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DEPARTMENT OF MATHEMATICS
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REPORT 137

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**ON MALLIAVIN CALCULUS AND
APPROXIMATION OF STOCHASTIC INTEGRALS
FOR LÉVY PROCESSES**

EIJA LAUKKARINEN



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Editor: Pekka Koskela
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
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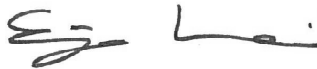
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Jyväskylä, October 2012

A handwritten signature in black ink, appearing to read 'E. Geiss', with a stylized flourish at the end.

List of included articles

This dissertation consists of an introductory part and the following publications:

- [GL] C. Geiss and E. Laukkarinen. Denseness of certain smooth Lévy functionals in $\mathbb{D}_{1,2}$. *Probability and Mathematical Statistics*, 31(1):1–15, 2011.
- [L] E. Laukkarinen. On Malliavin calculus for functionals of Lévy processes.
- [GGL] C. Geiss, S. Geiss and E. Laukkarinen. A note on Malliavin fractional smoothness for Lévy processes and approximation. To appear in *Potential Analysis*.

The author of this dissertation has actively taken part in the research of the papers [GGL] and [GL].

Introduction

This thesis comprehends Malliavin calculus for Lévy processes based on Itô's chaos decomposition, fractional smoothness and approximation of stochastic integrals. Our interest is in functionals of Lévy processes such as $f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}})$, where $X_{t_k} - X_{t_{k-1}}$, $k = 1, \dots, m$, are increments of a Lévy process and f is a Borel function. An explicit formulation of the Malliavin derivative is given using a difference quotient and weak derivative of the functional. In particular, $f(X_1)$ is considered. The Skorohod integral is expressed using pathwise integration for a class of stochastic fields.

Certain stochastic integrals are approximated in $L_2(\mathbb{P})$ by their left Riemann sums and also with the optimal choice of a discrete time process. For example, the stochastic integral $\int_{(0,1]} \varphi_{t-} dX_t$ arising from the Galtchouk-Kunita-Watanabe representation

$$f(X_1) = c + \int_{(0,1]} \varphi_{t-} dX_t + \mathcal{N}$$

is approximated. The convergence rate of the approximation error under certain discretizations is related to Malliavin fractional smoothness of the integral.

1 Malliavin calculus

Malliavin calculus or *stochastic calculus of variations* merges differential calculus and probability theory. Initially Paul Malliavin [39, 38] gave the basis for the theory while investigating the smoothness of the density of a random variable providing a probabilistic proof for Hörmander's theorem in 1978. The differential calculus on the Wiener space was further developed by several mathematicians such as Stroock [58, 59], Bismut [11] and Watanabe [62] and was applied again in studying the regularity of probability laws of solutions of stochastic differential equations driven by Brownian motion.

Later Malliavin calculus was applied to computing the trading strategy of contingent claims in complete markets. *The Clark-Ocone formula* by Ocone [45], an explicit interpretation of *the Clark representation formula* [14, 15], was used by Ocone and Karatzas [46] in 1991. Since then more applications in finance have been discovered, such as computation of *greeks* by Fournié et al. [20] in 1999.

Meanwhile there was increasing interest in using Lévy process based models in finance as well as investigation of smoothness of densities of solutions of stochastic differential equations driven by Lévy processes. The question whether the theory of Malliavin calculus could be extended to Lévy processes

with jumps gave rise. In the first attempts the Malliavin derivative was defined as a *stochastic gradient* by Bass and Cranston [4], Bichteler et al. [10] and Norris [41]. Another approach developed by Carlen and Pardoux [13], E-Khatib and Privault [19], Malliavin and Thalmaier [40] and others is based on the concept of *pathwise instantaneous derivative* or *true derivative*.

A third orientation uses *Itô chaos decomposition* by Itô [33], and it has been the tool for Nualart and Vives [44], Privault [52], Lee and Shih [35, 34], Løkka [37], Øksendal and Proske [47], DiNunno et al. [18, 17], Solé et al. [56], Alós et al. [2], Applebaum [3] and many others. For the Brownian motion the chaos decomposition based definition and the stochastic gradient are analogous, but in general they differ.

One more way of creating Malliavin calculus for Lévy processes is using *Teugels martingales* based on power jump processes, where the existence of all moments for the process is required. This chaotic representation was shown by Nualart and Schoutens [43], and Malliavin calculus based on it has been studied by authors such as Leon et al. [36] and Davis and Johansson [16] as well as Benth et al. [7] and Solé et al. [57], who compare the two chaos expansion based approaches.

In this thesis we consider Malliavin calculus which is founded on Itô's chaos decomposition.

1.1 Lévy processes

Considering stochastic processes starting at zero with independent and stationary increments, the only such process with continuous paths is the Brownian motion. This process is the underlying stochastic process in Paul Malliavin's calculus. When the trajectories are not necessarily continuous, but almost surely right-continuous with left limits, then such processes are called Lévy processes.

Each Lévy process admits a Lévy measure, which expresses the jump intensity of the process. Given a Lévy process $X = (X_t)_{t \geq 0}$, we write $\Delta X_t := X_t - \lim_{0 \leq s \uparrow t} X_s$. The Lévy measure of X is the Borel measure $\nu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ given by

$$\nu(A) := \mathbb{E} \# \{t \in (0, 1] : \Delta X_t \in A \setminus \{0\}\} \quad \text{for all } A \in \mathcal{B}(\mathbb{R}).$$

The Lévy-Itô decomposition states that when $X = (X_t)_{t \geq 0}$ is a Lévy process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then there exist $\gamma \in \mathbb{R}$, $\sigma \geq 0$, a standard Brownian motion W and a Poisson random measure N on $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ with intensity $dt\nu(dx)$ such that

$$X_t = \gamma t + \sigma W_t + \iint_{(0,t] \times \{x:|x|>1\}} x N(dt, dx) + \iint_{(0,t] \times \{x:0<|x|\leq 1\}} x \tilde{N}(dt, dx)$$

a.s. for all $t \geq 0$, where $\tilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx)$.

1.2 Itô's chaos decomposition

In this subsection we shortly explain the chaos decomposition shown by Itô [33]. Let

$$L_2(\mathbb{P}) := L_2(\Omega, \mathcal{F}^X, \mathbb{P}),$$

where \mathcal{F}^X is the completion of the σ -algebra generated by X . We define measures $\mu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ and $\mathfrak{m} : \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \rightarrow [0, \infty]$ by

$$\begin{aligned} \mu(dx) &:= \sigma^2 \delta_0(dx) + x^2 \nu(dx), \\ \mathfrak{m}(dt, dx) &:= dt\mu(dx). \end{aligned}$$

For sets $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mathfrak{m}(B) < \infty$ we let

$$M(B) := \sigma \int_{\{t \in \mathbb{R}_+ : (t, 0) \in B\}} dW_t + \lim_{n \rightarrow \infty} \iint_{\{(t, x) \in B : 1/n < |x| < n\}} x d\tilde{N}(t, x)$$

where the convergence is taken in $L_2(\mathbb{P})$. The measure M is an independent random measure with $\mathbb{E}M(B_1)M(B_2) = \mathfrak{m}(B_1 \cap B_2)$ for all $B_1, B_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mathfrak{m}(B_1) < \infty$ and $\mathfrak{m}(B_2) < \infty$.

We write $L_2(\mathfrak{m}^{\otimes 0}) := \mathbb{R}$ and $L_2(\mathfrak{m}^{\otimes n}) := L_2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^n, \mathfrak{m}^{\otimes n})$ for $n = 1, 2, \dots$. We define the multiple integral of order n ,

$$I_n : L_2(\mathfrak{m}^{\otimes n}) \rightarrow L_2(\mathbb{P})$$

as follows: Set $I_0(f_0) := f_0$ for $f_0 \in \mathbb{R}$. Let $n \geq 1$. Any mapping in $L_2(\mathfrak{m}^{\otimes n})$ can be approximated by simple functions of the form

$$\sum_{k=1}^m a_k \mathbb{1}_{B_1^k}(t_1, x_1) \otimes \cdots \otimes \mathbb{1}_{B_n^k}(t_n, x_n),$$

where $a_k \in \mathbb{R}$, $B_i^k \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$, $\mathfrak{m}(B_i^k) < \infty$ and $B_i^k \cap B_j^k = \emptyset$ for $k = 1, \dots, m$, $i, j = 1, \dots, n$, $j \neq i$ and $m = 1, 2, \dots$. For a simple function the multiple integral is defined by

$$I_n \left(\sum_{k=1}^m a_k \mathbb{1}_{B_1^k} \otimes \cdots \otimes \mathbb{1}_{B_n^k} \right) := \sum_{k=1}^m a_k M(B_1^k) \cdots M(B_n^k).$$

We denote by \tilde{f}_n the symmetrization of f_n ,

$$\tilde{f}_n((t_1, x_1), \dots, (t_n, x_n)) := \frac{1}{n!} \sum_{\pi \in \pi_n} f_n((t_{\pi(1)}, x_{\pi(1)}), \dots, (t_{\pi(n)}, x_{\pi(n)})),$$

where π_n is the set of all permutations $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. We have that $I_n(f_n) = I_n(\tilde{f}_n)$ and $\|I_n(f_n)\|_{L_2(\mathbb{P})}^2 = n! \|\tilde{f}_n\|_{L_2(\mathfrak{m}^{\otimes n})}^2$ for any simple function $f_n \in L_2(\mathfrak{m}^{\otimes n})$. By the denseness of simple functions in $L_2(\mathfrak{m}^{\otimes n})$ and continuity of I_n we may define $I_n(f_n)$ for any $f_n \in L_2(\mathfrak{m}^{\otimes n})$ as the $L_2(\mathbb{P})$ -limit of $I_n(f_n^{(m)})$, where $(f_n^{(m)})_{m=1}^\infty$ are simple functions converging to f_n in $L_2(\mathfrak{m}^{\otimes n})$.

See [33] for further properties of I_n .

Theorem 1 (Theorem 2, [33]). *Let $F \in L_2(\mathbb{P})$. Then there exist functions $f_n \in L_2(\mathfrak{m}^{\otimes n})$, $n = 0, 1, 2, \dots$, such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad a.s..$$

Furthermore,

$$\|F\|_{L_2(\mathbb{P})} = \sqrt{\sum_{n=0}^{\infty} n! \|\tilde{f}_n\|_{L_2(\mathfrak{m}^{\otimes n})}^2}.$$

1.3 The Malliavin derivative and known results

The Malliavin Sobolev space $\mathbb{D}_{1,2}$ is the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$ such that

$$\|F\|_{\mathbb{D}_{1,2}} := \sqrt{\sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{L_2(\mathfrak{m}^{\otimes n})}^2} < \infty.$$

For $F \in \mathbb{D}_{1,2}$ the Malliavin derivative $D : \mathbb{D}_{1,2} \rightarrow L_2(\mathfrak{m} \otimes \mathbb{P})$ is defined by

$$D_{t,x}F := \sum_{n=1}^{\infty} n I_{n-1} \left(\tilde{f}_n(\cdot, (t, x)) \right) \quad \text{in } L_2(\mathfrak{m} \otimes \mathbb{P}).$$

Our aim is to give an explicit representation for DF when F is a functional of increments of the Lévy process. If X is the Brownian motion, then the Malliavin derivative reduces to $D_{\cdot,0}$, and it is well known that

$$D_{t,0}F := \sum_{k=1}^m \frac{\partial}{\partial x_k} f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \mathbb{1}_{(t_{k-1}, t_k]}(t)$$

for $F = f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}})$ with $f \in C_p^\infty(\mathbb{R}^m)$ (i.e. f is infinitely many times differentiable in all coordinates and is of at most polynomial growth).

The Malliavin derivative $D_{t,x}$ for $x \neq 0$ has been shown to relate to a shift transformation or a difference quotient: Nualart and Vives [44] consider the chaos expansion with respect to \tilde{N} instead of M on the Poisson space: For a measure space $(T, \mathcal{B}, \lambda)$ satisfying certain conditions let

$$\Omega := \{\omega = \sum_{i \in I} \delta_{z_i} : I \subseteq \mathbb{N}, z_i \in T\}$$

and $N(A)(\omega) := \omega(A)$ for $A \in \mathcal{B}$ and choose \mathbb{P} such that N is a Poisson random measure on $(\Omega, \sigma(N), \mathbb{P})$. Nualart and Vives [44] show that the Malliavin derivative \mathcal{D} satisfies

$$\mathcal{D}_z F(\omega) = F(\omega + \delta_z) - F(\omega).$$

An analogous transformation is also used by Picard [50, 49, 51]. Løkka [37] uses a general probability space and he showed that if $\sigma = 0$, X is a square integrable martingale and the distribution of X_1 is absolutely continuous, then

$$\mathcal{D}_{t,x} F = \sum_{k=1}^m f(X_{t_1} + x \mathbb{1}_{(0,t_1]}(t), \dots, X_{t_m} + x \mathbb{1}_{(0,t_m]}) - F$$

for $F = f(X_{t_1}, \dots, X_{t_m})$ with $f \in C_c^\infty(\mathbb{R}^m)$ (i.e. f is smooth and has compact support).

Solé et al. [56] use the chaos decomposition with respect to the measure M . They construct a canonical probability space and show that for $x \neq 0$

$$D_{t,x} F(\omega) = \frac{F(\omega_{t,x}) - F(\omega)}{x},$$

where the shift transformation $\omega_{t,x}$ can be interpreted as adding a jump of size x at time t to the trajectory.

The crucial difference in using the measure \tilde{N} instead of M in the pure jump case is in multiplying the Malliavin derivative by x . Essentially, $\mathcal{D}_{t,x} F = x D_{t,x} F$.

1.4 Finding explicit representations for the Malliavin derivative on a general probability space

We convert above representation properties to any complete probability space generated by a Lévy process for certain functionals of the Lévy process using a difference quotient of the functional.

Definition 1. For $f : \mathbb{R}^m \rightarrow \mathbb{R}$ we denote by Δ_x^i the difference quotient in the i th coordinate,

$$\Delta_x^i f(x_1, \dots, x_m) := \frac{f(x_1, \dots, x_{i-1}, x_i + x, x_{i+1}, \dots, x_m) - f(x_1, \dots, x_m)}{x},$$

for $x \neq 0$. If $f \in L_1^{loc}(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), dx)$ and there exists a function $h_i \in L_1^{loc}(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), dx)$ such that

$$\int_{\mathbb{R}^m} f(x) \frac{\partial}{\partial x_i} \varphi(x) dx = - \int_{\mathbb{R}^m} h_i(x) \varphi(x) dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^m),$$

then we say that f has a weak derivative in the direction i and write

$$\Delta_0^i f := h_i.$$

If $m = 1$, we also use the notation $\Delta f := \Delta^1 f$.

Definition 2. Given a Borel function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, we say that the random variable $f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}})$ is in the domain of \mathbf{D} , $\text{Dom}(\mathbf{D})$, if

- (i) in case $\sigma \neq 0$, the function f has weak derivatives of order one in all coordinates, i.e. $\Delta_0^k f$ exists, and $\Delta_0^k f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \in L_2(\mathbb{P})$ for all $k = 1, \dots, m$ and
- (ii) $\Delta_x^k f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \mathbb{1}_{\mathbb{R}_0}(x) \in L_2(\mu \otimes \mathbb{P})$ for all $k = 1, \dots, m$.

For $f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \in \text{Dom}(\mathbf{D})$ we define

$$\begin{aligned} & \mathbf{D}_{t,x} f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \\ & := \begin{cases} 0, & \text{for } x = 0 \text{ if } \sigma = 0 \\ \sum_{k=1}^m \Delta_x^k f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \mathbb{1}_{(t_{k-1}, t_k]}(t), & \text{otherwise.} \end{cases} \end{aligned}$$

If $f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \in \text{Dom}(\mathbf{D})$, then

$$\mathbf{D}f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \in L_2(\mathfrak{m} \otimes \mathbb{P}).$$

Definition 3 (Smooth random variables, \mathcal{S}). We call a random variable F smooth, if there exists a set of time points $\tau = \{0 \leq t_0 < t_1 < \dots < t_m < \infty\}$ and a function $f \in C_c^\infty(\mathbb{R}^m)$ such that

$$F = f(X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_m} - X_{t_{m-1}}) \quad \text{a.s.}$$

We denote the set of smooth random variables by \mathcal{S} .

Theorem 2 ([GL], Theorem 4.1 and Lemma 3.1). *The set of smooth random variables \mathcal{S} is dense in $\mathbb{D}_{1,2}$, $\mathcal{S} \subseteq \text{Dom}(\mathbf{D})$ and*

$$DF = \mathbf{D}F \quad \mathfrak{m} \otimes \mathbb{P}\text{-a.e. for all } F \in \mathcal{S}.$$

Remark 1. *Note that for $F \in \mathcal{S}$ the representation $F = f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}})$ is not unique. However, if $F = f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) = g(X_{s_1} - X_{s_0}, \dots, X_{s_k} - X_{s_{k-1}}) \in \mathcal{S}$, then*

$$\mathbf{D}f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) = \mathbf{D}g(X_{s_1} - X_{s_0}, \dots, X_{s_k} - X_{s_{k-1}})$$

in $L_2(\mathfrak{m} \otimes \mathbb{P})$.

It is now possible to define the Malliavin derivative on the set of smooth random variables using the operator \mathbf{D} and obtain an equivalent definition by taking its closure. The relation $\mathbf{D} = D$ yields a criterion for $f(X_1) \in \mathbb{D}_{1,2}$ for a Borel function f .

Proposition 1 ([L], Corollary 3.1). *Let $f(X_1) \in L_2(\mathbb{P})$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function. Then $f(X_1) \in \mathbb{D}_{1,2}$ if and only if $f(X_1) \in \text{Dom}(\mathbf{D})$. Moreover, if $f(X_1) \in \mathbb{D}_{1,2}$, then*

$$Df(X_1) = \mathbf{D}f(X_1) \quad \mathfrak{m} \otimes \mathbb{P}\text{-a.e.}$$

From Proposition 1 one might make the conjecture that $\text{Dom}(\mathbf{D}) \subset \mathbb{D}_{1,2}$ and $\mathbf{D}F = DF$ for any $F \in \text{Dom}(\mathbf{D})$. For simplicity, the claim is proved only for $f(X_1)$ in [L].

When X is the Brownian motion it is well known that the Malliavin derivative admits the chain rule: let $F \in \mathbb{D}_{1,2}$ and $\varphi \in C^1(\mathbb{R})$ be Lipschitz continuous. Then $\varphi(F) \in \mathbb{D}_{1,2}$ and $D_{t,0}\varphi(F) = \varphi'(F)D_{t,0}F$ $\mathfrak{m} \otimes \mathbb{P}$ -a.e. Solé et al. [56] generalize the chain rule in the following way: Let $F \in \mathbb{D}_{1,2}$ be measurable with respect to the completion of the σ -algebra generated by the Brownian motion part of the Lévy process and $G \in L_2(\mathbb{P})$ be measurable with respect to the completion of the σ -algebra generated by the jump part. Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable in the first variable such that the mapping $(x_1, x_2) \mapsto \frac{\partial}{\partial x_1}\varphi(x_1, x_2)$ is bounded. Suppose $\varphi(F, G) \in L_2(\mathbb{P})$. Then

$$D_{\cdot,0}\varphi(F, G) = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, (\cdot, 0)))$$

is defined in $L_2(\mathfrak{m} \otimes \mathbb{P})$ and

$$D_{\cdot,0}\varphi(F, G) = \frac{\partial}{\partial x_1}\varphi(F, G)D_{\cdot,0}F \quad \mathfrak{m} \otimes \mathbb{P} - \text{a.e.}$$

From the difference quotient formula in Proposition 1 one can immediately see that the chain rule does not apply for $x \neq 0$. However, we have the following equation.

Proposition 2 ([GL], Proposition 5.1). *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz continuous and $F \in \mathbb{D}_{1,2}$. Then $\varphi(F) \in \mathbb{D}_{1,2}$ and*

$$D_{t,x}\varphi(F) = \begin{cases} GD_{t,0}F & \text{for } x = 0 \\ \frac{\varphi(F+xD_{t,x}F) - \varphi(F)}{x} & \text{for } x \neq 0 \end{cases}$$

$m \otimes \mathbb{P}$ -a.e., where G is a random variable bounded by the Lipschitz constant of φ .

If the function φ in Proposition 2 is continuously differentiable, then it holds that $G = \varphi'(F)$. This can be seen by the same procedure as in [42, Proposition 1.2.2].

1.5 A chaos expansion with Hermite polynomials

We present a chaos decomposition for $f(X_1) \in L_2(\mathbb{P})$ using Hermite polynomials H_k , that is $H_0(x) := 1$ and $H_k(x) := \frac{(-1)^k}{k!} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$ for $k = 1, 2, \dots$

Proposition 3 ([L], Proposition 2.1). *Assume $f(X_1) = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$ and let $Y_1 := X_1 - \sigma W_1$.*

(i) *There exist functions G_k , $k = 0, 1, 2, \dots$, such that $G_k(Y_1) \in L_2(\mathbb{P})$ and*

$$f(X_1) = \sum_{k=0}^{\infty} G_k(Y_1) H_k(W_1) \sigma^k \quad \text{in } L_2(\mathbb{P}).$$

(ii) *There exist symmetric functions $g_n \in L_2(\mu^{\otimes n})$ such that*

$$\tilde{f}_n((t_1, x_1), \dots, (t_n, x_n)) = g_n(x_1, \dots, x_n) \mathbb{1}_{(0,1]^{\otimes n}}(t_1, \dots, t_n)$$

and

$$G_k(Y_1) = \sum_{m=0}^{\infty} \frac{(m+k)!}{m!} I_m(g_{m+k}(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1]^{\otimes m} \times \mathbb{R}_0}), \quad k = 0, 1, \dots$$

(iii) *If $f \in C_c^\infty(\mathbb{R})$, then the functions G_k are obtained from*

$$G_k = \int_{\mathbb{R}} \frac{d^k}{dx^k} f(\sigma x + \cdot) \mathbb{P}_{W_1}(dx) \quad \text{for } k = 0, 1, \dots$$

The fact that $\tilde{f}_n = g_n \mathbb{1}_{(0,1]}^{\otimes n}$ was shown by Baumgartner [5] and follows also from the results in [GL]. According to [L, Lemma 3.1], if the sum $D_{t,x}^0 f(X_1) := \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot, (t, x))) \mathbb{1}_{\mathbb{R}_+ \times \{0\}}(t, x)$ converges in $L_2(\mathfrak{m} \otimes \mathbb{P})$, then

$$D^0 f(X_1) = \sum_{k=1}^{\infty} G_k(Y_1) H'_k(W_1) \sigma^{k-1} \mathbb{1}_{(0,1] \times \{0\}} \quad \text{in } L_2(\mathfrak{m} \otimes \mathbb{P}).$$

1.6 The Skorohod integral

The adjoint of the Malliavin derivative is the Skorohod integral and it is commonly denoted by δ . If $u \in L_2(\mathfrak{m} \otimes \mathbb{P})$ and there exists $H \in L_2(\mathbb{P})$ such that

$$(u, DG)_{L_2(\mathfrak{m} \otimes \mathbb{P})} = (H, G)_{L_2(\mathbb{P})} \quad \text{for all } G \in \mathbb{D}_{1,2},$$

then $u \in \text{Dom}(\delta)$ and $\delta(u) = H$. If u is predictable, then the Skorohod integral coincides with the Itô integral.

The forward integral is defined pathwise and - like the Skorohod integral - it extends the Itô integral to anticipating integrands (see Russo and Vallois [53] for continuous integrators). We are interested in the relation of the Skorohod integral and a pathwise integral with respect to the random measure M .

Considering certain pathwise integrable random fields, the relation between the Skorohod integral and a pathwise integral is shown by Alós et al. [2, Corollary 2.9] in the canonical probability space under the assumption $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$. Øksendal and Zhang [48, Lemma 2.1] consider the relation for a class of pathwise integrable random fields in the pure jump case. In [48] the space Ω is the continuous dual of the Schwartz space and X is an $L_2(\mathbb{P})$ -martingale. We show the relation on a general probability space and express the Skorohod integral using the pathwise integral on a dense subset of $\text{Dom}(\delta)$.

Let $m \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^{m+1})$ such that the set $\{x : f(y, x) \neq 0 \text{ for some } y \in \mathbb{R}^m\}$ is bounded, $0 \leq t_0 < t_1 < \dots < t_m < \infty$ and $k \in \{1, \dots, m\}$. Denote

$$u(t, x) := f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t), \quad (1)$$

$$bu(t, x) := f(X_{t_1} - X_{t_0}, \dots, X_{t_k} - X_{t_{k-1}} - x, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t)$$

and

$$\begin{aligned} & \Delta^- bu(t, x) \\ & := \Delta_x^k f(X_{t_1} - X_{t_0}, \dots, X_{t_k} - X_{t_{k-1}} - x, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t). \end{aligned}$$

For $u(t, 0) = bu(t, 0)$ we define the pathwise integral as:

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times \{0\}} bu(t, x) M(d^p t, dx) \\ & := f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, 0) \sigma(W_{t_k} - W_{t_{k-1}}) \end{aligned}$$

a.s. Moreover, we define

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times |x| > \varepsilon} bu(t, x) M(d^p t, dx) \\ & := \sum_{|\Delta X_t| > \varepsilon} bu(t, \Delta X_t) \Delta X_t - \int_{R_+} \int_{|x| > \varepsilon} bu(t, x) x dt \nu(dx). \end{aligned}$$

The right-hand side of the above equation is well defined since the sum is a.s. finite.

By [L, Lemma 4.1] we have $u \in \text{Dom}(\delta)$, and [L, Proposition 4.1] states that the linear span of random fields of the form (1) is dense in $\text{Dom}(\delta)$. The Skorohod integral of u can be expressed using the pathwise integral in the following way.

Theorem 3 ([L], Theorem 4.1). *For compact sets $U_\varepsilon \subset \mathbb{R} \setminus \{0\}$ such that $U_\varepsilon \subseteq U_{\varepsilon'}$ for $\varepsilon' \leq \varepsilon$ and $\bigcup_{\varepsilon > 0} U_\varepsilon = \mathbb{R} \setminus \{0\}$, it holds that*

$$\begin{aligned} \delta(u) = \lim_{\varepsilon \rightarrow 0} & \left(\iint_{\mathbb{R}_+ \times (U_\varepsilon \cup \{0\})} bu(t, x) M(d^p t, dx) \right. \\ & \left. - \iint_{\mathbb{R}_+ \times (U_\varepsilon \cup \{0\})} \Delta^- bu(t, x) \mathfrak{m}(dt, dx) \right), \end{aligned}$$

where the limit is taken in $L_2(\mathbb{P})$.

2 Fractional smoothness and approximation

Fractional smoothness of random variables is considered here in terms of fractional order Sobolev spaces obtained by real interpolation. These spaces have been considered by Watanabe [62] and Hirsch [30] on the Wiener space and Adams [1] for the usual Sobolev spaces. Letting γ denote the standard normal distribution, Geiss and Geiss [21] and S. Geiss and Hujo [25] found out that the interpolation spaces between the weighted Sobolev space $\mathbb{D}_{1,2}(\gamma)$ and $L_2(\gamma)$ are connected to the convergence rate of the approximation of stochastic integrals driven by (geometric) Brownian motion.

In mathematical finance the models of perfect hedging are based on trading continuously. In practice continuous trading is infeasible. When discretizing a trading strategy, an error occurs. The error has been measured in

the weak sense by, for instance, Bertsimas, Kogan and Lo [9], Hayashi and Mykland [29] as well as S. Geiss and Toivola [26] for continuous processes and by Tankov and Voltchkova [60] for jump processes. Approximation in $L_p(\mathbb{P})$ for $p > 2$ was considered by S. Geiss and Toivola [27]. In this work we measure the error in $L_2(\mathbb{P})$ as has been done by Zhang [63], Gobet and Temam [28], S. Geiss [23, 24], Geiss and Geiss [21, 22], S. Geiss and Hujo [25], Hujo [31, 32] and Seppälä [55] for continuous processes and Brodén and Tankov [12] for jump processes. We consider stochastic integrals driven by an exponential Lévy process or a Lévy process and show that the convergence rate depends on the choice of discretization time points and the fractional smoothness of the stochastic integral.

Consider a Lévy process X which is an $L_2(\mathbb{P})$ -martingale and its Doléans-Dade exponential $S = \mathcal{E}(X)$,

$$S_t = 1 + \int_{(0,t]} S_{r-} dX_r.$$

Then a random variable $f(S_T) \in L_2(\mathbb{P})$ admits the orthogonal Galtchouk-Kunita-Watanabe representation

$$f(S_T) = V_0 + \int_{(0,T]} \varphi_{t-} dS_t + \mathcal{N}.$$

We are interested in the quantitative Riemann approximation of the stochastic integral $\int_{(0,T]} \varphi_t dS_t$ and its relation to fractional smoothness. When X is the Brownian motion, then $\mathcal{N} = 0$. We consider discrete time points $0 = t_0 < t_1 < \dots < t_n = T$ and measure the error in $L_2(\mathbb{P})$. The convergence rate of the approximation error is r , if

$$\left\| \int_{(0,T]} \varphi_{t-} dS_t - \sum_{k=1}^n \varphi_{t_{k-1}} (S_{t_k} - S_{t_{k-1}}) \right\|_{L_2(\mathbb{P})} \sim_c n^{-r}$$

for all n .

First results of the convergence rate were obtained for the Brownian motion. Zhang [63] showed that if f is absolutely continuous and is of polynomial growth, then the convergence rate is $r = 1/2$. Gobet and Temam [28] proved that for $f = \mathbb{1}_{[K,\infty)}$ it holds $r = \frac{1}{4}$ and for $f(x) = (x - K)_+^a$, we have $r = \frac{1}{4} + \frac{a}{2}$, $a \in (0, \frac{1}{2})$. Both Zhang [63] as well as Gobet and Temam [28] used equidistant time nets and the essential observation is that the more smooth the pay-off function f is, the more accurate is the approximation. S. Geiss [23] showed that using non-equidistant time nets may improve the approximation rate.

Geiss and Geiss [21] showed that whenever the pay-off function f is in a Besov space $B_{2,q}^\theta(\gamma)$, the optimal convergence rate $r = \frac{1}{2}$ can be obtained using certain non-equidistant time nets. Besov spaces $B_{2,q}^\theta(\gamma) = (L_2(\gamma), \mathbb{D}_{1,2}(\gamma))_{\theta,q}$ are obtained by real interpolation between $L_2(\gamma)$ and the weighted Sobolev space $\mathbb{D}_{1,2}(\gamma)$, where γ is the distribution of S_T . By S. Geiss and Hujo [25] a characterization for $f \in B_{2,q}^\theta(\gamma)$ is given by means of the behaviour of the approximation.

These results were obtained in the Brownian motion setting. First approximation results concerning the discontinuous exponential Lévy model came from Brodén and Tankov [12]. They use equidistant time-nets and compare the convergence rates of the optimal trading strategy and delta-hedging and show that under certain assumptions the optimal strategy leads to faster convergence. Brodén and Tankov [12] compute under various assumptions the convergence rate and show that it depends on the small-jump behaviour of the Lévy process when $\sigma = 0$: for the optimal trading strategy they state some integrability and smoothness conditions under which $r = \frac{1}{2}$. For the digital option $\mathbb{1}_{[K,\infty)}(S_T)$ they show the convergence rate $r = \frac{3}{2}\alpha^{-1} - \frac{1}{2}$ for $\nu(dx) = k(x)|x|^{-1-\alpha}dx$ and $\alpha \in (\frac{3}{2}, 2)$ for a class of functions k .

We connect the convergence rate in the Lévy process setting to Malliavin fractional smoothness. The underlying process in [GGL] is the Lévy process X itself or its stochastic exponential S . The convergence results are analogous for the two processes. Therefore we omit here the case of the stochastic exponential S .

