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Research Article

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Jacobian of weak limits of Sobolev homeomorphisms

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Abstract: Let $\Omega$ be a domain in $\mathbb{R}^n$, where $n = 2, 3$. Suppose that a sequence of Sobolev homeomorphisms $f_k : \Omega \to \mathbb{R}^n$ with positive Jacobian determinants, $J(x, f_k) > 0$, converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, for some $p \geq 1$, to a mapping $f$. We show that $J(x, f) \geq 0$ a.e. in $\Omega$. Generalizations to higher dimensions are also given.

Keywords: Sobolev homeomorphism, weak limits, Jacobian

MSC 2010: Primary 26B10; secondary 46E35

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1 Introduction

The main goal of this note is to establish when the sign of the Jacobian is preserved under $W^{1,p}$-weak convergence. Such a question pops out naturally in the variational approach to Geometric Function Theory (GFT) \[2, 14, 22\] and Nonlinear Elasticity (NE) \[1, 4, 6, 19, 24, 25\]. Both theories GFT and NE deal with minimizing sequences of Sobolev homeomorphisms. In the context of NE, one typically deals with two-dimensional or three-dimensional models and require that the deformation gradients belong to $M^{m \times n}_{\text{re}}$, where $M^{m \times n}_{\text{re}} = \{\text{real } m \times n \text{ matrices}\}$, and $M^{m \times n}_{\text{re}} = \{A \in M^{m \times n} : \det A > 0\}$. The infimum of the energy is not attained, in general, at a homeomorphism; interpenetration of matter may occur. Even in a special case of Dirichlet energy injectivity is often lost when passing to the weak limit of the minimizing sequence, \[3, 13, 15, 16\]. Further examinations are needed to know the properties of such singular minimizers.

Throughout this text $\Omega$ will be a domain in $\mathbb{R}^n$. The class of Sobolev mappings $f : \Omega \to \mathbb{R}^n$ with nonnegative Jacobian determinant, $J(x, f) = \det Df(x) \geq 0$ almost everywhere, is closed under the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^n)$ provided $p \geq n$ (see \[14, \text{Theorem } 8.4.2\]). However, if $p < n$, passing to the weak $W^{1,p}$-limit of a sequence with nonnegative Jacobians one may lose the sign of the Jacobian. Indeed, there exists a sequence of Sobolev mappings $f_k : \Omega \to \mathbb{R}^n$ with $J(x, f_k) > 0$ almost everywhere such that the sequence converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p < n$, to the mapping $f(x) = (-x_1, x_2, \ldots, x_n)$, see \[14, \text{p. } 181\]. Moreover, following the construction in \[18\] such mappings $f_k$ can be made continuous. However, it is not obvious at all as to whether one can make a similar example with $f_k$ being homeomorphisms. This is the subject of our result here. Here $[\frac{q}{2}]$ denotes the integer part, i.e. $[\frac{2}{3}] = 1$, $[\frac{2}{2}] = 1$ and so on.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be a domain and let $p \geq 1$ for $n \in \{2, 3\}$ and $p > [\frac{q}{2}]$ for $n \geq 4$. Suppose that a sequence of Sobolev homeomorphisms $f_k : \Omega \to \mathbb{R}^n$ with $J(x, f_k) \geq 0$ converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$ to a mapping $f$ and further assume that $J(x, f_k)$ is not a.e. zero. Then $J(x, f) \geq 0$ a.e. in $\Omega$.

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It is worth noting that in Theorem 1 the Jacobian \( J(x, f) \) can have very different behavior than the Jacobians in the sequence without knowing that \( J(x, f_k) > 0 \) on a set of positive measure. Indeed, there exists a sequence of Sobolev homeomorphisms \( f_k \) with \( J(x, f_k) = 0 \) a.e., converging weakly in \( W^{1, p}(\Omega, \mathbb{R}^n) \), \( 1 \leq p < n \), to the mapping \( f(x) = x \). Let us briefly sketch this using the construction from [10]: we cover \( \Omega \) by cubes of diameter less than \( \frac{1}{k} \) and on each cube we follow the construction from [10] to obtain a homeomorphism with zero Jacobian a.e. It is possible to make the \( W^{1, p} \)-norm of the sequence uniformly bounded and hence find a weakly convergent subsequence. Furthermore, it follows from the construction that the sequence \( f_k \) converges uniformly to the identity. This also shows that there is a sequence with \( J(x, f_k) = 0 \) a.e. converging weakly in \( W^{1, p}(\Omega, \mathbb{R}^n) \), \( 1 \leq p < n \), to \( f(x) = (-x_1, x_2, \ldots, x_n) \).

