Research Article

Stanislav Hencl and Jani Onninen*

Jacobian of weak limits of Sobolev homeomorphisms

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Abstract: Let $\Omega$ be a domain in $\mathbb{R}^n$, where $n = 2, 3$. Suppose that a sequence of Sobolev homeomorphisms $f_k : \Omega \to \mathbb{R}^n$ with positive Jacobian determinants, $J(x, f_k) > 0$, converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, for some $p \geq 1$, to a mapping $f$. We show that $J(x, f) \geq 0$ a.e. in $\Omega$. Generalizations to higher dimensions are also given.

Keywords: Sobolev homeomorphism, weak limits, Jacobian

MSC 2010: Primary 26B10; secondary 46E35

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1 Introduction

The main goal of this note is to establish when the sign of the Jacobian is preserved under $W^{1,p}$-weak convergence. Such a question pops out naturally in the variational approach to Geometric Function Theory (GFT) [2, 14, 22] and Nonlinear Elasticity (NE) [1, 4, 6, 19, 24, 25]. Both theories GFT and NE deal with minimizing sequences of Sobolev homeomorphisms. In the context of NE, one typically deals with two-dimensional or three-dimensional models and require that the deformation gradients belong to $M^{n \times n}_{\text{real}}$, where $M^{n \times n}_{\text{real}} = \{\text{real } n \times n \text{ matrices}\}$, and $M^{n \times n}_{\text{sym}} = \{A \in M^{n \times n}_{\text{real}} : \det A > 0\}$. The infimum of the energy is not attained, in general, at a homeomorphism; interpenetration of matter may occur. Even in a special case of Dirichlet energy injectivity is often lost when passing to the weak limit of the minimizing sequence, [3, 13, 15, 16]. Further examinations are needed to know the properties of such singular minimizers.

Throughout this text $\Omega$ will be a domain in $\mathbb{R}^n$. The class of Sobolev mappings $f : \Omega \to \mathbb{R}^n$ with nonnegative Jacobian determinant, $J(x, f) = \det Df(x) \geq 0$ almost everywhere, is closed under the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^n)$ provided $p \geq n$ (see [14, Theorem 8.4.2]). However, if $p < n$, passing to the weak $W^{1,p}$-limit of a sequence with nonnegative Jacobians one may lose the sign of the Jacobian. Indeed, there exists a sequence of Sobolev mappings $f_k : \Omega \to \mathbb{R}^n$ with $J(x, f_k) > 0$ almost everywhere such that the sequence converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p < n$, to the mapping $f(x) = (-x_1, x_2, \ldots, x_n)$, see [14, p. 181]. Moreover, following the construction in [18] such mappings $f_k$ can be made continuous. However, it is not obvious at all as to whether one can make a similar example with $f_k$ being homeomorphisms. This is the subject of our result here. Here $[\frac{a}{b}]$ denotes the integer part, i.e. $[\frac{2}{3}] = 1$, $[\frac{3}{2}] = 1$ and so on.

Theorem 1. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $p \geq 1$ for $n \in \{2, 3\}$ and $p > [\frac{n}{2}]$ for $n \geq 4$. Suppose that a sequence of Sobolev homeomorphisms $f_k : \Omega \to \mathbb{R}^n$ with $J(x, f_k) \geq 0$ converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$ to a mapping $f$ and further assume that $f(x, f_k)$ is not a.e. zero. Then $J(x, f) \geq 0$ a.e. in $\Omega$.

*Corresponding author: Jani Onninen: Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA; and Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland, e-mail: jkonnine@syr.edu

Stanislav Hencl: Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ 186 00 Prague 8, Czech Republic, e-mail: hencl@karlin.mff.cuni.cz