The rest of this section involves approximation of a stochastic integral

$$F = \int_{(0,1]} \varphi_{t-} dX_t,$$

where the integrand is of the form

$$\varphi_t = \sum_{n=1}^{\infty} I_n \left(g_n \mathbb{1}_{(0,t]}^{\otimes n} \right), \quad t \in [0, 1),$$

with $g_n \in L_2(\mu^{\otimes n})$. It is notable that this is always the case when φ is the integrand from the Galtchouk-Kunita-Watanabe projection

$$f(X_1) = c + \int_{(0,1]} \varphi_{t-} dX_t + \mathcal{N}.$$

For a time net $\tau = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ we denote by $a(F; \tau)$ the $L_2(\mathbb{P})$ -approximation error,

$$a(F; \tau) := \left\| F - \sum_{k=1}^n \varphi_{t_{k-1}} (X_{t_k} - X_{t_{k-1}}) \right\|_{L_2(\mathbb{P})}.$$

The approximation number $a(F; \tau)$ corresponds to $a_X^{opt}(F; \tau)$ in [GGL]. Here the integral is approximated with its left Riemann sum, which also minimizes $\|F - \sum v_{k-1}(X_{t_k} - X_{t_{k-1}})\|_{L_2(\mathbb{P})}$ over square integrable predictable discrete time processes v , when the discretization time points are fixed. Therefore the left Riemann sum gives optimal approximation. When the underlying stochastic process is the stochastic exponential, then the left Riemann sum is no longer optimal, but the convergence rate remains the same.

2.1 Approximation

When the underlying process is continuous, then it is known from Geiss and Geiss [21] that the convergence rate is never better than $\frac{1}{2}$, unless φ is a deterministic constant. The following theorem states that the best possible convergence rate for any Lévy process is also $r = \frac{1}{2}$.

Theorem 4 (Theorem 5, [GGL]). *Unless there are $a, b \in \mathbb{R}$ such that $F = a + bX_1$ a.s., we have*

$$\liminf_{n \rightarrow \infty} \sqrt{n} \left[\inf_{\#\tau=n+1} a(F; \tau) \right] > 0.$$

Let $H(F; t) := \sqrt{\frac{d}{dt} \mathbb{E}|\varphi_t^2|}$. When the underlying process is continuous, S. Geiss [23] showed that

$$\frac{1}{c} a(F; \tau) \leq \left(\sum_{k=1}^n \int_{t_{k-1}}^{t_k} (t_k - t) H^2(F; t) dt \right)^{\frac{1}{2}} \leq c a(F; \tau) \quad (2)$$

for a constant $c \geq 1$ not depending on τ or F . In [GGL] we see that the approximation number is related to an integral of H also for processes with jumps.

Theorem 5 (Theorem 3, [GGL]). *It holds that*

$$a(F; \tau) = \mu(\mathbb{R}) \left(\sum_{k=1}^n \int_{t_{k-1}}^{t_k} (t_k - t) H^2(F; t) dt \right)^{\frac{1}{2}}.$$

If the underlying process is the stochastic exponential S instead of the Lévy process X , the equation of Theorem 5 turns into an equivalence like (2) with constant $c = c(\tau)$ such that $c(\tau_n) \rightarrow 1$ if $\sup_{t_{k-1}, t_k \in \tau_n} |t_k - t_{k-1}| \rightarrow 0$ as $n \rightarrow \infty$.

According to Seppälä [55, Theorem 2.4], the optimal convergence rate $r = \frac{1}{2}$ for $(\inf_{\tau_n} a(F; \tau_n))_{n=1}^{\infty}$ is obtained if and only if $\int_0^1 H(F; t) dt < \infty$. In

this case one gets the optimal rate using the regular time nets generated by $H(F; \cdot)$, i.e. the nets $\tau_n = \{0 = t_0^n < t_1^n < \dots < t_n^n = 1\}$, which satisfy

$$\int_{t_{k-1}^n}^{t_k^n} H(F; t) dt = \frac{1}{n} \int_0^1 H(F; t) dt \quad \text{for all } k = 1, \dots, n.$$

The function $H(F; \cdot)$ is increasing, which causes the need for the time points t_k^n to be concentrated close to 1. The regular time nets generated by $t \mapsto (1-t)^{\theta-1}$, where $\theta \in (0, 1]$, are the following nets τ_n^θ .

Definition 4. For $\theta \in (0, 1]$ let us denote by τ_n^θ the time net which consists of the time points $t_k = 1 - (1 - \frac{k}{n})^{\frac{1}{\theta}}$, $k = 0, \dots, n$.

If $H(F; t) \sim (1-t)^{\theta-1}$, then the optimal convergence rate is again achieved with the time nets τ_n^θ . S. Geiss [23] showed this before the results of Seppälä [55]. These time nets give the optimal convergence rate also in the case that F has certain Malliavin fractional smoothness (see [21] for continuous processes and [GGL] for processes with jumps).

2.2 Fractional smoothness and its connection to approximation

The convergence rate of the approximation relates to fractional smoothness in terms of Besov spaces. This observation was made first for approximation in $L_2(\mathbb{P})$ by Geiss and Geiss [21] and S. Geiss and Hujo [25] and Seppälä [55]. S. Geiss and Toivola considered weak convergence [26] and $L_p(\mathbb{P})$ convergence for $p > 2$ [27]. Seppälä [54] studied the convergence rate for stochastic integrals with no fractional smoothness in the usual sense. All these papers assume the underlying stochastic process to be continuous.

The Besov spaces $B_{2,q}^\theta$ are defined as interpolation spaces between $\mathbb{D}_{1,2}$ and $L_2(\mathbb{P})$. We use the *K-method* of real interpolation to describe the spaces $B_{2,q}^\theta$.

For $F \in L_2(\mathbb{P})$ and $t > 0$, the *K-functional* is defined by

$$K(F, t; L_2(\mathbb{P}), \mathbb{D}_{1,2}) := \inf \{ \|F_1\|_{L_2(\mathbb{P})} + t \|F_2\|_{\mathbb{D}_{1,2}} : F = F_1 + F_2 \}.$$

Given $\theta \in (0, 1)$ and $q \in [1, \infty]$, we let $B_{2,q}^\theta$ be the space of all $F \in L_2(\mathbb{P})$ such that

$$\|F\|_{B_{2,q}^\theta} := \left\| t^{-\theta} K(F, t; L_2(\mathbb{P}), \mathbb{D}_{1,2}) \right\|_{L_q((0, \infty), \frac{dt}{t})} < \infty.$$

The spaces $B_{2,q}^\theta$ are intermediate spaces of $\mathbb{D}_{1,2}$ and $L_2(\mathbb{P})$ and they have a lexicographical order,

$$\mathbb{D}_{1,2} \subseteq B_{2,q'}^\theta \subseteq B_{2,q}^\theta \subseteq B_{2,q'}^{\theta'} \subseteq L_2(\mathbb{P})$$

whenever $0 < \theta' < \theta < 1$ and $1 \leq q' \leq q \leq \infty$. We refer to Triebel [61], Bennet and Sharpley [6] or Bergh and Löfström [8] for more information about interpolation.

The following theorem was formulated for continuous processes by S. Geiss and Hujo [25], and it shows that the more smooth F is with respect to θ , the better is the approximation for the equidistant time net. The relation is shown for Lévy processes with jumps in [GGL].

Theorem 6 (Theorem 6, [GGL]). *Let $\theta \in (0, 1)$, $q \in [1, \infty]$ and τ_n be equidistant with $\#\tau_n = n + 1$. Then $F \in \mathbb{B}_{2,q}^\theta$ if and only if*

$$\left\| \left(n^{\frac{\theta}{2} - \frac{1}{q}} a(F; \tau_n) \right)_{n=1}^\infty \right\|_{\ell_q} < \infty.$$

For $q = \infty$ the theorem exposes the convergence rate $r = \frac{\theta}{2}$ for $F \in \mathbb{B}_{2,\infty}^\theta$. If $F \notin \mathbb{B}_{2,q}^\theta$ for any $(\theta, q) \in (0, 1) \times [1, \infty]$ then $a(F; \tau_n)$ converges to zero slower than n^{-r} for any $r > 0$. The convergence rate in the case of having no fractional smoothness has been investigated for continuous processes by Seppälä [54], who uses more general interpolation spaces.

Like for the Brownian motion (Geiss and Geiss [21]), the convergence rate can be improved by choosing appropriate non-equidistant time nets (see Theorem 7). On the other hand the convergence can be arbitrarily slow despite optimizing over time nets: when X is the Brownian motion, it was shown by Hujo [31] that for any sequence of positive real numbers $\beta = (\beta_n)_{n=1}^\infty$ with $\beta_n \downarrow 0$ there exists $f_\beta \in L_2(\gamma)$ such that

$$\inf_{\#\tau=n+1} a(f_\beta(X_1); \tau) \geq \beta_n \quad \text{for all } n.$$

The following theorem states that the optimal convergence rate $r = \frac{1}{2}$ can be attained for $F \in \mathbb{B}_{2,2}^\theta$ by using the time nets τ_n^θ from Definition 4. The less smooth the integral F is, the more we need the discretization time points to be concentrated close to 1. The observation was made first for continuous processes by S. Geiss [23] and Geiss and Geiss [21] and for processes with jumps in [GGL]. Seppälä [54] shows for the continuous underlying that this convergence rate is achievable also if $F \notin \mathbb{B}_{2,2}^\theta$ for any $\theta \in (0, 1)$, but F is contained in a more general interpolation space.

Theorem 7 (Theorem 7, [GGL]). *Let $\theta \in (0, 1]$, τ_n^θ be from Definition 4 and write $\mathbb{B}_{2,2}^1 := \mathbb{D}_{1,2}$. Then $F \in \mathbb{B}_{2,2}^\theta$ if and only if $\sup_n \sqrt{na}(F; \tau_n^\theta) < \infty$. If $F \in \mathbb{B}_{2,2}^\theta$, then*

$$\lim_{n \rightarrow \infty} \sqrt{na}(F; \tau_n^\theta) = \sqrt{\frac{1}{2\theta} \int_0^1 (1-t)^{1-\theta} H^2(F; t) dt}.$$

The fractional smoothness of $f(X_1) = c + \int_{(0,1]} \varphi_{t-} dX_t + \mathcal{N}$ is related to the smoothness of the term $F = \int_{(0,1]} \varphi_{t-} dX_t$ in the Galtchouk-Kunita-Watanabe decomposition:

$$f(X_1) \in \mathbb{B}_{2,q}^\theta \text{ implies } F \in \mathbb{B}_{2,q}^\theta$$

for $\theta \in (0, 1)$ and $q \in [1, \infty]$ and

$$f(X_1) \in \mathbb{D}_{1,2} \text{ implies } F \in \mathbb{D}_{1,2}$$

by [GGL, Lemma 3]. The integral part F may indeed have better smoothness than $f(X_1)$: Suppose X is tempered α -stable with $\alpha \in (1, \frac{3}{2})$ such that $\nu(dx) = d|x|^{-1-\alpha}(1+|x|)^{-m}dx$ for some $m > 2 - \alpha$ and $d > 0$. By [L, Example 3.1] and [GGL, Proposition 1] we have

$$\mathbb{1}_{[K,\infty)}(X_1) \notin \mathbb{D}_{1,2} \quad \text{and} \quad \int_{(0,1]} \varphi_{t-} dX_t \in \mathbb{D}_{1,2}.$$

3 Conclusions

Malliavin calculus for Lévy processes has been considered in this thesis on a general probability space. Some results from special probability spaces (canonical, dual of the Schwarz space) have been converted to a general probability space for functionals of the Lévy process. It was shown that the Malliavin derivative defined for smooth functionals using a difference quotient yields a definition which is equivalent to the definition based on Itô's chaos decomposition.

Some approximation results of stochastic integrals were generalized from the Brownian motion setting to the general Lévy process setting, provided that the underlying process is an $L_2(\mathbb{P})$ -martingale. Connections between $L_2(\mathbb{P})$ -approximation and Malliavin fractional smoothness were found. The fractional smoothness of functionals of a Lévy process as well as the fractional smoothness of stochastic integrals seem to depend heavily on the jump intensity of small jumps. However, so far only special cases have been investigated.

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Christel Geiss and Eija Laukkarinen

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DENSENESS OF CERTAIN SMOOTH LÉVY FUNCTIONALS IN $\mathbb{D}_{1,2}$

BY

CHRISTEL GEISS* AND EIJA LAUKKARINEN** (JYVÄSKYLÄ)

Abstract. The Malliavin derivative for a Lévy process (X_t) can be defined on the space $\mathbb{D}_{1,2}$ using a chaos expansion or in the case of a pure jump process also *via* an increment quotient operator. In this paper we define the Malliavin derivative operator D on the class \mathcal{S} of smooth random variables $f(X_{t_1}, \dots, X_{t_n})$, where f is a smooth function with compact support. We show that the closure of $L_2(\mathbb{P}) \supseteq \mathcal{S} \xrightarrow{D} L_2(\mathfrak{m} \otimes \mathbb{P})$ yields to the space $\mathbb{D}_{1,2}$. As an application we conclude that Lipschitz functions operate on $\mathbb{D}_{1,2}$.

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1. INTRODUCTION

In the recent years Malliavin calculus for Lévy processes has been developed using various types of chaos expansions. For example, Lee and Shih [5] applied a white noise approach, León et al. [6] worked with certain strongly orthogonal martingales, Løkka [7] and Di Nunno et al. [2] considered multiple integrals with respect to the compensated Poisson random measure and Solé et al. [11] used the chaos expansion proved by Itô [4].

This chaos representation from Itô applies to any square integrable functional of a general Lévy process. It uses multiple integrals like in the well-known Brownian motion case but with respect to an independent random measure associated with the Lévy process. Solé et al. propose in [12] a canonical space for a general Lévy process. They define for random variables on the canonical space the increment quotient operator

$$\Psi_{t,x}F(\omega) = \frac{F(\omega_{t,x}) - F(\omega)}{x}, \quad x \neq 0,$$

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in a pathwise sense, where, roughly speaking, $\omega_{t,x}$ can be interpreted as the outcome of adding at time t a jump of the size x to the path ω . They show that on the canonical Lévy space the Malliavin derivative $D_{t,x}F$ defined *via* the chaos expansion due to Itô and $\Psi_{t,x}F$ coincide a.e. on $\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega$ (where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$) whenever $F \in L_2$ and $\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}_0} |\Psi_{t,x}F|^2 d\mathfrak{m}(t,x) < \infty$ (see Section 2 for the definition of \mathfrak{m}). On the other hand, on the Wiener space, the Malliavin derivative is introduced as an operator D mapping smooth random variables of the form $F = f(W(h_1), \dots, W(h_n))$ into $L_2(\Omega; H)$, i.e.

$$DF = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(W(h_1), \dots, W(h_n)) h_i$$

(see, for example, [8]). Here f is a smooth function mapping from \mathbb{R}^n into \mathbb{R} such that all its derivatives have at most polynomial growth, and $\{W(h), h \in H\}$ is an isonormal Gaussian family associated with a Hilbert space H . The closure of the domain of the operator D is the space $\mathbb{D}_{1,2}$.

In the present paper we proceed in a similar way for a Lévy process $(X_t)_{t \geq 0}$. We will define a Malliavin derivative on a class of smooth random variables and determine its closure. The class of smooth random variables we consider consists of elements of the form $F = f(X_{t_1}, \dots, X_{t_n})$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function with compact support.

Analogously to results of Solé et al. [12] about the canonical Lévy space the Malliavin derivative $DF \in L_2(\mathfrak{m} \otimes \mathbb{P})$, defined *via* chaos expansion, can be expressed explicitly as a two-parameter operator $D_{t,x}$. For certain smooth random variables of the form $F = f(X_{t_1}, \dots, X_{t_n})$ we have

$$\begin{aligned} D_{t,x}f(X_{t_1}, \dots, X_{t_n}) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \mathbb{1}_{[0,t_i] \times \{0\}}(t,x) \\ &\quad + \Psi_{t,x}f(X_{t_1}, \dots, X_{t_n}) \mathbb{1}_{\{x \neq 0\}}(x) \end{aligned}$$

for $\mathfrak{m} \otimes \mathbb{P}$ -a.e. (t, x, ω) . Here $\Psi_{t,x}$ for $x \neq 0$ is given by

$$\begin{aligned} &\Psi_{t,x}f(X_{t_1}, \dots, X_{t_n}) \\ &:= \frac{f(X_{t_1} + x \mathbb{1}_{[0,t_1]}(t), \dots, X_{t_n} + x \mathbb{1}_{[0,t_n]}(t)) - f(X_{t_1}, \dots, X_{t_n})}{x}. \end{aligned}$$

Our main result is that the smooth random variables $f(X_{t_1}, \dots, X_{t_n})$ are dense in the space $\mathbb{D}_{1,2}$ defined *via* the chaos expansion. This implies that defining D as an operator on the smooth random variables as in Definition 3.2 below and taking the closure leads to the same result as defining D using Itô's chaos expansion (see Definition 2.1).

The paper is organized as follows. In Section 2 we shortly recall Itô's chaos expansion, the definition of the Malliavin derivative and some related facts. The

third and fourth sections focus on the introduction of the Malliavin derivative operator on smooth random variables and the determination of its closure. Applying the denseness result from the previous section we show in Section 5 that Lipschitz functions map from $\mathbb{D}_{1,2}$ into $\mathbb{D}_{1,2}$.

2. THE MALLIAVIN DERIVATIVE VIA ITÔ'S CHAOS EXPANSION

We assume a càdlàg Lévy process $X = (X_t)_{t \geq 0}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy triplet (γ, σ^2, ν) , where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is the Lévy measure. Then X has the Lévy–Itô decomposition

$$X_t = \gamma t + \sigma W_t + \int_{[0,t] \times \{|x| \geq 1\}} x dN(t,x) + \int_{[0,t] \times \{0 < |x| < 1\}} x d\tilde{N}(t,x),$$

where W denotes a standard Brownian motion, N is the Poisson random measure associated with the process X and \tilde{N} the compensated Poisson random measure, $d\tilde{N}(t,x) = dN(t,x) - dt d\nu(x)$. Consider the measures μ on $\mathcal{B}(\mathbb{R})$,

$$d\mu(x) := \sigma^2 d\delta_0(x) + x^2 d\nu(x),$$

and \mathfrak{m} on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$, where $\mathbb{R}_+ := [0, \infty)$,

$$d\mathfrak{m}(t,x) := dt d\mu(x).$$

For $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mathfrak{m}(B) < \infty$ let

$$M(B) = \sigma \int_{\{t \in \mathbb{R}_+ : (t,0) \in B\}} dW_t + \lim_{n \rightarrow \infty} \int_{\{(t,x) \in B : 1/n < |x| < n\}} x d\tilde{N}(t,x),$$

where the convergence is taken in the space $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Now $\mathbb{E}M(B_1)M(B_2) = \mathfrak{m}(B_1 \cap B_2)$ for all B_1, B_2 with $\mathfrak{m}(B_1) < \infty$ and $\mathfrak{m}(B_2) < \infty$. For $n = 1, 2, \dots$ let us write

$$L_2^n := L_2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^{\otimes n}, \mathfrak{m}^{\otimes n}).$$

For $f \in L_2^n$ Itô [4] defines a multiple integral $I_n(f)$ with respect to the random measure M . It follows that $I_n(f) = I_n(\tilde{f})$ a.s., where \tilde{f} is the symmetrization of f ,

$$\tilde{f}(z_1, \dots, z_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} f(z_{\pi(1)}, \dots, z_{\pi(n)})$$

for all $z_i = (t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}$, and \mathcal{S}_n denotes the set of all permutations on $\{1, \dots, n\}$.

Let $(\mathcal{F}_t^X)_{t \geq 0}$ be the augmented natural filtration of X . Then $(\mathcal{F}_t^X)_{t \geq 0}$ is right continuous ([9], Theorem I 4.31). Set $\mathcal{F}^X := \bigvee_{t \geq 0} \mathcal{F}_t^X$. By Theorem 2 of Itô [4] the chaos decomposition

$$L_2 := L_2(\Omega, \mathcal{F}^X, \mathbb{P}) = \bigoplus_{n=0}^{\infty} I_n(L_2^n)$$

holds, where $I_0(L_2^0) := \mathbb{R}$ and $I_n(L_2^n) := \{I_n(f_n) : f_n \in L_2^n\}$ for $n = 1, 2, \dots$. For $F \in L_2$ the representation

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

with $I_0(f_0) = \mathbb{E}F$ a.s. is unique if the functions f_n are symmetric. Furthermore,

$$\|F\|_{L_2}^2 = \sum_{n=0}^{\infty} n! \|\tilde{f}_n\|_{L_2^n}^2.$$

DEFINITION 2.1. Let $\mathbb{D}_{1,2}$ be the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2$ such that

$$\|F\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{L_2^n}^2 < \infty.$$

Set $L_2(\mathfrak{m} \otimes \mathbb{P}) := L_2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mathfrak{m} \otimes \mathbb{P})$. The Malliavin derivative $D : \mathbb{D}_{1,2} \rightarrow L_2(\mathfrak{m} \otimes \mathbb{P})$ is defined by

$$(2.1) \quad D_{t,x}F := \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n((t,x), \cdot)), \quad (t,x,\omega) \in \mathbb{R}_+ \times \mathbb{R} \times \Omega.$$

We consider (as Solé et al. [12]) the operators $D_{\cdot,0}$ and $D_{\cdot,x}$, $x \neq 0$, and their domains $\mathbb{D}_{1,2}^0$ and $\mathbb{D}_{1,2}^J$. For $\sigma > 0$ assume that $\mathbb{D}_{1,2}^0$ consists of random variables $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2$ such that

$$\|F\|_{\mathbb{D}_{1,2}^0}^2 := \|F\|_{L_2}^2 + \sum_{n=1}^{\infty} n \cdot n! \|\tilde{f}_n \mathbb{1}_{(\mathbb{R}_+ \times \{0\}) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|_{L_2^n}^2 < \infty.$$

For $\nu \neq 0$, let $\mathbb{D}_{1,2}^J$ be the set of $F \in L_2$ such that

$$\|F\|_{\mathbb{D}_{1,2}^J}^2 := \|F\|_{L_2}^2 + \sum_{n=1}^{\infty} n \cdot n! \|\tilde{f}_n \mathbb{1}_{(\mathbb{R}_+ \times \mathbb{R}_0) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|_{L_2^n}^2 < \infty,$$

where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. If both $\sigma > 0$ and $\nu \neq 0$, then

$$(2.2) \quad \mathbb{D}_{1,2} = \mathbb{D}_{1,2}^0 \cap \mathbb{D}_{1,2}^J.$$

In case $\nu = 0$, $D_{\cdot,0}$ coincides with the classical Malliavin derivative D^W (see, for example, [8]) except for a multiplicative constant, $D_t^W F = \sigma D_{t,0} F$.

In the next lemma we formulate a denseness result which will be used to determine the closure of the Malliavin operator from Definition 3.2 below.

LEMMA 2.1. *Let $\mathcal{L} \subseteq L_2$ be the linear span of random variables of the form*

$$M(T_1 \times A_1) \dots M(T_n \times A_n), \quad n = 1, 2, \dots$$

where the A_i 's are finite intervals of the form $(a_i, b_i]$ and the T_i 's are finite disjoint intervals of the form $T_i = (s_i, t_i]$. Then \mathcal{L} is dense in L_2 , $\mathbb{D}_{1,2}$, $\mathbb{D}_{1,2}^0$ and $\mathbb{D}_{1,2}^J$.

P r o o f. 1° First we consider the class of all linear combinations of

$$M(B_1) \dots M(B_n) = I_n(\mathbb{1}_{B_1 \times \dots \times B_n}),$$

$n = 1, 2, \dots$, where the sets $B_i \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ are disjoint and fulfill the condition $\mathfrak{m}(B_i) < \infty$. It follows from the completeness of the multiple integrals in L_2 (see [4], Theorem 2) that this class is dense in L_2 . Especially, the class of all linear combinations of $\mathbb{1}_{B_1 \times \dots \times B_n}$ with disjoint sets B_1, \dots, B_n of finite measure \mathfrak{m} is dense in $L_2^n = L_2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^{\otimes n}, \mathfrak{m}^{\otimes n})$. Let \mathcal{H}_n be the linear span of $\mathbb{1}_{(T_1 \times A_1) \times \dots \times (T_n \times A_n)}$, where $A_i = (a_i, b_i]$ and $T_i = (s_i, t_i]$. One can easily see that \mathcal{H}_n is dense in L_2^n as well. Indeed, because \mathfrak{m} is a Radon measure, there are compact sets $C_i \subseteq B_i$ such that $\mathfrak{m}(B_i \setminus C_i)$ is sufficiently small to get

$$\|\mathbb{1}_{B_1 \times \dots \times B_n} - \mathbb{1}_{C_1 \times \dots \times C_n}\|_{L_2^n} < \varepsilon$$

for some given $\varepsilon > 0$. Since the compact sets (C_i) are disjoint, one can find disjoint bounded open sets $U_i \supseteq C_i$ such that $\|\mathbb{1}_{C_1 \times \dots \times C_n} - \mathbb{1}_{U_1 \times \dots \times U_n}\|_{L_2^n} < \varepsilon$. For any bounded open set $U_i \subseteq (0, \infty) \times \mathbb{R}$ one can find a sequence of ‘half-open rectangles’ $Q_{i,k} = (s_k^i, t_k^i] \times (a_k^i, b_k^i] = T_k^i \times A_k^i$ such that $U_i = \bigcup_{k=1}^{\infty} Q_{i,k}$ (taking half-open rectangles $Q_x \subseteq U_i$ with rational ‘end points’ containing the point $x \in U_i$ gives $U_i = \bigcup_{Q_x \subseteq U_i} Q_x$).

Hence for sufficiently large K_i ’s we have

$$\|\mathbb{1}_{U_1 \times \dots \times U_n} - \mathbb{1}_P\|_{L_2^n} < \varepsilon, \quad \text{where } P := \bigcup_{k=1}^{K_1} Q_{1,k} \times \dots \times \bigcup_{k=1}^{K_n} Q_{n,k}$$

and where the $Q_{i,1}, \dots, Q_{i,K_i}$ can now be chosen such that they are disjoint. This implies that the linear span of $\mathbb{1}_{Q_1 \times \dots \times Q_n}$, where the Q_i ’s are of the form $T_i \times A_i$, is dense in L_2^n .

2° For the convenience of the reader we recall the idea of the proof of Lemma 2 in [4] to show that the intervals T_i can be chosen disjoint. Consider the situation (all other cases can be treated similarly) where for the set

$$(T_1 \times A_1) \times \dots \times (T_n \times A_n)$$

we have $T_1 = T_2$. To shorten the notation we write

$$Q := (T_3 \times A_3) \times \dots \times (T_n \times A_n).$$

Choosing an equidistant partition $(E_j)_{j=1}^k$ of T_1 we have

$$\mathbb{1}_{(T_1 \times A_1) \times (T_1 \times A_2) \times Q} = \sum_{j \neq l} \mathbb{1}_{(E_j \times A_1) \times (E_l \times A_2) \times Q} + \sum_{j=1}^k \mathbb{1}_{(E_j \times A_1) \times (E_j \times A_2) \times Q}.$$

It can be easily checked that $\|\sum_{j=1}^k \mathbb{1}_{(E_j \times A_1) \times (E_j \times A_2) \times Q}\|_{L_2^n} \rightarrow 0$ as $k \rightarrow \infty$.

3° The denseness of \mathcal{H}_n in L_2^n implies that \mathcal{L} is dense in L_2 and $\mathbb{D}_{1,2}$. The remaining cases follow from the fact that

$$\|f_n \mathbb{I}_{(\mathbb{R}_+ \times \{0\}) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|_{L_2^n} \leq \|f_n\|_{L_2^n}$$

and

$$\|f_n \mathbb{I}_{(\mathbb{R}_+ \times \mathbb{R}_0) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|_{L_2^n} \leq \|f_n\|_{L_2^n}. \quad \blacksquare$$

3. THE MALLIAVIN DERIVATIVE AS OPERATOR ON \mathcal{S}

Let $C_c^\infty(\mathbb{R}^n)$ denote the space of smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with compact support.

DEFINITION 3.1. A random variable of the form $F = f(X_{t_1}, \dots, X_{t_n})$, where $f \in C_c^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, and $t_1, \dots, t_n \geq 0$, is said to be a *smooth random variable*. The set of all smooth random variables is denoted by \mathcal{S} .

DEFINITION 3.2. For $F = f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}$ we define the *Malliavin derivative operator D* as a map from \mathcal{S} into $L_2(\mathfrak{m} \otimes \mathbb{P})$ by

$$\begin{aligned} & D_{t,x} f(X_{t_1}, \dots, X_{t_n}) \\ := & \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \mathbb{I}_{[0,t_i] \times \{0\}}(t, x) \\ & + \frac{f(X_{t_1} + x \mathbb{I}_{[0,t_1]}(t), \dots, X_{t_n} + x \mathbb{I}_{[0,t_n]}(t)) - f(X_{t_1}, \dots, X_{t_n})}{x} \mathbb{I}_{\mathbb{R}_0}(x) \end{aligned}$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

The following lemma holds true:

LEMMA 3.1. *We have $DF = DF$ in $L_2(\mathfrak{m} \otimes \mathbb{P})$ for all $F \in \mathcal{S}$.*

Since for $f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}$ we get

$$\mathbb{E} \int_{\mathbb{R}_+} |D_{t,0} f(X_{t_1}, \dots, X_{t_n})|^2 dt < \infty$$

and

$$\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}_0} |D_{t,x} f(X_{t_1}, \dots, X_{t_n})|^2 d\mathfrak{m}(t, x) < \infty,$$

Lemma 3.1 follows for the canonical Lévy space from Propositions 3.5 and 5.5 in [12].

A proof of Lemma 3.1 for the situation where the Lévy process (X_t) is a square integrable pure jump process which has an absolutely continuous distribution can be found in [7].

An outline of the proof in the general case is given in the Appendix. Like in [7], Proposition 8, one can derive from the proof an explicit form for the functions (f_n) of the chaos expansion $f(X_{t_1}, \dots, X_{t_k}) = \sum_{n=0}^{\infty} I_n(f_n)$,

$$f_n((s_1, x_1), \dots, (s_n, x_n)) \\ = \mathbb{E} \sum_{I \subset \{1, \dots, n\} \cup \emptyset} \frac{(-1)^{n-|I|}}{n!} \frac{f(X_{t_1} + \sum_{i \in I} x_i \mathbb{1}_{[0, t_1]}(s_i), \dots, X_{t_k} + \sum_{i \in I} x_i \mathbb{1}_{[0, t_k]}(s_i))}{x_1 \dots x_n},$$

with the convention that to get $f_n((s_1, x_1), \dots, (s_i, 0), \dots, (s_n, x_n))$ one has to take the limit $\lim_{|x_i| \downarrow 0} f_n((s_1, x_1), \dots, (s_n, x_n))$.

Especially, since any $F \in L_2 \supseteq \mathcal{S}$ has a unique chaos expansion, we conclude that also $\mathbf{D}F$ does not depend on the representation $F = f(X_{t_1}, \dots, X_{t_n})$. Using the equality of \mathbf{D} and D on \mathcal{S} and the fact that \mathcal{S} is closed with respect to multiplication we are now able to reformulate Proposition 5.1 of [12] for our situation:

COROLLARY 3.1. *For F and G in \mathcal{S} we have*

$$D_{t,x}(FG) = GD_{t,x}F + FD_{t,x}G + xD_{t,x}FD_{t,x}G$$

for $\mathfrak{m} \otimes \mathbb{P}$ -a.e. $(t, x, \omega) \in \mathbb{R}_+ \times \mathbb{R} \times \Omega$.

4. THE CLOSURE OF THE MALLIAVIN DERIVATIVE OPERATOR

The operator $\mathbf{D} : \mathcal{S} \rightarrow L_2(\mathfrak{m} \otimes \mathbb{P})$ is *closable* if for any sequence $(F_n) \subseteq \mathcal{S}$ which converges to 0 in L_2 such that $\mathbf{D}(F_n)$ converges in $L_2(\mathfrak{m} \otimes \mathbb{P})$ it follows that $(\mathbf{D}F_n)$ converges to 0 in $L_2(\mathfrak{m} \otimes \mathbb{P})$. As we know from the previous section that D and \mathbf{D} coincide on $\mathcal{S} \subseteq \mathbb{D}_{1,2}$, it is clear that \mathbf{D} is closable and the closure of the domain of definition of \mathbf{D} with respect to the norm

$$\|F\|_{\mathbf{D}} := [\mathbb{E}|F|^2 + \mathbb{E}\|\mathbf{D}F\|_{L_2(\mathfrak{m})}^2]^{1/2}$$

is contained in $\mathbb{D}_{1,2}$. What remains to show is that the closure is equal to $\mathbb{D}_{1,2}$.