Recently it was shown in [12] and [5] that a Jacobian of a Sobolev homeomorphism can change sign in dimension \( n \geq 4 \) for \( 1 \leq p < \frac{n}{2} \).

### 2 Preliminaries

#### 2.1 Degree and Jacobian

There are two basic approaches to the notion of local degree for a mapping, the algebraic (see e.g. Dold [7]) and the analytic (see e.g. Lloyd [17]). Both of these notions try to capture the idea of counting the preimages of a target point. For a continuous mapping \( f : \Omega \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \setminus f(\partial \Omega) \) the degree of \( f \) at \( y \), with respect to \( \Omega \) is denoted by \( \deg(f, \Omega, y) \). If \( f : \Omega \to \mathbb{R}^n \) is a homeomorphism, then \( \deg(f, \Omega, y) \) is either 0 or \( -1 \) for all \( y \in f(\Omega) \), see e.g. [17, Section IV.5] or [21, Section II.2.4, Theorem 3]. We say that a homeomorphism \( f \) is sense-preserving if \( \deg(f, \Omega, y) = 1 \). For a linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) with \( \det A \neq 0 \), it is easy to check from the definition that

\[
\deg(\Lambda, \Omega, y) = \text{sgn} \det A. \tag{1}
\]

We recall the following corollary [2, Corollary 2.8.2]. Given a homeomorphism \( f : \Omega \to \mathbb{R}^n \) suppose that \( f \) is differentiable at \( x \), with \( f(x, f) \neq 0 \). Then we have

\[
\deg(f, \Omega, f(x)) = \text{sgn} J(x, f). \tag{2}
\]

We will use the fact that the topological degree is stable under homotopy. That is for every continuous mapping \( H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) such that \( y \notin H(\partial \Omega, t) \) for all \( t \in [0, 1] \) we have

\[
\deg(H(\cdot, 0), \Omega, y) = \deg(H(\cdot, 1), \Omega, y). \tag{3}
\]

#### 2.2 Differentiability of Sobolev mappings

A Sobolev homeomorphism \( f \in W^{1, p}(\Omega, \mathbb{R}^n) \) is differentiable almost everywhere if \( p > n - 1, n \geq 3 \), and \( p \geq 1 \) for \( n = 2 \), see [9, 20, 26]. We will also need a generalization of the concept of differentiability, which is obtained by replacing the ordinary limit by an approximate limit, see e.g. [8, Section 6.1.3]. It is known that a Sobolev mapping \( f \in W^{1, 1}_{\text{loc}}(\Omega, \mathbb{R}^n) \) is \textit{approximatively differentiable} almost everywhere, see e.g. [8, Section 6.1.2, Theorem 2]. Moreover, such a mapping is \( L^1 \)-\textit{differentiable} almost everywhere [27]; that is, for almost every \( x \in \Omega \) we have

\[
\lim_{r \to 0} \int_{B(x, r)} \left\{ \frac{|f(x) - f(x_0) - Df(x_0)(x - x_0)|}{r} \right\} \, dx = 0. \tag{4}
\]

Hereafter, the notation \( \int_{B(x, r)} \) means the integral average over the \( n \)-dimensional ball

\[
B(x, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \}.
\]

In order to illustrate our ideas and for reader’s comprehension, we first prove Theorem 1 in the simpler cases \( p \geq 1, n = 2 \); and \( p > n - 1, n \geq 3 \), where we can avoid some technicalities.
3 Proof of Theorem 1 for $p > n - 1$, $n \geq 3$, and $p \geq 1$, $n = 2$

Each homeomorphism $f_j$ is either sense-preserving or sense-reversing. Under our assumptions there exists a point $x_j$ such that $f_j$ is differentiable at $x_j$, see Section 2.2, and $J(x_j, f) > 0$. By (2) we know that the degree of $f_j$ is one and hence each $f_j$ is sense-preserving.

