Convex analysis and dual problems

Salla Kupiainen

Matematiikan pro gradu

Jyväskylän yliopisto
Matematiikan ja tilastotieteen laitos
Kevät 2018

Tiivistelmä: Salla Kupiainen, *Convex analysis and dual problems*, matematiikan pro gradu -tutkielma, 54. s., Jyväskylän yliopisto, Matematiikan ja tilastotieteen laitos, kevät 2018.

Tässä tutkielmassa tarkastellaan valittujen variaatiolaskennan ongelmien ja näiden duaaliongelmien välisiä suhteita. Tutkielmassa esitetään aiheen yleinen teoria ja annetaan esimerkkejä sovelluksista.

Tutkielman ensimmäisessä osassa määritellään konveksin analyysin keskeiset käsitteet konveksi joukko, konveksi funktio ja konjugaattifunktio, sekä tarkastellaan konveksin funktion jatkuvuutta ja subdifferentioituvuutta reaaliarvoisessa normiavaruudessa.

Tutkielman toisessa osassa määritellään heikon konvergenssin käsite ja L^p-avaruudet. Avaruuden $L^p(\Omega)$ refleksiivisyys todistetaan tapauksessa $1 < p < \infty$. Toisen osan päätteeksi todistetaan, että refleksiivisen Banach-avaruuden rajoitetulla jonolla on heikosti suppeneva osajono.

Kolmannessa osassa määritellään primaali- ja duaaliongelma ja tarkastellaan näiden välisiä suhteita. Tutkielmassa keskitytään primaaliongelmiin, joiden objektifunktio on refleksiivisessä Banach-avaruudessa määritelty reaaliarvoinen, konveksi ja alhaalta puolijatkuva funktio. Tutkielmassa osoitetaan, että primaaliongelmalla on ratkaisu tapauksissa, joissa funktion lähtöjoukko on rajoitettu tai funktio on koersiivinen. Ratkaisu on yksikäsitteinen, mikäli objektifunktio on aidosti konveksi. Duaaliongelman ratkaisun olemassaolo näytetään tapauksessa, jossa primaaliongelma on stabiili ja sillä on vähintään yksi tunnettu ratkaisu. Edellisessä tilanteessa primaali- ja duaaliongelman ääriarvot ovat samat. Lopuksi osoitetaan, että mikäli primaali- ja duaaliongelmalla on ratkaisu ja ongelmien ääriarvot ovat samat, linkittyvät ongelmien ratkaisupisteet toisiinsa erityisellä suhteella.

Neljännessä osassa määritellään Sobolev-avaruudet ja osoitetaan, että $W^{k,p}(\Omega)$, missä $1 < p < \infty$ ja $k \in \mathbb{N}$, on refleksiivinen Banach-avaruus. Todistuksissa hyödynnetään tutkielman toisessa osassa saatuja tuloksia.

Tutkielman viimeisessä osassa tarkastellaan kolmea variaatiolaskennan ongelmaa: epälineaarinen Dirichlet'n ongelma, Stokesin ongelma ja Mossolovin ongelma. Jokaisen ongelman osalta muodostetaan primaaliongelma, primaaliongelmalle konstruoidaan duaaliongelma, osoitetaan primaali- ja duaaliongelmien ratkaisujen olemassaolo ja ääriarvojen yhtäsuuruus sekä näytetään millaisen muodon ratkaisupisteiden välinen suhde lopulta saa. Lisäksi osoitetaan, että epälineaarisen Dirichletin ongelman primaaliongelman ääriarvopiste on alkuperäisen ongelman heikko ratkaisu. Stokesin ongelman tapauksessa näytetään, että primaaliongelman ja duaaliongelman ratkaisuista muodostettu pari on alkuperäisen ongelman ratkaisu.

Contents

1. Introduction

In this thesis, we study relations of given variational problems and their corresponding dual problem. The main purpose is to present the general theory and to give examples.

In section 2 we introduce some basic concepts of convex analysis. The first part of the section deals with convex sets. In the second part of the section we introduce convex functions. A function $F : V \to \bar{\mathbb{R}}$, where V is a real vector space, is convex if for every $u, v \in V$, we have

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v) \quad \text{for all } \lambda \in [0,1].$$

We define the conjugate function of F, $F^* : V^* \to \bar{\mathbb{R}}$, as follows

$$F^*(u^*) = \sup_{u \in V} \{ \langle u, u^* \rangle - F(u) \}.$$

In the beginning of section 3 we give the definition of weak convergence. Later in the section we introduce L^p spaces and in Theorem 3.21 we show that $L^p(\Omega)$ is reflexive when $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$. In Theorem 3.22 we show that a bounded sequence of a reflexive Banach space has a weakly converging subsequence. The proof is given in the case of $L^p(\Omega)$, $1 < p < \infty$.

In section 4 we consider the primal problem

$$(\mathscr{P}) \qquad \inf_{u \in A} F(u),$$

where $F : A \to \mathbb{R}$ is a proper convex lower semi-continuous function and A a non-empty closed convex subset of reflexive Banach space V. In Theorem 4.7 we show that if A is bounded or F is coercive over A, then $\mathscr{P}$ has at least one solution. Moreover, the solution is unique if F is strictly convex in A. Given a normed space Y and a function $\Phi : V \times Y \to \bar{\mathbb{R}}$ such that $\Phi(u, 0) = F(u)$, we have for every $p \in Y$ a perturbed problem

$$(\mathscr{P}_p) \quad \inf_{u \in V} \Phi(u, p).$$

Let $\Phi^* : V^* \times Y^* \to \bar{\mathbb{R}}$ be the conjugate function of Φ. Finally, we have the dual problem of $\mathscr{P}$

$$(\mathscr{P}^*) \quad \sup_{p^* \in Y^*} \{-\Phi^*(0, p^*)\}.$$

In Theorem 4.9 we show that if $\mathscr{P}$ is stable and has a solution, then $\mathscr{P}^*$ has at least one solution and

$$-\infty < \inf \mathscr{P} = \sup \mathscr{P}^* < \infty.$$

In Theorem 4.12 we show that $\mathscr{P}$ is stable if Φ is convex, $\inf \mathscr{P}$ is finite and there exists $u_0 \in V$ such that $p \mapsto \Phi(u_0, p)$ is finite and continuous at $0 \in Y$. In Theorem 4.13 we show that the solutions of $\mathscr{P}$ and $\mathscr{P}^*$ are linked by the extremality relation

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = 0.$$

In section 5 we introduce Sobolev spaces and show that $W^{k,p}(\Omega)$, where $1 < p < \infty$, $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, is a reflexive Banach space. In section 6 we give examples of applications of duality. Our first example is the non-linear Dirichlet Problem

$$(1) \qquad \begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f, & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \infty$, $f \in L^q(\Omega)$ and $q = p/(p-1)$. In Lemma 6.2 we show that $u \in W_0^{1,p}(\Omega)$ is a weak solution to equation (1) if it is a minimizer of the functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p \, dx - \int_\Omega f(x)u(x) \, dx.$$

In Theorem 6.3 we show that the problem

$$(\mathscr{P}) \qquad \inf_{u \in W_0^{1,p}(\Omega)} I(u)$$

has a unique solution. The dual problem of $\mathscr{P}$ is of the form

$$(\mathscr{P}^*) \qquad \sup_{\substack{r^* \in L^q(\Omega)^n \\ \text{div } r^* = f}} \left[-\frac{1}{q}\int_\Omega |r^*(x)|^q \, dx \right].$$

In Theorem 6.6 we show that $\mathscr{P}^*$ possess a unique solution and

$$\max \mathscr{P}^* = \min \mathscr{P}.$$

Our second example is the Stokes problem: Given $f \in L^2(\Omega)^n$, we consider the following system

$$(2) \qquad \begin{cases} -\Delta u + \nabla p = f, & \text{in } \Omega; \\ \text{div } u = 0, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $u = (u_1, ..., u_n) : \Omega \to \mathbb{R}^n$ and $p : \Omega \to \mathbb{R}$. Let $W = \{v \in H_0^1(\Omega)^n, \ \text{div } v = 0\}$. Then W is a Hilbert space with the inner product

$$((u, v)) = \sum_{1 \le i,j \le n} (D_i u_j, D_i v_j) = \sum_{1 \le i,j \le n} \int_\Omega D_i u_j(x) D_i v_j(x) \, dx.$$

In Lemma 6.7 we show that $u \in W$ is a weak solution of equation (2), if it is a minimizer of the functional

$$I(u) = \frac{1}{2}||u||^2_{H^1(\Omega)^n} - (f, u) = \frac{1}{2}\left[\sum_{i=1}^n \sum_{j=1}^n ||D_i u_j||_2^2\right] - \int_\Omega f(x)u(x) \, dx.$$

In Theorem 6.8 we show that the primal problem

$$(\mathscr{P}) \qquad \inf_{u \in W} I(u)$$

possesses a unique solution. The dual problem of $\mathscr{P}$ is

$$(\mathscr{P}^*) \qquad \sup_{p^* \in L^2(\Omega)} \left\{ -\frac{1}{2}||v(p^*)||^2_{H^1(\Omega)^n} \right\},$$

where $v(p^*) \in H_0^1(\Omega)^n$ satisfies

$$((v(p^*), w)) = (f, w) + (p^*, \text{div } w), \qquad \text{for all } w \in H_0^1(\Omega)^n.$$

In Theorem 6.10 we show that if $\mathscr{P}^*$ is proper, then it has a solution. In Theorem 6.12 we show that problem (2) possesses a solution $(\bar{u}, \bar{p}^*)$, where $\bar{u}$ is a solution of the primal problem $\mathscr{P}$ and $\bar{p}^*$ is a solution of the dual problem $\mathscr{P}^*$. Moreover

$$\inf \mathscr{P} = \sup \mathscr{P}^*.$$

Our last example is Mossolov's problem

$$(\mathscr{P}) \qquad \inf_{u\in H_0^1(\Omega)} \left\{ \frac{\alpha}{2}\int_\Omega |\nabla u(x)|^2\, dx + \beta \int_\Omega |\nabla u(x)|\, dx - \int_\Omega f(x)u(x)\, dx \right\},$$

where α and β are positive constants and $f \in L^2(\Omega)$ is given. In Theorem 6.13 we show that $\mathscr{P}$ has a unique solution. The dual problem of $\mathscr{P}$ is

$$(\mathscr{P}^*) \qquad \sup_{\substack{p^*\in L^2(\Omega)^n \\ |p^*(x)|\le\beta \text{ a.e.}}} \left\{ -\frac{1}{2\alpha}||f - \operatorname{div} p^*||^2_{H^{-1}(\Omega)} \right\}.$$

In Theorem 6.16 we show that $\mathscr{P}^*$ has at least one solution and

$$\min \mathscr{P} = \max \mathscr{P}^*.$$

2. Convex Analysis

In this section we introduce some basic concepts of convex analysis.

2.1. Convex sets.

Let V be a real vector space.

Definition 2.1. A set $A \subset V$ is said to be *convex* if for every two points u and v in A the line segment $[u, v]$ is contained in A, that is,

$$[u, v] = \{\lambda u + (1-\lambda)v : \lambda \in [0, 1]\} \subset A, \quad \text{for all } u, v \in A.$$

The whole space V and the empty set $\emptyset$ are convex.

Definition 2.2. Let $A \subset V$ and $u_1, ..., u_n \in A$. The sum

$$\lambda_1 u_1 + \cdots + \lambda_n u_n, \quad \text{where } \lambda_1, ..., \lambda_n \ge 0 \text{ and } \sum_{i=1}^n \lambda_i = 1,$$

is said to be a *convex combination* of $u_1, ..., u_n$.

Proposition 2.3. *Let $A \subset V$ be a convex set. Then A contains all of the convex combinations of its elements.*

Proof. If $u_1, u_2 \in A$ and $\lambda_1, \lambda_2 \ge 0$ such that $\lambda_1 + \lambda_2 = 1$, then by the definition of convexity

$$\lambda_1 u_1 + \lambda_2 u_2 \in A.$$

Let $m \in \mathbb{N}$, $m > 2$. We make an induction hypothesis that all the convex combinations of less than m elements of A are contained in A. Let

$$u = \lambda_1 u_1 + \cdots + \lambda_m u_m, \quad \text{such that } \lambda_1, ..., \lambda_m \ge 0 \text{ and } \sum_{i=1}^m \lambda_i = 1,$$

be a convex combination of $u_1, ..., u_m \in A$. Suppose that $\lambda_1 = 1$. Then we have $\lambda_2 = \cdots = \lambda_m = 0$ and $u = u_1 \in A$. Thus we may assume that $0 < \lambda_1 < 1$. Let

$$v = \lambda_2' u_2 + \cdots + \lambda_m' u_m, \quad \text{where } \lambda_i' = \frac{\lambda_i}{1-\lambda_1}.$$

Then $\lambda_i' \ge 0$ for $i = 2, ..., m$ and

$$\lambda_2' + \cdots + \lambda_m' = 1.$$

Thus v is a convex combination of $m-1$ elements of A. Hence $v \in A$. Since $u = (1-\lambda_1)v + \lambda_1 u_1$, it follows that $u \in A$. □

Proposition 2.4. *The intersection of an arbitrary collection of convex sets is convex.*

Proof. Let $\{A_\alpha\}$, $\alpha \in I$, where I is the index set, be an arbitrary collection of convex sets. Let u and v be two elements of the intersection

$$A = \bigcap_{\alpha \in I} A_\alpha.$$

For all $\alpha \in I$ the line segment $[u, v]$ belongs to A_α and therefore it belongs to the intersection A. $\square$

Proposition 2.5. *Let V and W be two real vector spaces, $A \subset V$ a convex set and L a linear mapping from V into W. Then $L(A)$ is convex in W.*

Proof. Fix $\lambda \in [0, 1]$ and let x and y be two elements of $L(A)$. There exists u and v in A such that $x = Lu$ and $y = Lv$. Then by the linearity of L

$$\begin{aligned} \lambda x + (1-\lambda)y &= \lambda L(u) + (1-\lambda)L(v) = L(\lambda u) + L((1-\lambda)v) \\ &= L(\lambda u + (1-\lambda)v). \end{aligned}$$

Since A is convex, $\lambda u + (1-\lambda)v$ belongs to A for all $\lambda \in [0, 1]$. Therefore $L(\lambda u + (1-\lambda)v)$ is an element of $L(A)$. This implies that $L(A)$ is convex. $\square$

2.2. **Convex functions.**

Definition 2.6. *The epigraph* of a function $F : V \to \bar{\mathbb{R}}$ is the set

$$\text{epi } F = \{(u, a) \in V \times \mathbb{R} : F(u) \leq a\}.$$

The epigraph is the set of points of $V \times \mathbb{R}$ which lie above the graph of F. The projection of epi F to V is the set

$$\text{dom } F := \{u \in V : F(u) < +\infty\}.$$

We say that it is *the effective domain* of F.

Definition 2.7. Let A be a convex subset of V and $F : A \to \bar{\mathbb{R}}$ a function. F is said to be *convex* if for every u and v in A

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v) \quad \text{for all } \lambda \in [0, 1], \tag{3}$$

whenever the right-hand side is defined. Inequality (3) must therefore be valid unless $F(u) = -F(v) = \pm\infty$. F is said to be *strictly convex* if it is convex and for every $u, v \in A$, $u \neq v$

$$F(\lambda u + (1-\lambda)v) < \lambda F(u) + (1-\lambda)F(v) \quad \text{for all } \lambda \in (0, 1),$$

whenever the right-hand side is defined.

Let $A \subset V$ and $F : A \to \mathbb{R}$ be a function. We can associate with F the function $\tilde{F}$ on V by setting

$$\tilde{F}(u) = \begin{cases} F(u) & \text{if } u \in A; \\ +\infty & \text{if } u \notin A. \end{cases}$$

$\tilde{F}$ is convex if and only if A is a convex set and F is a convex function. This way we only need to concern functions defined on the whole space V.

Definition 2.8. We say that a convex function $F : V \to \bar{R}$ is *proper*, if there is $u \in V$ such that $F(u)$ is finite and F nowhere takes the value $-\infty$.

Definition 2.9. A function $F : V \to \bar{\mathbb{R}}$ is said to be *concave* if $-F$ is convex.

Proposition 2.10. *Let $F : V \to \bar{R}$ be a convex function. Then the sublevel sets*

$$E_a := \{u : F(u) \leq a\}$$

are convex for all $a \in \bar{\mathbb{R}}$.

Proof. Fix $a \in \bar{\mathbb{R}}$ and $\lambda \in [0, 1]$. Let $u, v \in E_a$. If $F(u) = -F(v) = \pm\infty$, then

$$F(\lambda u + (1 - \lambda)v) \leq \infty,$$

meaning $\lambda u + (1 - \lambda)v \in E_a$. Else, we have

$$\begin{aligned} F(\lambda u + (1 - \lambda)v) &\leq \lambda F(u) + (1 - \lambda)F(v) \\ &\leq \lambda a + (1 - \lambda)a \\ &= a, \end{aligned}$$

which shows that $\lambda u + (1 - \lambda)v \in E_a$. Thus E_a is convex. □

Definition 2.11. A function $F : \mathbb{R} \to \mathbb{R}$ is *non-decreasing* if $F(a) \leq F(b)$ for every $a, b \in \mathbb{R}$, $a < b$. F is *increasing* if $F(a) < F(b)$ for every $a, b \in \mathbb{R}$, $a < b$.

Proposition 2.12. *Let $F : V \to \mathbb{R}$ be a convex function and $G : \mathbb{R} \to \mathbb{R}$ a non-decreasing convex function. Then the composition $G \circ F : V \to \mathbb{R}$ is a convex function. If F is strictly convex and G is an increasing convex function, then the composition is strictly convex.*

Proof. Let $u, v \in V$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} G(F(\lambda u + (1 - \lambda)v)) &\leq G(\lambda F(u) + (1 - \lambda)F(v)) \\ &\leq \lambda G(F(u)) + (1 - \lambda)G(F(v)) \end{aligned}$$

and hence $G \circ F$ is convex. If F is strictly convex and G is increasing, then for $u \neq v$ we have that

$$F(\lambda u + (1 - \lambda)v) < \lambda F(u) + (1 - \lambda)F(v)$$

and that

$$G(F(\lambda u + (1 - \lambda)v)) < G(\lambda F(u) + (1 - \lambda)F(v)) \quad \text{for all } \lambda \in (0, 1).$$

Hence the composition $G \circ F$ is a strictly convex function. □

Proposition 2.13. *Let $A \subset \mathbb{R}$ be an open interval and $F : A \to \mathbb{R}$ a twice continuously differentiable function. If F'' is non-negative in A, then F is convex. If F'' is positive in A, then F is strictly convex.*

Proof. Since F'' is non-negative, F' is non-decreasing on A. For $x, y \in A$, $x < y$ and $\lambda \in [0, 1]$, we denote $z = \lambda x + (1 - \lambda)y$. By the Fundamental Theorem of Calculus, we have that

$$F(z) - F(x) = \int_x^z F'(t)\ dt \leq F'(z)(z - x),$$

and that

$$F(y) - F(z) = \int_z^y F'(t)\ dt \geq F'(z)(y - z).$$

Thus

$$F(z) \leq (1 - \lambda)F'(z)(y - x) + F(x)$$

and

$$F(z) \leq -\lambda F'(z)(y-x) + F(y).$$

Multiplying both sides of the first inequality by λ, those of the second one by $(1-\lambda)$ and combining the two resulting inequalities together, we have

$$F(z) = \lambda F(z) + (1-\lambda)F(z) \leq \lambda F(x) + (1-\lambda)F(y),$$

which gives us that F is convex. If F'' is positive, then F' is increasing on A. Let $x, y \in A$, $x < y$, $\lambda \in (0,1)$ and $z = \lambda x + (1-\lambda)y$. We then have

$$F(z) - F(x) = \int_x^z F'(t)\ dt < F'(z)(z-x),$$
$$F(y) - F(z) = \int_z^y F'(t)\ dt > F'(z)(y-z).$$

By repeating the same argument as in the proof of convexity, we obtain that

$$F(z) < \lambda F(x) + (1-\lambda)F(y),$$

which shows that F is strictly convex. □

From now on , we assume that V is a real normed vector space with a norm $||\cdot||_V$. We say that a sequence (u_j) in V converges to $u \in V$, that is,

$$u_j \to u \quad \text{in } V \quad \text{if} \quad ||u_j - u||_V \to 0 \quad \text{as} \quad j \to \infty.$$

Definition 2.14. A function $F : V \to \bar{\mathbb{R}}$ is said to be *lower semi-continuous* on V if for all $u \in V$ and all sequences (u_i) in V converging to u, we have

$$\underline{\lim}_{u_i \to u} F(u_i) \geq F(u). \tag{4}$$

A continuous function is lower semi-continuous.

Proposition 2.15. *Function $F : V \to \bar{R}$ is lower semi-continuous if and only if the sublevel set*

$$E_a := \{u \in V : F(u) \leq a\} \tag{5}$$

is closed for all $a \in \mathbb{R}$.

Proof. Suppose F is lower semi-continuous. Fix $a \in \mathbb{R}$. Let (u_j) be a sequence in E_a converging to $u \in V$. Then

$$F(u) \leq \underline{\lim}_{j \to \infty} F(u_j) \leq a,$$

from which follow that $u \in E_a$ and that E_a is closed.

