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**Research Article**

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# Jacobian of weak limits of Sobolev homeomorphisms

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**Abstract:** Let $\Omega$ be a domain in $\mathbb{R}^n$, where $n = 2, 3$. Suppose that a sequence of Sobolev homeomorphisms $f_k\colon \Omega \to \mathbb{R}^n$ with positive Jacobian determinants, $J(x, f_k) > 0$, converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, for some $p \geqslant 1$, to a mapping $f$. We show that $J(x, f) \geqslant 0$ a.e. in $\Omega$. Generalizations to higher dimensions are also given.

**Keywords:** Sobolev homeomorphism, weak limits, Jacobian

**MSC 2010:** Primary 26B10; secondary 46E35

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## 1 Introduction

The main goal of this note is to establish when the sign of the Jacobian is preserved under $W^{1,p}$-weak convergence. Such a question pops out naturally in the variational approach to Geometric Function Theory (GFT) [2, 14, 22] and Nonlinear Elasticity (NE) [1, 4, 6, 19, 24, 25]. Both theories GFT and NE deal with minimizing sequences of Sobolev homeomorphisms. In the context of NE, one typically deals with two-dimensional or three-dimensional models and require that the deformation gradients belong to $M_+^{n\times n}$, where $M^{m\times n} = \{\text{real } m \times n \text{ matrices}\}$, and $M_+^{n\times n} = \{A \in M^{n\times n} : \det A > 0\}$. The infimum of the energy is not attained, in general, at a homeomorphism; interpenetration of matter may occur. Even in a special case of Dirichlet energy injectivity is often lost when passing to the weak limit of the minimizing sequence, [3, 13, 15, 16]. Further examinations are needed to know the properties of such singular minimizers.

Throughout this text $\Omega$ will be a domain in $\mathbb{R}^n$. The class of Sobolev mappings $f\colon \Omega \to \mathbb{R}^n$ with nonnegative Jacobian determinant, $J(x, f) = \det Df(x) \geqslant 0$ almost everywhere, is closed under the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^n)$ provided $p \geqslant n$ (see [14, Theorem 8.4.2]). However, if $p < n$, passing to the weak $W^{1,p}$-limit of a sequence with nonnegative Jacobians one may lose the sign of the Jacobian. Indeed, there exists a sequence of Sobolev mappings $f_k\colon \Omega \to \mathbb{R}^n$ with $J(x, f_k) > 0$ almost everywhere such that the sequence converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p < n$, to the mapping $f(x) = (-x_1, x_2, \dots, x_n)$, see [14, p. 181]. Moreover, following the construction in [18] such mappings $f_k$ can be made continuous. However, it is not obvious at all as to whether one can make a similar example with $f_k$ being homeomorphisms. This is the subject of our result here. Here $[\frac{n}{2}]$ denotes the integer part, i.e. $[\frac{2}{2}] = 1$, $[\frac{3}{2}] = 1$ and so on.

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $p \geqslant 1$ for $n \in \{2, 3\}$ and $p > [\frac{n}{2}]$ for $n \geqslant 4$. Suppose that a sequence of Sobolev homeomorphisms $f_k\colon \Omega \to \mathbb{R}^n$ with $J(x, f_k) \geqslant 0$ converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$ to a mapping $f$ and further assume that $J(x, f_k)$ is not a.e. zero. Then $J(x, f) \geqslant 0$ a.e. in $\Omega$.*

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It is worth noting that in Theorem 1 the Jacobian $J(x, f)$ can have very different behavior than the Jacobians in the sequence without knowing that $J(x, f_k) > 0$ on a set of positive measure. Indeed, there exists a sequence of Sobolev homeomorphisms $f_k$ with $J(x, f_k) = 0$ a.e., converging weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $1 \leqslant p < n$, to the mapping $f(x) = x$. Let us briefly sketch this using the construction from [10]: we cover $\Omega$ by cubes of diameter less than $\frac{1}{k}$ and on each cube we follow the construction from [10] to obtain a homeomorphism with zero Jacobian a.e. It is possible to make the $W^{1,p}$-norm of the sequence uniformly bounded and hence find a weakly convergent subsequence. Furthermore, it follows from the construction that the sequence $f_k$ converges uniformly to the identity. This also shows that there is a sequence with $J(x, f_k) = 0$ a.e. converging weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $1 \leqslant p < n$, to $f(x) = (-x_1, x_2, \dots, x_n)$.