As $f_j \to f$ in $L^p$, $p > 1$, we know that $\int |Df_j|^p$ is uniformly bounded and hence we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$|Df_j|^p \overset{w^*}{\to} \mu$$

in measures.

Moreover, for $p = 1$ we can use De La Vale Poupin characterization of weak convergence in $L^1$ and we can find an continuous convex function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$\frac{\Phi(t)}{t} \text{ is increasing, } \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty \text{ and } \int_{\Omega} \Phi(|Df_j|) \leq 1.$$ (5)

It follows that we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$\Phi(|Df_j|) \overset{w^*}{\to} \mu$$

in measures.

It is well known that for almost every $x_\ast \in \Omega$ we have

$$M_\mu(x_\ast) := \sup_{r > 0} \frac{\mu(B(x_\ast, r) \cap \Omega)}{|B(x_\ast, r)|} < \infty.$$ (6)

Let $\delta > 0$. For the contrary we suppose that there is $x_\ast \in \Omega$ such that (4) and (6) hold at $x_\ast$ and $J(x_\ast, f) < 0$.

Without loss of generality we may and do assume that

$$Df(x_\ast) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{pmatrix}.$$ (7)

Using (4) we can find $0 < r_1$ small enough such that for all $0 < r < r_1$ we have

$$\int_{B(x_\ast, r)} \left| \frac{f(x) - f(x_\ast) - Df(x_\ast)(x - x_\ast)}{r} \right| \, dx < \frac{\delta^n}{2}.$$ (8)

Since the sequence of mappings $f_j$ converges to $f$ weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, we have that the sequence of mappings $f_j$ converges to $f$ strongly in $L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$. Now, we may pick up an index $j_0$ large enough such that for all $j \geq j_0$,

$$\int_{\Omega} |f(x) - f_j(x)| \, dx < |B(0, 1)| r_1 \delta^n.$$ (5)

The last two inequalities imply that for all $0 < r < r_1$ we have

$$\int_{B(x_\ast, r)} \left| \frac{f_j(x) - f_j(x_\ast) - Df_j(x_\ast)(x - x_\ast)}{r} \right| \, dx < \delta^n.$$ (8)

Our next goal is to prove the following:

(i) If $p > n - 1$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j_0$,

$$\delta^{n-1-p} \leq C \int_{B(x_\ast, r)} |Df|^p.$$
(ii) if \( n = 2 \) and \( p = 1 \), there exist a constant \( C \) and such that for all \( 0 < r < r_1 \) and \( j \geq j \), there is a set 
\( A \subset B(x_0, r) \) such that 
\[
|A| < C \delta |B(x_0, r)| \quad \text{and} \quad r^2 \leq C \int_A |Df_j|.
\]

These would lead to a desired contradiction. Indeed, choose \( 0 < r < r_1 \) such that \( \mu(\partial B(x_0, r)) = 0 \) and in case (i) we obtain after passing to a limit in \( j \) that 
\[
\delta^{n-1-p} \leq C \lim_{j \to \infty} \int_{B(x_0, r)} |Df_j|^p = C \mu(B(x_0, r) \cap \Omega)/|B(x_0, r)| \leq CM\mu(x_0).
\]

After passing \( \delta \to 0^+ \) we obtain a contradiction with (6). In case (ii) we can use Jensen’s inequality and (5) to obtain 
\[
\int_{B(x_0, r)} \Phi(|Df_j|) \geq \frac{|A|}{r^2} \int_A \Phi(|Df_j|) \geq \frac{|A|}{r^2} \Phi\left( \frac{\int_A |Df_j|}{|A|} \right) \geq C \delta \Phi\left( \frac{C}{\delta} \right).
\]

Similarly as above we obtain in the limit that 
\[
C \delta \Phi\left( \frac{C}{\delta} \right) \leq CM\mu(x_0)
\]

and now passing to a limit \( \delta \to 0^+ \) we obtain a contradiction using (5).