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It is worth noting that in Theorem 1 the Jacobian \( f(x, f) \) can have very different behavior than the Jacobians in the sequence without knowing that \( f(x, f_k) > 0 \) on a set of positive measure. Indeed, there exists a sequence of Sobolev homeomorphisms \( f_k \) with \( f(x, f_k) = 0 \) a.e., converging weakly in \( W^{1,p}(\Omega, \mathbb{R}^n) \), \( 1 < p < n \), to the mapping \( f(x) = x \). Let us briefly sketch this using the construction from [10]: we cover \( \Omega \) by cubes of diameter less than \( \frac{1}{2} \) and on each cube we follow the construction from [10] to obtain a homeomorphism with zero Jacobian a.e. It is possible to make the \( W^{1,p} \)-norm of the sequence uniformly bounded and hence find a weakly convergent subsequence. Furthermore, it follows from the construction that the sequence \( f_k \) converges uniformly to the identity. This also shows that there is a sequence with \( f(x, f_k) = 0 \) a.e. converging weakly in \( W^{1,p}(\Omega, \mathbb{R}^n) \), \( 1 < p < n \), to \( f(x) = (x_1, x_2, \ldots, x_n) \).

Recently it was shown in [12] and [5] that a Jacobian of a Sobolev homeomorphism can change sign in dimension \( n \geq 4 \) for \( 1 \leq p < \frac{n}{2} \).

## 2 Preliminaries

### 2.1 Degree and Jacobian

There are two basic approaches to the notion of local degree for a mapping, the algebraic (see e.g. Dold [7]) and the analytic (see e.g. Lloyd [17]). Both of these notions try to capture the idea of counting the preimages of a target point. For a continuous mapping \( f : \Omega \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \setminus f(\partial\Omega) \) the degree of \( f \) at \( y \), with respect to \( \Omega \) is denoted by \( \text{deg}(f, \Omega, y) \). If \( f : \Omega \to \mathbb{R}^n \) is a homeomorphism, then \( \text{deg}(f, \Omega, y) \) is either 1 or \(-1\) for all \( y \in f(\Omega) \), see e.g. [17, Section IV.5] or [21, Section II.2.4, Theorem 3]. We say that a homeomorphism \( f \) is sense-preserving if \( \text{deg}(f, \Omega, y) \equiv 1 \). For a linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) with \( \det A \neq 0 \), it is easy to check from the definition that

\[
\text{deg}(A, \Omega, y) = \text{sgn det } A. \tag{1}
\]

We recall the following corollary [2, Corollary 2.8.2]. Given a homeomorphism \( f : \Omega \to \mathbb{R}^n \) suppose that \( f \) is differentiable at \( x \), with \( f(x, f) \neq 0 \). Then we have

\[
\text{deg}(f, \Omega, f(x)) = \text{sgn } J(x, f). \tag{2}
\]

We will use the fact that the topological degree is stable under homotopy. That is for every continuous mapping \( H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) such that \( y \notin H(\partial\Omega, t) \) for all \( t \in [0, 1] \) we have

\[
\text{deg}(H(\cdot, 0), \Omega, y) = \text{deg}(H(\cdot, 1), \Omega, y). \tag{3}
\]

### 2.2 Differentiability of Sobolev mappings

A Sobolev homeomorphism \( f \in W^{1,p}(\Omega, \mathbb{R}^n) \) is differentiable almost everywhere if \( p > n - 1 \), \( n \geq 3 \), and \( p \geq 1 \) for \( n = 2 \), see [9, 20, 26]. We will also need a generalization of the concept of differentiability, which is obtained by replacing the ordinary limit by an approximate limit, see e.g. [8, Section 6.1.3]. It is known that a Sobolev mapping \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \) is \( \text{approximatively differentiable} \) almost everywhere, see e.g. [8, Section 6.1.2, Theorem 2]. Moreover, such a mapping is \( L^1 \)-differentiable almost everywhere [27]; that is, for almost every \( x \in \Omega \) we have

\[
\lim_{r \to 0} \int_{B(x,r)} \left\{ \left| f(x) - f(x_r) - Df(x)(x - x_r) \right| \right\} dx = 0. \tag{4}
\]

Hereafter, the notation \( \int_{B(x,r)} \) means the integral average over the \( n \)-dimensional ball

\[
B(x, r) = \{ x \in \mathbb{R}^n : |x - x_r| < r \}.
\]

In order to illustrate our ideas and for reader’s comprehension, we first prove Theorem 1 in the simpler cases \( p \geq 1, n = 2 \); and \( p > n - 1, n \geq 3 \), where we can avoid some technicalities.
3 Proof of Theorem 1 for $p > n - 1$, $n \geq 3$, and $p \geq 1$, $n = 2$

Each homeomorphism $f_j$ is either sense-preserving or sense-reversing. Under our assumptions there exists a point $x_j$ such that $f_j$ is differentiable at $x_j$, see Section 2.2, and $J(x_j, f_j) > 0$. By (2) we know that the degree of $f_j$ is one and hence each $f_j$ is sense-preserving.