Recently it was shown in [12] and [5] that a Jacobian of a Sobolev homeomorphism can change sign in dimension $n \geq 4$ for $1 \leq p < [\frac{n}{2}]$.

## 2 Preliminaries

### 2.1 Degree and Jacobian

There are two basic approaches to the notion of local degree for a mapping, the algebraic (see e.g. Dold [7]) and the analytic (see e.g. Lloyd [17]). Both of these notions try to capture the idea of counting the preimages of a target point. For a continuous mapping $f\colon \Omega \to \mathbb{R}^n$ and $y_\circ \in \mathbb{R}^n \setminus f(\partial\Omega)$ the degree of $f$ at $y_\circ$ with respect to $\Omega$ is denoted by $\deg(f, \Omega, y_\circ)$. If $f\colon \Omega \to \mathbb{R}^n$ is a homeomorphism, then $\deg(f, \Omega, y_\circ)$ is either 1 or $-1$ for all $y_\circ \in f(\Omega)$, see e.g. [17, Section IV.5] or [21, Section II.2.4, Theorem 3]. We say that a homeomorphism $f$ is *sense-preserving* if $\deg(f, \Omega, y_\circ) \equiv 1$. For a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\det A \neq 0$, it is easy to check from the definition that

$$\deg(A, \Omega, y_\circ) = \operatorname{sgn} \det A. \tag{1}$$

We recall the following corollary [2, Corollary 2.8.2]. Given a homeomorphism $f\colon \Omega \to \mathbb{R}^n$ suppose that $f$ is differentiable at $x_\circ$ with $J(x_\circ, f) \neq 0$. Then we have

$$\deg(f, \Omega, f(x_\circ)) = \operatorname{sgn} J(x_\circ, f). \tag{2}$$

We will use the fact that the topological degree is stable under homotopy. That is for every continuous mapping $H\colon \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ and $y_\circ \in \mathbb{R}^n$ such that $y_\circ \notin H(\partial\Omega, t)$ for all $t \in [0, 1]$ we have

$$\deg(H(\,\cdot\,, 0), \Omega, y_\circ) = \deg(H(\cdot, 1), \Omega, y_\circ). \tag{3}$$

### 2.2 Differentiability of Sobolev mappings

A Sobolev homeomorphism $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ is differentiable almost everywhere if $p > n - 1$, $n \geqslant 3$, and $p \geqslant 1$ for $n = 2$, see [9, 20, 26]. We will also need a generalization of the concept of differentiability, which is obtained by replacing the ordinary limit by an approximate limit, see e.g. [8, Section 6.1.3]. It is known that a Sobolev mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is *approximatively differentiable* almost everywhere, see e.g. [8, Section 6.1.2, Theorem 2]. Moreover, such a mapping is $L^1$-*differentiable* almost everywhere [27]; that is, for almost every $x_\circ \in \Omega$ we have

$$\lim_{r\to 0} \fint_{B(x_\circ, r)} \left| \frac{f(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \right| \mathrm{d}x = 0. \tag{4}$$

Hereafter, the notation $\fint_{B(x_\circ,r)}$ means the integral average over the $n$-dimensional ball

$$B(x_\circ, r) = \{x \in \mathbb{R}^n : |x - x_\circ| < r\}.$$

In order to illustrate our ideas and for reader's comprehension, we first prove Theorem 1 in the simpler cases $p \geq 1$, $n = 2$; and $p > n - 1$, $n \geq 3$, where we can avoid some technicalities.

## 3 Proof of Theorem 1 for $p > n - 1$, $n \geqslant 3$, and $p \geqslant 1$, $n = 2$

Each homeomorphism $f_j$ is either sense-preserving or sense-reversing. Under our assumptions there exists a point $x_j$ such that $f_j$ is differentiable at $x_j$, see Section 2.2, and $J(x_j, f_j) > 0$. By (2) we know that the degree of $f_j$ is one and hence each $f_j$ is sense-preserving.

