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Research Article

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Products of Snowflaked Euclidean Lines Are Not Minimal for Looking Down

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Abstract: We show that products of snowflaked Euclidean lines are not minimal for looking down. This question was raised in *Fractured fractals and broken dreams*, Problem 11.17, by David and Semmes. The proof uses arguments developed by Le Donne, Li and Rajala to prove that the Heisenberg group is not minimal for looking down. By a method of shortcuts, we define a new distance d such that the product of snowflaked Euclidean lines looks down on (\mathbb{R}^N, d) , but not vice versa.

Keywords: Ahlfors-regularity, biLipschitz pieces, BPI-spaces

MSC: Primary 26B05; Secondary 28A80

1 Introduction

The concept of BPI space (Big Pieces of Itself) was introduced by David and Semmes in [4] in order to provide a framework in which to work with self-similarity in metric spaces setting. A BPI space is more or less a metric space in which any two balls contain big pieces that look almost the same up to scaling and bounded distortions. They also introduced a notion of BPI equivalence in order to understand and classify BPI geometries. Two BPI spaces are BPI equivalent if they possess pieces of positive measure that are biLipschitz equivalent. With the aim of classifying BPI spaces that are not BPI equivalent, David and Semmes defined a notion of looking down between BPI spaces of the same dimension. A natural question arises when working with looking down BPI spaces: what are the most primitive BPI spaces? Such BPI spaces are called minimal for looking down (see Section 2 for the definitions).

By using ideas of [7], where it is proved that the Heisenberg group is not minimal for looking down, we prove the following theorem, which gives an answer to Problem 11.17 in [4].

Theorem 1.1. *Given $s_1, \dots, s_N \in (0, 1]$, the space $(\mathbb{R}^N, \sum_{k=1}^N |x_k - y_k|^{s_k})$ is minimal for looking down if and only if $s_k = 1$ for all k .*

The distance defined above will be denoted by d_s , where s stands for $(s_1, \dots, s_N) \in (0, 1]^N$. It is the ℓ^1 product distance of the snowflaked distances $|\cdot|^{s_k}$ defined on \mathbb{R} : if $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ are in \mathbb{R}^N , then

$$d_s(x, y) = \sum_{k=1}^N |x_k - y_k|^{s_k}.$$

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Kirchheim proved in [5] that Euclidean spaces are minimal for looking down, that is, if $s = (1, \dots, 1)$, then (\mathbb{R}^N, d_s) is minimal for looking down. To prove Theorem 1.1, it is thus sufficient to show that if $s_k < 1$ for some k , then the space (\mathbb{R}^N, d_s) is not minimal for looking down.

From now on we fix an integer $N \geq 1$ and an N -tuple $s = (s_1, \dots, s_N) \in (0, 1]^N$ such that $s_k < 1$ for some k . We denote by s the *minimum snowflaking factor*, i.e. $s = \min \{s_k, 1 \leq k \leq N\}$, and by L the *minimally snowflaked layer*, that is the subset of $\{1, \dots, N\}$ where the snowflaking factor is minimum:

$$L = \{k \in \{1, \dots, N\}, s_k = s\}.$$

The strategy to prove Theorem 1.1 is the following. First we look at a one dimensional problem. We construct a quotient semi-distance $d_{\mathcal{R}}$ on \mathbb{R} associated with an equivalence relation \mathcal{R} using the shortening technique developed in [7]. The construction of $d_{\mathcal{R}}$ is made in a self-similar way so that the subspace $[0, 1]$ of \mathbb{R} , endowed with this semi-distance $d_{\mathcal{R}}$ is a BPI space. To be more precise, the quotient space $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is a BPI space. The semi-distance $d_{\mathcal{R}}$ satisfies $d_{\mathcal{R}} \leq |\cdot|^s$, where $|\cdot|$ is the Euclidean distance on \mathbb{R} , and s the minimum snowflaking factor. Moreover any Lipschitz function from $([0, 1], d_{\mathcal{R}})$ to $(\mathbb{R}, |\cdot|^s)$ is constant.

Then we look at the N -dimensional problem. We slightly modify the distance d_s in the minimally snowflaked layer L by replacing the terms $|\cdot|^s$ with the semi-distance $d_{\mathcal{R}}$. This gives a new semi-distance $d_{s,\mathcal{R}}$ on \mathbb{R}^N . As a product of bounded BPI spaces is a BPI space, the quotient space $([0, 1]^N/d_{s,\mathcal{R}}, \bar{d}_{s,\mathcal{R}})$ is a BPI space. Suppose then that $([0, 1]^N, d_{s,\mathcal{R}})$ looks down on (\mathbb{R}^N, d_s) . There exists a Lipschitz map $g : (A, d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ such that $\mathcal{H}^\alpha(g(A)) > 0$, where α is the Ahlfors dimension of (\mathbb{R}^N, d_s) and A a closed subset of $[0, 1]^N$. By a blow-up argument, we prove that there exists a Lipschitz map $f : ([0, 1]^N, d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ whose image has positive measure, which is in contradiction with the property on Lipschitz functions from $([0, 1], d_{\mathcal{R}})$ to $(\mathbb{R}, |\cdot|^s)$.

Section 2 deals with definitions related to BPI spaces, quotient semi-distance, etc. In Section 3, we prove that the product of two BPI spaces, both bounded or both unbounded, is a BPI space. In Section 4, we construct the semi-distance $d_{\mathcal{R}}$ on \mathbb{R} and we prove that the metric space $([0, 1], d_{\mathcal{R}})$ is a BPI space. In Section 5, we prove that every Lipschitz function from $([0, 1], d_{\mathcal{R}})$ to $(\mathbb{R}, |\cdot|^s)$ is constant. Finally, in Sections 6 and 7, we conclude by a blow-up process that the space (\mathbb{R}^N, d_s) is not minimal for looking down.

2 Preliminaries

In what follows, $\mathbb{N} = \{1, 2, 3, \dots\}$. By a measure m on a metric space (X, d) we always mean an outer measure such that Borel sets are m -measurable. Recall that an outer measure m on a set X is a map $m : \mathcal{P}(X) \rightarrow [0, \infty]$ defined on all subsets of X , such that $m(\emptyset) = 0$, $m(A) \leq m(B)$ for all A, B subsets of X with $A \subset B$, and for all countable sequences $(A_n)_{n \in \mathbb{N}}$ of subsets of X ,

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n).$$

Any metric space (X, d) can be endowed with a one-parameter family of natural measures: for all $\alpha > 0$, we define the α -dimensional Hausdorff measure \mathcal{H}_d^α (or just \mathcal{H}^α when the distance is implicit) as follows: for all $A \subset X$,

$$\mathcal{H}_d^\alpha(A) = \liminf_{\delta > 0} \left\{ \sum_{n=1}^{\infty} (\text{diam}_d(A_n))^\alpha, A \subset \bigcup_{n=1}^{\infty} A_n, \text{diam}_d(A_n) \leq \delta \right\}.$$

Definition 2.1 (Ahlfors regularity). *Let m be a measure on a complete metric space (X, d) , and $\alpha > 0$. We say that the metric measure space (X, d, m) is Ahlfors regular of dimension α (or Ahlfors α -regular) if there exists a constant $K > 0$ such that for all $x \in X$ and $r \in (0, \text{diam}(X))$,*

$$K^{-1}r^\alpha \leq m(B(x, r)) \leq Kr^\alpha.$$

In Definition 2.1 and later on we follow the convention of [4] where each ball $B(x, r)$ is implicitly assumed to have finite radius even if the range of radii would permit $r = \infty$. The following well-known lemma (see [4, Lemma 1.2]) allows us to talk about Ahlfors regularity on a metric space (X, d) .

Lemma 2.2. *If (X, d, m) is Ahlfors regular of dimension α , then so is $(X, d, \mathcal{H}^\alpha)$, and there exists a constant $K > 0$ such that for all Borel sets $B \subset X$, $K^{-1}m(B) \leq \mathcal{H}^\alpha(B) \leq Km(B)$.*

It is well-known that closed and bounded sets of an Ahlfors regular space are compact, because closed subsets are totally bounded, and Ahlfors regular spaces are assumed to be complete.

Recall that a C -biLipschitz map $f : (X, d) \rightarrow (Y, \rho)$ between two metric spaces is a map $f : X \rightarrow Y$ such that for all $x, y \in X$,

$$C^{-1}d(x, y) \leq \rho(f(x), f(y)) \leq Cd(x, y).$$

Definition 2.3 (BPI space). *An Ahlfors α -regular metric space (X, d) is a BPI space of dimension α if there exist $C > 0, \theta > 0$ such that for all $x, y \in X$ and all $r, t \in (0, \text{diam}_d X]$, there exist a closed subset $A \subset B(x, r)$ with $\mathcal{H}^\alpha(A) \geq \theta r^\alpha$ and a C -biLipschitz embedding $f : (A, r^{-1}d) \rightarrow (B(y, t), t^{-1}d)$.*

We next define an equivalence relation for BPI spaces.

Definition 2.4 (BPI equivalence). *Two BPI spaces (X, d) and (Y, ρ) of the same dimension α are BPI equivalent if there exist $C > 0, \theta > 0$ such that for all $x \in X, y \in Y$ and all $r \in (0, \text{diam}_d X], t \in (0, \text{diam}_\rho Y]$, there exist a closed subset $A \subset B_d(x, r)$ with $\mathcal{H}_d^\alpha(A) \geq \theta r^\alpha$ and a C -biLipschitz embedding $f : (A, r^{-1}d) \rightarrow (B_\rho(y, t), t^{-1}\rho)$.*

This is an equivalence relation (see [4, Chapter 7]). The following definition allows us to compare BPI spaces of the same dimension that are not BPI equivalent.

Definition 2.5 (Looking down). *Let (X, d) and (Y, ρ) be two BPI spaces of the same dimension α . We say that X looks down on Y if there exist a closed subset $A \subset X$ and a Lipschitz map $g : A \rightarrow Y$ such that $\mathcal{H}_\rho^\alpha(g(A)) > 0$.*

This is a partial order on the set of equivalence classes of BPI spaces with the equivalence relation "looking down equivalence" (two BPI spaces are *looking down equivalent* if each looks down on the other, see [4, Chapter 11]). We propose here a slightly different definition of minimal for looking down from the one given in [4] by David and Semmes.

Definition 2.6 (Minimal for looking down). *A BPI space X is minimal for looking down if for any BPI space Y such that X looks down on Y , then Y looks down on X .*

The original definition of David and Semmes says that a BPI space X is minimal for looking down if for any BPI space Y such that X looks down on Y , then X and Y are BPI equivalent. Notice that BPI equivalence implies looking down equivalence, but the converse is false [6]. Definition 2.6 is thus weaker, but more natural, since with this definition, a BPI space that is minimal for looking down is a BPI space minimal for the partial order "looking down". Notice also that since our definition is weaker, Theorem 1.1 answers Problem 11.17 in [4] also with the original definition of minimality for looking down.

Next, following [3, Definition 3.1.12], we define the notion of quotient semi-distance, which is useful in the shortening technique used in Section 4. Given an equivalence relation \mathcal{R} on a metric space (X, d) we can construct the *quotient semi-distance* $d_{\mathcal{R}}$ defined on X by

$$d_{\mathcal{R}}(x, y) = \inf \left\{ \sum_{k=0}^n d(x_k, y_k), x \mathcal{R} x_0, y_k \mathcal{R} x_{k+1}, y_n \mathcal{R} y \right\}.$$

The quotient semi-distance $d_{\mathcal{R}}$ is a *semi-distance*, that is $d_{\mathcal{R}}$ is nonnegative, symmetric, satisfies the triangle inequality, is zero on the diagonal of $X \times X$ but can be zero also outside the diagonal.