As $f_j \to f$ in $L^p$, $p > 1$, we know that $\int_{\Omega} |Df_j|^p$ is uniformly bounded and hence we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$|Df_j|^p \overset{w^*}{\to} \mu \text{ in measures.}$$

Moreover, for $p = 1$ we can use De La Vale Poussin characterization of weak convergence in $L^1$ and we can find an continuous convex function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$\frac{\Phi(t)}{t} \text{ is increasing, } \lim_{t \to \infty} \Phi(t) = \infty \text{ and } \int_{\Omega} \Phi(|Df_j|) \leq 1. \quad (5)$$

It follows that we can find a Radon measure $\mu$ and a subsequence (which we will denote again as $f_j$) such that

$$\Phi(|Df_j|) \overset{w^*}{\to} \mu \text{ in measures.}$$

It is well known that for almost every $x_j \in \Omega$ we have

$$M_\mu(x_j) := \sup_{r > 0} \frac{\mu(B(x_j, r) \cap \Omega)}{|B(x_j, r)|} < \infty. \quad (6)$$

Let $\delta > 0$. For the contrary we suppose that there is $x_j \in \Omega$ such that (4) and (6) hold at $x_j$ and

$$J(x_j, f) < 0. \quad (7)$$

Without loss of generality we may and do assume that

$$Df(x_j) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{pmatrix}. \quad (8)$$

Using (4) we can find $0 < r_1$ small enough such that for all $0 < r < r_1$ we have

$$\int_{B(x_j, r)} \left| \frac{f(x) - f(x_j) - Df(x_j)(x - x_j)}{r} \right| \, dx < \frac{\delta^n}{2}. \quad (9)$$

Since the sequence of mappings $f_j$ converges to $f$ weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, we have that the sequence of mappings $f_j$ converges to $f$ strongly in $L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$. Now, we may pick up an index $j_1$ large enough such that for all $j \geq j_1$, $\delta^n/2$.

The last two inequalities imply that for all $0 < r < r_1$ we have

$$\int_{B(x_j, r)} \left| \frac{f_j(x) - f(x_j) - Df(x_j)(x - x_j)}{r} \right| \, dx < \delta^n. \quad (10)$$

Our next goal is to prove the following:

(i) if $p > n - 1$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j_1$, $\delta^n$.

$$\delta^{n-1-p,n} \leq C \int_{B(x_j, r)} |Df_j|^p, \quad \delta^{n-1-p,n} \leq C \int_{B(x_j, r)} |Df_j|^p, \quad (11)$$
(ii) if $n = 2$ and $p = 1$, there exist a constant $C$ and such that for all $0 < r < r_1$ and $j \geq j_i$ there is a set $A \subset B(x_i, r)$ such that

$$|A| < C\delta |B(x_i, r)| \quad \text{and} \quad r^2 \leq C \int_A |Df_j|.$$ 

These would lead to a desired contradiction. Indeed, choose $0 < r < r_1$ such that $\mu(\partial B(x_i, r)) = 0$ and in case (i) we obtain after passing to a limit in $j$ that

$$\delta^{n-1-p} \leq C \lim_{j \to \infty} \frac{1}{|B(x_i, r)|} \int_{B(x_i, r)} |Df_j|^p = C \frac{\mu(B(x_i, r) \cap \Omega)}{|B(x_i, r)|} \leq CM\mu(x_i).$$

After passing $\delta \to 0+$ we obtain a contradiction with (6). In case (ii) we can use Jensen’s inequality and (5) to obtain

$$\int_{B(x_i, r)} \Phi(|Df_j|) \geq \frac{|A|}{r^2} \int_A \Phi(|Df_j|) \geq \frac{|A|}{r^2} \Phi \left( \int_A |Df_j| \right) \geq \frac{|A|}{r^2} \Phi \left( C^2 \frac{\delta^2}{|A|} \right) \geq C \delta \Phi \left( \frac{C^2}{\delta} \right).$$

Similarly as above we obtain in the limit that

$$C \delta \Phi \left( \frac{C}{\delta} \right) \leq C M \mu(x_i)$$

and now passing to a limit $\delta \to 0+$ we obtain a contradiction using (5).