**Proof of (i).** We simplify the notation and write 
\[
\varphi_j(x) = |f_j(x) - f(x_0) - Df(x_0)(x - x_0)| \quad \text{and} \quad B_s = B(x_0, s).
\]

In the following we use the notation \( \mathcal{H}^k(A) \) for the \( k \)-dimensional Hausdorff measure of the set \( A \). We claim that the set of radii 
\[
I_G = \{ s \in [0, r] : \mathcal{H}^{n-1}(x \in \partial B_s : \varphi_j(x) \geq \delta r) < 5^n \delta^{n-1} \mathcal{H}^{n-1}(\partial B_s) \}
\]

has measure at least \( \frac{3r}{2} \), i.e. \( |I_G| \geq \frac{3r}{2} \), otherwise 
\[
\int \frac{\varphi_j(x)}{r} \, dx \geq \frac{1}{|B_s|} \int_0^r 5^n \delta^{n-1} \mathcal{H}^{n-1}(\partial B_s) \frac{\delta r}{r} \, ds = 5^n \delta^n \frac{|B_1|}{|B_s|}
\]

which contradicts (8).

On the other hand, the key point in our argument is that for \( x_0 \in \Omega \), and for every \( s \in (0, r) \) we can find \( \beta = \beta(s) \in \partial B_s \) such that 
\[
\varphi_j(\beta) \geq \frac{4}{5^j} s \quad \text{for every} \quad j = 1, 2, \ldots.
\]

Finding such a point \( \beta \) is the only place where we use the homeomorphism assumption of \( f_j \). Suppose on the contrary that (9) fails for every \( \beta \in \partial B_s \) and for some \( j \in \{ 1, 2, \ldots \} \). For \( x \in \partial B_s \) and \( t \in [0, 1] \) we consider the following homotopy: 
\[
H(x, t) := (1 - t)(f_j(x) - f(x_0)) + tDf(x_0)(x - x_0).
\]

By (7) we know that \( Df(x_0) \) is an isometry and thus \( |Df(x_0)| = |z| \). Furthermore, if (9) does not hold, then for all \( x \in \partial B_s \) we have 
\[
|H(x, t)| = |Df(x_0)(x - x_0)|(1 - t) + tDf(x_0)(x - x_0)| \geq s - (1 - t) \frac{4}{5} s > 0.
\]

It follows that \( H(x, t) \neq 0 \) for every \( x \in \partial B_s \) and all \( t \in [0, 1] \). Thus, by (3) and (1), 
\[
\deg(f_j, B_s, f(x_0)) = \text{sgn } \det(Df(x_0)) = -1.
\]

This contradicts the fact that \( f_j \) is sense-preserving.
We apply the Sobolev embedding theorem [8, Theorem 3 (i), p. 143] on the \((n - 1)\)-dimensional spheres. This way for almost every \(s \in (0, r)\) and for all \(z_1, z_2 \in \partial B(x, s)\) we have

\[
|f_j(z_1) - f_j(z_2)| \leq C(n, p)|z_1 - z_2|^{1 - \frac{p}{n+1}} \left( \int_{\partial B_s} |Df_j|^p \right)^{\frac{1}{p}}. \tag{10}
\]

Now let us fix \(s \in I_G\) so that (10) is satisfied on the sphere \(\partial B_s\). Since \(s \in I_G\), we find \(\alpha = a(s) \in \partial B_s\) satisfying

\[
\varphi_j(\alpha) < \delta r \quad \text{and} \quad |\alpha - \beta| \leq C_0 \delta s,
\]

where \(C_0\) is some fixed constant (which depends only on \(n\)). Combining this with (9) we have found \(\alpha, \beta \in \partial B_s\) such that

\[
\frac{4}{5}s - \delta r - 2C_0 \delta s \leq |\varphi_j(\beta)| - |\varphi_j(\alpha)| - 2|\alpha - \beta| \leq |f_j(\alpha) - f_j(\beta)|.
\]

This together with (10) implies that for \(s \in I_G \cap [\frac{r}{2}, r]\) and \(\delta\) small enough

\[
C_0^p \leq \left( \frac{4}{5}s - \delta r - 2C_0 \delta s \right)^p \leq C(n, p)(\delta s)^{p-n+1} \int_{\partial B_s} |Df_j|^p. \tag{11}
\]

Integrating inequality (11) over the set \(I_G \cap [\frac{r}{2}, r]\) we obtain (i), finishing the proof of Theorem 1 in the case \(p > n - 1\).