THEOREM 4.1. *The closure of \mathcal{S} with respect to the norm $\|\cdot\|_{\mathbf{D}} = \|\cdot\|_{\mathbb{D}_{1,2}}$ is the space $\mathbb{D}_{1,2}$.*

Theorem 4.1 implies that the Malliavin derivative D defined *via* Itô's chaos expansion and the closure of the operator $L_2 \supseteq \mathcal{S} \xrightarrow{D} L_2(\mathfrak{m} \otimes \mathbb{P})$ coincide. Before we start with the proof we formulate a lemma for later use.

LEMMA 4.1. For $\varphi \in C_c^\infty(\mathbb{R})$ and partitions $\pi_n := \{s = t_0^n < t_1^n < \dots < t_n^n = u\}$ of the interval $[s, u]$ it follows for $\psi(x) := x\varphi(x)$ that

$$\begin{aligned} \mathbb{D}_{1,2} - \lim_{|\pi_n| \rightarrow 0} \left(\sum_{j=1}^n \psi(X_{t_j^n} - X_{t_{j-1}^n}) - \mathbb{E} \sum_{j=1}^n \psi(X_{t_j^n} - X_{t_{j-1}^n}) \right) \\ = \int_{(s,u] \times \mathbb{R}} \varphi(x) dM(t, x), \end{aligned}$$

where $|\pi_n| := \max_{1 \leq i \leq n} |t_i^n - t_{i-1}^n|$.

Proof. To keep the notation simple, we drop the n of the partition points t_j^n . Notice that

$$\int_{(s,u] \times \mathbb{R}} \varphi(x) dM(t, x) = I_1(\mathbb{I}_{(s,u]} \otimes \varphi).$$

We set

$$G^n := \sum_{j=1}^n \psi(X_{t_j} - X_{t_{j-1}}) - \mathbb{E} \sum_{j=1}^n \psi(X_{t_j} - X_{t_{j-1}})$$

and

$$G := \int_{(s,u] \times \mathbb{R}} \varphi(x) dM(t, x).$$

In general, $\psi(X_{t_j} - X_{t_{j-1}}) \notin \mathcal{S}$ but we can conclude from Lemma 3.1 that

$$D_{t,x} \psi(X_{t_j} - X_{t_{j-1}}) = D_{t,x} \psi(X_{t_j} - X_{t_{j-1}})$$

$\mathfrak{m} \otimes \mathbb{P}$ -a.e. using a suitable approximation of $\psi(X_{t_j} - X_{t_{j-1}})$ by a sequence of smooth random variables from \mathcal{S} . So we can write $D_{t,x} G^n$ explicitly as

$$\begin{aligned} D_{t,x} G^n &= \sum_{j=1}^n \psi'(X_{t_j} - X_{t_{j-1}}) \mathbb{I}_{(t_{j-1}, t_j] \times \{0\}}(t, x) \\ &\quad + \sum_{j=1}^n \frac{\psi(X_{t_j} - X_{t_{j-1}} + x) - \psi(X_{t_j} - X_{t_{j-1}})}{x} \mathbb{I}_{(t_{j-1}, t_j] \times \mathbb{R}_0}(t, x). \end{aligned}$$

Moreover, we have $D_{t,x} I_1(\mathbb{I}_{(s,u]} \otimes \varphi) = \mathbb{I}_{(s,u]}(t) \varphi(x)$ \mathfrak{m} -a.e. Using the general fact that for any $F \in \mathbb{D}_{1,2}$ with expectation zero the inequality

$$\|F\|_{\mathbb{D}_{1,2}}^2 \leq 2 \|DF\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2$$

holds true, we obtain

$$\begin{aligned}
\|G - G^n\|_{\mathbb{D}_{1,2}}^2 &\leq 2\|DG - DG^n\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 \\
&= 2\sigma^2 \mathbb{E} \int \sum_{\mathbb{R}_+} \sum_{j=1}^n \mathbb{1}_{(t_{j-1}, t_j]}(t) [\varphi(0) - \psi'(X_{t_j} - X_{t_{j-1}})]^2 dt \\
&\quad + 2\mathbb{E} \int \sum_{\mathbb{R}_+ \times \mathbb{R}_0} \sum_{j=1}^n \mathbb{1}_{(t_{j-1}, t_j]}(t) [\psi(x) - \psi(X_{t_j} - X_{t_{j-1}} + x) \\
&\quad + \psi(X_{t_j} - X_{t_{j-1}})]^2 dt d\nu(x) \\
&\rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$ because of dominated convergence and the a.s. càdlàg property of the paths of (X_t) . ■

Proof of Theorem 4.1. According to Lemma 2.1 it is sufficient to show that an expression like $M(T_1 \times A_1) \dots M(T_n \times A_n)$, where the A_i 's are bounded Borel sets and the T_i 's finite disjoint intervals, can be approximated in $\mathbb{D}_{1,2}$ by a sequence $(F_k)_k \subseteq \mathcal{S}$.

1° In this step we want to show that it is enough to approximate

$$(4.1) \quad I_1(\mathbb{1}_{T_1} \otimes \varphi_1) \dots I_1(\mathbb{1}_{T_n} \otimes \varphi_n)$$

by $(F_k)_k \subseteq \mathcal{S}$, where $\varphi_i \in C_c^\infty(\mathbb{R})$. Since the intervals T_i are disjoint, the definition of the multiple integral implies that

$$M(T_1 \times A_1) \dots M(T_n \times A_n) = I_n(\mathbb{1}_{T_1 \times A_1} \otimes \dots \otimes \mathbb{1}_{T_n \times A_n}) \quad \text{a.s.}$$

By the same reason,

$$I_1(\mathbb{1}_{T_1} \otimes \varphi_1) \dots I_1(\mathbb{1}_{T_n} \otimes \varphi_n) = I_n((\mathbb{1}_{T_1} \otimes \varphi_1) \otimes \dots \otimes (\mathbb{1}_{T_n} \otimes \varphi_n)) \quad \text{a.s.}$$

We have

$$\begin{aligned}
&\|I_n(\mathbb{1}_{(T_1 \times A_1) \times \dots \times (T_n \times A_n)}) - I_n((\mathbb{1}_{T_1} \otimes \varphi_1) \otimes \dots \otimes (\mathbb{1}_{T_n} \otimes \varphi_n))\|_{\mathbb{D}_{1,2}}^2 \\
&\leq (n+1)! \|\mathbb{1}_{(T_1 \times A_1) \times \dots \times (T_n \times A_n)} - (\mathbb{1}_{T_1} \otimes \varphi_1) \otimes \dots \otimes (\mathbb{1}_{T_n} \otimes \varphi_n)\|_{L_2^n}^2 \\
&\leq (n+1)! |T_1| \dots |T_n| \|\mathbb{1}_{A_1 \times \dots \times A_n} - \varphi_1 \otimes \dots \otimes \varphi_n\|_{L_2^n(\mu^{\otimes n})}^2.
\end{aligned}$$

The last expression can be made arbitrarily small by choosing φ_i such that the expression $\|\mathbb{1}_{A_i} - \varphi_i\|_{L_2^1(\mu)}$ is small. Indeed, for each i there are compact sets $C_1^i \subseteq C_2^i \subseteq \dots \subseteq A_i$ and open sets $U_1^i \supseteq U_2^i \supseteq \dots \supseteq A_i$ such that

$$\mu(U_k^i \setminus C_k^i) \rightarrow 0$$

as $k \rightarrow \infty$. By the C^∞ Urysohn lemma ([3], p. 237) there is for each k a function $\varphi_k^i \in C_c^\infty(\mathbb{R})$ such that $0 \leq \varphi_k^i \leq 1$, $\varphi_k^i = 1$ on C_k^i and $\text{supp}(\varphi_k^i) \subset U_k^i$. Then

$$\|\mathbb{1}_{A_i} - \varphi_k^i\|_{L^1_2(\mu)}^2 \leq \mu(U_k^i \setminus C_k^i) \rightarrow 0$$

as $k \rightarrow \infty$.

2° Now we use Lemma 4.1 to approximate the expression (4.1) by a sequence $(F_k)_k \subseteq \mathcal{S}$. For $i = 1, \dots, n$ set $\psi_i(x) := x\varphi_i(x)$ and

$$G_i^k := \sum_{j=1}^k \mathbb{1}_{\{t_j, t_{j-1} \in \bar{T}_i\}} \psi_i(X_{t_j} - X_{t_{j-1}}) - \mathbb{E} \sum_{j=1}^k \mathbb{1}_{\{t_j, t_{j-1} \in \bar{T}_i\}} \psi_i(X_{t_j} - X_{t_{j-1}}).$$

The partition $\pi_k = \{0 \leq t_0^k \leq \dots \leq t_k^k\}$ can be chosen such that all end points of the closed intervals \bar{T}_i belong to π_k . Put

$$f_k(X_{t_0}, \dots, X_{t_k}) := \prod_{i=1}^n G_i^k$$

and notice that $f_k \in C^\infty(\mathbb{R}^{k+1})$. Let us choose functions $\beta_m \in C_c^\infty(\mathbb{R})$ such that $0 \leq \beta_m \leq 1$ and $\beta_m(x) = 1$ for $|x| \leq m$, the support of β_m is contained in $\{x; |x| \leq m+2\}$ and $\|\beta_m'\|_\infty \leq 1$. Setting $x_{-1} := 0$ and

$$\alpha_m(x_0, \dots, x_k) := \prod_{i=0}^k \beta_m(x_i - x_{i-1}),$$

we have $f_k(x)\alpha_m(x) \in C_c^\infty(\mathbb{R}^{k+1})$. By dominated convergence one can show that

$$\mathbb{D}_{1,2} - \lim_{m \rightarrow \infty} f_k(X_{t_0}, \dots, X_{t_k})\alpha_m(X_{t_0}, \dots, X_{t_k}) = f_k(X_{t_0}, \dots, X_{t_k}).$$

Because the intervals (T_i) are disjoint, it follows that the product rule holds in our case:

$$(4.2) \quad D \prod_{i=1}^n G_i^k = \sum_{i=1}^n G_1^k \dots G_{i-1}^k (DG_i^k) G_{i+1}^k \dots G_n^k \quad \mathbb{m} \otimes \mathbb{P}\text{-a.e.}$$

Indeed, because of $D_{t,x}G_i^k = (D_{t,x}G_i^k)\mathbb{1}_{T_i}(t)$ we have

$$x(D_{t,x}G_i^k)\mathbb{1}_{T_i}(t)(D_{t,x}G_j^k)\mathbb{1}_{T_j}(t) = 0 \quad \mathbb{m} \otimes \mathbb{P}\text{-a.e.}$$

for any $i \neq j$. Equation (4.2) follows then by induction. Let

$$G_i := I_1(\mathbb{1}_{T_i} \otimes \varphi_i).$$

We observe that G_1^k, \dots, G_n^k as well as $G_1^k, \dots, G_{i-1}^k, DG_i^k, G_{i+1}^k, \dots, G_n^k$ are mutually independent by construction. Hence to show L_2 -convergence of these

products it is enough to prove L_2 -convergence for each factor. From Lemma 4.1 we obtain $G_i^k \rightarrow G_i$ in $\mathbb{D}_{1,2}$ for all $i = 1, \dots, n$, so that

$$\begin{aligned} L_2(\mathfrak{m} \otimes \mathbb{P}) - \lim_{|\pi_k| \rightarrow 0} G_1^k \dots G_{i-1}^k (DG_i^k) G_{i+1}^k \dots G_n^k \\ = G_1 \dots G_{i-1} (DG_i) G_{i+1} \dots G_n. \end{aligned}$$

Consequently, we have found a sequence $(F_k) \subseteq \mathcal{S}$ given by

$$F_k = f_k(X_{t_0}, \dots, X_{t_k}) \alpha_{m_k}(X_{t_0}, \dots, X_{t_k}),$$

where the m_k 's are chosen in a suitable way, that converges to expression (4.1) in $\mathbb{D}_{1,2}$. ■

COROLLARY 4.1. *The set \mathcal{S} of smooth random variables is dense in $L_2, \mathbb{D}_{1,2}^0$ and $\mathbb{D}_{1,2}^J$.*

PROOF. The denseness in L_2 is clear. To show that \mathcal{S} is dense in $\mathbb{D}_{1,2}^0$ assume $F \in \mathbb{D}_{1,2}^0$ has the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$. For a given $\varepsilon > 0$ fix N_ε such that $\|\sum_{n=N_\varepsilon}^{\infty} I_n(f_n)\|_{\mathbb{D}_{1,2}^0} < \varepsilon$. From $F \in L_2$ we conclude

$$F^{N_\varepsilon} := \sum_{n=0}^{N_\varepsilon} I_n(f_n) \in \mathbb{D}_{1,2}.$$

By Theorem 4.1 we can find a sequence $(F_k) \subseteq \mathcal{S}$ converging to F^{N_ε} in $\mathbb{D}_{1,2}$, and therefore also in $\mathbb{D}_{1,2}^0$. In the same way one can see that \mathcal{S} is dense in $\mathbb{D}_{1,2}^J$. ■

5. LIPSCHITZ FUNCTIONS OPERATE ON $\mathbb{D}_{1,2}$

LEMMA 5.1. *Assume that $g : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_g .*

(a) *If $\sigma > 0$, then $g(F) \in \mathbb{D}_{1,2}^0$ for all $F \in \mathbb{D}_{1,2}^0$ and*

$$(5.1) \quad D_{t,0}g(F) = G D_{t,0}F \quad dt \otimes \mathbb{P}\text{-a.e.},$$

where G is a random variable which is a.s. bounded by L_g .

(b) *If $\nu \neq 0$, then $g(F) \in \mathbb{D}_{1,2}^J$ for all $F \in \mathbb{D}_{1,2}^J$, where*

$$(5.2) \quad D_{t,x}g(F) = \frac{g(F + xD_{t,x}F) - g(F)}{x}$$

for $\mathfrak{m} \otimes \mathbb{P}$ -a.e. $(t, x, \omega) \in \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega$.

Proof. (a) We will adapt the proof of Proposition 1.2.4 in [8] to our situation. Corollary 4.1 implies that there exists a sequence $(F_n) \subseteq \mathcal{S}$ of the form $F_n = f_n(X_{t_1}, \dots, X_{t_n})$ which converges to F in $\mathbb{D}_{1,2}^0$. Like in [8], we choose a non-negative $\psi \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(\psi) \subseteq [-1, 1]$ and $\int_{\mathbb{R}} \psi(x) dx = 1$ and define the approximation of unity $\psi_m(x) := m\psi(mx)$. Then $g_m := g * \psi_m$ is smooth and converges uniformly to g as $m \rightarrow \infty$. Moreover, $\|g'_m\|_\infty \leq L_g$. Hence $g_m(F_n) - g_m(0) \in \mathcal{S}$ and $(g_n(F_n))$ converges to $g(F)$ in L_2 . Moreover,

$$\mathbb{E} \int_{\mathbb{R}_+} |D_{t,0} g_n(F_n)|^2 dt \leq L_g^2 \|F_n\|_{\mathbb{D}_{1,2}^0}^2.$$

Since $(g_n(F_n))$ converges to $g(F)$ in L_2 and

$$\sup_n \|g_n(F_n)\|_{\mathbb{D}_{1,2}^0}^2 < \infty,$$

Lemma 1.2.3 in [8] states that $g(F) \in \mathbb{D}_{1,2}^0$ and that $(D_{\cdot,0} g_n(F_n))$ converges to $D_{\cdot,0} g(F)$ in the weak topology of $L_2(\Omega; L_2(\mathbb{R}_+ \times \{0\}))$. The obvious inequality $\mathbb{E}|g'_n(F_n)|^2 \leq L_g^2$ implies the existence of a subsequence $(g'_{n_k}(F_{n_k}))_k$ which converges to some $G \in L_2$ in the weak topology of L_2 . One can show that $|G| \leq L_g$ a.s. Hence for any element $\alpha \in L_\infty(\Omega; L_2(\mathbb{R}_+ \times \{0\}))$ we have

$$\lim_{k \rightarrow \infty} \mathbb{E} \int_{\mathbb{R}_+} g'_{n_k}(F_{n_k})(D_{t,0} F_{n_k}) \alpha(t) dt = \mathbb{E} \left(G \int_{\mathbb{R}_+} (D_{t,0} F) \alpha(t) dt \right).$$

Consequently, $D_{t,0} g(F) = G D_{t,0} F dt \otimes \mathbb{P}$ -a.e.

(b) Let $(F_n)_n \subseteq \mathcal{S}$ be a sequence such that $\mathbb{D}_{1,2}^J - \lim F_n = F$. Since the expression

$$Z(t, x) := \frac{g(F + x D_{t,x} F) - g(F)}{x} \mathbb{I}_{\mathbb{R}_+ \times \mathbb{R}_0}(t, x)$$

is in $L_2(\mathfrak{m} \otimes \mathbb{P})$, it is enough to show that the sequence $(Dg_n(F_n) \mathbb{I}_{\mathbb{R}_+ \times \mathbb{R}_0})$ converges in $L_2(\mathfrak{m} \otimes \mathbb{P})$ to Z , where (g_n) is the sequence constructed in (a). Choose $T > 0$ and $L > 0$ large enough and $\delta > 0$ sufficiently small such that

$$\limsup_n \mathbb{E} \int_{([0,T] \times \{\delta \leq |x| \leq L\})^c} |Z(t, x)|^2 + |D_{t,x} g_n(F_n)|^2 d\mathfrak{m}(t, x) < \varepsilon.$$

Then, for $n \geq n_0$,

$$\begin{aligned} & \|Z - Dg_n(F_n) \mathbb{I}_{\mathbb{R}_+ \times \mathbb{R}_0}\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 \\ & \leq \varepsilon + 2\mathbb{E} \int_{[0,T] \times \{\delta \leq |x| \leq L\}} \left| Z(t, x) - \frac{g(F_n + x D_{t,x} F_n) - g(F_n)}{x} \right|^2 d\mathfrak{m}(t, x) \\ & \quad + 8\delta^{-2} T \mu(\{\delta \leq |x| \leq L\}) \|g - g_n\|_\infty^2. \end{aligned}$$

Hence we obtain (5.2) from the Lipschitz continuity of g and the uniform convergence of g_n to g . ■

PROPOSITION 5.1. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz continuous. Then $F \in \mathbb{D}_{1,2}$ implies $g(F) \in \mathbb{D}_{1,2}$, where $Dg(F)$ is given by (5.1) and (5.2).*

PROOF. The assertion is an immediate consequence of Lemma 5.1 and the equality (2.2). ■

6. APPENDIX

PROOF OF LEMMA 3.1. We denote by $J_n(f_n)$ the multiple integral

$$\int_{\mathbb{R}_+ \times \mathbb{R}} \int_{[0, t_n] \times \mathbb{R}} \dots \int_{[0, t_2] \times \mathbb{R}} f_n((t_1, x_1), \dots, (t_n, x_n)) dM(t_1, x_1) \dots dM(t_n, x_n),$$

where for the definition of a stochastic integral with respect to M we refer to [1]. We have

$$(6.1) \quad I_n(\tilde{f}_n) = n! J_n(\tilde{f}_n).$$

Let us first prove on \mathcal{S} a Clark–Ocone–Haussman type formula for the operator D . By the Fourier inversion formula (see, for example, [1]) we infer for $f \in C_c^\infty(\mathbb{R}^k)$ that

$$f(X_{t_1}, \dots, X_{t_k}) = \int_{\mathbb{R}^k} \hat{f}(u) \exp\left(2\pi i \sum_{j=1}^k u_j X_{t_j}\right) du = \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u, T)} Y_T(u) du,$$

where $e^{\eta(u, t)} = \mathbb{E} \exp\left(2\pi i \sum_{j=1}^k (u_j X_{t_j \wedge t})\right)$ and

$$Y_t(u) = \exp\left(2\pi i \sum_{j=1}^k u_j X_{t_j \wedge t} - \eta(u, t)\right) \quad \text{for } 0 \leq t \leq T := \max\{t_1, \dots, t_k\}.$$

We rewrite $Y_T(u)$ by Itô's formula using $\xi(u, s) := 2\pi i \sum_{j=1}^k u_j \mathbb{I}_{[0, t_j]}(s)$ and get

$$(6.2) \quad \begin{aligned} f(X_{t_1}, \dots, X_{t_k}) &= \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u, T)} du \\ &+ \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u, T)} \left(\int_0^T Y_{s-}(u) \xi(u, s) \sigma dW_s \right) du \\ &+ \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u, T)} \left(\int_{(0, T] \times \mathbb{R}_0} Y_{s-}(u) (e^{x \xi(u, s)} - 1) d\tilde{N}(s, x) \right) du. \end{aligned}$$

It follows by Fubini's theorem that

$$\int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u, T)} du = \mathbb{E} \int_{\mathbb{R}^k} \hat{f}(u) \exp\left(2\pi i \sum_{j=1}^k u_j X_{t_j}\right) du = \mathbb{E} f(X_{t_1}, \dots, X_{t_k}).$$

Now we deal with the second term on the right-hand side of (6.2). Using the fact that the process $(Y_t)_{t \in [0, T]}$ is a square integrable martingale, we infer by the conditional theorem of Fubini (see, e.g., [1]) and Fubini's theorem for stochastic integrals (see, e.g., [10]) that it can be written as

$$\int_0^T \mathbb{E} \left[\int_{\mathbb{R}^k} Y_T(u) \hat{f}(u) e^{\eta(u, T)} \xi(u, s) du \middle| \mathcal{F}_{s^-} \right] \sigma dW_s.$$

Applying Theorem 8.22 (e) of [3] and the Fourier inversion formula we rewrite the inner integral as follows:

$$\begin{aligned} \int_{\mathbb{R}^k} Y_T(u) \hat{f}(u) e^{\eta(u, T)} \xi(u, s) du \\ &= \sum_{j=1}^k \mathbb{I}_{[0, t_j]}(s) \int_{\mathbb{R}^k} 2\pi i u_j \hat{f}(u) \exp \left(2\pi i \sum_{j=1}^k u_j X_{t_j} \right) du \\ &= \sum_{j=1}^k \mathbb{I}_{[0, t_j]}(s) \frac{\partial f}{\partial x_j} (X_{t_1}, \dots, X_{t_k}). \end{aligned}$$

Similarly, one can write the last term on the right-hand side of (6.2) as

$$\int_{(0, T] \times \mathbb{R}_0} \mathbb{E} \left[\int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u, T)} Y_T(u) (e^{x\xi(u, s)} - 1) du \middle| \mathcal{F}_{s^-} \right] d\tilde{N}(s, x),$$

where

$$\begin{aligned} &\int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u, T)} Y_T(u) (e^{x\xi(u, s)} - 1) du \\ &= \int_{\mathbb{R}^k} \hat{f}(u) \left(\exp \left[2\pi i \sum_{j=1}^k u_j (X_{t_j} + x \mathbb{I}_{[0, t_j]}(s)) \right] - \exp \left(2\pi i \sum_{j=1}^k u_j X_{t_j} \right) \right) du \\ &= f(X_{t_1} + x \mathbb{I}_{[0, t_1]}(s), \dots, X_{t_k} + x \mathbb{I}_{[0, t_k]}(s)) - f(X_{t_1}, \dots, X_{t_k}). \end{aligned}$$

Consequently, for $F = f(X_{t_1}, \dots, X_{t_k}) \in \mathcal{S}$ the Clark–Ocone–Haussman type formula holds true:

$$(6.3) \quad F = \mathbb{E}F + \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{E} [D_{t,x} F | \mathcal{F}_{t^-}] dM(t, x).$$

Since $D_{t,x} f(X_{t_1}, \dots, X_{t_k}) \in \mathcal{S}$ for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, iterating equation (6.3) we obtain

$$f(X_{t_1}, \dots, X_{t_k}) = \mathbb{E}f(X_{t_1}, \dots, X_{t_k}) + \sum_{n=1}^{\infty} J_n(\mathbb{E}D^n f(X_{t_1}, \dots, X_{t_k})),$$

where $D^n := D \dots D$.

Notice that $\mathbb{E}D^n f(X_{t_1}, \dots, X_{t_k})$ is a symmetric function on $(\mathbb{R}_+ \times \mathbb{R})^n$. The relation (6.1) between the multiple and the iterated integral and equation (2.1) together with $D_{t,x}f(X_{t_1}, \dots, X_{t_k}) \in L_2(\mathfrak{m} \otimes \mathbb{P})$ imply that

$$\begin{aligned} D_{t,x}f(X_{t_1}, \dots, X_{t_k}) &= \sum_{n=1}^{\infty} J_{n-1}(\mathbb{E}D^{n-1}D_{t,x}f(X_{t_1}, \dots, X_{t_k})) \\ &= D_{t,x}f(X_{t_1}, \dots, X_{t_k}) \quad \mathfrak{m} \otimes \mathbb{P}\text{-a.e.} \quad \blacksquare \end{aligned}$$

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Department of Mathematics
University of Innsbruck
Innsbruck, Austria
E-mail: christel.geiss@uibk.ac.at

Department of Mathematics and Statistics
University of Jyväskylä
Jyväskylä, Finland
E-mail: eija.laukkarinen@jyu.fi

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On Malliavin calculus for functionals of Lévy processes

Eija Laukkarinen

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Eija Laukkarinen

Abstract

The goal of this paper is to develop a stochastic calculus for random elements whose randomness originates from finitely many increments of a Lévy process. Using an explicit formula for the Malliavin derivative of we give a characterization for Malliavin smoothness of $f(X_1)$. The Skorohod integral is expressed via a pathwise integral with respect to a random measure generated by the Lévy process.

Keywords Lévy process, Malliavin calculus

Mathematics Subject Classification (2010) 60G51, 60H05, 60H07

1 Introduction

In recent years Malliavin calculus for Lévy processes has been introduced using chaos expansions (Applebaum [4], Di Nunno et al. [7], Løkka [12], Nualart and Vives [15], Solé et al. [18] and others). When an explicit form of the derivative is needed, the infinite series representation becomes laborious. In many applications such as finance and backward stochastic differential equations we would need to work with random variables of the form

$$f(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}), \quad (1)$$

where X is a Lévy process, f is a Borel function and $0 \leq t_0 < t_1 < \dots < t_n < \infty$. In this paper we study Malliavin calculus for random variables of the form $f(X_1)$ and the Skorohod integral for random fields whose randomness originates from $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$.

It is well known that for the Brownian motion W it holds that $f(W_1)$ is in the Malliavin Sobolev space $\mathbb{D}_{1,2}$ if and only if f belongs to the weighted Sobolev space $W^{1,2}(\mathbb{R}^n, N(0, 1))$ (see for instance [13, Proposition V 2.3.1]). We relate Malliavin differentiability to the properties of $f : \mathbb{R} \rightarrow \mathbb{R}$ for any Lévy process X . We also investigate the Skorohod integral and give the relation between the Skorohod integral and the pathwise integral on a

dense subset of $\text{Dom}(\delta)$. The relation between the two integrals for certain integrable mappings can also be found in papers of Alós et al. [2, Corollary 2.9] and Øksendal and Zhang [16, Lemma 2.1]. In these works the relation is shown for certain forward integrable processes, whereas we consider the relation for certain Skorohod integrable processes.

1.1 The setting

Let $X = (X_t)_{t \geq 0}$ be a Lévy process with càdlàg paths on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_{t \geq 0}$ denote the natural filtration of X augmented with the null sets of \mathcal{F} . Denote $\mathcal{F}^X := \bigvee_{t \geq 0} \mathcal{F}_t$. By the Lévy-Itô decomposition there exist $\gamma \in \mathbb{R}$, $\sigma \geq 0$, a standard Brownian motion W and a Poisson random measure N on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that

$$X_t = \gamma t + \sigma W_t + \iint_{(0,t] \times \{|x| > 1\}} x N(ds, dx) + \iint_{(0,t] \times \{0 < |x| \leq 1\}} x \tilde{N}(ds, dx).$$

Here $\tilde{N}(ds, dx) = N(ds, dx) - ds\nu(dx)$ is the compensated Poisson random measure and $\nu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ is the Lévy measure of X satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty$ and $\nu(B) = \mathbb{E}N((0, 1] \times B)$ when $0 \notin \bar{B}$.

We consider the following measures μ and \mathfrak{m} defined as

$$\begin{aligned} \mu : \mathcal{B}(\mathbb{R}) &\rightarrow [0, \infty], & \mu(dx) &:= \sigma^2 \delta_0(dx) + x^2 \nu(dx), \\ \mathfrak{m} : \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) &\rightarrow [0, \infty], & \mathfrak{m}(dt, dx) &:= dt \mu(dx). \end{aligned}$$

For sets $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mathfrak{m}(B) < \infty$, a random measure M is defined by

$$M(B) := \sigma \int_{\{t \in \mathbb{R}_+ : (t, 0) \in B\}} dW_t + \lim_{n \rightarrow \infty} \iint_{\{(t, x) \in B : \frac{1}{n} < |x| < n\}} x d\tilde{N}(t, x),$$

where the convergence is taken in $L_2(\mathbb{P})$. Then $\mathbb{E}M(B_1)M(B_2) = \mathfrak{m}(B_1 \cap B_2)$ for all B_1, B_2 with $\mathfrak{m}(B_1) < \infty$ and $\mathfrak{m}(B_2) < \infty$. For $n = 1, 2, \dots$ write

$$L_2(\mathfrak{m}^{\otimes n}) = L_2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^{\otimes n}, \mathfrak{m}^{\otimes n})$$

and set $L_2(\mathfrak{m}^{\otimes 0}) := \mathbb{R}$. A function $f_n : (\mathbb{R}_+ \times \mathbb{R})^n \rightarrow \mathbb{R}$ is said to be symmetric, if it coincides with its symmetrization \tilde{f}_n ,

$$\tilde{f}_n((t_1, x_1), \dots, (t_n, x_n)) = \frac{1}{n!} \sum_{\pi} f_n((t_{\pi(1)}, x_{\pi(1)}), \dots, (t_{\pi(n)}, x_{\pi(n)})),$$

where the sum is taken over all permutations $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$.

We let I_n denote the multiple integral of order n defined by Itô [10]. For pairwise disjoint $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $m(B_i) < \infty$ the integral of $\mathbb{1}_{B_1} \otimes \dots \otimes \mathbb{1}_{B_n}$ is defined by

$$I_n(\mathbb{1}_{B_1} \otimes \dots \otimes \mathbb{1}_{B_n}) := M(B_1) \cdots M(B_n).$$

It is then extended to a linear and continuous operator $I_n : L_2(\mathfrak{m}^{\otimes n}) \rightarrow L_2(\mathbb{P})$ and it holds that $I_n(f_n) = I_n(\tilde{f}_n)$ for all $f_n \in L_2(\mathfrak{m}^{\otimes n})$. We let $I_0(f_0) := f_0$ for $f_0 \in \mathbb{R}$.

According to [10, Theorem 2], letting $I_n(L_2(\mathfrak{m}^{\otimes n})) := \{I_n(f_n) : f_n \in L_2(\mathfrak{m}^{\otimes n})\}$ for $n = 0, 1, 2, \dots$ it holds that

$$L_2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} I_n(L_2(\mathfrak{m}^{\otimes n}))$$

and the functions f_n in the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ in $L_2(\mathbb{P})$ are unique when they are chosen to be symmetric. It then holds that

$$\|F\|_{L_2(\mathbb{P})}^2 = \sum_{n=0}^{\infty} n! \|\tilde{f}_n\|_{L_2(\mathfrak{m}^{\otimes n})}^2.$$

1.2 Notation

Here we introduce some frequently used notation:

- $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$.
- $C_c^\infty(\mathbb{R}^m)$ is the space of real-valued functions on \mathbb{R}^m which are infinitely many times differentiable in all coordinates and have compact support.
- $C_b^\infty(\mathbb{R}^m)$ is the space of bounded smooth functions such that all the partial derivatives are bounded.

2 On the chaos expansion of $f(X_1)$

In this section we investigate chaos representations for random variables of the form $f(X_1) = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$, where f is a Borel function. If $f \in C_c^\infty(\mathbb{R})$, then according to Geiss et al. [9, page 7] it holds that

$$f_n((t_1, x_1), \dots, (t_n, x_n)) = \frac{1}{n!} \mathbb{E} \Delta_{x_1} \cdots \Delta_{x_n} f(X_1) \mathbb{1}_{(0,1]^{\otimes n}}(t_1, \dots, t_n) \quad \mathfrak{m}^{\otimes n} - a.e., \quad (2)$$

where Δ_x is defined in Definition 2.1 below. Since $C_c^\infty(\mathbb{R})$ is a dense subset of $L_2(\mathbb{P}_{X_1})$, we can see by approximation that

$$f_n((t_1, x_1), \dots, (t_n, x_n)) = g_n(x_1, \dots, x_n) \mathbb{1}_{(0,1]^{\otimes n}}(t_1, \dots, t_n) \quad \mathbb{m}^{\otimes n} - a.e. \quad (3)$$

for some symmetric function $g_n \in L_2(\mu^{\otimes n})$ for any $f(X_1) \in L_2(\mathbb{P})$.