**Proof of (i).** We simplify the notation and write

$$\varphi_j(x) = |f_j(x) - f(x_\ast) - Df(x_\ast)(x - x_\ast)| \quad \text{and} \quad B_\ast = B(x_\ast, s).$$

In the following we use the notation $\mathcal{H}^{k}(A)$ for the $k$-dimensional Hausdorff measure of the set $A$. We claim that the set of radii

$$I_G = \{ s \in (0, r) : \mathcal{H}^{n-1}(\{ x \in \partial B_\ast : \varphi_j(x) \geq \delta r \}) < 5^n \delta^{n-1} \mathcal{H}^{n-1}(\partial B_\ast) \}$$

has measure at least $\frac{3r}{4}$, i.e. $|I_G| \geq \frac{3r}{4}$, otherwise

$$\int_{B_\ast} \frac{\varphi_j(x)}{r} \, dx \geq \frac{1}{|B_\ast|} \int_0^{\frac{r}{\delta}} 5^n \delta^{n-1} \mathcal{H}^{n-1}(\partial B_\ast) \frac{\delta r}{r} \, ds = 5^n \delta^n \frac{|B_\ast|}{|B_\ast|}$$

which contradicts (8).

On the other hand, the key point in our argument is that for $x_\ast \in \Omega$, and for every $s \in (0, r)$ we can find $\beta = \beta(s) \in \partial B_\ast$ such that

$$\varphi_j(\beta) \geq \frac{4}{5} s \quad \text{for every} \quad j = 1, 2, \ldots.$$ \hfill (9)

Finding such a point $\beta$ is the only place where we use the homeomorphism assumption of $f_j$. Suppose on the contrary that (9) fails for every $\beta \in \partial B_\ast$ and for some $j \in \{ 1, 2, \ldots \}$. For $x \in \partial B_\ast$ and $t \in [0, 1]$ we consider the following homotopy:

$$H(x, t) := (1 - t)(f_j(x) - f(x_\ast)) + t Df(x_\ast)(x - x_\ast).$$

By (7) we know that $Df(x_\ast)$ is an isometry and thus $|Df(x_\ast)| = |z|$. Furthermore, if (9) does not hold, then for all $x \in \partial B_\ast$ we have

$$|H(x, t)| \geq |Df(x_\ast)(x - x_\ast)| - (1 - t)|f_j(x) - f(x_\ast)| - Df(x_\ast)(x - x_\ast) \geq s - (1 - t) \frac{4}{5} s > 0.$$

It follows that $H(x, t) \neq 0$ for every $x \in \partial B_\ast$ and all $t \in [0, 1]$. Thus, by (3) and (1),

$$\deg(f_j, B_\ast, f(x_\ast)) = \text{sgn} \det(Df(x_\ast)) = -1.$$

This contradicts the fact that $f_j$ is sense-preserving.
We apply the Sobolev embedding theorem [8, Theorem 3 (i), p. 143] on the \((n - 1)\)-dimensional spheres. This way for almost every \(s \in (0, r)\) and for all \(z_1, z_2 \in \partial B(x, s)\) we have

\[
|f_j(z_1) - f_j(z_2)| \leq C(n, p)|z_1 - z_2|^{1 - \frac{n}{2p}}\left(\int_{\partial B} |Df_j|^p\right)^{\frac{1}{p}}. \tag{10}
\]

Now let us fix \(s \in I_G\) so that (10) is satisfied on the sphere \(\partial B_s\). Since \(s \in I_G\), we find \(a = a(s) \in \partial B_s\) satisfying

\[
\varphi_j(a) < \delta r \quad \text{and} \quad |a - \beta| \leq C_0 \delta s,
\]

where \(C_0\) is some fixed constant (which depends only on \(n\)). Combining this with (9) we have found \(a, \beta \in \partial B_s\) such that

\[
\frac{4}{5}s - \delta r - 2C_0 \delta s \leq |\varphi_j(\beta) - |\varphi_j(a)| - 2|a - \beta| \leq |f_j(\alpha) - f_j(\beta)|.
\]

This together with (10) implies that for \(s \in I_G \cap \left[\frac{7}{8}, r\right]\) and \(\delta\) small enough

\[
Cs^p \leq \left(\frac{4}{5}s - \delta r - 2C_0 \delta s\right)^p \leq C(n, p)(\delta s)^{p-n+1} \int_{\partial B_s} |Df_j|^p. \tag{11}
\]

Integrating inequality (11) over the set \(I_G \cap \left[\frac{7}{8}, r\right]\) we obtain (i), finishing the proof of Theorem 1 in the case \(p > n - 1\).