As $f_j \rightharpoonup f$ in $L^p$, $p > 1$, we know that $\int_\Omega |Df|^p$ is uniformly bounded and hence we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$|Df_j|^p \overset{w^*}{\rightharpoonup} \mu \text{ in measures.}$$

Moreover, for $p = 1$ we can use De La Vale Pousin characterization of weak convergence in $L^1$ and we can find an continuous convex function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$\frac{\Phi(t)}{t} \text{ is increasing,} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty \quad \text{and} \quad \int_\Omega \Phi(|Df_j|) \leq 1. \tag{5}$$

It follows that we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$\Phi(|Df_j|) \overset{w^*}{\rightharpoonup} \mu \text{ in measures.}$$

It is well known that for almost every $x_\circ \in \Omega$ we have

$$M\mu(x_\circ) := \sup_{r>0} \frac{\mu(B(x_\circ, r) \cap \Omega)}{|B(x_\circ, r)|} < \infty. \tag{6}$$

Let $\delta > 0$. For the contrary we suppose that there is $x_\circ \in \Omega$ such that (4) and (6) hold at $x_\circ$ and

$$J(x_\circ, f) < 0.$$

Without loss of generality we may and do assume that

$$Df(x_\circ) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}. \tag{7}$$

Using (4) we can find $0 < r_1$ small enough such that for all $0 < r < r_1$ we have

$$\fint_{B(x_\circ,r)} \left| \frac{f(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \right| \mathrm{d}x < \frac{\delta^n}{2}.$$

Since the sequence of mappings $f_j$ converges to $f$ weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, we have that the sequence of mappings $f_j$ converges to $f$ strongly in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Now, we may pick up an index $j_\circ$ large enough such that for all $j \geq j_\circ$,

$$\int_\Omega |f(x) - f_j(x)| \, \mathrm{d}x < |B(0,1)| r^{n+1} \frac{\delta^n}{2}.$$

The last two inequalities imply that for all $0 < r < r_1$ we have

$$\fint_{B(x_\circ,r)} \left| \frac{f_j(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \right| \mathrm{d}x < \delta^n. \tag{8}$$

Our next goal is to prove the following:

(i) if $p > n - 1$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j_\circ$,

$$\delta^{n-1-p} r^n \leqslant C \int_{B(x_\circ,r)} |Df_j|^p,$$

(ii) if $n = 2$ and $p = 1$, there exist a constant $C$ and such that for all $0 < r < r_1$ and $j \geq j_\circ$ there is a set $A \subset B(x_\circ, r)$ such that

$$|A| < C\delta|B(x_\circ, r)| \quad \text{and} \quad r^2 \leqslant C \int_A |Df_j|.$$

These would lead to a desired contradiction. Indeed, choose $0 < r < r_1$ such that $\mu(\partial B(x_\circ, r)) = 0$ and in case (i) we obtain after passing to a limit in $j$ that

$$\delta^{n-1-p} \leq C \lim_{j\to\infty} \fint_{B(x_\circ,r)} |Df_j|^p = C\frac{\mu(B(x_\circ, r) \cap \Omega)}{|B(x_\circ, r)|} \leq CM\mu(x_\circ).$$

After passing $\delta \to 0+$ we obtain a contradiction with (6). In case (ii) we can use Jensen's inequality and (5) to obtain

$$\fint_{B(x_\circ,r)} \Phi(|Df_j|) \geq \frac{|A|}{r^2} \fint_A \Phi(|Df_j|) \geq \frac{|A|}{r^2}\Phi\Big(\fint_A |Df_j|\Big) \geq \frac{|A|}{r^2}\Phi\Big(\frac{Cr^2}{|A|}\Big) \geq C\delta\Phi\Big(\frac{C}{\delta}\Big).$$

Similarly as above we obtain in the limit that

$$C\delta\Phi\Big(\frac{C}{\delta}\Big) \leq CM\mu(x_\circ)$$

and now passing to a limit $\delta \to 0+$ we obtain a contradiction using (5).