Suppose that E_a is closed for all $a \in \mathbb{R}$. Fix $u \in V$ and let (u_j) be a sequence in V converging to u. Now we have two cases: $F(u) = \infty$ and $F(u) < \infty$. For the first case $F(u) = \infty$, we claim that for every $a \in \mathbb{R}$, there exists $N_a \in \mathbb{N}$ such that

$$F(u_j) > a \quad \text{for all } j \geq N_a.$$

We argue by contradiction. Suppose that the claim does not hold. Then there is a subsequence (u_{j_k}) of (u_j) and there is $b \in \mathbb{R}$ such that

$$F(u_{j_k}) \leq b, \quad \text{for all } j_k \in \mathbb{N}.$$

This means that $u_{j_k} \in E_b$ for all $j_k \in \mathbb{N}$. Since E_b is closed and the subsequence (u_{j_k}) converges to u in V, we have $u \in E_b$. This means that $F(u) \leq b$. This contradicts to our assumption that $F(u) = \infty$. Thus in this case F is lower semi-continuous at u.

For the second case $F(u) < \infty$. We argue by contradiction. We assume that

$$m := \varliminf_{j\to\infty} F(u_j) < F(u) = M.$$

Now for every $\varepsilon > 0$, there is a subsequence $(u_{j_k}) \subset (u_j)$ such that

$$F(u_{j_k}) < m + \varepsilon.$$

This means that $u_{j_k} \in E_{m+\varepsilon}$ for all $k \in \mathbb{N}$. Let $\varepsilon = (M - m)/2$. Now $u_{j_k} \to u$, as $j_k \to \infty$. Since E_a is closed for all $a \in \mathbb{R}$, we have $u \in E_{m+(M-m)/2}$ and

$$F(u) \le m + \frac{M-m}{2} < M.$$

This is a contradiction. Hence we have

$$\varliminf_{j\to\infty} F(u_j) \ge M.$$

This implies that F is lower semi-continuous at u. □

Proposition 2.16. *A function $F : V \to \bar{\mathbb{R}}$ is convex if and only if its epigraph epi F is convex.*

Proof. Suppose F is convex. Let $(u, a), (v, b) \in \text{epi } F$. Then $F(u) \le a < \infty$ and $F(v) \le b < \infty$. By the convexity of F, for all $\lambda \in [0, 1]$, we have

$$F(\lambda u + (1-\lambda)v) \le \lambda F(u) + (1-\lambda)F(v) \le \lambda a + (1-\lambda)b \tag{6}$$

from which follows that

$$\big(\lambda u + (1-\lambda)v, \lambda a + (1-\lambda)b\big) = \lambda(u, a) + (1-\lambda)(v, b) \in \text{epi } F.$$

Therefore epi F is convex.

Assume then that epi F is convex. The projection of epi F to V is dom F, which is convex since epi F is convex and convexity is preserved by linear mappings. We first show that F is convex in dom F. Indeed, for $u, v \in \text{dom } F$, we have $\big(u, F(u)\big) \in \text{epi } F$ and $\big(v, F(v)\big) \in \text{epi } F$. Since epi F is convex, then for any $\lambda \in [0, 1]$, we have

$$\lambda\big(u, F(u)\big) + (1-\lambda)\big(v, F(v)\big) = \big(\lambda u + (1-\lambda)v, \lambda F(u) + (1-\lambda)F(v)\big) \in \text{epi } F,$$

which means that

$$F\big(\lambda u + (1-\lambda)v\big) \le \lambda F(u) + (1-\lambda)F(v).$$

This shows that F is convex in dom F. Since dom F is convex and $F = +\infty$ in $V \setminus \text{dom } F$, we know that F is convex in V. This finishes the proof. □

Proposition 2.17. *A function $F : V \to \bar{\mathbb{R}}$ is lower semi-continuous if and only if its epigraph epi F is closed.*

Proof. Define a function $\varphi : V \times \mathbb{R} \to \bar{\mathbb{R}}$ by setting $\varphi(u, a) = F(u) - a$. We claim that function F is lower semi-continuous on V if and only if φ is lower semi-continuous on $V \times \mathbb{R}$. Indeed, let $\big((u_j, a_j)\big)$ be a sequence in $V \times \mathbb{R}$ converging to $(u, a) \in V \times \mathbb{R}$. This means that

$$||u_j - u||_V \to 0 \quad \text{and} \quad |a_j - a| \to 0 \quad \text{as } j \to \infty.$$

Assume first that F is lower semi-continuous. Then

$$\begin{aligned}\varphi(u,a) = F(u) - a &\le \varliminf_{j\to\infty} F(u_j) - a \\ &= \varliminf_{j\to\infty} \Big(F(u_j) - a_j\Big) = \varliminf_{j\to\infty} \varphi(u_j, a_j),\end{aligned}$$

which shows that φ is lower semi-continuous. Conversely, we assume that φ is lower semi-continuous. By our assumption

$$F(u) - a = \varphi(u,a) \le \varliminf_{j\to\infty} \varphi(u_j, a_j) = \varliminf_{j\to\infty} \Big(F(u_j) - a_j\Big).$$

Therefore

$$F(u) \le \varliminf_{j\to\infty} \Big(F(u_j) - a_j\Big) + a = \varliminf_{j\to\infty} F(u_j),$$

which shows that F is lower semi-continuous. This proves the claim.

We notice that

$$\varphi^{-1}((-\infty, 0]) = \{(u,a) \in V \times \mathbb{R} : \varphi(u,a) \le 0\} = \text{epi } F$$

and that

$$\varphi^{-1}((-\infty, r]) \text{ is the translation of epi } F \text{ by vector } (0,r) \in V \times \mathbb{R}.$$

Since the translate of a closed set is closed, the sublevel sets $\varphi^{-1}((-\infty, r])$ of φ are closed if and only if $\varphi^{-1}((-\infty, 0]) =$ epi F is closed. Recall that Proposition 2.15 says that $\varphi : V \times \mathbb{R} \to \bar{\mathbb{R}}$ is lower semi-continuous if and only if the sublevel set $\varphi^{-1}((-\infty, r])$ is closed for all $r \in \mathbb{R}$. Thus $F : V \to \bar{\mathbb{R}}$ is lower semi-continuous if and only if epi F is closed. □

Proposition 2.18.

i) Let $\{F_i\}$, $i \in I$, be any family of convex functions in V. Let $F(x) = \sup_{i\in I} F_i(x)$. Then F is convex.

ii) Let $\{F_i\}$, $i \in I$, be any family of lower semi-continuous functions in V. Let $F(x) = \sup_{i\in I} F_i(x)$. Then F is lower semi-continuous.

Proof. i) Proposition 2.16 states that a function is convex if and only if its epigraph is convex. Therefore for every $i \in I$, epi F_i is convex. By Proposition 2.4 we have that the intersection of an arbitrary collection of convex sets is convex. We notice that

$$\text{epi } F = \bigcap_{i\in I} \text{epi } F_i.$$

Hence epi F is convex. Then F is convex, by Proposition 2.16. ii) Proposition 2.17 states that a function is lower semi-continuous if and only if its epigraph is closed. For every $i \in I$, epi F_i is closed. Since the intersection of an arbitrary collection of closed sets is closed, epi F is closed. Therefore, by Proposition 2.17, F is lower semi-continuous. □

2.3. **Continuity of convex functions.**

Lemma 2.19. *Let $F : V \to \bar{\mathbb{R}}$ be a convex function. If there exists a neighborhood W of $u \in V$ such that*

$$F(v) \le M < \infty \quad \textit{for all } v \in W,$$

then F is continuous at u.

Proof. By translation we may assume that $u = 0$ and $F(u) = 0$. Since W is a neighborhood of 0, there exists a real number $r > 0$ such that $B(0, r) \subset W$. Let $\varepsilon \in (0, 1)$. If $v \in B(0, \varepsilon r)$, we have

$$F(v) = F\Big((1-\varepsilon)0 + \varepsilon\frac{v}{\varepsilon}\Big) \leq (1-\varepsilon)F(0) + \varepsilon F(v/\varepsilon) \leq \varepsilon M.$$

Writing

$$0 = \frac{\varepsilon}{1+\varepsilon}\frac{-v}{\varepsilon} + \frac{1}{1+\varepsilon}v,$$

we also have

$$0 = F(0) \leq \frac{\varepsilon}{1+\varepsilon}F(-v/\varepsilon) + \frac{1}{1+\varepsilon}F(v),$$

from which follows that

$$F(v) \geq -\varepsilon F(-v/\varepsilon) \geq -\varepsilon M.$$

Thus $|F(v)| \leq \varepsilon M$ for every $v \in B(0, \varepsilon r)$. Therefore F is continuous at 0. □

Proposition 2.20. *Let $F : V \to \bar{\mathbb{R}}$ be a convex function. The following statements are equivalent to each other:*

(i) There exists a non-empty open set U on which F is not everywhere equal to $-\infty$ and

$$F(u) < a < \infty \qquad \text{for all } u \in U.$$

(ii) F is proper and continuous in the interior of its effective domain.

Proof. Suppose (ii) is true. Since F is proper, $\operatorname{int}(\operatorname{dom} F) \neq \emptyset$ and F nowhere takes the value $-\infty$. Let $u \in \operatorname{int}(\operatorname{dom} F)$. Since F is continuous in $\operatorname{int}(\operatorname{dom} F)$, there exists a neighborhood U of u and $M < \infty$ such that $F(v) < M$ for all $v \in U$. Thus (ii) implies (i).

Suppose then that (i) is true. Then $U \subset \operatorname{int}(\operatorname{dom} F)$. By assumption, there exists $u \in U$ such that $F(u) > -\infty$. From Lemma 2.19, we have that F is continuous at u and hence bounded in a neighborhood of u.

We claim that $F(v) > -\infty$ for all $v \in \operatorname{int}(\operatorname{dom} F)$. Indeed, suppose that there is $v \in \operatorname{int}(\operatorname{dom} F)$ such that $F(v) = -\infty$. Then by the convexity of F

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v) = -\infty \qquad \text{for all } \lambda \in (0, 1)$$

and in particular, $F(w) = -\infty$ for all w in the open line segment (u, v). This contradicts the fact that u has a neighborhood in which F is finite. Hence F is proper.

Fix $u_0 \in \operatorname{int}(\operatorname{dom} F)$. Then there is $\rho > 1$ such that $u_1 = u + \rho(u_0 - u) \in \operatorname{int}(\operatorname{dom} F)$. Now define $h : V \to V$ by setting

$$h(v) = \frac{\rho-1}{\rho}v + \frac{1}{\rho}u_1, \quad v \in V.$$

Then we have $h(u) = u_0$ and $h(U)$ is open set. It is easy to see that h is invertible. For $w \in h(U)$

$$w = \frac{\rho-1}{\rho}h^{-1}(w) + \frac{1}{\rho}u_1$$

and hence by the convexity of F

$$F(w) \leq \frac{\rho-1}{\rho}F\big(h^{-1}(w)\big) + \frac{1}{\rho}F(u_1) \leq \frac{\rho-1}{\rho}a + \frac{1}{\rho}F(u_1) < \infty.$$

Therefore F is bounded from above in the neighborhood $h(U)$ of u_0. By Lemma 2.19, F is continuous at u_0. This shows that F is continuous in int(dom F). The proof is finished. □

2.4. **Conjugate function.** The vector space V^* of bounded linear functionals over V is said to be the (topological) dual of V and its elements are denoted by u^*. Notation $\langle u, u^* \rangle$ denotes the value of $u^* \in V^*$ at u that is, $\langle u, u^* \rangle = u^*(u)$.

The continuous affine functions over V are of the type $v \mapsto l(v) + \alpha$, where l is a continuous linear functional over V and $\alpha \in \mathbb{R}$.

We denote the set of functions $F : V \to \bar{\mathbb{R}}$ which are pointwise supremum of a family of continuous affine functions by $\Gamma(V)$. In addition, we denote

$$\Gamma_0(V) = \{f \in \Gamma(V) : \exists\ u_0 \in V \text{ such that } -\infty < f(u_0) < \infty\}.$$

THEOREM 2.21. *The following statements are equivalent to each other:*

(i) $F \in \Gamma(V)$

(ii) F *is a convex lower semi-continuous function from* V *to* $\bar{\mathbb{R}}$*. If* F *takes value* $-\infty$ *then* $F \equiv -\infty$*.*

For the proof of Theorem 2.21, we need the second geometric form of Hahn-Banach Theorem. For the proof we refer to [6, p. 58].

THEOREM 2.22 (Hahn-Banach, second geometric form)**.** *Let* V *be a real normed space. Let* $A \subset V$ *be a non-empty convex and compact set and* $B \subset V$ *be a non-empty convex closed set such that* $A \cap B = \emptyset$*. Then there exists a closed affine hyperplane* $\mathscr{H}$ *which strictly separates* A *and* B*, that is, if* $l(u) = \alpha$ *is the equation of* $\mathscr{H}$*, we have*

$$l(u) < \alpha \text{ for all } u \in A \quad \text{and} \quad l(v) > \alpha \text{ for all } v \in B.$$

Proof of Theorem 2.21. We first claim that continuous affine functions over V are convex and lower semi-continuous. Indeed, if G is a continuous affine function over V, then $G(u) = l(u) + \alpha$, where l is a continuous linear functional over V and $\alpha \in \mathbb{R}$. For $u, v \in V$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} G(\lambda u + (1-\lambda)v) &= l(\lambda u + (1-\lambda)v) + \alpha = \lambda[l(u) + \alpha] + (1-\lambda)[l(v) + \alpha] \\ &= \lambda G(u) + (1-\lambda)G(v), \end{aligned}$$

which shows that G is convex. Since G is continuous, it is lower semi-continuous. This proves the claim.

By Proposition 2.16 and Proposition 2.17 the epigraph of continuous affine function is closed and convex set. If $F : V \to \bar{R}$ is a pointwise supremum of non-empty family of continuous affine functions, then by Proposition 2.18 epi F is convex and closed. From Propositions 2.16 and 2.17 we have that F is convex and lower semi-continuous. Moreover, the pointwise supremum of an empty family is $-\infty$ and if the family under consideration is non-empty, F cannot take the value $-\infty$. Thus (ii) follows from (i).

Conversely, suppose that $F : V \to \bar{\mathbb{R}}$ is a convex lower semi-continuous function and that $F(u) > -\infty$ for all $u \in V$. We show that $F \in \Gamma(V)$. If $F \equiv \infty$, then it is the pointwise supremum of all continuous affine functions in V.

If $F \not\equiv \infty$, then for $\bar{u} \in V$ we fix a number $\bar{a}$ such that $\bar{a} < F(\bar{u})$. We know epi F is a closed convex set that does not contain the point $(\bar{u}, \bar{a})$. By Theorem 2.22, we can strictly separate epi F and the point $(\bar{u}, \bar{a})$ by a closed affine hyperplane $\mathscr{H}$ of $V \times \mathbb{R}$. $\mathscr{H}$ is of the form

$$\mathscr{H} = \{(u, a) \in V \times \mathbb{R} : l(u) + \alpha a = \beta\},$$

where l is a continuous linear functional over V and $\alpha, \beta \in \mathbb{R}$. We have

$$l(\bar{u}) + \alpha\bar{a} < \beta \quad \text{and} \quad l(u) + \alpha a > \beta, \quad \text{for all } (u, a) \in \text{epi } F.$$

Suppose $F(\bar{u}) < \infty$. Then $(\bar{u}, F(\bar{u})) \in \text{epi } F$. Thus

$$l(\bar{u}) + \alpha F(\bar{u}) > \beta > l(\bar{u}) + \alpha\bar{a}, \tag{7}$$

from which follows that $\alpha\big(F(\bar{u}) - \bar{a}\big) > 0$. Hence $\alpha > 0$. From the inequality (7), we obtain

$$\bar{a} < \frac{\beta}{\alpha} - \frac{1}{\alpha} l(\bar{u}) < F(\bar{u}). \tag{8}$$

Now suppose $F(\bar{u}) = \infty$. We have

$$l(\bar{u}) + \alpha F(\bar{u}) \geq \beta > l(\bar{u}) + \alpha\bar{a},$$

from which follows that $\alpha\big(F(\bar{u}) - \bar{a}\big) \geq 0$ and furthermore $\alpha \geq 0$. If $\alpha > 0$, we obtain (8). If $\alpha = 0$, we have

$$\beta - l(\bar{u}) > 0 \quad \text{and} \quad \beta - l(u) < 0, \quad \text{for all } u \in \text{dom } F.$$

Earlier in the proof, we showed that it is possible to construct a continuous affine function $h : V \to \mathbb{R}$ such that $h(u) < F(u)$ for every $u \in \text{dom}(F)$. For every $c > 0$, $h(\cdot) + c\big(\beta - l(\cdot)\big)$ is a continuous affine function everywhere less than F and therefore it only remains to choose c sufficiently large so that

$$h(\bar{u}) + c\big(\beta - l(\bar{u})\big) > \bar{a}.$$

Finally, we have proved that for every $\bar{u} \in V$ and $\bar{a} \in \mathbb{R}$ such that $\bar{a} < F(\bar{u})$, there exists continuous affine function $m : V \to \mathbb{R}$ such that

$$m(u) \leq F(u), \quad \text{for all } u \in V \quad \text{and} \quad \bar{a} < m(\bar{u}) < F(\bar{u}).$$

Thus F is a pointwise supremum of family of continuous affine functions. □

Definition 2.23. Let $F : V \to \bar{\mathbb{R}}$ be a function. Define $F^* : V^* \to \bar{\mathbb{R}}$ as follows

$$F^*(u^*) = \sup_{u \in V} \{\, \langle u, u^* \rangle - F(u)\}.$$

We say F^* is the *polar* or *conjugate* function of F.

Definition 2.24. Let $F : V \to \bar{\mathbb{R}}$ be a function. Define $F^{**} : V \to \bar{\mathbb{R}}$ as follows

$$F^{**}(u) = \sup_{u^* \in V^*} \{\langle u, u^* \rangle - F^*(u^*)\}.$$

We say F^{**} is the *bipolar* of F.

Lemma 2.25. *Let $F : V \to \bar{\mathbb{R}}$ be a function. Then $F^* \in \Gamma(V^*)$ and $F^{**} \in \Gamma(V)$.*

Proof. If dom $F = \emptyset$, then $F^* \equiv -\infty$. If dom $F \neq \emptyset$, then F^* is the pointwise supremum of the family of continuous affine functions

$$\langle u, \cdot \rangle - F(u) \quad \text{for } u \in \text{dom } F.$$

Hence $F^* \in \Gamma(V^*)$.

Clearly $F^{**} \leq F$. In fact, the bipolar of F is the pointwise supremum of maximal continuous affine functions everywhere less than F. Hence, F^{**} is the largest minorant of F in $\Gamma(V)$. □

2.5. **Subdifferentiability.**

Definition 2.26. Let $F : V \to \bar{\mathbb{R}}$ be a function and $l : V \to \bar{\mathbb{R}}$ be a continuous affine function everywhere less than F, that is, $l(v) \leq F(v)$ for all $v \in V$. We say that l is *exact* at the point $u \in V$ if $l(u) = F(u)$.

Definition 2.27. A function $F : V \to \bar{\mathbb{R}}$ is said to be *subdifferentiable* at the point $u \in V$, if there exists a continuous affine function $l : V \to \bar{\mathbb{R}}$ exact at u. Let l be of the form

$$l(\cdot) = \langle \cdot , u^* \rangle - \alpha, \quad \alpha \in \mathbb{R}.$$

Then $u^* \in V^*$ is called a *subgradient* of F at u. The set of subgradients at u is called the *subdifferential* at u and is denoted $\partial F(u)$.

Proposition 2.28. *Let $F : V \to \bar{\mathbb{R}}$ be a function and $u \in V$. If $\partial F(u) \neq \emptyset$, then $F(u) = F^{**}(u)$. If $F(u) = F^{**}(u)$, then $\partial F(u) = \partial F^{**}(u)$.*

Proof. Let $u^* \in \partial F(u)$. There exists a continuous affine function l such that $l \leq F$ and $l(u) = F(u)$. Necessarily, $l(u)$ is finite and l is of the form

$$l(v) = \langle v, u^* \rangle - \big(\langle u, u^* \rangle - F(u)\big), \quad v \in V.$$

Since l is everywhere less than F, we have by the definition of F^*

$$\langle u, u^* \rangle - F(u) \geq F^*(u^*).$$

Again by the definition of the conjugate function,

$$\langle u, u^* \rangle - F(u) \leq F^*(u^*).$$

Thus

$$\langle u, u^* \rangle - F(u) = F^*(u^*) \quad \text{and} \quad l(v) = \langle v, u^* \rangle - F^*(u^*), \quad v \in V.$$

Therefore for all $v \in V$

$$l(v) \leq F^{**}(v) \leq F(v),$$

from which follows that $F(u) = F^{**}(u)$.

By the definition of bipolar, we have that a continuous affine function

$$v \mapsto \langle v, u^* \rangle - \alpha$$

is everywhere less than F if and only if it is less than F^{**}. Hence, if $F(u) = F^{**}(u)$, we have that $u^* \in \partial F(u)$ if and only if $u^* \in \partial F^{**}(u)$. This proves the Proposition. □

Proposition 2.29. *Let $F : V \to \bar{\mathbb{R}}$ be a function and F^* the conjugate function of F. Then $u^* \in \partial F(u)$ if and only if*

$$F(u) + F^*(u^*) = \langle u, u^* \rangle.$$

Proof. Suppose $u^* \in \partial F(u)$. Then by the proof of Proposition 2.28, we have that

$$F(u) = l(u) = \langle u, u^* \rangle - F^*(u^*).$$

Suppose then that $F(u) + F^*(u^*) = \langle u, u^* \rangle$. It follows that the continuous affine function

$$\langle \cdot, u^* \rangle + F(u) - \langle u, u^* \rangle$$

is everywhere less than F and exact at u. Hence $u^* \in \partial F(u)$. □

Proposition 2.30. *Let $F : V \to \bar{\mathbb{R}}$ be a convex function which is finite and continuous at a point $u \in V$. Then $\partial F(v) \neq \emptyset$ for all $v \in$ int(dom F).*

For the proof of Proposition 2.30, we need the first geometric form of Hahn-Banach Theorem. For the proof we refer to [6, p. 58].