**Proof of (ii).** We proceed as above. For \(s \in I_G\) we can find \(\beta = \beta(s) \in \partial B_s\) so that (9) holds. In fact we consider the measurable set

\[
A := \{x \in B_r : \varphi_j(x) > \delta r\}.
\]

By Chebyshev’s inequality and (8) we obtain

\[
|A| \leq \frac{1}{\delta r} \int_{\partial B_s} |\varphi_j(x)| \, dx \leq \frac{1}{\delta r} \delta^2 r^2 r = C\delta |B_r|.
\]

Let \(s \in I_G \cap \left[\frac{7}{8}, r\right]\). The point \(\beta \in \partial B_s\) with (9) clearly belongs to \(A \cap \partial B_s\) and the closest point \(a\) on the relative boundary of \(\partial B_s \cap A\) satisfies

\[
|\varphi_j(a)| = \delta r
\]

by the definition of \(A\). It follows that for every \(s \in I_G \cap \left[\frac{7}{8}, r\right]\) we have

\[
s \leq C \int_{\partial B_s \cap A} |Df_j|.
\]

Integrating this over \(I_G \cap \left[\frac{7}{8}, r\right]\) we obtain

\[
r^2 \leq C \int_A |Df_j|
\]

finishing the proof of (ii).

The above proof was based on the Sobolev embedding theorem on spheres and therefore does not work for \(p < n - 1\). To overcome these difficulties we follow Hencl and Malý [11] and use the theory of linking numbers and its topological invariance. For the convenience of the reader we recall the needed properties of linking numbers here.

## 4 Linking number

We use the notation \(B_d\) for the unit ball in \(\mathbb{R}^d\) and \(S_{d-1}\) for the unit sphere. By \(\overline{B}_d(c, r)\) we denote the closed ball with center \(c\) and radius \(r > 0\).
Let \( n, t, q \) be positive integers with \( t + q = n - 1 \). Let us consider the mapping \( \Phi(\xi, \eta) : \mathbb{B}_{t+1} \times \mathbb{B}_{q+1} \to \mathbb{R}^n \) defined coordinatewise as \( \Phi(\xi, \eta) = x \), where
\[
\begin{align*}
x_1 &= (2 + \eta_1)\xi_1, \\
\vdots \\
x_{t+1} &= (2 + \eta_1)\xi_{t+1}, \\
x_{t+2} &= \eta_2, \\
\vdots \\
x_{t+q+1} &= \eta_{q+1}.
\end{align*}
\]
Denote by \( \mathcal{A} \) the anuloid
\[
\Phi(\mathcal{A} \times \mathbb{B}_{q+1}) = \left\{ x \in \mathbb{R}^n : \left( \sqrt{x_1^2 + \cdots + x_{t+1}^2} - 2 \right)^2 + x_{t+2}^2 + \cdots + x_n^2 < 1 \right\}.
\]
Of course, given \( x \in \mathcal{A} \) we can find a unique \( \xi \in \mathcal{S}_t \) and \( \eta \in \mathbb{B}_{q+1} \) such that \( \Phi(\xi, \eta) = x \). We will denote these as \( \xi(x) \) and \( \eta(x) \).