A set $(x_0, y_0, \dots, x_n, y_n)$ such that $x \mathcal{R} x_0$, $y_k \mathcal{R} x_{k+1}$ and $y_n \mathcal{R} y$ is called an *itinerary* between x and y . That is, one is allowed to take a shortcut by teleporting itself between x and x_0 , between y_k and x_{k+1} for all k and between y_n and y . An itinerary $(x_0, y_0, \dots, x_n, y_n)$ is *shorter* than another itinerary $(x'_0, y'_0, \dots, x'_n, y'_n)$ if

$$\sum_{k=0}^n d(x_k, y_k) \leq \sum_{k=0}^n d(x'_k, y'_k).$$

The next lemma gives a way to construct Lipschitz maps and similitudes for quotient semi-distances.

Lemma 2.7. *Let (X, d) , (Y, ρ) be two metric spaces, \mathcal{R}_X an equivalence relation on X and \mathcal{R}_Y an equivalence relation on Y . Suppose that there exist a map $f : X \rightarrow Y$ and $\lambda > 0$ such that for all $x, y \in X$, $\rho(f(x), f(y)) = \lambda d(x, y)$ and $x \mathcal{R}_X y \Rightarrow f(x) \mathcal{R}_Y f(y)$. Then, for all $x, y \in X$,*

$$\rho_{\mathcal{R}_Y}(f(x), f(y)) \leq \lambda d_{\mathcal{R}_X}(x, y).$$

Moreover, if f is bijective and $x \mathcal{R}_X y \Leftrightarrow f(x) \mathcal{R}_Y f(y)$, then $\rho_{\mathcal{R}_Y}(f(x), f(y)) = \lambda d_{\mathcal{R}_X}(x, y)$.

Proof. Let $\varepsilon > 0$ and $(x_0, y_0, \dots, x_n, y_n)$ be an itinerary from x to y such that

$$\lambda^{-1} \sum_{k=0}^n \rho(f(x_k), f(y_k)) = \sum_{k=0}^n d(x_k, y_k) \leq d_{\mathcal{R}_X}(x, y) + \varepsilon.$$

Since, $(f(x_0), f(y_0), \dots, f(x_n), f(y_n))$ is an itinerary from $f(x)$ to $f(y)$, then $\rho_{\mathcal{R}_Y}(f(x), f(y)) \leq \lambda(d_{\mathcal{R}_X}(x, y) + \varepsilon)$, and the result holds for $\varepsilon \rightarrow 0$. If f is bijective and $x \mathcal{R}_X y \Leftrightarrow f(x) \mathcal{R}_Y f(y)$, it is sufficient to apply the foregoing to f^{-1} . □

If d is a semi-distance on a space X , we denote by $(X/d, \bar{d})$ the quotient metric space, which is the space of all equivalence classes for the relation $x \sim y \Leftrightarrow d(x, y) = 0$. The natural distance \bar{d} on X/d is defined by $\bar{d}(\pi(x), \pi(y)) = d(x, y)$, where $\pi : X \rightarrow X/d$ is the canonical projection. One can easily check that \bar{d} is well defined and is a distance on X/d .

Finally, we recall some basic facts about the Hausdorff distance. If (X, d) is a metric space, we denote by $\mathcal{C}(X)$ the set of all compact subsets of X . The ε -neighborhood of a set $A \subset X$, denoted by A^ε , is $\{x \in X, d(x, A) < \varepsilon\}$. On $\mathcal{C}(X)$, we consider the Hausdorff distance d_H defined for $A, B \in \mathcal{C}(X)$ by

$$d_H(A, B) = \inf \{ \varepsilon > 0, A \subset B^\varepsilon, B \subset A^\varepsilon \}.$$

The space $(\mathcal{C}(X), d_H)$ is a metric space, which is compact if X is compact (Blaschke Theorem, see for instance [3, Theorem 7.3.8]). We sometimes write $K_n \xrightarrow{H} K$ to say that $(K_n)_{n \in \mathbb{N}}$ converges to K in the Hausdorff distance. Moreover, if (X, d) is Ahlfors α -regular, then the α -dimensional Hausdorff measure is upper semi-continuous on $(\mathcal{C}(X), d_H)$.

Lemma 2.8. *Let (X, d) be an Ahlfors regular metric space of dimension α . Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact sets that converges to $K \in \mathcal{C}(X)$ in d_H . Then,*

$$\limsup_{n \rightarrow \infty} \mathcal{H}^\alpha(K_n) \leq \mathcal{H}^\alpha(K).$$

Proof. For all $n \in \mathbb{N}$, we set $f_n = \mathbb{1}_{K_n \setminus K}$. The convergence of K_n to K in the Hausdorff distance implies that (f_n) converges pointwise to 0. In fact, if $x \in K$, then for all n , $f_n(x) = 0$. If $x \notin K$, then there exists $r > 0$ such that $B(x, r) \subset X \setminus K$. Fix $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d_H(K_n, K) < r/2$. Then, for all $n \geq n_0$, $x \notin K_n$, thus $f_n(x) = 0$. Moreover, since (K_n) converges, it is a bounded sequence, so there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $K_n \subset K^\varepsilon \subset \bar{K}^\varepsilon$, which is a compact set since closed and bounded subsets of an Ahlfors regular space are compact. Then, for all $n \in \mathbb{N}$, $f_n \leq \mathbb{1}_{\bar{K}^\varepsilon}$ and the latter function is integrable with respect to \mathcal{H}^α ,

since compact subsets of an Ahlfors α -regular space have finite α -dimensional Hausdorff measure. By the dominated convergence theorem,

$$\mathcal{H}^\alpha(K_n \setminus K) = \int_X f_n d\mathcal{H}^\alpha \xrightarrow[n \rightarrow \infty]{} 0.$$

We conclude by writing $\mathcal{H}^\alpha(K_n) \leq \mathcal{H}^\alpha(K_n \setminus K) + \mathcal{H}^\alpha(K)$. □

We will need the following proposition, whose proof can be found in [1, Proposition 4.4.14].

Proposition 2.9. *Let $(K_n)_{n \in \mathbb{N}}, K$ be compact sets of a metric space (X, d) . If $K_n \xrightarrow[n \rightarrow \infty]{H} K$ then*

1. *for all $x \in K$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in K_n$, and $d(x_n, x) \xrightarrow[n \rightarrow \infty]{} 0$,*
2. *for all x such that $x = \lim_{k \rightarrow \infty} x_{n_k}$ where $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in K_n$, we have $x \in K$.*

Moreover, the converse is true if X is compact.

3 Product of BPI spaces

We will prove the following

Theorem 3.1. *Let (X, d) and (Y, ρ) be two BPI spaces of dimension α and β . If X and Y are both bounded or both unbounded, then the product $X \times Y$ endowed with a product distance is a BPI space of dimension $\alpha + \beta$.*

By a *product distance* on the product of two metric spaces $(X, d), (Y, \rho)$ we mean a distance denoted by $\|\cdot\|(d, \rho)$ and defined for all $(x, y), (x', y') \in X \times Y$ by

$$\|\cdot\|(d, \rho)((x, y), (x', y')) = \|(d(x, x'), \rho(y, y'))\|,$$

where $\|\cdot\|$ is a norm on \mathbb{R}^2 . By the equivalence of norms in finite-dimensional vector spaces, all the product distances are biLipschitz equivalent, and since being a BPI space is invariant under biLipschitz maps, it is sufficient to prove Theorem 3.1 for one specific product distance.

In this section, we fix two BPI spaces (X, d) and (Y, ρ) . Let α denote the dimension of X and β the dimension of Y . Let d_∞ be the product distance $\|\cdot\|(d, \rho)$, where $\|\cdot\|_\infty$ is the sup norm on \mathbb{R}^2 . We remark that if $(x, y) \in X \times Y$, and $r \in (0, \max\{\text{diam}_d X, \text{diam}_\rho Y\}]$, then

$$B_{d_\infty}((x, y), r) = B_d(x, r) \times B_\rho(y, r). \quad (3.1)$$

First of all, we need to prove that $(X \times Y, d_\infty)$ is Ahlfors regular of dimension $\alpha + \beta$. To do so, the Hausdorff measure $\mathcal{H}_{d_\infty}^{\alpha+\beta}$ seems to be natural. The problem is that this measure behaves badly with product sets. We define another measure m on the product $X \times Y$ that is better for measuring product sets. We denote by \mathcal{B}_X and \mathcal{B}_Y the Borel σ -algebra of X and Y . For all $A \subset X \times Y$, let

$$m(A) = \inf \left\{ \sum_{k=1}^{\infty} \mathcal{H}_d^\alpha(A_k) \mathcal{H}_\rho^\beta(B_k), A \subset \bigcup_{k=1}^{\infty} A_k \times B_k, A_k \in \mathcal{B}_X, B_k \in \mathcal{B}_Y \right\}.$$

In general it is not true that $\mathcal{H}^{\alpha+\beta} = m$.

Proposition 3.2. *The measure m is an outer measure on $X \times Y$ such that for all $A \in \mathcal{B}_X, B \in \mathcal{B}_Y$,*

$$m(A \times B) = \mathcal{H}_d^\alpha(A) \mathcal{H}_\rho^\beta(B).$$

Moreover, Borel sets of $(X \times Y, d_\infty)$ are m -measurable.

Proof. See [2, Theorem 6.2]. □

Proof of Theorem 3.1. In this proof, any constant that refers to properties of X or Y is denoted with either an X or a Y in the subscript. First we prove that $(X \times Y, d_\infty)$ is Ahlfors regular of dimension $\alpha + \beta$.

Suppose that X and Y are both unbounded. For all $(x, y) \in X \times Y$ and all $r > 0$,

$$m\left(B_{d_\infty}((x, y), r)\right) = m\left(B_d(x, r) \times B_\rho(y, r)\right) = \mathcal{H}_d^\alpha(B_d(x, r))\mathcal{H}_\rho^\beta(B_\rho(y, r)),$$

with (3.1) and Proposition 3.2, hence

$$(K_X K_Y)^{-1} r^{\alpha+\beta} \leq m\left(B_{d_\infty}((x, y), r)\right) \leq K_X K_Y r^{\alpha+\beta}. \quad (3.2)$$

By Lemma 2.2, $(X \times Y, d_\infty)$ is Ahlfors regular of dimension $\alpha + \beta$.

If X and Y are both bounded, then the estimate (3.2) holds for all $(x, y) \in X \times Y$ and all $r \in (0, \min\{\text{diam}_d X, \text{diam}_\rho Y\}]$. By modifying the Ahlfors regularity constants K_X, K_Y , it also holds for all $r \in (0, \max\{\text{diam}_d X, \text{diam}_\rho Y\}]$, thus $(X \times Y, d_\infty)$ is Ahlfors $(\alpha + \beta)$ -regular.

Fix now two points $(x, y), (x', y') \in X \times Y$ and two radii $r \in (0, \text{diam}_d(X)], t \in (0, \text{diam}_\rho(Y)]$. Let $A_X \subset B_d(x, r)$, $A_Y \subset B_\rho(y, r)$ be the two big pieces and $f_X : (A_X, r^{-1}d) \rightarrow (B_d(x', t), t^{-1}d)$, $f_Y : (A_Y, r^{-1}\rho) \rightarrow (B_\rho(y', t), t^{-1}\rho)$ the two C_X, C_Y -biLipschitz maps given by the definition of a BPI space. By Lemma 2.2, and Proposition 3.2 (A_X, A_Y are Borel sets), there exists a constant $\theta > 0$ such that $\mathcal{H}^{\alpha+\beta}(A_X \times A_Y) \geq \theta r^{\alpha+\beta}$. Finally, the map

$$f : (A_X \times A_Y, r^{-1}d_\infty) \rightarrow (B_{d_\infty}((x', y'), t), t^{-1}d_\infty)$$

defined by $f(x, y) = (f_X(x), f_Y(y))$ is a C -biLipschitz map, for a constant C depending only on C_X and C_Y . \square

We can now prove that (\mathbb{R}^N, d_s) is a BPI space of dimension $\alpha = \sum s_k^{-1}$.