THEOREM 2.31 (Hahn-Banach, first geometric form)**.** *Let V be a real normed space. Let $A \subset V$ be a open non-empty convex set and $B \subset V$ be a non-empty convex set such that $A \cap B = \emptyset$. Then there exists a closed affine hyperplane $\mathscr{H}$ which separates A and B, that is, if $l(u) = \alpha$ is the equation of $\mathscr{H}$, we have*

$$l(u) \leq \alpha \text{ for all } u \in A \quad \text{ and } \quad l(v) \geq \alpha \text{ for all } v \in B.$$

Proof of Proposition 2.30. Since F is finite and continuous at u, it is bounded from above in a neighborhood of u. By Proposition 2.20, we have that F is finite and continuous at each point of int(dom F). Hence we only need to show that $\partial F(u) \neq \emptyset$.

Since F is convex, epi F is a convex subset of $V \times \mathbb{R}$. Since F is continuous, the interior of epi F is non-empty. The point $(u, F(u))$ belongs to the boundary of epi F. By Theorem 2.31 we can separate it from the open non-empty convex set int(epi F) by a closed affine hyperplane

$$\mathscr{H} = \{(v, a) \in V \times \mathbb{R} : \ \langle v, u^* \rangle + \ \alpha a = \beta\}, \quad u^* \in V^* \text{ and } \alpha, \beta \in \mathbb{R}.$$

We have

$$\begin{aligned} &\langle v, u^* \rangle + \ \alpha a \geq \beta \quad \text{for all } (v, a) \in \text{ epi } F \\ \text{and} \quad &\langle u, u^* \rangle + \ \alpha F(u) = \beta. \end{aligned} \tag{9}$$

We claim that $\alpha \neq 0$. Indeed, if $\alpha = 0$, then $\langle v - u, u^* \rangle \geq 0$ for all $v \in$ dom F. Since dom F is a neighborhood of u, there exists a real number $r > 0$ such that $B(u, r) \subset$ dom F. Let $v \in V$ such that $||v||_V < 1$. Then $u \pm rv \in B(u, r)$ and

$$\begin{aligned} \langle u + rv - u, u^* \rangle \geq 0 \quad &\Leftrightarrow \quad ru^*(v) \geq 0 \\ \langle u - rv - u, u^* \rangle \geq 0 \quad &\Leftrightarrow \quad -ru^*(v) \geq 0. \end{aligned}$$

Therefore

$$u^*(v) = 0 \quad \text{ for all } v \in B(0, 1),$$

from which follows that $u^* \equiv 0$. This is impossible, since the linear form of the equation of the hyperplane is non-zero. This proves the claim.

By assumption F is finite and continuous at u. Hence there exists $0 < M < \infty$ such that $F(u) < M$. Then

$$\alpha(M - F(u)) = \langle u, u^* \rangle + \alpha M - \langle u, u^* \rangle - \alpha F(u) \geq \beta - \beta = 0.$$

Thus we have $\alpha > 0$. Dividing (9) by α, we obtain for all $v \in$ dom F

$$\beta/\alpha - \langle v, u^*/\alpha \rangle \leq F(v) \quad \text{and} \quad \beta/\alpha - \langle u, u^*/\alpha \rangle = F(u).$$

Combining these, we have

$$\langle v - u, -u^*/\alpha \rangle + F(u) \leq F(v) \quad \text{ for all } v \in V.$$

Now $v \mapsto \langle v - u, -u^*/\alpha \rangle + F(u)$ is a continuous affine function everywhere less than F and exact at a point u. Hence F is subdifferentiable at u and $-u^*/\alpha \in \partial F(u)$. □

3. Weak convergence and reflexive Banach spaces

3.1. **Weak convergence.**

Definition 3.1. A sequence (u_j) in V *converges weakly* to $u \in V$ if

$$\varphi(u_j) \to \varphi(u) \quad \text{as } j \to \infty \qquad \text{for all } \varphi \in V^*.$$

In this case we write $u_j \rightharpoonup u$ in V.

Proposition 3.2. *Let (u_j) be a sequence in V converging strongly to $u \in V$, that is*

$$||u_j - u||_V \to 0 \quad \text{as } j \to \infty.$$

Then $u_j \rightharpoonup u$ in V.

Proof. Let $\varphi \in V^*$. Then there exists a positive real number $M < \infty$ such that $||\varphi||_{V^*} \leq M$. Hence

$$|\varphi(u_j) - \varphi(u)| = |\varphi(u_j - u)| \leq ||\varphi||_{V^*}||u_j - u||_V \leq M||u_j - u||_V \to 0$$

as $j \to \infty$. □

Definition 3.3. A normed vector space $(V, ||\cdot||_V)$ is said to be *complete* if every Cauchy sequence (u_j) in V converges strongly to some $u \in V$.

Definition 3.4. A complete normed vector space $(V, ||\cdot||_V)$ is called a *Banach space.*

Lemma 3.5 (Mazur's Lemma)**.** *Let V be Banach space and (u_n) a sequence in V converging weakly to $\bar{u}$ in V. Then for any n, there is $N = N(n) \in \mathbb{N}$ and $\lambda_k \geq 0$, $k = n, \cdots, N$ with*

$$\sum_{k=n}^{N} \lambda_k = 1, \qquad \text{such that} \qquad v_n = \sum_{k=n}^{N} \lambda_k u_k \qquad \text{converges strongly to } \bar{u} \text{ in } V.$$

Proof. Refer to [9, p. 120]. □

Proposition 3.6. *Let V be Banach space and $A \subset V$ a closed convex set. Then A is weakly closed.*

Proof. Let (u_j) be a sequence in A converging weakly to $u \in V$. By Lemma 3.5 there exists a sequence of convex combinations $\{v_n\}$ of $\{u_j\}$ converging strongly to u. By Proposition 2.3 $v_n \in A$ for all n. Since A is closed, it follows that $u \in A$. Thus A is weakly closed. □

Definition 3.7. A function $F : V \to \bar{\mathbb{R}}$ is said to be *weakly lower semi-continuous* on V if for all $u \in V$ and all sequences (u_i) in V converging weakly to u, we have

$$\varliminf_{i \to \infty} F(u_i) \geq F(u).$$

The proof of the following Lemma is similar to that of Proposition 2.17.

Lemma 3.8. *A function $F : V \to \bar{\mathbb{R}}$ is weakly lower semi-continuous if and only if its epigraph epi F is weakly closed.*

Proposition 3.9. *Let $F : V \to \bar{\mathbb{R}}$ be a convex and lower semi-continuous function. Then F is weakly lower semi-continuous.*

Proof. By Propositions 2.16 and 2.17 function is convex and lower semi-continuous if and only if its epigraph is closed and convex set. Since F is convex and lower semi-continuous, then by Proposition 3.6 epi F is weakly closed. By Lemma 3.8 F is weakly lower semi-continuous. □

3.2. L^p **spaces.** In this section we let Ω be a bounded domain in $\mathbb{R}^n$. We denote the Lebesgue measure of a set $A \subset \mathbb{R}^n$ by $m(A)$.

Definition 3.10. Let $1 \leq p < \infty$. The set $\mathcal{L}^p(\Omega)$ consists of all measurable functions $f : \Omega \to \bar{\mathbb{R}}$ such that $|f|^p$ is integrable, that is

$$\int_\Omega |f|^p \, dx < \infty.$$

The set $\mathcal{L}^\infty(\Omega)$ consists of all measurable functions $f : \Omega \to \bar{\mathbb{R}}$ such that

$$\sup \, \Big\{ t \geq 0 : m\big(\{x \in A : |f(x)| > t\}\big) > 0 \Big\} < \infty.$$

Definition 3.11. Let $1 \leq p \leq \infty$ and $f, g \in \mathcal{L}^p(\Omega)$. Then

$$g \sim f \quad \text{if and only if} \quad g(x) = f(x) \quad \text{for almost every } x \in \Omega.$$

The equivalence class of an element f is denoted by

$$[f] = \{g \in \mathcal{L}^p(\Omega) : g \sim f\}.$$

Definition 3.12. Let $1 \leq p \leq \infty$. We set

$$L^p(\Omega) = \{[f] : f \in \mathcal{L}^p(\Omega)\}.$$

Definition 3.13. Let $1 \leq p < \infty$ and $f : \Omega \to \bar{\mathbb{R}}$ be a measurable function. Denote

$$||f||_p = \Big(\int_\Omega |f|^p \, dx \Big)^{1/p}$$

and

$$||f||_\infty = \sup \, \big\{ t \geq 0 : m(\{x \in A : |f(x)| > t\} > 0) \big\}.$$

Proposition 3.14 (Young's inequality)**.** *Let p be a real number such that $1 < p < \infty$. Then for non-negative numbers a and b, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $q = \frac{p}{p-1}$.

Proof. If either $a = 0$ or $b = 0$, the inequality is trivial. We may therefore assume that $a, b > 0$. We notice that the function $f(t) = \log t$ is a concave function in $(0, \infty)$. Thus

$$\log \Big(\frac{a^p}{p} + \frac{b^q}{q} \Big) \geq \frac{1}{p} \log a^p + \frac{1}{q} \log b^q,$$

from which it follows that

$$\log(ab) \leq \log \Big(\frac{a^p}{p} + \frac{b^q}{q} \Big),$$

that is,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Definition 3.15. Let $1 < p, q < \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We say that p and q are *conjugate exponents.*

Proposition 3.16 (Hölder's inequality)**.** *Let p and q be conjugate exponents such that $1 \le p, q \le \infty$. Then*

$$\int_\Omega |uv| \; dx \le ||u||_p ||v||_q$$

for all $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$.

Proof. Suppose first that $p = 1$ and $q = \infty$. Then

$$\int_\Omega |uv| \; dx \le ||v||_\infty \int_\Omega |u| \; dx = ||v||_\infty ||u||_1.$$

Suppose $1 < p < \infty$ and $1 < q < \infty$. If $||u||_p = 0$, then $u(x) = 0$ for almost every $x \in \Omega$. In this case

$$\int_\Omega |uv| \; dx = 0$$

and the inequality is trivial. We may therefore assume that $||u||_p, ||v||_q > 0$. Denote for $x \in \Omega$

$$a_x := \frac{|u(x)|}{||u||_p} \quad \text{and} \quad b_x := \frac{|v(x)|}{||v||_q}.$$

Then by Young's inequality

$$\frac{|u(x)||v(x)|}{||u||_p ||v||_q} = a_x b_x \le \frac{{a_x}^p}{p} + \frac{{b_x}^q}{q} = \frac{|u(x)|^p}{p||u||_p^p} + \frac{|v(x)|^q}{q||v||_q^q}$$

for every $x \in \Omega$. Integrating over Ω, we obtain that

$$\begin{aligned} ||u||_p^{-1} ||v||_q^{-1} \int_\Omega |uv| \; dx &\le \frac{1}{p} ||u||_p^{-p} \int_\Omega |u|^p \; dx + \frac{1}{q} ||v||_q^{-q} \int_\Omega |v|^q \; dx \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

which gives us

$$\int_\Omega |uv| \; dx \le ||u||_p ||v||_q.$$

□

Proposition 3.17 (Minkowski's inequality)**.** *Let $1 \le p < \infty$ be a real number and $u, v \in L^p(\Omega)$. Then*

$$||u + v||_p \le ||u||_p + ||v||_p.$$

Proof. The case $p = 1$ follows from the triangle inequality. Let $1 < p < \infty$. Then

$$||u + v||_p^p = \int_\Omega |u + v|^p \; dx \le 2^p \int_\Omega |u|^p + |v|^p \; dx = 2^p ||u||_p^p + 2^p ||v||_p^p < \infty,$$

from which follows that $u + v \in L^p(\Omega)$. We have for almost every $x \in \Omega$

$$|u(x)+v(x)|^p = |u(x)+v(x)||u(x)+v(x)|^{p-1} \le |u(x)||u(x)+v(x)|^{p-1} + |v(x)||u(x)+v(x)|^{p-1}$$

and hence by Hölder's inequality

$$\begin{aligned} ||u + v||_p^p &\le \int_\Omega \left(|u||u + v|^{p-1} + |v||u + v|^{p-1} \right) dx \\ &\le ||u||_p \Big(\int_\Omega |u + v|^p \Big)^{(p-1)/p} + ||v||_p \Big(\int_\Omega |u + v|^p \Big)^{(p-1)/p} \\ &= ||u||_p ||u + v||_p^{p-1} + ||v||_p ||u + v||_p^{p-1}. \end{aligned}$$

This proves the proposition. ∎

Remark 3.1. For $1 < p < \infty$, we have $||u + v||_p = ||u||_p + ||v||_p$ if and only if $u = \lambda v$ for some $\lambda \in \mathbb{R}$.

Proposition 3.18. *Let $1 \leq p < \infty$. Then $L^p(\Omega)$ is a Banach space.*

Proof. We show that $L^p(\Omega)$ is complete. Let (f_j) be a Cauchy sequence in $L^p(\Omega)$. We will show that there is $f \in L^p(\Omega)$ such that

$$||f_j - f||_p \to 0 \quad \text{as } j \to \infty.$$

Since (f_j) is a Cauchy sequence, there exists a subsequence $(f_{j_k}) \subset (f_j)$ such that

$$||f_{j_{k+1}} - f_{j_k}||_p < 2^{-k},$$

for every $k \in \mathbb{N}$. We define

$$g_l(x) = \sum_{k=1}^{l} |f_{j_{k+1}}(x) - f_{j_k}(x)| \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} |f_{j_{k+1}}(x) - f_{j_k}(x)|, \quad l \in \mathbb{N}.$$

For every $l \in \mathbb{N}$ the function g_l is measurable and non-negative. Clearly, (g_l) is an increasing sequence and

$$g_l(x) \to g(x) \quad \text{as } l \to \infty \quad \text{for almost every } x \in \Omega.$$

By Minkowski's inequality

$$||g_l||_p \leq \sum_{k=1}^{l} ||f_{j_{k+1}} - f_{j_k}||_p \leq \sum_{k=1}^{l} 2^{-k} \leq 1,$$

for all $l \in \mathbb{N}$. Using the monotone convergence theorem [4, p. 186], we have

$$\int_\Omega g(x)^p \, dx = \lim_{l \to \infty} \int_\Omega g_l(x)^p \leq 1.$$

Thus $g \in L^p(\Omega)$. Therefore $0 \leq g(x) < \infty$ for almost every $x \in \Omega$.

Since $g(x) < \infty$ for almost every $x \in \Omega$, the series

$$\sum_{k=1}^{\infty} \left[f_{j_{k+1}}(x) - f_{j_k}(x) \right]$$

is absolutely convergent. Hence it is convergent for almost every $x \in \Omega$. This means that

$$f_{j_l}(x) = f_{j_1}(x) + \sum_{k=1}^{l-1} \big(f_{j_{k+1}}(x) - f_{j_k}(x)\big) \to f_{j_1}(x) + \sum_{k=1}^{\infty} \big(f_{j_{k+1}}(x) - f_{j_k}(x)\big)$$

as $l \to \infty$ for almost every $x \in \Omega$. Therefore (f_{j_k}) is a converging sequence for almost every $x \in \Omega$. Define

$$f(x) = \begin{cases} f_{j_1}(x) + \sum_{k=1}^{\infty} \big(f_{j_{k+1}}(x) - f_{j_k}(x)\big), & \text{when the limit exists;} \\ 0, & \text{elsewhere.} \end{cases}$$

Then f is measurable and $f \in L^p(\Omega)$.

Let $\varepsilon > 0$. Since (f_j) is Cauchy in $L^p(\Omega)$ there exists $N \in \mathbb{N}$ such that

$$||f_j - f_i||_p < \varepsilon, \quad \text{when } i, j \geq N.$$

Let $j \geq N$. Finally, using Fatou's Lemma [4, p. 243], we have

$$\int_\Omega |f(x) - f_j(x)|^p \; dx \leq \liminf_{k\to\infty} \int_\Omega |f_{j_k}(x) - f_j(x)|^p \; dx < \varepsilon^p.$$

This implies that $f_j \to f$ in $L^p(\Omega)$ as $j \to \infty$. Thus $L^p(\Omega)$ is complete. □

Remark 3.2. The following result can be obtained from the proof of Proposition 3.18: If $f_j \to f$ in $L^p(\Omega)$, $1 \leq p < \infty$, then there exists a subsequence $(f_{j_k}) \subset (f_j)$ such that $f_{j_k}(x) \to f(x)$ for almost every $x \in \Omega$.

The notation $\Omega' \subset\subset \Omega$ means that $\overline{\Omega'} \subset \Omega$.

Definition 3.19. Let $1 \leq p \leq \infty$ and $u : \Omega \to \mathbb{R}$ be a measurable function. The function u is said to be *locally p-integrable* if

$$u \in L^p(\Omega') \quad \text{for all} \quad \Omega' \subset\subset \Omega.$$

We write $u \in L^p_{\text{loc}}(\Omega)$.

By a locally integrable function on Ω we refer to a function of class $L^1_{\text{loc}}(\Omega)$. The convergence in $L^p_{\text{loc}}(\Omega)$ is understood as convergence in $L^p(\Omega')$ for each $\Omega' \subset\subset \Omega$.

Remark 3.3. Let $u \in L^p(\Omega)$ and $\Omega' \subset\subset \Omega$. From the monotonicity of the integral it follows that

$$\int_{\Omega'} |u(x)|^p \; dx \leq \int_\Omega |u(x)|^p \; dx < \infty, \quad 1 \leq p < \infty.$$

Hence $u \in L^p_{\text{loc}}(\Omega)$.

Proposition 3.20. *Let p and q be conjugate exponents such that $1 < p, q < \infty$. If $v \in L^q(\Omega)$, then*

$$\varphi(u) = \int_\Omega u(x)v(x) \; dx$$

defines a bounded linear functional $\varphi : L^p(\Omega) \to \mathbb{R}$, and

$$||\varphi||_{L^p(\Omega)^*} = ||v||_{L^q(\Omega)},$$

where $L^p(\Omega)^*$ is the dual space of $L^p(\Omega)$.

Proof. By Hölder's inequality, we have that

$$|\varphi(u)| \leq \int_\Omega |u(x)||v(x)| \; dx \leq ||u||_{L^p(\Omega)}||v||_{L^q(\Omega)},$$

which implies that φ is a bounded functional on $L^p(\Omega)$ and

$$||\varphi||_{L^p(\Omega)^*} \leq ||v||_{L^q(\Omega)}.$$

Next we prove the reverse inequality. We may assume that $v \not\equiv 0$. Let

$$u(x) = \big[\text{sgn } v(x)\big] \left(\frac{|v(x)|}{||v||_{L^q(\Omega)}} \right)^{q/p},$$

then $u \in L^p(\Omega)$ and

$$||u||^p_{L^p(\Omega)} = \int_\Omega \big|\text{sgn } v(x)\big|^p \left(\frac{|v(x)|}{||v||_{L^q(\Omega)}} \right)^q dx = 1.$$

Note that $q/p = q - 1$ and that

$$\varphi(u) = \int_{\Omega} \big[\text{sgn } v(x)\big] \left(\frac{|v(x)|}{||v||_{L^q(\Omega)}} \right)^{q-1} v(x)\ dx = \frac{1}{||v||^{q-1}_{L^q(\Omega)}} ||v||^q_{L^q(\Omega)} = ||v||_{L^q(\Omega)}.$$

Then we arrive at

$$||v||_{L^q(\Omega)} = |\varphi(u)| \leq ||\varphi||_{L^p(\Omega)^*} ||u||_p = ||\varphi||_{L^p(\Omega)^*}.$$

This proves the proposition. □

THEOREM 3.21. *Let p and q be conjugate exponents such that $1 < p, q < \infty$. Define a mapping $J : L^q(\Omega) \to L^p(\Omega)^*$ as follows: for $v \in L^q(\Omega)$, $J(v) \in L^p(\Omega)^*$ is defined as*

$$\langle J(v), u \rangle = \int_{\Omega} u(x)v(x)\ dx, \quad \forall u \in L^p(\Omega).$$

Then J is an isometric isomorphism from $L^q(\Omega)$ onto $L^p(\Omega)^*$.

Proof. Clearly, J is linear and by Proposition 3.20 J is an isometric mapping from $L^q(\Omega)$ onto $L^p(\Omega)^*$. Since J is isometric, it is necessarily injective. Therefore, in order to show that J is isomorphism, we only need to show that J is surjective.