\(\square\)

**Proof of (ii).** We proceed as above. For \(s \in I_G\) we can find \(\beta = \beta(s) \in \partial B_s\) so that (9) holds. In fact we consider the measurable set

\[
A := \{x \in B_r : \varphi_j(x) > \delta r\}.
\]

By Chebyshev’s inequality and (8) we obtain

\[
|A| \leq \frac{1}{\delta r} \int_{\partial B_s} |\varphi_j(x)| \, dx \leq \frac{1}{\delta r} \delta^2 r^2 r = C\delta |B_r|.
\]

Let \(s \in I_G \cap [\frac{r}{2}, r]\). The point \(\beta \in \partial B_s\) with (9) clearly belongs to \(A \cap \partial B_s\) and the closest point \(\alpha\) on the relative boundary of \(\partial B_s \cap A\) satisfies

\[
|\varphi_j(\alpha)| = \delta r
\]

by the definition of \(A\). It follows that for every \(s \in I_G \cap [\frac{r}{2}, r]\) we have

\[
s \leq C \int_{\partial B_s \cap A} |Df_j|.
\]

Integrating this over \(I_G \cap [\frac{r}{2}, r]\) we obtain

\[
r^2 \leq C \int_A |Df_j|
\]

finishing the proof of (ii).

\(\square\)

The above proof was based on the Sobolev embedding theorem on spheres and therefore does not work for \(p < n - 1\). To overcome these difficulties we follow Hencl and Malý [11] and use the theory of linking numbers and its topological invariance. For the convenience of the reader we recall the needed properties of linking numbers here.

### 4 Linking number

We use the notation \(B_d\) for the unit ball in \(\mathbb{R}^d\) and \(S_{d-1}\) for the unit sphere. By \(\overline{B}_d(c, r)\) we denote the closed ball with center \(c\) and radius \(r > 0\).
Let $n$, $t$, $q$ be positive integers with $t + q = n - 1$. Let us consider the mapping $\Phi(\xi, \eta) : \mathbb{B}_{t+1} \times \mathbb{B}_{q+1} \to \mathbb{R}^n$ defined coordinatewise as $\Phi(\xi, \eta) = x$, where

$$
x_1 = (2 + \eta_1)\xi_1, \\
\vdots \\
x_{t+1} = (2 + \eta_1)\xi_{t+1}, \\
x_{t+2} = \eta_2, \\
\vdots \\
x_{t+q+1} = \eta_{q+1}.
$$

Denote by $A$ the anuloid

$$
\Phi(S_t \times B_{q+1}) = \{ x \in \mathbb{R}^n : \left( \sqrt{x_1^2 + \cdots + x_{t+1}^2} - 2 \right)^2 + x_{t+2}^2 + \cdots + x_n^2 < 1 \}.
$$

Of course, given $x \in \overline{A}$ we can find a unique $\xi \in S_t$ and $\eta \in B_{q+1}$ such that $\Phi(\xi, \eta) = x$. We will denote these as $\xi(x)$ and $\eta(x)$.

A link is a pair $(\varphi, \psi)$ of parametrized surfaces $\varphi : S_t \to \mathbb{R}^n$, $\psi : S_q \to \mathbb{R}^n$. The linking number of the link $(\varphi, \psi)$ is defined as the topological degree

$$
L(\varphi, \psi) = \deg(L, A, 0),
$$

where the mapping $L = L_{\varphi, \psi} : \overline{A} \to \mathbb{R}^n$ is defined as

$$
L(x) = \varphi(\xi(x)) - \psi(-\eta(x)),
$$

or equivalently

$$
L(\Phi(\xi, \eta)) = \varphi(\xi) - \psi(-\eta), \quad \xi \in S_t, \eta \in B_{q+1},
$$

where $\psi$ is an arbitrary continuous extension of $\psi$ to $\mathbb{B}_{q+1}$ (of course, the degree does not depend on the way how we extend $\psi$, it depends only on the values on the boundary $\partial A = \Phi(S_t \times S_q)$). Geometrically speaking, for $t = q = 1$, the linking number is the number of loops of a curve $\varphi$ around a curve $\psi$ counting orientation into account as $+1$ or $-1$. For the introductions to the linking number in $\mathbb{R}^3$ and its application to the theory of knots see [23].