Proposition 3.3. *The metric space (\mathbb{R}^N, d_s) is a BPI space.*

Proof. It is easy to check that if (X, d) is an unbounded BPI space of dimension α , then for $0 < s < 1$, (X, d^s) is an unbounded BPI space of dimension α/s . The space $(\mathbb{R}, |\cdot|)$ is an unbounded BPI space of dimension 1, thus $(\mathbb{R}, |\cdot|^{s_k})$ is an unbounded BPI space of dimension $1/s_k$ for all k . By Theorem 3.1, (\mathbb{R}^N, d_s) is a BPI space of dimension $\alpha = \sum s_k^{-1}$, since d_s is the ℓ^1 product distance of the distances $|\cdot|^{s_k}$. \square

4 Construction of a quotient semi-distance

We construct a semi-distance $d_{\mathcal{R}}$ on \mathbb{R} using a shortening technique, by following the article [7]. First we define an equivalence relation \mathcal{R} on \mathbb{R} in a self-similar way. This corresponds to the shortcuts. We then look at the quotient semi-distance $d_{\mathcal{R}}$. By construction, $d_{\mathcal{R}} \leq |\cdot|^s$, where s is the minimum snowflaking factor. For compactness reason, it is more convenient to work with the subset $[0, 1]$ of \mathbb{R} , endowed with the semi-distance $d_{\mathcal{R}}$. By using a theorem in [7], the quotient space $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is Ahlfors regular of dimension $\alpha = 1/s$. We finally prove that the space $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is a BPI space.

4.1 Motivation: the philosophy of shortcuts

In $(\mathbb{R}, |\cdot|^s)$, the triangle inequality can be improved if the points are chosen correctly. This is the general idea of the shortcuts' method. Let us explain this. In what follows, every ball is a ball for $|\cdot|^s$. When the centre of a ball of radius r does not matter, we just write B_r .

Let $x, y, p, q \in \mathbb{R}$ be four distinct points. By the triangle inequality,

$$|x - y|^s \leq |x - p|^s + |p - q|^s + |q - y|^s. \quad (4.1)$$

Without loss of generality, we may assume that $|x - p| \leq |y - q|$. Suppose that there exists $r > 0$ such that $q \in B(p, r)$ and $x, y \notin B(p, r)$. We write $|p - q|^s = C^s r$ where $C \in (0, 1)$. We have $|x - p| \geq r^{1/s} \geq (1 - C)r^{1/s}$ and $|y - q| \geq |y - p| - |p - q| \geq (1 - C)r^{1/s}$, so

$$|p - q| \leq \frac{C}{1 - C}|x - p|.$$

By using the inequality $(1 + t)^s \leq 1 + st$ for $t \geq 0$, we have

$$\begin{aligned} |x - y|^s &\leq (|x - p| + |p - q| + |q - y|)^s \\ &\leq \left(\frac{1}{1 - C}|x - p| + |q - y|\right)^s \\ &= |q - y|^s \left(1 + \frac{1}{1 - C} \frac{|p - x|}{|q - y|}\right)^s \\ &\leq |q - y|^s + \frac{s}{1 - C} |q - y|^{s-1} |p - x| \\ &\leq \frac{s}{1 - C} |x - p|^s + |q - y|^s. \end{aligned} \quad (4.2)$$

If $C \leq 1 - s$, then (4.2) leads to an improvement of (4.1): the term $|p - q|^s$ disappears. The ball $B(p, (1 - s)^s r)$ seems to be invisible when going from x to y . This motivates the introduction of a shortcut between p and q by identifying them.

Definition 4.1. A metric space (X, d) for which there exists $\lambda \in (0, 1)$ such that for all $r > 0$, and all balls B_r of radius r , there are two points $p, q \in B_r$ satisfying $d(p, q) \geq \lambda r$ and

$$d(x, y) \leq d(x, p) + d(q, y), \quad \forall x, y \notin B_r \quad (4.3)$$

is called a space with λ -invisible pieces. We say that $(p, q) \in B_r \times B_r$ is λ -invisible outside B_r .

We proved above that $(\mathbb{R}, |\cdot|^s)$ has λ -invisible pieces for all $0 < \lambda < (1 - s)^s$. In [7], it was proven that the Heisenberg group as well as any snowflake of an Ahlfors regular space have invisible pieces.

4.2 Construction of the shortcuts

Following [7], we construct an equivalence relation \mathcal{R} that corresponds to the shortcuts.

Let c be a fixed constant. Let \mathcal{N}_1 be a 4λ -separated set, and a $c\lambda$ -net for X , i.e.

$$\forall x, y \in \mathcal{N}_1 \text{ distinct, } d(x, y) \geq 4\lambda \text{ and } X = \bigcup_{x \in \mathcal{N}_1} B_d(x, c\lambda). \quad (4.4)$$

Then for all $x \in \mathcal{N}_1$, choose two points $p_x, q_x \in B_d(x, \lambda)$ such that (p_x, q_x) is λ -invisible outside $B_d(x, \lambda)$. Define the shortcuts of level 1 as $\mathcal{S}_1 = \{(p_x, q_x), x \in \mathcal{N}_1\} \cup \{(q_x, p_x), x \in \mathcal{N}_1\} \subset X \times X$. By induction, let \mathcal{N}_n be a $4\lambda^n$ -separated set that is also a $c\lambda^n$ -net of X , such that

$$\mathcal{N}_n \subset X \setminus \bigcup_{\substack{(p, q) \in \mathcal{S}_k \\ 1 \leq k \leq n-1}} \{p, q\}^{4\lambda^n}, \quad (4.5)$$

where $\{p, q\}^{4\lambda^n}$ is the $4\lambda^n$ -neighborhood of $\{p, q\}$, i.e. $\{p, q\}^{4\lambda^n} = \{x \in X, d(\{p, q\}, x) < 4\lambda^n\}$.

For all $x \in \mathcal{N}_n$, choose (p_x, q_x) λ -invisible outside $B_d(x, \lambda^n)$, and then define the level n shortcuts as $\mathcal{S}_n = \{(p_x, q_x), x \in \mathcal{N}_n\} \cup \{(q_x, p_x), x \in \mathcal{N}_n\} \subset X \times X$. Finally, define the set of all shortcuts:

$$\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}_n.$$

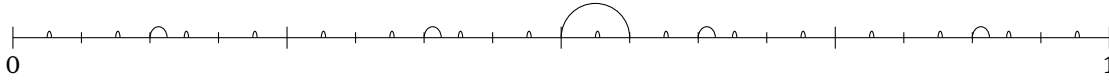


Figure 1: The shortcuts $\mathcal{S}_1, \mathcal{S}_2$ and \mathcal{S}_3 in $[0, 1]$ for $l = 2, h = 1/4$.

Definition 4.2. A set \mathcal{S} in $X \times X$ constructed as above is called a set of shortcuts. An element $(p, q) \in \mathcal{S}$ is called a shortcut between p and q . The integer n such that $(p, q) \in \mathcal{S}_n$ is called the level of the shortcut (p, q) .

With a set of shortcuts \mathcal{S} , we define an equivalence relation \mathcal{R} on X :

$$x\mathcal{R}y \Leftrightarrow (x, y) \in \mathcal{S} \text{ or } x = y. \quad (4.6)$$

The following proposition is proved in [7, Section 3.3].

Proposition 4.3. Let $\lambda \in (0, 1)$ and (X, d) be an Ahlfors α -regular space with λ -invisible pieces. If \mathcal{S} is a set of shortcuts on X and \mathcal{R} the equivalence relation defined as in (4.6), then the quotient metric space $(X/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is Ahlfors α -regular.

In our case, we apply Proposition 4.3 to $(X, d) = (\mathbb{R}, |\cdot|^s)$. It is an Ahlfors regular space of dimension $1/s$ with λ -invisible pieces, for all $0 < \lambda < (1 - s)^s$. We construct the set of shortcuts \mathcal{S} in a self-similar way, so that the space $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is a BPI space of dimension $\alpha = 1/s$.

Let $l \in \mathbb{N}, h = 1/2^l$ and $\mu = h^s$. For all $n \in \mathbb{N}$, we define the level n shortcuts (see Figure 1)

$$\mathcal{S}_n = \left\{ \left(h^{n-1} \left(m + \frac{1}{2} \right), h^{n-1} \left(m + \frac{1}{2} \right) + h^{n+1} \right), m \in \mathbb{Z} \right\} \subset \mathbb{R} \times \mathbb{R}.$$

Let $\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}_n$. We will see that for l, c large enough, \mathcal{S} is a set of shortcuts.

Following the notations defined above, we set $\mathcal{N}_n = \left\{ h^{n-1} \left(m + \frac{1}{2} \right), m \in \mathbb{Z} \right\}$. If $x, y \in \mathcal{N}_n$ are distinct, then $|x - y|^s \geq \mu^{n-1}$. Moreover, for all $x \in \mathbb{R}$, there exists $m \in \mathbb{Z}$ such that $|x - h^{n-1}(m + 1/2)|^s = \mu^{n-1} |x/h^{n-1} - 1/2 - m|^s \leq 2^{-s} \mu^{n-1}$. Since \mathcal{N}_n must be a $4\lambda^n$ -separated set and a $c\lambda^n$ -net for X , we obtain the inequalities

$$4\lambda^n \leq \mu^{n-1} \leq 2^s c\lambda^n. \quad (4.7)$$

If $x \in \mathcal{N}_n$, the closest point p to x for which there exists q such that $(p, q) \in \bigcup_{1 \leq j \leq n-1} \mathcal{S}_j$ satisfies $|x - p|^s = \mu^{n-1} (1/2 - h)^s = \mu^{n-2} (2^{l-1} - 1)^s$. The condition (4.5) implies that

$$4\lambda^n \leq \mu^{n-2} (2^{l-1} - 1)^s. \quad (4.8)$$

Finally, every couple $(p_m, q_m) \in \mathcal{S}_n$ has to be λ -invisible outside $B_{|\cdot|^s}(h^{n-1}(m + 1/2), \lambda^n)$, so

$$\lambda^{n+1} \leq \mu^{n+1} \leq (1 - s)^s \lambda^n. \quad (4.9)$$

We see that if $\lambda = \mu$ and l, c are sufficiently large such that

$$\frac{1}{2^s c} \leq \mu \leq \min \left\{ \frac{1}{4}, (1 - s)^s, \frac{1}{2} (2^{l-1} - 1)^{s/2} \right\},$$

then the conditions (4.7), (4.8) and (4.9) are satisfied. Let us henceforth fix the constants l and c . With this set of shortcuts \mathcal{S} , we define as above the equivalence relation \mathcal{R} on \mathbb{R} by

$$x\mathcal{R}y \Leftrightarrow (x, y) \in \mathcal{S} \text{ or } x = y.$$

By Proposition 4.3, the space $(\mathbb{R}/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is Ahlfors regular of dimension $\alpha = 1/s$.

In the sequel, it will be interesting to work on $[0, 1]$ endowed with this semi-distance $d_{\mathcal{R}}$. A priori, if $\mathcal{R}_{[0,1]}$ denotes the equivalence relation \mathcal{R} restricted on the subset $[0, 1] \times [0, 1]$ (that is one takes only shortcuts in $[0, 1]$), then $d_{\mathcal{R}} \leq d_{\mathcal{R}_{[0,1]}}$, because there are more itineraries for the relation \mathcal{R} than for the relation $\mathcal{R}_{[0,1]}$. Actually we will see in Corollary 4.9 that for all $x, y \in [0, 1]$, $d_{\mathcal{R}}(x, y) = d_{\mathcal{R}_{[0,1]}}(x, y)$.