*Proof of* (i). We simplify the notation and write

$$\varphi_j(x) = |f_j(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)| \quad \text{and} \quad B_s = B(x_\circ, s).$$

In the following we use the notation $\mathcal{H}^k(A)$ for the $k$-dimensional Hausdorff measure of the set $A$. We claim that the set of radii

$$I_G = \{s \in [0, r] : \mathcal{H}^{n-1}(\{x \in \partial B_s : \varphi_j(x) \geqslant \delta r\}) < 5^n\delta^{n-1}\mathcal{H}^{n-1}(\partial B_s)\}$$

has measure at least $\frac{3r}{4}$, i.e. $|I_G| \geqslant \frac{3r}{4}$, otherwise

$$\fint_{B_r} \Big|\frac{\varphi_j(x)}{r}\Big| \,dx \geq \frac{1}{|B_r|}\int_0^{\frac{r}{4}} 5^n\delta^{n-1}\mathcal{H}^{n-1}(\partial B_s)\Big|\frac{\delta r}{r}\Big| \,ds = 5^n\delta^n\frac{|B_{\frac{r}{4}}|}{|B_r|}$$

which contradicts (8).

On the other hand, the key point in our argument is that for $x_\circ \in \Omega_\circ$ and for every $s \in (0, r)$ we can find $\beta = \beta(s) \in \partial B_s$ such that

$$\varphi_j(\beta) \geqslant \frac{4}{5}s \quad \text{for every } j = 1, 2, \dots . \tag{9}$$

Finding such a point $\beta$ is the only place where we use the homeomorphism assumption of $f_j$. Suppose on the contrary that (9) fails for every $\beta \in \partial B_s$ and for some $j \in \{1, 2, \dots\}$. For $x \in \partial B_s$ and $t \in [0, 1]$ we consider the following homotopy:

$$H(x, t) := (1 - t)\big(f_j(x) - f(x_\circ)\big) + tDf(x_\circ)(x - x_\circ).$$

By (7) we know that $Df(x_\circ)$ is an isometry and thus $|Df(x_\circ)z| = |z|$. Furthermore, if (9) does not hold, then for all $x \in \partial B_s$ we have

$$|H(x, t)| \geqslant |Df(x_0)(x - x_0)| - (1 - t)|f_j(x) - f(x_0) - Df(x_0)(x - x_0)| \geq s - (1 - t)\frac{4}{5}s > 0.$$

It follows that $H(x, t) \neq 0$ for every $x \in \partial B_s$ and all $t \in [0, 1]$. Thus, by (3) and (1),

$$\deg(f_j, B_s, f(x_\circ)) = \operatorname{sgn}\det(Df(x_\circ)) = -1.$$

This contradicts the fact that $f_j$ is sense-preserving.

We apply the Sobolev embedding theorem [8, Theorem 3 (i), p. 143] on the $(n-1)$-dimensional spheres. This way for almost every $s \in (0, r_\circ)$ and for all $z_1, z_2 \in \partial B(x_\circ, s)$ we have

$$|f_j(z_1) - f_j(z_2)| \leqslant C(n,p)|z_1 - z_2|^{1-\frac{n-1}{p}} \Big( \int\limits_{\partial B_s} |Df_j|^p \Big)^{\frac{1}{p}}. \tag{10}$$

Now let us fix $s \in I_G$ so that (10) is satisfied on the sphere $\partial B_s$. Since $s \in I_G$, we find $\alpha = \alpha(s) \in \partial B_s$ satisfying

$$\varphi_j(\alpha) < \delta r \quad \text{and} \quad |\alpha - \beta| \leqslant C_0 \delta s,$$

where $C_0$ is some fixed constant (which depends only on $n$). Combining this with (9) we have found $\alpha, \beta \in \partial B_s$ such that

$$\frac{4}{5}s - \delta r - 2C_0\delta s \leqslant |\varphi_j(\beta)| - |\varphi_j(\alpha)| - 2|\alpha - \beta| \leqslant |f_j(\alpha) - f_j(\beta)|.$$

This together with (10) implies that for $s \in I_G \cap [\frac{r}{2}, r]$ and $\delta$ small enough

$$Cs^p \leqslant \Big(\frac{4}{5}s - \delta r - 2C_0\delta s\Big)^p \leqslant C(n,p)(\delta s)^{p-n+1} \int\limits_{\partial B_s} |Df_j|^p. \tag{11}$$

Integrating inequality (11) over the set $I_G \cap [\frac{r}{2}, r]$ we obtain (i), finishing the proof of Theorem 1 in the case $p > n-1$. □

*Proof of* (ii). We proceed as above. For $s \in I_G$ we can find $\beta = \beta(s) \in \partial B_s$ so that (9) holds. In fact we consider the measurable set

$$A := \{x \in B_r : \varphi_j(x) > \delta r\}.$$

By Chebyshev's inequality and (8) we obtain

$$|A| \leq \frac{1}{\delta r} \int\limits_{B_r} |\varphi_j(x)|\, dx \leq \frac{1}{\delta r} \delta^2 r^2 r = C\delta |B_r|.$$