Let $\mathscr{A}$ denote the set of measurable subsets of Ω. Suppose first that $m(\Omega) < \infty$ and let $F : L^p(\Omega) \to \mathbb{R}$ be a bounded linear functional on $L^p(\Omega)$. If $A \in \mathscr{A}$, then

$$\int_{\Omega} |\chi_A(x)|^p\ dx \leq \int_{\Omega} 1\ dx \leq m(\Omega) < \infty,$$

which implies that $\chi_A \in L^p(\Omega)$ for every $A \in \mathscr{A}$. Therefore we may define a function $\nu : \mathscr{A} \to \mathbb{R}$ by setting $\nu(A) = F(\chi_A)$. Let $A_1, A_2, ... \in \mathscr{A}$ be disjoint sets such that

$$A = \bigcup_{i=1}^{\infty} A_i.$$

Denoting

$$f(x) = \chi_A(x) \quad \text{ and } \quad f_j(x) = \sum_{i=1}^{j} \chi_{A_i}(x), \quad j \in \mathbb{N},$$

we have that $|f_j| \leq 1$ for all $j \in \mathbb{N}$ and

$$\chi_A = \chi_{\bigcup_{i=1}^{\infty} A_i} = \sum_{i=1}^{\infty} \chi_{A_i}.$$

Since $f_j(x) \to f(x)$ for every $x \in \Omega$, $||f_j||_p \to ||f||_p$ as $j \to \infty$ by the dominated convergence theorem. It then follows that

$$(10) \qquad ||\chi_A - \sum_{i=1}^{j} \chi_{A_i}||_p \to 0 \quad \text{ as } j \to \infty.$$

From (10) and the fact that F is a bounded linear functional on $L^p(\Omega)$, we have

$$\nu(A) = F(\chi_A) = F\Big(\sum_{i=1}^{\infty} \chi_{A_i} \Big) = \sum_{i=1}^{\infty} F(\chi_{A_i}) = \sum_{i=1}^{\infty} \nu(A_i),$$

which implies that ν is a signed measure on $\mathscr{A}$. If $m(A) = 0$, then $\nu(A) = F(\chi_A) = 0$ by the linearity of F. Thus ν is absolutely continuous with respect to Lebesgue measure and by the Radon-Nikodym Theorem [4, p. 196] there exists a function $v \in L^1(\Omega)$ such that

$$\nu(A) = F(\chi_A) = \int_\Omega v(x)\chi_A(x)\ dx \quad \text{for every } A \in \mathscr{A}.$$

First, let $\phi : \Omega \to \mathbb{R}$ be a simple function. There are constants $c_1, ..., c_l$ and measurable sets $C_1, ..., C_l$ such that

$$\phi(x) = \sum_{i=1}^{l} c_i \chi_{C_i}(x) \quad \text{for all } x \in \Omega.$$

Thus

$$\begin{aligned} F(\phi) &= F\big(\sum_{i=1}^{l} c_i\chi_{C_i}\big) = \sum_{i=1}^{l} c_i F(\chi_{C_i}) = \sum_{i=1}^{l} c_i \int_\Omega v(x)\chi_{C_i}(x)\ dx \\ &= \int_\Omega v(x) \sum_{i=1}^{l} c_i\chi_{C_i}(x)\ dx = \int_\Omega v(x)\phi(x)\ dx. \end{aligned} \tag{11}$$

Second, let $\phi : \Omega \to [0, \infty)$ be a bounded measurable function. Then $\phi \in L^p(\Omega)$. There are simple functions (ϕ_i) such that

$$0 \le \phi_i \le \phi_{i+1} \le \phi \quad \text{and} \quad \lim_{i\to\infty} \phi_i(x) = \phi(x) \quad \text{for every } x \in \Omega.$$

Since $|\phi_i - \phi|^p \le |\phi|^p$ and $|\phi|^p \in L^1(\Omega)$, by the dominated convergence theorem

$$\Big(\int_\Omega |\phi - \phi_i|^p\ dx\Big)^{1/p} \to 0 \quad \text{as } i \to \infty.$$

For every $i \in \mathbb{N}$ it holds

$$|\phi_i(x)v(x)| \le |\phi(x)v(x)| \le ||\phi||_\infty |v(x)| \in L^1(\Omega).$$

Thus by the dominated convergence theorem, (11) and the continuity of F

$$F(\phi) = \lim_{i\to\infty} F(\phi_i) = \lim_{i\to\infty} \int_\Omega v(x)\phi_i(x)\ dx = \int_\Omega v(x)\phi(x)\ dx. \tag{12}$$

Third, let $\phi : \Omega \to \mathbb{R}$ be a bounded measurable function. Let $\phi^+ : \Omega \to [0, \infty)$ denote the positive part of ϕ and $\phi^- : \Omega \to [0, \infty)$ the negative part of ϕ. By (12) it follows that

$$\begin{aligned} F(\phi) = F(\phi^+ - \phi^-) &= F(\phi^+) - F(\phi^-) = \int_\Omega v(x)\phi^+(x) - \int_\Omega v(x)\phi^-(x)\ dx \\ &= \int_\Omega v(x)\big(\phi^+(x) - \phi^-(x)\big)\ dx = \int_\Omega v(x)\phi(x)\ dx. \end{aligned}$$

Next we will show that $v \in L^q(\Omega)$. Define functions $h : \Omega \to \mathbb{R}$, $h = |v|^{q-2}v$ and $h_j : \Omega \to \mathbb{R}$ as follows

$$h_j(x) = \begin{cases} h(x), & \text{if } |h(x)| \le j; \\ 0, & \text{else.} \end{cases}$$

For every $j > 0$, h_j is a bounded measurable function and hence

$$\Big|\int_\Omega v(x)h_j(x)\ dx\Big| = |F(h_j)| \le ||F||_{L^p(\Omega)^*} ||h_j||_p < \infty. \tag{13}$$

On the other hand, we have

$$\int_\Omega v(x)h_j(x)\ dx = \int_{\Omega\cap\{|h|\le j\}} |v(x)|^q\ dx$$

and

$$\begin{aligned} ||h_j||_p &= \Big(\int_\Omega |h_j(x)|^p\ dx\Big)^{1/p} = \Big(\int_{\Omega\cap\{|h|\le j\}} (|v(x)|^{q-1})^{\frac{q}{q-1}}\ dx\Big)^{\frac{q-1}{q}} \\ &= \Big(\int_{\Omega\cap\{|h|\le j\}} |v(x)|^q\ dx\Big)^{\frac{q-1}{q}}, \quad \text{for every } j > 0. \end{aligned}$$

Therefore, (13) can be written as

$$\Big(\int_{\Omega\cap\{|h|\le j\}} |v(x)|^q\ dx\Big)^{\frac{1}{q}} \le ||F||_{L^p(\Omega)^*}, \quad \forall j > 0.$$

Since $v \in L^1(\Omega)$, it follows that $|v(x)| < \infty$ for almost every $x \in \Omega$. Thus $||v||_q \le ||F||_{L^p(\Omega)^*}$ and $v \in L^q(\Omega)$.

Define a functional $\tilde{F} : L^p(\Omega) \to \mathbb{R}$ as follows

$$\tilde{F}(u) = \int_\Omega v(x)u(x)\ dx.$$

By Proposition 3.20 $\tilde{F} \in L^p(\Omega)^*$ and $||\tilde{F}||_{L^p(\Omega)^*} = ||v||_{L^q(\Omega)}$. In addition, we have $\tilde{F}(\varphi) = F(\varphi)$ for every $\varphi \in L^\infty(\Omega)$. Let $u \in L^p(\Omega)$. For every $j \in \mathbb{N}$, define $u_j : \Omega \to \mathbb{R}$ by setting

$$u_j(x) = \begin{cases} j, & \text{if } u(x) > j; \\ u(x), & \text{if } |u(x)| \le j; \\ -j, & \text{if } u(x) < -j. \end{cases}$$

Then $u_j \in L^\infty(\Omega)$ and $u_j \to u$ in $L^p(\Omega)$ as $j \to \infty$. We have

$$\tilde{F}(u) = \lim_{j\to\infty} \tilde{F}(u_j) = \lim_{j\to\infty} F(u_j) = F(u),$$

since $\tilde{F}$ and F are continuous. Thus $\tilde{F}(u) = F(u)$ for every $u \in L^p(\Omega)$ and

$$F(u) = \int_\Omega v(x)u(x)\ dx, \quad \text{for every } u \in L^p(\Omega).$$

Suppose there are $v_1, v_2 \in L^q(\Omega)$ such that

$$F(u) = \int_\Omega v_1(x)u(x)\ dx \quad \text{and} \quad F(u) = \int_\Omega v_2(x)u(x)\ dx, \quad \text{for every } u \in L^p(\Omega).$$

It follows that

$$\int_\Omega \big(v_1(x) - v_2(x)\big)u(x)\ dx = 0, \quad \text{for every } u \in L^p(\Omega)$$

and furthermore, $v_1(x) = v_2(x)$ for almost every $x \in \Omega$. Thus the function v is unique.

Suppose then that $m(\Omega) = \infty$. Let $i \in \mathbb{N} \setminus \{0\}$ and denote $\Omega_i = B(0, i) \cap \Omega$. For every i we have $m(\Omega_i) < \infty$, $\Omega_1 \subset \Omega_2 \subset \cdots$ and

$$\bigcup_{i=1}^\infty \Omega_i = \Omega.$$

Define functions $F_i : L^p(\Omega_i) \to \mathbb{R}$ as $F_i(u) = F(\tilde{u})$, where

$$\tilde{u}(x) = \begin{cases} u(x), & \text{if } x \in \Omega_i \ ; \\ 0, & \text{else.} \end{cases}$$

Since $u \in L^p(\Omega_i)$, it follows that $\tilde{u} \in L^p(\Omega)$. Let $u \in L^p(\Omega_i)$ such that $||u||_{L^p(\Omega_i)} \leq 1$. Then

$$F_i(u) = F(\tilde{u}) \leq ||\tilde{u}||_p ||F||_{L^p(\Omega)^*} \leq ||F||_{L^p(\Omega)^*}.$$

Hence $F_i \in L^p(\Omega_i)^*$ and $||F_i||_{L^p(\Omega_i)^*} \leq ||F||_{L^p(\Omega)^*}$.

Fix $i \in \mathbb{N} \setminus \{0\}$. There exists a unique $v_i \in L^q(\Omega_i)$ such that

$$F_i(u) = \int_{\Omega_i} v_i(x)u(x)\ dx, \qquad \text{for all } u \in L^p(\Omega_i)$$

and $||F_i||_{L^p(\Omega)^*} = ||v_i||_{L^q(\Omega_i)}$. Let $i > j$. By the uniqueness of v_j, we have $v_i(x) = v_j(x)$ for all $x \in \Omega_j \subset \Omega_i$. Thus there is v such that

$$v(x) = \lim_{i\to\infty} v_i(x), \qquad \text{for almost every } x \in \Omega.$$

By the monotone convergence theorem

$$\int_\Omega |v(x)|^q\ dx = \lim_{i\to\infty} \int_\Omega |v_i(x)|^q\ dx = \lim_{i\to\infty} ||F_i||^q_{L^p(\Omega)^*} \leq ||F||^q_{L^p(\Omega)^*}.$$

Hence $v \in L^q(\Omega)$ and $||v||_q \leq ||F||_{L^p(\Omega)^*}$. By the dominated convergence theorem

$$||v_i - v||_q \to 0 \quad \text{as} \quad i \to \infty.$$

Let $u \in L^p(\Omega)$ and denote $u_i = u\chi_{\Omega_i}$. Again by the dominated convergence theorem

$$||u_i - u||_p \to 0 \quad \text{as} \quad i \to \infty.$$

Finally

$$\begin{aligned} &\int_\Omega |v_i(x)u_i(x) - v(x)u(x)|\ dx \\ &\qquad \leq \int_\Omega |v_i(x)u_i(x) - v_i(x)u(x)| + |v_i(x)u(x) - v(x)u(x)|\ dx \\ &\qquad \leq \int_\Omega ||v_i||_q||u_i - u||_p + ||v_i - v||_q||u||_p\ dx \\ &\qquad \to 0 \quad \text{as } i \to \infty \end{aligned}$$

and therefore

$$F(u) = \lim_{i\to\infty} F(u_i) = \lim_{i\to\infty} F_i(u_i) = \lim_{i\to\infty} \int_\Omega v_i(x)u_i(x)\ dx = \int_\Omega v(x)u(x)\ dx.$$

The proof is complete. □

Remark 3.4. As a consequence of Theorem 3.21, a sequence (u_j) in $L^p(\Omega)$, $1 < p < \infty$, converges weakly to $u \in L^p(\Omega)$ if

$$\lim_{j\to\infty} \int_\Omega u_j(x)v(x)\ dx = \int_\Omega u(x)v(x)\ dx \qquad \text{for all } v \in L^q(\Omega).$$

3.3. **Weak convergence in reflexive Banach space.** Let V be a normed space. Define a mapping $J : V \to V^{**}$ as follows: for $u \in V$, $J(u) \in V^{**}$ is defined as

$$\langle J(u), u^* \rangle = \langle u, u^* \rangle, \qquad \text{for every } u^* \in V^*.$$

Space V is said to be *reflexive* if the mapping J is an isometric isomorphism. Theorem 3.21 states that $L^p(\Omega)$, $1 < p < \infty$, is reflexive.

THEOREM 3.22. *Let V be a reflexive Banach space and (u_j) a bounded sequence of V, that is, $u_j \in V$ for all j and there exists a real number $M > 0$ such that*

$$||u_j||_V \leq M \qquad \textit{for all } j \in \mathbb{N}.$$

Then there exists a subsequence $(u_{j_k}) \subset (u_j)$ that converges weakly in V.

Proof. We will only prove the theorem for the case $V = L^p(\Omega)$, where $1 < p < \infty$. Let $p \in (1, \infty)$ and suppose (u_j) is a bounded sequence of $L^p(\Omega)$. We may assume that

$$||u_j||_p \leq 1 \qquad \text{for all } j \in \mathbb{N}.$$

Let U be the closure of the linear span of $\{u_1, u_2, ...\}$. Now, U is a closed linear subspace of the reflexive space $L^p(\Omega)$, from which follows that U is reflexive [7, p. 192]. The set

$$\Big\{ u : u = \sum_{i=1}^{k} q_i u_i, \quad q_i \in \mathbb{Q}, \ k \in \mathbb{N} \Big\}$$

is numerable and dense in U. Hence U is separable. From the reflexivity of U, it follows that $U = U^{**}$ and thus U^{**} is separable. Since U^{**} is the dual of U^*, we have that U^* is also separable [9, p. 126].

Let $\{\varphi_1, \varphi_2, ...\}$ be a countable dense set in U^*. Fix $k \in \mathbb{N}$. By the analytic form of Hahn-Banach Theorem [9, p. 106] there exists $\psi_k \in L^p(\Omega)^*$ such that

$$\langle u, \varphi_k \rangle = \langle u, \psi_k \rangle \qquad \text{for every } u \in U.$$

Hence we may assume $\varphi_k \in L^p(\Omega)^* = L^q(\Omega)$, where q is the conjugate exponent of p. Define for every $j, k \in \mathbb{N}$

$$L_j(\varphi_k) = \langle u_j, \varphi_k \rangle = \int_\Omega u_j(x) \varphi_k(x) \, dx.$$

By Hölder's inequality, we have

$$|L_j(\varphi_k)| \leq ||u_j||_p ||\varphi_k||_q \leq ||\varphi_k||_q, \qquad \text{for every } j, k \in \mathbb{N}. \tag{14}$$

From (14) we have that the sequence $\big(L_j(\varphi_1)\big)$ is bounded in $\mathbb{R}$. Hence there exists a converging subsequence $\big(L_j^{(1)}(\varphi_1)\big)$ with a corresponding subsequence $(u_j^{(1)}) \subset (u_j)$. Consequently, with (14) we see that the sequence $\big(L_j^{(1)}(\varphi_2)\big)$ is bounded in $\mathbb{R}$ and therefore has a converging subsequence $\big(L_j^{(2)}(\varphi_2)\big)$ with a corresponding subsequence $(u_j^{(2)}) \subset (u_j^{(1)})$. Continuing the process, we achieve for every $m \in \mathbb{N}$ sequences $\big(L_j^{(m)}\big)$ and $(u_j^{(m)})$, for which the limit

$$\lim_{j \to \infty} L_j^{(m)}(\varphi_k) = \lim_{j \to \infty} \langle u_j^{(m)}, \varphi_k \rangle$$

exists for every $k \leq m$. Moreover, for diagonal sequences $\big(L_j^{(j)}\big)$ and $(u_j^{(j)})$, the limit

$$\lim_{j \to \infty} L_j^{(j)}(\varphi_k) = \lim_{j \to \infty} \langle u_j^{(j)}, \varphi_k \rangle$$

exists for every element φ_k of the countable dense set $\{\varphi_1, \varphi_2, ...\}$.

Let $\varphi \in U^*$ and $\varepsilon > 0$. The set $\{\varphi_1, \varphi_2, ...\}$ is dense in U^* and therefore there exists a function φ_k such that

$$||\varphi - \varphi_k||_q \leq \varepsilon.$$

The sequence $\left(L_j^{(j)}(\varphi_k)\right)$ is a Cauchy sequence and hence there is an index J_ε such that

$$|L_i^{(i)}(\varphi_k) - L_j^{(j)}(\varphi_k)| < \varepsilon \quad \text{whenever } i, j \geq J_\varepsilon.$$

Now

$$\begin{aligned}
|L_i^{(i)}(\varphi) - L_j^{(j)}(\varphi)| &\leq |L_i^{(i)}(\varphi) - L_i^{(i)}(\varphi_k)| \\
&\quad + |L_i^{(i)}(\varphi_k) - L_j^{(j)}(\varphi_k)| + |L_j^{(j)}(\varphi_k) - L_j^{(j)}(\varphi)| \\
&\leq ||\varphi - \varphi_k||_q ||u_i^{(i)}||_p + \varepsilon + ||\varphi - \varphi_k||_q ||u_j^{(j)}||_p \\
&\leq 3\varepsilon, \quad \text{whenever } i, j \geq J_\varepsilon.
\end{aligned}$$

It follows that the sequence $\left(L_j^{(j)}(\varphi)\right)$ is a Cauchy sequence in $\mathbb{R}$ and hence the limit

$$\lim_{j\to\infty} L_j^{(j)}(\varphi) = \lim_{j\to\infty} \langle u_j^{(j)}, \varphi \rangle \tag{15}$$

exists for every $\varphi \in U^*$.

Define for every $j \in \mathbb{N}$ a function $v_j : U^* \to \mathbb{R}$ by setting

$$v_j(\varphi) := \langle u_j^{(j)}, \varphi \rangle$$

and a function $v : U^* \to \mathbb{R}$ by

$$v(\varphi) := \lim_{j\to\infty} v_j(\varphi).$$

We claim that $v \in U^{**}$. Indeed, v_j is a linear function for every $j \in \mathbb{N}$. Thus v is a linear function. Since the limit (15) exists for every $\varphi \in U^*$, we have

$$\sup_j |v_j(\varphi)| < \infty \quad \text{for all } \varphi \in U^*.$$

By Banach-Steinhaus Theorem [7, p. 203]

$$\sup_j ||v_j|| < \infty,$$

where $|| \cdot ||$ is the operator norm. This implies that v is bounded and the claim is true.

Now, U is reflexive and hence there exists an element $u \in U$ such that

$$v(\varphi) = \varphi(u) \quad \text{for every } \varphi \in U^*.$$

It follows that

$$\lim_{j\to\infty} \langle u_j^{(j)}, \varphi \rangle = \langle u, \varphi \rangle \quad \text{for every } \varphi \in U^*.$$

Define $T : U \to L^p(\Omega)$, $T(u) = u$. Let $\psi \in L^p(\Omega)^*$. It follows that $\psi \circ T \in U^*$. We have

$$\langle u_j^{(j)}, \psi \rangle = \langle u_j^{(j)}, \psi \circ T \rangle \to \langle u, \psi \circ T \rangle = \langle u, \psi \rangle$$

as $j \to \infty$. Thus $u_j^{(j)} \rightharpoonup u$ in $L^p(\Omega)$ and the proof is done. $\square$

4. Duality in convex optimization

4.1. **The primal problem and the dual problem.** Let $F : V \to \mathbb{R}$ be a function. We consider the minimization problem

$$(\mathscr{P}) \qquad \inf_{u \in V} F(u).$$

This problem will be termed as the *primal problem* and we refer to it as $\mathscr{P}$. We denote the infimum by $\inf \mathscr{P}$ and any element u of V such that $F(u) = \inf \mathscr{P}$ will be called a solution of $\mathscr{P}$.

Definition 4.1. The problem $\mathscr{P}$ is said to be *non-trivial* if there exists $u_0 \in V$ such that $F(u_0) < \infty$.

Suppose we are given a normed space Y with dual Y^* and a function $\Phi : V \times Y \to \bar{\mathbb{R}}$ such that

$$\Phi(u, 0) = F(u).$$

For every $p \in Y$

$$(\mathscr{P}_p) \quad \inf_{u \in V} \Phi(u, p)$$

is said to be the *perturbed problem* of $\mathscr{P}$ with respect to the given perturbations.

Let $\Phi^* : V^* \times Y^* \to \bar{\mathbb{R}}$ be the conjugate function of Φ. Then $\Phi^* \in \Gamma(V^* \times Y^*)$. The problem

$$(\mathscr{P}^*) \quad \sup_{p^* \in Y^*} \{-\Phi^*(0, p^*)\}$$

is said to be the *dual problem* of $\mathscr{P}$ with respect to the given perturbations. The supremum of $\mathscr{P}^*$ is denoted by $\sup \mathscr{P}^*$ and any element p^* of Y^* such that $-\Phi^*(0, p^*) = \sup \mathscr{P}^*$ is termed a solution of $\mathscr{P}^*$. The problem $\mathscr{P}^*$ is said to be non-trivial if there exists $p^* \in Y^*$ such that $-\Phi^*(0, p^*) > -\infty$.