The canonical link is the pair $(\mu, \nu)$, where

$$
\mu(\xi) = \Phi(\xi, 0), \quad \xi \in S_t, \\
\nu(\eta) = \Phi(e_1, \eta), \quad \eta \in S_q.
$$

For example in dimension $n = 3$ we get that

$$
\mu(S_3) = \{ x \in \mathbb{R}^3 : x_3 = 0, \quad x_1^2 + x_2^2 = 4 \}, \\
\nu(S_1) = \{ x_2 = 0, \quad (x_1 - 2)^2 + x_3^2 = 1 \}.
$$

It is well known that the linking number is a topological invariant. The simple proof of the following proposition can be found in [11].

**Proposition 2.** Let $n$, $t$, $q$ be positive integers with $t + q = n - 1$. Let $f : B_n(4) \to \mathbb{R}^n$ be a homeomorphism. Then $L(f \circ \mu, f \circ \nu)$ is $1$ if $f$ is sense preserving and $-1$ if $f$ is sense reversing.

Analogously, we can pick $a \in \mathbb{B}_{q+1}(0, \frac{1}{10})$ and $b \in \mathbb{B}_{t+1}(e_1, \frac{1}{10}) \cap \mathbb{B}_{t+1}$ and consider the pair

$$
\mu_a(\xi) = \Phi(\xi, a), \quad \xi \in S_t, \\
\nu_b(\eta) = \Phi(b, \eta), \quad \eta \in S_q.
$$

Similarly to the previous proposition we have:

**Proposition 3.** Let $n$, $t$, $q$ be positive integers with $t + q = n - 1$, $a \in \mathbb{B}_{q+1}(0, \frac{1}{10})$ and $b \in \mathbb{B}_{t+1}(e_1, \frac{1}{10}) \cap \mathbb{B}_{t+1}$. Let $f : B_n(4) \to \mathbb{R}^n$ be a homeomorphism. Then $L(f \circ \mu_a, f \circ \nu_b)$ is $1$ if $f$ is sense preserving and $-1$ if $f$ is sense reversing.
5 Proof of Theorem 1 for $p > \left[ \frac{n}{2} \right], n \geq 3,$ and $p \geq 1, n = 3$

Our argument is similar to the proof given in Section 3 and therefore some details are only sketched. By $\mu$ we again denote the $w^*$ limit of (some subsequence) $\int |Df|^p$ for $p > \left[ \frac{n}{2} \right]$ and of $\int \Phi(|Df|)$ for $p = 1$ and $n = 3$.

By $C_1$ and $C_2$ we denote a fixed constants whose exact value will be determined later. We fix $\delta > 0$ and we choose a point $x_0$ such that (4) and (6) hold and without loss of generality we assume that the derivative of $f$ at $x_0$ is given by (7).

We fix $r_1 > 0$ such that for all $0 < r < r_1$ we have

$$\int_{B(x_0, 4r)} \left| \frac{f(x) - f(x_0) - Df(x_0)(x - x_0)}{r} \right| \, dx < C_1 \delta^n$$

and again for all $j \geq j$, we obtain

$$\int_{B(x_0, 4r)} \left| \frac{f_j(x) - f(x_0) - Df(x_0)(x - x_0)}{r} \right| \, dx < C_1 \delta^n. \quad (12)$$

We fix $t, q \leq \left[ \frac{n}{2} \right]$ such that $t + q = n - 1$ (e.g. $t = q = \frac{n - 1}{2}$ for $n$ odd and $t = \frac{n - 2}{2}, q = \frac{n}{2}$ for $n$ even). Our goal is to prove the following:

(i) if $p > \left[ \frac{n}{2} \right]$ and $n \geq 3$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j$,

$$d_{\text{min}}(t, q) - p, r^n \leq C \int_{B(x_0, 4r)} |Df|^p,$$

(ii) if $p = 1$ and $n = 3$, we have $A \subset B(x_0, 4r)$ such that

$$|A| < C_2 \delta |B(x_0, 4r)| \quad \text{and} \quad r^3 \leq C \int_A |Df|.$$

Analogously to reasoning in Section 3 we obtain a contradiction using $\min(t, q) - p < 0$ for $p > \left[ \frac{n}{2} \right]$ and (5) for $p = 1$ and $n = 3$.

Proof of (i). Without loss of generality we will assume that $x_0 = 0$. We write

$$\varphi_j(x) = |f_j(rx) - f(0) - Df(0)rx|.$$

Let us fix $y \in \mu_a(S_t)$ and denote

$$B_{\mu_a(S_t)}(y, \delta) = \{x \in \mu_a(S_t) : |x - y| < \delta\},$$

the ball of radius $\delta$ on the link $\mu_a(S_t)$. We can clearly choose a constant $C_1$ small enough at the beginning of the proof so that (12) implies that the set of good links

$$I_a = \left\{ a \in \mathbb{B}_{q+1}\left(0, \frac{1}{10}\right) : \mathcal{H}(x \in \mu_a(S_t) : \varphi_j(x) \geq \delta r) < \mathcal{H}(B_{\mu_a(S_t)}(y, \delta)) \right\},$$

$$I_b = \left\{ b \in \mathbb{B}_{t+1}\left(e_1, \frac{1}{10}\right) \cap \mathbb{B}_{t+1} : \mathcal{H}(x \in \nu_b(S_q) : \varphi_j(x) \geq \delta r) < \mathcal{H}(B_{\nu_b(S_q)}(y, \delta)) \right\}$$

has measure at least

$$\mathcal{H}^{q+1}(I_a) > \frac{1}{2} \mathbb{B}_{q+1}\left(0, \frac{1}{10}\right) \quad \text{and} \quad \mathcal{H}^{t+1}(I_b) > \frac{1}{2} \mathbb{B}_{t+1}\left(e_1, \frac{1}{10}\right) \cap \mathbb{B}_{t+1}.$$

The key point of our argument is that for every $a \in \mathbb{B}_{q+1}(0, \frac{1}{10})$ and every $b \in \mathbb{B}_{t+1}(e_1, \frac{1}{10}) \cap \mathbb{B}_{t+1}$ we can find $\xi \in S_t$ and $\eta \in S_q$ such that

$$\varphi_j(\mu_a(\xi)) = |f_j(r\mu_a(\xi)) - f(0) - Df(0)r\mu_a(\xi)| > \frac{r}{10} \quad \text{or} \quad \varphi_j(\nu_b(\eta)) = |f_j(r\nu_b(\eta)) - f(0) - Df(0)r\nu_b(\eta)| > \frac{r}{10}. \quad (13)$$
We prove the observation by contradiction and we suppose that (13) does not hold. We define
\[ f_s(x) = (1 - s)(f(0) + Df(0)x) + sf_j(rx) \]
and we consider the homotopy \( H(\overline{\mathbb{A}} \times [0, 1]) \rightarrow \mathbb{R}^n \) defined as
\[ H(\Phi(\xi, \eta), s) = (f_s \circ \mu_a)(\xi) - (f_s \circ \nu_b)(-\eta), \]
where \((f_s \circ \nu_b)\) denotes a continuous extension of \(f_s \circ \nu_b\) to \(\mathbb{R}_{q+1}\) as in the definition of the linking number, which in addition depends continuously on \(s\). From [11] we know that the mapping \(f_j \in W^{1,p}, p > \frac{r}{2}\), with nonnegative and nonzero Jacobian is sense preserving. By Proposition 3 we get that
\[ \deg(H(x, 1), A, 0) = 1. \]
On the other hand
\[ \deg(H(x, 0), A, 0) = -1 \]
since the affine mapping \(f(0) + Df(0)x\) is sense reversing. To obtain a contradiction (with the preservation of the degree under homotopy) it is now enough to show that for every \(\xi \in \mathbb{S}_t\), for every \(\eta \in \mathbb{S}_q\) and for every \(s \in [0, 1]\) we have \(H(\Phi(\xi, \eta), s) \neq 0\). It is easy to see that
\[ \text{dist}(f_0) \circ \mu_a)(\mathbb{S}_t), (f_0 \circ \nu_b)(\mathbb{S}_q) \geq \text{dist}(f_0 \circ \mu(\mathbb{S}_t), (f_0 \circ \nu)(\mathbb{S}_q)) - \frac{6r}{10} \geq \frac{3r}{10}. \]
Since (13) does not hold, we obtain from the definition of \(f_s\) that
\[ \text{dist}(f_s \circ \mu_a)(\mathbb{S}_t), (f_s \circ \nu_b)(\mathbb{S}_q) \geq \frac{3r}{10} - \frac{r}{10} - \frac{r}{10} \]
which implies \(H(\Phi(\xi, \eta), s) \neq 0\).