We let H_k be the Hermite polynomial of order k , that is $H_0(x) := 1$ and $H_k(x) := \frac{(-1)^k}{k!} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$ for $k = 1, 2, \dots$. Write $Y_t := X_t - \sigma W_t$. The following proposition presents a decomposition for $f(X_1) \in L_2(\mathbb{P})$ using Hermite polynomials.

Proposition 2.1. *Let $f(X_1) = \sum_{n=0}^{\infty} I_n(g_n \mathbb{1}_{(0,1]^{\otimes n}}) \in L_2(\mathbb{P})$ with g_n symmetric. Then there exist functions $G_k : \mathbb{R} \rightarrow \mathbb{R}$ such that $G_k(Y_1) \in L_2(\mathbb{P})$,*

$$G_k(Y_1) = \sum_{m=0}^{\infty} \frac{(m+k)!}{m!} I_m(g_{m+k}(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1]^{\otimes m} \times \mathbb{R}_0}), \quad k = 0, 1, \dots,$$

and

$$f(X_1) = \sum_{k=0}^{\infty} G_k(Y_1) H_k(W_1) \sigma^k \quad \text{in } L_2(\mathbb{P}).$$

For the proof we introduce a difference quotient and weak derivative.

Definition 2.1. *For $f : \mathbb{R}^m \rightarrow \mathbb{R}$ we denote by Δ_x^i the difference quotient with respect to the i th coordinate,*

$$\Delta_x^i f(x_1, \dots, x_m) := \frac{f(x_1, \dots, x_{i-1}, x_i + x, x_{i+1}, \dots, x_m) - f(x_1, \dots, x_m)}{x},$$

for $x \neq 0$. If $f \in L_1^{loc}(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), dx)$ and there exists a function $h_i \in L_1^{loc}(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), dx)$ such that

$$\int_{\mathbb{R}^m} f(x) \frac{\partial}{\partial x_i} \varphi(x) dx = - \int_{\mathbb{R}^m} h_i(x) \varphi(x) dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^m),$$

then we say that f has a weak derivative in the direction i and write

$$\Delta_0^i f := h_i.$$

If $m = 1$, we also use the notation $\Delta f := \Delta^1 f$.

The following lemma will be used as a technical tool in this article.

Lemma 2.1. *Let $n, k \geq 0$, $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ and $f \in L_2(\mathfrak{m}^{\otimes n})$, $g \in L_2(\mathfrak{m}^{\otimes k})$. Then*

$$I_n(f \mathbb{1}_A^{\otimes n}) I_k(g \mathbb{1}_{A^c}^{\otimes k}) = I_{n+k}(f \mathbb{1}_A^{\otimes n} \otimes g \mathbb{1}_{A^c}^{\otimes k}) \quad a.s.$$

Proof. Let $f_m = \sum_{i=1}^m a_i^m \otimes_{j=1}^n \mathbb{1}_{A_{ij}^m}$ and $g_m = \sum_{i=1}^m b_i^m \otimes_{j=1}^k \mathbb{1}_{B_{ij}^m}$ be sequences of simple functions converging to f in $L_2(\mathfrak{m}^{\otimes n})$ and g in $L_2(\mathfrak{m}^{\otimes k})$, respectively. Then $f_m \mathbb{1}_A^{\otimes n} \rightarrow f \mathbb{1}_A^{\otimes n}$ in $L_2(\mathfrak{m}^{\otimes n})$, $g_m \mathbb{1}_{A^c}^{\otimes k} \rightarrow g \mathbb{1}_{A^c}^{\otimes k}$ in $L_2(\mathfrak{m}^{\otimes k})$ and $I_n(f \mathbb{1}_A^{\otimes n})$ and $I_k(g \mathbb{1}_{A^c}^{\otimes k})$ are independent. This yields

$$\begin{aligned} I_n(f \mathbb{1}_A^{\otimes n}) I_k(g \mathbb{1}_{A^c}^{\otimes k}) &= \lim_{m \rightarrow \infty} I_n(f_m \mathbb{1}_A^{\otimes n}) I_k(g_m \mathbb{1}_{A^c}^{\otimes k}) \\ &= \lim_{m \rightarrow \infty} I_{n+k}(f_m \mathbb{1}_A^{\otimes n} \otimes g_m \mathbb{1}_{A^c}^{\otimes k}) \\ &= I_{n+k}(f \mathbb{1}_A^{\otimes n} \otimes g \mathbb{1}_{A^c}^{\otimes k}) \end{aligned}$$

in $L_2(\mathbb{P})$. □

Lemma 2.2. *Let $n \geq 1$ and $g_n \in L_2(\mu^{\otimes n})$ be symmetric. Then*

$$I_n(g_n \mathbb{1}_{(0,1]}^{\otimes n}) = \sum_{k=0}^n \frac{n!}{(n-k)!} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right) H_k(W_1) \sigma^k,$$

where the sum is orthogonal.

Proof. Using $\mathbb{R}^n = \bigcup_{k=0}^n \{x : \#\{i : x_i = 0\} = k\}$ and Lemma 2.1 we get

$$\begin{aligned} I_n(g_n \mathbb{1}_{(0,1]}^{\otimes n}) &= \sum_{k=0}^n I_n \left(g_n \mathbb{1}_{(0,1]}^{\otimes n} \mathbb{1}_{\{x : \#\{i : x_i = 0\} = k\}} \right) \\ &= \sum_{k=0}^n \binom{n}{k} I_n \left(g_n \left(\mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \otimes \mathbb{1}_{(0,1] \times \{0\}}^{\otimes k} \right) \right) \\ &= \sum_{k=0}^n \binom{n}{k} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right) I_k \left(\mathbb{1}_{(0,1] \times \{0\}}^{\otimes k} \right) \\ &= \sum_{k=0}^n \frac{n!}{(n-k)!} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right) H_k(W_1) \sigma^k, \end{aligned}$$

where in the last equation we used [14, Proposition 1.1.4]. The orthogonality follows from the orthogonality of the sequence $H_k(W_1)$, $k = 0, 1, \dots$, which is independent from the sequence $I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right)$, $k = 0, 1, \dots$ □

Proof of Proposition 2.1. Suppose first that $f \in C_c^\infty(\mathbb{R})$. Then $f(X_1) = \sum_{n=0}^{\infty} I_n(f_n)$ where $f_n = g_n \mathbb{1}_{(0,1]}^{\otimes n}$ for $g_n(x_1, \dots, x_n) = \frac{1}{n!} \mathbb{E} \Delta_{x_1} \cdots \Delta_{x_n} f(X_1)$ by equation (2). It holds that the function

$$G_k := \int_{\mathbb{R}} \frac{d^k}{dx^k} f(\sigma x + \cdot) \mathbb{P}_{W_1}(dx) \quad \text{for } k = 0, 1, \dots$$

satisfies the assumption of Lemma A.1: Choose $K > 0$ such that $f(x) = 0$ for $|x| > K$. Since for all k , $\frac{d^k}{dx^k} f$ is bounded and bounded functions are in $L_1(\mathbb{P}_{W_1})$ we have by [8, Theorem 2.27 (b)] that

$$\begin{aligned} \left| \frac{d^j}{dy^j} G_k(y) \right| &= \left| \int_{-K}^K \left(\frac{d^{k+j}}{dx^{k+j}} f(x) \right) e^{-\frac{(x-y)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} dx \right| \\ &\leq \sup_{x \in \mathbb{R}} \left| \frac{d^{k+j}}{dx^{k+j}} f(x) \right| e^{\frac{K|y|}{\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}. \end{aligned}$$

By Lemma A.1 we have $G_k(Y_1) = \sum_{m=0}^{\infty} I_m \left(g_{m,k} \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes m} \right)$ with

$$\begin{aligned} g_{m,k}(x_1, \dots, x_m) &= \frac{1}{m!} \mathbb{E} \Delta_{x_1} \cdots \Delta_{x_m} G_k(Y_1) \\ &= \frac{1}{m!} \mathbb{E} \Delta_{x_1} \cdots \Delta_{x_m} \underbrace{\Delta_0 \cdots \Delta_0}_k f(X_1) \\ &= \frac{n!}{(n-k)!} g_n(x_1, \dots, x_{n-k}, \underbrace{0, \dots, 0}_k) \end{aligned}$$

for $n = m + k$ and $x_1, \dots, x_{n-k} \in \mathbb{R}_0$. Thus

$$G_k(Y_1) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right)$$

and by Lemma 2.2 we have

$$\begin{aligned} f(X_1) &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{n!}{(n-k)!} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right) H_k(W_1) \sigma^k \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right) H_k(W_1) \sigma^k \\ &= \sum_{k=0}^{\infty} G_k(Y_1) H_k(W_1) \sigma^k \end{aligned}$$

in $L_2(\mathbb{P})$.

If $f \notin C_c^\infty(\mathbb{R})$, then we find a sequence $(f^{(j)})_{j=1}^\infty \subset C_c^\infty(\mathbb{R})$ such that $f^{(j)} \rightarrow f$ in $L_2(\mathbb{P}_{X_1})$ (see [8, Proposition (7.9) using (8.18)]) with

$$f^{(j)}(X_1) = \sum_{k=0}^{\infty} G_k^{(j)}(Y_1) H_k(W_1) \sigma^k = \sum_{n=0}^{\infty} I_n(g_n^{(j)} \mathbb{1}_{(0,1]^{\otimes n}}).$$

Thus

$$0 \leftarrow \mathbb{E}|f^{(j)}(X_1) - f^{(i)}(X_1)|^2 = \sum_{k=0}^{\infty} \mathbb{E}|G_k^{(j)}(Y_1) - G_k^{(i)}(Y_1)|^2 \mathbb{E}|H_k(W_1) \sigma^k|$$

whence there exist $G_k(Y_1) \in L_2(\mathbb{P})$, $k = 0, 1, \dots$, such that

$$f(X_1) = \sum_{k=0}^{\infty} G_k(Y_1) H_k(W_1) \sigma^k.$$

The convergence $g_n^{(j)} \rightarrow g_n$ in $L_2(\mathfrak{m}^{\otimes n})$ implies

$$G_k(Y_1) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} I_{n-k}(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0^{\otimes(n-k)}}), \quad k = 0, 1, \dots$$

□

3 The Malliavin derivative

We denote by $\mathbb{D}_{1,2}$ the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$ such that

$$\|F\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1)! \|f_n\|_{L_2(\mathfrak{m}^{\otimes n})}^2 < \infty.$$

Let us denote $L_2(\mathfrak{m} \otimes \mathbb{P}) := L_2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mathfrak{m} \otimes \mathbb{P})$ and define the Malliavin derivative $D : \mathbb{D}_{1,2} \rightarrow L_2(\mathfrak{m} \otimes \mathbb{P})$ by letting

$$D_{t,x} F = \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot, (t, x))) \quad \text{in } L_2(\mathfrak{m} \otimes \mathbb{P}).$$

Next we define a set of smooth random variables. For smooth random variables we have an explicit representation of the Malliavin derivative.

Definition 3.1 (Smooth random variables \mathcal{S}). *We call a random variable F smooth, if there exists*

- a set of time points $\tau = \{0 \leq t_0 < t_1 < \dots < t_m < \infty\}$ and
- a function $f \in C_b^\infty(\mathbb{R}^m)$

such that

$$F = f(X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_m} - X_{t_{m-1}}) \quad a.s. \quad (4)$$

We denote the set of smooth random variables by \mathcal{S} .

Note that the set τ and the function f in the above definition are not unique.

According to Geiss et al. [9, Theorem 4.1] smooth random variables (for which $t_0 = 0$ and $f \in C_c^\infty(\mathbb{R}^m)$) are dense in $\mathbb{D}_{1,2}$ and from [9, Lemma 3.1] one immediately obtains the following representation of the Malliavin derivative for smooth random variables.

Proposition 3.1. *The set \mathcal{S} of smooth random variables is dense in $\mathbb{D}_{1,2}$ and for $F = f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \in \mathcal{S}$ it holds that*

$$D_{t,x}F = \sum_{i=1}^m \Delta_x^i f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \mathbb{1}_{(t_{i-1}, t_i]}(t) \quad (5)$$

$\mathfrak{m} \otimes \mathbb{P}$ -a.e.

Proof. The proposition coincides with the claims in [9] for the subset of \mathcal{S} where $t_0 = 0$ and $f \in C_c^\infty(\mathbb{R}^m)$. Let $F \in \mathcal{S}$ with $f \in C_b^\infty(\mathbb{R}^m)$ and $F_k = F\alpha_k(X_{t_0} - X_0, \dots, X_{t_m} - X_{t_{m-1}})$, where $\alpha_k \in C_c^\infty(\mathbb{R}^{m+1})$ is such that $\alpha_k(x) = 1$ for $|x| \leq k$, $0 \leq \alpha_k(x) \leq 1$ and $|\frac{\partial}{\partial x_j} \alpha_k(x)| \leq 1$ for all x and j . Then $F_k \rightarrow F$ in $L_2(\mathbb{P})$ and $DF_k \rightarrow \sum_{i=1}^m \Delta_x^i f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \mathbb{1}_{(t_{i-1}, t_i]}(t)$ in $L_2(\mathfrak{m} \otimes \mathbb{P})$. Hence the proposition follows by the closability of D . \square

We distinguish two components of the Malliavin derivative: the derivative with respect to the Brownian motion part and the derivative with respect to the jump part of the Lévy process. We let $\mathbb{D}_{1,2}^0$ and $\mathbb{D}_{1,2}^{\mathbb{R}_0}$ be subspaces of $L_2(\mathbb{P})$ equipped with the norms

$$\|F\|_{\mathbb{D}_{1,2}^0} := \sqrt{\|F\|_{L_2(\mathbb{P})}^2 + \sum_{n=1}^{\infty} n \cdot n! \left\| \tilde{f}_n \left(\mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}^{\otimes (n-1)} \otimes \mathbb{1}_{\mathbb{R}_+ \times \{0\}} \right) \right\|_{L_2(\mathfrak{m}^{\otimes n})}^2}$$

and

$$\|F\|_{\mathbb{D}_{1,2}^{\mathbb{R}_0}} := \sqrt{\|F\|_{L_2(\mathbb{P})}^2 + \sum_{n=1}^{\infty} n \cdot n! \left\| \tilde{f}_n \left(\mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}^{\otimes(n-1)} \otimes \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}_0} \right) \right\|_{L_2(\mathfrak{m}^{\otimes n})}^2}$$

respectively.

Let us define the operators

$$D_{t,x}^0 F := \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot, (t, x))) \mathbb{1}_{\mathbb{R}_+ \times \{0\}}(t, x) \quad \text{for } F \in \mathbb{D}_{1,2}^0$$

and

$$D_{t,x}^{\mathbb{R}_0} F = \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot, (t, x))) \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}_0}(t, x) \quad \text{for } F \in \mathbb{D}_{1,2}^{\mathbb{R}_0},$$

where the convergence of the sums is taken in $L_2(\mathfrak{m} \otimes \mathbb{P})$. The operators D^0 and $D^{\mathbb{R}_0}$ are closed: let $A \in \{\{0\}, \mathbb{R}_0\}$ and $(F_k) \subset \mathbb{D}_{1,2}^A$ such that $F_k = \sum_{n=0}^{\infty} I_n(f_n^{(k)}) \rightarrow 0$ in $L_2(\mathbb{P})$ and $D^A F_k \rightarrow u$ in $L_2(\mathfrak{m} \otimes \mathbb{P})$. There exist $f_n \in L_2(\mathfrak{m}^{\otimes n})$, $n = 1, 2, \dots$, such that f_n is symmetric in the first $n-1$ pairs of variables and $u(t, x) = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, (t, x)))$ in $L_2(\mathfrak{m} \otimes \mathbb{P})$ (see Remark 4.1 below). We have

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \left\| \tilde{f}_n^{(k)} \left(\mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}^{\otimes(n-1)} \otimes \mathbb{1}_{\mathbb{R}_+ \times A} \right) - f_n \right\|_{L_2(\mathfrak{m}^{\otimes n})} \\ &\geq \lim_{k \rightarrow \infty} \left| \left\| \tilde{f}_n^{(k)} \left(\mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}^{\otimes(n-1)} \otimes \mathbb{1}_{\mathbb{R}_+ \times A} \right) \right\|_{L_2(\mathfrak{m}^{\otimes n})} - \|f_n\|_{L_2(\mathfrak{m}^{\otimes n})} \right| \\ &= \|f_n\|_{L_2(\mathfrak{m}^{\otimes n})} \end{aligned}$$

since $\left\| \tilde{f}_n^{(k)} \left(\mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}^{\otimes(n-1)} \otimes \mathbb{1}_{\mathbb{R}_+ \times A} \right) \right\|_{L_2(\mathfrak{m}^{\otimes n})} \leq \left\| \tilde{f}_n^{(k)} \right\|_{L_2(\mathfrak{m}^{\otimes n})} \rightarrow 0$. Hence $u = 0$ in $L_2(\mathfrak{m} \otimes \mathbb{P})$.

Clearly $\mathbb{D}_{1,2} = \mathbb{D}_{1,2}^0 \cap \mathbb{D}_{1,2}^{\mathbb{R}_0}$ and $D = D^0 + D^{\mathbb{R}_0}$. In Propositions 3.2 and 3.3 we use the operator Δ to state a necessary and sufficient condition on a Borel function f such that $f(X_1)$ is in $\mathbb{D}_{1,2}^0$ or $\mathbb{D}_{1,2}^{\mathbb{R}_0}$, respectively.

3.1 The derivative D^0

Proposition 3.2. *Assume $\sigma \neq 0$. Let $f(X_1) \in L_2(\mathbb{P})$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function. Then $f(X_1) \in \mathbb{D}_{1,2}^0$ if and only if $\Delta_0 f$ exists and $\Delta_0 f(X_1) \in L_2(\mathbb{P})$. If $f(X_1) \in \mathbb{D}_{1,2}^0$, then*

$$D_{t,x}^0 f(X_1) = \Delta_0 f(X_1) \mathbb{1}_{(0,1] \times \{0\}}(t, x) \quad \mathfrak{m} \otimes \mathbb{P}\text{-a.e.} \quad (6)$$

For the proof we need the following lemma.

Lemma 3.1. *Let $f(X_1) = \sum_{k=0}^{\infty} G_k(Y_1)H_k(W_1)\sigma^k \in \mathbb{D}_{1,2}^0$. Then*

$$D_{t,x}^0 f(X_1) = \sum_{k=1}^{\infty} G_k(Y_1)H'_k(W_1)\sigma^{k-1} \mathbb{1}_{(0,1] \times \{0\}}(t, x) \quad \text{in } \mathfrak{m} \otimes \mathbb{P}\text{-a.e.}$$

Proof. Consider the representation $f(X_1) = \sum_{n=0}^{\infty} I_n(g_n \mathbb{1}_{(0,1]}^{\otimes n})$, where the functions $g_n \in L_2(\mu^{\otimes n})$ are symmetric. By Lemma 2.2 we have

$$\begin{aligned} D^0 f(X_1) &= \sum_{n=1}^{\infty} n I_{n-1} \left(g_n(\cdot, 0) \mathbb{1}_{(0,1]}^{\otimes(n-1)} \right) \mathbb{1}_{(0,1] \times \{0\}} \\ &= \sum_{n=1}^{\infty} n \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!} I_{n-1-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_{k+1}) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-1-k)} \right) H_k(W_1) \sigma^k \mathbb{1}_{(0,1] \times \{0\}} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{n!}{(n-k)!} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right) H'_k(W_1) \sigma^{k-1} \mathbb{1}_{(0,1] \times \{0\}} \end{aligned}$$

in $L_2(\mathfrak{m} \otimes \mathbb{P})$ since $H_{k-1} = H'_k$. Denoting

$$F_{n,k} := \frac{n!}{(n-k)!} I_{n-k} \left(g_n(\cdot, \underbrace{0, \dots, 0}_k) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes(n-k)} \right)$$

we get

$$\begin{aligned} & \mathbb{E} \left| \sum_{k=1}^K \sum_{n=k}^{\infty} F_{n,k} H'_k(W_1) \sigma^{k-1} - \sum_{n=1}^{\infty} \sum_{k=1}^n F_{n,k} H'_k(W_1) \sigma^{k-1} \right|^2 \\ &= \mathbb{E} \left| \sum_{n=1}^{\infty} \sum_{k=(n \wedge K)+1}^n F_{n,k} H'_k(W_1) \sigma^{k-1} \right|^2 \\ &= \mathbb{E} \sum_{n=1}^{\infty} \sum_{k=(n \wedge K)+1}^n \mathbb{E} |F_{n,k}|^2 \mathbb{E} |H'_k(W_1) \sigma^{k-1}|^2 \\ &\rightarrow 0 \end{aligned}$$

as $K \rightarrow \infty$ since $\sum_{k=(n \wedge K)+1}^n \mathbb{E} |F_{n,k}|^2 \mathbb{E} |H'_k(W_1) \sigma^{k-1}|^2$ decreases monotonously for all n as K increases. By Proposition 2.1 we have $G_k(Y_1) = \sum_{n=k}^{\infty} F_{n,k}$ in $L_2(\mathbb{P})$, so that

$$D_{t,x}^0 f(X_1) = \sum_{k=1}^{\infty} G_k(Y_1) H'_k(W_1) \sigma^{k-1} \mathbb{1}_{(0,1] \times \{0\}}(t, x) \quad \text{in } L_2(\mathfrak{m} \otimes \mathbb{P}).$$

□

Proof of Proposition 3.2. "Only if": Suppose

$$f(X_1) = \sum_{k=0}^{\infty} G_k(Y_1) H_k(W_1) \sigma^k \in \mathbb{D}_{1,2}^0.$$

We define functions h and f_m as orthogonal sums in $L_2(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathbb{P}_{W_1} \otimes \mathbb{P}_{Y_1})$ by

$$h(x, y) := \sum_{k=1}^{\infty} G_k(y) H'_k(x) \sigma^{k-1}$$

and

$$f_m(x, y) := \sum_{k=0}^m G_k(y) H_k(x) \sigma^k.$$

By orthogonality in $L_2(\mathbb{P}_{W_1} \otimes \mathbb{P}_{Y_1})$ and monotone convergence we get

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} |f(\sigma x + y)|^2 \mathbb{P}_{W_1}(dx) \mathbb{P}_{Y_1}(dy) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} |G_k(y)|^2 |H_k(x)|^2 \sigma^{2k} \mathbb{P}_{W_1}(dx) \mathbb{P}_{Y_1}(dy) \\ &= \int_{\mathbb{R}} \sum_{k=0}^{\infty} \int_{\mathbb{R}} |G_k(y)|^2 |H_k(x)|^2 \sigma^{2k} \mathbb{P}_{W_1}(dx) \mathbb{P}_{Y_1}(dy) \\ &< \infty. \end{aligned}$$

Thus

$$y \mapsto \|f(\sigma \cdot + y)\|_{L_2(\mathbb{P}_{W_1})} \in L_2(\mathbb{P}_{Y_1})$$

and

$$y \mapsto \sqrt{\sum_{k=0}^{\infty} |G_k(y)|^2 \|H_k\|_{L_2(\mathbb{P}_{W_1})}^2 \sigma^{2k}} \in L_2(\mathbb{P}_{Y_1}).$$

Since $\|f_m(\cdot, y)\|_{L_2(\mathbb{P}_{W_1})} = \sqrt{\sum_{k=0}^m |G_k(y)|^2 \|H_k\|_{L_2(\mathbb{P}_{W_1})}^2 \sigma^{2k}}$ for all $y \in \mathbb{R}$, we obtain using dominated convergence that

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(\sigma x + y) - f_m(x, y)|^2 \mathbb{P}_{W_1}(dx) \mathbb{P}_{Y_1}(dy) \\ &= \int_{\mathbb{R}} \left(\lim_{m \rightarrow \infty} \int_{\mathbb{R}} |f(\sigma x + y) - f_m(x, y)|^2 \mathbb{P}_{W_1}(dx) \right) \mathbb{P}_{Y_1}(dy). \end{aligned}$$

Hence there exists a set $A_1 \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}_{Y_1}(A_1) = 1$ and $f_m(\cdot, y) \rightarrow f(\sigma \cdot + y)$ in $L_2(\mathbb{P}_{W_1})$ for all $y \in A_1$. Analogously we find a set A_2 such that $\frac{\partial}{\partial x} f_m(\cdot, y) \rightarrow \sigma h(\cdot, y)$ in $L_2(\mathbb{P}_{W_1})$ for all $y \in A_2$. Fix $y \in A := A_1 \cap A_2$. Let $\varphi \in C_c^\infty(\mathbb{R})$ and write $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Since $\frac{\varphi'(\sigma \cdot + y)}{p}$ and $\frac{\varphi(\sigma \cdot + y)}{p}$ are bounded functions, we get

$$\begin{aligned}
\int_{\mathbb{R}} f(x) \varphi'(x) dx &= \int_{\mathbb{R}} f(\sigma x + y) \varphi'(\sigma x + y) \sigma dx \\
&= \int_{\mathbb{R}} f(\sigma x + y) \frac{\varphi'(\sigma x + y)}{p(x)} p(x) \sigma dx \\
&= \lim_{m \rightarrow \infty} \int_{\mathbb{R}} f_m(x, y) \frac{\varphi'(\sigma x + y)}{p(x)} p(x) \sigma dx \\
&= - \lim_{m \rightarrow \infty} \int_{\mathbb{R}} \frac{\partial}{\partial x} f_m(x, y) \frac{\varphi(\sigma x + y)}{p(x)} p(x) dx \\
&= - \int_{\mathbb{R}} \sigma h(x, y) \frac{\varphi(\sigma x + y)}{p(x)} p(x) dx \\
&= - \int_{\mathbb{R}} h\left(\frac{x-y}{\sigma}, y\right) \varphi(x) dx.
\end{aligned}$$

Furthermore, using Hölder's inequality we get

$$\begin{aligned}
\int_{-n}^n \left| h\left(\frac{x-y}{\sigma}, y\right) \right| dx &= \int_{\frac{-n-y}{\sigma}}^{\frac{n-y}{\sigma}} |\sigma h(x, y)| \frac{1}{\sqrt{p(x)}} \sqrt{p(x)} dx \\
&\leq \left(\int_{\frac{-n-y}{\sigma}}^{\frac{n-y}{\sigma}} |\sigma h(x, y)|^2 p(x) dx \right)^{\frac{1}{2}} \left(\int_{\frac{-n-y}{\sigma}}^{\frac{n-y}{\sigma}} \frac{1}{p(x)} dx \right)^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

This implies $h\left(\frac{\cdot - y}{\sigma}, y\right) \in L_1^{loc}(dx)$ and $\Delta_0 f = h\left(\frac{\cdot - y}{\sigma}, y\right)$ for all $y \in A$.

Consequently, $h(x, y) = \Delta_0 f(\sigma x + y)$ for $\mathbb{P}_{W_1} \otimes \mathbb{P}_{Y_1}$ -a.e. (x, y) and by Lemma 3.1 it holds that

$$D^0 f(X_1) = h(W_1, Y_1) \mathbb{1}_{(0,1] \times \{0\}} = \Delta_0 f(X_1) \mathbb{1}_{(0,1] \times \{0\}} \quad \text{in } L_2(\mathfrak{m} \otimes \mathbb{P}).$$

"If": Assume f has a weak derivative and $\Delta_0 f(X_1) \in L_2(\mathbb{P})$. Assume first that f has compact support and choose $K > 0$ such that $f(x) = 0$ for $|x| > K$. Denote by q the continuous density of X_1 ,

$$q(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-y)^2}{2\sigma^2}} \mathbb{P}_{Y_1}(dy) \in \left(0, \frac{1}{\sqrt{2\pi\sigma^2}}\right] \quad \text{for all } x \in \mathbb{R}.$$

Then

$$\int_{\mathbb{R}} |f(x)|^2 dx \leq \sup_{|z| \leq K} \frac{1}{q(z)} \int_{\mathbb{R}} |f(x)|^2 q(x) dx < \infty$$

and

$$\int_{\mathbb{R}} |\Delta_0 f(x)|^2 dx \leq \sup_{|z| \leq K} \frac{1}{q(z)} \int_{\mathbb{R}} |\Delta_0 f(x)|^2 q(x) dx < \infty.$$

Hence $f \in W^{1,2}(\mathbb{R}) = \{f \in L_2(dx) : \exists \Delta_0 f \in L_2(dx)\}$. By [1, Theorem 3.22] there exists a sequence $(f_k) \subset C_c^\infty(\mathbb{R})$ converging to f in $W^{1,2}(\mathbb{R})$. Then

$$\begin{aligned} & \|f_k(X_1) - f(X_1)\|_{L_2(\mathbb{P})}^2 + \|D^0 f_k(X_1) - \Delta_0 f(X_1) \mathbb{1}_{(0,1] \times \{0\}}\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 \\ &= \int_{\mathbb{R}} |f_k(x) - f(x)|^2 q(x) dx + \int_{\mathbb{R}} |f'_k(x) - \Delta_0 f(x)|^2 q(x) dx \\ &\leq \frac{1}{\sqrt{2\pi\sigma^2}} \left(\int_{\mathbb{R}} |f_k(x) - f(x)|^2 dx + \int_{\mathbb{R}} |f'_k(x) - \Delta_0 f(x)|^2 dx \right) \\ &\rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Since D^0 is closed, it follows that $f(X_1) \in \mathbb{D}_{1,2}^0$ and $D^0 f(X_1) = \Delta_0 f(X_1) \mathbb{1}_{(0,1] \times \{0\}}$ in $L_2(\mathfrak{m} \otimes \mathbb{P})$.

In case f does not have compact support, define for all $K = 1, 2, \dots$ a function $g_K := f\varphi_K$, where $\varphi_K \in C_c^\infty(\mathbb{R})$, $|\varphi_K|, |\varphi'_K| \in [0, 1]$, $\varphi_K(x) = 1$ when $|x| < K - 2$ and $\varphi_K(x) = 0$ when $|x| > K$. Then $g_K(X_1) \in L_2(\mathbb{P})$ and $\Delta_0 g_K(X_1) \in L_2(\mathbb{P})$, thus

$$g_K(X_1) \in \mathbb{D}_{1,2}^0 \quad \text{and} \quad D^0 g_K(X_1) = \Delta_0 g_K(X_1) \mathbb{1}_{(0,1] \times \{0\}}.$$

Furthermore,

$$\begin{aligned} & \|g_K(X_1) - f(X_1)\|_{L_2(\mathbb{P})}^2 + \|D^0 g_K(X_1) - \Delta_0 f(X_1) \mathbb{1}_{(0,1]}\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 \\ &\leq \|f(X_1) \mathbb{1}_{[-K, K]^c}(X_1)\|_{L_2(\mathbb{P})}^2 \\ &\quad + 2\sigma^2 (\|\Delta_0 f(X_1) \mathbb{1}_{[-K, K]^c}(X_1)\|_{L_2(\mathbb{P})}^2 + \|f(X_1) \varphi'_K(X_1)\|_{L_2(\mathbb{P})}^2) \\ &\rightarrow 0 \end{aligned}$$

as $K \rightarrow \infty$. Again, the closedness of D^0 assures that $f(X_1) \in \mathbb{D}_{1,2}^0$ and $D^0 f(X_1) = \Delta_0 f(X_1) \mathbb{1}_{(0,1] \times \{0\}}$. □

3.2 The derivative $D^{\mathbb{R}_0}$

On the canonical Lévy space the operator $D^{\mathbb{R}_0}$ can be defined using a difference quotient with respect to ω (see for instance Solé et al. [18, Section 5]). For the pure jump process in a canonical space, the following proposition is a special case of [18, Proposition 5.4].