A link is a pair \((\varphi, \psi)\) of parametrized surfaces \( \varphi : \mathcal{S}_t \to \mathbb{R}^n, \psi : \mathcal{S}_q \to \mathbb{R}^n \). The linking number of the link \((\varphi, \psi)\) is defined as the topological degree
\[
\mathcal{L}(\varphi, \psi) = \deg(L, \mathcal{A}, 0),
\]
where the mapping \( L = L_{\varphi, \psi} : \mathcal{A} \to \mathbb{R}^n \) is defined as
\[
L(x) = \varphi(\xi(x)) - \psi(-\eta(x)),
\]
or equivalently
\[
L(\Phi(\xi, \eta)) = \varphi(\xi) - \psi(-\eta), \quad \xi \in \mathcal{S}_t, \quad \eta \in \mathbb{B}_{q+1},
\]
where \( \psi \) is an arbitrary continuous extension of \( \psi \) to \( \mathbb{B}_{q+1} \) (of course, the degree does not depend on the way how we extend \( \psi \), it depends only on the values on the boundary \( \partial \mathcal{A} = \Phi(\mathcal{S}_t \times \mathcal{S}_q) \)). Geometrically speaking, for \( t = q = 1 \), the linking number is the number of loops of a curve \( \varphi \) around a curve \( \psi \) counting orientation into account as \(+1\) or \(-1\). For the introductions to the linking number in \( \mathbb{R}^3 \) and its application to the theory of knots see [23].

The canonical link is the pair \((\mu, \nu)\), where
\[
\begin{align*}
\mu(\xi) &= \Phi(\xi, 0), \quad \xi \in \mathcal{S}_t, \\
\nu(\eta) &= \Phi(e_1, \eta), \quad \eta \in \mathcal{S}_q.
\end{align*}
\]
For example in dimension \( n = 3 \) we get that
\[
\begin{align*}
\mu(\mathcal{S}_t) &= \{ x \in \mathbb{R}^3 : x_3 = 0, \ x_1^2 + x_2^2 = 4 \}, \\
\nu(\mathcal{S}_q) &= \{ x_2 = 0, \ (x_1 - 2)^2 + x_2^2 = 1 \}.
\end{align*}
\]
It is well known that the linking number is a topological invariant. The simple proof of the following proposition can be found in [11].

**Proposition 2.** Let \( n, t, q \) be positive integers with \( t + q = n - 1 \). Let \( f : \mathbb{B}_n(4) \to \mathbb{R}^n \) be a homeomorphism. Then \( \mathcal{L}(f \circ \mu, f \circ \nu) \) is \( 1 \) if \( f \) is sense preserving and \(-1\) if \( f \) is sense reversing.

Analogously, we can pick \( a \in \mathbb{B}_{q+1}(0, 1/10) \) and \( b \in \mathbb{B}_{t+1}(e_1, 1/10) \cap \mathbb{B}_{t+1} \) and consider the pair
\[
\begin{align*}
\mu_a(\xi) &= \Phi(\xi, a), \quad \xi \in \mathcal{S}_t, \\
\nu_b(\eta) &= \Phi(b, \eta), \quad \eta \in \mathcal{S}_q.
\end{align*}
\]
Similarly to the previous proposition we have:

**Proposition 3.** Let \( n, t, q \) be positive integers with \( t + q = n - 1 \). Let \( f : \mathbb{B}_n(4) \to \mathbb{R}^n \) be a homeomorphism. Then \( \mathcal{L}(f \circ \mu_a, f \circ \nu_b) \) is \( 1 \) if \( f \) is sense preserving and \(-1\) if \( f \) is sense reversing.
5 Proof of Theorem 1 for $p > \left[ \frac{n}{2} \right]$, $n \geq 3$, and $p \geq 1$, $n = 3$

Our argument is similar to the proof given in Section 3 and therefore some details are only sketched. By $\mu$ we again denote the $w^*$ limit of (some subsequence) $\int |Df|^p$ for $p > \left[ \frac{n}{2} \right]$ and of $\int \Phi(|Df|)$ for $p = 1$ and $n = 3$.

By $C_1$ and $C_2$ we denote a fixed constants whose exact value will be determined later. We fix $\delta > 0$ and we choose a point $x_0$ such that (4) and (6) hold and without loss of generality we assume that the derivative of $f$ at $x_0$ is given by (7).