By Proposition 4.3 and by what we observed above, the space $([0, 1]/d_{\mathcal{R}_{[0,1]}}, \bar{d}_{\mathcal{R}_{[0,1]}})$ is also Ahlfors regular of dimension α .

Until the end of the article, the symbol \mathcal{R} always denotes the equivalence relation on \mathbb{R} we just constructed.

4.3 The metric space $([0, 1], d_{\mathcal{R}})$ is a BPI space

In order to have compactness, we will prove that the subspace $[0, 1]$, endowed with $d_{\mathcal{R}}$ is a BPI space. To do so, we prove that $d_{\mathcal{R}} = d_{\mathcal{R}_{[0,1]}}$ on $[0, 1]$ (Corollary 4.9), thus $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is Ahlfors regular of dimension $\alpha = 1/s$. Then, we prove that any two balls possess big pieces that are biLipschitz equivalent for the rescaled distances.

The notion of BPI space has been defined for a metric space, but $d_{\mathcal{R}}$ is only a semi-distance on $[0, 1]$. Let $B(\pi(x), r) \subset ([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ be a ball. By definition, we have $B(\pi(x), r) = \pi(B(x, r))$ where $\pi : ([0, 1], d_{\mathcal{R}}) \rightarrow ([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is the canonical projection, and $B(x, r) \subset ([0, 1], d_{\mathcal{R}})$. Moreover, π is an isometry by definition. Notice also that a biLipschitz map $f : (A_1, d_{\mathcal{R}}) \rightarrow (A_2, d_{\mathcal{R}})$ with $A_1, A_2 \subset [0, 1]$, corresponds to a biLipschitz map $\tilde{f} : (\pi(A_1), \bar{d}_{\mathcal{R}}) \rightarrow (\pi(A_2), \bar{d}_{\mathcal{R}})$ satisfying $\{\tilde{f}(x)\} = \pi(f(\pi^{-1}(\{x\})))$. To prove that $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is a BPI space, it is thus sufficient to prove that there exist constants C, θ so that for each pair of balls $B(x, r), B(y, t)$ in $([0, 1], d_{\mathcal{R}})$, there is a closed subset $A \subset B(x, r)$ with $\mathcal{H}_{d_{\mathcal{R}}}^{\alpha}(A) \geq \theta r^{\alpha}$ and a C -biLipschitz map $f : (A, r^{-1}d_{\mathcal{R}}) \rightarrow (B(y, t), t^{-1}d_{\mathcal{R}})$.

Until the end of the section, each ball $B(x, r)$ is a ball for the semi-distance $d_{\mathcal{R}}$. For this paragraph, we introduce the following definition for notational convenience.

Definition 4.4. An interval $I = [a, b]$ is called an interval without shortcut at the ends if for all $x \in \mathbb{R}$, $(a, x) \notin \mathcal{S}$ and $(x, b) \notin \mathcal{S}$.

Lemma 4.5. For all $x \in \mathbb{R}$, and $r > 0$, there exists $I \subset B(x, r)$ an interval of the form $I = [h^n m, h^n(m+1)]$, without shortcut at the ends, where $n = 1 + \lceil \frac{\ln r}{\ln \mu} \rceil$ and $m \in \mathbb{Z}$.

Proof. Since $d_{\mathcal{R}} \leq |\cdot|^s$, we have $]x-r^{1/s}, x+r^{1/s}[\subset B(x, r)$. By definition of n , $h^{n-1} \leq r^{1/s}$, so $B(x, r)$ contains an interval of (Euclidean) length h^{n-1} . In this interval, we can find $h^{-1} - 1$ intervals of the form $[h^n m, h^n(m+1)]$, where $m \in \mathbb{Z}$. Among these intervals, at least one suits. \square

Let us make the following easy remark, that will be useful later: if $(p, q) \in \mathcal{S}$ with $p < q$, is a shortcut of level less than or equal to n , then there exists $m \in \mathbb{Z}$ such that $p = h^n m$ (and then $q = h^{n+1}(m^l + 1)$). Moreover, the converse is true: if $(p, q) \in \mathcal{S}$, $p < q$, and $p = h^n m$, then the level of the shortcut (p, q) is less than or equal to n . We can also say something about q : if $(p, q) \in \mathcal{S}$, $p < q$ and $q = h^{n+1} m$, then the level of the shortcut (p, q) is less than or equal to n .

Lemma 4.6. Let $n \geq 0$, $m \in \mathbb{Z}$ and $I = [h^n m, h^n(m+1)]$ be an interval without shortcut at the ends. Then, I contains no shortcut of level less than or equal to n . Moreover, for all $(p, q) \in \mathcal{S}$, $p \in I \Leftrightarrow q \in I$.

Proof. The fact that I contains no shortcut of level less than or equal to n is easy with the remark made above, and with the assumption that I is an interval without shortcut at the ends.

Then, let $(p, q) \in \mathcal{S}$, $p < q$ be a shortcut of level $n' > n$. Write $p = h^{n'} m'$, $q = h^{n'} m' + h^{n'+1}$ with $m' \in \mathbb{Z}$. Suppose that $p \in I$, that is $h^n m < h^{n'} m' < h^n(m+1)$, i.e. $m < h^{n'-n} m' < m+1$. If one cuts the interval

$[m, m + 1]$ into equal intervals of length $h^{n'-n}$, then one sees that $|m + 1 - h^{n'-n}m'| \geq h^{n'-n} > h^{n'-n+1}$, so $m < h^{n'-n}m' + h^{n'-n+1} < m + 1$, which means that $q \in I$. A similar argument proves that $q \in I \Rightarrow p \in I$. \square

Lemma 4.7. *Let $n \geq 0$, $m \in \mathbb{Z}$ and $I = [h^n m, h^n(m + 1)]$ be an interval without shortcut at the ends. Then, either $[h^n(m - 1), h^n m]$ or $[h^n(m + 1), h^n(m + 2)]$ is an interval without shortcut at the ends.*

Proof. Suppose not. Then, there exist $x, y \in \mathbb{R}$ such that $(h^n(m - 1), x) \in \mathcal{S}$ and $(y, h^n(m + 2)) \in \mathcal{S}$. Many variations are possible:

1. If $h^n(m - 1) < x$ and $h^n(m + 2) < y$, then these two shortcuts have level less than or equal to n .
2. If $h^n(m - 1) < x$ and $y < h^n(m + 2)$, then the level of $(h^n(m - 1), x)$ is less than or equal to n and the level of $(y, h^n(m + 2))$ is less than or equal to $n - 1$.
3. If $x < h^n(m - 1)$ and $h^n(m + 2) < y$, then the level of $(x, h^n(m - 1))$ is less than or equal to n and the level of $(h^n(m + 2), y)$ is less than or equal to n .
4. If $x < h^n(m - 1)$ and $y < h^n(m + 2)$, then these two shortcuts have level less than or equal to $n - 1$.

In each case, this is impossible, because the distance between the two shortcuts is too small. \square

Lemma 4.8. *Let $n \geq 0$, $m \in \mathbb{Z}$ and $I = [h^n m, h^n(m + 1)]$ be an interval without shortcut at the ends. Denote by \mathcal{R}_I the equivalence relation on I which is the restriction of \mathcal{R} to the subset $I \times I$. Then, for all $x, y \in I$,*

$$d_{\mathcal{R}}(x, y) = d_{\mathcal{R}_I}(x, y).$$

Proof. The inequality $d_{\mathcal{R}} \leq d_{\mathcal{R}_I}$ is easy. Let us prove the other inequality. Let $x, y \in I$. Let $(x_0, y_0, \dots, x_n, y_n)$ be an itinerary from x to y . Suppose that this itinerary gets out of I . We will construct another itinerary that stays in I and that is shorter. Let $a = \min \{k \in \{0, \dots, n\}, x_k \notin I\}$ and $b = \max \{k \in \{0, \dots, n\}, x_k \notin I\}$. Since $x \in I$ and $x \mathcal{R} x_0$, by Lemma 4.6, $a \geq 1$.

We remark that $y_{a-1} \notin I$ and $y_b \in I$. Indeed, $y_{a-1} \mathcal{R} x_a$, but by Lemma 4.6, since $x_a \notin I$, we have $y_{a-1} \notin I$. The same argument works for y_b . We consider separately two cases.

1. If y_{a-1} and x_b are in the same connected component of $\mathbb{R} \setminus I$, say in $(-\infty, h^n m)$, then

$$|x_{a-1} - y_b| \leq |x_{a-1} - h^n m| + |h^n m - y_b| < |x_{a-1} - y_{a-1}| + |x_b - y_b|.$$

Thus, the itinerary

$$(x_0, y_0, \dots, x_{a-1}, y_b, \dots, x_n, y_n)$$

stays in I , and is shorter than $(x_0, y_0, \dots, x_n, y_n)$. Similar construction works if the points are in $(h^n(m + 1), \infty)$.

2. If y_{a-1} and x_b are not in the same connected component of $\mathbb{R} \setminus I$, then either there exists $j \in \{a, \dots, b - 1\}$ such that x_j and y_j are in different components, or there exists $j \in \{a, \dots, b\}$ such that y_{j-1} and x_j are in different components. In the first case, we easily notice that

$$|x_{a-1} - y_b| \leq h^n < |x_j - y_j|$$

and thus again

$$(x_0, y_0, \dots, x_{a-1}, y_b, \dots, x_n, y_n)$$

stays in I , and is shorter than $(x_0, y_0, \dots, x_n, y_n)$.

In the second case, the itinerary follows a shortcut that steps over I , see Figure 2. We may suppose that $y_{a-1} < h^n m < h^n(m + 1) < x_b$ and that also the step happens from the left to right, that is, there exists an integer $j \in \{a, \dots, b\}$ such that $y_{j-1} < h^n m < h^n(m + 1) < x_j$. Let J be an interval without shortcut at the ends, adjacent to I , given by Lemma 4.7. Without loss of generality, we may suppose that $J = I + h^n$. The following remark is easy: for all $(p, q) \in \mathcal{S}$, $(p, q) \in I \times I \Leftrightarrow (p + h^n, q + h^n) \in J \times J$.

Since J has no shortcut at the ends, there exists i such that $x_i \notin J \cup I$, $y_i \in J$. Define the new itinerary in three parts.

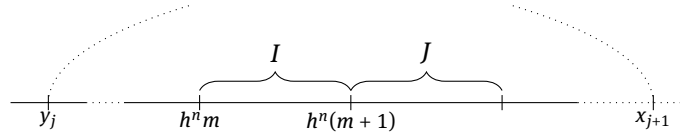


Figure 2: The shortcut (y_j, x_{j+1}) across I .

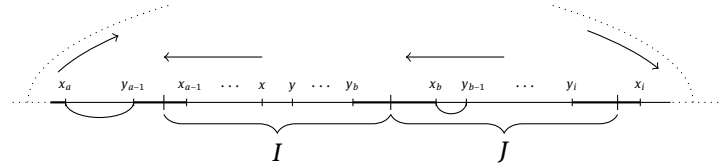


Figure 3: The original itinerary.

- (i) The first part is $(x_0, y_0, \dots, x_{a-1}, h^n m)$.
- (ii) The second part is $(h^n m, x_b - h^n, y_{b-1} - h^n, x_{b-1} - h^n, \dots, y_i - h^n, h^n(m + 1))$.
- (iii) The last part is $(h^n(m + 1), y_b, \dots, x_n, y_n)$.