Proposition 3.3. *Let $f(X_1) \in L_2(\mathbb{P})$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function. Then*

$$\mathbb{E} \iint_{(0,1] \times \mathbb{R}_0} |\Delta_x f(X_1)|^2 \mathfrak{m}(dt, dx) < \infty$$

if and only if $f(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0}$. If $f(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0}$, then

$$D_{t,x}^{\mathbb{R}_0} f(X_1) = \Delta_x f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}(t, x)$$

in $L_2(\mathfrak{m} \otimes \mathbb{P})$.

Proof. Consider the chaos expansion $f(X_1) = \sum_{n=0}^{\infty} I_n(f_n)$. Assume first that f is bounded. Let $\varepsilon > 0$. We show that

$$\Delta_x f(X_1) \mathbb{1}_{(0,1] \times \{|x| > \varepsilon\}}(t, x) = \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot, (t, x))) \mathbb{1}_{\{|x| > \varepsilon\}} \quad \mathfrak{m} \otimes \mathbb{P}\text{-a.e.} \quad (7)$$

Since f is bounded, the random field $(t, x, \omega) \mapsto \Delta_x f(X_1) \mathbb{1}_{(0,1] \times \{|x| > \varepsilon\}}(t, x)$ is in $L_2(\mathfrak{m} \otimes \mathbb{P})$ and it has a chaos expansion

$$\Delta_x f(X_1) \mathbb{1}_{(0,1] \times \{|x| > \varepsilon\}}(t, x) = \sum_{n=1}^{\infty} n I_{n-1}(h_n(\cdot, (t, x))) \quad \text{in } L_2(\mathfrak{m} \otimes \mathbb{P}),$$

where $h_n \in L_2(\mathfrak{m}^{\otimes n})$ is symmetric in the first $n - 1$ pairs of variables (see Remark 4.1 below).

Denote $\eta := \nu_{\{|x| > \varepsilon\}} + \delta_0$. Since $\mathbb{P}_{X_1} * \eta$ is a Radon measure, we can choose $f_k \in C_c^\infty(\mathbb{R})$ such that $f_k \rightarrow f$ in $L_2(\mathbb{P}_{X_1} * \eta)$ as $k \rightarrow \infty$ ([8, Proposition 7.9 using the C^∞ Urysohn Lemma 8.18]). Then

$$\begin{aligned} & \mathbb{E} \iint_{(0,1] \times \{|x| > \varepsilon\}} |\Delta_x f_k(X_1) - \Delta_x f(X_1)|^2 \mathfrak{m}(dt, dx) \\ & \leq 2 \mathbb{E} \int_{\{|x| > \varepsilon\}} (|f_k(X_1 + x) - f(X_1 + x)|^2 + |f_k(X_1) - f(X_1)|^2) \nu(dx) \\ & \leq (2 + 2\nu(|x| > \varepsilon)) \|f_k - f\|_{L_2(\mathbb{P}_{X_1} * \eta)}^2 \\ & \rightarrow 0 \end{aligned} \quad (8)$$

as $k \rightarrow \infty$. Proposition 3.1 implies that equation (7) holds for f_k . Using (8) and the fact that $f_k(X_1) \rightarrow f(X_1)$ in $L_2(\mathbb{P})$ we get

$$h_n((t_1, x_1), \dots, (t_n, x_n)) = \tilde{f}_n((t_1, x_1), \dots, (t_n, x_n)) \mathbb{1}_{\{|x_n| > \varepsilon\}}$$

$\mathfrak{m}^{\otimes n}$ -a.e. Thus (7) holds. The proposition follows for bounded f from (7) by letting $\varepsilon \rightarrow 0$.

When f is not bounded, consider $f_k := (-k) \vee f \wedge k$. If

$$\Delta_x f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}(t, x) \in L_2(\mathfrak{m} \otimes \mathbb{P}),$$

then from $|\Delta_x f_k(y)| \leq |\Delta_x f(y)|$ for all $x \in \mathbb{R}_0, y \in \mathbb{R}$ we obtain $f_k(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0}$ and $D^{\mathbb{R}_0} f_k(X_1) \rightarrow \Delta f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}$ in $L_2(\mathfrak{m} \otimes \mathbb{P})$ by dominated convergence. It follows that $f(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0}$ and $D^{\mathbb{R}_0} f(X_1) = \Delta f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}$.

Assume $f(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0}$. Then by [9, Lemma 5.1(b)], for all $k \geq 1$ it holds that $f_k(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0}$ and for $x \neq 0$ we have

$$D_{t,x}^{\mathbb{R}_0} f_k(X_1) = \frac{(-k) \vee [f(X_1) + x D_{t,x}^{\mathbb{R}_0} f(X_1)] \wedge k - ((-k) \vee f(X_1) \wedge k)}{x}$$

$\mathfrak{m} \otimes \mathbb{P}$ -a.e., so that $|D_{t,x}^{\mathbb{R}_0} f_k(X_1)| \leq |D_{t,x}^{\mathbb{R}_0} f(X_1)|$ $\mathfrak{m} \otimes \mathbb{P}$ -a.e. and hence $f_k(X_1) \rightarrow f(X_1)$ in $\mathbb{D}_{1,2}^{\mathbb{R}_0}$. Since f_k is bounded, we have

$$\begin{aligned} D_{t,x}^{\mathbb{R}_0} f_k(X_1) &= \frac{(-k) \vee f(X_1 + x) \wedge k - ((-k) \vee f(X_1) \wedge k)}{x} \mathbb{1}_{(0,1] \times \mathbb{R}_0}(t, x) \\ &\rightarrow \Delta_x f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}(t, x) \end{aligned}$$

$\mathfrak{m} \otimes \mathbb{P}$ -a.e. and we get $D_{t,x}^{\mathbb{R}_0} f(X_1) = \Delta_x f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}(t, x)$ for $\mathfrak{m} \otimes \mathbb{P}$ -a.e. (t, x, ω) . \square

3.3 Characterization for $f(X_1) \in \mathbb{D}_{1,2}$

Corollary 3.1. *Let $f(X_1) \in L_2(\mathbb{P})$. Then $f(X_1) \in \mathbb{D}_{1,2}$ if and only if*

- (a) *in case $\sigma \neq 0$, $\Delta_0 f$ exists and $\Delta_0 f(X_1) \in L_2(\mathbb{P})$ and*
- (b) *$\Delta f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0} \in L_2(\mathfrak{m} \otimes \mathbb{P})$.*

If $f(X_1) \in \mathbb{D}_{1,2}$, then $Df(X_1) = \Delta f(X_1) \mathbb{1}_{(0,1] \times \mathbb{R}}$ $\mathfrak{m} \otimes \mathbb{P}$ -a.e.

In the following example we see that in case X_1 has a density, which is (locally) bounded from below and above by a positive constant, then $\mathbb{1}_{[K,\infty)}(X_1)$ is Malliavin differentiable if and only if the Lévy process X is of bounded variation.

Example 3.1. (a) Let $K \in \mathbb{R}$ and assume that X_1 has a density q which is bounded from below and above by a positive constant on $[K - \varepsilon, K + \varepsilon]$

for some $\varepsilon > 0$. Let $f = \mathbb{1}_{[K,\infty)}$. If $\sigma > 0$, then $\mathbb{1}_{[K,\infty)}(X_1) \notin \mathbb{D}_{1,2}^0$ by Proposition 3.2. Let us check the condition of Proposition 3.3:

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times \mathbb{R}_0} \mathbb{E} |\Delta_x f(X_1) \mathbb{1}_{(0,1]}(t)|^2 \mathfrak{m}(dt, dx) \\ &= \int_{(-\infty, 0)} \mathbb{E} \mathbb{1}_{[K, K-x)}(X_1) \nu(dx) + \int_{(0, \infty)} \mathbb{E} \mathbb{1}_{[K-x, K)}(X_1) \nu(dx) \\ &\sim_c \int_{0 < |x| \leq \varepsilon} |x| \nu(dx) + \int_{|x| > \varepsilon} \mathbb{E} |\mathbb{1}_{[K,\infty)}(X_1 + x) - \mathbb{1}_{[K,\infty)}(X_1)| \nu(dx), \end{aligned}$$

where $A \sim_c B$ signifies $\frac{1}{c}B \leq A \leq cB$ and the constant $c \geq 1$ depends on $\sup_{x \in [K-\varepsilon, K+\varepsilon]} |q(x)|$ and $\inf_{x \in [K-\varepsilon, K+\varepsilon]} |q(x)|$. Thus

$$\mathbb{1}_{[K,\infty)}(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0} \quad \Leftrightarrow \quad \int_{0 < |x| \leq 1} |x| \nu(dx) < \infty$$

and

$$\mathbb{1}_{[K,\infty)}(X_1) \in \mathbb{D}_{1,2} \quad \Leftrightarrow \quad \sigma = 0 \text{ and } \int_{0 < |x| \leq 1} |x| \nu(dx) < \infty.$$

Note that the process X has trajectories of finite variation if and only if $\sigma = 0$ and $\int_{0 < |x| \leq 1} |x| \nu(dx) < \infty$ (see [17, Theorem 21.9 and Definition 11.9]).

- (b) In the following we use the idea of Avikainen [5, 6] to express bounded variation functions with the help of signed measures. Suppose X_1 has a bounded density q and $f : \mathbb{R} \rightarrow \mathbb{R}$ is non-zero, of bounded variation, right-continuous and vanishes at $-\infty$. By [8, Theorem 3.29] there exists a finite signed Borel-measure η (i.e. $\eta = \eta^+ - \eta^-$ and $|\eta| = \eta^+(\mathbb{R}) + \eta^-(\mathbb{R}) < \infty$) on \mathbb{R} such that $f(x) = \eta((-\infty, x])$. We use $f(X_1) = \eta((-\infty, X_1]) = \int_{\mathbb{R}} \mathbb{1}_{[y,\infty)}(X_1) \eta(dy)$ and get

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times \mathbb{R}_0} \mathbb{E} |\Delta_x f(X_1) \mathbb{1}_{(0,1]}(t)|^2 \mathfrak{m}(dt, dx) \\ &= \int_{\mathbb{R}_0} \mathbb{E} \left| \int_{\mathbb{R}} (\mathbb{1}_{[y,\infty)}(X_1 + x) - \mathbb{1}_{[y,\infty)}(X_1)) \eta(dy) \right|^2 \nu(dx) \\ &\leq |\eta|(\mathbb{R}) \int_{\mathbb{R}} \int_{\mathbb{R}_0} \mathbb{E} |\mathbb{1}_{[y,\infty)}(X_1 + x) - \mathbb{1}_{[y,\infty)}(X_1)|^2 \nu(dx) |\eta|(dy) \\ &\leq |\eta|(\mathbb{R}) \int_{\mathbb{R}} \int_{\mathbb{R}_0} ((\|q\|_{\infty} |x|) \wedge 1) \nu(dx) |\eta|(dy) \\ &\leq c \int_{\mathbb{R}_0} (|x| \wedge 1) \nu(dx), \end{aligned}$$

where $c > 0$ depends on $|\eta|(\mathbb{R})$ and $\|q\|_\infty$. Proposition 3.3 yields

$$f(X_1) \in \mathbb{D}_{1,2}^{\mathbb{R}_0} \quad \text{if} \quad \int_{0 < |x| \leq 1} |x| \nu(dx) < \infty.$$

4 The Skorohod integral

The adjoint operator δ of the Malliavin derivative $D : \mathbb{D}_{1,2} \rightarrow L_2(\mathfrak{m} \otimes \mathbb{P})$ is called the Skorohod integral (see, for instance, [14]). For $u \in L_2(\mathfrak{m} \otimes \mathbb{P})$ it holds that $u \in \text{Dom}(\delta)$ if and only if there exists a random variable $H \in L_2(\mathbb{P})$ such that

$$(u, DG)_{L_2(\mathfrak{m} \otimes \mathbb{P})} = \mathbb{E}HG \quad \text{for all } G \in \mathbb{D}_{1,2}.$$

If $u \in \text{Dom}(\delta)$, then H is unique and $H =: \delta(u)$.

Remark 4.1. For each $u \in L_2(\mathfrak{m} \otimes \mathbb{P})$ there exist $f_n \in L_2(\mathfrak{m}^{\otimes n})$, $n = 1, 2, \dots$, such that f_{n+1} is symmetric in the first n pairs of variables and $u(t, x) = \sum_{n=0}^{\infty} I_n(f_{n+1}(\cdot, (t, x)))$ (see Nualart and Vives [15, Section 4]). Then $\|u\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 = \sum_{n=1}^{\infty} (n-1)! \|f_n\|_{L_2(\mathfrak{m}^{\otimes n})}^2$. From the equation

$$(u, nI_{n-1}(\tilde{g}_n))_{L_2(\mathfrak{m} \otimes \mathbb{P})} = n!(\tilde{f}_n, \tilde{g}_n)_{L_2(\mathfrak{m}^{\otimes n})} = \mathbb{E}I_n(f_n)I_n(g_n)$$

for all $n \geq 1$ and $g_n \in L_2(\mathfrak{m}^{\otimes n})$ one concludes (analogously to [14, Proposition 1.3.7]) that $u \in \text{Dom}(\delta)$ if and only if

$$\|u\|_{\text{Dom}(\delta)} := \left(\|u\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 + \sum_{n=1}^{\infty} n! \|\tilde{f}_n\|_{L_2(\mathfrak{m}^{\otimes n})}^2 \right)^{\frac{1}{2}} < \infty.$$

If $u \in \text{Dom}(\delta)$, then

$$\delta(u) = \sum_{n=1}^{\infty} I_n(f_n). \quad (9)$$

Let $\mathbb{D}_{1,2}(L_2(\mathfrak{m}))$ denote the space of random fields $u \in L_2(\mathfrak{m} \otimes \mathbb{P})$ such that $u(t, x) \in \mathbb{D}_{1,2}$ for \mathfrak{m} -a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and there exists a measurable version of Du verifying $\|Du\|_{L_2(\mathfrak{m}^{\otimes 2} \otimes \mathbb{P})} < \infty$. It holds that $\mathbb{D}_{1,2}(L_2(\mathfrak{m})) \subseteq \text{Dom}(\delta)$, and for $u, v \in \mathbb{D}_{1,2}(L_2(\mathfrak{m}))$ one has

$$\begin{aligned} \mathbb{E}\delta(u)\delta(v) &= \mathbb{E} \iint_{\mathbb{R}_+ \times \mathbb{R}} u(t, x)v(t, x)\mathfrak{m}(dt, dx) \\ &\quad + \mathbb{E} \iint_{\mathbb{R}_+ \times \mathbb{R}} \iint_{\mathbb{R}_+ \times \mathbb{R}} D_{t,x}u(s, y)D_{s,y}v(t, x)\mathfrak{m}(dt, dx)\mathfrak{m}(ds, dy) \end{aligned} \quad (10)$$

(see, for instance, [15, Theorem 4.1] or [4, Equation (5.31)], where the random measure $\tilde{N}(dt, dx)$ is used instead of $x\tilde{N}(dt, dx)$).

4.1 Relation between the Skorohod integral and a pathwise integral

In the following we aim to express the Skorohod integral using pathwise integration. Alós et al. [2, Corollary 2.9] have shown a relation between the pathwise integral and the Skorohod integral for a certain class of pathwise integrable random fields. The relation is also considered by Øksendal and Zhang [16, Lemma 2.1]. For later purpose, we show this relation here on a dense subset of $\text{Dom}(\delta)$.

Definition 4.1. We let $\mathcal{S}(L_2(\mathfrak{m}))$ denote the linear span of random fields of the form

$$u(t, x) = f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t) \quad (11)$$

with $m \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^{m+1})$ such that the set $\{x : f(y, x) \neq 0 \text{ for some } y \in \mathbb{R}^m\}$ is bounded, $0 \leq t_0 < t_1 < \dots < t_m < \infty$ and $k \in \{1, \dots, m\}$.

Lemma 4.1. $\mathcal{S}(L_2(\mathfrak{m})) \subseteq \mathbb{D}_{1,2}(L_2(\mathfrak{m}))$.

Proof. Consider u as in (11). Then

$$\begin{aligned} \|u\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 &= \int_{(t_{k-1}, t_k]} \int_{\mathbb{R}} \mathbb{E} |f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, x)|^2 \mathfrak{m}(dt, dx) \\ &\leq (t_k - t_{k-1}) \|f\|_\infty^2 \mu(\{x \in \mathbb{R} : f(y, x) \neq 0 \text{ for some } y \in \mathbb{R}^m\}) \\ &< \infty, \end{aligned}$$

so that $u \in L_2(\mathfrak{m} \otimes \mathbb{P})$. Clearly $u(t, x) \in \mathcal{S} \subseteq \mathbb{D}_{1,2}$ for all (t, x) and since there exists a constant $c > 0$ such that $|\Delta_y^i f(z, x)|^2 \leq c(1 \wedge \frac{1}{|y|^2})$ for all $x, y \in \mathbb{R}$, $z \in \mathbb{R}^m$, $i = 1, \dots, m$, we have

$$\begin{aligned} &\|Du\|_{L_2(\mathfrak{m}^{\otimes 2} \otimes \mathbb{P})}^2 \\ &= \iint_{(t_{k-1}, t_k] \times \mathbb{R}} \sum_{i=1}^m \iint_{(t_{i-1}, t_i] \times \mathbb{R}} \mathbb{E} |\Delta_y^i f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, x)|^2 \times \\ &\quad \mathfrak{m}(ds, dy) \mathfrak{m}(dt, dx) \\ &\leq c(t_k - t_{k-1}) \mu(\{x \in \mathbb{R} : f(z, x) \neq 0 \text{ for some } z \in \mathbb{R}^m\}) \\ &\quad \times (t_m - t_0) \left(\sigma^2 + \int_{\mathbb{R}} (y^2 \wedge 1) \nu(dy) \right) \\ &< \infty, \end{aligned}$$

so that $u \in \mathbb{D}_{1,2}(L_2(\mathfrak{m}))$. □

Definition 4.2. For $u \in \mathcal{S}(L_2(\mathfrak{m}))$ with representation like in (11) we write

$$\flat u(t, x) := f(X_{t_1} - X_{t_0}, \dots, X_{t_k} - X_{t_{k-1}} - x, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t)$$

and

$$\begin{aligned} \Delta^- \flat u(t, x) &:= \Delta_x^k f(X_{t_1} - X_{t_0}, \dots, X_{t_k} - X_{t_{k-1}} - x, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t) \\ &= \begin{cases} \frac{\partial}{\partial x_k} f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t), & x = 0 \\ \frac{u(t, x) - \flat u(t, x)}{x}, & x \neq 0. \end{cases} \end{aligned}$$

The expressions $\flat u(t, x)$ and $\Delta^- \flat u(t, x)$ depend on the function f in (8), so that we always mean $\flat_f u(t, x)$ and $\Delta_f^- \flat_f u(t, x)$ rather than $\flat u(t, x)$ and $\Delta^- \flat u(t, x)$. To keep the notation simple we omit the dependence on f in the notation.

The independence of the Brownian motion and jump part of the Lévy process together with [14, formula (1.44)] give

$$\begin{aligned} \delta(u \mathbb{1}_{\mathbb{R}_+ \times \{0\}}) &= f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, 0) \sigma(W_{t_k} - W_{t_{k-1}}) \\ &\quad - \int_{\mathbb{R}_+} \Delta^- \flat u(t, 0) \sigma^2 dt. \end{aligned} \quad (12)$$

Noting that $\flat u(t, 0) = u(t, 0)$ we denote the first term on the right hand side of (12) by

$$\iint_{\mathbb{R}_+ \times \{0\}} \flat u(t, x) M(d^p t, dx).$$

Moreover, we use the notation

$$\begin{aligned} &\iint_{\mathbb{R}_+ \times |x| > \varepsilon} \flat u(t, x) M(d^p t, dx) \\ &:= \sum_{|\Delta X_t| > \varepsilon} (\flat u(t, \Delta X_t)) \Delta X_t - \int_{\mathbb{R}_+} \int_{|x| > \varepsilon} \flat u(t, x) x dt \nu(dx), \end{aligned}$$

where the sum is a.s. finite.

Theorem 4.1. For $u \in \mathcal{S}(L_2(\mathfrak{m}))$ and compact sets $U_\varepsilon \subset \mathbb{R}_0$ such that $U_\varepsilon \subseteq U_{\varepsilon'}$ for $\varepsilon' \leq \varepsilon$ and $\bigcup_{\varepsilon > 0} U_\varepsilon = \mathbb{R}_0$, it holds that

$$\begin{aligned} \delta(u) &= \lim_{\varepsilon \rightarrow 0} \left(\iint_{\mathbb{R}_+ \times (U_\varepsilon \cup \{0\})} \flat u(t, x) M(d^p t, dx) \right. \\ &\quad \left. - \iint_{\mathbb{R}_+ \times (U_\varepsilon \cup \{0\})} \Delta^- \flat u(t, x) \mathfrak{m}(dt, dx) \right), \end{aligned} \quad (13)$$

where the limit is taken in $L_2(\mathbb{P})$.

For the proof we present the following anticipating integration by parts formula for the compound Poisson process.

Lemma 4.2. *Let $(Y_t)_{t \geq 0}$ be a compound Poisson process with Lévy measure $\bar{\nu}$, $T > 0$ and $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be such that $\mathbb{E} \int_{\mathbb{R}} |\phi(Y_T, x)| \bar{\nu}(dx) < \infty$. Then*

$$T \mathbb{E} \int_{\mathbb{R}} \phi(Y_T, x) \bar{\nu}(dx) = \mathbb{E} \sum_{0 < t \leq T, |\Delta Y_t| > 0} \phi(Y_T - \Delta Y_t, \Delta Y_t).$$

Proof. There exists a Poisson process $(N_t)_{t \geq 0}$ with intensity $\lambda = \bar{\nu}(\mathbb{R}) \in (0, \infty)$ and an independent sequence $(F_i)_{i=1}^{\infty}$ of independent random variables with $F_i \sim \bar{\nu}/\lambda$ for all $i = 1, 2, \dots$ such that

$$Y_t = \sum_{i=1}^{N_t} F_i \quad a.s. \text{ for all } t \geq 0.$$

Hence

$$\begin{aligned} T \mathbb{E} \int_{\mathbb{R}} \phi(Y_T, x) \bar{\nu}(dx) &= T \sum_{n=0}^{\infty} \mathbb{P}(N_T = n) \mathbb{E} \int_{\mathbb{R}} \phi \left(\sum_{i=1}^n F_i, x \right) \bar{\nu}(dx) \\ &= T \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \mathbb{E} \phi \left(\sum_{i=1}^n F_i, F_{n+1} \right) \lambda \\ &= \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^{n+1}}{n!} \mathbb{E} \phi \left(\sum_{i=1}^n F_i, F_{n+1} \right) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(N_T = n+1) (n+1) \mathbb{E} \phi \left(\sum_{i=1}^n F_i, F_{n+1} \right). \end{aligned}$$

Since $\phi \left(\sum_{i=1}^{n+1} F_i - F_j, F_j \right)$ has the same distribution for all $j = 1, \dots, n+1$, it holds that

$$\begin{aligned} T \mathbb{E} \int_{\mathbb{R}} \phi(Y_T, x) \bar{\nu}(dx) &= \sum_{n=0}^{\infty} \mathbb{P}(N_T = n+1) \sum_{j=1}^{n+1} \mathbb{E} \phi \left(\sum_{i=1}^{n+1} F_i - F_j, F_j \right) \\ &= \mathbb{E} \sum_{j=1}^{N_T} \phi \left(\sum_{i=1}^{N_T} F_i - F_j, F_j \right) \\ &= \mathbb{E} \sum_{0 < t \leq T, |\Delta Y_t| > 0} \phi(Y_T - \Delta Y_t, \Delta Y_t). \end{aligned}$$

□

Proof of Theorem 4.1. If $\nu(\mathbb{R}) = 0$, then the assertion follows from equation (12). Assume $\nu(\mathbb{R}) \in (0, \infty]$ and let $u \in \mathcal{S}(L_2(\mathfrak{m}))$ be given by

$$u(t, x) = \sum_{i=1}^K f_{i,0}(X_{r_1^i} - X_{r_0^i}, \dots, X_{r_{n_i}^i} - X_{r_{n_i-1}^i}, x) \mathbb{1}_{(r_{m_i-1}^i, r_{m_i}^i]}(t).$$

Since by Proposition 3.1 the set \mathcal{S} is dense in $\mathbb{D}_{1,2}$, it holds that $H = \delta(u)$ if and only if

$$(u, DG)_{L_2(\mathfrak{m} \otimes \mathbb{P})} = \mathbb{E}HG \quad \text{for all } G \in \mathcal{S}.$$

Let $G = g_0(X_{s_1} - X_{s_0}, \dots, X_{s_n} - X_{s_{n-1}}) \in \mathcal{S}$. Next we find new expressions for u and G to unify the time nets which determine the increments in the functional. Write

$$\begin{aligned} \tau &= \{r_k^i, s_j, k = 1, \dots, n_i, j = 1, \dots, n, i = 1, \dots, K\} \\ &= \{0 \leq t_0 < t_1 < \dots < t_m < \infty\}, \end{aligned}$$

where $s_j = t_{k_j}$ for some k_j for all j and $r_j^i = t_{k_j^i}$ for some k_j^i for all j and i . Then

$$\begin{aligned} G &= g_0 \left(\sum_{k=k_0+1}^{k_1} (X_{t_k} - X_{t_{k-1}}), \dots, \sum_{k=k_{n-1}+1}^{k_n} (X_{t_k} - X_{t_{k-1}}) \right) \\ &= g(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}) \end{aligned}$$

for

$$g(x_1 - x_0, \dots, x_m - x_{m-1}) = g_0 \left(\sum_{k=k_0+1}^{k_1} (x_k - x_{k-1}), \dots, \sum_{k=k_{n-1}+1}^{k_n} (x_k - x_{k-1}) \right).$$

Moreover,

$$u = \sum_{i=1}^K \sum_{k=k_{m_i-1}+1}^{k_{m_i}} f_i(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]},$$

for

$$\begin{aligned} &f_i(x_1 - x_0, \dots, x_m - x_{m-1}, x) \\ &= f_{i,0} \left(\sum_{k=k_0^i+1}^{k_1^i} (x_k - x_{k-1}), \dots, \sum_{k=k_{n-1}^i+1}^{k_n^i} (x_k - x_{k-1}), x \right). \end{aligned}$$

The mappings $\flat u$ and $\Delta^- \flat u$ do not depend on the choice between f_i and $f_{i,0}$. Hence it is sufficient to consider

$$u(t, x) = f(X_{t_1} - X_{t_0}, \dots, X_{t_m} - X_{t_{m-1}}, x) \mathbb{1}_{(t_{k-1}, t_k]}(t).$$

To shorten the notation we write

$$\mathbf{X}^{\tau(i,j)} := (X_{t_i} - X_{t_{i-1}}, \dots, X_{t_j} - X_{t_{j-1}}) \quad \text{for } 1 \leq i \leq j \leq m.$$

Let $U \subset \mathbb{R}_0$ be compact with $\nu(U) > 0$ and

$$Y_t := \iint_{(0,t] \times U} x N(ds, dx).$$

Then Y and $Z := X - Y$ are independent Lévy processes, and from the independence of their increments we conclude

$$\begin{aligned} & (u \mathbb{1}_{\mathbb{R}_+ \times U}, DG)_{L_2(\mathfrak{m} \otimes \mathbb{P})} \\ &= \mathbb{E} \left(\int_{\mathbb{R}} \iint_{(t_{k-1}, t_k] \times U} \left[f(\mathbf{X}^{\tau(1,k-1)}, Z_{t_k} - Z_{t_{k-1}} + y, \mathbf{X}^{\tau(k+1,m)}, x) \times \right. \right. \\ & \quad \left. \left. \Delta_x^k g(\mathbf{X}^{\tau(1,k-1)}, Z_{t_k} - Z_{t_{k-1}} + y, \mathbf{X}^{\tau(k+1,m)}) \right] \mathfrak{m}(dt, dx) \mathbb{P}_{Y_{t_k - t_{k-1}}}(dy) \right)^*. \end{aligned} \tag{14}$$

To compute $^* \dots ^*$ we keep random variables fixed and denote $f_k(y, x) := f(\dots, \cdot + y, \dots, x)$, $g_k(y) := g(\dots, \cdot + y, \dots)$ and $T := t_k - t_{k-1}$. Then we have

$$\begin{aligned} ^* \dots ^* &= T \mathbb{E} \int_U f_k(Y_T, x) (g_k(Y_T + x) - g_k(Y_T)) x \nu(dx) \\ &= T \mathbb{E} \int_U f_k(Y_T, x) g_k(Y_T + x) x \nu(dx) - \mathbb{E} g_k(Y_T) T \int_U f_k(Y_T, x) x \nu(dx). \end{aligned}$$

The process Y is a compound Poisson process and using Lemma 4.2 we get

$$\begin{aligned} & T \mathbb{E} \int_U f_k(Y_T, x) g_k(Y_T + x) x \nu(dx) \\ &= \mathbb{E} \sum_{0 < t \leq T, |\Delta Y_t| > 0} f_k(Y_T - \Delta Y_t, \Delta Y_t) g_k(Y_T) \Delta Y_t \\ &= \mathbb{E} \sum_{t_{k-1} < t \leq t_k, \Delta X_t \in U} f_k(Y_T - \Delta X_t, \Delta X_t) g_k(Y_T) \Delta X_t. \end{aligned}$$

From equation (14) and from the independence of the increments and the independence of Y and Z we get

$$\begin{aligned}
& (u\mathbb{1}_{\mathbb{R}_+ \times U}, DG)_{L_2(\mathfrak{m} \otimes \mathbb{P})} \\
&= \mathbb{E} \left[G \times \right. \\
& \quad \left(\sum_{t_{k-1} < t \leq t_k, \Delta X_t \in U} f(\mathbf{X}^{\tau(1,k-1)}, X_{t_k} - X_{t_{k-1}} - \Delta X_t, \mathbf{X}^{\tau(k+1,m)}, \Delta X_t) \Delta X_t \right. \\
& \quad \left. - \int_{(t_{k-1}, t_k]} \int_U f(\mathbf{X}^{\tau(1,m)}, x) x dt \nu(dx) \right) \Big] \\
&= \mathbb{E} G \left(\iint_{\mathbb{R}_+ \times U} bu(t, x) M(d^p t, dx) - \int_{\mathbb{R}_+} \int_U (u(t, x) - bu(t, x)) x dt \nu(dx) \right).
\end{aligned}$$

Hence

$$\delta(u\mathbb{1}_{\mathbb{R}_+ \times U}) = \iint_{\mathbb{R}_+ \times U} bu(t, x) M(d^p t, dx) - \int_{\mathbb{R}_+} \int_U \Delta^- bu(t, x) x^2 dt \nu(dx). \tag{15}$$

Using the same reasoning as in proof of Lemma 4.1 we see that $u\mathbb{1}_{\mathbb{R}_+ \times U_\varepsilon} \in \mathbb{D}_{1,2}(L_2(\mathfrak{m}))$. From equation (10) we get

$$\begin{aligned}
& \|u\mathbb{1}_{\mathbb{R}_+ \times U_\varepsilon} - u\mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}_0}\|_{\text{Dom}(\delta)}^2 \\
& \leq 2 \|u\mathbb{1}_{\mathbb{R}_+ \times (\mathbb{R}_0 \setminus U_\varepsilon)}\|_{L_2(\mathfrak{m} \otimes \mathbb{P})}^2 + \|D(u\mathbb{1}_{\mathbb{R}_+ \times (\mathbb{R}_0 \setminus U_\varepsilon)})\|_{L_2(\mathfrak{m} \otimes^2 \mathbb{P})}^2 \\
& \rightarrow 0
\end{aligned}$$

as $\varepsilon \rightarrow 0$ since $u \in \mathbb{D}_{1,2}(L_2(\mathfrak{m}))$. The claim follows from equations (12) and (15) and the above convergence. \square

Definition 4.3. We denote by \mathcal{L} the linear span of mappings of the form

$$u(t, x) = F\mathbb{1}_{(a,b]}(t)\varphi(x),$$

where $F \in \mathcal{S}$ and $\varphi \in C_c^\infty(\mathbb{R})$.

Finally, we complement Theorem 4.1 by

Proposition 4.1. The set $\mathcal{L} \subseteq \mathcal{S}(L_2(\mathfrak{m}))$ is dense in $\text{Dom}(\delta)$.

In the proof we use the following lemma.