By (13) and the symmetry we may assume without loss of generality that
\[ \tilde{I}_a = \{ a \in I_a : \text{there exists } \xi \in \mathbb{S}_t \text{ such that } \varphi_j(\mu_a(\xi)) > \frac{r}{10} \} \]
satisfies \(3\tilde{I}^{q+1}(\tilde{I}_a) > \frac{1}{2}[\mathbb{B}_{q+1}(0, \frac{1}{2})]\). Since \(p > \frac{r}{2}\), we can use the Sobolev embedding theorem on the \(t\)-dimensional space \(\mu_a(\mathbb{S}_t)\) and we have for almost every \(a \in \tilde{I}_a\) and for all \(z_1, z_2 \in \mu_a(\mathbb{S}_t)\),
\[ |f_j(z_1) - f_j(z_2)| \leq C|z_1 - z_2|^{1 - \frac{p}{2}} \left( \int_{\mu_a(\mathbb{S}_t)} |Df_j|^p \right)^{\frac{1}{p}}. \tag{14} \]
Now let us fix \(a \in \tilde{I}_a\) so that (14) is satisfied and find \(\xi \in \mathbb{S}_t\) so that for \(\beta = \mu_a(\xi)\) we have \(\varphi_j(\beta) > \frac{r}{10}\) as in the definition of \(\tilde{I}_a\). Using \(a \in I_a\) we find \(a \in \mu_a(\mathbb{S}_t)\) satisfying \(\varphi_j(a) < \delta r\) and \(|a - \beta| \leq \delta\).
Thus we have found \(a, \beta \in \mu_a(\mathbb{S}_t)\) such that
\[ \frac{r}{10} - 3\delta r \leq |\varphi_j(\beta) - |\varphi_j(a)| - 2r|a - \beta| \leq |f_j(ra) - f_j(r\beta)|. \]
This together with (14) implies that for almost every \(a \in \tilde{I}_a\) and \(\delta\) small enough we have
\[ r^t \leq C\delta^{p-t} \int_{\mu_a(\mathbb{S}_t)} |Df_j|^p. \tag{15} \]
Integrating inequality (15) over the set \(\tilde{I}_A\) we obtain (i).

**Proof of (ii).** If \(p = 1\) and \(n = 3\), then for each \(a \in \tilde{I}_a\) we can find \(\xi \in \mathbb{S}(t)\) so that (13) holds for \(\beta = \mu_a(\xi)\). The measurable set
\[ A := \{ x \in B(x, 4r) : \varphi_j(x) > \delta r \} \]
satisfies \(|A| \leq C\delta(B(x, 4r))\) by inequality (12) and Chebyshev’s inequality. Now clearly \(\xi\) with (13) satisfies \(r\beta = \mu_a(\xi) \in A\). In (14) and (15) instead of integrating over the entire \(\mu_a(\mathbb{S}_t)\) we integrate only over the set \(\mu_a(\mathbb{S}_t) \cap A\). Integrating over \(\tilde{I}_a\) we obtain the desired conclusion.
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