In Figures 3 and 4, we represent the original and modified itineraries, the thickest parts are the parts of the itinerary where one has to walk, and the arcs are the shortcuts.

This modified itinerary stays in I , and is shorter than the original one. This construction works similarly if $x_b < h^n m < h^n(m + 1) < y_{a-1}$, or if $J = I - h^n$.

We have proved that for any itinerary from x to y , there exists a shorter itinerary between x and y that stays in I . Therefore, $d_{\mathcal{R}_I} \leq d_{\mathcal{R}}$. □

Corollary 4.9. For all $x, y \in [0, 1]$, $d_{\mathcal{R}}(x, y) = d_{\mathcal{R}_{[0,1]}}(x, y)$.

We can now prove that $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is a BPI space of dimension $\alpha = 1/s$. By Proposition 4.3, we know that it is Ahlfors regular of dimension α . Let $x, y \in [0, 1]$, $r, t \in (0, \text{diam}_{d_{\mathcal{R}}}([0, 1]))$, $n = 1 + \lceil \ln r / \ln \mu \rceil$ and $n' = 1 + \lceil \ln t / \ln \mu \rceil$. Let $I = [h^n m, h^n(m + 1)] \subset B(x, r)$ and $I' = [h^{n'} m', h^{n'}(m' + 1)] \subset B(y, t)$ be given by Lemma 4.5. We remark that $n, n' \geq 1$, since $r, t \leq 1$.

Define $f : I \rightarrow I'$ by $f(x) = h^{n'-n}x + h^{n'}(m' - m)$. Then, f is bijective and compatible with the shortcuts, that is $x\mathcal{R}_I y \Leftrightarrow f(x)\mathcal{R}_{I'} f(y)$, and $|f(x) - f(y)|^s = \mu^{n'-n}|x - y|^s$, so by Lemma 2.7, for all $x, y \in I$,

$$d_{\mathcal{R}_{I'}}(f(x), f(y)) = \mu^{n'-n}d_{\mathcal{R}_I}(x, y),$$

thus by Lemma 4.8,

$$d_{\mathcal{R}}(f(x), f(y)) = \mu^{n'-n}d_{\mathcal{R}}(x, y).$$

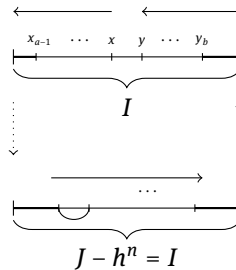


Figure 4: The modified itinerary, which stays in I .

The inequalities $\frac{\ln(t/r)}{\ln \mu} - 1 \leq n' - n \leq \frac{\ln(t/r)}{\ln \mu} + 1$ imply that $\frac{t}{r}\mu \leq \mu^{n'-n} \leq \frac{t}{r}\mu^{-1}$. Therefore, $f : (I, r^{-1}d_{\mathcal{R}}) \rightarrow (I', t^{-1}d_{\mathcal{R}})$ is a μ^{-1} -biLipschitz map.

We finally have to estimate $\mathcal{H}_{d_{\mathcal{R}}}^{\alpha}(I)$. The map $g : I \rightarrow [0, 1]$ defined by $g(x) = h^{-n}x - m$ is a bijection that satisfies

$$d_{\mathcal{R}}(g(x), g(y)) = \mu^{-n}d_{\mathcal{R}}(x, y),$$

by Lemma 2.7 and Lemma 4.8 since g is bijective and compatible with the shortcuts. Then, $\mathcal{H}_{d_{\mathcal{R}}}^{\alpha}(I) = \mu^{n\alpha}\mathcal{H}_{d_{\mathcal{R}}}^{\alpha}([0, 1])$. By definition of n , $\mu^{n\alpha} \geq \mu^{2\alpha}r^{\alpha}$. Finally, if we set $\theta = \mu^{2\alpha}\mathcal{H}_{d_{\mathcal{R}}}^{\alpha}([0, 1])$, then $\mathcal{H}_{d_{\mathcal{R}}}^{\alpha}(I) \geq \theta r^{\alpha}$.

We have proved that $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ is a BPI space of dimension $\alpha = 1/s$.

5 Lipschitz functions between $([0, 1], d_{\mathcal{R}})$ and $(\mathbb{R}, |\cdot|^s)$.

The following proposition deals with Lipschitz functions from $([0, 1], d_{\mathcal{R}})$ to $(\mathbb{R}, |\cdot|^s)$.

Proposition 5.1. *Any Lipschitz function $f : ([0, 1], d_{\mathcal{R}}) \rightarrow (\mathbb{R}, |\cdot|^s)$ is constant.*

Proof. Suppose not. Let $f : ([0, 1], d_{\mathcal{R}}) \rightarrow (\mathbb{R}, |\cdot|^s)$ be a non constant Lipschitz map. Since $d_{\mathcal{R}} \leq |\cdot|^s$, $f : ([0, 1], |\cdot|^s) \rightarrow (\mathbb{R}, |\cdot|^s)$ is Lipschitz, and hence $f : ([0, 1], |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$ is also a Lipschitz map. By the Rademacher Theorem, f is differentiable almost everywhere. Since f is a non constant Lipschitz map, f' does not vanish almost everywhere. Hence, there exists a subset A of $[0, 1]$ with positive measure such that for all $x \in A$, $f'(x) \neq 0$. For $x_0 \in A$ and $p \in [0, 1]$,

$$\left| \frac{1}{\varepsilon} \left(f(x_0 + \varepsilon p) - f(x_0) \right) - f'(x_0)p \right|^s \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (5.1)$$

Let $p, q \in [0, 1]$ with $p\mathcal{R}q$ and $p \neq q$, and $\varepsilon > 0$ sufficiently small so that $x_0 + \varepsilon p, x_0 + \varepsilon q \in [0, 1]$. Then,

$$\begin{aligned} |f'(x_0)p - f'(x_0)q|^s &\leq \left| f'(x_0)p - \frac{1}{\varepsilon} \left(f(x_0 + \varepsilon p) - f(x_0) \right) \right|^s \\ &\quad + \varepsilon^{-s} |f(x_0 + \varepsilon p) - f(x_0 + \varepsilon q)|^s + \left| \frac{1}{\varepsilon} \left(f(x_0 + \varepsilon q) - f(x_0) \right) - f'(x_0)q \right|^s. \end{aligned} \quad (5.2)$$

By (5.1), the first and third terms of (5.2) tend to 0 when $\varepsilon \rightarrow 0$. Since f is Lipschitz, there exists $L > 0$ such that $\varepsilon^{-s} |f(x_0 + \varepsilon p) - f(x_0 + \varepsilon q)|^s \leq L\varepsilon^{-s}d_{\mathcal{R}}(x_0 + \varepsilon p, x_0 + \varepsilon q)$.

For all $n \in \mathbb{N}$, $m \in \mathbb{Z}$, the map $x \rightarrow h^n(x + m)$ from $(\mathbb{R}, d_{\mathcal{R}})$ to $(\mathbb{R}, d_{\mathcal{R}})$ is a h^{sn} -Lipschitz map. Indeed, it follows from Lemma 2.7, since $x\mathcal{R}y \Rightarrow h^n(x + m)\mathcal{R}h^n(y + m)$. Then,

$$\begin{aligned} h^{-sn}d_{\mathcal{R}}(x_0 + h^n p, x_0 + h^n q) &\leq d_{\mathcal{R}}(x_0/h^n + p - m, x_0/h^n + q - m) \\ &\leq d_{\mathcal{R}}(x_0/h^n - m + p, p) + d_{\mathcal{R}}(p, q) + d_{\mathcal{R}}(q, q + x_0/h^n - m). \end{aligned} \quad (5.3)$$

Since $d_{\mathcal{R}} \leq |\cdot|^s$, the first and third terms of (5.3) are less than or equal to

$$\left| \frac{x_0}{h^n} - m \right|^s = \left| 2^{ln}x_0 - m \right|^s.$$

The following is a well-known fact of base-2 expansion of real numbers: There exists a Borel set $U \subset [0, 1]$ with $\mathcal{L}^1(U) = 1$ such that for every point $x \in U$, there exists a sequence $(n_i)_{i \in \mathbb{N}}$ of integers such that $n_i \rightarrow \infty$ when $i \rightarrow \infty$, and for all $i \in \mathbb{N}$, $x_{ln_i+1} = \dots = x_{2ln_i} = 0$, where $x = \sum_{k \geq 0} x_k 2^{-k}$ is the standard binary representation of x . In fact, if $x \in U$, then for all $i \in \mathbb{N}$,

$$\begin{aligned} 2^{ln_i}x &= m_i + \sum_{k=2ln_i+1}^{\infty} x_k 2^{ln_i-k}, \quad \text{where } m_i \in \mathbb{Z} \\ &\leq m_i + 2^{ln_i} \sum_{k=2ln_i+1}^{\infty} 2^{-k} \end{aligned}$$

$$= m_i + \frac{1}{2^{ln_i}},$$

so $2^{ln_i} - m_i \xrightarrow{i \rightarrow \infty} 0$.

Let us finish the proof of the proposition. Since A has positive measure, there exists a point $x_0 \in A \cap U$. With the inequalities (5.2), (5.3), choosing $n = n_i$, $m = m_i$ and letting $i \rightarrow \infty$, we can conclude that

$$|f'(x_0)p - f'(x_0)q|^s \leq Ld_{\mathcal{R}}(p, q),$$

but this is impossible since $f'(x_0) \neq 0$ (recalling that $p\mathcal{R}q$ and $p \neq q$ so that $d_{\mathcal{R}}(p, q) = 0$ but $|p - q| > 0$). Thus, f is constant. \square

6 Blow-up

In the sequel, topological properties (closed sets, compact sets, etc) are related to the Euclidean topology on \mathbb{R}^N , which is the same as the topology induced by d_s . The balls will be balls for the distance d_s and the measure \mathcal{H}^α will always refer to $\mathcal{H}_{d_s}^\alpha$, where $\alpha = \sum s_k^{-1}$ is the dimension of the BPI space (\mathbb{R}^N, d_s) . Recall that s is the minimum snowflaking factor, and L the minimally snowflaked layer. We define a semi-distance $d_{s,\mathcal{R}}$ on \mathbb{R}^N by modifying d_s on the minimally snowflaked layer L :

$$\forall x, y \in \mathbb{R}^N, d_{s,\mathcal{R}}(x, y) = \sum_{k \in L} d_{\mathcal{R}}(x_k, y_k) + \sum_{k \notin L} |x_k - y_k|^{s_k},$$

where $d_{\mathcal{R}}$ is the semi-distance defined on \mathbb{R} in Section 4, by the shortcuts method. The topology induced by $d_{s,\mathcal{R}}$, for which a basis is given by the open balls $\{y \in \mathbb{R}^N, d_{s,\mathcal{R}}(x, y) < r\}$, is the Euclidean topology.

Proposition 6.1. *The quotient space $([0, 1]^N/d_{s,\mathcal{R}}, \bar{d}_{s,\mathcal{R}})$ is a BPI space of dimension α .*

Proof. The quotient space $([0, 1]^N/d_{s,\mathcal{R}}, \bar{d}_{s,\mathcal{R}})$ is the product of $([0, 1], |\cdot|^{s_k})$ for all $k \notin L$ and $([0, 1]/d_{\mathcal{R}}, \bar{d}_{\mathcal{R}})$ for all $k \in L$, which are bounded BPI spaces. Thus, we may apply Theorem 3.1 to conclude. \square

We want to prove the following theorem, by a blow-up technique.