Lemma 4.3 (Lemma 4.1 [9]). *Consider $I_1(\mathbb{1}_{(a,b]}\varphi)$ where $0 \leq a < b < \infty$ and $\varphi \in C_c^\infty(\mathbb{R})$. Set*

$$F_k := (G_k - \mathbb{E}G_k) \quad \text{for} \quad G_k := \sum_{i=1}^k \varphi(X_{t_i^k} - X_{t_{i-1}^k})(X_{t_i^k} - X_{t_{i-1}^k}),$$

where $a = t_0^k < t_1^k < \dots < t_k^k = b$ and $\sup_i |t_i^k - t_{i-1}^k| \rightarrow 0$ as $k \rightarrow \infty$. Then $\mathcal{S} \ni F_k \rightarrow I_1(\mathbb{1}_{(a,b]}\varphi)$ in $L_2(\mathbb{P})$ as $k \rightarrow \infty$.

Proof of Proposition 4.1. (a) We first observe that for $n \geq 1$ the linear span of functions of the form

$$\prod_{i=1}^n \mathbb{1}_{(a_i, b_i]} \varphi_i,$$

where $(a_i, b_i] \cap (a_j, b_j] = \emptyset$ for $i \neq j$ and $\varphi_i \in C_c^\infty(\mathbb{R})$, is dense in $L_2(\mathfrak{m}^{\otimes n})$. By [9, Lemma 2.1] the linear span of functions of the form $\prod_{i=1}^n \mathbb{1}_{(a_i, b_i] \times (c_i, d_i]}$ is dense in $L_2(\mathfrak{m}^{\otimes n})$. Since μ is regular, we find for any $\varepsilon > 0$ an open set $U_i \supset (c_i, d_i]$ and a compact set $V_i \subset (c_i, d_i]$ such that $\mu(U_i \setminus V_i) < \varepsilon$. Using the C^∞ Urysohn Lemma ([8, 8.18]) we see that there exist functions $\varphi_i^k \in C_c^\infty(\mathbb{R})$ such that $\varphi_i^k \rightarrow \mathbb{1}_{(c_i, d_i]}$ in $L_2(\mu)$.

(b) To prove the proposition it is sufficient to show that for any n , $I_n(f_{n+1})$ can be approximated in $\text{Dom}(\delta)$ by mappings from \mathcal{L} . Note that $\|I_n(f_{n+1})\|_{\text{Dom}(\delta)}^2 \leq 2(n+1)! \|f_{n+1}\|_{L_2(\mathfrak{m}^{\otimes (n+1)})}^2$. Thus by part (a) of the proof it is sufficient to approximate $u = I_n(g_{n+1})$ by mappings in \mathcal{L} , where $g_{n+1} = \prod_{i=1}^{n+1} \mathbb{1}_{(a_i, b_i]} \varphi_i$. Set F_k^i as in Lemma 4.3 with $F_k^i \rightarrow I_1(\mathbb{1}_{(a_i, b_i]} \varphi_i)$. Then $F_k := \prod_{i=1}^n F_k^i \rightarrow \prod_{i=1}^n I_1(\mathbb{1}_{(a_i, b_i]} \varphi_i)$ in $L_2(\mathbb{P})$ by independence and it holds that $u_k := F_k \mathbb{1}_{(a_{n+1}, b_{n+1}]} \varphi_{n+1} \in \mathcal{L}$. Consider the chaos representation

$$F_k = \sum_{n=0}^{\infty} I_n(f_n^k \mathbb{1}_{((a_{n+1}, b_{n+1}] \times \mathbb{R})^c}^{\otimes n}).$$

Using equation (9) and Lemma 2.1 we get

$$\begin{aligned} \delta(u_k) &= \sum_{n=0}^{\infty} I_{n+1}(f_n^k \mathbb{1}_{((a_{n+1}, b_{n+1}] \times \mathbb{R})^c}^{\otimes n} \otimes \mathbb{1}_{(a_{n+1}, b_{n+1}]} \varphi_{n+1}) \\ &= F_k I_1(\mathbb{1}_{(a_{n+1}, b_{n+1}]} \varphi_{n+1}) \end{aligned}$$

and $\delta(u) = I_{n+1}(g_{n+1}) = \prod_{i=1}^{n+1} I_1(\mathbb{1}_{(a_i, b_i]} \varphi_i)$. The independence of the increments of X implies that F_k and $I_1(\mathbb{1}_{(a_{n+1}, b_{n+1}]} \varphi_{n+1})$ are independent and Lemma 4.3 gives

$$\|u_k - u\|_{\text{Dom}(\delta)}^2 = \|u_k - u\|_{L_2(\mathfrak{m}^{\otimes \mathbb{P}})}^2 + \|\delta(u_k) - \delta(u)\|_{L_2(\mathbb{P})}^2$$

$$\begin{aligned}
&= 2\mathbb{E} \left| F_k - \prod_{i=1}^n I_1(\mathbb{1}_{(a_i, b_i]} \varphi_i) \right|^2 \|\mathbb{1}_{(a_{n+1}, b_{n+1}]} \varphi_{n+1}\|_{L_2(\mathfrak{m})}^2 \\
&\rightarrow 0
\end{aligned}$$

as $k \rightarrow \infty$. □

A Appendix

Lemma A.1. *Let $f \in C^\infty(\mathbb{R})$ be such that $\sup_{x \in \mathbb{R}} (1 + |x|)^m \left| \frac{d^k}{dx^k} f(x) \right| < \infty$ for all $k, m = 0, 1, 2, \dots$ and let $Y_t = X_t - \sigma W_t$. Then $f(Y_1) = \sum_{n=0}^{\infty} I_n(f_n)$ with*

$$f_n((t_1, x_1), \dots, (t_n, x_n)) = \frac{1}{n!} \mathbb{E} \Delta_{x_1} \cdots \Delta_{x_n} f(Y_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes n}((t_1, x_1), \dots, (t_n, x_n))$$

$\mathfrak{m}^{\otimes n}$ -a. e.

Proof. The proof follows the steps of proof of [9, Lemma 3.1]: By the Fourier inversion theorem (see, for instance, [8, Theorem 8.26, using Corollary 8.23]) we get

$$f(Y_1) = \int_{\mathbb{R}} \hat{f}(u) e^{2\pi i u Y_1} du = \int_{\mathbb{R}} \hat{f}(u) (\mathbb{E} e^{2\pi i u Y_1}) \frac{e^{2\pi i u Y_1}}{\mathbb{E} e^{2\pi i u Y_1}} du.$$

Fix $u \in \mathbb{R}$ and define an $L_2(\mathbb{P})$ -martingale M by letting $M_t := \frac{e^{2\pi i u Y_t}}{\mathbb{E} e^{2\pi i u Y_t}}$ for $t \geq 0$. Write $g(x) := \frac{e^{2\pi i u x} - 1}{x}$. Then

$$\begin{aligned}
M_t &= \exp \left\{ -t \int_{\mathbb{R}_0} (g(x) - 2\pi i u \mathbb{1}_{\{|x| \leq 1\}}) x \nu(dx) + \int_{(0,t] \times \{|x| > 1\}} 2\pi i u x N(ds, dx) \right. \\
&\quad \left. + \int_{(0,t] \times \{|x| \leq 1\}} 2\pi i u x \tilde{N}(ds, dx) \right\} \\
&=: e^{Z_t}.
\end{aligned}$$

Using Itô's formula [3, Theorem 4.4.7] we get

$$\begin{aligned}
M_t - 1 &= - \int_0^t \int_{\mathbb{R}_0} e^{Z_{s^-}} (g(x) - 2\pi i u \mathbb{1}_{\{|x| \leq 1\}}) x ds \nu(dx) \\
&\quad + \iint_{(0,t] \times \{|x| > 1\}} e^{Z_{s^-}} g(x) x N(ds, dx) \\
&\quad + \iint_{(0,t] \times \{|x| \leq 1\}} e^{Z_{s^-}} g(x) x \tilde{N}(ds, dx) \\
&\quad + \int_0^t \int_{\{|x| \leq 1\}} e^{Z_{s^-}} (g(x) - 2\pi i u) x ds \nu(dx) \\
&= \iint_{(0,t] \times \mathbb{R}_0} M_{s^-} g(x) x \tilde{N}(ds, dx).
\end{aligned}$$

Next we define an $L_2(\mathbb{P})$ -martingale \tilde{M} . We have $\kappa := \|g \mathbb{1}_{\mathbb{R}_0}\|_{L_2(\mu)}^2 < \infty$, so that $\frac{1}{n!} (\mathbb{1}_{(0,t]} g \mathbb{1}_{\mathbb{R}_0})^{\otimes n} \in L_2(\mathfrak{m}^{\otimes n})$ with

$$\left\| I_n \left(\frac{1}{n!} (\mathbb{1}_{(0,t]} g \mathbb{1}_{\mathbb{R}_0})^{\otimes n} \right) \right\|_{L_2(\mathbb{P})}^2 = \frac{1}{n!} (t\kappa)^n.$$

Hence \tilde{M} , defined by

$$\tilde{M}_t := 1 + \sum_{n=1}^{\infty} I_n \left(\frac{1}{n!} (\mathbb{1}_{(0,t]} g \mathbb{1}_{\mathbb{R}_0})^{\otimes n} \right)$$

is an $L_2(\mathbb{P})$ -martingale with $\|\tilde{M}_t\|_{L_2(\mathbb{P})}^2 = e^{t\kappa}$. Since

$$\sum_{n=0}^{\infty} (n+1)! \left\| \frac{1}{n!} (\mathbb{1}_{(0,t]} g \mathbb{1}_{\mathbb{R}_0})^{\otimes n} \right\|_{L_2(\mathfrak{m}^{\otimes n})}^2 = (1 + t\kappa) e^{t\kappa},$$

we have $\tilde{M}_t \in \mathbb{D}_{1,2}$ and from the Clark-Ocone formula [19, Theorem 10] we obtain

$$\tilde{M}_t = 1 + \int_{(0,t] \times \mathbb{R}_0} \mathbb{E} \left[D_{s,x} \tilde{M}_t | \mathcal{F}_{s^-} \right] M(ds, dx) = 1 + \int_{(0,t] \times \mathbb{R}_0} \tilde{M}_{s^-} g(x) M(ds, dx).$$

Writing $D_t := M_t - \tilde{M}_t$ it holds that

$$h(t) := \|D_t\|_{L_2(\mathbb{P})}^2 = \left\| \int_{(0,t] \times \mathbb{R}_0} D_{s^-} g(x) M(ds, dx) \right\|_{L_2(\mathbb{P})}^2$$

$$\begin{aligned}
&= \kappa \int_0^t \mathbb{E} |D_{s-}|^2 ds \\
&= \kappa \int_0^t h(s) ds,
\end{aligned}$$

so that $h(t) = 0$ and consequently $\tilde{M} = M$.

Now, writing $A_K(u) = \sum_{n=K+1}^{\infty} I_n \left(\frac{1}{n!} \left(\frac{e^{2\pi i u x} - 1}{x} \mathbb{1}_{(0,1] \times \mathbb{R}_0} \right)^{\otimes n} \right)$, we get

$$\begin{aligned}
&f(Y_1) \\
&= \int_{\mathbb{R}} \hat{f}(u) (\mathbb{E} e^{2\pi i u Y_1}) \left(\sum_{n=0}^K I_n \left(\frac{1}{n!} \left(\frac{e^{2\pi i u x} - 1}{x} \mathbb{1}_{(0,1] \times \mathbb{R}_0} \right)^{\otimes n} \right) + A_K(u) \right) du,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{E} \left[\int_{\mathbb{R}} \hat{f}(u) (\mathbb{E} e^{2\pi i u Y_1}) A_K(u) du \right]^2 \\
&\leq \int_{\mathbb{R}} |\hat{f}(u) (\mathbb{E} e^{2\pi i u Y_1})| du \int_{\mathbb{R}} |\hat{f}(u) (\mathbb{E} e^{2\pi i u Y_1})| |\mathbb{E} |A_K(u)|^2 du \\
&\rightarrow 0
\end{aligned}$$

as $K \rightarrow \infty$ since \hat{f} in $L_1(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$. As a consequence,

$$\begin{aligned}
&f(Y_1) \xrightarrow{K \rightarrow \infty} \int_{\mathbb{R}} \hat{f}(u) (\mathbb{E} e^{2\pi i u Y_1}) \sum_{n=0}^K I_n \left(\frac{1}{n!} \left(\frac{e^{2\pi i u x} - 1}{x} \mathbb{1}_{(0,1] \times \mathbb{R}_0} \right)^{\otimes n} \right) du \\
&= \sum_{n=0}^K \int_{\mathbb{R}} \hat{f}(u) (\mathbb{E} e^{2\pi i u Y_1}) \iint_{\{0 < t_1 < \dots < t_n \leq 1\} \times \mathbb{R}_0^n} \left(\frac{e^{2\pi i u x} - 1}{x} \right)^{\otimes n} dM^{\otimes n} du \\
&= \sum_{n=0}^K \iint_{\{0 < t_1 < \dots < t_n \leq 1\} \times \mathbb{R}_0^n} \mathbb{E} \int_{\mathbb{R}} \hat{f}(u) e^{2\pi i u Y_1} \left(\frac{e^{2\pi i u x} - 1}{x} \right)^{\otimes n} dudM^{\otimes n} \\
&= \sum_{n=0}^K \iint_{\{0 < t_1 < \dots < t_n \leq 1\} \times \mathbb{R}_0^n} \mathbb{E} \int_{\mathbb{R}} \hat{f}(u) \Delta_{x_1} \cdots \Delta_{x_n} e^{2\pi i u Y_1} dudM^{\otimes n} \\
&= \sum_{n=0}^K \iint_{\{0 < t_1 < \dots < t_n \leq 1\} \times \mathbb{R}_0^n} \mathbb{E} \Delta_{x_1} \cdots \Delta_{x_n} f(Y_1) dM^{\otimes n} \\
&= \sum_{n=0}^K I_n(f_n)
\end{aligned}$$

in $L_2(\mathbb{P})$ for

$$\begin{aligned} & f_n((t_1, x_1), \dots, (t_n, x_n)) \\ &= \frac{1}{n!} \mathbb{E} \Delta_{x_1} \cdots \Delta_{x_n} f(Y_1) \mathbb{1}_{(0,1] \times \mathbb{R}_0}^{\otimes n}((t_1, x_1), \dots, (t_n, x_n)). \end{aligned}$$

□

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**A note on Malliavin fractional smoothness for Lévy
processes and approximation**

Christel Geiss, Stefan Geiss and Eija Laukkarinen

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A note on Malliavin fractional smoothness for Lévy processes and approximation

Christel Geiss · Stefan Geiss · Eija Laukkarinen

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Abstract Assume a Lévy process $(X_t)_{t \in [0,1]}$ that is an L_2 -martingale and let Y be either its stochastic exponential or X itself. For certain integrands φ we investigate the behavior of

$$\left\| \int_{(0,1]} \varphi_t dX_t - \sum_{k=1}^N v_{k-1} (Y_{t_k} - Y_{t_{k-1}}) \right\|_{L_2},$$

where v_{k-1} is $\mathcal{F}_{t_{k-1}}$ -measurable, in dependence on the fractional smoothness in the Malliavin sense of $\int_{(0,1]} \varphi_t dX_t$. A typical situation where these techniques apply occurs if the stochastic integral is obtained by the Galtchouk-Kunita-Watanabe decomposition of some $f(X_1)$. Moreover, using the example $f(X_1) = \mathbb{1}_{(K,\infty)}(X_1)$ we show how fractional smoothness depends on the distribution of the Lévy process.

Keywords Lévy process · Besov spaces · Approximation · Stochastic integrals

Mathematics Subject Classification (2010) 60G51 · 60H07 · 46E35 · 41A05 · 41A25

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Christel Geiss
Department of Mathematics, University of Innsbruck
A-6020 Innsbruck, Technikerstraße 13/7, Austria
E-mail: christel.geiss@uibk.ac.at

Stefan Geiss
Department of Mathematics, University of Innsbruck
A-6020 Innsbruck, Technikerstraße 13/7, Austria
E-mail: stefan.geiss@uibk.ac.at

Eija Laukkarinen
Department of Mathematics and Statistics
University of Jyväskylä, P.O. Box 35 (MAD)
FIN-40014 Jyväskylä, Finland
E-mail: eija.laukkarinen@jyu.fi

1 Introduction

We consider the quantitative Riemann approximation of stochastic integrals driven by Lévy processes and its relation to the fractional smoothness in the Malliavin sense. Besides the interest on its own, the problem is of interest for numerical algorithms and for Stochastic Finance. To explain the latter aspect, assume a price process $(S_t)_{t \in [0,1]}$ given under the martingale measure by a diffusion

$$S_t = s_0 + \int_0^t \sigma(S_r) dW_r,$$

where W is the Brownian motion and where usual conditions on σ are imposed. For a polynomially bounded Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$ we obtain a representation

$$f(S_1) = V_0 + \int_0^1 \varphi_t dS_t \quad (1)$$

where $(\varphi_t)_{t \in [0,1]}$ is a continuous adapted process which can be obtained via the gradient of a solution to a parabolic backward PDE related to σ with terminal condition f . The process $(\varphi_t)_{t \in [0,1]}$ is interpreted as a trading strategy. In practice one can trade only finitely many times which corresponds to a replacement of the stochastic integral in (1) by the sum $\sum_{k=1}^N \varphi_{t_{k-1}}(S_{t_k} - S_{t_{k-1}})$ with $0 = t_0 < t_1 < \dots < t_N = 1$. The error

$$\int_0^1 \varphi_t dS_t - \sum_{k=1}^N \varphi_{t_{k-1}}(S_{t_k} - S_{t_{k-1}}) \quad (2)$$

caused by this replacement is often measured in L_2 and has been studied by various authors, for example by Zhang [21], Gobet and Temam [11], S. Geiss [8], S. Geiss and Hujo [9] and C. Geiss and S. Geiss [7]. For results concerning L_p with $p \in (2, \infty)$ we refer to [20], the weak convergence is considered in [10] and [19] and by other authors. In particular, if S is the Brownian motion or the geometric Brownian motion, S. Geiss and Hujo investigated in [9] the relation between the Malliavin fractional smoothness of $f(S_1)$ and the L_2 -rate of the discretization error (2).

It is natural to extend these results to Lévy processes. A first step was done by M. Brodén and P. Tankov [5] (see Remark 3). The aim of this paper is to develop results of [9] into the following directions:

(a) The Brownian motion and the geometric Brownian motion are generalized to Lévy processes $(X_t)_{t \in [0,1]}$ that are L_2 -martingales and their Doléans-Dade exponentials $S = \mathcal{E}(X)$,

$$S_t = 1 + \int_{(0,t]} S_{u-} dX_u,$$

respectively. For certain stochastic integrals (see Section 2.4 below)

$$F = \int_{(0,1]} \varphi_{s-} dX_s \quad (3)$$

and for $Y \in \{X, \mathcal{E}(X)\}$ we study the connection of the Malliavin fractional smoothness of F (introduced by the real interpolation method) and the behavior of

$$a_Y^{\text{opt}}(F; (t_k)_{k=0}^N) = \inf \left\| F - \sum_{k=1}^N v_{k-1}(Y_{t_k} - Y_{t_{k-1}}) \right\|_{L_2}, \quad (4)$$

where the infimum is taken over $\mathcal{F}_{t_{k-1}}$ -measurable v_{k-1} such that $\mathbb{E}v_{k-1}^2(Y_{t_k} - Y_{t_{k-1}})^2 < \infty$ and where $0 = t_0 < \dots < t_N = 1$ is a deterministic time-net.

(b) In contrast to [9], where the reduction of the stochastic approximation problem to a deterministic one is based on Itô's formula and was done in [8, 7], we prove an analogous reduction in Theorems 3 and 4 by techniques based on the Itô chaos decomposition.

(c) One more principal difference to [9] is the fact that Lévy processes in general do not satisfy the representation property and therefore there are $F \in L_2$ that cannot be approximated by sums of the form $\sum_{k=1}^N v_{k-1}(Y_{t_k} - Y_{t_{k-1}})$ in L_2 . As a consequence we have to use the (orthogonal) Galtchouk-Kunita-Watanabe projection that projects L_2 onto the subspace $I(X)$ of stochastic integrals $\int_{(0,1]} \lambda_s dX_s$ with $\mathbb{E} \int_0^1 |\lambda_s|^2 ds < \infty$ that can be defined in our setting as the L_2 -closure of

$$\left\{ \sum_{k=1}^N v_{a_{k-1}}(X_{a_k} - X_{a_{k-1}}) : v_{a_{k-1}} \in L_2(\mathcal{F}_{a_{k-1}}), \begin{array}{l} 0 = a_0 < \dots < a_N = 1 \\ N = 1, 2, \dots \end{array} \right\} \quad (5)$$

to deal with our approximation problem.

The paper is organized as follows. In Section 2 we recall some facts about Lévy processes and Besov spaces. The Besov spaces are used to describe Malliavin fractional smoothness. In Section 3 we investigate the discrete time approximation. The basic statement is Theorem 3 that reduces the stochastic approximation problem to a deterministic one in case of the Riemann-approximation (2) (which we call simple approximation in the sequel). The difference between the simple and optimal approximation (4) is shown in Theorem 4 to be sufficiently small. Theorem 5 provides a lower bound for the optimal L_2 -approximation. Finally, Theorems 6 and 7 give the connection to the Besov spaces defined by real interpolation. We conclude with Section 4 where we use the example $f(x) = \mathbb{1}_{(K, \infty)}(x)$ to demonstrate how the fractional smoothness depends on the underlying Lévy process.

2 Preliminaries

2.1 Notation

Throughout this paper we will use for $A, B, C \geq 0$ and $c \geq 1$ the notation

$$\begin{aligned} A \sim_c B & \quad \text{for} \quad \frac{1}{c}B \leq A \leq cB, \\ A = B \pm C & \quad \text{for} \quad B - C \leq A \leq B + C. \end{aligned} \quad (6)$$

The phrase *càdlàg* stands for a path which is right-continuous and has left limits. Given $q \in [1, \infty]$, the sequence space ℓ_q consists of all $\alpha = (\alpha_N)_{N \geq 1} \subseteq \mathbb{R}$ such that $\|\alpha\|_{\ell_q} := (\sum_{N=1}^{\infty} |\alpha_N|^q)^{1/q} < \infty$ for $q < \infty$ and $\|\alpha\|_{\ell_\infty} := \sup_{N \geq 1} |\alpha_N| < \infty$, respectively.

2.2 Lévy processes

We follow the setting and presentation of [17, Section 1.1] and assume a *square integrable mean zero Lévy process* $X = (X_t)_{t \in [0,1]}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]})$ satisfying the usual assumptions, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is complete where the filtration $(\mathcal{F}_t)_{t \in [0,1]}$ is the augmented natural filtration of X and therefore right-continuous and $\mathcal{F} := \mathcal{F}_1$ is assumed without loss of generality. The Lévy measure ν with $\nu(\{0\}) = 0$ satisfies

$$\int_{\mathbb{R}} x^2 \nu(dx) < \infty$$

by the square integrability of X (see [16, Theorem 25.3]). Let N be the associated Poisson random measure and $d\tilde{N}(t, x) = dN(t, x) - dt d\nu(x)$ be the compensated Poisson random measure. The Lévy-Itô decomposition (see [16, Theorem 19.2]) can be written under our assumptions as

$$X_t = \sigma W_t + \int_{(0,t] \times \mathbb{R} \setminus \{0\}} x \tilde{N}(ds, dx).$$

We introduce the finite measures μ on $\mathcal{B}(\mathbb{R})$ and \mathfrak{m} on $\mathcal{B}([0,1] \times \mathbb{R})$ by

$$\begin{aligned} \mu(dx) &:= \sigma^2 \delta_0(dx) + x^2 \nu(dx), \\ \mathfrak{m}(dt, dx) &:= dt \mu(dx), \end{aligned}$$

where we agree about $\mu(\mathbb{R}) > 0$ to avoid pathologies. For $B \in \mathcal{B}((0,1] \times \mathbb{R})$ we define the random measure

$$M(B) := \sigma \int_{\{t \in (0,1] : (t,0) \in B\}} dW_t + \int_{B \cap ((0,1] \times (\mathbb{R} \setminus \{0\}))} x \tilde{N}(dt, dx)$$

and let

$$L_2^n := L_2([0,1] \times \mathbb{R})^n, \mathcal{B}([0,1] \times \mathbb{R})^n, \mathfrak{m}^{\otimes n} \quad \text{for } n \geq 1.$$

By [12, Theorem 2] there is the chaos decomposition

$$L_2 := L_2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{n=0}^{\infty} I_n(L_2^n),$$

where $I_0(L_2^0)$ is the space of the a.s. constant random variables and $I_n(L_2^n) := \{I_n(f_n) : f_n \in L_2^n\}$ for $n = 1, 2, \dots$ and $I_n(f_n)$ denotes the multiple integral w.r.t. the random measure M . For properties of the multiple integral see [12, Theorem 1]. Especially, $\|I_n(f_n)\|_{L_2}^2 = n! \|f_n\|_{L_2^n}^2$ and

$$\|F\|_{L_2}^2 = \sum_{n=0}^{\infty} n! \|\tilde{f}_n\|_{L_2^n}^2$$

with \tilde{f}_n being the symmetrization of f_n , i.e.

$$\tilde{f}_n(z_1, \dots, z_n) = \frac{1}{n!} \sum f_n(z_{\pi(1)}, \dots, z_{\pi(n)})$$

for all $z_i = (t_i, x_i) \in [0, 1] \times \mathbb{R}$, where the sum is taken over all permutations π of $\{1, \dots, n\}$. For $F \in L_2$ the L_2 -representation

$$F = \sum_{n=0}^{\infty} I_n(\tilde{f}_n),$$

with $I_0(f_0) = \mathbb{E}F$ a.s. is unique (note that $I_n(f_n) = I_n(\tilde{f}_n)$ a.s.).

2.3 Doléans-Dade stochastic exponential

Definition 1 For $0 \leq a \leq t \leq 1$ we let

$$S_t^a := 1 + \sum_{n=1}^{\infty} \frac{I_n(\mathbf{1}_{(a,t]}^{\otimes n})}{n!},$$

where we can assume that all paths of $(S_t^a)_{t \in [a, 1]}$ are càdlàg for any fixed $a \in [0, 1]$. In particular, we let $S = (S_t)_{t \in [0, 1]} := (S_t^0)_{t \in [0, 1]}$.

The following lemma is standard and we omit its proof.

Lemma 1 For $0 \leq a \leq t \leq 1$ one has that

- (i) $S_t^a = 1 + \int_{(a,t]} S_{u-}^a dX_u$ a.s.,
- (ii) $S_t = S_t^a S_a$ a.s.,
- (iii) S_t^a is independent from \mathcal{F}_a and $\mathbb{E}(S_t^a)^2 = e^{\mu(\mathbb{R})(t-a)}$.

2.4 The space \mathbb{M} of the random variables to approximate

We will approximate random variables $F \in L_2$ from a space \mathbb{M} introduced below. By (10) we see that this approach is analogous to the Brownian motion case considered in [7] and [9], where the representation $F = \mathbb{E}F + \int_{(0,1]} \varphi_s dB_s$ was used together with the regularity assumption that $(\varphi_s)_{s \in [0,1]}$ is a martingale or close to a martingale. This regularity assumption is relevant for the approach in this paper as well.

Definition 2 The closed subspace $\mathbb{M} \subseteq L_2$ consists of all mean zero $F \in L_2$ such that there exists a representation

$$F = \sum_{n=1}^{\infty} I_n(f_n)$$

with symmetric f_n such that there are $h_0 \in \mathbb{R}$ and symmetric $h_n \in L_2(\mu^{\otimes n})$ for $n \geq 1$ with

$$f_n((t_1, x_1), \dots, (t_n, x_n)) = h_{n-1}(x_1, \dots, x_{n-1}) \quad \text{for } 0 < t_1 < \dots < t_n < 1.$$

The orthogonal projection onto \mathbb{M} is denoted by $\Pi : L_2 \rightarrow \mathbb{M} \subseteq L_2$.

Let us summarize some facts about the space \mathbb{M} :

(a) **Representation of Π .** For

$$G = \sum_{n=0}^{\infty} I_n(\alpha_n) \in L_2$$

with symmetric $\alpha_n \in L_2^n$ one computes the functions h_n of the projection $F = \Pi(G)$ by

$$\begin{aligned} & h_{n-1}(x_1, \dots, x_{n-1}) \\ &= n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} \int_{\mathbb{R}} \alpha_n((t_1, x_1), \dots, (t_{n-1}, x_{n-1}), (t_n, x_n)) \\ & \quad \times \frac{\mu(dx_n)}{\mu(\mathbb{R})} dt_1 \dots dt_n \quad \text{for } n \geq 1. \end{aligned} \tag{7}$$

(b) **Integral representation of the elements of \mathbb{M} .** Given $F \in \mathbb{M}$ with a representation like in Definition 2 (the functions h_n are unique as elements of $L_2(\mu^{\otimes n})$), we define the martingale $\varphi = (\varphi_t)_{t \in [0,1]}$ by the L_2 -sum

$$\varphi_t := h_0 + \sum_{n=1}^{\infty} (n+1) I_n \left(h_n \mathbf{1}_{(0,t]}^{\otimes n} \right), \tag{8}$$

which we will assume to be path-wise *càdlàg*. It follows that

$$\begin{aligned} \|\varphi_t\|_{L_2}^2 &= h_0^2 + \sum_{n=1}^{\infty} (n+1)^2 n! t^n \|h_n\|_{L_2(\mu^{\otimes n})}^2 \\ &= h_0^2 + \frac{1}{\mu(\mathbb{R})} \sum_{n=1}^{\infty} (n+1)^2 n! t^n \|f_{n+1}\|_{L_2^{n+1}}^2 \\ &= h_0^2 + \frac{1}{\mu(\mathbb{R})} \sum_{n=1}^{\infty} t^n (n+1) \|I_{n+1}(f_{n+1})\|_{L_2}^2 \end{aligned}$$

so that

$$\mu(\mathbb{R}) \sup_{t \in [0,1]} \|\varphi_t\|_{L_2}^2 + \|F\|_{L_2}^2 = \sum_{n=0}^{\infty} (n+1) \|I_n(f_n)\|_{L_2}^2. \quad (9)$$

Moreover, for $t \in [0, 1]$ we get that, a.s.,

$$F_t := \mathbb{E}(F | \mathcal{F}_t) = \int_{(0,t]} \varphi_{s-} dX_s. \quad (10)$$

In other words, (10) characterizes the elements from \mathbb{M} if φ is defined by symmetric $(h_n)_{n=0}^{\infty}$ as in (8) with $\sum_{n=0}^{\infty} (n+1)! \|h_n\|_{L_2(\mu^{\otimes n})}^2 < \infty$.

(c) **Examples for elements of \mathbb{M} .**

(c1) One class of examples is taken from Lemma 4 below: Let $\Pi_X : L_2 \rightarrow I(X) \subseteq L_2$ be the orthogonal projection onto $I(X)$ defined in (5) and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function with $f(X_1) \in L_2$, then

$$\Pi_X(f(X_1)) = \Pi(f(X_1)).$$

This means the elements of \mathbb{M} occur naturally when applying the Galtchouk-Kunita-Watanabe projection. It should be noted, that in the case that $\sigma = 0$ and $\nu = \alpha \delta_{x_0}$ with $\alpha > 0$ and $x_0 \in \mathbb{R} \setminus \{0\}$ we have a chaos decomposition of the form $f(X_1) = \mathbb{E}f(X_1) + \sum_{n=1}^{\infty} \beta_n I_n(\mathbf{1}_{(0,1]}^{\otimes n})$ with $\beta_n \in \mathbb{R}$, so that already $f(X_1) \in \mathbb{M}$.