Let $s \in I_G \cap [\frac{r}{2}, r]$. The point $\beta \in \partial B_s$ with (9) clearly belongs to $A \cap \partial B_s$ and the closest point $\alpha$ on the relative boundary of $\partial B_s \cap A$ satisfies

$$|\varphi_j(\alpha)| = \delta r$$

by the definition of $A$. It follows that for every $s \in I_G \cap [\frac{r}{2}, r]$ we have

$$s \leqslant C \int\limits_{\partial B_s \cap A} |Df_j|.$$

Integrating this over $I_G \cap [\frac{r}{2}, r]$ we obtain

$$r^2 \leqslant C \int\limits_A |Df_j|$$

finishing the proof of (ii). □

The above proof was based on the Sobolev embedding theorem on spheres and therefore does not work for $p < n-1$. To overcome these difficulties we follow Hencl and Malý [11] and use the theory of linking numbers and its topological invariance. For the convenience of the reader we recall the needed properties of linking numbers here.

# 4 Linking number

We use the notation $\mathbb{B}_d$ for the unit ball in $\mathbb{R}^d$ and $\mathbb{S}_{d-1}$ for the unit sphere. By $\overline{\mathbb{B}}_d(c, r)$ we denote the closed ball with center $c$ and radius $r > 0$.

Let $n, t, q$ be positive integers with $t + q = n - 1$. Let us consider the mapping $\Phi(\xi, \eta)\colon \overline{\mathbb{B}}_{t+1} \times \overline{\mathbb{B}}_{q+1} \to \mathbb{R}^n$ defined coordinatewise as $\Phi(\xi, \eta) = x$, where

$$\begin{aligned} x_1 &= (2 + \eta_1)\xi_1, \\ &\vdots \\ x_{t+1} &= (2 + \eta_1)\xi_{t+1}, \\ x_{t+2} &= \eta_2, \\ &\vdots \\ x_{t+q+1} &= \eta_{q+1}. \end{aligned}$$

Denote by $\mathbb{A}$ the anuloid

$$\Phi(\mathbb{S}_t \times \mathbb{B}_{q+1}) = \Big\{x \in \mathbb{R}^n : \Big(\sqrt{x_1^2 + \cdots + x_{t+1}^2} - 2\Big)^2 + x_{t+2}^2 + \cdots + x_n^2 < 1\Big\}.$$

Of course, given $x \in \overline{\mathbb{A}}$ we can find a unique $\xi \in \mathbb{S}_t$ and $\eta \in \overline{\mathbb{B}}_{q+1}$ such that $\Phi(\xi, \eta) = x$. We will denote these as $\xi(x)$ and $\eta(x)$.

A *link* is a pair $(\varphi, \psi)$ of parametrized surfaces $\varphi\colon \mathbb{S}_t \to \mathbb{R}^n$, $\psi\colon \mathbb{S}_q \to \mathbb{R}^n$. The *linking number* of the link $(\varphi, \psi)$ is defined as the topological degree

$$\mathcal{L}(\varphi, \psi) = \deg(L, \mathbb{A}, 0),$$

where the mapping $L = L_{\varphi,\psi} : \overline{\mathbb{A}} \to \mathbb{R}^n$ is defined as

$$L(x) = \varphi(\xi(x)) - \bar{\psi}(-\eta(x)),$$

or equivalently

$$L(\Phi(\xi, \eta)) = \varphi(\xi) - \bar{\psi}(-\eta), \quad \xi \in \mathbb{S}_t,\ \eta \in \mathbb{B}_{q+1},$$

where $\bar{\psi}$ is an arbitrary continuous extension of $\psi$ to $\overline{\mathbb{B}}_{q+1}$ (of course, the degree does not depend on the way how we extend $\psi$, it depends only on the values on the boundary $\partial\mathbb{A} = \Phi(\mathbb{S}_t \times \mathbb{S}_q)$). Geometrically speaking, for $t = q = 1$, the linking number is the number of loops of a curve $\varphi$ around a curve $\psi$ counting orientation into account as $+1$ or $-1$. For the introductions to the linking number in $\mathbb{R}^3$ and its application to the theory of knots see [23].