Theorem 6.2. *Let $A \subset [0, 1]^N$ be a closed subset, and $g : (A, d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ a Lipschitz map such that $\mathcal{H}^\alpha(g(A)) > 0$. Then, there exists a Lipschitz map $f : ([0, 1]^N, d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ such that $\mathcal{H}^\alpha(f([0, 1]^N)) > 0$.*

From now on, we fix a closed (and thus compact) subset A of $[0, 1]^N$, and an L -Lipschitz map $g : (A, d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$.

For $\delta > 0$, we define $\text{dil}_\delta(z) = (\delta^{1/s_1}z_1, \dots, \delta^{1/s_N}z_N)$ where $z = (z_1, \dots, z_N) \in \mathbb{R}^N$. The map dil_δ is a similitude for d_s : for all $x, y \in \mathbb{R}^N$,

$$d_s(\text{dil}_\delta(x), \text{dil}_\delta(y)) = \delta d_s(x, y).$$

For all $j = (j_1, \dots, j_N) \in \mathbb{Z}^N$ and $i \in \mathbb{N}$, we set

$$I_j^i = \text{dil}_{h^{is}}([0, 1]^N + j) = \prod_{k=1}^N [h^{i \frac{s}{s_k}} j_k, h^{i \frac{s}{s_k}} (j_k + 1)].$$

The I_j^i will be called "cubes" in the sequel even though it would be more correct to call them parallelepipeds. The family $\{I_j^i, i \in \mathbb{N}, j \in \mathbb{Z}^N\}$ is not a family of nested cubes, which means that two cubes $I_j^i, I_{j'}^{i'}$ with $i \neq i'$ might overlap.

If we set $f_{i,j}(x) = \text{dil}_{h^{is}}(x + j)$, then $I_j^i = f_{i,j}([0, 1]^N)$. We also define

$$\text{Adm} = \left\{ (i, j) \in \mathbb{N} \times \mathbb{Z}^N, I_j^i \subset [0, 1]^N \text{ and } f_{i,j} : ([0, 1]^N, d_{s,\mathcal{R}}) \rightarrow (I_j^i, h^{-is}d_{s,\mathcal{R}}) \text{ is an isometry} \right\},$$

the set of indices for which the corresponding cube is included in the unit cube and is similar to it for the distance $d_{s,\mathcal{R}}$.

Recall that by construction of $d_{\mathcal{R}}$, for all $(i, j) \in \mathbb{N} \times \mathbb{Z}^N$ such that $I_j^i \subset [0, 1]^N$, there exists $j' \in \mathbb{Z}^N$ such that $(i+1, j') \in \text{Adm}$ and $I_{j'}^{i+1} \subset I_j^i$.

For all $(i, j) \in \text{Adm}$, such that $A \cap I_j^i \neq \emptyset$, choose $z_{i,j} \in A \cap I_j^i$ and define

$$\begin{aligned} g_{i,j} : (f_{i,j}^{-1}(A \cap I_j^i), d_{s,\mathcal{R}}) &\longrightarrow (\mathbb{R}^N, d_s) \\ x &\longmapsto \text{dil}_{h^{-is}}(g(f_{i,j}(x)) - g(z_{i,j})). \end{aligned}$$

Lemma 6.3. *The maps $\{g_{i,j}, (i, j) \in \text{Adm}\}$ are Lipschitz, with uniformly bounded Lipschitz constants.*

Proof. Let $(i, j) \in \text{Adm}$ such that $A \cap I_j^i \neq \emptyset$. For all $x, y \in f_{i,j}^{-1}(A \cap I_j^i)$,

$$\begin{aligned} d_s(g_{i,j}(x), g_{i,j}(y)) &= h^{-is} d_s(g(f_{i,j}(x)), g(f_{i,j}(y))) \\ &\leq Lh^{-is} d_{s,\mathcal{R}}(f_{i,j}(x), f_{i,j}(y)) \\ &\leq Ld_{s,\mathcal{R}}(x, y). \end{aligned} \quad \square$$

For any set $E \subset \mathbb{R}^N$, all $m \in \mathbb{N}$ and all indices $i \in \mathbb{N}$, we set

$$\mathcal{E}_m^i(E) = \left\{ j \in \mathbb{Z}^N, \frac{\mathcal{H}^\alpha(E \cap I_j^i)}{\mathcal{H}^\alpha(I_j^i)} < 1 - \frac{1}{m} \right\}.$$

In order to prove Theorem 6.2, we need to find a sequence $(i_m, j_m)_{m \in \mathbb{N}} \in \text{Adm}^{\mathbb{N}}$ that satisfies two properties, explained in the following proposition.

Proposition 6.4. *There exists $c > 0$ such that for all $m \in \mathbb{N}$, there exists $(i_m, j_m) \in \text{Adm}$ such that*

$$j_m \notin \mathcal{E}_m^{i_m}(A), \quad \text{i.e.} \quad \frac{\mathcal{H}^\alpha(A \cap I_{j_m}^{i_m})}{\mathcal{H}^\alpha(I_{j_m}^{i_m})} \geq 1 - \frac{1}{m}, \quad (6.1)$$

and

$$h^{-i_m s \alpha} \mathcal{H}^\alpha(g(A \cap I_{j_m}^{i_m})) \geq c. \quad (6.2)$$

The property (6.1) will imply that the sequence of compact sets $(f_{i_m, j_m}^{-1}(A \cap I_{j_m}^{i_m}))_{m \in \mathbb{N}}$ converges to $[0, 1]^N$ in the Hausdorff distance, whereas the property (6.2) will imply that $\mathcal{H}^\alpha(f([0, 1]^N)) > 0$, where f is the blow-up map. The proof of Proposition 6.4 requires some lemmas.

The next lemma proves that in a small ball $B(x, r)$ where x is a point of density of a set E in \mathbb{R}^N , a "good" cover of $B(x, r)$ by cubes has a small number of cubes that have density in E not close to 1.

Lemma 6.5. *Let x be a point of density for a subset E of \mathbb{R}^N . For all $\varepsilon > 0$ and all $m \in \mathbb{N}$, there exists $r_0 > 0$ such that for all $r \in (0, r_0)$, for all $i \in \mathbb{N}$ and all coverings $\{I_j^i\}_{j \in J^i}$ of $B(x, r)$ such that*

$$\frac{\mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right)}{\mathcal{H}^\alpha(B(x, r))} \leq 1 + \frac{\varepsilon}{m},$$

then

$$0 \leq \frac{\#\{J^i \cap \mathcal{E}_m^i(E)\}}{\#J^i} \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$, $m \in \mathbb{N}$ and $\eta > 0$. Since x is a point of density for E , there exists $r_0 > 0$ such that for all $r \in (0, r_0)$,

$$1 - \eta \leq \frac{\mathcal{H}^\alpha(E \cap B(x, r))}{\mathcal{H}^\alpha(B(x, r))} \leq 1.$$

Fix $r \in (0, r_0)$. Let $i \in \mathbb{N}$ and $\{I_j^i\}_{j \in J^i}$ be a covering of $B(x, r)$ such that

$$\frac{\mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right)}{\mathcal{H}^\alpha(B(x, r))} \leq 1 + \frac{\varepsilon}{m}, \quad (6.3)$$

then,

$$\begin{aligned} (1 - \eta)\mathcal{H}^\alpha(B(x, r)) &\leq \mathcal{H}^\alpha(E \cap B(x, r)) \\ &\leq \mathcal{H}^\alpha\left(E \cap \bigcup_{j \in J^i} I_j^i\right) \\ &\leq \sum_{j \in J^i} \mathcal{H}^\alpha(E \cap I_j^i) \\ &= \sum_{j \in J^i \cap \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E \cap I_j^i) + \sum_{j \in J^i \setminus \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E \cap I_j^i) \\ &\leq (1 - 1/m) \sum_{j \in J^i \cap \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(I_j^i) + \sum_{j \in J^i \setminus \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(I_j^i). \end{aligned}$$

For all $j \in J^i$, $\mathcal{H}^\alpha(I_j^i) = \frac{1}{\#J^i} \mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right)$. Using $\#J^i = \#(J^i \cap \mathcal{E}_m^i(E)) + \#(J^i \setminus \mathcal{E}_m^i(E))$ we get

$$\begin{aligned} (1 - \eta)\mathcal{H}^\alpha(B(x, r)) &\leq \frac{1}{\#J^i} \mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right) \left((1 - 1/m) \#(J^i \cap \mathcal{E}_m^i(E)) + \#(J^i \setminus \mathcal{E}_m^i(E)) \right) \\ &\leq \mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right) \left(1 - \frac{\#(J^i \cap \mathcal{E}_m^i(E))}{m\#J^i} \right). \end{aligned}$$

Using now the inequality (6.3), we get

$$(1 - \eta) \leq (1 + \varepsilon/m) \left(1 - \frac{\#(J^i \cap \mathcal{E}_m^i(E))}{m\#J^i} \right)$$

which is the same as

$$\frac{\#(J^i \cap \mathcal{E}_m^i(E))}{\#J^i} \leq m \left(1 - \frac{1 - \eta}{1 + \varepsilon/m} \right).$$

Set $\eta = (\varepsilon/m)^2$ to get the lemma. \square

With this lemma, it is easy to get local information on the sum of the measures of all the $E \cap I_j^i$ such that I_j^i has density in E not close to 1.

Corollary 6.6. *Let x be a point of density for a subset E of \mathbb{R}^N . For all $\varepsilon > 0$, and all $m \in \mathbb{N}$, there exists $r_0 > 0$ such that for all $r \in (0, r_0)$, for all $i \in \mathbb{N}$ and all coverings $\{I_j^i\}_{j \in J^i}$ of $B(x, r)$ such that*

$$\frac{\mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right)}{\mathcal{H}^\alpha(B(x, r))} \leq 1 + \frac{\varepsilon}{m},$$

then

$$\sum_{j \in J^i \cap \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E \cap I_j^i) \leq \varepsilon \left(1 + \frac{\varepsilon}{m} \right) \mathcal{H}^\alpha(B(x, r)).$$

Proof. Let $\varepsilon > 0$, $m \in \mathbb{N}$. Let $r_0 > 0$ be given by Lemma 6.5. For all $r \in (0, r_0)$, all $i \in \mathbb{N}$ and all coverings $\{I_j^i\}_{j \in J^i}$ such that

$$\frac{\mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right)}{\mathcal{H}^\alpha(B(x, r))} \leq 1 + \frac{\varepsilon}{m},$$

we have

$$\begin{aligned}
\sum_{j \in J^i \cap \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E \cap I_j^i) &\leq \sum_{j \in J^i \cap \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(I_j^i) \\
&= \frac{1}{\#J^i} \mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right) \#J^i \cap \mathcal{E}_m^i(E) \\
&\leq \varepsilon \mathcal{H}^\alpha\left(\bigcup_{j \in J^i} I_j^i\right) \quad \text{by Lemma 6.5} \\
&\leq \varepsilon \left(1 + \frac{\varepsilon}{m}\right) \mathcal{H}^\alpha(B(x, r)). \quad \square
\end{aligned}$$

By a compactness argument, we can now deduce global information on the sum of the measures of the $E \cap I_j^i$ where I_j^i is a cube with density in E not close to 1.