(c2) There are also examples of $F \in \mathbb{M}$ that cannot be obtained as projections $\Pi_X(f(X_1)) = \Pi(f(X_1))$ as in (c1). To construct such an example we will decompose the Lévy process into a sum of two independent Lévy processes and use only one of them in the integrand:

Example 1 Assume that $X_t = N_t^1 - N_t^2$, where $N^i = (N_t^i)_{t \in [0,1]}$ are independent Poisson processes with intensity $\lambda > 0$. Then

$$F := \int_{(0,1]} (N_{t-}^1 - \lambda t) dX_t \in \mathbb{M}$$

and there is no Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(X_1) \in L_2$ and

$$\Pi(f(X_1)) = F \quad a.s.$$

Proof For the Lévy measure of X we get $\nu = \lambda(\delta_{\{-1\}} + \delta_{\{1\}})$. Because X_1 takes only values in the set of integers we can apply Lemma 5 to a *bounded* Borel function f without assuming smoothness to get that

$$\varphi_1 = \frac{1}{2} [f(X_1 + 1) - f(X_1 - 1)] =: \psi_f(X_1)$$

and

$$\Pi(f(X_1)) = \int_{(0,1]} \varphi_{t-} dX_t \quad a.s. \quad \text{with} \quad \varphi_t := \mathbb{E}(\psi_f(X_1) | \mathcal{F}_t).$$

For a fixed $T \in (0, 1)$ and a bounded f as above we have

$$\begin{aligned} \|F - \Pi(f(X_1))\|_{L_2}^2 &= \mu(\mathbb{R}) \int_{(0,1]} \mathbb{E} [(N_{t-}^1 - \lambda t) - \varphi_{t-}]^2 dt \\ &= \mu(\mathbb{R}) \int_{(0,1]} \mathbb{E} [(N_t^1 - \lambda t) - \varphi_t]^2 dt \\ &\geq \mu(\mathbb{R})(1 - T) \mathbb{E} [(N_T^1 - \lambda T) - \varphi_T]^2. \end{aligned}$$

If we can show that

$$\inf_{f \text{ bounded}} \mathbb{E} [(N_T^1 - \lambda T) - \mathbb{E}(\psi_f(X_1) | \mathcal{F}_T)]^2 > 0,$$

then we get the assertion of our example. Because $\mathbb{E}(\psi_f(X_1) | \mathcal{F}_T)$ is a functional of $X_T = N_T^1 - N_T^2$ it is sufficient to check that

$$\mathbb{E} [N_T^1 - \mathbb{E}(N_T^1 | \sigma(N_T^1 - N_T^2))]^2 > 0$$

which follows by the independence of the Poisson processes N^1 and N^2 .

2.5 Real interpolation

Now we recall some facts about the real interpolation method.

Definition 3 For Banach spaces $X_1 \subseteq X_0$, where X_1 is continuously embedded into X_0 , we define for $u > 0$ the *K-functional*

$$K(u, x; X_0, X_1) := \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + u\|x_1\|_{X_1}\}.$$

For $\theta \in (0, 1)$ and $q \in [1, \infty]$ the real interpolation space $(X_0, X_1)_{\theta, q}$ consists of all elements $x \in X_0$ such that $\|x\|_{(X_0, X_1)_{\theta, q}} < \infty$ where

$$\|x\|_{(X_0, X_1)_{\theta, q}} := \begin{cases} \left[\int_0^\infty [u^{-\theta} K(u, x; X_0, X_1)]^q \frac{du}{u} \right]^{\frac{1}{q}}, & q \in [1, \infty) \\ \sup_{u>0} u^{-\theta} K(u, x; X_0, X_1), & q = \infty. \end{cases}$$

The spaces $(X_0, X_1)_{\theta, q}$ equipped with $\|\cdot\|_{(X_0, X_1)_{\theta, q}}$ become Banach spaces and form a lexicographical scale, i.e. for any $0 < \theta_1 < \theta_2 < 1$ and $q_1, q_2 \in [1, \infty]$ it holds that

$$X_0 \supseteq (X_0, X_1)_{\theta_1, q_1} \supseteq (X_0, X_1)_{\theta_2, q_2} \supseteq (X_0, X_1)_{\theta_2, \min\{q_1, q_2\}} \supseteq X_1.$$

For more information the reader is referred to [3, 4].

2.6 Besov spaces obtained by real interpolation

We recall the construction of Sobolev spaces based on the chaos expansion and the construction of Besov spaces (or spaces of random variables of fractional smoothness) based on real interpolation. We introduce two variants of the Besov spaces, a direct one in Definition 4 and an abstract one in Definition 5. The purpose of the abstract variant is twofold: firstly, it is needed to transfer the results from [9] to our setting in the proofs of Theorems 6 and 7, and secondly, the abstract variant indicates a way for further generalizations.

Definition 4 Let $\mathbb{D}_{1,2}$ be the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2$ such that

$$\|F\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1) \|I_n(f_n)\|_{L_2}^2 < \infty.$$

Moreover,

$$\mathbb{B}_{2,q}^{\theta} := \begin{cases} (L_2, \mathbb{D}_{1,2})_{\theta, q} : \theta \in (0, 1), q \in [1, \infty] \\ \mathbb{D}_{1,2} : \theta = 1, q = 2 \end{cases}.$$

Definition 5 For a sequence of Banach spaces $E = (E_n)_{n=0}^{\infty}$ with $E_n \neq \{0\}$ we let $\ell_2(E)$ and $d_{1,2}(E)$ be the Banach spaces of all $a = (a_n)_{n=0}^{\infty} \in E$ such that

$$\|a\|_{\ell_2(E)} := \left(\sum_{n=0}^{\infty} \|a_n\|_{E_n}^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|a\|_{d_{1,2}(E)} := \left(\sum_{n=0}^{\infty} (n+1) \|a_n\|_{E_n}^2 \right)^{\frac{1}{2}},$$

respectively, are finite. Moreover, for $\theta \in (0, 1)$ and $q \in [1, \infty]$ we let

$$\mathbb{B}_{2,q}^{\theta}(E) := \begin{cases} (\ell_2(E), d_{1,2}(E))_{\theta, q} : \theta \in (0, 1), q \in [1, \infty] \\ d_{1,2}(E) : \theta = 1, q = 2 \end{cases}.$$

It can be shown that (cf. [9, Remark A.1])

$$\|a\|_{\mathbb{B}_{2,2}^{\theta}(E)}^2 \sim_{c_{\theta}^2} \sum_{n=0}^{\infty} (n+1)^{\theta} \|a_n\|_{E_n}^2.$$

To describe the interpolation spaces $\mathbb{B}_{2,q}^\theta(E)$ we use two types of functions. The first one is a generating function for $(\|a_n\|_{E_n}^2)_{n=0}^\infty$, i.e. for $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ we let

$$T_a(t) := \sum_{n=0}^{\infty} \|a_n\|_{E_n}^2 t^n.$$

The second function will be used to describe our stochastic approximation in a deterministic way: For $a \in \ell_2(E)$ and a deterministic time-net $\tau = (t_k)_{k=0}^N$ with $0 = t_0 \leq \dots \leq t_N = 1$ we let

$$A(a, \tau) := \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t)(T_a)''(t) dt \right)^{\frac{1}{2}}.$$

For the formulation of the next two theorems which will connect approximation properties with fractional smoothness special time nets are needed. Given $\theta \in (0, 1]$ and $N \geq 1$, we let τ_N^θ be the time-net

$$t_k^{N,\theta} := 1 - \left(1 - \frac{k}{N}\right)^{\frac{1}{\theta}} \quad \text{for } k = 0, 1, \dots, N \quad (11)$$

for which one has (see [10, relation (4)])

$$\frac{|t_k^{N,\theta} - t|}{(1-t)^{1-\theta}} \leq \frac{|t_k^{N,\theta} - t_{k-1}^{N,\theta}|}{(1-t_{k-1}^{N,\theta})^{1-\theta}} \leq \frac{1}{\theta N} \quad \text{for } k = 1, \dots, N \quad (12)$$

and $t \in [t_{k-1}^{N,\theta}, t_k^{N,\theta}]$. For $\theta = 1$ we obtain equidistant time-nets. The following two theorems are taken from [9]. For the convenience of the reader we comment about the proofs in Remark 1 below.

Theorem 1 ([9]) *For $\theta \in (0, 1)$, $q \in [1, \infty]$ and $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ one has*

$$\|a\|_{\mathbb{B}_{2,q}^\theta(E)} \sim_c \|a\|_{\ell_2(E)} + \left\| \left(N^{\frac{\theta}{2} - \frac{1}{q}} A(a, \tau_N^1) \right)_{N=1}^\infty \right\|_{\ell_q}$$

where $c \in [1, \infty)$ depends at most on (θ, q) and the expressions may be infinite.

Theorem 2 ([9]) *For $\theta \in (0, 1]$ and $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ the following assertions are equivalent:*

- (i) $a \in \mathbb{B}_{2,2}^\theta(E)$.
- (ii) $\int_0^1 (1-t)^{1-\theta} T_a''(t) dt < \infty$.
- (iii) There exists a constant $c > 0$ such that

$$A(a, \tau_N^\theta) \leq \frac{c}{\sqrt{N}} \quad \text{for } N = 1, 2, \dots$$

Remark 1 We fix $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ and (θ, q) according to Theorems 1 and 2. Then we let $\beta_n := \|a_n\|_{E_n}$ and define $f = \sum_{n=0}^\infty \beta_n h_n \in L_2(\mathbb{R}, \gamma)$, where γ is the standard Gaussian measure and $(h_n)_{n=0}^\infty$ the orthonormal basis of Hermite polynomials. As before, let

$$A(\beta, \tau) := \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t)(T_\beta)''(t) dt \right)^{\frac{1}{2}} \quad \text{with} \quad T_\beta(t) := \sum_{n=0}^\infty \beta_n^2 t^n.$$

Omitting the notation (E) in the case $E = (\mathbb{R}, \mathbb{R}, \dots)$, we have $\|a\|_{\ell_2(E)} = \|\beta\|_{\ell_2}$ and $\|a\|_{d_{1,2}(E)} = \|\beta\|_{d_{1,2}}$. Moreover, [9, Theorem 2.2] gives $\|a\|_{\mathbb{B}_{2,q}^\theta(E)} \sim_{c(\theta,q)} \|\beta\|_{\mathbb{B}_{2,q}^\theta}$ for $\theta \in (0, 1)$ and $q \in [1, \infty]$ because of $T_a = T_\beta$. Hence [9, Lemmas 3.9 and 3.10, Theorem 3.5 (X=W)] imply Theorem 1 of this paper. The equivalence of (i) and (iii) of Theorem 2 follows in the same way by [9, Lemmas 3.9 and 3.10, Theorem 3.2 (X=W)]. Finally, the equivalence of (i) and (ii) of Theorem 2 is a consequence of the proof of [9, Theorem 3.2 (X=W)].

3 Approximation of stochastic integrals

In the sequel we will use

$$\mathcal{T}_N := \{\tau = (t_k)_{k=0}^N : 0 = t_0 < \dots < t_N = 1\} \quad \text{and} \quad \mathcal{T} := \bigcup_{N=1}^\infty \mathcal{T}_N$$

as sets of deterministic time-nets and define $|\tau| := \max_{1 \leq k \leq N} |t_k - t_{k-1}|$. We will consider the following approximations of a random variable $F \in \mathbb{M}$ with respect to the processes X and S :

Definition 6 For $N \geq 1$, $Y \in \{X, S\}$, $F = \int_{(0,1]} \varphi_s dX_s \in \mathbb{M}$, $A = (A_k)_{k=1}^N \subseteq \mathcal{F}$ and $\tau \in \mathcal{T}_N$ we let

- (i) $a_X^{\text{sim}}(F; \tau) := \left\| F - \sum_{k=1}^N \varphi_{t_{k-1}} (X_{t_k} - X_{t_{k-1}}) \right\|_{L_2}$,
- (ii) $a_S^{\text{sim}}(F; \tau, A) := \left\| F - \sum_{k=1}^N \varphi_{t_{k-1}} \mathbb{1}_{A_k} (S_{t_k}^{t_{k-1}} - 1) \right\|_{L_2}$,
- (iii) $a_Y^{\text{opt}}(F; \tau) := \inf \left\| F - \sum_{k=1}^N v_{k-1} (Y_{t_k} - Y_{t_{k-1}}) \right\|_{L_2}$, where the infimum is taken over all $\mathcal{F}_{t_{k-1}}$ -measurable $v_{k-1} : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{E}|v_{k-1}(Y_{t_k} - Y_{t_{k-1}})|^2 < \infty$.

Remark 2 (i) The definition of a_S^{sim} takes into account the additional sets $(A_k)_{k=1}^N$ to avoid problems with the case that S vanishes. These extra sets A in $a_S^{\text{sim}}(F; \tau, A)$ play different roles in Theorem 3, Theorem 4, and in Theorems 5, 6 and 7. To recover a more standard form of a_S^{sim} assume that $(S_t)_{t \in [0,1]}$ and $(S_{t-})_{t \in [0,1]}$ are positive so that we can write

$$F = \int_{(0,1]} \psi_u (S_u dX_u) \quad \text{with} \quad \psi_u := \frac{\varphi_u}{S_u}$$

and obtain that

$$\begin{aligned} F - \sum_{k=1}^N \varphi_{t_{k-1}} (S_{t_k}^{t_{k-1}} - 1) &= F - \sum_{k=1}^N \psi_{t_{k-1}} S_{t_{k-1}} (S_{t_k}^{t_{k-1}} - 1) \\ &= F - \sum_{k=1}^N \psi_{t_{k-1}} (S_{t_k} - S_{t_{k-1}}) \end{aligned}$$

which is what one expects.

(ii) In the sequel the crucial assumption will be

$$\Omega = \{S_t \neq 0\} \quad \text{for all } t \in [0, 1].$$

This can be achieved by the condition $\nu((-\infty, -1]) = 0$ which implies the almost sure positivity of S and we can adjust S on a set of measure zero; see [13, Theorem I.4.61] and [16, Theorem 19.2].

(iii) Because of the martingale property of $(\varphi_t)_{t \in [0,1]}$ it is easy to check that

$$a_X^{\text{opt}}(F; \tau) = \left\| F - \sum_{k=1}^N \varphi_{t_{k-1}} (X_{t_k} - X_{t_{k-1}}) \right\|_{L_2}$$

so that $a_X^{\text{sim}} = a_X^{\text{opt}}$.

The theorem below gives a description of the simple approximation by a function $H_Y(t)$ that describes, in some sense, the curvature of $F \in \mathbb{M}$ with respect to Y .

Theorem 3 *Let $F \in \mathbb{M}$,*

$$H_Y^2(t) := \mu(\mathbb{R}) \sum_{n=1}^{\infty} n n! t^{n-1} \|A_n^Y\|_{L_2(\mu^{\otimes n})}^2$$

with

$$\begin{aligned} A_n^Y(x_1, \dots, x_n) &:= \begin{cases} (n+1)h_n(x_1, \dots, x_n) & : Y = X \\ (n+1)h_n(x_1, \dots, x_n) - h_{n-1}(x_1, \dots, x_{n-1}) & : Y = S \end{cases} \end{aligned}$$

Then, for $\tau \in \mathcal{T}$, one has

$$\begin{aligned} a_X^{\text{opt}}(F; \tau) &= \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) H_X^2(t) dt \right)^{\frac{1}{2}}, \\ a_S^{\text{sim}}(F; \tau, \Omega^N) &\sim_c \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) H_S^2(t) dt \right)^{\frac{1}{2}}, \end{aligned}$$

where in the last equivalence $|\tau| < 1/\mu(\mathbb{R})$ and $c := (1 - \sqrt{\mu(\mathbb{R})|\tau|})^{-1}$ and $\Omega^N = (\Omega, \dots, \Omega)$.

Proof Case $Y = X$: We get that

$$\begin{aligned} \mathbb{E}|\varphi_t - \varphi_{t_{k-1}}|^2 &= \sum_{n=1}^{\infty} (t^n - t_{k-1}^n)(n+1)^2 n! \|h_n\|_{L_2(\mu^{\otimes n})}^2 \\ &= \sum_{n=1}^{\infty} (n+1)^2 n n! \int_{t_{k-1}}^t u^{n-1} du \|h_n\|_{L_2(\mu^{\otimes n})}^2 \\ &= \frac{1}{\mu(\mathbb{R})} \int_{t_{k-1}}^t H_X^2(u) du \end{aligned}$$

which implies for $a_X^{\text{sim}}(F; \tau) = a_X^{\text{opt}}(F; \tau) =: a_X(F; \tau)$ that

$$\begin{aligned} |a_X(F; \tau)|^2 &= \mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E}|\varphi_t - \varphi_{t_{k-1}}|^2 dt \\ &= \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - u) H_X^2(u) du. \end{aligned}$$

Case $Y = S$: Here we get that

$$\begin{aligned} &a_S^{\text{sim}}(F; \tau, \Omega^N) \\ &= \left(\mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E} \left| \varphi_t - \varphi_{t_{k-1}} S_{t_{k-1}}^{t_k-1} \right|^2 dt \right)^{\frac{1}{2}} \\ &= \left(\mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E} \left[\left| \varphi_t - \varphi_{t_{k-1}} - \int_{(t_{k-1}, t]} \varphi_u dX_u \right. \right. \right. \\ &\quad \left. \left. \left. + \left[\int_{(t_{k-1}, t]} \varphi_u dX_u - \varphi_{t_{k-1}} (S_{t_{k-1}}^{t_k-1} - 1) \right] \right|^2 dt \right) \right]^{\frac{1}{2}} \\ &= \left(\mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E} \left[\left| \varphi_t - \varphi_{t_{k-1}} - \int_{(t_{k-1}, t]} \varphi_u dX_u \right|^2 dt \right) \right]^{\frac{1}{2}} \\ &\quad \pm \left(\mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E} \left[\left| \int_{(t_{k-1}, t]} \varphi_u dX_u - \varphi_{t_{k-1}} (S_{t_{k-1}}^{t_k-1} - 1) \right|^2 dt \right) \right]^{\frac{1}{2}} \end{aligned}$$

where the notation \pm was introduced in (6) and

$$\begin{aligned} &\left(\mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E} \left[\left| \int_{(t_{k-1}, t]} \varphi_u dX_u - \varphi_{t_{k-1}} (S_{t_{k-1}}^{t_k-1} - 1) \right|^2 dt \right) \right]^{\frac{1}{2}} \\ &\leq \sqrt{|\tau|} \left(\mu(\mathbb{R}) \sum_{k=1}^N \mathbb{E} \left[\left| \int_{(t_{k-1}, t_k]} \varphi_u dX_u - \varphi_{t_{k-1}} (S_{t_k}^{t_{k-1}} - 1) \right|^2 \right) \right]^{\frac{1}{2}} \\ &= \sqrt{|\tau| \mu(\mathbb{R})} a_S^{\text{sim}}(F; \tau, \Omega^N). \end{aligned}$$

Here we used $S_{t-}^{t_{k-1}} = S_t^{t_{k-1}}$ a.s. for $t \in (t_{k-1}, t_k]$ and the martingale property of $\int_{(t_{k-1}, t]} \varphi_u - dX_u - \varphi_{t_{k-1}}(S_t^{t_{k-1}} - 1)$. Finally,

$$\begin{aligned} & \left(\mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E} \left[\varphi_t - \varphi_{t_{k-1}} - \int_{(t_{k-1}, t]} \varphi_u - dX_u \right]^2 dt \right)^{\frac{1}{2}} \\ &= \left(\mu(\mathbb{R}) \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \mathbb{E} [(\varphi_t - \varphi_{t_{k-1}}) - (F_t - F_{t_{k-1}})]^2 dt \right)^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t H_S^2(u) du dt \right)^{\frac{1}{2}}. \end{aligned}$$

□

The next theorem states that the simple and optimal approximation are equivalent whenever $A_k := \{S_{t_{k-1}} \neq 0\}$ is taken.

Theorem 4 For $F \in \mathbb{M}$ and $\tau \in \mathcal{T}$ one has that

$$|a_S^{\text{sim}}(F; \tau, A) - a_S^{\text{opt}}(F; \tau)| \leq c[|\tau| \|F\|_{L_2} + \sqrt{|\tau|} a_X^{\text{opt}}(F; \tau)]$$

where $c > 0$ depends on μ only and $A_k := \{S_{t_{k-1}} \neq 0\}$.

Proof (a) In the first step we determine an optimal sequence of $(v_k)_{k=1}^{N-1}$. For $0 \leq a < b \leq 1$ we get from Lemma 1 that

$$\begin{aligned} & \inf \left\{ \left\| v(S_b - S_a) - \int_{(a,b]} \varphi_u - dX_u \right\|_{L_2} : \begin{array}{l} v \text{ is } \mathcal{F}_a\text{-measurable} \\ \mathbb{E}|v(S_b - S_a)|^2 < \infty \end{array} \right\} \\ &= \inf \left\{ \left\| v S_a(S_b^a - 1) - \int_{(a,b]} \varphi_u - dX_u \right\|_{L_2} : \begin{array}{l} v \text{ is } \mathcal{F}_a\text{-measurable} \\ \mathbb{E}|v S_a|^2 < \infty \end{array} \right\} \\ &= \inf \left\{ \left\| \bar{v} \mathbb{1}_{\{S_a \neq 0\}}(S_b^a - 1) - \int_{(a,b]} \varphi_u - dX_u \right\|_{L_2} : \begin{array}{l} \bar{v} \text{ is } \mathcal{F}_a\text{-measurable} \\ \mathbb{E}|\bar{v}|^2 < \infty \end{array} \right\}. \end{aligned}$$

The infimum is obtained with

$$\bar{v} =$$

$$\frac{\mathbb{E} \left(\int_a^b \varphi_t - S_{t-}^a dt | \mathcal{F}_a \right)}{\mathbb{E} \left(\int_a^b (S_{t-}^a)^2 dt | \mathcal{F}_a \right)} = \frac{\mathbb{E} \left(\int_a^b \varphi_t S_t^a dt | \mathcal{F}_a \right)}{\int_a^b \mathbb{E}(S_t^a)^2 dt} =: \frac{\mathbb{E} \left(\int_a^b \varphi_t S_t^a dt | \mathcal{F}_a \right)}{\kappa(a, b)}$$

and

$$v := \begin{cases} \frac{1}{S_a \kappa(a, b)} \mathbb{E} \left(\int_a^b \varphi_t S_t^a dt | \mathcal{F}_a \right) & : S_a \neq 0 \\ 0 & : S_a = 0 \end{cases}$$

where we used that

$$\varphi_{t-} = \varphi_t \quad \text{a.s. and} \quad S_{t-}^a = S_t^a \quad \text{a.s. on } (a, b]. \quad (13)$$

(b) Now it holds that

$$\begin{aligned}
 & |a_S^{\text{sim}}(F; \tau, A) - a_S^{\text{opt}}(F; \tau)| \\
 &= \left\| \left\| F - \mathbb{E}F - \sum_{k=1}^N \varphi_{t_{k-1}} \mathbf{1}_{A_k} (S_{t_k}^{t_{k-1}} - 1) \right\|_{L_2} \right. \\
 &\quad \left. - \left\| F - \mathbb{E}F - \sum_{k=1}^N v_{k-1} (S_{t_k} - S_{t_{k-1}}) \right\|_{L_2} \right\| \\
 &\leq \left\| \sum_{k=1}^N [\varphi_{t_{k-1}} - v_{k-1} S_{t_{k-1}}] (S_{t_k}^{t_{k-1}} - 1) \mathbf{1}_{A_k} \right\|_{L_2} \\
 &= \left(\sum_{k=1}^N \left\| [\varphi_{t_{k-1}} - v_{k-1} S_{t_{k-1}}] \mathbf{1}_{A_k} \right\|_{L_2}^2 [e^{\mu(\mathbb{R})(t_k - t_{k-1})} - 1] \right)^{\frac{1}{2}}.
 \end{aligned}$$

Moreover (using again (13)) we have

$$\begin{aligned}
 & \left\| [\varphi_{t_{k-1}} - v_{k-1} S_{t_{k-1}}] \mathbf{1}_{A_k} \right\|_{L_2} \\
 &\leq \left\| \varphi_{t_{k-1}} \left(1 - \frac{t_k - t_{k-1}}{\kappa(t_{k-1}, t_k)} \right) \mathbf{1}_{A_k} \right\|_{L_2} \\
 &\quad + \left\| \frac{\mathbf{1}_{A_k}}{\kappa(t_{k-1}, t_k)} \mathbb{E} \left(\int_{t_{k-1}}^{t_k} (\varphi_t - \varphi_{t_{k-1}}) (S_t^{t_{k-1}} - 1) dt \middle| \mathcal{F}_{t_{k-1}} \right) \right\|_{L_2}.
 \end{aligned}$$

The first term on the right-hand side can be bounded from above by $\mu(\mathbb{R})(t_k - t_{k-1}) \|\varphi_{t_{k-1}} \mathbf{1}_{A_k}\|_{L_2}$. For the second term we let $a = t_{k-1} < t_k = b$ and $\lambda_t = \mathbf{1}_{A_k}(\varphi_t - \varphi_{t_{k-1}})$ and obtain

$$\begin{aligned}
 & \mathbb{E} \left(\int_a^b \lambda_t (S_t^a - 1) dt \middle| \mathcal{F}_a \right) \\
 &\leq \left(\mathbb{E} \left(\int_a^b |\lambda_t|^2 dt \middle| \mathcal{F}_a \right) \right)^{\frac{1}{2}} \left(\mathbb{E} \left(\int_a^b (S_t^a - 1)^2 dt \middle| \mathcal{F}_a \right) \right)^{\frac{1}{2}} \\
 &= \left(\mathbb{E} \left(\int_a^b |\lambda_t|^2 dt \middle| \mathcal{F}_a \right) \right)^{\frac{1}{2}} \left(\int_a^b \|S_t^a - 1\|_2^2 dt \right)^{\frac{1}{2}} \\
 &\leq \left(\mathbb{E} \left(\int_a^b |\lambda_t|^2 dt \middle| \mathcal{F}_a \right) \right)^{\frac{1}{2}} \sqrt{\frac{\mu(\mathbb{R})}{2}} \kappa(a, b)
 \end{aligned}$$

where the last inequality follows from

$$\int_a^b \|S_t^a - 1\|_2^2 dt = \int_a^b \mu(\mathbb{R}) \kappa(a, t) dt$$

$$\begin{aligned} &\leq \int_a^b \mu(\mathbb{R}) \kappa(a, t) \left(\frac{d}{dt} \kappa(a, t) \right) dt \\ &= \frac{\mu(\mathbb{R})}{2} \kappa(a, b)^2. \end{aligned}$$

Hence

$$\begin{aligned} &\|[\varphi_{t_{k-1}} - v_{k-1} S_{t_{k-1}}] \mathbb{1}_{A_k}\|_{L_2} \\ &\leq \mu(\mathbb{R})(t_k - t_{k-1}) \|\varphi_{t_{k-1}} \mathbb{1}_{A_k}\|_{L_2} \\ &\quad + \sqrt{\frac{\mu(\mathbb{R})}{2}} \left(\int_{t_{k-1}}^{t_k} \|\mathbb{1}_{A_k}(\varphi_t - \varphi_{t_{k-1}})\|_{L_2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Using $e^{\mu(\mathbb{R})(t_k - t_{k-1})} - 1 \leq \mu(\mathbb{R})e^{\mu(\mathbb{R})}(t_k - t_{k-1})$ we conclude with

$$\begin{aligned} &|a_S^{\text{sim}}(F; \tau, A) - a_S^{\text{opt}}(F; \tau)| \\ &\leq \left(\sum_{k=1}^N [\mu(\mathbb{R})(t_k - t_{k-1}) \|\varphi_{t_{k-1}} \mathbb{1}_{A_k}\|_{L_2}]^2 \mu(\mathbb{R}) e^{\mu(\mathbb{R})}(t_k - t_{k-1}) \right)^{\frac{1}{2}} \\ &+ \left(\sum_{k=1}^N \left[\frac{\mu(\mathbb{R})}{2} \int_{t_{k-1}}^{t_k} \|\mathbb{1}_{A_k}(\varphi_t - \varphi_{t_{k-1}})\|_{L_2}^2 dt \right] \mu(\mathbb{R}) e^{\mu(\mathbb{R})}(t_k - t_{k-1}) \right)^{\frac{1}{2}} \\ &\leq |\tau| \mu(\mathbb{R}) e^{\mu(\mathbb{R})/2} \|F\|_{L_2} + \sqrt{|\tau|} \sqrt{\frac{\mu(\mathbb{R})}{2}} e^{\mu(\mathbb{R})/2} a_X^{\text{opt}}(F; \tau). \end{aligned}$$

□

Now we show that $1/\sqrt{N}$ is the lower bound for our approximation if time-nets of cardinality $N + 1$ are used.

Theorem 5 *Let $F \in \mathbb{M}$ and $Y \in \{X, S\}$, where in the case $X = S$ we assume that $\Omega = \{S_t \neq 0\}$ for all $t \in [0, 1]$. Unless there are $a, b \in \mathbb{R}$ such that $F = a + bY_1$ a.s., one has that*

$$\liminf_{N \rightarrow \infty} \sqrt{N} \left[\inf_{\tau_N \in \mathcal{T}_N} a_Y^{\text{opt}}(F; \tau_N) \right] > 0.$$

Proof Case $Y = X$: We have $H_X(t) = 0$ for some $t \in (0, 1)$ if and only if $h_n = 0$ $\mu^{\otimes n}$ a.e. for all $n = 1, 2, \dots$ which implies that $F = I_1(f_1) = I_1(h_0) = h_0 X_1$. This means that our assumption on F implies that $H_X(t) > 0$ for all $t \in (0, 1)$. Consequently, Theorem 3 gives for any fixed $s \in (0, 1)$ that

$$N |a_X^{\text{opt}}(F; \tau_N)|^2 = N \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) H_X^2(t) dt$$

$$\begin{aligned}
 &\geq N \int_s^1 \left[\sum_{k=1}^N (t_k - t) \mathbf{1}_{[t_{k-1}, t_k)}(t) H_X^2(s) \right] dt \\
 &= \frac{1}{2} H_X^2(s) N \sum_{k=1}^N (t_k \vee s - t_{k-1} \vee s)^2 \\
 &\geq \frac{1}{2} H_X^2(s) (1 - s)^2
 \end{aligned}$$

which proves the statement for $Y = X$.

Case $Y = S$: Similarly as in the previous case our assumption on F implies that $H_S(t) > 0$ for all $t \in (0, 1)$. In fact, assuming that $H_S(t) = 0$ for some $t \in (0, 1)$ implies

$$(n+1)h_n(x_1, \dots, x_n) = h_{n-1}(x_1, \dots, x_{n-1}) \quad \mu^{\otimes n}\text{-a.e.}$$

for all $n = 1, 2, \dots$. By induction we derive that

$$h_n = \frac{h_0}{(n+1)!} \quad \mu^{\otimes n}\text{-a.e.} \quad \text{for } n \geq 0$$

so that $f_n = h_0/n!$ $\mu^{\otimes n}$ -a.e. for $n \geq 1$. This would give that $F = h_0(S_1 - 1)$ a.s.

Hence applying Theorem 3 as in the case $Y = X$ implies that there is an $\varepsilon > 0$ such that

$$\sqrt{N} a_S^{\text{sim}}(F; \tau_N, \Omega^N) \geq \varepsilon > 0 \quad \text{for all } \tau_N \in \mathcal{T}_N \text{ with } |\tau_N| \leq \frac{1}{2\mu(\mathbb{R})}.$$

For an arbitrary $N \geq 1$ and $\tau_N \in \mathcal{T}_N$ Theorem 4 gives

$$a_S^{\text{opt}}(F; \tau_N) \geq a_S^{\text{sim}}(F; \tau_N, \Omega^N) - c_{(4)} [|\tau_N| \|F\|_{L_2} + \sqrt{|\tau_N|} a_X^{\text{opt}}(F; \tau_N)].$$

Letting $\tilde{\tau}_N := \tau_N \cup \{k/N : k = 1, \dots, N-1\} \in \bigcup_{k=N}^{2N-1} \mathcal{T}_k$, $N \geq 2\mu(\mathbb{R}) \vee 2$ implies $|\tilde{\tau}_N| \leq 1/N \leq 1/(2\mu(\mathbb{R}))$ and

$$\begin{aligned}
 &\sqrt{N} a_S^{\text{opt}}(F; \tau_N) \\
 &\geq \sqrt{N} a_S^{\text{opt}}(F; \tilde{\tau}_N) \\
 &\geq \sqrt{N} \frac{\varepsilon}{\sqrt{2N}} - c_{(4)} \sqrt{N} \left[|\tilde{\tau}_N| \|F\|_{L_2} + \sqrt{|\tilde{\tau}_N|} a_X^{\text{opt}}(F; \tilde{\tau}_N) \right] \\
 &\geq \frac{\varepsilon}{\sqrt{2}} - c_{(4)} \left[\frac{\|F\|_{L_2}}{\sqrt{N}} + a_X^{\text{opt}}(F; (k/N)_{k=0}^N) \right].
 \end{aligned}$$

The convergence $a_X^{\text{opt}}(F; (k/N)_{k=0}^N) \rightarrow 0$ as $N \rightarrow \infty$ follows from Theorem 3 because of $\int_0^1 (1-t) H_X^2(t) dt < \infty$ which can be seen by considering the trivial time-net $\{0, 1\}$. Consequently,

$$\liminf_{N \rightarrow \infty} \sqrt{N} \left[\inf_{\tau_N \in \mathcal{T}_N} a_S^{\text{opt}}(F; \tau_N) \right] \geq \frac{\varepsilon}{\sqrt{2}}.$$

□

Now we relate the approximation properties to the Besov regularity. We recall that the nets τ_N^θ were introduced in (11) and that for $\theta = 1$ we obtain the equidistant nets.