Definition 4.2. Let $L : V \to Y$ be a continuous linear operator. The function $L^* : Y^* \to V^*$ is said to be the *transpose* of L if

$$\langle v, L^* y^* \rangle = \langle Lv, y^* \rangle, \qquad \text{for every } v \in V \text{ and } y^* \in Y^*.$$

Example 4.3. Let $\Lambda : V \to Y$ be a continuous linear mapping with transpose $\Lambda^* \in \mathscr{L}(Y^*, V^*)$. Let $L : V \to \bar{\mathbb{R}}$ be of the following form

$$L(u) = F(u) + G(\Lambda u),$$

where $F : V \to \mathbb{R}$ and $G : Y \to \mathbb{R}$. Our primal problem $\mathscr{P}$ is of the form

$$\inf_{u \in V} [F(u) + G(\Lambda u)].$$

We perturb our problem by defining the function Φ in the following way:

$$\Phi(u, p) = F(u) + G(\Lambda u - p).$$

For the conjugate function of Φ, we have

$$\begin{aligned}
\Phi^*(u^*, p^*) &= \sup_{\substack{u \in V \\ p \in Y}} \big[\langle u^*, u \rangle + \langle p^*, p \rangle - \Phi(u, p)\big] \\
&= \sup_{u \in V} \sup_{p \in Y} \big[\langle u^*, u \rangle + \langle p^*, p \rangle - F(u) - G(\Lambda u - p)\big].
\end{aligned}$$

For a fixed u, we set $q = \Lambda u - p$. Then

$$\begin{aligned}
\Phi^*(0,p^*) &= \sup_{u\in V}\sup_{q\in Y}\big[\langle p^*, \Lambda u - q\rangle - F(u) - G(q)\big] \\
&= \sup_{u\in V}\sup_{q\in Y}\big[\langle \Lambda^* p^*, u\rangle - F(u) + \langle -p^*, q\rangle - G(q)\big] \\
&= \sup_{u\in V}\Big[\langle \Lambda^* p^*, u\rangle - F(u) + \sup_{q\in Y}\big[\langle -p^*, q\rangle - G(q)\big]\Big] \\
&= \sup_{u\in V}\Big[\langle \Lambda^* p^*, u\rangle - F(u) + G^*(-p^*)\Big] \\
&= F^*(\Lambda^* p^*) + G^*(-p^*)
\end{aligned}$$

where F^* is the conjugate function of F and G^* is the conjugate function of G. Finally, our dual problem of $\mathscr{P}$ with respect to given perturbations takes the form

$$\sup_{p^*\in Y^*}\big[-F^*(\Lambda^* p^*) - G^*(-p^*)\big].$$

4.2. **Relationship between the primal problem and its dual problem.**

Proposition 4.4.

$$-\infty \le \sup \mathscr{P}^* \le \inf \mathscr{P} \le \infty.$$

Proof. Let $p^* \in Y^*$. By definition

$$\begin{aligned}
\Phi^*(0,p^*) &= \sup_{\substack{u\in V\\ p\in Y}}\big[\langle p, p^*\rangle - \Phi(u,p)\big] \\
&\ge \langle 0, p^*\rangle - \Phi(u,0) \\
&= -\Phi(u,0)
\end{aligned}$$

for all $u \in V$. Therefore for all $u \in V$ and $p^* \in Y^*$

$$-\Phi^*(0,p^*) \le \Phi(u,0)$$

and in particular $\sup \mathscr{P}^* \le \inf \mathscr{P}$. ☐

Remark 4.1. If $\mathscr{P}$ and $\mathscr{P}^*$ are non-trivial, then

$$-\infty < \sup \mathscr{P}^* \le \inf \mathscr{P} < \infty.$$

4.2.1. *Primal Problem - Existence of Solution.* In this section we let V be a reflexive Banach space, $A \subset V$ a non-empty closed convex set and $F : A \to \mathbb{R}$ convex, lower semi-continuous and proper function. Our primal problem is the minimization problem

$$(\mathscr{P}) \qquad \inf_{u\in A} F(u). \tag{16}$$

Problem (16) is identical with the problem

$$(\mathscr{P}^*) \qquad \inf_{u\in V} \tilde{F}(u),$$

where

$$\tilde{F}(u) = \begin{cases} F(u) & \text{if } u \in A; \\ +\infty & \text{if } u \notin A. \end{cases}$$

THEOREM 4.5. *The solutions of* (16) *is a closed convex set contained in A and possibly empty.*

Proof. We denote $m = \inf \mathscr{P}$. Apart from the trivial cases where $m = +\infty$ or $m = -\infty$, the set of solutions is the sublevel set

$$E_m = \{u \in V : \tilde{F}(u) \leq m\}.$$

$\tilde{F}$ is convex and lower semi-continuous. By Propositions 2.10 and 2.15, we have that E_m is convex and closed. □

Definition 4.6. A function $F : A \to \mathbb{R}$ is *coercive* over A if

$$\lim_{||u|| \to \infty} F(u) = +\infty \quad \text{for } u \in A.$$

THEOREM 4.7. *Let V be a reflexive Banach space, $A \subset V$ a non-empty closed convex set and $F : A \to \mathbb{R}$ a proper convex lower semi-continuous function.*

(17) *If the set A is bounded*

or

(18) *if F is coercive over A,*

then the minimization problem (16) *has at least one solution. Moreover, the solution is unique if F is strictly convex in A.*

Proof. Let (u_j) be a minimizing sequence of problem (16), that is, $u_j \in A$ for all $j \in \mathbb{N}$ and

$$\lim_{j \to \infty} F(u_j) = \inf_{v \in A} F(v) := m.$$

Since F is proper there exists $u_0 \in A$ such that $F(u_0) < +\infty$. Therefore the problem (16) is non-trivial and $m < +\infty$.

We claim that (u_j) is a bounded sequence in V. Indeed, if A is bounded, there exists a real number $M > 0$ such that $||u_j||_V \leq M$ for all j. Let F be coercive over A and suppose that the sequence (u_j) is unbounded. There is a subsequence $(u_{j_k}) \subset (u_j)$ such that

$$||u_{j_k}||_V \to \infty, \quad \text{as } j_k \to \infty.$$

Since F is coercive

$$\lim_{j \to \infty} F(u_j) = \lim_{j_k \to \infty} F(u_{j_k}) = +\infty.$$

This is a contradiction and therefore (u_j) has to be bounded. This proves the claim.

V is a reflexive Banach space and (u_j) is a bounded sequence in V. Hence by Theorem 3.22 there exists a subsequence $(u_{j_n}) \subset (u_j)$ in A, which converges weakly to an element $u \in V$. By Proposition 3.6 A is weakly closed. Therefore $u \in A$.

Since F is convex lower semi-continuous on A, by Proposition 3.9 it is weakly lower semi-continuous in A. We have

$$F(u) \leq \varliminf_{j_n \to \infty} F(u_{j_n}) = m. \tag{19}$$

This implies that u is a solution of (16). Moreover, F is proper convex function and therefore $-\infty < F(u) \leq m$.

Suppose F is strictly convex in A. We prove that the solution is unique. If u_1 and u_2 are two different solutions of the minimization problem (16), then by Theorem 4.5 their convex combination $\frac{1}{2}(u_1 + u_2)$ is also a solution. It follows that

$$F(\frac{u_1 + u_2}{2}) < \frac{1}{2}F(u_1) + \frac{1}{2}F(u_2) = m.$$

This is a contradiction. Thus the solution is unique. □

4.3. **Stability criterion.** Consider the primal problem

$$(\mathscr{P}) \qquad \inf_{u\in V} F(u) = \inf_{u\in V}\{\Phi(u,0)\}$$

and its dual problem with respect to given perturbations

$$(\mathscr{P}^*) \qquad \sup_{p^*\in Y^*}\{-\Phi^*(0,p^*)\}.$$

In addition, in the rest of the section we assume in general that

$$\Phi \in \Gamma_0(V\times Y).$$

For $p\in Y$ set

$$h(p) = \inf \mathscr{P}_p = \inf_{u\in V}\Phi(u,p). \tag{20}$$

Notice that $h(0) = \inf \mathscr{P}$. We consider the conjugate function of h:

$$\begin{aligned} h^*(p^*) &= \sup_{p\in Y}[\langle p^*,p\rangle - h(p)] \\ &= \sup_{p\in Y}[\langle p^*,p\rangle - \inf_{u\in V}\Phi(u,p)] \\ &= \sup_{p\in Y}[\langle p^*,p\rangle + \sup_{u\in V}\{-\Phi(u,p)\}] \\ &= \sup_{p\in Y}\sup_{u\in V}[\langle p^*,p\rangle - \Phi(u,p)] = \Phi^*(0,p^*). \end{aligned}$$

Therefore

$$\sup \mathscr{P}^* = \sup_{p^*\in Y^*}\{-\Phi^*(0,p^*)\} = \sup_{p^*\in Y^*}\{-h^*(p^*)\} = h^{**}(0).$$

Definition 4.8. Problem $\mathscr{P}$ is said to be *stable* if $h(0)$ is finite and h is subdifferentiable at 0.

The stability of problem $\mathscr{P}^*$ is defined in the same manner, just replacing $\mathscr{P}$ by $\mathscr{P}^*$ and h by h^{**} in the definition. In general, a stable problem has the following property: if we perturb our problem only a bit, then the solutions of the perturbed problem should not differ too much from the original problem.

THEOREM 4.9. *Let $\Phi\in\Gamma_0(V\times Y)$. If $\mathscr{P}$ is stable and has a solution, then $\mathscr{P}^*$ has at least one solution and*

$$-\infty < \inf\mathscr{P} = \sup\mathscr{P}^* < \infty.$$

Proof. Since $\mathscr{P}$ is stable, $h(0)$ is finite and $\partial h(0)\neq\emptyset$. By Proposition 2.28, we have that $h^{**}(0) = h(0)\in\mathbb{R}$, that is

$$-\infty < \inf\mathscr{P} = h(0) = h^{**}(0) = \sup\mathscr{P}^* < \infty.$$

In addition, $\partial h^{**}(0) = \partial h(0) \neq \emptyset$. By Lemma 4.10 below, $\mathscr{P}^*$ has at least one solution. □

Lemma 4.10. *The set of solutions of $\mathscr{P}^*$ is identical to $\partial h^{**}(0)$.*

Proof. $p^* \in Y^*$ is a solution of $\mathscr{P}^*$ if and only if

$$-\Phi^*(0, p^*) = \sup_{q^* \in Y^*} \{-\Phi^*(0, q^*)\}.$$

This can be written as

$$-h^*(p^*) = \sup_{q^* \in Y^*} \{-h^*(q^*)\} = \sup_{q^* \in Y^*} \{\langle 0, q^* \rangle - h^*(q^*)\} = h^{**}(0).$$

By Proposition 2.29

$$h^{**}(0) + h^*(p^*) = 0 \quad \text{if and only if} \quad p^* \in \partial h^{**}(0).$$

□

Lemma 4.11. *Let Φ be convex. Then the function $h : Y \to \bar{\mathbb{R}}$ defined as in (20) is convex.*

Proof. Let $p, q \in Y$ and $\lambda \in [0, 1]$. We need to show

$$h(\lambda p + (1 - \lambda)q) \leq \lambda h(p) + (1 - \lambda)h(q),$$

whenever the right-hand side is defined. If $h(p) = \infty$ or $h(q) = \infty$, the inequality is obvious. We can therefore assume that $h(p) < \infty$ and $h(q) < \infty$. For every $a > h(p)$ and $b > h(q)$ there exists $u, v \in V$ such that

$$h(p) \leq \Phi(u, p) \leq a$$
$$h(q) \leq \Phi(v, q) \leq b.$$

It follows that

$$\begin{aligned} h(\lambda p + (1 - \lambda)q) &= \inf_{w \in V} \Phi(w, \lambda p + (1 - \lambda)q) \\ &\leq \Phi(\lambda u + (1 - \lambda)v, \lambda p + (1 - \lambda)q) \\ &\leq \lambda \Phi(u, p) + (1 - \lambda)\Phi(v, q) \\ &\leq \lambda a + (1 - \lambda)b. \end{aligned}$$

The above inequality holds for all $a > h(p)$ and $b > h(q)$. Hence we have

$$h(\lambda p + (1 - \lambda)q) \leq \lambda h(p) + (1 - \lambda)h(q).$$

This proves the Lemma. □

THEOREM 4.12 (A stability criterion)**.** *Let us assume that Φ is convex,* $\inf \mathscr{P}$ *is finite and there exists $u_0 \in V$ such that $p \mapsto \Phi(u_0, p)$ is finite and continuous at $0 \in Y$. Then $\mathscr{P}$ is stable.*

Proof. By assumption $h(0) = \inf \mathscr{P}$ is finite. The function $p \mapsto \Phi(u_0, p)$ is convex and continuous at $0 \in Y$. Therefore there exists a neighborhood $U \subset Y$ of 0, on which the function is bounded above, that is

$$\Phi(u_0, p) \leq M < \infty \quad \text{for all } p \in U.$$

By Lemma 4.11 h is convex. In addition, we have

$$h(p) = \inf_{u \in V} \Phi(u, p) \leq \Phi(u_0, p) \leq M \quad \text{for all } p \in U$$

and hence by Lemma 2.19, h is continuous at 0. Proposition 2.30 then implies that h is subdifferentiable at 0. Thus $\mathscr{P}$ is stable. □

4.4. **Extremality relation.**

THEOREM 4.13. *If $\mathscr{P}$ and $\mathscr{P}^*$ have solutions and*

$$-\infty < \inf \mathscr{P} = \sup \mathscr{P}^* < \infty,$$

then all solutions $\bar{u}$ of $\mathscr{P}$ and $\bar{p}^*$ of $\mathscr{P}^*$ are linked by the extremality relation

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = 0. \tag{21}$$

Conversely if $\bar{u} \in V$ and $\bar{p}^* \in Y^*$ satisfy the extremality relation (21)*, then $\bar{u}$ is a solution of $\mathscr{P}$ and $\bar{p}^*$ is a solution of $\mathscr{P}^*$ and*

$$-\infty < \inf \mathscr{P} = \sup \mathscr{P}^* < \infty.$$

Proof. If $\bar{u}$ is a solution of $\mathscr{P}$ and $\bar{p}^*$ is a solution of $\mathscr{P}^*$ then by our assumption

$$\Phi(\bar{u}, 0) = \inf \mathscr{P} = \sup \mathscr{P}^* = -\Phi^*(0, \bar{p}^*).$$

Therefore

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = 0.$$

Conversely, suppose that $\bar{u}$ and $\bar{p}^*$ satisfy (21). By Proposition 4.4

$$-\Phi^*(0, p^*) \le \sup \mathscr{P}^* \le \inf \mathscr{P} \le \Phi(u, 0), \qquad \text{for all } u \in V \text{ and } p^* \in Y^*.$$

Since the pair $(\bar{u}, \bar{p}^*)$ satisfies (21), we have

$$\Phi(\bar{u}, 0) = \inf_{u \in V} \Phi(u, 0) = \inf \mathscr{P},$$

$$-\Phi^*(0, \bar{p}^*) = \sup_{p^* \in Y^*} \{-\Phi^*(0, p^*)\} = \sup \mathscr{P}^*$$

and

$$-\infty < \sup \mathscr{P}^* = \inf \mathscr{P} < \infty.$$

□

Remark 4.2. The extremality relation (21) can be written as

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = \langle \bar{u}, 0 \rangle + \langle 0, \bar{p}^* \rangle = \langle (\bar{u}, 0), (0, \bar{p}^*) \rangle.$$

By Proposition 2.29 this is the same as $(0, \bar{p}^*) \in \partial\Phi(\bar{u}, 0)$.

In summary, we have the following result.

THEOREM 4.14. *Assume that V is a reflexive Banach space, $\Phi \in \Gamma_0(V \times Y)$, there exists $u_0 \in V$ such that $p \mapsto \Phi(u_0, p)$ is finite and continuous at $0 \in Y$ and that $u \mapsto \Phi(u, 0)$ is coercive over V. Under these conditions, $\mathscr{P}$ and $\mathscr{P}^*$ each have at least one solution,*

$$\inf \mathscr{P} = \sup \mathscr{P}^*$$

and extremality relation (21) *is satisfied.*

Proof. The existence of solution of $\mathscr{P}$ follows from Theorem 4.7. The stability criterion implies that $\mathscr{P}$ is stable and therefore by Theorem 4.9 we have that $\mathscr{P}^*$ has at least one solution and

$$-\infty < \inf \mathscr{P} = \sup \mathscr{P}^* < \infty.$$

The extremality relation follows from Theorem 4.13. □

5. Sobolev spaces

Definition 5.1. Let $\Omega \subset \mathbb{R}^n$. We say that the boundary of Ω, denoted by $\partial\Omega$, is $\mathscr{C}^k$ if for each point $u_0 \in \partial\Omega$ there exists $r > 0$ and a function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, $\gamma \in C^k(\mathbb{R}^{n-1})$ such that

$$\Omega \cap B(u_0, r) = \{u \in B(u_0, r) : u_n > \gamma(u_1, u_2, \cdots, u_{n-1})\}.$$

Definition 5.2. We say that a domain Ω is of class $\mathscr{C}^k$, if its boundary $\partial\Omega$ is $\mathscr{C}^k$.

In this section, we let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, that is, Ω is $\mathscr{C}^k$ for some $k \geq 1$.

5.1. Regularization.

Let ρ be a non-negative smooth function on $\mathbb{R}^n$ vanishing outside the unit ball and satisfying

$$\int_{\mathbb{R}^n} \rho(x)\ dx = 1.$$

The function ρ is called a *mollifier*.

Definition 5.3. Let $u \in L^1_{\text{loc}}(\Omega)$ and $h > 0$. The *regularization* of u is the convolution

$$u_h(x) = h^{-n} \int_\Omega \rho\Big(\frac{x-y}{h}\Big) u(y)\ dy,$$

provided $h < \text{dist}(x, \partial\Omega)$.

Lemma 5.4. *Let $\Omega' \subset\subset \Omega$ and $h < dist(\Omega', \partial\Omega)$. If $u \in L^1_{loc}(\Omega)$, then $u_h \in C^\infty(\Omega')$.*

Proof. Fix $i \in \{1, ..., n\}$ and denote

$$\rho_h(x) = h^{-n}\rho(\frac{x}{h}).$$

Let $u \in L^1_{\text{loc}}(\Omega)$ and $x \in \Omega'$. By the Fundamental Theorem of Calculus and Fubini's Theorem, we have

$$\begin{aligned}
D_i u_h(x) &= \lim_{k\to 0^+} \frac{u_h(x+ke_i) - u_h(x)}{k} \\
&= \lim_{k\to 0^+} \frac{1}{k} \int_\Omega u(y)\big[\rho_h(x-y+ke_i) - \rho_h(x-y)\big]\ dy \\
&= \lim_{k\to 0^+} \frac{1}{k} \int_\Omega u(y) \int_0^k D_i\rho_h(x-y+te_i)\ dt\ dy \\
&= \lim_{k\to 0^+} \frac{1}{k} \int_0^k \int_\Omega u(y) D_i\rho_h(x-y+te_i)\ dy\ dt \\
&= \lim_{k\to 0^+} \frac{1}{k} \int_0^k g(t)\ dt.
\end{aligned}$$

Now if $0 \leq t < \frac{1}{2}\ \text{dist}(x, \partial\Omega')$, then $x + te_i \in \Omega'$ and

$$B(x,h) \cup B(x+te_i, h) \subset \{x :\ \text{dist}(x, \Omega') < h\} =: \Omega''.$$

Therefore, with a small enough k

$$\begin{aligned}|g(t)-g(0)| &\leq \int_{\Omega} |u(y)||D_i\rho_h(x-y+te_i)-D_i\rho_h(x-y)|\ dy \\ &= \int_{\Omega''} |u(y)||D_i\rho_h(x-y+te_i)-D_i\rho_h(x-y)|\ dy \\ &\leq \int_{\Omega''} |u(y)| \max_{\Omega''}\{|D_i(D_i\rho_h)|\}|t|\ dy \\ &\leq M|t|||u||_{L^1(\Omega'')},\end{aligned}$$

from which follows that

$$\begin{aligned}\Big|\frac{1}{k}\int_0^k g(t)\ dt - g(0)\Big| &\leq \frac{1}{k}\int_0^k |g(t)-g(0)|\ dt \\ &\leq M\frac{1}{k}||u||_{L^1(\Omega'')}\int_0^k |t|\ dt \to 0, \quad \text{as } k\to 0^+.\end{aligned}$$

In conclusion

$$D_iu_h(x) = g(0) = \int_{\Omega'} u(y)D_i\rho_h(x-y)\ dy = \big(u\ *\ D_i\rho_h\big)(x) \quad \text{for all } i\in\{1,...,n\}.$$

□

Lemma 5.5. *If $u \in C(\Omega)$, then $u_h \to u$ uniformly on any domain $\Omega' \subset\subset \Omega$.*

Proof. We have

$$\begin{aligned}u_h(x) &= h^{-n}\int_{|x-y|\leq h} \rho\Big(\frac{x-y}{h}\Big)u(y)\ dy \\ &= \int_{|z|\leq 1} \rho(z)u(x-zh)\ dz.\end{aligned}$$

Now if $\Omega' \subset\subset \Omega$ and $h < \frac{1}{2}\text{dist}(\Omega', \partial\Omega)$

$$\begin{aligned}|u(x)-u_h(x)| &= \left|u(x) - \int_{|z|\leq 1} \rho(z)u(x-zh)\ dz\right| \\ &= \left|\int_{|z|\leq 1} \rho(z)u(x)\ dz - \int_{|z|\leq 1} \rho(z)u(x-zh)\ dz\right| \\ &\leq \int_{|z|\leq 1} \rho(z)|u(x)-u(x-zh)|\ dz \\ &\leq \sup_{|z|\leq 1} |u(x)-u(x-zh)|.\end{aligned}$$

Since u is continuous over Ω, it is uniformly continuous over any compact subset of Ω. In particular u is uniformly continuous over the set

$$\Omega'' := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega') \leq h\}.$$

Fix $\varepsilon > 0$, there exists a positive real number δ, depending only on ε, such that

$$\sup_{|z|\leq 1} |u(x)-u(x-zh)| < \varepsilon \quad \text{for all } x \in \Omega' \text{ when } h < \delta$$

and hence

$$\sup_{\Omega'} |u - u_h| \leq \sup_{\Omega'} \sup_{|z| \leq 1} |u(x) - u(x - zh)| \to 0 \quad \text{as } h \to 0.$$

□

Definition 5.6. Let $u : \Omega \to \mathbb{R}$ be a function. Denote

$$\operatorname{spt}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We say that $\operatorname{spt}(u)$ is the *support* of u.