Lemma 6.7. *Suppose that E is a compact subset of \mathbb{R}^N such that $\mathcal{H}^\alpha(E) > 0$. Then, for all $\varepsilon > 0$ and all $m \in \mathbb{N}$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,*

$$\sum_{j \in \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E \cap I_j^i) \leq \varepsilon.$$

Proof. Take a bounded set K compactly containing E . Let $\varepsilon > 0$ and $m \in \mathbb{N}$. By Ahlfors-regularity there exists a constant $C > 0$ such that $\mathcal{H}^\alpha(B(x, 3r)) \leq C \mathcal{H}^\alpha(B(x, r))$ for all $x \in \mathbb{R}^N$ and all $r > 0$. Let $\delta > 0$ be such that

$$\varepsilon = \delta + C\delta \left(1 + \frac{\delta}{m}\right) \mathcal{H}^\alpha(K).$$

Denote by E' the set of points of density in E . Recall that $\mathcal{H}^\alpha(E \setminus E') = 0$. By inner regularity of \mathcal{H}^α , there exists a compact set $E'' \subset E'$ such that $\mathcal{H}^\alpha(E' \setminus E'') \leq \delta$. For all $x \in E''$, there exists $r_0(x) > 0$ given by Lemma 6.5. For all $x \in E''$, choose $r_x \in (0, r_0(x)/3)$ such that $B(x, r_x) \subset K$. Then, $\{B(x, r_x)\}_{x \in E''}$ covers E'' . By compactness, there exists a finite subfamily $\{B(x_k, r_k)\}_{1 \leq k \leq n}$ of $\{B(x, r_x)\}_{x \in E''}$ that covers E'' . By Vitali Covering Lemma, we can extract another subfamily of disjoint balls, say $\{B(x_k, r_k)\}_{1 \leq k \leq n_0}$ such that

$$E'' \subset \bigcup_{k=1}^{n_0} B(x_k, 3r_k).$$

Then choose i_0 sufficiently large such that for all $i \geq i_0$ and all $1 \leq k \leq n_0$,

$$\frac{\mathcal{H}^\alpha\left(\bigcup_{j \in J_k^i} I_j^i\right)}{\mathcal{H}^\alpha(B(x_k, 3r_k))} \leq 1 + \frac{\delta}{m},$$

where $J_k^i = \{j \in \mathbb{Z}^N, B(x_k, 3r_k) \cap I_j^i \neq \emptyset\}$. Then,

$$\begin{aligned}
\sum_{j \in \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E \cap I_j^i) &\leq \mathcal{H}^\alpha(E \setminus E') + \mathcal{H}^\alpha(E' \setminus E'') + \sum_{j \in \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E'' \cap I_j^i) \\
&\leq \delta + \sum_{k=1}^{n_0} \sum_{j \in J_k^i \cap \mathcal{E}_m^i(E)} \mathcal{H}^\alpha(E'' \cap I_j^i) \\
&\leq \delta + \sum_{k=1}^{n_0} \delta \left(1 + \frac{\delta}{m}\right) \mathcal{H}^\alpha(B(x_k, 3r_k)) \quad \text{by Corollary 6.6} \\
&\leq \delta + C\delta \left(1 + \frac{\delta}{m}\right) \sum_{k=1}^{n_0} \mathcal{H}^\alpha(B(x_k, r_k)) \quad \text{by Ahlfors regularity} \\
&\leq \delta + C\delta \left(1 + \frac{\delta}{m}\right) \mathcal{H}^\alpha\left(\bigcup_{k=1}^{n_0} B(x_k, r_k)\right) \quad \text{since the balls are disjoint}
\end{aligned}$$

$$\leq \delta + C\delta \left(1 + \frac{\delta}{m}\right) \mathcal{H}^\alpha(K) = \varepsilon,$$

which proves the lemma. □

Proof of Proposition 6.4. Suppose that Proposition 6.4 is false. For all $c > 0$, there exists $m \in \mathbb{N}$ such that for all $(i, j) \in \text{Adm}$, either $j \in \mathcal{E}_m^i(A)$ or $h^{-is\alpha} \mathcal{H}^\alpha(g(A \cap I_j^i)) < c$.

Fix $c > 0$ and the corresponding integer m . By Lemma 6.7, there exists an increasing sequence $(i_n)_{n \in \mathbb{N}}$ of integers such that for all $n \in \mathbb{N}$,

$$\sum_{j \in \mathcal{E}_m^{i_n}(A)} \mathcal{H}^\alpha(A \cap I_j^{i_n}) \leq \frac{1}{2^n}.$$

Fix $n_0 \in \mathbb{N}$, define $\mathcal{F}_{n_0} = \emptyset$ and iteratively for $n > n_0$

$$\mathcal{F}_n = \left\{ j \in \mathbb{Z}^N \mid (i_n, j) \in \text{Adm}, h^{-i_n s \alpha} \mathcal{H}^\alpha(g(A \cap I_j^{i_n})) < c, \text{ and } I_j^{i_n} \cap \left(\bigcup_{n'=n_0}^{n-1} \bigcup_{j' \in \mathcal{F}_{n'}} I_{j'}^{i_{n'}} \right) = \emptyset \right\}.$$

Now we set

$$B = A \setminus \bigcup_{n=n_0}^{\infty} \left(\bigcup_{\substack{j \in \mathcal{E}_m^{i_n}(A) \\ (i_n, j) \in \text{Adm}}} I_j^{i_n} \cup \bigcup_{j \in \mathcal{F}_n} I_j^{i_n} \right).$$

Let us prove that $\mathcal{H}^\alpha(B) = 0$. Suppose that $\mathcal{H}^\alpha(B) > 0$. Choose a point of density $x \in B$. We prove that there exists $\varepsilon > 0$ and a sequence $(r_n)_{n \in \mathbb{N}}$ such that $r_n \rightarrow 0$ and for all $n \in \mathbb{N}$,

$$\frac{\mathcal{H}^\alpha(B \cap B(x, r_n))}{\mathcal{H}^\alpha(B(x, r_n))} \leq 1 - \varepsilon, \tag{6.4}$$

which contradicts the fact that x is a point of density. Let $r_n = 2N h^{(i_n-1)s}$. By definition of r_n ,

$$\prod_{k=1}^N [x_k - h^{(i_n-1)s/s_k}, x_k + h^{(i_n-1)s/s_k}] \subset \overline{B(x, r_n/2)},$$

thus if we set $j_n = (\lfloor x_1/h^{(i_n-1)s/s_1} \rfloor, \dots, \lfloor x_N/h^{(i_n-1)s/s_N} \rfloor)$, then

$$I_{j_n}^{i_n-1} \subset \prod_{k=1}^N [x_k - h^{(i_n-1)s/s_k}, x_k + h^{(i_n-1)s/s_k}] \subset \overline{B(x, r_n/2)}.$$

For all $n \in \mathbb{N}$, there exists $j'_n \in \mathbb{Z}^N$ such that $(i_n, j'_n) \in \text{Adm}$ and $I_{j'_n}^{i_n} \subset I_{j_n}^{i_n-1}$. Now there are two cases: either $j'_n \in \mathcal{E}_m^{i_n}(A)$ or $h^{-i_n s \alpha} \mathcal{H}^\alpha(g(A \cap I_{j'_n}^{i_n})) < c$.

1. If $j'_n \in \mathcal{E}_m^{i_n}(A)$, then $B \cap I_{j'_n}^{i_n} = \emptyset$. Thus, $B \cap B(x, r_n) = B \cap (B(x, r_n) \setminus I_{j'_n}^{i_n})$, and we can write

$$\frac{\mathcal{H}^\alpha(B \cap B(x, r_n))}{\mathcal{H}^\alpha(B(x, r_n))} \leq \frac{\mathcal{H}^\alpha(B(x, r_n) \setminus I_{j'_n}^{i_n})}{\mathcal{H}^\alpha(B(x, r_n))} \leq 1 - \frac{\mathcal{H}^\alpha(I_{j'_n}^{i_n})}{C r_n^\alpha} = 1 - h^{s\alpha} \frac{\mathcal{H}^\alpha([0, 1]^N)}{C(2N)^\alpha},$$

where C is a constant given by Ahlfors regularity. This proves (6.4) in this case.

2. If $h^{-i_n s \alpha} \mathcal{H}^\alpha(g(A \cap I_{j'_n}^{i_n})) < c$, then either $j'_n \in \mathcal{F}_n$ or $j'_n \notin \mathcal{F}_n$. If $j'_n \in \mathcal{F}_n$, then $B \cap I_{j'_n}^{i_n} = \emptyset$ and we conclude as in the first case. If $j'_n \notin \mathcal{F}_n$, then

$$I_{j'_n}^{i_n} \cap \left(\bigcup_{n'=n_0}^{n-1} \bigcup_{j' \in \mathcal{F}_{n'}} I_{j'}^{i_{n'}} \right) \neq \emptyset.$$

Thus, there exists $n' \in \{n_0, \dots, n-1\}$ and $j' \in \mathcal{F}_{n'}$ such that $I_{j'_n}^{i_n} \cap I_{j'}^{i_{n'}} \neq \emptyset$. By construction, $B \cap I_{j'}^{i_{n'}} = \emptyset$. If one translates the cube $I_{j'_n}^{i_n}$ so that the image lies in $I_{j'}^{i_{n'}}$ and intersects $I_{j'_n}^{i_n}$, then the translated cube is in $B(x, r_n)$ but not in B , and has the same measure that $I_{j'_n}^{i_n}$, so the same estimate as in the first case holds (see Figure 5).

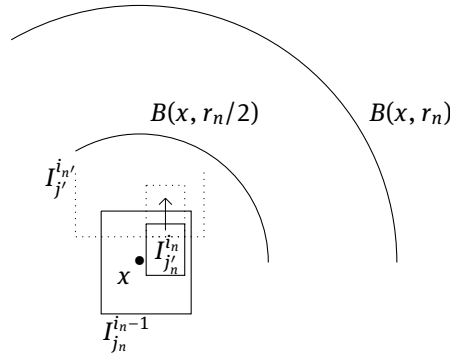


Figure 5: Construction of a cube isometric to $I_{j_n}^{i_n}$ that does not intersect $B \cap B(x, r_n)$.

Thus, $\mathcal{H}^\alpha(B) = 0$, and since g is Lipschitz, $\mathcal{H}^\alpha(g(B)) = 0$. With these we can write

$$\begin{aligned} \mathcal{H}^\alpha(g(A)) &\leq \mathcal{H}^\alpha(g(A \setminus B)) + \mathcal{H}^\alpha(g(A \cap B)) \\ &\leq \sum_{n=n_0}^{\infty} \left(\sum_{\substack{j \in \mathcal{E}_m^{i_n}(A) \\ (i_n, j) \in \text{Adm}}} \mathcal{H}^\alpha(g(A \cap I_j^{i_n})) + \sum_{j \in \mathcal{F}_n} \mathcal{H}^\alpha(g(A \cap I_j^{i_n})) \right) \\ &\leq \sum_{n=n_0}^{\infty} \left(\sum_{\substack{j \in \mathcal{E}_m^{i_n}(A) \\ (i_n, j) \in \text{Adm}}} L^\alpha \mathcal{H}^\alpha(A \cap I_j^{i_n}) + c \sum_{j \in \mathcal{F}_n} h^{i_n s \alpha} \right) \\ &\leq \sum_{n=n_0}^{\infty} \left(L^\alpha \sum_{j \in \mathcal{E}_m^{i_n}(A)} \mathcal{H}^\alpha(A \cap I_j^{i_n}) + c \sum_{j \in \mathcal{F}_n} \frac{\mathcal{H}^\alpha(I_j^{i_n})}{\mathcal{H}^\alpha([0, 1]^N)} \right) \\ &\leq L^\alpha \sum_{n=n_0}^{\infty} \frac{1}{2^n} + \frac{c}{\mathcal{H}^\alpha([0, 1]^N)} \sum_{n=n_0}^{\infty} \sum_{j \in \mathcal{F}_n} \mathcal{H}^\alpha(I_j^{i_n}) \\ &\leq L^\alpha \frac{1}{2^{n_0-1}} + c \frac{\mathcal{H}^\alpha([0, 1]^N)}{\mathcal{H}^\alpha([0, 1]^N)}, \end{aligned}$$

the last inequality is true since the cubes $(I_j^{i_n})_{n \geq n_0, j \in \mathcal{F}_n}$ have disjoint interiors by construction. Since the last inequality is true for all $c > 0$ and all $n_0 \in \mathbb{N}$, then $\mathcal{H}^\alpha(g(A)) = 0$, which contradicts the assumption on A . \square

Until the end of this section, we fix a constant $c > 0$ and a sequence $\{(i_m, j_m) \in \text{Adm}, m \in \mathbb{N}\}$ given by Proposition 6.4. For all $m \in \mathbb{N}$, set $f_m = f_{i_m, j_m}$ and $K_m = f_m^{-1}(A \cap I_{j_m}^{i_m})$. For all $m \in \mathbb{N}$, K_m is a compact of $[0, 1]^N$.