The *canonical link* is the pair $(\mu, \nu)$, where

$$\begin{aligned} \mu(\xi) &= \Phi(\xi, 0), & \xi &\in \mathbb{S}_t, \\ \nu(\eta) &= \Phi(\mathbf{e}_1, \eta), & \eta &\in \mathbb{S}_q. \end{aligned}$$

For example in dimension $n = 3$ we get that

$$\begin{aligned} \mu(\mathbb{S}_1) &= \{x \in \mathbb{R}^3 : x_3 = 0,\ x_1^2 + x_2^2 = 4\}, \\ \nu(\mathbb{S}_1) &= \{x_2 = 0,\ (x_1 - 2)^2 + x_3^2 = 1\}. \end{aligned}$$

It is well known that the linking number is a topological invariant. The simple proof of the following proposition can be found in [11].

**Proposition 2.** *Let $n, t, q$ be positive integers with $t + q = n - 1$. Let $f : \mathbb{B}_n(4) \to \mathbb{R}^n$ be a homeomorphism. Then $\mathcal{L}(f \circ \mu, f \circ \nu)$ is 1 iff $f$ is sense preserving and $-1$ iff $f$ is sense reversing.*

Analogously, we can pick $a \in \overline{\mathbb{B}}_{q+1}(0, \frac{1}{10})$ and $b \in \overline{\mathbb{B}}_{t+1}(\mathbf{e}_1, \frac{1}{10}) \cap \overline{\mathbb{B}}_{t+1}$ and consider the pair

$$\begin{aligned} \mu_a(\xi) &= \Phi(\xi, a), & \xi &\in \mathbb{S}_t, \\ \nu_b(\eta) &= \Phi(b, \eta), & \eta &\in \mathbb{S}_q. \end{aligned}$$

Similarly to the previous proposition we have:

**Proposition 3.** *Let $n, t, q$ be positive integers with $t + q = n - 1$, $a \in \overline{\mathbb{B}}_{q+1}(0, \frac{1}{10})$ and $b \in \overline{\mathbb{B}}_{t+1}(\mathbf{e}_1, \frac{1}{10}) \cap \overline{\mathbb{B}}_{t+1}$. Let $f : \mathbb{B}_n(4) \to \mathbb{R}^n$ be a homeomorphism. Then $\mathcal{L}(f \circ \mu_a, f \circ \nu_b)$ is 1 iff $f$ is sense preserving and $-1$ iff $f$ is sense reversing.*

## 5 Proof of Theorem 1 for $p > [\frac{n}{2}]$, $n \geqslant 3$, and $p \geq 1$, $n = 3$

Our argument is similar to the proof given in Section 3 and therefore some details are only sketched. By $\mu$ we again denote the $w^*$ limit of (some subsequence) $\int |Df_j|^p$ for $p > [\frac{n}{2}]$ and of $\int \Phi(|Df_j|)$ for $p = 1$ and $n = 3$.

By $C_1$ and $C_2$ we denote a fixed constants whose exact value will be determined later. We fix $\delta > 0$ and we choose a point $x_0$ such that (4) and (6) hold and without loss of generality we assume that the derivative of $f$ at $x_\circ$ is given by (7).

We fix $r_1 > 0$ such that for all $0 < r < r_1$ we have

$$\fint_{B(x_\circ,4r)} \left|\frac{f(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r}\right| \mathrm{d}x < C_1 \frac{\delta^n}{2}$$

and again for all $j \geq j_\circ$ we obtain

$$\fint_{B(x_\circ,4r)} \left|\frac{f_j(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r}\right| \mathrm{d}x < C_1 \delta^n. \tag{12}$$

We fix $t, q \leqslant [\frac{n}{2}]$ such that $t + q = n - 1$ (e.g. $t = q = \frac{n-1}{2}$ for $n$ odd and $t = \frac{n-2}{2}$, $q = \frac{n}{2}$ for $n$ even). Our goal is to prove the following:

(i) if $p > [\frac{n}{2}]$ and $n \geqslant 3$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j_\circ$,

$$\delta^{\min\{t,q\}-p} r^n \leqslant C \int_{B(x_\circ,4r)} |Df_j|^p,$$

(ii) if $p = 1$ and $n = 3$, we have $A \subset B(x_\circ, 4r)$ such that

$$|A| < C_2 \delta |B(x_\circ, 4r)| \quad \text{and} \quad r^3 \leqslant C \int_A |Df_j|.$$

Analogously to reasoning in Section 3 we obtain a contradiction using $\min\{t, q\} - p < 0$ for $p > [\frac{n}{2}]$ and (5) for $p = 1$ and $n = 3$.