We fix $r_1 > 0$ such that for all $0 < r < r_1$ we have

$$\iint_{B(x_0, 4r)} \left| \frac{f(x) - f(x_0) - Df(x_0)(x - x_0)}{r} \right| \, dx < C_1 \delta^n$$

and again for all $j \geq j_*$ we obtain

$$\iint_{B(x_0, 4r)} \left| \frac{f_j(x) - f(x_0) - Df(x_0)(x - x_0)}{r} \right| \, dx < C_1 \delta^n. \quad (12)$$

We fix $t$, $q \leq \left[ \frac{n}{2} \right]$ such that $t + q = n - 1$ (e.g. $t = q = \frac{n-1}{2}$ for $n$ odd and $t = \frac{n-2}{2}$, $q = \frac{n}{2}$ for $n$ even). Our goal is to prove the following:

(i) if $p > \left[ \frac{n}{2} \right]$ and $n \geq 3$, then there exists a constant $C$ (depending only on $p$ and $n$) such that for all $0 < r < r_1$ and $j \geq j_*$,

$$\delta_{\text{min}}(t, q) \cdot r^n \leq C \iint_{B(x_0, 4r)} |Df|^p,$$

(ii) if $p = 1$ and $n = 3$, we have $A \subset B(x_0, 4r)$ such that

$$|A| < C_2 \delta |B(x_0, 4r)| \quad \text{and} \quad r^3 \leq C \iint_{A} |Df|.$$

Analogously to reasoning in Section 3 we obtain a contradiction using $\min(t, q) - p < 0$ for $p > \left[ \frac{n}{2} \right]$ and (5) for $p = 1$ and $n = 3$.

Proof of (i). Without loss of generality we will assume that $x_0 = 0$. We write

$$\varphi_j(x) = |f_j(rx) - f(0) - Df(0)rx|.$$

Let us fix $y \in \mu_\alpha(S_t)$ and denote

$$B_{\mu_\alpha(S_t)}(y, \delta) = \{ x \in \mu_\alpha(S_t) : |x - y| < \delta \},$$

the ball of radius $\delta$ on the link $\mu_\alpha(S_t)$. We can clearly choose a constant $C_1$ small enough at the beginning of the proof so that (12) implies that the set of good links

$$I_a = \left\{ a \in \overline{B}_{q+1}(0, \frac{1}{10}) : \mathcal{H}^{q+1}(x \in \mu_\alpha(S_t) : \varphi_j(x) \geq \delta r) < \mathcal{H}^{q+1}(B_{\mu_\alpha(S_t)}(y, \delta)) \right\},$$

$$I_b = \left\{ b \in \overline{B}_{q+1}(e_1, \frac{1}{10}) \cap \mathcal{H}^{q+1}(x \in \Phi_b(S_q) : \varphi_j(x) \geq \delta r) < \mathcal{H}^{q+1}(B_{\Phi_b(S_q)}(y, \delta)) \right\}$$

has measure at least

$$\mathcal{H}^{q+1}(I_a) > \frac{1}{2} \mathcal{H}^{q+1}(0, \frac{1}{10}) \quad \text{and} \quad \mathcal{H}^{q+1}(I_b) > \frac{1}{2} \mathcal{H}^{q+1}(e_1, \frac{1}{10}) \cap \mathcal{H}^{q+1}(e_1, \frac{1}{10}).$$

The key point of our argument is that for every $a \in \overline{B}_{q+1}(0, \frac{1}{10})$ and every $b \in \overline{B}_{q+1}(e_1, \frac{1}{10}) \cap \mathcal{H}^{q+1}(e_1, \frac{1}{10})$ we can find $\xi \in S_t$ and $\eta \in S_q$ such that

$$\varphi_j(\mu_\alpha(\xi)) = |f_j(r_\mu_\alpha(\xi)) - f(0) - Df(0)r_\mu_\alpha(\xi)| > \frac{r}{10}$$