Definition 5.7. We denote

$$C_0(\Omega) = \{u \in C(\Omega) : \operatorname{spt}(u) \subset \Omega \text{ and } \operatorname{spt}(u) \text{ is compact}\}$$

and

$$C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega).$$

Lemma 5.8. *Let $u \in L^p(\Omega)$, $1 \leq p < \infty$, and $\varepsilon > 0$. Then there exists a function $v \in C_0(\mathbb{R}^n)$ such that*

$$||u - v||_p < \varepsilon.$$

Proof. Let $u \in L^p(\Omega)$. We extend u to be a function of $L^p(\mathbb{R}^n)$ by setting $u(x) = 0$ for every $x \in \mathbb{R}^n \setminus \Omega$. Let $\varepsilon > 0$. Since simple functions are dense in $L^p(\mathbb{R}^n)$, there exists a simple function $w : \mathbb{R}^n \to \mathbb{R}$, constants $c_i \in \mathbb{R} \setminus \{0\}$ and sets C_i, $i \in \{1, ..., l\}$, such that

$$||u - w||_p \leq \varepsilon \quad \text{and} \quad w(x) = \sum_{i=1}^{l} c_i \chi_{C_i}(x) \quad \text{for every } x \in \mathbb{R}^n.$$

Since Ω is bounded, it follows that C_i are bounded for every $i \in \{1, ..., l\}$. Therefore for every i there exists a compact set $K_i \subset C_i$ such that

$$m(C_i \setminus K_i) < \Big(\frac{\varepsilon}{|c_i| 2l}\Big)^p.$$

Moreover, for every K_i there exists a function $\varphi_i \in C_0(\mathbb{R}^n)$, $0 \leq \varphi_i \leq 1$, such that

$$||\chi_{K_i} - \varphi_i||_p < \frac{\varepsilon}{|c_i| 2l}.$$

Now

$$
\begin{aligned}
\Big|\Big|u-\sum_{i=1}^{l}c_i\varphi_i\Big|\Big|_p &\le ||u-w||_p+\Big|\Big|w-\sum_{i=1}^{l}c_i\varphi_i\Big|\Big|_p\\
&\le \varepsilon+\sum_{i=1}^{l}|c_i|||\chi_{C_i}-\varphi_i||_p\\
&\le \varepsilon+\sum_{i=1}^{l}|c_i|\Big[||\chi_{C_i}-\chi_{K_i}||_p+||\chi_{K_i}-\varphi_i||_p\Big]\\
&< \varepsilon+\sum_{i=1}^{l}|c_i|\left[\left(\int_{\mathbb{R}^n}|\chi_{C_i}(x)-\chi_{K_i}(x)|^p\ dx\right)^{1/p}+\frac{\varepsilon}{|c_i|2l}\right]\\
&\le \varepsilon+\sum_{i=1}^{l}|c_i|\left[m(C_i\setminus K_i)^{1/p}+\frac{\varepsilon}{|c_i|2l}\right]\\
&< \varepsilon+\sum_{i=1}^{l}|c_i|\left[\frac{\varepsilon}{|c_i|2l}+\frac{\varepsilon}{|c_i|2l}\right]=\varepsilon+\sum_{i=1}^{l}\frac{\varepsilon}{l}=2\varepsilon,
\end{aligned}
$$

where the second inequality follows from Minkowski's inequality, Proposition 3.17. The function $v:=\sum_{i=1}^{l}c_i\varphi_i$ is a continuous function defined on $\mathbb{R}^n$ and hence the claim is true. □

In the following theorem we don't require the boundedness of Ω.

Proposition 5.9. *Let $u\in L^p_{loc}(\Omega)$, $1\le p<\infty$. Then u_h converges to u in $L^p_{loc}(\Omega)$.*

Proof. By Hölder's inequality, we have

$$
\begin{aligned}
u_h(x) &= \int_{|z|\le 1}\rho(z)u(x-zh)\ dz\\
&= \int_{|z|\le 1}\Big(\rho(z)^{1-1/p}\Big)\Big(\rho(z)^{1/p}u(x-zh)\Big)\ dz\\
&\le \left(\int_{|z|\le 1}\rho(z)\ dz\right)^{1/q}\left(\int_{|z|\le 1}\rho(z)|u(x-zh)|^p\ dz\right)^{1/p}\\
&= \left(\int_{|z|\le 1}\rho(z)|u(x-zh)|^p\ dz\right)^{1/p}.
\end{aligned}
$$

Let $\Omega'\subset\subset\Omega$ and $h<\frac{1}{2}\ \mathrm{dist}(\Omega',\partial\Omega)$. Then

$$
\begin{aligned}
\int_{\Omega'}|u_h(x)|^p\ dx &\le \int_{\Omega'}\left(\int_{|z|\le 1}\rho(z)|u(x-zh)|^p\ dz\right)\ dx\\
&= \int_{|z|\le 1}\rho(z)\left(\int_{\Omega'}|u(x-zh)|^p\ dx\right)\ dz\\
&\le \int_{\Omega''}|u(x)|^p\ dx,
\end{aligned}
$$

where $\Omega'' = \{x \in \Omega : \text{dist}(x, \Omega') < h\}$. This implies that

$$||u_h||_{L^p(\Omega')} \leq ||u||_{L^p(\Omega'')}.$$

Fix $\varepsilon > 0$. Now, by Lemma 5.8 there exists a function $v \in C_0(\mathbb{R}^n)$ such that

$$||u - v||_{L^p(\Omega'')} \leq \frac{\varepsilon}{3}.$$

In addition, for a small enough h we have by Lemma 5.5

$$||v - v_h||_{L^p(\Omega')} \leq \frac{\varepsilon}{3}.$$

Therefore

$$\begin{aligned} ||u - u_h||_{L^p(\Omega')} &\leq ||u - v||_{L^p(\Omega')} + ||v - v_h||_{L^p(\Omega')} + ||v_h - u_h||_{L^p(\Omega')} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + ||u - v||_{L^p(\Omega'')} \\ &\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

for small enough h. Since this is true for all $\Omega' \subset\subset \Omega$, it follows that u_h converges to u in $L^p_{\text{loc}}(\Omega)$. □

Remark 5.1. From the previous Proposition we also have the following result: if $u \in L^p(\Omega)$, then u_h converges to u in $L^p(\Omega)$.

5.2. **Sobolev spaces.** Let α be a multi-index, that is $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \in \mathbb{N} \cup \{0\}$. We denote

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

For a function $u : \Omega \to \mathbb{R}$, define

$$D^\alpha u := \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}} u(x).$$

We denote

$$D_i u(x) := \frac{\partial}{\partial x_i} u(x)$$

and

$$\nabla u(x) = \left(\frac{\partial}{\partial x_1} u(x), \frac{\partial}{\partial x_2} u(x), \cdots, \frac{\partial}{\partial x_n} u(x) \right)$$

is the gradient of u at x.

Definition 5.10. Let u be a locally integrable function on Ω. Locally integrable function v is called the α^{th} *weak derivative* of u if it satisfies

$$\int_\Omega \varphi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \, dx \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

We write $v = D^\alpha u$.

Definition 5.11 (Sobolev space)**.** Let $p \geq 1$, k be a non-negative integer and

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}.$$

The space $W^{k,p}(\Omega)$ is called a *Sobolev space* and the formula

$$||u||_{W^{k,p}(\Omega)} = ||u||_{k,p;\Omega} := \sum_{|\alpha| \leq k} ||D^\alpha u||_{L^p(\Omega)}$$

defines a norm on $W^{k,p}(\Omega)$.

Remark 5.2. Let

$$||u||'_{W^{k,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha|\le k} |D^\alpha u|^p \, dx \right)^{1/p}.$$

Then $||\cdot||'_{W^{k,p}(\Omega)}$ is also a norm on $W^{k,p}(\Omega)$. It is easy to prove out that $||\cdot||_{W^{k,p}(\Omega)}$ and $||\cdot||'_{W^{k,p}(\Omega)}$ are equivalent.

Proposition 5.12. *Let $1 \le p \le \infty$ and k be a non-negative integer. Then $W^{k,p}(\Omega)$ is a Banach space.*

Proof. We show that $W^{k,p}(\Omega)$ is complete. Let $\varepsilon > 0$ and (u_j) be a Cauchy sequence in $W^{k,p}(\Omega)$. Fix a multi-index α such that $|\alpha| \le k$. Since (u_j) is a Cauchy sequence there exists an index $J \in \mathbb{N}$ such that

$$||u_j - u_i||_{W^{k,p}(\Omega)} < \varepsilon \quad \text{whenever } i, j \ge J.$$

We have

$$||u_j - u_i||_p \le ||u_j - u_i||_{W^{k,p}(\Omega)} < \varepsilon$$

and

$$||D^\alpha u_j - D^\alpha u_i||_p \le ||u_j - u_i||_{W^{k,p}(\Omega)} < \varepsilon$$

whenever $i, j \ge J$. This means that (u_j) and $\big(D^\alpha u_j\big)$ are Cauchy sequences in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete there exists functions $u \in L^p(\Omega)$ and $u^\alpha \in L^p(\Omega)$ such that

$$||u_j - u||_p \to 0 \quad \text{and} \quad ||D^\alpha u_j - u^\alpha||_p \to 0 \quad \text{as } j \to \infty.$$

Next we show that $u \in W^{k,p}(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$. Since $u_j \in W^{k,p}(\Omega)$ for every j, we have

$$\int_\Omega u_j D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u_j \varphi \, dx.$$

By Hölder's inequality we obtain

$$\int_\Omega (u_j - u) D^\alpha \varphi \, dx \le \int_\Omega |u_j - u||D^\alpha \varphi| \, dx \le ||u_j - u||_p ||D^\alpha \varphi||_q$$

and

$$\int_\Omega (D^\alpha u_j - u^\alpha)\varphi \, dx \le \int_\Omega |D^\alpha u_j - u^\alpha||\varphi| \, dx \le ||D^\alpha u_j - u^\alpha||_p ||\varphi||_q,$$

where q is the conjugate exponent of p. Hence

$$\int_\Omega u D^\alpha \varphi \, dx = \lim_{j\to\infty} \int_\Omega u_j D^\alpha \varphi \, dx = \lim_{j\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_j \varphi \, dx = (-1)^{|\alpha|} \int_\Omega u^\alpha \varphi \, dx,$$

from which follows that $D^\alpha u = u^\alpha$ and $u \in W^{k,p}(\Omega)$. Finally

$$||u_j - u||_{W^{k,p}(\Omega)} \to 0 \quad \text{as } j \to \infty$$

and the proof is complete. □

Proposition 5.13. *Let $u \in L^1_{loc}(\Omega)$, α a multi-index and suppose that $D^\alpha u$ exists. Then if $h < dist(x, \partial\Omega)$, we have*

$$D^\alpha u_h(x) = \big(D^\alpha u\big)_h(x).$$

Proof.

$$\begin{aligned} D^\alpha u_h(x) &= h^{-n}\int_\Omega D^\alpha_x\rho\Big(\frac{x-y}{h}\Big)u(y)\ dy \\ &= (-1)^{|\alpha|}h^{-n}\int_\Omega D^\alpha_y\rho\Big(\frac{x-y}{h}\Big)u(y)\ dy \\ &= h^{-n}\int_\Omega \rho\Big(\frac{x-y}{h}\Big)D^\alpha u(y)\ dy = \big(D^\alpha u\big)_h(x). \end{aligned}$$

□

Definition 5.14. Let $\Omega \subset \mathbb{R}^n$ and $\mathscr{U}$ be an open cover of Ω. We say that a countable collection of functions

$$\varphi_j \in C_0^\infty(\mathbb{R}^n)$$

is a *partition of unity* subordinate to the covering $\mathscr{U}$ if

(1) for every compact set $K \subset \Omega$ the intersection $\mathrm{spt}(\varphi_j) \cap K$ is non-empty for only a finite number of φ_j;
(2) $0 \le \varphi_j \le 1$ in Ω for every j;
(3) for every φ_j there exists an open $U \in \mathscr{U}$ such that $\mathrm{spt}(\varphi_j) \subset U$;
(4) $\sum_j \varphi_j(x) = 1$ for every $x \in \Omega$.

Proposition 5.15. *Let $1 \le p < \infty$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let $u \in W^{k,p}(\Omega)$ and $\varepsilon > 0$. Define for $i = 1, 2, ...$

$$U_i := \{x \in \Omega : d(x, \partial\Omega) > \frac{1}{i}\}.$$

We notice that U_i is bounded for every $i \in \mathbb{N}$, $U_1 \subset\subset U_2 \subset\subset \cdots \subset\subset \Omega$ and

$$\Omega = \bigcup_{i=1}^\infty U_i.$$

We set

$$G_i = U_{i+1} \setminus \overline{U}_{i-1}, \quad U_0 = \emptyset, \quad i \in \mathbb{N}.$$

Now the collection $\{G_i\}$ is an open cover of Ω and hence there exists a partition of unity $\mathscr{F}$ subordinate to the covering $\{G_i\}$ [10, p. 53]. We denote

$$\mathscr{F}_i = \{\varphi \in \mathscr{F} :\ \mathrm{spt}(\varphi) \subset G_i,\ \mathrm{spt}(\varphi) \cap U_i \neq \emptyset\}.$$

Since $U_i \subset\subset \Omega$ and $\mathscr{F}$ is a partition of unity, we have $\#\mathscr{F}_i < \infty$. Moreover, the sum

$$\varphi_i := \sum_{\varphi \in \mathscr{F}_i} \varphi, \quad i \in \mathbb{N},$$

is finite. This implies that $\varphi_i \in C_0^\infty(\Omega)$, $\mathrm{spt}(\varphi_i) \subset G_i$ and $\sum_{i=1}^\infty \varphi_i(x) = 1$ for every $x \in \Omega$.

Now $\mathrm{spt}(u\varphi_i) \subset U_{i+1}$ and $u\varphi_i \in W^{k,p}(\Omega)$. If $h < \mathrm{dist}(\overline{U}_{i+1}, \partial\Omega)$, we have by Proposition 5.13

$$D^{|\alpha|}(u\varphi_i)_h = \big(D^{|\alpha|}(u\varphi_i)\big)_h$$

and hence by Proposition 5.9

$$(u\varphi_i)_h \to u\varphi_i \quad \text{and} \quad D^{|\alpha|}(u\varphi_i)_h \to D^{|\alpha|}(u\varphi_i) \quad \text{in } L^p(\Omega) \quad \text{as } h \to 0,$$

that is,

$$||(u\varphi_i)_h - u\varphi_i||_{W^{k,p}(\Omega)} \to 0 \quad \text{as } h \to 0.$$

Therefore, for every $i \in \mathbb{N}$, there exists $h_i < \operatorname{dist}(\Omega_i, \partial\Omega_{i+1})$ such that

$$||(u\varphi_i)_{h_i} - u\varphi_i||_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2^i}.$$

Define

$$\varphi := \sum_{i=1}^{\infty} (u\varphi_i)_{h_i}.$$

It follows that $\varphi \in C^\infty(\Omega)$ and finally,

$$\begin{aligned} ||u - \varphi||_{W^{k,p}(\Omega)} &= \Big|\Big| \sum_{i=1}^{\infty} \Big(u\varphi_i - (u\varphi_i)_{h_i} \Big) \Big|\Big|_{W^{k,p}(\Omega)} \\ &\le \sum_{i=1}^{\infty} \big|\big| u\varphi_i - (u\varphi_i)_{h_i} \big|\big|_{W^{k,p}(\Omega)} \\ &< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon, \end{aligned}$$

which completes the proof. □

Remark 5.3. Let $1 \le p < \infty$ and $k \in \mathbb{N}$. Define

$$\begin{aligned} H^{k,p}(\Omega) = \{ &u \in L^p(\Omega) : \text{ there exists a sequence } (\varphi_j) \text{ in } C^\infty(\Omega) \text{ such that} \\ &\varphi_j \to u \text{ in } L^p(\Omega) \text{ and } D^\alpha \varphi_j \to D^\alpha u \text{ in } L^p(\Omega) \text{ for every } \alpha \text{ such that } |\alpha| \le k \}. \end{aligned}$$

Then, by Proposition 5.15 and the completeness of $W^{k,p}(\Omega)$, it follows that $H^{k,p}(\Omega) = W^{k,p}(\Omega)$.

Definition 5.16.

$$\begin{aligned} W_0^{k,p}(\Omega) = \{ u \in W^{k,p}(\Omega) : &\text{ there exists a sequence } (\varphi_j) \text{ in } C_0^\infty \cap W^{k,p}(\Omega) \\ &\text{such that } \varphi_j \to u \text{ in } W^{k,p}(\Omega) \quad \text{as } j \to \infty. \} \end{aligned}$$

The space $(W_0^{k,p}(\Omega), ||\cdot||_{k,p;\Omega})$ is a Banach space since it is a closed subspace of $(W^{k,p}(\Omega), ||\cdot||_{k,p;\Omega})$.

Proposition 5.17 (Poincaré inequality)**.** *Let $1 \le p < \infty$. Then for all $u \in W_0^{1,p}(\Omega)$*

$$||u||_p \le C||\nabla u||_p, \tag{22}$$

where C is a positive constant depending only on Ω and p.

Proof. Since Ω is bounded, $\Omega \subset [-M, M]^n$ for some $M > 0$. Let $u \in C_0^\infty(\Omega)$ and extend u to be zero outside Ω. By the Fundamental Theorem of Calculus, we have

$$|u(x)| = \Big| \int_{-M}^{x_1} D_1 u(t, x')\, dt \Big| \le \int_{-M}^{M} |D_1 u(t, x')|\, dt,$$

where $x' = (x_2, x_3, ..., x_n)$. By Hölder's inequality, it follows that

$$|u(x)|^p \le \left(\int_{-M}^{M} |D_1 u(t,x')| \ dt \right)^p \le \left(\int_{-M}^{M} 1^{\frac{p}{p-1}} \ dt \right)^{p-1} \left(\int_{-M}^{M} |D_1 u(t,x')|^p \ dt \right)$$
$$= (2M)^{p-1} \int_{-M}^{M} |D_1 u(t,x')|^p \ dt.$$

Integrating over x_1 we obtain

$$\int_{-M}^{M} |u(x)|^p \ dx_1 \le (2M)^{p-1} \int_{-M}^{M} \int_{-M}^{M} |D_1 u(t,x')|^p \ dt \ dx_1$$
$$= (2M)^p \int_{-M}^{M} |D_1 u(t,x')|^p \ dt.$$

Therefore

$$\int_{\Omega} |u(x)|^p \ dx = \int_{-M}^{M} \cdots \int_{-M}^{M} |u(x)|^p \ dx_1 \cdots dx_n$$
$$\le (2M)^p \int_{-M}^{M} \cdots \int_{-M}^{M} |D_1 u(t, x_2, ..., x_n)|^p \ dt \ dx_2 \cdots dx_n$$
$$= (2M)^p \int_{\Omega} |D_1 u(x)|^p \ dx$$
$$\le (2M)^p \int_{\Omega} |\nabla u(x)|^p \ dx$$

and hence

$$||u||_p \le C ||\nabla u||_p \quad \text{for every } u \in C_0^\infty(\Omega).$$

Let then $u \in W_0^{1,p}(\Omega)$. Now there exists a sequence (φ_j) in $C_0^\infty(\Omega)$ such that $\varphi_j \to u$ in $W^{1,p}(\Omega)$ as $j \to \infty$. Since

$$\big| ||v||_p - ||w||_p \big| \le ||v - w||_p \quad \text{for all } v, w \in L^p(\Omega),$$

we have

$$||\varphi_j||_p \to ||u||_p \quad \text{and} \quad ||\nabla \varphi_j||_p \to ||\nabla u||_p \quad \text{as } j \to \infty$$

and hence

$$||u||_p = \lim_{j\to\infty} ||\varphi_j||_p \le C \lim_{j\to\infty} ||\nabla \varphi_j||_p = C \ ||\nabla u||_p.$$

□

Remark 5.4. Let

$$||u||''_{W^{1,p}(\Omega)} := ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}.$$

Then$||u||''_{W^{1,p}(\Omega)}$ is equivalent to $||u||_{W^{1,p}(\Omega)}$. It follows from the previous proposition that $||\nabla u||_p$ and $||u||_{W^{1,p}(\Omega)}$ are equivalent norms on $W_0^{1,p}(\Omega)$.

Proposition 5.18. *Let $1 < p < \infty$ and $k \in \mathbb{N}$. Then $W^{k,p}(\Omega)$ is reflexive.*

Proof. Fix k and let

$$N = N(n,k) = \sum_{0 \le |\alpha| \le k} 1$$

be the number of multi-indexes α satisfying $0 \le |\alpha| \le k$. For $1 \le p \le \infty$, denote the product space

$$L^p(\Omega)^N = \Big(L^p(\Omega)\Big)^N = \underbrace{L^p(\Omega) \times ... \times L^p(\Omega)}_{N}.$$

The product norm of $u = (u_1, ..., u_N)$ in $L^p(\Omega)^N$ is given by

$$||u||_{L^p(\Omega)^N} = \begin{cases} \sum_{i=1}^N ||u_i||_p, & \text{if } 1 \le p < \infty; \\ \max_{1 \le i \le N} ||u_i||_\infty, & \text{if } p = \infty. \end{cases}$$

From Theorem 3.21 it follows that $L^p(\Omega)$ is reflexive when $1 < p < \infty$. Therefore $L^p(\Omega)^N$ is reflexive when $1 < p < \infty$. Since closed subspace of a reflexive space is reflexive, it suffices to find an isomorphism between $W^{k,p}(\Omega)$ and closed subspace of $L^p(\Omega)^N$.