*Proof of* (i). Without loss of generality we will assume that $x_\circ = 0$. We write

$$\varphi_j(x) = |f_j(rx) - f(0) - Df(0)rx|.$$

Let us fix $y \in \mu_a(\mathbb{S}_t)$ and denote

$$B_{\mu_a(\mathbb{S}_t)}(y, \delta) = \{x \in \mu_a(\mathbb{S}_t) : |x - y| < \delta\},$$

the ball of radius $\delta$ on the link $\mu_a(\mathbb{S}_t)$. We can clearly choose a constant $C_1$ small enough at the beginning of the proof so that (12) implies that the set of good links

$$I_a = \left\{a \in \overline{\mathbb{B}}_{q+1}\left(0, \frac{1}{10}\right) : \mathcal{H}^t(x \in \mu_a(\mathbb{S}_t) : \varphi_{j_\circ}(x) \geqslant \delta r) < \mathcal{H}^t(B_{\mu_a(\mathbb{S}_t)}(y, \delta))\right\},$$
$$I_b = \left\{b \in \overline{\mathbb{B}}_{t+1}\left(\mathbf{e}_1, \frac{1}{10}\right) \cap \overline{\mathbb{B}}_{t+1} : \mathcal{H}^q(x \in \nu_b(\mathbb{S}_q) : \varphi_{j_\circ}(x) \geqslant \delta r) < \mathcal{H}^q(B_{\nu_b(\mathbb{S}_q)}(y, \delta))\right\}$$

has measure at least

$$\mathcal{H}^{q+1}(I_a) > \frac{1}{2}\left|\mathbb{B}_{q+1}\left(0, \frac{1}{10}\right)\right| \quad \text{and} \quad \mathcal{H}^{t+1}(I_b) > \frac{1}{2}\left|\mathbb{B}_{t+1}\left(\mathbf{e}_1, \frac{1}{10}\right) \cap \mathbb{B}_{t+1}\right|.$$

The key point of our argument is that for every $a \in \overline{\mathbb{B}}_{q+1}(0, \frac{1}{10})$ and every $b \in \overline{\mathbb{B}}_{t+1}(\mathbf{e}_1, \frac{1}{10}) \cap \overline{\mathbb{B}}_{t+1}$ we can find $\xi \in \mathbb{S}_t$ and $\eta \in \mathbb{S}_q$ such that

$$\begin{aligned} \varphi_j(\mu_a(\xi)) &= \left|f_j(r\mu_a(\xi)) - f(0) - Df(0)r\mu_a(\xi)\right| > \frac{r}{10} \quad \text{or} \\ \varphi_j(\nu_b(\eta)) &= \left|f_j(r\nu_b(\eta)) - f(0) - Df(0)r\nu_b(\eta)\right| > \frac{r}{10}. \end{aligned} \tag{13}$$

We prove the observation by contradiction and we suppose that (13) does not hold. We define

$$f_s(x) = (1-s)(f(0) + Df(0)rx) + sf_j(rx)$$

and we consider the homotopy $H(\overline{\mathbb{A}} \times [0,1]) \to \mathbb{R}^n$ defined as

$$H(\Phi(\xi,\eta), s) = (f_s \circ \mu_a)(\xi) - \overline{(f_s \circ \nu_b)}(-\eta),$$

where $\overline{(f_s \circ \nu_b)}$ denotes a continuous extension of $f_s \circ \nu_b$ to $\overline{\mathbb{B}}_{q+1}$ as in the definition of the linking number, which in addition depends continuously on $s$. From [11] we know that the mapping $f_j \in W^{1,p}$, $p > [\frac{n}{2}]$, with nonnegative and nonzero Jacobian is sense preserving. By Proposition 3 we get that