Lemma 6.8. *There exists a subsequence of $(K_m)_{m \in \mathbb{N}}$ that converges to $[0, 1]^N$ in the Hausdorff distance.*

Proof. By Proposition 6.4,

$$\mathcal{H}^\alpha(K_m) = h^{-i_m s \alpha} \mathcal{H}^\alpha(A \cap I_{j_m}^{i_m}) = \mathcal{H}^\alpha([0, 1]^N) \frac{\mathcal{H}^\alpha(A \cap I_{j_m}^{i_m})}{\mathcal{H}^\alpha(I_{j_m}^{i_m})} \xrightarrow{m \rightarrow \infty} \mathcal{H}^\alpha([0, 1]^N). \tag{6.5}$$

For all $m \in \mathbb{N}$, K_m is a compact of $[0, 1]^N$, so by the Blaschke Theorem, we can suppose that $K_m \xrightarrow{m \rightarrow \infty} K$. By (6.5) and Lemma 2.8, $\mathcal{H}^\alpha(K) = \mathcal{H}^\alpha([0, 1]^N)$. Then, $K = [0, 1]^N$, because a compact set strictly contained in $[0, 1]^N$ has Hausdorff α -measure strictly less than $\mathcal{H}^\alpha([0, 1]^N)$. \square

We can now prove Theorem 6.2.

Proof of Theorem 6.2. The proof is pretty much the same as the proof of Ascoli-Arzelà Theorem. Let E be a dense subset of $[0, 1]^N$. By Proposition 2.9, for all $x \in E$, there exists a sequence (x_m) such that for all $m \in \mathbb{N}$, $x_m \in K_m$, and $d_s(x_m, x) \rightarrow 0$. For all $x \in E$, we fix a sequence as above.

For all $m \in \mathbb{N}$, we set $g_m = g_{i_m, j_m}$ and $z_m = z_{i_m, j_m}$ the chosen point in $A \cap I_{j_m}^{i_m}$. We now prove that there exists a compact K such that for all $m \in \mathbb{N}$, $g_m(K_m) \subset K$. Let $m \in \mathbb{N}$ and $x \in K_m$. Then,

$$\begin{aligned} d_s(g_m(x), 0) &\leq h^{-i_m s} d_s(g(f_m(x)) - g(z_m), 0) \\ &\leq h^{-i_m s} d_s(g(f_m(x)), g(z_m)) \\ &\leq L h^{-i_m s} d_{s, \mathbb{R}}(f_m(x), z_m) \\ &\leq L h^{-i_m s} d_s(f_m(x), z_m) \\ &\leq L h^{-i_m s} \text{diam}_{d_s} I_{j_m}^{i_m}, \end{aligned}$$

but $\text{diam}_{d_s} I_{j_m}^{i_m} = N h^{i_m s}$; then $d_s(g_m(x), 0) \leq LN$. Thus, $g_m(K_m) \subset \overline{B}(0, LN)$, which is compact. The usual Cantor diagonalization argument allows us to choose a subsequence (that we will still denote by m) such that $(g_m(x_m))_{m \in \mathbb{N}}$ converges (for the distance d_s) for all $x \in E$ (that is, for all the sequences (x_m) fixed above). Denote $f(x)$ the limit. The map $f : (E, d_{s, \mathbb{R}}) \rightarrow (\mathbb{R}^N, d_s)$ is Lipschitz. In fact, for all $x, y \in E$, let $(x_m), (y_m)$ the two fixed sequences converging to x, y for the distance d_s . Then, for all $m \in \mathbb{N}$,

$$\begin{aligned} d_{s, \mathbb{R}}(f(x), f(y)) &\leq d_{s, \mathbb{R}}(f(x), g_m(x_m)) + d_{s, \mathbb{R}}(g_m(x_m), g_m(y_m)) + d_{s, \mathbb{R}}(g_m(y_m), f(y)) \\ &\leq d_s(f(x), g_m(x_m)) + L d_s(x_m, y_m) + d_s(g_m(y_m), f(y)) \\ &\leq d_s(f(x), g_m(x_m)) + L(d_s(x_m, x) + d_s(x, y) + d_s(y, y_m)) + d_s(g_m(y_m), f(y)) \\ &\leq L d_s(x, y). \end{aligned}$$

By a standard argument, there is a unique extension of f (that we still denote by f) defined on $[0, 1]^N$ such that $f : ([0, 1]^N, d_{s, \mathbb{R}}) \rightarrow (\mathbb{R}^N, d_s)$ is Lipschitz. Moreover, (g_m) "converges pointwise" in the following sense: for all $x \in [0, 1]^N$ and for all sequences (x_m) (not only the ones fixed above) such that $x_m \in K_m$ and $d_{s, \mathbb{R}}(x_m, x) \xrightarrow{m \rightarrow \infty} 0$, we have $d_s(g_m(x_m), f(x)) \xrightarrow{m \rightarrow \infty} 0$. In fact, by density of E , for all $p \in \mathbb{N}$, there exists $x^{(p)} \in E$ such that $d_s(x^{(p)}, x) \leq 1/2^p$. For all p , denote by $(x_m^{(p)})_{m \in \mathbb{N}}$ the sequence converging to $x^{(p)}$ that we choose above. Then,

$$\begin{aligned} d_s(g_m(x_m), f(x)) &\leq d_s(g_m(x_m), g_m(x_m^{(p)})) + d_s(g_m(x_m^{(p)}), f(x^{(p)})) + d_s(f(x^{(p)}), f(x)) \\ &\leq L d_{s, \mathbb{R}}(x_m, x_m^{(p)}) + d_s(g_m(x_m^{(p)}), f(x^{(p)})) + L d_{s, \mathbb{R}}(x^{(p)}, x) \end{aligned}$$

and the result holds by letting $m \rightarrow \infty$ and then $p \rightarrow \infty$.

We now want to prove that $\mathcal{H}^\alpha(f([0, 1]^N)) > 0$. Recalling that

$$g_m(x) = \text{dil}_{h^{-i_m s}}(g(f_m(x)) - g(z_m)),$$

we get by Proposition 6.4, that

$$\mathcal{H}^\alpha(g_m(K_m)) = h^{-i_m s \alpha} \mathcal{H}^\alpha(g(f_m(K_m))) = h^{-i_m s \alpha} \mathcal{H}^\alpha(g(A \cap I_{j_m}^{i_m})) \geq c.$$

Therefore, by Lemma 2.8 it is sufficient to prove that $d_H(g_m(K_m), f([0, 1]^N)) \xrightarrow{m \rightarrow \infty} 0$, because then

$$0 < c \leq \limsup_{m \rightarrow \infty} \mathcal{H}^\alpha(g_m(K_m)) \leq \mathcal{H}^\alpha(f([0, 1]^N)).$$

Let us remark that since $g_m(K_m) \subset \overline{B}(0, Ln)$, then $f([0, 1]^N) \subset \overline{B}(0, LN)$ by pointwise convergence, thus we can use the equivalence in Proposition 2.9 to prove that $g_m(K_m) \xrightarrow{H} f([0, 1]^N)$. The first point of Proposition 2.9 is true by pointwise convergence of g_m to f . For the second point, let $y = \lim_{k \rightarrow \infty} g_{m_k}(x_{m_k})$ where, $\{x_m\}$ is a sequence such that for all $m \in \mathbb{N}$, $x_m \in K_m$. We want to prove that $y \in f([0, 1]^N)$. Without loss of generality, since $K_m \subset [0, 1]^N$ we can suppose that $x_{m_k} \xrightarrow{k \rightarrow \infty} x \in [0, 1]^N$. Then, $g_{m_k}(x_{m_k}) \rightarrow f(x) = y$ by pointwise convergence.

This proves that $d_H(g_m(K_m), f([0, 1]^N)) \xrightarrow{m \rightarrow \infty} 0$, and thus $\mathcal{H}^\alpha(f([0, 1]^N)) \geq c > 0$. \square

7 (\mathbb{R}^N, d_s) is not minimal for looking down

We know that (\mathbb{R}^N, d_s) looks down on $([0, 1]^N/d_{s,\mathcal{R}}, \overline{d}_{s,\mathcal{R}})$, since $d_{s,\mathcal{R}} \leq d_s$ and the projection $\pi : ([0, 1]^N, d_{s,\mathcal{R}}) \rightarrow ([0, 1]^N/d_{s,\mathcal{R}}, \overline{d}_{s,\mathcal{R}})$ is an isometry.

Proposition 7.1. *A Lipschitz map $f : ([0, 1]^N, d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ satisfies $\mathcal{H}^\alpha(f([0, 1]^N)) = 0$.*

Proof. Let $x, y \in \mathbb{R}^N$ and $i \in L$. Let π_i be the canonical projection on the i -th coordinate. We denote by $\gamma : [1, N] \rightarrow \mathbb{R}^N$ the following piecewise linear curve: if $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ then for all $k \in \{1, \dots, N\}$

$$\gamma|_{[k,k+1]}(t) = (y_1, \dots, y_{k-1}, x_k + (t-k)(y_k - x_k), x_{k+1}, \dots, x_n).$$

γ is a geodesic between x and y with the distance induced by the classical norm $\|\cdot\|_1$ in \mathbb{R}^N . For all $k \in \{1, \dots, N\}$ the map

$$\begin{aligned} \varphi_k : ([0, 1], d_k) &\longrightarrow (\mathbb{R}, |\cdot|^s) \\ t &\longmapsto (\pi_i \circ f)(\gamma(k+t)) \end{aligned}$$

is a Lipschitz map since $\gamma|_{[k,k+1]}$ and π_i are Lipschitz. Here, $d_k = |\cdot|^s$ if $k \notin L$, and $d_k = d_{\mathcal{R}}$ if $k \in L$. The case $k \notin L$ implies by a standard argument that φ_k is constant, since $s < s_k$. The case $k \in L$ also implies that φ_k is constant by Proposition 5.1. We conclude that $\pi_i \circ f$ is constant along the curve γ and thus $\pi_i(f(x)) = \pi_i(f(y))$, which proves that $\mathcal{H}^\alpha(f([0, 1]^N)) = 0$. \square

Suppose that $([0, 1]^N/d_{s,\mathcal{R}}, \overline{d}_{s,\mathcal{R}})$ looks down on (\mathbb{R}^N, d_s) . Then, there exist a closed subset $A \subset [0, 1]^N/d_{s,\mathcal{R}}$ and an L -Lipschitz map $g : (A, \overline{d}_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ such that $\mathcal{H}_{d_s}^\alpha(g(A)) > 0$. The map $\tilde{g} = g \circ \pi : (\pi^{-1}(A), d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ is Lipschitz, $\mathcal{H}_{d_s}^\alpha(\tilde{g}(A)) > 0$ and $\pi^{-1}(A)$ is compact. Thus, we can apply Theorem 6.2. There exists a Lipschitz map $f : ([0, 1]^N, d_{s,\mathcal{R}}) \rightarrow (\mathbb{R}^N, d_s)$ such that $\mathcal{H}^\alpha(f([0, 1]^N)) > 0$, which contradicts Proposition 7.1.

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