generality that

\[ \hat{I}_a = \{ a \in I_a : \text{there exists } \xi \in S_t \text{ such that } \varphi_j(\mu_a(\xi)) > \frac{r}{10} \} \]

satisfies \(\delta (\hat{I}_a) > \frac{1}{2} \| B_{q+1}^p(0, \frac{1}{10}) \|\). Since \(p > \left[\frac{q}{2}\right] \geq t\), we can use the Sobolev embedding theorem on the \(t\)-dimensional space \(r \mu_a(S_t)\) and we have for almost every \(a \in \hat{I}_a\) and for all \(z_1, z_2 \in r \mu_a(S_t)\),

\[ |f_j(z_1) - f_j(z_2)| \leq C |z_1 - z_2|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{r \mu_a(S_t)} |Df_j|^p \right)^{\frac{1}{p}}. \]

(14)

Now let us fix \(a \in \hat{I}_a\) so that \((14)\) is satisfied and find \(\xi \in S_t\) so that for \(\beta = \mu_a(\xi)\) we have \(\varphi(\beta) > \frac{r}{10}\) as in the definition of \(\hat{I}_a\). Using \(a \in \hat{I}_a\) we find \(a \in \mu_a(S_t)\) satisfying

\[ \varphi_j(a) < \delta r \quad \text{and} \quad |a - \beta| \leq \delta. \]

Thus we have found \(a, \beta \in \mu_a(S_t)\) such that

\[ \frac{r}{10} - 3 \delta r \leq |\varphi_j(\beta)| - |\varphi_j(a)| - 2r |a - \beta| \leq |f_j(ra) - f_j(r\beta)|. \]

This together with \((14)\) implies that for almost every \(a \in \hat{I}_a\) and \(\delta\) small enough we have

\[ r^t \leq C \delta^{p-t} \int_{r \mu_a(S_t)} |Df_j|^p. \]

(15)

Integrating inequality \((15)\) over the set \(\hat{I}_a\) we obtain (i).

**Proof of (ii).** If \(p = 1\) and \(n = 3\), then for each \(a \in \hat{I}_a\) we can find \(\xi \in S(t)\) so that \((13)\) holds for \(\beta = \mu_a(\xi)\). The measurable set

\[ A := \{ x \in B(x, 4r) : \varphi_j(x) > \delta r \} \]

satisfies \(|A| \leq C \delta |B(x, 4r)|\) by inequality \((12)\) and Chebyshev’s inequality. Now clearly \(\xi\) with \((13)\) satisfies \(r \beta = r \mu_a(\xi) \in A\). In \((14)\) and \((15)\) instead of integrating over the entire \(r \mu_a(S_t)\) we integrate only over the set \(r \mu_a(S_t) \cap A\). Integrating over \(\hat{I}_a\) we obtain the desired conclusion. \(\square\)
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