Denote by $\alpha_1, \alpha_2, ..., \alpha_N$ the multi-indexes satisfying $0 \le |\alpha_i| \le k$. If $1 < p < \infty$, the isomorphism is given by the mapping $\Phi : W^{k,p}(\Omega) \to L^p(\Omega)^N$,

$$\Phi(u) = (D^{\alpha_1} u, D^{\alpha_2} u, ..., D^{\alpha_N} u).$$

This proves the claim. □

6. Applications of duality

In this section we consider $\Omega \subset \mathbb{R}^n$ to be a bounded smooth domain.

6.1. **The non-linear Dirichlet problem.** Let p and q be conjugate exponents such that $1 < p, q < \infty$. Given $f \in L^q(\Omega)$ we consider the following Dirichlet problem

$$\begin{cases} -\text{div}\big(|\nabla u|^{p-2}\nabla u\big) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Definition 6.1. We say that $u \in W_0^{1,p}(\Omega)$ is a *weak solution* of (23), if for any $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\varphi(x)\ dx = \int_\Omega f(x)\varphi(x)\ dx.$$

Lemma 6.2. *Function $u \in W_0^{1,p}(\Omega)$ is a weak solution to equation* (23) *if and only if it is a minimizer of the functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$*

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p\ dx - \int_\Omega f(x)u(x)\ dx, \tag{24}$$

that is

$$I(u) = \inf_{v \in W_0^{1,p}(\Omega)} I(v).$$

Proof. Suppose $u \in W_0^{1,p}(\Omega)$ is a minimizer of I. Fix $\varphi \in C_0^\infty(\Omega)$ and define a function $h : \mathbb{R} \to \mathbb{R}$ by setting

$$h(t) = I(u + t\varphi).$$

Since u is a minimizer,

$$h(0) = I(u) \le I(u + t\varphi) = h(t) \quad \text{for all } t \in \mathbb{R}.$$

This implies that $h'(0) = 0$. We have

$$h'(t) = \int_\Omega |\nabla(u(x) + t\varphi(x))|^{p-2}\nabla(u(x) + t\varphi(x)) \cdot \nabla\varphi(x)\ dx - \int_\Omega f(x)\varphi(x)\ dx.$$

Therefore

$$0 = h'(0) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\varphi(x)\ dx - \int_\Omega f(x)\varphi(x)\ dx,$$

which shows that u is a weak solution to equation (23).

Suppose then that u is a weak solution to (23). Fix $w \in W_0^{1,p}(\Omega)$. By Young's inequality, we have

$$\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\big[u(x) - w(x)\big]\ dx \\
&\quad = \int_\Omega |\nabla u(x)|^p\ dx - \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla w(x)\ dx \\
&\quad \geq \int_\Omega |\nabla u(x)|^p - \int_\Omega \left[\frac{1}{q}|\nabla u(x)|^p + \frac{1}{p}|\nabla w(x)|^p\right] dx \\
&\quad = \frac{1}{p}\int_\Omega \left[|\nabla u(x)|^p - |\nabla w(x)|^p\right] dx.
\end{aligned}$$

Since u is a weak solution,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\big[u(x) - w(x)\big]\ dx = \int_\Omega f(x)(u(x) - w(x))\ dx.$$

We have

$$\frac{1}{p}\int_\Omega \left[|\nabla u(x)|^p - |\nabla w(x)|^p\right] dx - \int_\Omega f(x)(u(x) - w(x))\ dx \leq 0,$$

which is the same as

$$I(u) \leq I(w).$$

Thus u is a minimizer of I. □

THEOREM 6.3. *The problem*

$$\inf_{u \in W_0^{1,p}(\Omega)} I(u)$$

has a unique solution.

Proof. By Propositions 5.12 and 5.18, $W_0^{1,p}(\Omega)$ is a reflexive Banach space when $1 < p < \infty$. We will apply Theorem 4.7 to prove the Theorem. We need to verify that I is strictly convex, lower semi-continuous, coercive and proper over $W_0^{1,p}(\Omega)$.

First, we show that I is strictly convex. Since $u \mapsto \int_\Omega fu\ dx$ is linear, it suffices to show that the mapping

$$u \mapsto \int_\Omega |\nabla u|^p$$

is strictly convex. The function $f(\xi) = |\xi|^p$ defined on $\mathbb{R}^n$ is strictly convex. Thus $u \mapsto \int_\Omega |\nabla u|^p$ is strictly convex.

Second, we claim that I is continuous. Fix $u \in W_0^{1,p}(\Omega)$ and let $\varepsilon > 0$. For every $v \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned} |I(u) - I(v)| &= \left| \frac{1}{p}\left(\int_\Omega |\nabla u(x)|^p \, dx - \int_\Omega |\nabla v(x)|^p \, dx \right) + \int_\Omega f(x)[v(x) - u(x)] \, dx \right| \\ &\leq \frac{1}{p}\Big| ||\nabla u||_p^p - ||\nabla v||_p^p \Big| + ||f||_q ||v - u||_p. \end{aligned}$$

There exists $\delta_1 > 0$ such that

$$|x^p - y^p| < \frac{\varepsilon p}{2} \qquad \text{for all } x, y \geq 0 \text{ with } |x - y| < \delta_1.$$

Hence

$$\frac{1}{p}\Big| ||\nabla u||_p^p - ||\nabla v||_p^p \Big| + ||f||_q ||v - u||_p < \frac{1}{p}\frac{\varepsilon p}{2} + ||f||_q \frac{\varepsilon}{2||f||_q} = \varepsilon,$$

whenever

$$\Big| ||\nabla u||_p - ||\nabla v||_p \Big| \leq ||\nabla u - \nabla v||_p < \delta_1 \quad \text{ and } \quad ||u - v||_p < \frac{\varepsilon}{2||f||_q}.$$

Thus $|I(u) - I(v)| < \varepsilon$ for all $v \in W_0^{1,p}(\Omega)$ such that

$$||u - v||_{W_0^{1,p}(\Omega)} = ||u - v||_p + ||\nabla u - \nabla v||_p < \delta := 2 \min\left\{ \delta_1, \frac{\varepsilon}{2||f||_q} \right\}.$$

This proves our claim. Since continuous functions are lower semi-continuous, I is lower semi-continuous.

Third, the coerciveness of I follows from the Poincaré inequality and Hölder's inequality:

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^p \, dx - \int_\Omega f(x)u(x) \, dx \\ &\quad \geq \frac{1}{C^p}||u||_p^p - \left(\int_\Omega |f(x)|^q \, dx \right)^{1/q} \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} \\ &\quad = \frac{1}{C^p}||u||_p^p - ||f||_q ||u||_p \to \infty \quad \text{ as } ||u||_p \to \infty. \end{aligned}$$

Finally, let $u \equiv 0$. Then

$$I(u) = \int_\Omega |\nabla u|^p \, dx - \int_\Omega f(x)u(x) \, dx = 0 < \infty,$$

from which follows that I is proper. Therefore, the strictly convex functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ satisfies all the assumptions of Theorem 4.7. We conclude by Theorem 4.7 that I has a unique minimizer. This proves the Theorem. □

We proceed by constructing a dual problem for our minimization problem

$$\inf_{u \in W_0^{1,p}(\Omega)} I(u).$$

We do this as in Example 4.3. Set

$$V = W_0^{1,p}(\Omega), \quad Y = L^p(\Omega)^n \text{ and } \Lambda = \nabla : W_0^{1,p}(\Omega) \to L^p(\Omega)^n.$$

It then follows that

$$V^* = \big(W_0^{1,p}(\Omega)\big)^* = W^{-1,q}(\Omega), \quad Y^* = \big(L^p(\Omega)^n\big)^* = L^q(\Omega)^n$$
$$\text{and } \ \Lambda^* = -\ \text{div} : L^q(\Omega)^n \to W^{-1,q}(\Omega).$$

Define $F : V \to \bar{\mathbb{R}}$ and $G : Y \to \bar{\mathbb{R}}$ by setting

$$F(u) = -(f, u) = -\int_\Omega f(x)u(x)\ dx$$

and

$$G(r) = \frac{1}{p}\int_\Omega |r(x)|^p\ dx.$$

The primal problem takes the form

$$(25) \qquad (\mathscr{P}) \quad \inf_{u\in W_0^{1,p}(\Omega)} I(u) = \inf_{u\in W_0^{1,p}(\Omega)} \big[F(u) + G(\Lambda u)\big].$$

In order to construct the dual problem, we consider the conjugate function of F and G. For the conjugate function of F, we have

$$\begin{aligned} F^*(u^*) &= \sup_{u\in V}\{\langle u^*, u\rangle - F(u)\} = \sup_{u\in V}\{\langle u^*, u\rangle + (f, u)\} \\ &= \sup_{u\in V}\{\langle u^*, u\rangle + \langle f, u\rangle\} \\ &= \sup_{u\in V}\ \langle u^* + f, u\rangle \\ &= \begin{cases} 0, & \text{if } u^* + f = 0; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

For the conjugate function of G, we have

$$\begin{aligned} G^*(r^*) &= \sup_{r\in Y}\{\langle r^*, r\rangle - G(r)\} = \sup_{r\in Y}\{\langle r^*, r\rangle - \frac{1}{p}\int_\Omega |r(x)|^p\ dx\} \\ &= \sup_{r\in Y}\left\{\sum_{i=1}^n \int_\Omega r_i^*(x)r_i(x)\ dx - \frac{1}{p}\int_\Omega |r(x)|^p\ dx\right\} \\ &= \sup_{r\in Y}\left\{\int_\Omega r^*(x)\cdot r(x)\ dx - \frac{1}{p}\int_\Omega |r(x)|^p\ dx\right\}. \end{aligned}$$

By the Young's inequality

$$\int_\Omega r^*(x)\cdot r(x)\ dx \le \int_\Omega |r^*(x)||r(x)|\ dx \le \int_\Omega \left[\frac{1}{q}|r^*(x)|^q + \frac{1}{p}|r(x)|^p\right]\ dx.$$

Thus

$$G^*(r^*) = \frac{1}{q}\int_\Omega |r^*(x)|^q\ dx.$$

The dual problem of $\mathscr{P}$ is of the form

$$(26) \qquad (\mathscr{P}^*) \quad \sup_{\substack{r^*\in L^q(\Omega)^n \\ \text{div } r^* = f}} \Big[-G^*(-r^*)\Big] = \sup_{\substack{r^*\in L^q(\Omega)^n \\ \text{div } r^* = f}} \left[-\frac{1}{q}\int_\Omega |r^*(x)|^q\ dx\right].$$

Lemma 6.4. *G^* is lower semi-continuous and strictly convex over $L^q(\Omega)^n$.*

Proof. The proof is analogous to the proof of strict convexity and lower semi-continuity of the functional I in Theorem 6.3. We first claim that G^* is continuous. Indeed, fix $r^* \in Y^*$ and let $\varepsilon > 0$. For all $s^* \in Y^*$

$$\begin{aligned}|G^*(r^*) - G^*(s^*)| &= \left| \frac{1}{q} \int_\Omega |r^*(x)|^q \, dx - \frac{1}{q} \int_\Omega |s^*(x)|^q \, dx \right| \\ &= \frac{1}{q} \Big| ||r^*||_{L^q(\Omega)^n} - ||s^*||_{L^q(\Omega)^n} \Big|.\end{aligned}$$

There exists $\delta > 0$ such that

$$|x^q - y^q| < q\varepsilon \quad \text{for all } x, y \geq 0 \text{ with } |x - y| < \delta.$$

Hence

$$\frac{1}{q} \Big| ||r^*||^q_{L^q(\Omega)^n} - ||s^*||^q_{L^q(\Omega)^n} \Big| < \frac{1}{q} q\varepsilon = \varepsilon$$

whenever

$$\Big| ||r^*||_{L^q(\Omega)^n} - ||s^*||_{L^q(\Omega)^n} \Big| \leq ||r^* - s^*||_{L^q(\Omega)^n} < \delta.$$

It follows that G^* is continuous and in particular it is lower semi-continuous. Second, we claim that G^* is strictly convex. This follows from the fact that the function $\xi \mapsto |\xi|^q$ is strictly convex on $\mathbb{R}^n$. □

Lemma 6.5. *$\mathscr{P}$ is stable.*

Proof. Theorem 6.3 states that $\mathscr{P}$ has a unique solution and from the proof of Theorem 4.7 we have that $\inf \mathscr{P}$ is finite. By Theorem 2.21, $F \in \Gamma_0(V)$ and $G \in \Gamma_0(Y)$. Thus $\Phi(u, r) = F(u) + G(\Lambda u - r) \in \Gamma_0(V \times Y)$. In particular, Φ is convex over $V \times Y$. There exists $u_0 \in V$ such that $F(u_0) < \infty$, $G(\Lambda u_0) < \infty$, G being continuous at Λu_0. Hence Theorem 4.12 implies that $\mathscr{P}$ is stable. □

THEOREM 6.6. *Problem $\mathscr{P}$ in (25) has $\mathscr{P}^*$ in (26) as its dual problem. $\mathscr{P}$ possesses a unique solution $\bar{u}$ and $\mathscr{P}^*$ a unique solution $\bar{r}^*$. We have*

$$\max \mathscr{P}^* = \min \mathscr{P}$$

and the following extremality relation

$$\bar{r}_i^*(x) = - \, |\nabla \bar{u}(x)|^{p-2} D_i \bar{u}(x) \quad \text{a.e. } x \in \Omega.$$

Proof. The existence of a unique solution to problem $\mathscr{P}$ was shown in Theorem 6.3. By Lemma 6.5, $\mathscr{P}$ is stable and by Theorem 4.9, $\mathscr{P}^*$ has a solution and

$$-\infty < \inf \mathscr{P} = \sup \mathscr{P}^* < \infty.$$

Since G^* is strictly convex, the solution of $\mathscr{P}^*$ is unique.

By Theorem 4.13 our unique solutions $\bar{u}$ and $\bar{r}^*$ satisfy the extremality relation

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{r}^*) = 0, \tag{27}$$

which can be written as

$$\begin{aligned}0 &= F(\bar{u}) + G(\Lambda\bar{u}) + F^*(\Lambda^*\bar{r}^*) + G^*(-\bar{r}^*) \\ &= \big[F(\bar{u}) + F^*(\Lambda^*\bar{r}^*) - \langle \Lambda^*\bar{r}^*, \bar{u}\rangle\big] + \big[G(\Lambda\bar{u}) + G^*(-\bar{r}^*) - \langle -\bar{r}^*, \Lambda\bar{u}\rangle\big].\end{aligned}$$

By the definition of conjugate function, we have

$$G(\Lambda\bar{u}) + G^*(-\bar{r}^*) - \langle -\bar{r}^*, \Lambda\bar{u}\rangle \geq 0.$$

Hence

$$F(\bar{u}) + F^*(\Lambda^*\bar{r}^*) - \langle \Lambda^*\bar{r}^*, \bar{u}\rangle = 0 \tag{28}$$

$$\Leftrightarrow \quad \Lambda^*\bar{r}^* + f = 0 \quad \Leftrightarrow \quad \text{div } \bar{r}^* = f$$

and

$$G(\Lambda\bar{u}) + G^*(-\bar{r}^*) + \langle \bar{r}^*, \Lambda\bar{u}\rangle = 0 \tag{29}$$

$$\begin{aligned}
&\Leftrightarrow \quad \frac{1}{p}\int_\Omega |\nabla \bar{u}(x)|^p \, dx + \frac{1}{q}\int_\Omega |\bar{r}^*(x)|^{\frac{p}{p-1}} \, dx = -\int_\Omega \bar{r}^*(x)\nabla\bar{u}(x) \, dx \\
&\Leftrightarrow \quad -\bar{r}^*(x)\nabla\bar{u}(x) = \frac{1}{p}|\nabla\bar{u}(x)|^p + \frac{1}{q}|\bar{r}^*(x)|^{\frac{p}{p-1}} \quad \text{for a.e. } x \in \Omega \\
&\Leftrightarrow \quad -\bar{r}^*(x) = |\nabla\bar{u}(x)|^{p-2}\nabla\bar{u}(x) \quad \text{for a.e. } x \in \Omega.
\end{aligned}$$

□

6.2. **The Stokes problem.** Given $f \in L^2(\Omega)^n$, we consider the following system

$$\begin{cases} -\Delta u + \nabla p = f, & \text{in } \Omega; \\ \text{div } u = 0, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{30}$$

where $u = (u_1, ..., u_n) : \Omega \to \mathbb{R}^n$ and $p : \Omega \to \mathbb{R}$. We use the notation $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ for

$$W = \{v \in H_0^1(\Omega)^n, \ \text{div } v = 0\}.$$

It follows that W is a Hilbert space with the norm

$$||u||_{H^1(\Omega)^n} = \left(\int_\Omega \sum_{1\le i,j\le n} |D_i u_j|^2 \, dx\right)^{1/2}$$

and inner product

$$((u,v)) = \sum_{1\le i,j\le n} (D_i u_j, D_i v_j) = \sum_{1\le i,j\le n} \int_\Omega D_i u_j(x) D_i v_j(x) \, dx.$$

If $v \in W$, then problem (30) has a variational formulation:

$$\sum_{1\le i,j\le n} \int_\Omega D_i u_j(x) D_i v_j(x) \, dx - \int_\Omega \text{div } v(x) p(x) \, dx = \int_\Omega f(x)\cdot v(x) \, dx$$

$$\Leftrightarrow \quad ((u,v)) = (f,v).$$

Lemma 6.7. *Function $u \in W$ is a weak solution of equation (30), that is,*

$$((u,v)) = (f,v) \quad \textit{for all } v \in W \tag{31}$$

if and only if it is a minimizer of the functional

$$I(u) = \frac{1}{2}||u||^2_{H^1(\Omega)^n} - (f,u) = \frac{1}{2}\left[\sum_{i=1}^n\sum_{j=1}^n ||D_i u_j||_2^2\right] - \int_\Omega f(x)u(x) \, dx.$$

Proof. The proof is similar to that of Lemma 6.2. Suppose u is a minimizer of I. Let $v \in W$ and define a function $h : \mathbb{R} \to \mathbb{R}$ as follows

$$h(t) = I(u + tv).$$

By assumption

$$h(0) = I(u) \le I(u + tv) = h(t) \quad \text{for all } t \in \mathbb{R}.$$

This implies that $h'(0) = 0$. We have

$$h'(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega (D_i u_j(x) + tD_i v_j(x))D_i v_j(x) - \int_\Omega f(x)v(x)\ dx.$$

Therefore

$$h(0) = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega D_i u_j(x)D_i v_j(x) - \int_\Omega f(x)v(x)\ dx = 0,$$

which is the same as (31).

Suppose then that u satisfies (31). Fix $w \in W$. We have

$$((u, u - w)) = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \frac{1}{2}\Big[|D_i u_j(x)|^2 + |D_i(u_j(x) - w_j(x))|^2 - |D_i w_j(x)|^2\Big]\ dx.$$

Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \frac{1}{2}\Big[|D_i u_j(x)|^2 - |D_i w_j(x)|^2\Big]\ dx - ((u, u - w))$$
$$= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega |D_i(u_j(x) - w_j(x))|^2\ dx \le 0.$$

Since u satisfies (31), we have

$$((u, u - w)) = \int_\Omega f(x)\big(u(x) - w(x)\big)\ dx.$$

Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \frac{1}{2}\Big[|D_i u_j(x)|^2 - |D_i w_j(x)|^2\Big]\ dx - \int_\Omega f(x)\big(u(x) - w(x)\big)\ dx \le 0,$$

which implies that

$$I(u) \le I(w).$$

Thus u is a minimizer of I. □

Set

$$V = H_0^1(\Omega)^n, \quad Y = L^2(\Omega) \ \text{ and } \ \Lambda = \text{ div } : H_0^1(\Omega)^n \to L^2(\Omega).$$

It follows that

$$V^* = \big(H_0^1(\Omega)^n\big)^* = H^{-1}(\Omega)^n, \quad Y^* = L^2(\Omega) \ \text{ and } \ \Lambda^* = -\nabla : L^2(\Omega) \to H^{-1}(\Omega)^n.$$

Define $F : V \to \bar{\mathbb{R}}$ and $G : Y \to \bar{\mathbb{R}}$ by setting

$$F(u) = \frac{1}{2}||u||^2_{H^1(\Omega)^n} - (f, u)$$

and

$$G(p) = \begin{cases} 0 & \text{if } p = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Our primal problem has the form

$$(\mathscr{P}) \qquad \inf_{u \in W} I(u) = \inf_{u \in H_0^1(\Omega)^n} [F(u) + G(\Lambda u)].$$

THEOREM 6.8. *The primal problem $\mathscr{P}$ possesses a unique solution.*

Proof. The proof is similar to that of Theorem 6.3. By Propositions 5.12 and 5.18, $W \subset H_0^1(\Omega)^n$ is a reflexive Banach space. Therefore, in order to apply Theorem 4.7 to prove the Theorem, we need to verify that I is strictly convex, lower semi-continuous, coercive and proper over W.