or

$$\varphi_j(v_b(\eta)) = |f_j(r_\Phi(\eta)) - f(0) - Df(0)r_\Phi(\eta)| > \frac{r}{10}, \quad (13)$$
We prove the observation by contradiction and we suppose that (13) does not hold. We define
\[ f_s(x) = (1 - s)(f(0) + Df(0)rx) + sf_j(rx) \]
and we consider the homotopy \( H(\mathbb{A} \times [0, 1]) \to \mathbb{R}^n \) defined as
\[ H(\Phi(\xi, \eta), s) = (f_s \circ \mu_a)(\xi) - (f_s \circ \nu_b)(-\eta), \]
where \( (f_s \circ \nu_b) \) denotes a continuous extension of \( f_s \circ \nu_b \) to \( \mathbb{R}^{q+1} \) as in the definition of the linking number, which in addition depends continuously on \( s \). From [11] we know that the mapping \( f_j \in W^{1,p}, p > \left[ \frac{q}{2} \right] \), with nonnegative and nonzero Jacobian is sense preserving. By Proposition 3 we get that
\[ \text{deg}(H(x, 1), A, 0) = 1. \]
On the other hand
\[ \text{deg}(H(x, 0), A, 0) = -1 \]
since the affine mapping \( f(0) + Df(0)rx \) is sense reversing. To obtain a contradiction (with the preservation of the degree under homotopy) it is now enough to show that for every \( \xi \in S_t \), for every \( \eta \in S_q \) and for every \( s \in [0, 1] \) we have \( H(\Phi(\xi, \eta), s) \neq 0 \). It is easy to see that
\[ \text{dist}((f_0 \circ \mu_a)(S_t), (f_0 \circ \nu_b)(S_q)) \geq \text{dist}((f_0 \circ \mu)(S_t), (f_0 \circ \nu)(S_q)) - \frac{6r}{10} \geq \frac{3r}{10}. \]
Since (13) does not hold, we obtain from the definition of \( f_s \) that
\[ \text{dist}((f_s \circ \mu_a)(S_t), (f_s \circ \nu_b)(S_q)) \geq \frac{3r}{10} - \frac{r}{10} - \frac{r}{10} \]
which implies \( H(\Phi(\xi, \eta), s) \neq 0 \).

By (13) and the symmetry we may assume without loss of generality that
\[ \tilde{I}_a = \left\{ a \in I_a : \text{there exists } \xi \in S_t \text{ such that } \varphi_j(\mu_a(\xi)) > \frac{r}{10} \right\} \]
satisfies \( \tilde{I}_a \supset \tilde{I}_a \supset \tilde{I}_a \) and for every \( a \in \tilde{I}_a \), we have for almost every \( a \in \tilde{I}_a \) and for every \( z_1, z_2 \in \mu_a(S_t) \),
\[ |f_j(z_1) - f_j(z_2)| \leq C|z_1 - z_2|^{1 + \frac{1}{p}} \left( \int_{\mu_a(S_t)} |Df_j|^p \right)^{\frac{1}{p}}. \] (14)

Now let us fix \( a \in \tilde{I}_a \) so that (14) is satisfied and find \( \xi \in S_t \) so that for \( \beta = \mu_a(\xi) \) we have \( \varphi_j(\beta) > \frac{r}{10} \) as in the definition of \( \tilde{I}_a \). Using \( a \in I_a \) we find \( a \in \mu_a(S_t) \) satisfying
\[ \varphi_j(a) < \delta \text{ and } |a - \beta| < \delta. \]
Thus we have found \( a, \beta \in \mu_a(S_t) \) such that
\[ \frac{r}{10} - 3\delta r \leq |\varphi_j(\beta) - |\varphi_j(a)| - 2r|a - \beta| \leq |f_j(ra) - f_j(rb)|. \]
This together with (14) implies that for almost every \( a \in \tilde{I}_a \) and \( \delta \) small enough we have
\[ r^t \leq C\delta^{p-t} \int_{\mu_a(S_t)} |Df_j|^p. \] (15)

Integrating inequality (15) over the set \( \tilde{I}_a \) we obtain (i).

**Proof of (ii).** If \( p = 1 \) and \( n = 3 \), then for each \( a \in \tilde{I}_a \) we can find \( \xi \in S(t) \) so that (13) holds for \( \beta = \mu_a(\xi) \). The measurable set
\[ A := \{ x \in B(x, 4r) : \varphi_j(x) > \delta r \} \]
satisfies \( |A| \leq C\delta B(x, 4r) \) by inequality (12) and Chebyshev’s inequality. Now clearly \( \xi \) with (13) satisfies \( r\beta = r\mu_a(\xi) \in A \). In (14) and (15) instead of integrating over the entire \( r\mu_a(S_t) \) we integrate only over the set \( r\mu_a(S_t) \cap A \). Integrating over \( \tilde{I}_a \) we obtain the desired conclusion.
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