$$\deg(H(x,1), \mathbb{A}, 0) = 1.$$

On the other hand

$$\deg(H(x,0), \mathbb{A}, 0) = -1$$

since the affine mapping $f(0) + Df(0)rx$ is sense reversing. To obtain a contradiction (with the preservation of the degree under homotopy) it is now enough to show that for every $\xi \in \mathbb{S}_t$, for every $\eta \in \mathbb{S}_q$ and for every $s \in [0,1]$ we have $H(\Phi(\xi,\eta),s) \neq 0$. It is easy to see that

$$\operatorname{dist}((f_0 \circ \mu_a)(\mathbb{S}_t), (f_0 \circ \nu_b)(\mathbb{S}_q)) \geqslant \operatorname{dist}((f_0 \circ \mu)(\mathbb{S}_t), (f_0 \circ \nu)(\mathbb{S}_q)) - \frac{6r}{10} \geq \frac{3r}{10}.$$

Since (13) does not hold, we obtain from the definition of $f_s$ that

$$\operatorname{dist}((f_s \circ \mu_a)(\mathbb{S}_t), (f_s \circ \nu_b)(\mathbb{S}_q)) \geqslant \frac{3r}{10} - \frac{r}{10} - \frac{r}{10}$$

which implies $H(\Phi(\xi,\eta),s) \neq 0$.

By (13) and the symmetry we may assume without loss of generality that

$$\tilde{I}_a = \left\{ a \in I_a : \text{there exists } \xi \in \mathbb{S}_t \text{ such that } \varphi_j(\mu_a(\xi)) > \frac{r}{10} \right\}$$

satisfies $\mathcal{H}^{q+1}(\tilde{I}_a) > \frac{1}{4}|\mathbb{B}_{q+1}(0, \frac{1}{10})|$. Since $p > [\frac{n}{2}] \geq t$, we can use the Sobolev embedding theorem on the $t$-dimensional space $r\mu_a(\mathbb{S}_t)$ and we have for almost every $a \in \tilde{I}_a$ and for all $z_1, z_2 \in r\mu_a(\mathbb{S}_t)$,

$$|f_j(z_1) - f_j(z_2)| \leqslant C|z_1 - z_2|^{1-\frac{t}{p}} \Bigg( \int\limits_{r\mu_a(\mathbb{S}_t)} |Df_j|^p \Bigg)^{\frac{1}{p}}. \tag{14}$$

Now let us fix $a \in \tilde{I}_a$ so that (14) is satisfied and find $\xi \in \mathbb{S}_t$ so that for $\beta = \mu_a(\xi)$ we have $\varphi_j(\beta) > \frac{r}{10}$ as in the definition of $\tilde{I}_a$. Using $a \in I_a$ we find $\alpha \in \mu_a(\mathbb{S}_t)$ satisfying

$$\varphi_j(\alpha) < \delta r \quad \text{and} \quad |\alpha - \beta| \leqslant \delta.$$

Thus we have found $\alpha, \beta \in \mu_a(\mathbb{S}_t)$ such that

$$\frac{r}{10} - 3\delta r \leqslant |\varphi_j(\beta)| - |\varphi_j(\alpha)| - 2r|\alpha - \beta| \leqslant |f_j(r\alpha) - f_j(r\beta)|.$$

This together with (14) implies that for almost every $a \in \tilde{I}_a$ and $\delta$ small enough we have

$$r^t \leqslant C\delta^{p-t} \int\limits_{r\mu_a(\mathbb{S}_t)} |Df_j|^p. \tag{15}$$

Integrating inequality (15) over the set $\tilde{I}_A$ we obtain (i). □

*Proof of* (ii). If $p = 1$ and $n = 3$, then for each $a \in \tilde{I}_a$ we can find $\xi \in \mathbb{S}(t)$ so that (13) holds for $\beta = \mu_a(\xi)$. The measurable set

$$A := \{x \in B(x_\circ, 4r) : \varphi_j(x) > \delta r\}$$

satisfies $|A| \leq C\delta|B(x_\circ, 4r)|$ by inequality (12) and Chebyshev's inequality. Now clearly $\xi$ with (13) satisfies $r\beta = r\mu_a(\xi) \in A$. In (14) and (15) instead of integrating over the entire $r\mu_a(\mathbb{S}_t)$ we integrate only over the set $r\mu_a(\mathbb{S}_t) \cap A$. Integrating over $\tilde{I}_a$ we obtain the desired conclusion. □

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