The strict convexity follows from the fact that the function $x \mapsto x^2$ is strictly convex over $\mathbb{R}$. This can be shown with Proposition 2.13. Since I is continuous in W, it is lower semi-continuous. We have

$$(f, u) \leq ||f||_{L^2(\Omega)^n} ||u||_{L^2(\Omega)^n} = \sum_{j=1}^{n} ||f_j||_2 ||u_j||_2$$

and hence

$$\begin{aligned} I(u) &= \frac{1}{2} ||u||^2_{H^1(\Omega)^n} - (f, u) \\ &\geq \frac{1}{2} ||u||^2_{H^1(\Omega)^n} - \sum_{j=1}^{n} ||f_j||_2 ||u_j||_2 \\ &\geq \frac{1}{2} ||u||^2_{H^1(\Omega)^n} - C \sum_{j=1}^{n} ||f_j||_2 ||u||_{H^1(\Omega)^n} \to \infty \quad \text{ as } ||u||_{H^1(\Omega)^n} \to \infty, \end{aligned}$$

for some positive constant C. This implies that I is coercive. Finally, let $u = \bar{0} = (0, ..., 0)$. Then $u \in W$ and $I(u) = 0 < \infty$ meaning that I is proper. $\square$

For the conjugate function of F, we have

$$\begin{aligned} F^*(\Lambda^* p^*) &= \sup_{v \in V} \{\langle \Lambda^* p^*, v \rangle - F(v) - G(\Lambda v)\} \\ &= \sup_{v \in H_0^1(\Omega)^n} \left\{ (p^*, \text{div } v) + (f, v) - \frac{1}{2} ||v||^2_{H^1(\Omega)^n} - G(\Lambda v) \right\}. \end{aligned}$$

Lemma 6.9. *$v(p^*) \in H_0^1(\Omega)^n$ is a maximizer of*

$$(p^*, \mathit{div}\ v) + (f, v) - \frac{1}{2} ||v||^2_{H^1(\Omega)^n}$$

if it satisfies

$$(32) \qquad ((v(p^*), w)) = (f, w) + (p^*, \mathit{div}\ w), \quad \textit{for all } w \in H_0^1(\Omega)^n.$$

Proof. Denote

$$I(v) = (p^*, \text{div } v) + (f, v) - \frac{1}{2} ||v||^2_{H^1(\Omega)^n}.$$

Fix $p^* \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)^n$ such that it satisfies (32). Then

$$((u, u - w)) = (f, u - w) + (p^*, \text{div}(u - w)) \quad \text{for all } w \in H_0^1(\Omega)^n.$$

Now

$$((u, u-w)) \leq (f, u-w) + (p^*, \operatorname{div}(u-w)) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega |D_i(u_j - w_j)|^2 \, dx.$$

In the proof of Lemma 6.7, we showed that

$$\frac{1}{2}\Big[||u||^2_{H^1(\Omega)^n} - ||w||^2_{H^1(\Omega)^n}\Big] = ((u, u-w)) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega |D_i(u_j - w_j)|^2 \, dx.$$

Therefore

$$\frac{1}{2}\Big[||u||^2_{H^1(\Omega)^n} - ||w||^2_{H^1(\Omega)^n}\Big] \leq (f, u-w) + (p^*, \operatorname{div}(u-w)),$$

which can be written as

$$(p^*, \operatorname{div} w) + (f, w) - \frac{1}{2}||w||^2_{H^1(\Omega)^n} \leq (p^*, \operatorname{div} u) + (f, u) - \frac{1}{2}||u||^2_{H^1(\Omega)^n}.$$

Thus $I(w) \leq I(u)$ for every $w \in H_0^1(\Omega)^n$ and u is a maximizer. □

By the Lemma 6.9, the supremum is attained at a point $v(p^*) \in W$ such that $v(p^*)$ satisfies (32). Hence

$$\begin{aligned} F^*(\Lambda^* p^*) &= \sup_{v \in H^1(\Omega)^n} \big\{(p^*, \operatorname{div} v) + (f, v) - \frac{1}{2}||v||^2_{H^1(\Omega)^n} - G(\Lambda v)\big\} \\ &= \Big\{\Big(\big(v(p^*), v(p^*)\big)\Big) - \frac{1}{2}||v(p^*)||^2_{H^1(\Omega)^n}\Big\} \\ &= \frac{1}{2}||v(p^*)||^2_{H^1(\Omega)^n}. \end{aligned}$$

For the conjugate function of G we have

$$G^*(p^*) = \sup_{p \in Y}\{\langle p^*, p\rangle - G(p)\} = \langle p^*, 0\rangle - 0 = 0.$$

The dual problem of $\mathscr{P}$ is

$$(\mathscr{P}^*) \qquad \sup_{p^* \in L^2(\Omega)} \Big\{ -\frac{1}{2}||v(p^*)||^2_{H^1(\Omega)^n}\Big\}.$$

THEOREM 6.10. *If $\mathscr{P}^*$ is proper, then the dual problem $\mathscr{P}^*$ possesses a solution.*

Proof. Let (p_m^*) in $L^2(\Omega)$ be an maximizing sequence of $\mathscr{P}^*$, that is,

$$-\frac{1}{2}||v(p_m^*)||_{H^1(\Omega)^n} \to \sup \mathscr{P}^* \quad \text{ as } m \to \infty.$$

By Theorem 6.8, problem $\mathscr{P}$ possesses a unique solution $\bar{u}$ and $\inf \mathscr{P} < \infty$. Since

$$\sup \mathscr{P}^* \leq \inf \mathscr{P} < \infty$$

and $\mathscr{P}^*$ is proper, we have that $\sup \mathscr{P}^*$ is finite. Moreover, the sequence $\big(v(p_m^*)\big)$ is bounded in $H_0^1(\Omega)^n$.

We claim that the sequence $\big(\nabla p_m^*\big)$ is bounded in $H^{-1}(\Omega)^n$. Indeed, let $\psi \in H_0^1(\Omega)^n$. Since $v(p_m^*)$ is bounded and satisfies (32) for every $m \in \mathbb{N}$, we have

$$\begin{aligned} \left|\int_\Omega \nabla p_m^*(x)\psi(x)\, dx\right| &= \Big|(f, \psi) - ((v(p_m^*), \psi))\Big| \\ &\leq ||f||_{H^{-1}(\Omega)^n}||\psi||_{H^1(\Omega)^n} + ||v(p_m^*)||_{H^1(\Omega)^n}||\psi||_{H^1(\Omega)^n} < M \end{aligned}$$

with some $M < \infty$. Thus for every $m \in \mathbb{N}$ the mapping

$$\psi \mapsto \int_\Omega \nabla p_m^*(x)\psi(x)\, dx = (\nabla p_m^*, \psi)$$

is a bounded linear functional on $H_0^1(\Omega)^n$. In particular, $\big(\nabla p_m^*\big)$ is bounded in $H^{-1}(\Omega)^n$ by Banach-Steinhaus Theorem [7, p. 203]. Hence the claim is true.

Since Ω is a bounded smooth domain, by [8, p. 14] there is a constant $C > 0$ only depending on Ω such that

$$||p_m^*||_{L^2(\Omega)/\mathbb{R}} \leq C||\nabla p_m^*||_{H^{-1}(\Omega)^n},$$

where

$$L^2(\Omega)/\mathbb{R} = \Big\{p \in L^2(\Omega) : \int_\Omega p(x)\, dx = 0\Big\}.$$

There is subsequence $(p_{m_i}^*) \subset (p_m^*)$ and $p^* \in L^2(\Omega)/\mathbb{R}$ such that

$$p_{m_i}^* \rightharpoonup p^* \quad \text{in } L^2(\Omega)/\mathbb{R} \;\text{ as } m_i \to \infty.$$

Since $H_0^1(\Omega)^n$ is reflexive, its dual space $H^{-1}(\Omega)^n$ is also reflexive. Now by Theorem 3.22, there is a subsequence of $(p_{m_i}^*)$, still denoted by itself, such that

$$\nabla p_{m_i}^* \rightharpoonup \nabla p^* \quad \text{in } H^{-1}(\Omega)^n \;\text{ as } m_i \to \infty.$$

Now, F is convex lower semi-continuous and by Proposition 3.9 it is weakly lower semi-continuous. Thus

$$-F(\Lambda^* p^*) \geq -\liminf_{m\to\infty} F(\Lambda^* p_m^*) = \limsup_{m\to\infty} -F(\Lambda^* p_m^*) = \sup \mathscr{P}^*,$$

which implies that p^* is a solution of $\mathscr{P}^*$. □

THEOREM 6.11. *$\mathscr{P}^*$ is proper.*

Proof. We show that there is an element $p^* \in L^2(\Omega)$ such that

$$||v(p^*)||_{H^1(\Omega)^n} < \infty,$$

where $v(p^*) \in W$ satisfies (32). Indeed, $v(p^*)$ is a weak solution of the Stokes equation

$$\begin{cases} -\Delta v = f - \nabla p^*, & \text{in } \Omega; \\ \text{div } v = 0, & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

For every $f \in L^2(\Omega)^n$ there exists $v \in H_0^1(\Omega)$ and $p^* \in L^2(\Omega)$ such that (32) is satisfied. For the proof we refer to [8, p. 31]. Thus $\mathscr{P}^*$ is proper. □

THEOREM 6.12. *Problem (30) possesses a solution $(\bar{u}, \bar{p}^*)$, where $\bar{u}$ is a solution of the primal problem $\mathscr{P}$ and $\bar{p}^*$ is a solution of the dual problem $\mathscr{P}^*$. Moreover*

$$\inf \mathscr{P} = \sup \mathscr{P}^*.$$

Proof. From Theorems 6.8 and 6.10, we have the existence of the pair $(\bar{u}, \bar{p}^*)$. By Lemma 4.10, the set of solutions of $\mathscr{P}^*$ is identical to $\partial h^{**}(0)$. Thus h^{**} is subdifferentiable at 0. Since $h^{**}(0)$ is also finite, by definition the problem $\mathscr{P}^*$ is stable.

We claim that h is lower semi-continuous at 0. Indeed,

$$h(0) = F(\bar{u}) \leq \inf_{u \in H_0^1(\Omega)^n} \{F(u) + G(\Lambda u - p)\} = h(p) \quad \text{for every } p \in L^2(\Omega).$$

Hence
$$h(0) \le \liminf_{p\to 0} h(p),$$
meaning h is lower semi-continuous at 0. Therefore, at the point 0 h coincides with its largest minorant of class $\Gamma(L^2(\Omega))$, which is h^{**}. It follows that
$$\inf \mathscr{P} = h(0) = h^{**}(0) = \sup \mathscr{P}^*.$$
By Theorem 4.13 we have the following extremality relation
$$F(\bar{u}) + F^*(\Lambda \bar{p}^*) = 0.$$
This can be written as
$$\begin{aligned} &\frac{1}{2}||\bar{u}||^2_{H^1(\Omega)^n} - (f,u) + \frac{1}{2}||v(\bar{p}^*)||^2_{H^1(\Omega)^n} = 0 \\ \Leftrightarrow \quad &\frac{1}{2}((\bar{u},\bar{u})) - (f,\bar{u}) + \frac{1}{2}((v(\bar{p}^*), v(\bar{p}^*))) = 0 \\ \Leftrightarrow \quad &-\frac{1}{2}(f,\bar{u}) + \frac{1}{2}(f, v(\bar{p}^*)) + \frac{1}{2}(\bar{p}^*, \text{div } v(\bar{p}^*)) = 0. \end{aligned}$$
Since div $v(\bar{p}^*) = 0$, we have
$$(f, v(\bar{p}^*) - \bar{u}) = 0.$$
It follows that $\bar{u} = v(\bar{p}^*)$ almost everywhere. Thus u satisfies (32) and the pair $(\bar{u}, \bar{p}^*)$ is the solution of (30). □

Remark 6.1. The solution of $\mathscr{P}^*$ is not unique. However, the solution is unique except for additive constants.

6.3. **Mossolov's problem.** Consider the minimization problem

$$\inf_{u\in H_0^1(\Omega)} \left\{ \frac{\alpha}{2}\int_\Omega |\nabla u(x)|^2\, dx + \beta \int_\Omega |\nabla u(x)|\, dx - \int_\Omega f(x)u(x)\, dx \right\}, \tag{33}$$

where α and β are positive constants and $f \in L^2(\Omega)$ is given. We set
$$V = H_0^1(\Omega), \quad Y = L^2(\Omega)^n \text{ and } \Lambda = \nabla : H_0^1(\Omega) \to L^2(\Omega)^n,$$
from which follows that
$$V^* = \left(H_0^1(\Omega)\right)^* = H^{-1}(\Omega), \quad Y^* = L^2(\Omega)^n \text{ and } \Lambda^* = -\text{div } : L^2(\Omega)^n \to H^{-1}(\Omega).$$
Define $F : V \to \bar{\mathbb{R}}$ and $G : Y \to \bar{\mathbb{R}}$ as follows:
$$F(u) = \frac{\alpha}{2}||u||^2_{H^1(\Omega)} - (f,u), \quad ||u||_{H^1(\Omega)} = ||\nabla u||_2,$$
and
$$G(p) = \beta \int_\Omega |p(x)|\, dx.$$
The primal problem
$$(\mathscr{P}) \qquad \inf_{u\in H_0^1(\Omega)} \left[F(u) + G(\Lambda u)\right]$$
is identical with (33).

THEOREM 6.13. *The primal problem $\mathscr{P}$ possesses a unique solution.*

Proof. By Propositions 5.12 and 5.18, $H_0^1(\Omega)$ is a reflexive Banach space. It is easy to verify that the function

$$u \mapsto F(u) + G(\nabla u)$$

is strictly convex, lower semi-continuous, coercive and proper over $H_0^1(\Omega)$. The result follows from the Theorem 4.7. □

For the conjugate function of F, we have

$$\begin{aligned} F^*(u^*) &= \sup_{u\in H_0^1(\Omega)} \big\{\langle u^*, u\rangle - F(u)\big\} \\ &= \sup_{u\in H_0^1(\Omega)} \Big\{\langle u^*, u\rangle + (f,u) - \frac{\alpha}{2}||u||^2_{H^1(\Omega)}\Big\} \\ &= \sup_{u\in H_0^1(\Omega)} \Big\{\langle u^* + f, u\rangle - \frac{\alpha}{2}||u||^2_{H^1(\Omega)}\Big\}. \end{aligned} \tag{34}$$

Lemma 6.14. *The maximum of (34) is attained at the point* $u \in H_0^1(\Omega)$, *which satisfies*

$$\alpha(\nabla u, \nabla w) = (u^*, w) + (f, w) \quad \textit{for every } w \in H_0^1(\Omega). \tag{35}$$

Proof. The proof is analogous to the proof of Lemma 6.9. Thus we omit the details. □

It follows that

$$\begin{aligned} F^*(u^*) &= \sup_{u\in H_0^1(\Omega)} \Big\{\langle u^* + f, u\rangle - \frac{\alpha}{2}||u||^2_{H^1(\Omega)}\Big\} \\ &= \langle u^* + f, u\rangle - \frac{1}{2}\langle u^* + f, u\rangle \\ &= \frac{1}{2}\langle u^* + f, u\rangle \frac{\langle u^* + f, u\rangle}{\langle u^* + f, u\rangle} \\ &= \frac{1}{2\alpha}\frac{\langle u^* + f, u\rangle^2}{||u||^2_{H^1(\Omega)}} = \frac{1}{2\alpha}||u^* + f||^2_{H^{-1}(\Omega)}, \end{aligned}$$

since

$$||u^*||_{H^{-1}(\Omega)} = \sup_{u\in H_0^1(\Omega)} \frac{|\langle u^*, u\rangle|}{||u||_{H^1(\Omega)}}.$$

For the conjugate function of G

$$\begin{aligned} G^*(p^*) &= \sup_{p\in L^2(\Omega)^n} \big\{\langle p^*, p\rangle - G(p)\big\} \\ &= \sup_{p\in L^2(\Omega)^n} \Big\{\langle p^*, p\rangle - \beta\int_\Omega |p(x)|\,dx\Big\} \\ &= \sup_{p\in L^2(\Omega)^n} \Big\{\int_\Omega p^*(x)p(x) - \beta|p(x)|\,dx\Big\} \\ &= \begin{cases} 0, & \text{if } |p^*(x)| \le \beta \text{ for almost every } x \in \Omega; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

The dual problem of $\mathscr{P}$ can be written as

$$(\mathscr{P}^*) \qquad \sup_{p^*\in L^2(\Omega)^n} \big[-F^*(\Lambda^* p^*) - G^*(-p^*)\big].$$

And further as

$$\sup_{\substack{p^* \in L^2(\Omega)^n \\ |p^*(x)| \leq \beta \text{ a.e.}}} \left\{ -\frac{1}{2\alpha} ||f - \text{div } p^*||^2_{H^{-1}(\Omega)} \right\}. \tag{36}$$

Lemma 6.15. *$\mathscr{P}$ is stable.*

Proof. Theorem 6.13 states that $\mathscr{P}$ has a unique solution and inf $\mathscr{P}$ is finite. The functions F and G are convex and lower semi-continuous over V and Y respectively and hence by Theorem 2.21 $\Phi(u,p) = F(u) + G(\Lambda u - p) \in \Gamma_0(V \times Y)$. In particular, Φ is convex over $V \times Y$. There exists $u_0 \in V$ such that $F(u_0) < \infty$, $G(\Lambda u_0) < \infty$, G being continuous at Λu_0. Thus by Theorem 4.12, $\mathscr{P}$ is stable. □

THEOREM 6.16. *Problem $\mathscr{P}$ in (33) has $\mathscr{P}^*$ in (36) as its dual problem. $\mathscr{P}$ possesses a unique solution $\bar{u}$, $\mathscr{P}^*$ has at least one solution $\bar{p}^*$ and*

$$\min \mathscr{P} = \max \mathscr{P}^*.$$

We have the following extremality relations:

$$\alpha(\nabla \bar{u}, \nabla w) = (u^*, w) + (f, w) \quad \textit{for every } w \in H^1_0(\Omega) \tag{37}$$

and

$$\beta |\nabla \bar{u}(x)| = -\bar{p}^*(x) \cdot \nabla \bar{u}(x) \quad \textit{for almost every } x \in \Omega. \tag{38}$$

Proof. The existence of a unique solution of $\mathscr{P}$ was shown in Theorem 6.13. By previous Lemma, $\mathscr{P}$ is stable and hence by Theorem 4.9, there exists a solution of $\mathscr{P}^*$ and

$$\min \mathscr{P} = \max \mathscr{P}^*.$$

By Theorem 4.13 the solutions satisfy the following extremality relations

$$F(\bar{u}) + F^*(\Lambda^* \bar{p}^*) = \langle \Lambda^* \bar{p}^*, \bar{u} \rangle \tag{39}$$

and

$$G(\Lambda \bar{u}) + G^*(-\bar{p}^*) = -\langle \bar{p}^*, \Lambda \bar{u} \rangle. \tag{40}$$

We can write

$$F^*(u^*) = \sup_{u \in H^1_0(\Omega)} \left\{ \langle u^* + f, u \rangle - \frac{\alpha}{2} ||u||^2_{H^1(\Omega)} \right\} = \frac{\alpha}{2} ||v(u^*)||^2_{H^1(\Omega)},$$

where $v(u^*)$ satisfies (35). Denote $v = v(-\text{div } \bar{p}^*)$. Relation (39) yields

$$\begin{aligned} & \frac{\alpha}{2} ||\bar{u}||^2_{H^1(\Omega)} - (f, \bar{u}) + \frac{\alpha}{2} ||v||^2_{H^1(\Omega)} = \langle -\text{div } \bar{p}^*, \bar{u} \rangle \\ \Leftrightarrow \quad & \frac{\alpha}{2} ||\bar{u}||^2_{H^1(\Omega)} - \alpha(\nabla v, \nabla \bar{u}) + \frac{\alpha}{2} ||v||^2_{H^1(\Omega)} = 0 \\ \Leftrightarrow \quad & \alpha(\nabla \bar{u} - \nabla v)^2 = 0 \quad \text{ for almost everywhere.} \end{aligned}$$

Since $\alpha > 0$, we have $\nabla \bar{u}(x) = \nabla v(x)$ for almost every $x \in \Omega$. Thus $\bar{u}$ satisfies (35) with $u^* = -\text{div } \bar{p}^*$. We have showed (37).

Equation (40) can be written as

$$\int_\Omega \beta |\nabla \bar{u}(x)| + \bar{p}^*(x) \cdot \nabla \bar{u}(x) \, dx = 0.$$

Since $|\bar{p}|^* \leq \beta$, the integral is non-negative. Thus we have

$$\beta|\nabla\bar{u}(x)| = -\bar{p}^*(x)\cdot\nabla\bar{u}(x) \quad \text{for almost every } x \in \Omega.$$

□

References

[1] Andrew M. Bruckner, Judith B. Bruckner and Brian S. Thomson: *Real Analysis.* Prentice-Hall, 1997.

[2] Ivar Ekeland and Roger Temam: *Convex Analysis and Variational Problems.* North-Holland Publishing Company, 1976.

[3] David Gilbarg and Neil S. Trudinger: *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin-Heidelberg 1977, 1983.

[4] M. E. Munroe: *Introduction to Measure and Integration.* Addison-Wesley Publishing Company, 1959.

[5] R. Tyrrell Rockafellar: *Convex Analysis.* Princeton University Press, 1970.

[6] Walter Rudin: *Functional Analysis.* Tata McGraw-Hill Publishing Company, 1973.

[7] Angus E. Taylor: *Introduction to Functional Analysis.* John Wiley & Sons, 1958.

[8] Roger Temam: *Navier-Stokes Equations, Theory and Numerical Analysis.* North-Holland Publishing Company, England 1977.

[9] Kôsaku Yosida: *Functional Analysis.* Springer-Verlag, Berlin-Heidelberg 1965, 1968.

[10] William P. Ziemer: *Weakly Differentiable Functions.* Springer-Verlag, New York 1989.