Theorem 6 For $\theta \in (0, 1)$, $q \in [1, \infty]$, $Y \in \{X, S\}$ and $F \in \mathbb{M}$ the following assertions are equivalent:

- (i) $F \in \mathbb{B}_{2,q}^\theta$.
- (ii) $\left\| (N^{\frac{\theta}{2} - \frac{1}{q}} a_X^{\text{opt}}(F; \tau_N^1))_{N=1}^\infty \right\|_{\ell_q} < \infty$.

If $\Omega = \{S_t \neq 0\}$ for all $t \in [0, 1]$, then (i) and (ii) are equivalent to:

- (iii) $\left\| (N^{\frac{\theta}{2} - \frac{1}{q}} a_S^{\text{opt}}(F; \tau_N^1))_{N=1}^\infty \right\|_{\ell_q} < \infty$.
- (iv) $\left\| (N^{\frac{\theta}{2} - \frac{1}{q}} a_S^{\text{sim}}(F; \tau_N^1, \Omega^N))_{N=1}^\infty \right\|_{\ell_q} < \infty$.

For the proof the following lemma is needed.

Lemma 2 For $F \in \mathbb{M}$ and $t \in [0, 1)$ one has that

$$|H_S(t) - H_X(t)| \leq \mu(\mathbb{R}) \|\varphi_t\|_{L_2}.$$

Moreover,

$$\left| \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) H_S^2(t) dt \right)^{\frac{1}{2}} - \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) H_X^2(t) dt \right)^{\frac{1}{2}} \right| \leq \sqrt{\mu(\mathbb{R}) |\tau|} \|F\|_{L_2}.$$

Proof From the definition we get that

$$\begin{aligned} |H_S(t) - H_X(t)| &\leq \left(\mu(\mathbb{R}) \sum_{n=1}^\infty n n! t^{n-1} \|h_{n-1}\|_{L_2(\mu^{\otimes n})}^2 \right)^{\frac{1}{2}} \\ &= \left(\mu(\mathbb{R})^2 \sum_{n=1}^\infty (n-1)! t^{n-1} \|n h_{n-1}\|_{L_2(\mu^{\otimes (n-1)})}^2 \right)^{\frac{1}{2}} \\ &= \mu(\mathbb{R}) \|\varphi_t\|_{L_2}. \end{aligned}$$

Finally,

$$\begin{aligned} &\left| \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) H_S^2(t) dt \right)^{\frac{1}{2}} - \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) H_X^2(t) dt \right)^{\frac{1}{2}} \right| \\ &\leq \left(\sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) |H_S(t) - H_X(t)|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &\leq |\tau|^{\frac{1}{2}} |\mu(\mathbb{R})|^{\frac{1}{2}} \left(\int_0^1 \|\varphi_t\|_{L_2}^2 dt \mu(\mathbb{R}) \right)^{\frac{1}{2}} \\
 &= |\tau|^{\frac{1}{2}} |\mu(\mathbb{R})|^{\frac{1}{2}} \|F\|_{L_2}.
 \end{aligned}$$

□

Proof (of Theorem 6) (i) \iff (ii) follows from Theorem 1 and Theorem 3 because

$$H_X^2(t) = \frac{d^2}{dt^2} \left(\sum_{n=1}^{\infty} \|I_n(f_n)\|_{L_2}^2 t^n \right) \quad \text{if } F = \sum_{n=1}^{\infty} I_n(f_n). \quad (14)$$

(iii) \iff (iv) follows from Theorem 4 and (ii) \iff (iv) from Theorem 3 and Lemma 2. □

Theorem 7 (a) For $F \in \mathbb{M}$ and $\theta \in (0, 1]$ the following assertions are equivalent:

- (i) $F \in \mathbb{B}_{2,2}^\theta$.
 - (ii) $\sup_N N^{\frac{1}{2}} a_X^{\text{opt}}(F; \tau_N^\theta) < \infty$.
- If $\Omega = \{S_t \neq 0\}$ for all $t \in [0, 1]$, then (i) and (ii) are equivalent to:
- (iii) $\sup_N N^{\frac{1}{2}} a_S^{\text{opt}}(F; \tau_N^\theta) < \infty$.
 - (iv) $\sup_N N^{\frac{1}{2}} a_S^{\text{sim}}(F; \tau_N^\theta, \Omega^N) < \infty$.
- (b) If the assertions (i) - (ii) hold, then we have

$$\lim_{N \rightarrow \infty} N |a_X^{\text{opt}}(F; \tau_N^\theta)|^2 = \frac{1}{2\theta} \int_0^1 (1-t)^{1-\theta} H_X^2(t) dt$$

and if in addition $\Omega = \{S_t \neq 0\}$ for all $t \in [0, 1]$, then

$$\begin{aligned}
 \lim_{N \rightarrow \infty} N |a_S^{\text{opt}}(F; \tau_N^\theta)|^2 &= \lim_{N \rightarrow \infty} N |a_S^{\text{sim}}(F; \tau_N^\theta, \Omega^N)|^2 \\
 &= \frac{1}{2\theta} \int_0^1 (1-t)^{1-\theta} H_S^2(t) dt.
 \end{aligned}$$

Proof Part (a): (i) \iff (ii) follows from Theorems 2 and 3 because of (14).

(ii) \iff (iv) From [9, Lemma 3.8] and Theorem 3 it follows that the desired equivalence is equivalent to

$$\int_0^1 (1-t)^{1-\theta} H_X^2(t) dt < \infty \quad \text{if and only if} \quad \int_0^1 (1-t)^{1-\theta} H_S^2(t) dt < \infty. \quad (15)$$

In view of Lemma 2 it is therefore sufficient to check $\int_0^1 (1-t)^{1-\theta} \|\varphi_t\|_{L_2}^2 dt < \infty$ which follows from $\int_0^1 \|\varphi_t\|_{L_2}^2 \mu(\mathbb{R}) dt = \|F - \mathbb{E}F\|_{L_2}^2 < \infty$.

(iv) \iff (iii) follows from Theorem 4, $a_X^{\text{opt}}(F; \tau) \leq \|F\|_{L_2}$ and $|\tau_N^\theta| \leq 1/(\theta N)$ by (12).

Part (b): Let $\alpha(s) := 1 - (1 - s)^{\frac{1}{\theta}}$ and $H : [0, 1) \rightarrow [0, \infty)$ be non-decreasing and continuous such that $\int_0^1 (1 - t)^{1-\theta} H^2(t) dt < \infty$. For any $\delta \in (0, 1)$ and $\eta := \alpha^{-1}(\delta)$ we observe that

$$\begin{aligned} \frac{1}{2\theta} \int_0^\delta (1 - t)^{1-\theta} H^2(t) dt &= \frac{1}{2} \int_0^\delta \alpha'(\alpha^{-1}(t)) H^2(t) dt \\ &= \frac{1}{2} \int_0^\eta \alpha'(s) [H^2(\alpha(s)) \alpha'(s)] ds. \end{aligned}$$

Because

$$\alpha'(s) = \lim_{N \rightarrow \infty} \sum_{k=1}^N N \left[\alpha\left(\frac{k}{N} \wedge \eta\right) - \alpha\left(\frac{k-1}{N} \wedge \eta\right) \right] \mathbf{1}_{\left[\frac{k-1}{N}, \frac{k}{N}\right)}(s)$$

for $s \in [0, \eta)$ and all terms on the right-hand side are bounded by the Lipschitz constant of α on $[0, \eta]$, dominated convergence implies that

$$\begin{aligned} &\frac{1}{2\theta} \int_0^\delta (1 - t)^{1-\theta} H^2(t) dt \\ &= \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{k=1}^N \int_{\frac{k-1}{N} \wedge \eta}^{\frac{k}{N} \wedge \eta} N \left[\alpha\left(\frac{k}{N} \wedge \eta\right) - \alpha\left(\frac{k-1}{N} \wedge \eta\right) \right] \\ &\quad [H^2(\alpha(s)) \alpha'(s)] ds \\ &= \lim_{N \rightarrow \infty} N \sum_{k=1}^N H^2(t_{k-1}^{N,\theta}) \frac{(t_k^{N,\theta} \wedge \delta - t_{k-1}^{N,\theta} \wedge \delta)^2}{2} \\ &= \lim_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta} \wedge \delta}^{t_k^{N,\theta} \wedge \delta} (t_k^{N,\theta} \wedge \delta - t) H^2(t_{k-1}^{N,\theta}) dt \end{aligned}$$

where we use that H is uniformly continuous on $[0, \delta]$. From this we deduce that

$$\begin{aligned} &\liminf_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta}}^{t_k^{N,\theta}} (t_k^{N,\theta} - t) H^2(t) dt \\ &\geq \liminf_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta} \wedge \delta}^{t_k^{N,\theta} \wedge \delta} (t_k^{N,\theta} \wedge \delta - t) H^2(t_{k-1}^{N,\theta}) dt \\ &= \frac{1}{2\theta} \int_0^\delta (1 - t)^{1-\theta} H^2(t) dt \end{aligned}$$

for all $\delta \in (0, 1)$ and therefore

$$\liminf_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta}}^{t_k^{N,\theta}} (t_k^{N,\theta} - t) H^2(t) dt \geq \frac{1}{2\theta} \int_0^1 (1-t)^{1-\theta} H^2(t) dt.$$

On the other hand, (12) implies

$$\int_\delta^1 N \sum_{k=1}^N \left((t_k^{N,\theta} - t) \mathbb{1}_{[t_{k-1}^{N,\theta}, t_k^{N,\theta})}(t) \right) H^2(t) dt \leq \frac{1}{\theta} \int_\delta^1 (1-t)^{1-\theta} H^2(t) dt$$

for $\delta \in (0, 1)$. Choose δ such that the right hand side is less than $\varepsilon > 0$. We conclude (also using the previous computations of part (b) and the uniform continuity of H on $[0, \delta]$)

$$\begin{aligned} & \limsup_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta}}^{t_k^{N,\theta}} (t_k^{N,\theta} - t) H^2(t) dt \\ & \leq \limsup_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta} \wedge \delta}^{t_k^{N,\theta} \wedge \delta} (t_k^{N,\theta} - t) H^2(t) dt + \varepsilon \\ & = \lim_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta} \wedge \delta}^{t_k^{N,\theta} \wedge \delta} (t_k^{N,\theta} \wedge \delta - t) H^2(t) dt + \varepsilon \\ & = \frac{1}{2\theta} \int_0^\delta (1-t)^{1-\theta} H^2(t) dt + \varepsilon \\ & \leq \frac{1}{2\theta} \int_0^1 (1-t)^{1-\theta} H^2(t) dt + \varepsilon \end{aligned}$$

and

$$\limsup_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta}}^{t_k^{N,\theta}} (t_k^{N,\theta} - t) H^2(t) dt \leq \frac{1}{2\theta} \int_0^1 (1-t)^{1-\theta} H^2(t) dt.$$

Consequently,

$$\lim_{N \rightarrow \infty} N \sum_{k=1}^N \int_{t_{k-1}^{N,\theta}}^{t_k^{N,\theta}} (t_k^{N,\theta} - t) H^2(t) dt = \frac{1}{2\theta} \int_0^1 (1-t)^{1-\theta} H^2(t) dt.$$

It follows from (15) that for $H \in \{H_X, H_S\}$ our assumptions on H are satisfied. Hence Theorem 3 implies the limit expressions for a_X^{opt} and $a_S^{\text{sim}}(\cdot, \cdot, \Omega^N)$ (note that $c \rightarrow 1$ for $|\tau| \rightarrow 0$ in Theorem 3). The relation for a_S^{opt} follows from the one for $a_S^{\text{sim}}(\cdot, \cdot, \Omega^N)$, Theorem 4 and the fact that

$$\lim_{N \rightarrow \infty} \sqrt{N} \sqrt{|\tau_N^\theta|} a_X^{\text{opt}}(F; \tau_N^\theta) \leq \limsup_{N \rightarrow \infty} \sqrt{\frac{1}{\theta}} a_X^{\text{opt}}(F; \tau_N^\theta) = 0$$

where we have used (12) and, as in the proof of Theorem 5, the relation $\int_0^1 (1-t) H_X^2(t) dt < \infty$ together with Theorem 3. \square

Using the results from [15, Theorem 2.4] one can derive from Theorem 3 for example the following assertion.

Corollary 1 *For $F \in \mathbb{M}$ one has the following equivalences:*

(i) *There is a constant $c > 0$ such that*

$$\inf_{\tau_N \in \mathcal{T}_N} a_X^{\text{opt}}(F; \tau_N) \leq \frac{c}{\sqrt{N}} \quad \text{for } N = 1, 2, \dots \quad \text{iff } \int_0^1 H_X(t) dt < \infty.$$

(ii) *There is a constant $c > 0$ such that*

$$\inf_{\tau_N \in \mathcal{T}_N} a_S^{\text{sim}}(F; \tau_N, \Omega^N) \leq \frac{c}{\sqrt{N}} \quad \text{for } N = 1, 2, \dots \quad \text{iff } \int_0^1 H_S(t) dt < \infty.$$

4 Examples

4.1 Preparations

The following two lemmas provide information about the orthogonal projection $\Pi : L_2 \rightarrow \mathbb{M} \subseteq L_2$.

Lemma 3 *Given $G \in L_2$, $\theta \in (0, 1)$ and $q \in [1, \infty]$, one has that*

- (i) $G \in \mathbb{D}_{1,2}$ implies $\Pi(G) \in \mathbb{D}_{1,2}$,
- (ii) $G \in \mathbb{B}_{2,q}^\theta$ implies $\Pi(G) \in \mathbb{B}_{2,q}^\theta$.

Proof The lemma follows from the fact that for

$$G = \sum_{n=0}^{\infty} I_n(\alpha_n)$$

with symmetric $\alpha_n \in L_2^n$ the function h_n from Definition 2 computes as in (7) so that $\|f_n\|_{L_2^n} \leq \|\alpha_n\|_{L_2^n}$ where f_n is defined as in Definition 2. Hence, the statement can be derived (for example) from Theorem 1 using the monotonicity of A with respect to $\|a_n\|_{E_n}$ and the definition of $\mathbb{D}_{1,2}$. \square

Lemma 4 *For a Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(X_1) \in L_2$ there are symmetric $g_n \in L_2(\mu^{\otimes n})$ such that*

$$f(X_1) = \mathbb{E}f(X_1) + \sum_{n=1}^{\infty} I_n(g_n \mathbf{1}_{(0,1]}^{\otimes n}). \quad (16)$$

Moreover, it holds that $\Pi(f(X_1)) = \sum_{n=1}^{\infty} I_n(f_n)$ with symmetric f_n satisfying

$$\begin{aligned} f_n((t_1, x_1), \dots, (t_n, x_n)) &= h_{n-1}(x_1, \dots, x_{n-1}) \\ &:= \int_{\mathbb{R}} g_n(x_1, \dots, x_{n-1}, x) \frac{\mu(dx)}{\mu(\mathbb{R})} \end{aligned} \quad (17)$$

on $0 < t_1 < \dots < t_n < 1$ and $\Pi(f(X_1))$ is the orthogonal projection of $f(X_1)$ onto $I(X)$ defined in (5).

The representation (16) is proved in [1] and [2] and is based on invariance properties of $f(X_1)$ that transfer to the chaos representation. One could also use [6, Section 6].

Lemma 5 *Let $f \in C_b^\infty(\mathbb{R})$ and $f(X_1) = \sum_{n=1}^\infty I_n(g_n \mathbf{1}_{(0,1]}^{\otimes n}) \in \mathbb{D}_{1,2}$ with symmetric $g_n \in L_2(\mu^{\otimes n})$. Then the martingale $(\varphi_t)_{t \in [0,1]}$ given by (8) and (17) has a closure φ_1 , i.e. $\mathbb{E}(\varphi_1 | \mathcal{F}_t) = \varphi_t$ a.s., with*

$$\varphi_1 = \int_{\mathbb{R}} \left[\mathbf{1}_{\{x \neq 0\}} \frac{f(X_1 + x) - f(X_1)}{x} + \mathbf{1}_{\{x=0\}} f'(X_1) \right] \frac{\mu(dx)}{\mu(\mathbb{R})} \quad a.s.$$

Proof From [6, Proposition 5.1 and its proof] it is known that

$$\begin{aligned} \mathbf{1}_{\{x \neq 0\}} \frac{f(X_1 + x) - f(X_1)}{x} + \mathbf{1}_{\{x=0\}} f'(X_1) \\ = \sum_{n=1}^\infty n I_{n-1}(g_n(\cdot, x) \mathbf{1}_{(0,1]}^{\otimes(n-1)}) \quad \mu \otimes \mathbb{P} \quad a.e. \end{aligned} \quad (18)$$

Consequently, (17) implies that, a.s.,

$$\begin{aligned} & \int_{\mathbb{R}} \left[\mathbf{1}_{\{x \neq 0\}} \frac{f(X_1 + x) - f(X_1)}{x} + \mathbf{1}_{\{x=0\}} f'(X_1) \right] \frac{\mu(dx)}{\mu(\mathbb{R})} \\ &= \int_{\mathbb{R}} \left[\sum_{n=1}^\infty n I_{n-1} \left(g_n(\cdot, x) \mathbf{1}_{(0,1]}^{\otimes(n-1)} \right) \right] \frac{\mu(dx)}{\mu(\mathbb{R})} \\ &= \sum_{n=1}^\infty n I_{n-1} \left(\int_{\mathbb{R}} g_n(\cdot, x) \frac{\mu(dx)}{\mu(\mathbb{R})} \mathbf{1}_{(0,1]}^{\otimes(n-1)} \right) \\ &= \sum_{n=1}^\infty n I_{n-1} \left(h_{n-1} \mathbf{1}_{(0,1]}^{\otimes(n-1)} \right) \\ &=: \varphi_1 \end{aligned}$$

where the second equality follows by a standard Fubini argument. \square

Definition 7 For $\delta > 0$ we let

$$\psi(\delta) := \sup_{\lambda \in \mathbb{R}} \mathbb{P}(|X_1 - \lambda| \leq \delta).$$

Example 2 The small ball estimate

$$\psi(\delta) \leq c\delta \quad (19)$$

can be deduced if X_1 has a bounded density. As an example we use tempered α -stable processes with $\alpha \in (0, 2)$, given by the Lévy measure

$$\nu_\alpha(dx) := \frac{d}{|x|^{1+\alpha}} (1 + |x|)^{-m} \mathbf{1}_{\{x \neq 0\}} dx$$

with $d > 0$ and $m \in (2 - \alpha, \infty)$ being fixed parameters. Then [18, Theorem 5] implies that X_1 has a bounded density.

For $K \in \mathbb{R}$ and $\varepsilon \in (0, 1]$ we let $f_{K,\varepsilon} \in C_b^\infty(\mathbb{R})$ with $f_{K,\varepsilon}(x) = 0$ if $x \leq K$, $f_{K,\varepsilon}(x) = 1$ if $x \geq K + \varepsilon$, $0 \leq f_{K,\varepsilon}(x) \leq 1$ and $0 \leq f'_{K,\varepsilon}(x) \leq 2/\varepsilon$ for all $x \in \mathbb{R}$.

Lemma 6 For $K \in \mathbb{R}$ and $\varepsilon > 0$ we have that

$$\begin{aligned} \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \\ \leq 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx). \end{aligned}$$

Proof We get that

$$\begin{aligned} & \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \\ &= \mathbb{E} \int_{0 < |x| \leq \varepsilon} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \\ & \quad + \mathbb{E} \int_{\varepsilon < |x| < \infty} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \\ &\leq \frac{4}{\varepsilon^2} \mathbb{P}(X_1 \in [K - \varepsilon, K + 2\varepsilon]) \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) \\ & \quad + \int_{\varepsilon < x < \infty} \mathbb{P}(X_1 \leq K + \varepsilon, X_1 + x \geq K) \nu(dx) \\ & \quad + \int_{-\infty < x < -\varepsilon} \mathbb{P}(X_1 + x \leq K + \varepsilon, X_1 \geq K) \nu(dx) \\ &\leq 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) \\ & \quad + \int_{\varepsilon < x < \infty} \mathbb{P}(|X_1 - K| \leq x) \nu(dx) \\ & \quad + \int_{-\infty < x < -\varepsilon} \mathbb{P}(K \leq X_1 \leq K - 2x) \nu(dx) \\ &\leq 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx). \end{aligned}$$

□

Lemma 7 For $K \in \mathbb{R}$ and $\varepsilon > 0$ the following assertions are true:

$$(i) \quad \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \leq \nu(\mathbb{R})$$

(ii) If $\psi(\delta) \leq c\delta$, then

$$\begin{aligned} \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \\ \leq 9c \min \left\{ \frac{1}{\varepsilon} \int_{\mathbb{R}} x^2 \nu(dx), \int_{\mathbb{R}} |x| \nu(dx) \right\}. \end{aligned}$$

Proof (i) Using $\mu(dx) = x^2 \nu(dx)$ on $\mathbb{R} \setminus \{0\}$ one has that

$$\int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \leq \nu(\mathbb{R}).$$

(ii) If $\psi(\delta) \leq c\delta$, then we can bound the right-hand side in Lemma 6 by

$$\begin{aligned} & 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 d\nu(x) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx) \\ & \leq \frac{8c}{\varepsilon} \int_{\mathbb{R}} x^2 d\nu(x) + c \int_{\varepsilon < |x| < \infty} |x| \nu(dx) \\ & \leq \frac{8c}{\varepsilon} \int_{\mathbb{R}} x^2 \nu(dx) + \frac{c}{\varepsilon} \int_{\varepsilon < |x| < \infty} x^2 \nu(dx) \\ & \leq \frac{9c}{\varepsilon} \int_{\mathbb{R}} x^2 \nu(dx). \end{aligned}$$

Moreover,

$$\begin{aligned} & 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx) \\ & \leq 8c \int_{0 < |x| \leq \varepsilon} |x| \nu(dx) + c \int_{\varepsilon < |x| < \infty} |x| \nu(dx) \\ & \leq 8c \int_{\mathbb{R}} |x| \nu(dx). \end{aligned}$$

□

Lemma 8 Let $f(x) = \chi_{[K,\infty)}(x)$ for a $K \in \mathbb{R}$. Assume $\sigma = 0$, $\int_{\mathbb{R}} |x|^{\frac{3}{2}} \nu(dx) < \infty$ and assume that there is a $c > 0$ such that $\psi(\delta) \leq c\delta$ for all $\delta > 0$. Then one has that

$$\mathbb{E} \left| \int_{\mathbb{R} \setminus \{0\}} \left| \frac{f(X_1 + x) - f(X_1)}{x} \right| \mu(dx) \right|^2 \leq \frac{c}{2} \left(\int_{\mathbb{R}} |x|^{\frac{3}{2}} \nu(dx) \right)^2.$$

Proof For $d\nu_0(x) := |x|^{\frac{3}{2}} \nu(dx)$ we get that

$$\mathbb{E} \left| \int_{\mathbb{R} \setminus \{0\}} \left| \frac{f(X_1 + x) - f(X_1)}{x} \right| \mu(dx) \right|^2$$

$$\begin{aligned}
&\leq \mathbb{E} \left| \int_{\mathbb{R}} |f(X_1 + x) - f(X_1)| |x|^{-\frac{1}{2}} \nu_0(dx) \right|^2 \\
&\leq \nu_0(\mathbb{R}) \mathbb{E} \int_{\mathbb{R}} |f(X_1 + x) - f(X_1)|^2 |x|^{-1} \nu_0(dx) \\
&\leq \nu_0(\mathbb{R}) \int_{\mathbb{R}} \psi \left(\frac{|x|}{2} \right) |x|^{-1} \nu_0(dx) \\
&\leq \frac{c}{2} \nu_0(\mathbb{R})^2.
\end{aligned}$$

□

4.2 Examples

Throughout the whole subsection we fix a real number K and let

$$f(x) := \mathbf{1}_{(K, \infty)}(x).$$

(a) Without projection on \mathbb{M} : We will obtain the (fractional) smoothness of $\mathbf{1}_{(K, \infty)}(X_1)$ in dependence of distributional properties of X . Note that Lemma 3 ensures that $\Pi(\mathbf{1}_{(K, \infty)}(X_1))$ has at least the (fractional) smoothness of $\mathbf{1}_{(K, \infty)}(X_1)$. Our standing assumption, as mentioned before, is $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$. The case C_1 below confirms that for a compound Poisson process X we have $\mathbf{1}_{(K, \infty)}(X_1) \in \mathbb{D}_{1,2}$.

	σ	ψ	additional assumption on ν	Smoothness
C_1	$\sigma = 0$	arbitrary	$\int_{ x \leq 1} \nu(dx) < \infty$	$\mathbb{D}_{1,2}$
C_2	$\sigma = 0$	$\psi(\delta) \leq c\delta$	$\int_{ x \leq 1} x \nu(dx) < \infty$	$\mathbb{D}_{1,2}$
C_3	arbitrary	$\psi(\delta) \leq c\delta$		$\mathbb{B}_{2, \infty}^{\frac{1}{2}}$

To check this table assume that the chaos-decomposition of $f_{K, \varepsilon}(X_1)$ is described by symmetric $g_n^{K, \varepsilon} \in L_2(\mu^{\otimes n})$. From (18) we derive in the case $\sigma = 0$ that

$$\begin{aligned}
\sum_{n=1}^{\infty} n n! \|g_n^{K, \varepsilon}\|_{L_2(\mu^{\otimes n})}^2 &= \sum_{n=1}^{\infty} n^2 \int_{\mathbb{R}} (n-1)! \|g_n^{K, \varepsilon}(\cdot, x)\|_{L_2(\mu^{\otimes(n-1)})}^2 \mu(dx) \\
&= \sum_{n=1}^{\infty} n^2 \mathbb{E} \int_{\mathbb{R}} I_{n-1}(g_n^{K, \varepsilon}(\cdot, x) \mathbf{1}_{(0,1]}^{\otimes(n-1)})^2 \mu(dx) \\
&= \int_{\mathbb{R}} \mathbb{E} \left| \sum_{n=1}^{\infty} n I_{n-1}(g_n^{K, \varepsilon}(\cdot, x) \mathbf{1}_{(0,1]}^{\otimes(n-1)}) \right|^2 \mu(dx) \\
&= \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K, \varepsilon}(X_1 + x) - f_{K, \varepsilon}(X_1)}{x} \right|^2 \mu(dx)
\end{aligned}$$

so that

$$\|f_{K,\varepsilon}(X_1)\|_{\mathbb{D}_{1,2}}^2 \leq 1 + \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx).$$

Cases C_1 and C_2 : Exploiting Lemma 7 gives that

$$\sup_{m=1,2,\dots} \|f_{K,1/m}(X_1)\|_{\mathbb{D}_{1,2}} < \infty.$$

Moreover $\|f_{K,1/m}(X_1) - \chi_{(K,\infty)}(X_1)\|_{L_2} \rightarrow_m 0$ by dominated convergence so that C_1 and C_2 follow by a standard argument.

Case C_3 : As before we get from (18) that

$$\begin{aligned} & \|f_{K,\varepsilon}(X_1)\|_{\mathbb{D}_{1,2}}^2 \\ & \leq 1 + \int_{\mathbb{R}} \mathbb{E} \left| \mathbf{1}_{\{x \neq 0\}} \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} + \mathbf{1}_{\{x=0\}} f'_{K,\varepsilon}(X_1) \right|^2 \mu(dx). \end{aligned}$$

Exploiting Lemma 7 and the property $0 \leq f'_{K,\varepsilon}(x) \leq 2/\varepsilon$ we continue with

$$\begin{aligned} \|f_{K,\varepsilon}(X_1)\|_{\mathbb{D}_{1,2}}^2 & \leq 1 + \frac{9c}{\varepsilon} \int_{\mathbb{R}} x^2 d\nu(x) + \sigma^2 \frac{4}{\varepsilon^2} \psi\left(\frac{\varepsilon}{2}\right) \\ & \leq 1 + \frac{9c}{\varepsilon} \int_{\mathbb{R}} x^2 d\nu(x) + \sigma^2 \frac{2c}{\varepsilon}. \end{aligned}$$

On the other hand,

$$\|\chi_{(K,\infty)}(X_1) - f_{K,\varepsilon}(X_1)\|_{L_2} \leq \sqrt{\psi\left(\frac{\varepsilon}{2}\right)} \leq \sqrt{\frac{c\varepsilon}{2}}.$$

Estimating the K -functional $K(u, \mathbf{1}_{(K,\infty)}(X_1); L_2, \mathbb{D}_{1,2})$ by the help of the decomposition $\mathbf{1}_{(K,\infty)}(X_1) = \left[\mathbf{1}_{(K,\infty)}(X_1) - f_{K,\varepsilon}(X_1) \right] + f_{K,\varepsilon}(X_1)$ and optimizing over $\varepsilon > 0$ gives $\chi_{(K,\infty)}(X_1) \in \mathbb{B}_{2,\infty}^{\frac{1}{2}}$.

(b) After projection on \mathbb{M} : Here we have the following

Proposition 1 Assume that $\sigma = 0$, $0 < \int_{\mathbb{R}} |x|^{\frac{3}{2}} \nu(dx) < \infty$ and that $\psi(\delta) \leq c\delta$. Then one has for all $K \in \mathbb{R}$ that

$$II(\mathbf{1}_{(K,\infty)}(X_1)) \in \mathbb{D}_{1,2}.$$

Proof By the same reasoning as in the cases C_1 and C_2 it is sufficient to show that

$$\sup_{m=1,2,\dots} \|II(f_{K,1/m}(X_1))\|_{\mathbb{D}_{1,2}} < \infty.$$

By (9) and Lemma 5 it suffices to check that

$$\sup_{m=1,2,\dots} \mathbb{E} \left| \int_{\mathbb{R} \setminus \{0\}} \left[\frac{f_{K,\frac{1}{m}}(X_1 + x) - f_{K,\frac{1}{m}}(X_1)}{x} \right] d\mu(x) \right|^2 < \infty.$$

But this estimate follows from Lemma 8 and the representation

$$f_{K,\varepsilon}(x) = \int_{-\infty}^x f'_{K,\varepsilon}(y)dy = \int_{\mathbb{R}} \mathbb{1}_{[y,\infty)}(x) f'_{K,\varepsilon}(y)dy$$

and $\int_{\mathbb{R}} f'_{K,\varepsilon}(y)dy = 1$. □

Example 3 An example for Proposition 1 is obtained from Example 2. Considering

$$\nu_{\alpha}(dx) = \frac{d}{|x|^{1+\alpha}}(1 + |x|)^{-m} \mathbb{1}_{\{x \neq 0\}} dx$$

for $d > 0$, $\alpha \in (0, \frac{3}{2})$, $m \in (2 - \alpha, \infty)$ gives $\psi(\delta) \leq c\delta$ and $0 < \int_{\mathbb{R}} |x|^{\frac{3}{2}} d\nu_{\alpha}(x) < \infty$, where α turns out to be the Blumenthal-Gettoor index. Using the results of [14, Example 3.1] one can also show that $\mathbb{1}_{(K,\infty)}(X_1) \notin \mathbb{D}_{1,2}$ for $\alpha \geq 1$ so that the projection Π improves the smoothness of $\mathbb{1}_{(K,\infty)}(X_1)$ for $\alpha \in [1, \frac{3}{2})$.

Remark 3 Using a Fourier transform approach Brodén and Tankov [5] compute the discretization error under the historical measure for the delta hedging as well as for a strategy which is optimal under a given equivalent martingale measure. Using the equivalences of Theorem 6 (i) \iff (iv) and Theorem 7 (i) \iff (iv) one can also conclude about the fractional smoothness of the projection of the considered digital option from the computed convergence rate for equidistant time nets.

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